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Locally Non-compact Spaces and Continuity Principles

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Abstract

We give a constructive proof that Baire space embeds in any inhabited locally non-compact complete separable metric space, X, in such a way that every sequentially continuous function from Baire space to $\mathbb{Z}$ extends to a function from X to $\mathbb{R}$. As an application, we show that in the presence of certain choice and continuity principles, the statement "all functions from X to $\mathbb{R}$ is continuous" is false. This generalizes a result previously obtained by Ecardó and Streicher, in the context of "domain realizability", for the special case $X = \mathcal{C}[0,1]$.

1 Introduction

In a recent paper, [ES02], Escardó and Streicher analyse *continuity principles* in the context of so-called *domain realizability*, i.e. in realizability toposes constructed over domain-theoretic models of the untyped λ-calculus. In such models, the internal statement "all functions from Baire space to $\mathbb{N}$ are continuous" is known to be false (even though externally all morphisms from Baire space to $\mathbb{N}$ *are* continuous), because it conflicts with choice principles valid in the models. Escardó and Streicher show that, similarly, the internal statement "all functions from $\mathcal{C}[0,1]$ to $\mathbb{R}$ are continuous" is false. (Once again, externally, all morphisms from $\mathcal{C}[0,1]$ to $\mathbb{R}$ are continuous.) Their proof exploits specific features of the space $\mathcal{C}[0,1]$, and requires a concrete analysis of the nature of "realizers" of certain functions in the model. In this paper, we show instead how it is possible to derive the failure of continuity principles, for a wide range of analytic spaces, directly from the known failure for Baire space.

Working within the context of constructive mathematics [Bis67, BB85], we identify a property of complete separable metric spaces (CSMs) which we call *local non-compactness*. Our main result, Theorem 2.3, states that Baire space, which is itself locally non-compact, embeds in any inhabited locally non-compact CSM, X, in such a way that every sequentially continuous function from Baire space to $\mathbb{Z}$ extends to a function from X to $\mathbb{R}$. This result is proved in Section 3.

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In Section 4, we apply Theorem 2.3 to derive Escardó and Streicher's result that "all functions from $\mathcal{C}[0,1]$ to $\mathbb{R}$ are continuous" is false in domain realizability [ES02]. This is a simple consequence of the known result for Baire space, together with the fact that $\mathcal{C}[0,1]$ is easily shown to be locally non-compact. Furthermore, our approach establishes a more general result that for any inhabited locally non-compact X, the statement "all functions from X to $\mathbb{R}$ are continuous" is false in any topos in which certain choice and continuity principles are valid.

We believe that Theorem 2.3 may have other applications in computable and constructive analysis. Indeed, it may provide a useful general tool for establishing that properties of Baire space find themselves reflected in analogous properties of other locally non-compact spaces.

2 Locally non-compact metric spaces

Following Bishop [Bis67, BB85], we do mathematics using intuitionistic logic, and we assume $\mathsf{AC}_{0,0}$: the axiom of choice for properties $\forall x \in \mathbb{N}. \exists y \in \mathbb{N}. \varphi$. We shall not need dependent choice. For the development that follows, it does not matter whether real numbers are taken to be Cauchy sequences of rationals, with equality as an equivalence relation over them, or whether real numbers are taken to be equivalence classes of Cauchy sequences. The former is Bishop's approach to real numbers, the latter is the natural approach when reasoning in the internal logic of an elementary topos, where, because we assume $\mathsf{AC}_{0,0}$, the object $\mathbb{R}$ of equivalence classes of Cauchy sequences is isomorphic to the favoured object of Dedekind reals.

We assume familiarity with the constructive notions of metric space, Cauchy sequence and convergence. Because we consider several notions of continuity, we spell out each one of them. A function $f : X \to Y$ between metric spaces is:

- *uniformly continuous* when for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $x, x' \in X$, if $d(x, x') < \delta$ then $d(f(x), f(x')) < \varepsilon$.
- *pointwise continuous at* $x \in X$ when for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for all $x' \in X$, $d(x, x') < \delta$ implies $d(f(x), f(x')) < \varepsilon$. A function which is pointwise continuous at every point is *pointwise continuous.*
- *sequentially continuous* when it preserves limits of convergent sequences: if $\langle a_i \rangle_{i \in \mathbb{N}}$ converges to a in X then $\langle f(a_i) \rangle_{i \in \mathbb{N}}$ converges to $f(a)$ in Y.

For a metric space (X, d), we write $B(x, r)$ for the *open ball* centered at $x \in X$ with radius $r > 0$, and $\overline{B}(x, r)$ for the *closed ball.* We say that (X, d) is *separable* if it contains a countable dense subspace; and that it is *complete* if every Cauchy sequence converges. As is customary we abbreviate *complete separable metric space* as *CSM.*

In Section 3 we will need the "cone" and "hill" functions, which we define now. For a metric space X, $x \in X$, and $0 < q < r$ let $\mathsf{cone}(x, r) : X \to \mathbb{R}$ and $\mathsf{hill}(x, r, q) : X \to \mathbb{R}$ be defined as

$$\begin{aligned} \mathsf{cone}(x, r)(y) &= \max(0, 1 - r^{-1} \cdot d(x, y))) \,, \\ \mathsf{hill}(x, q, r)(y) &= \max(0, 1 - (r - q)^{-1} \cdot \max(0, d(x, y) - q)) \,. \end{aligned}$$

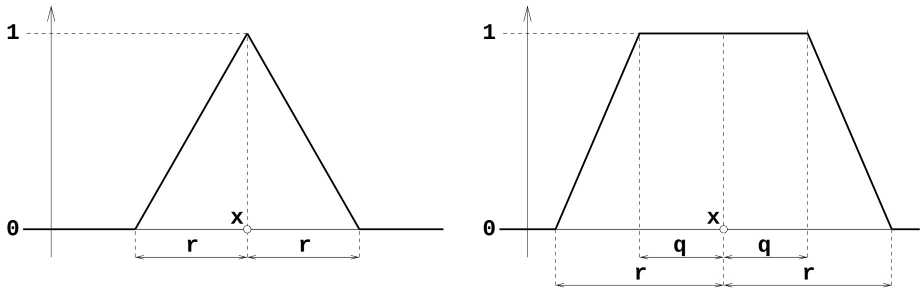


Figure 1: Graphs of $\mathsf{cone}(x, r)$ and $\mathsf{hill}(x, q, r)$

See Figure 1 for a picture of a cone and a hill.

