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Kähler-Einstein metrics on symmetric Fano T-varieties

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KÄHLER-EINSTEIN METRICS ON SYMMETRIC FANO T-VARIETIES

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ABSTRACT. We relate the global log canonical threshold of a variety with torus action to the global log canonical threshold of its quotient. We apply this to certain Fano varieties and use Tian's criterion to prove the existence of Kähler-Einstein metrics on them. In particular, we obtain simple examples of Fano threefolds being Kähler-Einstein but admitting deformations without Kähler-Einstein metric.

1. INTRODUCTION

This paper deals with the question of the existence of Kähler-Einstein metrics on Fano manifolds. While for the case of negative Ricci curvature and for the Ricci flat case the existence of Kähler-Einstein metrics is known for some time due to famous results of Aubin and Yau, the Fano case is still not completely understood. There are some known obstructions such as the reductivity of the automorphism group and the vanishing of the Futaki character [Fut83]. There is also a sufficient criterion due to Tian [Tia87], but for a long time there was no complete algebraic characterization of the Kähler-Einstein property, only recently it was proved that a certain notion of stability plays this rôle [CDS12, Tia12, CDS13]. And even this stability condition is hard to check in general.

However, for toric varieties the problem is completely solved. By a result of Wang and Zhu a toric Fano variety is Kähler-Einstein if and only if the Futaki character vanishes [WZ04]. Moreover, the Futaki character can be easily calculated as the barycenter of the polytope corresponding to the toric Fano manifold [Mab87]. These positive results suggest to also exploit lower dimensional torus actions for approaching the problem. Since Wang's and Zhu's result seems not to generalize well for non-toric situations, we follow the idea of an older paper by Batyrev and Selivanova [BS99]. They proved the existence of Kähler-Einstein metrics on *symmetric* toric Fano manifolds. It turns out that this result generalizes well. More precisely, high symmetry of the Fano manifold allows us to approach the question by solving a related problem on a torus quotient.

One motivation for considering lower dimensional torus actions in this context is the wish to study the behavior of the Kähler-Einstein property under deformations of the manifold, similar to the famous example of deformations of the Mukai-Umemura threefold studied in detail by Tian [Tia97] and Donaldson [Don08]. Since toric Fano manifolds are rigid [BB96], we cannot hope for such an example in the toric world.

Hence, we consider an effective action of an algebraic torus $T = (\mathbf{C}^*)^m$ on a normal algebraic variety X, possibly of higher dimension than T. The union of the maximal orbits will be denoted by X°. We also consider some rational quotient map $\pi : X \dashrightarrow Y$. The dimension of Y equals the *complexity* of the torus action which is defined as the difference of dimensions $\dim X - \dim T$. Let $\mathcal{N}(T)$ be the normalizer of the acting torus inside $\operatorname{Aut}(X)$. Then one obtains an action of $\mathcal{N}(T)$ on T by conjugation, which descends to an action on the characters $M = \operatorname{Hom}(T, \mathbf{C}^*)$. Assume that G is a finite subgroup of $\mathcal{N}(T)$. This induces a G-action on M. Following [BS99] the variety X is called *symmetric* with respect to the T-action if the identity is the only fixed point of such a G-action on M for some finite subgroup $G \subset \mathcal{N}(T)$.

An equivariant automorphism $\varphi \in \mathcal{N}(T)$ descends to an automorphism $\underline{\varphi}$ of $\mathrm{im}(\pi)$, via $\underline{\varphi}(y) := \pi(\varphi(x))$ where x is an arbitrary element in the preimage of y. Hence, we also obtain an action of G on $\mathrm{im}(\pi)$, which is assumed to extend to the whole quotient Y. A fiber of π is called *non-reduced*, if T acts with disconnected stabilizer on this fiber (indeed the fibers *are* non-reduced in the algebro-geometric sense).

Our first result is the following criterion for the existence of Kähler-Einstein metrics on T-varieties of complexity one.