We next define the concepts needed to formulate our main result, Theorem 2.3 below.

Definition 2.1 A sequence *without accumulation point* in a metric space (X, d) is a sequence $\langle a_i \rangle_{i \in \mathbb{N}}$ with the property that for every $x \in X$ there exist $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $d(x, a_i) > \varepsilon$ for all $i \geq m$.

Definition 2.2 A metric space (X, d) is *locally non-compact at* $x \in X$ if for every $\varepsilon > 0$ the open ball $B(x, \varepsilon)$ contains a sequence without accumulation point in X. It is *locally non-compact* if it is locally non-compact at every x.

Any infinite-dimensional separable Hilbert space is locally non-compact CSM, for example the space ℓ^2 of square-summable sequences; or the space $\mathcal{C}_\mathsf{u}[0, 1]$ of uniformly continuous maps $[0, 1] \to \mathbb{R}$, equipped with the supremum norm. The latter example generalizes as follows. An *ε-net* in a metric space X is a finite subset $N \subseteq X$ such that for every $x \in X$ there exists $y \in N$ for which $d(x, y) < \varepsilon$. A CSM is said to be *complete totally bounded (CTB)* if it has an ε-net for every $\varepsilon > 0$. An *injective sequence* $\langle a_i \rangle_{i \in \mathbb{N}}$ in X is a sequence for which $d(a_n, a_m) > 0$ whenever $n \neq m$. For any CTB space X containing a convergent injective sequence, it is straightforward to show that $\mathcal{C}_\mathsf{u}(X)$ is locally non-compact.

Another important example of a locally non-compact CSM is the space $\mathbb{R}^\mathbb{N}$ of infinite sequences of real numbers with metric

$$d(x, y) = \sum_{k=0}^{\infty} \min(1, |x_k - y_k|) \cdot 2^{-k}.$$

Baire space, which is also a locally non-compact CSM, can be defined as the subspace $\mathbb{Z}^\mathbb{N}$ of $\mathbb{R}^\mathbb{N}$.

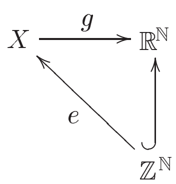
In the presence of Church's Thesis CT_0 [TvD88a, 4.3] the closed interval $[0, 1]$ gives a surprising example of a locally non-compact space. This is because CT_0 implies the existence of *strong Specker sequencess* [TvD88a, 6.4.7], which are nothing but bounded monotone sequences of reals without accumulation point. This example shows that it is possible for a CSM to be simultaneously CTB and locally non-compact.

Theorem 2.3 *If X is inhabited and locally non-compact then there exists a uniformly continuous embedding $e : \mathbb{Z}^{\mathbb{N}} \to X$ with the property that, for every sequentially continuous $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$, there exists a function $\overline{f} : X \to \mathbb{R}$ such that $f = \overline{f} \circ e$.*

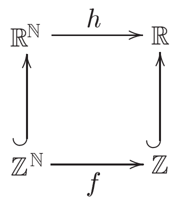
3 The Proof of Theorem 2.3

In this section we prove Theorem 2.3. Throughout we assume that X is an inhabited locally non-compact CSM with a countable dense subset $S \subseteq X$. The proof consists of two parts, which are stated in the following two propositions.

Proposition 3.1 *There exists a uniformly continuous embedding $e : \mathbb{Z}^{\mathbb{N}} \to X$ and a pointwise continuous map $g : X \to \mathbb{R}^{\mathbb{N}}$ such that the following diagram commutes.*



Proposition 3.2 *For every sequentially continuous $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ there exists a function $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ such that the following diagram commutes.*



Theorem 2.3 follows immediately from Propositions 3.1 and 3.2, because the map $\overline{f} = h \circ g$ is an extension of f along e.

3.1 Proof of Proposition 3.1

We begin by proving several lemmas that are needed for the proof of Proposition 3.1.

Lemma 3.3 *A sequence that has no accumulation points has an injective subsequence, which has no accumulation points.*

Proof. Suppose $\langle a_i \rangle_{i \in \mathbb{N}}$ is a sequence without accumulation points. By $\mathsf{AC}_{0,0}$ there is a choice function $c : \mathbb{N} \to \mathbb{N}$ which chooses for each $n \in \mathbb{N}$ some $c(n) > n$ such that there exists $\varepsilon > 0$ for which $d(a_n, a_m) > \varepsilon$ for all $m \geq c(n)$. Now the subsequence $\langle a_{c^n(0)} \rangle_{n \in \mathbb{N}}$ is injective, and it has no accumulation points because it is a subsequence of $\langle a_i \rangle_{i \in \mathbb{N}}$. ∎

Henceforth we assume that all sequences without accumulation point are injective.