Theorem 1.1. *Let X be a symmetric log terminal Fano T-variety of complexity one. If one of the following conditions is fulfilled:*

    (i) *there are three non-reduced fibers,*
   (ii) *there are two non-reduced fibers which are swapped by an element of G,*
  (iii) *G acts fixed-point-free on $Y = \mathbf{P}^1$,*

then X is Kähler-Einstein.

This result is used to prove the existence of a Kähler-Einstein metric on a certain blowup of the quadric threefold in two conics and on the hypersurface $V(xu^2 + yv^2 + zw^2) \subset \mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1,2)$. These give examples in the flavor of the Mukai-Umemura threefold [MU83], i.e. inside a family of Fano threefolds there is an element of high symmetry admitting a Kähler-Einstein metric and there are nearby elements with and without such a metric, see Section 6.

The main tools for proving Theorem 1.1 are global log canonical thresholds (glct) of varieties and pairs, which are the algebro-geometric counterparts of Tian's α-invariants. We are using the following theorem by Tian [Tia87, Theorem 2.1].

Tian's criterion. *Let X be a smooth (or orbifold) Fano variety and $G \subset \mathrm{Aut}(X)$ a finite symmetry group. If we have the bound*

$$\mathbf{glct}_G(X) > \frac{\dim(X)}{\dim(X) + 1},$$

for the global log canonical threshold, then X is Kähler-Einstein.

For a symmetric T-variety the global log canonical thresholds on X can be calculated as global log canonical thresholds of a pair on the quotient variety.

For a prime divisor Z on Y we consider its preimage $\pi^{-1}(Z)$. Here, the generic stabilizer on a component of $\pi^{-1}(Z)$ will be a finite abelian group $\mathbf{Z}/m\mathbf{Z}$. The maximal order obtained on the components is denoted by m_Z. This gives rise to a boundary divisor $B = \sum_Z \frac{m_Z - 1}{m_Z} \cdot Z$ on Y.

Theorem 1.2. *If X is a symmetric log terminal Fano T-variety and $\pi|_{X^\circ}$ is a morphism onto Y, then we have*

$$\mathbf{glct}_G(X) = \min\{1, \mathbf{glct}_G(Y, B)\}.$$

In particular, $\mathbf{glct}_G(X) \geq 1$ holds if and only if $\mathbf{glct}_G(Y, B) \geq 1$ holds.

If Y has only orbifold singularities, then the pair (Y, B) can be interpreted as a kind of orbifold quotient of X. Moreover, the above result somehow motivates the following conjecture.

Conjecture. *Assume that X fulfills the preconditions of Theorem 1.2 and the quotient (Y, B) admits an (orbifold) Kähler-Einstein metric, then X is Kähler-Einstein.*

Remark. The converse of the conjecture is not true as the quadric threefold shows. The quotient is $(\mathbf{P}^1, {}^1\!/_2 \cdot \{\mathbf{pt}\})$ which is not Kähler-Einstein by [RT11].

G will be called *valuable* for a pair (X, B) if $\mathbf{glct}_G(X, B) \geq 1$. In particular, Tian's criterion states the existence of a Kähler-Einstein metric for Fano varieties admitting a valuable subgroup of symmetries. By Theorem 1.2 This property is preserved by our quotient maps.

Remark. Also for another reason it makes sense to look for Kähler-Einstein metrics on symmetric Fano varieties. Since the Futaki character corresponds to an element of M which is fixed by torus equivariant automorphisms, it has to vanish on a symmetric manifold. Hence, Futaki's necessary criterion is automatically fulfilled.

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2. Log canonical thresholds

First, we shortly remind the notion of log pairs and some of their properties. A log pair (Y, B), consists of a normal variety Y and a **Q**-divisor B, which is called *boundary*. A canonical divisor of such a pair is simply the sum $K_X + B$ of a canonical divisor K_Y on Y and the boundary. Hence, we call a pair **Q**-Gorenstein if $K_Y + B$ is **Q**-Cartier. A log pair is called *smooth*, if Y is smooth and the boundary is a simply normal crossing divisor. A *log resolution* of (Y, B) is a birational proper morphism $\varphi : \widetilde{Y} \to Y$ such that $(\widetilde{Y}, \varphi^* B)$ is smooth. A pair is called *orbifold*, if Y has only quotient singularities and B is of the form $\sum_{i=1}^{\ell} \frac{m_i - 1}{m_i} \cdot Z_i$, with positive integers m_i.

Definition 2.1. Consider a Gorenstein pair (Y, B) and a log resolution $\varphi : \widetilde{Y} \to Y$. Then there are canonical divisors $K_{\widetilde{Y}}$, K_Y on $\widetilde{Y}$ and Y, respectively, such that $E := K_{\widetilde{Y}} - \varphi^*(K_Y + B)$, is supported only on the strict transform of supp B and the exceptional divisor. The pair is called *log canonical (l.c.)* if all the coefficients of E are at least -1.

The coefficients of E are called *discrepancies* of the morphism φ with respect to the pair (Y, B).

Let us denote the set of effective **Q**-divisors being linearly equivalent to a divisor D by $|D|_{\mathbf{Q}}$ and by $|D|_{\mathbf{Q}}^G$ the set of those divisors which are also G-invariant. Having this we are going to introduce the central notion of this article.

Definition 2.2. For a **Q**-Gorenstein pair (Y, B) and a finite subgroup $G \subset \mathrm{Aut}(Y, B)$ one defines its *global log canonical threshold* by

$$\mathbf{glct}_G(Y, B) = \sup \left\{ \lambda \mid (Y, B + \lambda D) \text{ is l.c. for all } D \in |-K_X - B|_{\mathbf{Q}}^G \right\}.$$

Remark. Since [DK01] appeared it is known, that global log canonical thresholds coincide with Tian's α-invariant from [Tia87]. For a proof see Demailly's appendix of [CS08].

Now, we will slightly generalize the notion of a pair and of a valuable subgroup of symmetries by allowing also $-\infty$ as a coefficient for boundary divisors. In this case we denote the open subset $Y \setminus \{-\infty\text{-locus of } B\} \subset Y$ by Y°.

Definition 2.3. A finite subgroup $G \subset \mathrm{Aut}(Y, B)$ is called *valuable* for (Y, B) if (Y, D) is log canonical for all G-invariant **Q**-divisors D, with $D \sim_{\mathbf{Q}} -K_Y$ and $D \geq B$.

Remark. For B finite this definition simply means that $\mathbf{glct}_G(Y, B) \geq 1$, i.e. the definition coincides with the original one from the introduction.

The notion of $(-\infty)$-coefficients looks a bit artificial. To illustrate the meaning we consider a G-invariant blowup $Y' \to Y$ with exceptional divisor E. Now, G is valuable for $(Y', -\infty \cdot E)$ if and only if the same is true for Y.

The notion of $(-\infty)$-coefficients allows us to generalize Theorem 1.2 to the case that $\pi|_{X^\circ}$ is not a morphism onto Y.

Theorem 2.4. *G is valuable for a symmetric T-variety X if and only if the same is true for the quotient (Y, B).*

Example 2.5. Consider $X = \mathbf{P}^1$ and $B = \frac{1}{2} \cdot \{0\} + \frac{1}{2} \cdot \{\infty\}$. The effective **Q**-divisors $D \sim_{\mathbf{Q}} -(K_{\mathbf{P}^1} + B)$ are exactly those of degree 1. Hence, the maximal possible coefficient of $(B + \lambda D)$ is $(\frac{1}{2} + \lambda)$ and the pair $(\mathbf{P}^1, B + \lambda D)$ is log canonical if and only if $\lambda \leq \frac{1}{2}$. One obtains $\mathbf{glct}_{\langle 1 \rangle}(\mathbf{P}^1, B) = \frac{1}{2}$.