Lemma 3.4 *If $\langle a_i \rangle_{i\in\mathbb{N}}$ is a sequence without accumulation point and $\langle b_i \rangle_{i\in\mathbb{N}}$ is a sequence satisfying $\lim_{i\to\infty} d(a_i, b_i) = 0$ then $\langle b_i \rangle_{i\in\mathbb{N}}$ is without accumulation point as well.*

Proof. Consider an arbitrary $x \in X$. There exists $\varepsilon > 0$ and $m \in \mathbb{N}$ such that $d(x, a_i) > \varepsilon$ for all $i \geq m$. There exists $n \in \mathbb{N}$ such that $d(a_i, b_i) < \varepsilon/2$ for all $i \geq n$. Then for all $i \geq \max(m, n)$ we have $d(x, b_i) \geq d(x, a_i) - d(a_i, b_i) > \varepsilon/2$. ∎

Lemma 3.5 *For every $x \in X$ and $\varepsilon > 0$ the open ball $B(x, \varepsilon)$ contains a sequence in S without an accumulation point in X.*

Proof. There exists a sequence $\langle a_i \rangle_{i\in\mathbb{N}}$ without accumulation point that lies in $B(x, \varepsilon/2)$. By $\mathsf{AC}_{0,0}$ there exists a sequence $\langle b_i \rangle_{i\in\mathbb{N}}$ in S such that $d(a_i, b_i) < \varepsilon \cdot 2^{-i-1}$ for every $i \in \mathbb{N}$. By Lemma 3.4 the sequence $\langle b_i \rangle_{i\in\mathbb{N}}$ is without accumulation point. It is contained in $B(x, \varepsilon)$ because $d(x, b_i) \leq d(x, a_i) + d(a_i, b_i) < \varepsilon/2 + \varepsilon \cdot 2^{-i-1} \leq \varepsilon$. ∎

Lemma 3.6 *Suppose $\langle w_i \rangle_{i\in\mathbb{N}}$ is a sequence without accumulation point. There exists a sequence $\langle \xi_i \rangle_{i\in\mathbb{N}}$ of positive real numbers such that $d(w_i, w_j) > 2(\xi_i + \xi_j)$ whenever $i \neq j$.*

Proof. By $\mathsf{AC}_{0,0}$ there exists a sequence $\langle m_i \rangle_{i\in\mathbb{N}}$ of natural numbers and a sequence $\langle \zeta_i \rangle_{i\in\mathbb{N}}$ of positive real numbers such that $d(w_i, w_j) > \zeta_i$ for all $j \geq m_i$. Let $\xi_i = \min(\{\zeta_i\} \cup \{d(w_i, w_j) \mid i \neq j \wedge j \leq m_i\})/5$. Now consider any distinct $i, j \in \mathbb{N}$. If $j \geq m_i$ then $d(w_i, w_j) > \zeta_i \geq 5\xi_i$, and otherwise $d(w_i, w_j) \geq 5\xi_i$. Similarly it follows that $d(w_i, w_j) \geq 5\xi_j$, therefore $d(w_i, w_j) \geq 5(\xi_i + \xi_j)/2 > 2(\xi_i + \xi_j)$. ∎

Lemma 3.7 *For any $v \in X$ and $\eta > 0$ there exists a sequence $\langle \varepsilon_i \rangle_{i\in\mathbb{Z}}$ of positive real numbers, and a sequence $\langle v_i \rangle_{i\in\mathbb{Z}}$ in S without accumulation point in X such that, for all $i, j \in \mathbb{Z}$:*

1. *$d(v, v_i) < \eta/3$,*
2. *$\varepsilon_i < \eta/3$ and $\varepsilon_i < 2^{-|i|}$,*
3. *$d(v_i, v_j) > 2(\varepsilon_i + \varepsilon_j)$ unless $i = j$,*
4. *for all $x \in X$, there exists a unique $k \in \mathbb{Z}$ such that $d(x, v_k) < 2\varepsilon_k$, or $d(x, v_i) > \varepsilon_i$ for all $i \in \mathbb{Z}$.*

Furthermore, we may assume that for every $i \in \mathbb{Z}$ there is $p \in \mathbb{N}$ such that $\varepsilon_i = 2^{-p}$.

Proof. Since X is locally non-compact at v, the ball $B(v, \eta/3)$ contains a sequence $\langle v_i \rangle_{i\in\mathbb{Z}}$ in S without accumulation point, as was proved in Lemma 3.5. Clearly it is the case that $d(v, v_i) < \eta/3$. By Lemma 3.6 there exists a sequence $\langle \xi_i \rangle_{i\in\mathbb{Z}}$ such that $d(v_i, v_j) > 2(\xi_i + \xi_j)$. If we set $\varepsilon_i = \min(\xi_i, \eta/3, 2^{-|i|+1})$ then the second condition is satisfied, as well as the third one because $i \neq j$ implies $d(v_i, v_j) > 2(\xi_i + \xi_j) \geq 2(\varepsilon_i + \varepsilon_j)$.

To see that the fourth requirement is satisfied, consider any $x \in X$. There exists $\zeta > 0$ and $n_1 \in \mathbb{N}$ such that $d(x, v_i) > \zeta$ whenever $|i| \geq n_1$. Because $\varepsilon_i < 2^{-|i|}$ there exists $n_2 \in \mathbb{N}$ such that $\varepsilon_i < \zeta$ whenever $|i| \geq n_2$. Define $n = \max(n_1, n_2)$ and observe that $|i| \geq n$ implies $d(x, v_i) > \zeta > \varepsilon_i$. For every $i \in \mathbb{Z}$ satisfying $|i| < n$, $d(x, v_i) > \varepsilon_i$ or $d(x, v_i) < 2\varepsilon_i$. Therefore, there exists $j \in \mathbb{Z}$ satisfying $|j| < n$ and $d(x, v_j) < 2\varepsilon_j$, or $d(x, i) > \varepsilon_i$ for all $i \in \mathbb{Z}$ with $|i| < n$. If the second case holds, then we may conclude that $d(x, v_i) > \varepsilon_i$ for all $i \in \mathbb{Z}$. In the first case, there is a unique j for which $d(x, v_j) < 2\varepsilon_j$, since another such j' implies $d(v_j, v_{j'}) < d(x, v_j) + d(x, v_{j'}) \leq 2(\varepsilon_j + \varepsilon_{j'})$, which contradicts $d(v_j, v_{j'}) > 2(\varepsilon_j + \varepsilon_{j'})$ unless $j = j'$.