Let G be the group consisting of the identity and the involution given by $x \mapsto 1/x$. Then G has exactly two fixed points: 1 and -1. All other orbits consist of two elements. Let's assume that D is G-invariant and chosen in such a way, that the biggest coefficient is maximal. Then D might be either concentrated at one of the fixed points or it might be equally distributed over x and $1/x$, for some point $x \in \mathbf{P}^1$. In the first case the maximal exponent of $(B + \lambda D)$ is $\max\{1/2, \lambda\}$ and in the second case it is $\lambda/2 + 1/2$. In both cases $(\mathbf{P}^1, B + \lambda D)$ is log canonical if and only if $\lambda \leq 1$. Hence, one obtains $\mathbf{glct}_G(\mathbf{P}^1, B) = 1$.

Propositio 2.6. *For a T-variety the global log canonical threshold can be calculated by considering only T-invariant divisors D.*

Proof. We argue exactly as in [CS08, Section 5]: a priori the value in Definition 2.2 may grow when considering only torus invariant divisors. Let us assume that there is an effective divisor D such that $(X, \lambda D)$ is not log canonical. Then we may choose a one-parameter subgroup $\gamma : \mathbf{C}^* \hookrightarrow T$ and consider $D_0^\gamma = \lim_{t \to 0} \gamma(t).D$. Now, we have

(i) $\gamma(t).D \sim D$ and $D_0^\gamma \sim D$,
(ii) $(X, \lambda \cdot \gamma(t).D)$ is isomorphic to $(X, \lambda D)$, hence not log canonical,
(iii) D_0^γ is invariant by the $\mathbf{C}^*$-action induced by γ.

From [DK01, Theorem 0.2.] it follows, that $(X, \lambda D_0^\gamma)$ is not log canonical. After iteration with a basis of the group of one-parameter subgroups of T we end up with a divisor D_0 which is T-invariant and $(X, \lambda D_0)$ is still not log canonical. Moreover, since the G action normalizes the torus action $g.D_0$ will be torus invariant as well and linearly equivalent to D for every element $g \in G$. Hence, the $\mathbf{Q}$-divisor $1/|G| \cdot (G.D_0)$ will be $(G \times T)$-invariant and again linearly equivalent to D. $\qquad\square$

3. Torus quotients and invariant divisors

For a semi-projective T-variety (i.e. projective over some affine variety) we obtain a quotient map by considering all the GIT-quotients corresponding to linearizations of an ample line bundle. These form an inverse system. The corresponding inverse limit comes with a rational map $X \dashrightarrow \varprojlim Y_u$. The limit may be reducible and non-normal, but in any case we might consider the normalization of the distinguished component, which is defined as the closure of the image of this map. For a detailed discussion of this construction, see [AH06] for the affine case and [KSZ91, CM07] for the toric situation. We will denote the normalization of the distinguished component simply by $X /\!/ T$. For projective varieties this construction coincides with the Chow quotient introduced by Kapranov, Sturmfels and Zelevinsky in [KSZ91], see [KSZ91, CM07, BHK12]. Hence, we will refer to the normalization of the distinguished component simply as the Chow quotient of X, even if the variety is not projective (e.g. affine).

Remark. In general it is not easy to obtain an explicit description of $X /\!/ T$. Nevertheless, in important cases the quotient is well known, e.g. for the case of rational T-varieties of complexity one it has to be $\mathbf{P}^1$. Even in higher complexity we sometimes obtain simply a projective space (e.g. for quadrics) or other well known varieties, as the moduli space $\overline{M}_{0,n}$ for the Grassmanian $G(2,n)$, see [Kap93]. Moreover, for toric varieties X_Σ the Chow quotient with respect to a subtorus action $T' \subset T$ is given as the coarsest common refinement of the cones in the image $P(\Sigma)$, where $P : N \to N''$ is given by the exact sequence

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

of co-character lattices of the tori T, T' and $T'' = T'/T$, see [KSZ91, AH06, CM07].

For the Chow quotient map $\pi : X \dashrightarrow Y = X /\!/ T$ we have the following properties:

(i) π is T-invariant, $\pi^* K(Y) = K(X)^T$ holds,
(ii) $\pi|_{X^\circ}$ is locally a geometric quotient onto its image. In particular, prime divisors on X° are mapped to prime divisors on Y,
(iii) there is a natural G-action on Y making π a G-equivariant map,

(iv) for $U \subset X$ open and affine there is a semi-projective open subset $V \subset Y$ and a projective morphism $V \twoheadrightarrow U/\!/T$,

We will also consider G-invariant, birational and projective modifications $Y \to X/\!/T$ together with the induced maps $\pi : X \dashrightarrow Y$. We call such a map simply a *rational quotient map* of X. All the properties mentioned above hold for such a Y as well. In the following we may assume that Y is chosen to be at least **Q**-factorial.

Remark. In [AH06] Altmann and Hausen introduced a description of affine T-varieties by so called *polyhedral divisors* on a quotient variety Y as above. Property (iv) of the Chow quotient implies that in the projective case we may cover X by affine open subsets corresponding to polyhedral divisors all living on $X/\!/T$ (this leads to the notion of a *divisorial fan* defined in [AHS08]). We will use this fact to apply several results from [AH06, PS11, LS10, AIP$^+$11] which are formulated in the language of polyhedral divisors.

We characterize two different types of T-invariant prime divisors on X:

- Those which intersect X° are called *vertical*, their images under π are prime divisors in Y (after taking the closure).
- The prime divisors inside $X \setminus X^\circ$ are called *horizontal*, their image under π is dense in Y.

The set of vertical divisors with image $Z \subset Y$ is denoted by $\mathcal{V}_Z$, the set of all vertical prime divisors by $\mathcal{V}$ and the set of all horizontal divisors by $\mathcal{H}$. Vertical prime divisors consist of closures of maximal orbits. Horizontal ones consist of orbit closures of dimension $(\dim T - 1)$. On maximal orbits we have finite stabilizer groups. Moreover, the order of the stabilizers is fixed on an open subset of a vertical prime divisor D. This generic order is denoted by $\mu(D)$, and will be called simply the *order of D*. The maximal order for all vertical prime divisors in $\mathcal{V}_Z$ is called *multiplicity of Z* and will be denoted by m_Z. Note, that due to the effectiveness of the torus action there are only finitely many prime divisors Z with multiplicity > 1. For a prime divisor Z on Y we have then the following pullback formula, see [AIP$^+$11, Section 7]:

$$\pi^* Z = \sum_{D \in \mathcal{V}_Z} \mu(D) \cdot D. \tag{1}$$

In particular, the pullback of a divisor on Y does not have a horizontal component.

Propositio 3.1 ([AIP$^+$11, Section 8]). *Let K_Y be a canonical divisor on Y. Then*

$$\pi^* K_Y - \sum_{D \in \mathcal{H}} D + \sum_{D \in \mathcal{V}} (\mu(D) - 1) \cdot D$$

defines a canonical divisor on X.

Example 3.2. Consider the $\mathbf{C}^*$-action on $\mathbf{P}^2$ given by

$$t.(x_1 : x_2 : x_3) = (tx_1 : tx_2 : x_3).$$

In this case the quotient is given by the map

$$\pi : \mathbf{P}^2 \dashrightarrow \mathbf{P}^1; \quad (x_1 : x_2 : x_3) \mapsto (x_1 : x_2).$$

It is defined outside the point $(0 : 0 : 1)$. For every point $y = (a : b) \in P^1$ the set $\mathcal{V}_{(a:b)}$ consists only of the invariant line $\{(a : b : s) \mid s \in \mathbf{C}\}$ and outside of $(a : b : 0)$ the stabilizer is trivial. Hence, $m_y = 1$. There is exactly one horizontal prime divisor, namely the line $[x_3 = 0]$ consisting only of fixed points.