Finally, observe that the lemma still holds if we make all the ε_i's smaller. By $\mathsf{AC}_{0,0}$, for each $i \in \mathbb{Z}$ there exists $p_i \in \mathbb{N}$ such that $2^{-p_i} < \varepsilon_i$. We may replace $\langle \varepsilon_i \rangle_{i\in\mathbb{Z}}$ with $\langle 2^{-p_i} \rangle_{i\in\mathbb{Z}}$. ∎

Let $\mathbb{Z}^*$ be the set of finite sequences of integers. If $a \in \mathbb{Z}^*$ and $j \in \mathbb{Z}$, we write aj for the sequence a followed by j. The empty sequence is denoted by $[\,]$ and the length of a is denoted by $|a|$.

Lemma 3.8 *There exist a family $\langle \delta(a) \rangle_{a\in\mathbb{Z}^*}$ of positive real numbers and a family $\langle w(a) \rangle_{a\in\mathbb{Z}^*}$ in S without accumulation point in X such that, for all $a \in \mathbb{Z}^*$ and $i, j \in \mathbb{Z}$:*

1. $d(w(a), w(aj)) < \delta(a)/3$,
2. $\delta(aj) < \delta(a)/3$ *and* $\delta(aj) < 2^{-|j|}$,
3. $d(w(ai), w(aj)) > 2(\delta(ai) + \delta(aj))$ *unless* $i = j$,
4. *for any $x \in X$, there exists a unique $k \in \mathbb{Z}$ such that $d(w(ak), x) < 2\delta(ak)$, or $d(w(ai), x) > \delta(ai)$ for all $i \in \mathbb{Z}$.*

Proof. The sort of proof that comes to mind first, namely an inductive construction of $\langle \delta(a) \rangle_{a\in\mathbb{Z}^*}$ and $\langle w(a) \rangle_{a\in\mathbb{Z}^*}$ by successive applications of Lemma 3.7, uses dependent choice. We present a more careful proof that relies on separability of X and only requires $\mathsf{AC}_{0,0}$.

Let $\langle s_i \rangle_{i\in\mathbb{N}}$ be an enumeration of S. Because $\mathsf{AC}_{0,0}$ implies choice from $\mathbb{N}\times\mathbb{N}$ to $\mathbb{N}^{\mathbb{Z}}$ there exist choice functions $c_\delta : \mathbb{N}\times\mathbb{N} \to \mathbb{N}^{\mathbb{Z}}$ and $c_w : \mathbb{N}\times\mathbb{N} \to \mathbb{N}^{\mathbb{Z}}$, such that for all $m, n \in \mathbb{N}$ the conditions of Lemma 3.7 are satisfied if we take $v = s_m$, $\eta = 2^{-n}$, $\varepsilon_i = 2^{-c_\delta(m,n)(i)}$, and $v_i = s_{c_w(m,n)(i)}$. Define mutually recursive functions $\kappa : \mathbb{Z}^* \to \mathbb{N}$ and $\lambda : \mathbb{Z}^* \to \mathbb{N}$ by

$$\begin{aligned} \kappa([\,]) &= 0\,, & \kappa(aj) &= c_w(\kappa(a), \lambda(a))(j)\,,\\ \lambda([\,]) &= 0\,, & \lambda(aj) &= c_\delta(\kappa(a), \lambda(a))(j)\,. \end{aligned}$$

Now for $a \in \mathbb{Z}^*$ let $w(a) = s_{\kappa(a)}$ and $\delta(a) = 2^{-\lambda(a)}$. The desired properties of $\langle \delta(a) \rangle_{a\in\mathbb{Z}^*}$ and $\langle w(a) \rangle_{a\in\mathbb{Z}^*}$ follow directly from Lemma 3.7 and the definition of κ and λ. ∎

Lemma 3.9 *Let $\langle\delta(a)\rangle_{a\in\mathbb{Z}^*}$ and $\langle w(a)\rangle_{a\in\mathbb{Z}^*}$ be as in Lemma 3.8. If $a \in \mathbb{Z}^*$ is a prefix of $b \in \mathbb{Z}^*$ then $B(w(b),\delta(b)) \subseteq B(w(a), 2\delta(a)/3)$.*

Proof. It follows from the first and the second condition of Lemma 3.8 that for any $a \in \mathbb{Z}^*$ and $j \in \mathbb{Z}$, $B(w(aj),\delta(aj)) \subseteq B(w(a), 2\delta(a)/3)$. The general case when $b = ab_i \cdots b_n$ is then witnessed by a chain of inclusions of the form $B(w(ab_i \cdots b_j), \delta(ab_i \cdots b_j)) \subseteq B(w(ab_i \cdots b_{j+1}), 2\delta(ab_i \cdots b_{j+1})/3)$. ■

Lemma 3.10 *Let $\langle\delta(a)\rangle_{a\in\mathbb{Z}^*}$ and $\langle w(a)\rangle_{a\in\mathbb{Z}^*}$ be as in Lemma 3.8. For every $i \in \mathbb{N}$ and $x \in X$, there exists a unique $a \in \mathbb{Z}^i$ such that $d(w(a),x) < \delta(a)$, or for all $a \in \mathbb{Z}^i$ it is the case that $d(w(a),x) > \delta(a)$.*

Proof. We prove the lemma by induction on $i \in \mathbb{N}$. The base case just claims that $d(w([\,]),x) < 2\delta([\,])$ or $d(w([\,]),x) > \delta([\,])$, which of course holds. To prove the induction step, suppose there exists a unique $a \in \mathbb{Z}^{i+1}$ such that $d(w(a),x) < 2\delta(a)$, or $d(w(a),x) > \delta(a)$ for all $a \in \mathbb{Z}^{i+1}$. In the first case we use the fourth condition of Lemma 3.8 to conclude that there exists a unique $b = aj \in \mathbb{Z}^{i+2}$ such that $d(w(b),x) < 2\delta(b)$, or that $d(w(b),x) > \delta(b)$ for all $b \in \mathbb{Z}^{i+2}$. In the second case, the first and the second condition of Lemma 3.8 imply, for all $aj \in \mathbb{Z}^{i+2}$,

$$d(w(aj),x) > d(w(a),x) - d(w(a),w(aj)) > \delta(a)/3 > \delta(aj) .$$

■

We prove Proposition 3.1. Let $\langle\delta(a)\rangle_{a\in\mathbb{Z}^*}$ and $\langle w(a)\rangle_{a\in\mathbb{Z}^*}$ be as in Lemma 3.8. Observe that $\delta(a) \leq \delta([\,]) \cdot 3^{-|a|} = 3^{-|a|}$ for all $a \in \mathbb{Z}^*$. Define the map $e : \mathbb{Z}^{\mathbb{N}} \to X$ by

$$e(\alpha) = \lim_{i\to\infty} w(\alpha\restriction_i) .$$

The first and the second condition of Lemma 3.8 imply that $\langle w(\alpha\restriction_i)\rangle_{i\in\mathbb{N}}$ satisfies $d(w(\alpha\restriction_i), w(\alpha\restriction_{i+i})) \leq \delta(\alpha\restriction_i) \leq 3^{-i}$, which means that it is a Cauchy sequence, hence e is well defined. It is the case that $d(e(\alpha), w(\alpha\restriction_i)) \leq 3^{-i+1}$. To see that e is uniformly continuous, consider any $\varepsilon > 0$. There exists $k \in \mathbb{N}$ such that $3^{-k+1} < \varepsilon/2$. If $d(\alpha,\beta) < 2^{-k}$ then $\alpha\restriction_k = \beta\restriction_k$, therefore

$$d(e(\alpha), e(\beta)) \leq d(e(\alpha), w(\alpha\restriction_k)) + d(e(\beta), w(\beta\restriction_k)) \leq 2 \cdot 3^{-k+1} < \varepsilon .$$

This shows that e is uniformly continuous. Next we define the map $g : X \to \mathbb{R}^{\mathbb{N}}$ as $g = \langle g_i\rangle_{i\in\mathbb{N}}$ where the value of $g_i : X \to \mathbb{R}$ at $x \in X$ is defined by the following two clauses:

1. if there exists a unique $a = a_0 a_1 \ldots a_i \in \mathbb{Z}^{i+1}$ such that $d(w(a),x) < 2\delta(a)$ then
$$g_i(x) = a_i \cdot \mathsf{hill}(2\delta(a)/3, \delta(a))(x) ,$$
2. if $d(w(a),x) > \delta(a)$ for all $a \in \mathbb{Z}^{i+1}$ then
$$g_i(x) = 0 .$$

Note that when both clauses apply they agree that $g_i(x) = 0$, and by Lemma 3.10 at least one of the cases always applies, hence g_i is well defined.

Let us prove that g_i is continuous at x. First consider the case when $d(x, w(a)) > \delta(a)$ for all $a \in \mathbb{Z}^{i+1}$. Because $\langle w(a) \rangle_{a \in \mathbb{Z}^*}$ is without accumulation point there exists $\rho > 0$ such that $d(x, w(a)) > \rho$ for all $a \in \mathbb{Z}^{i+1}$. By the second condition of Lemma 3.8, there exists $m \in \mathbb{N}$ such that $\delta(a) < \rho/2$ if $\|a\|_\infty = \max(|a_0|, \ldots, |a_i|) \geq m$. Since there are finitely many $a \in \mathbb{Z}^{i+1}$ such that $\|a\|_\infty < m$, we may define

$$\tau = \min(\{\rho\} \cup \{d(x, w(a)) - \delta(a) \mid a \in \mathbb{Z}^{i+1}, \|a\|_\infty < m\}) \ .$$

Let $y \in B(x, \tau/2)$ and $a \in \mathbb{Z}^{i+1}$. If $\|a\|_\infty \geq m$ then $d(w(a), y) \geq d(w(a), x) - d(x, y) > \rho - \tau/2 \geq \rho/2 > \delta(a)$. If $\|a\|_\infty < m$ then $d(w(a), y) \geq d(w(a), x) - d(x, y) > d(w(a), x) - \tau > \delta(a)$. In either case, $y \in B(x, \tau/2)$ implies $g_i(y) = 0$, which means that g_i is indeed continuous at x. Now consider the case when there exists $a \in \mathbb{Z}^{i+1}$ such that $d(w(a), x) < 2\delta(a)$. Then for every $y \in B(x, 2\delta(a) - d(x, w(a)))$ it holds that $d(w(a), y) < 2\delta(a)$, therefore g_i restricted to $B(x, \delta(a)/2)$ is equal to the continuous function $a_i \cdot \mathsf{hill}(2\delta(a)/3, \delta(a))$ restricted to the same ball.

Since every g_i is pointwise continuous, it is not hard to see that g is pointwise continuous as well. Finally, observe that, for any $\alpha \in \mathbb{Z}^{\mathbb{N}}$, Lemma 3.9 implies that, for all $i \in \mathbb{N}$,

$$d(e(\alpha), w(\alpha\restriction_i)) \leq 2\delta(\alpha\restriction_i)/3 \ ,$$

therefore,

$$g_i(e(\alpha)) = \alpha_i \cdot \mathsf{hill}(2\delta(\alpha\restriction_i)/3, \delta(\alpha\restriction_i))(e(\alpha)) = \alpha_i \ .$$

Hence $g(e(\alpha)) = \alpha$, which concludes the proof of Proposition 3.1.

3.2 Proof of Proposition 3.2

We move on to proving Proposition 3.2. Assume given a sequentially continuous $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$. We construct a function $h : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ extending f.

For $\gamma \in \mathbb{R}^{\mathbb{N}}$ and $\beta \in \mathbb{Z}^{\mathbb{N}}$, define a sequence $\langle h_i^\beta(\gamma) \rangle_{i \in \mathbb{N}}$ of real numbers by:

$$h_0^\beta(\gamma) = f(0^\omega) \ ,$$

$$h_{i+1}^\beta(\gamma) = h_i^\beta(\gamma) + (f(\beta_0 \ldots \beta_i 0^\omega) - h_i^\beta(\gamma)) \cdot \prod_{j=0}^{i} \left(\mathsf{cone}(\beta_j, 1/4)(\gamma_j)\right)^{2^{i-j}} \ .$$

We say that β is *adequate for* γ if, for all $i \in \mathbb{N}$,

$$\beta_i - 2/3 < \gamma_i < \beta_i + 2/3 \ .$$

By $\mathsf{AC}_{0,0}$, for every $\gamma \in \mathbb{R}^{\mathbb{N}}$, there exists $\beta \in \mathbb{Z}^{\mathbb{N}}$ adequate for γ.

Lemma 3.11 *If β and β' are both adequate for γ then $h_i^\beta(\gamma) = h_i^{\beta'}(\gamma)$.*

Proof. The proof proceeds by induction on i. Clearly $h_i^\beta(\gamma) = h_i^{\beta'}(\gamma)$ in the case that $\beta_j = \beta'_j$, for all $j < i$. Otherwise, without loss of generality, there exists $j < i$ such that $\beta_j < \beta'_j$. Then, as both β and β' are adequate for γ, it holds that $\gamma_j - 2/3 < \beta_j < \beta'_j < \gamma_j + 2/3$. Thus $\beta'_j = \beta_j + 1$ and $\beta_j + 1/3 < \gamma_j < \beta'_j - 1/3$, so $\mathsf{cone}(\beta_j, 1/4)(\gamma_j) = 0 = \mathsf{cone}(\beta'_j, 1/4)(\gamma_j)$. By induction hypothesis, $h_i^\beta(\gamma) = h_{i-1}^\beta(\gamma) = h_{i-1}^{\beta'}(\gamma) = h_i^{\beta'}(\gamma)$. ∎

The above lemma justifies the definition

$$h_i(\gamma) = h_i^\beta(\gamma), \text{ for any } \beta \text{ adequate for } \gamma \ .$$

The following technical lemma is in preparation for Lemma 3.13 below.

Lemma 3.12 *If $\langle \xi_i \rangle_{i\in\mathbb{N}}$ is a sequence in $[0,1]$ satisfying $\xi_{i+1} \le \xi_i^2$, for all $i \in \mathbb{N}$, then the infinite product $\prod_{j=0}^{\infty}(1-\xi_j)$ converges.*

Proof. We need to prove convergence of the sequence $\langle P_i \rangle_{i\in\mathbb{N}}$ of partial products

$$P_i = \prod_{j=0}^{i-1}(1-\xi_j) \ .$$

We show that $m \ge n$ implies $P_n - P_m \le (2/3)^n$. There are two cases. First, if $\xi_i > 1/3$ for all $i < n$, then

$$P_n - P_m \le P_n \le (2/3)^n \ .$$

In the second case there exists $k < n$ such that $\xi_k < 1/2$ and $\xi_i > 1/3$ for all $i < k$. Then, for all $i \ge k$, $P_i \le P_k \le (2/3)^k$ and $\xi_i < (1/2)^{2^{i-k}}$, so

$$P_i - P_{i+1} = P_i \cdot \xi_i < (2/3)^k \cdot (1/2)^{2^{i-k}} \le (2/3)^k \cdot (1/2)^{1+i-k} \ .$$

From this we derive

$$P_n - P_m < (2/3)^k \cdot \sum_{i=n}^{m-1} (1/2)^{1+i-k} < (2/3)^k \cdot (1/2)^{n-k} \le (2/3)^n \ .$$

■

Lemma 3.13 *For every $\gamma \in \mathbb{R}^{\mathbb{N}}$, the sequence $\langle h_i(\gamma) \rangle_{i\in\mathbb{N}}$ converges.*

Proof. Let β be adequate for γ. We must show that $\langle h_i^\beta(\gamma) \rangle_{i\in\mathbb{N}}$ converges. As f is sequentially continuous, there exists n such that $f(\beta_0 \dots \beta_{m-1} 0^\omega) = f(\beta)$ for all $m \ge n$. Then, for $m \ge n$, the equality

$$h_m^\beta(\gamma) = f(\beta) + (h_n^\beta(\gamma) - f(\beta)) \cdot \prod_{i=n}^{m-1} \left(1 - \prod_{j=0}^{i} \big(\mathsf{cone}(\beta_j, 1/4)(\gamma_j)\big)^{2^{i-j}}\right) \quad (1)$$

is easily shown by induction on m. Define

$$\xi_k = \prod_{j=0}^{n+k} \big(\mathsf{cone}(\beta_j, 1/4)(\gamma_j)\big)^{2^{n+k-j}} \ .$$

By Lemma 3.12, $\xi = \prod_{k=0}^{\infty}(1-\xi_k)$ exists, and so $\langle h_m^\beta(\gamma) \rangle_{m\in\mathbb{N}}$ converges to $\xi h_n^\beta(\gamma) + (1-\xi) f(\beta)$. ■

Finally, define

$$h(\gamma) = \lim_{i\to\infty} h_i(\gamma) \ .$$

Lemma 3.14 *For all $\alpha \in \mathbb{Z}^{\mathbb{N}}$, it holds that $h(\alpha) = f(\alpha)$.*

Proof. Trivially, $\beta = \alpha$ is (the only β) adequate for α. We must show that $\lim_{m\to\infty} h^{\beta}_m(\alpha) = f(\alpha)$. Let n be such that, for all $m \geq n$, it holds that $f(\beta_0 \dots \beta_m 0^{\omega}) = f(\beta)$. Then, by (1), we have $h^{\beta}_m(\gamma) = f(\beta) = f(\alpha)$ for all $m > n$, because $\mathsf{cone}(\beta_j, 1/4)(\alpha_j) = 1$ for all j and so $\xi = 0$. ∎

This completes the proof of Proposition 3.2. Observe that, in addition to showing the existence of h given f, the proof constructs a function mapping any sequentially-continuous function $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ to a corresponding $h_f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$.

Various strengthenings of Proposition 3.2 are possible, for which we do not include proofs as they are not required for our application. First, it is possible to generalize the proposition to apply to any continuous $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{R}$. Second, it can be shown that the function h_f constructed above is sequentially continuous. Note that with the assertion of sequential continuity added, Proposition 3.2 becomes a statement with content in classical mathematics; whereas, as currently stated, the result is a classical triviality.

4 Continuity and Choice Principles

Continuity principles are statements asserting that all functions between certain spaces are continuous. Nontrivial continuity principles are inconsistent with classical mathematics, but play an important rôle in Brouwer's intuitionistic mathematics. Interesting continuity principles are also a feature of the internal logic of many toposes. It is well-known that there are nontrivial interactions between continuity principles and choice principles. In this section we briefly survey a few such results. Our main contribution, Theorem 4.5, explains the failure of certain continuity principles in toposes based on domain realizability.

For CSM X and Y we consider the continuity principle:

All functions $f : X \to Y$ are pointwise continuous. $\quad (\mathsf{CP}(X, Y))$

For sets X and Y we consider the choice principle:

$$(\forall x \in X \,.\, \exists y \in Y \,.\, \varphi(x, y)) \implies \exists f \in Y^X \,.\, \forall x \in X \,.\, \varphi(x, f(x)) \,. \qquad (\mathsf{AC}(X, Y))$$

Thus $\mathsf{AC}_{0,0}$ is $\mathsf{AC}(\mathbb{N}, \mathbb{N})$. As is standard, we write $\mathsf{AC}_{1,0}$ for $\mathsf{AC}(\mathbb{N}^{\mathbb{N}}, \mathbb{N})$, and $\mathsf{AC}_{2,0}$ for $\mathsf{AC}(\mathbb{N}^{\mathbb{N}^{\mathbb{N}}}, \mathbb{N})$. Easily $\mathsf{AC}_{1,0}$ implies $\mathsf{AC}_{0,0}$, and $\mathsf{AC}_{2,0}$ implies $\mathsf{AC}_{1,0}$.

In Brouwer's intuitionism, and in the realizability topos $\mathsf{RT}(K_2)$ over Kleene's second algebra K_2 [KV65, Bau00], both $\mathsf{CP}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$ and $\mathsf{AC}_{1,0}$ are valid. These two principles interact well together:

Proposition 4.1 **($\mathsf{CP}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N}) + \mathsf{AC}_{1,0}$)** *For all CSM X, Y, the continuity principle $\mathsf{CP}(X, Y)$ holds.*

On the other hand, stronger forms of choice are *not* compatible with $\mathsf{CP}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$.

Proposition 4.2 **($\mathsf{CP}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$)** $\mathsf{AC}_{2,0}$ *does not hold.*

For proofs of Propositions 4.1 and 4.2, see Sections 7.2.7 and 9.6.10 of [TvD88b] respectively. Note that Section 7.2.7 of *ibid.* relies on the continuity principle $\mathsf{WC\text{-}N}$, which follows easily from $\mathsf{AC}_{1,0}$ and $\mathsf{CP}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$.

Given Proposition 4.2, it is natural to ask just how strong a continuity principle is still consistent with $\mathsf{AC}_{2,0}$. In this context, it is interesting to look at the realizability topos $\mathsf{RT}(D)$ where D is a universal Scott domain, as this validates choice between any two finite types [BB00, Bau00], in particular $\mathsf{AC}_{2,0}$. In $\mathsf{RT}(D)$, as a consequence of the existence of a continuous modulus of uniform continuity on $\mathbb{K}$, the continuity principle $\mathsf{CP}(\mathbb{K}, \mathbb{N})$ holds, where $\mathbb{K} \subseteq \mathbb{Z}^{\mathbb{N}}$ is *Cantor space*:

$$\mathbb{K} = \{\alpha \in \mathbb{Z}^{\mathbb{N}} \mid \forall i \in \mathbb{N} \,.\, 0 \leq \alpha_i \leq 1\} \,.$$

(In fact a stronger continuity principle holds: all functions from $\mathbb{K}$ to $\mathbb{N}$ are uniformly continuous.) Again, $\mathsf{CP}(\mathbb{K}, \mathbb{N})$ interacts nicely with $\mathsf{AC}_{1,0}$. Recall, from Section 2, the notion of CTB space. We say that a CSM X is *locally CTB* if every point in X has a CTB neighbourhood.

Proposition 4.3 ($\mathsf{CP}(\mathbb{K}, \mathbb{N}) + \mathsf{AC}_{1,0}$) *For all locally CTB X and CSM Y, the continuity principle $\mathsf{CP}(X, Y)$ holds.*

This result follows from Section 7.4.4 of [TvD88b]. Observe that, since $\mathbb{K}$ is itself (locally) CTB, the principle $\mathsf{CP}(\mathbb{K}, \mathbb{N})$ is itself a special case of the general continuity principle established.

The notion of locally CTB space provides one possible constructive formulation of local compactness (indeed, in the presence of Brouwer's Fan Theorem, it implies that the Heine-Borel property holds locally [TvD88a, 7.4.10]). Thus domain realizability shows that $\mathsf{AC}_{2,0}$ is consistent with a continuity principle for "locally compact" spaces. As the main result of this section we show that, in contrast, extending the continuity principle to any single inhabited locally non-compact space is inconsistent with $\mathsf{AC}_{2,0}$.

To obtain inconsistency, we require only a very weak continuity principle to hold. Define the *one-point compactification of* $\mathbb{N}$ to be the subspace $\mathbb{N}^+ \subseteq \mathbb{K}$:

$$\mathbb{N}^+ = \{\alpha \in \mathbb{K} \mid \forall n \in \mathbb{N} \,.\, (\alpha_n = 1 \implies \forall m > n \,.\, \alpha_m = 0)\} \,.$$

As $\mathbb{N}^+$ is a retract of $\mathbb{K}$, it holds that $\mathsf{CP}(\mathbb{K}, \mathbb{N})$ implies $\mathsf{CP}(\mathbb{N}^+, \mathbb{N})$. Thus $\mathsf{CP}(\mathbb{N}^+, \mathbb{N})$ holds in domain realizability. Once again, $\mathsf{CP}(\mathbb{N}^+, \mathbb{N})$ enjoys a pleasant interaction with $\mathsf{AC}_{1,0}$:

Proposition 4.4 ($\mathsf{CP}(\mathbb{N}^+, \mathbb{N}) + \mathsf{AC}_{1,0}$) *For all CSMs X and Y, it holds that all functions from X to Y are sequentially continuous.*

Proof. Because $\mathbb{N}^+$ is a retract of $\mathbb{N}^{\mathbb{N}}$ we have $\mathsf{AC}(\mathbb{N}^+, \mathbb{N})$. Let $f : X \to Y$ be a function and $\langle a_i \rangle_{i \in \mathbb{N}}$ a sequence in X converging to x. We want to show that $\langle f(a_i) \rangle_{i \in \mathbb{N}}$ converges to $f(x)$. First we construct a function $g : \mathbb{N}^+ \to X$ such that $g(0^\omega) = x$ and $g(0^n 1 0^\omega) = a_n$ for all $n \in \mathbb{N}$. We define $g(\alpha) = \lim_{n \to \infty} h(\alpha, n)$ where $h(\alpha, n) = x$ if $\alpha\restriction_n = 0^n$ and $h(\alpha, n) = a_m$ if $\alpha\restriction_n = 0^m 1 0^{n-m-1}$. Now let $\varepsilon > 0$. For every $\alpha \in \mathbb{N}^+$, it holds that $d(f(x), g(\alpha)) < \varepsilon$ or $d(f(x), g(\alpha)) > \varepsilon/2$. By $\mathsf{AC}(\mathbb{N}^+, \mathbb{N})$ there exists a function $c : \mathbb{N}^+ \to \{0, 1\}$ such that, for all $\alpha \in \mathbb{N}^+$, if $c(\alpha) = 1$ then $d(f(x), g(\alpha)) < \varepsilon$, and if $c(\alpha) = 0$ then $d(f(x), g(\alpha)) > \varepsilon/2$. By $\mathsf{CP}(\mathbb{N}^+, \mathbb{N})$ there exists $m \in \mathbb{N}$ such that $\alpha\restriction_m = 0^m$ implies $c(\alpha) = c(0^\omega) = 1$. This means that, for all $n \geq m$, we have $d(f(x), f(a_n)) = d(f(x), g(0^n 1 0^\omega)) < \varepsilon$. Therefore $\lim_{n \to \infty} f(a_n) = f(x)$. ∎

Theorem 4.5 ($\mathsf{CP}(\mathbb{N}^+, \mathbb{N}) + \mathsf{AC}_{2,0}$) *For any inhabited locally non-compact CSM X, the continuity principle $\mathsf{CP}(X, \mathbb{R})$ is not true.*

Proof. We derive a contradiction from the assumption that all functions $X \to \mathbb{R}$ are pointwise continuous. Let $f : \mathbb{Z}^{\mathbb{N}} \to \mathbb{Z}$ be any function. As $\mathsf{AC}_{2,0}$ holds, so does $\mathsf{AC}_{1,0}$, so, by Proposition 4.4, f is sequentially continuous. By Theorem 2.3, we obtain a uniformly continuous embedding $e : \mathbb{Z}^{\mathbb{N}} \to X$ together with a function $\overline{f} : X \to \mathbb{R}$ such that $f = \overline{f} \circ e$. By assumption, $\overline{f}$ is pointwise continuous, therefore $f = \overline{f} \circ e$ is pointwise continuous, too. We have derived $\mathsf{CP}(\mathbb{Z}^{\mathbb{N}}, \mathbb{N})$. But, by Proposition 4.2, this contradicts $\mathsf{AC}_{2,0}$. ■

Theorem 4.5 is a generalization of [ES02], where it is proved that $\mathsf{RT}(D)$ validates the statement "not all functions $\mathcal{C}[-1,1] \to \mathbb{R}$ are pointwise continuous". This is so because in $\mathsf{RT}(D)$ it is the case that $\mathcal{C}[-1,1] = \mathcal{C}_{\mathsf{u}}[-1,1]$ and, as remarked in Section 2, $\mathcal{C}_{\mathsf{u}}[-1,1]$ is an inhabited locally non-compact CSM.

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