Example 3.3 (A hypersurface of bidegree $(1, 2)$). The hypersurface

$$X = V(u_0 v_0^2 + u_1 v_1^2 + u_2 v_2^2) \subset \mathbf{P}^2 \times \mathbf{P}^2$$

of bidegree $(1,2)$ is a smooth Fano threefold. It admits a 2-torus action given by the weight matrix

$$\begin{array}{cccccc} u_0 & u_1 & u_2 & v_0 & v_1 & v_2 \end{array}$$
$$\left(\begin{array}{cccccc} 2 & 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 0 & -1 & 0 \end{array} \right)$$

The quotient map is given by

$$\pi : X \dashrightarrow \mathbf{P}^1; \quad (u_0 : u_1 : u_2, \, v_0 : v_1 : v_2) \mapsto (u_0 v_0^2 : u_1 v_1^2).$$

There are no horizontal prime divisors. For $y = (0 : 1)$ the vertical prime divisors $\mathcal{V}_y$ are the hyperplanes $[u_0 = 0]$ and $[v_0 = 0]$ having generic stabilizer of orders 1 and 2, respectively. Hence, one obtains multiplicity $m_y = 2$. One obtains the same for the points $(1 : 0)$ and $(1 : -1)$. For all other points $y = (a : b) \in \mathbf{P}^1$ the unique vertical prime divisor in $\mathcal{V}_y$ is $[bu_0 v_0^2 - au_1 v_1^2 = 0]$, having a trivial generic stabilizer.

We have an action of $S_3 \subset \mathcal{N}(T)$ on X given by permuting the indices 0, 1, 2 of the variables u_i and v_i. Look at the cyclic permutation $0 \mapsto 1 \mapsto 2 \mapsto 0$. The induced lattice homomorphism is given by the matrix $\left(\begin{smallmatrix} 0 & -1 \\ 1 & -1 \end{smallmatrix} \right)$ which has not real eigenvalues. Hence, X is symmetric and from Theorem 1.1 it follows the existence of a Kähler-Einstein metric, since the fibers over 0, ∞ and -1 are non-reduced.

The action of S_3 on $\mathbf{P}^1$ is induced by the permutations of the points 0, -1, ∞.

4. Toroidal resolutions and discrepancies

The definition of log canonicity of a T-variety X a priori enforces to consider a log resolution, but it turns out that it is enough to resolve to a locally toric situation. For this we consider log-resolutions $\psi : \widetilde{Y} \to Y$ for a pair (Y, Δ) on the quotient Y and lift it to a so called *toroidal resolutions* $\widetilde{X}$ of X. Such a construction is given in [LS10].

Given a T-pair (X, D) with a rational quotient map $X \dashrightarrow Y$. By projecting the vertical part of D this gives rise to a divisor on Y

$$\mathrm{supp}_Y D := \pi(\mathrm{supp}_{\mathrm{vert}} D).$$

Propositio 4.1. *Given an affine T-pair (X, D) and a rational quotient map $X \dashrightarrow Y$. Then, there are a birational T-invariant projective morphism $\varphi : \widetilde{X} \to X$, an open subset $U \subset \widetilde{X}$ and a log resolution $\widetilde{Y} \to Y$ of $(Y, \mathrm{supp}_Y D)$ having the following properties*

(i) *There is a good quotient morphism $\widetilde{\pi} : \widetilde{X} \to \widetilde{Y}$ fitting into the following commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\widetilde{\pi}} & \widetilde{Y} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \psi} \\ X & \xdashrightarrow{\pi} & Y \end{array}$$

   In particular, we have $\widetilde{X} /\!/ T = \widetilde{Y}$.

(ii) *$(\widetilde{X}, U)$ is an toroidal embedding, i.e. at every point $x \in \widetilde{X}$ there is a toric variety Z with torus H, such that $(\widetilde{X}, U)$ is locally formally isomorphic to (Z, H) at some point $z \in Z$,*

(iii) *$\varphi|_U$ is an isomorphism onto its image,*

(iv) *$\varphi^* D$ is supported on $\widetilde{X} \setminus U$. Hence, locally it looks like a toric divisor in a toric variety Z.*

The corresponding construction is given in Section 2 of [LS10], by using the language of polyhedral divisors. Using the results of [CP12] we can rephrase it as follows.

We resolve the indeterminacy of $\pi : X \dashrightarrow Y$ by considering the normalization of the closure of the graph of π in $X \times Y$. This is again a T-variety. We denote it by W. The projection to the second factor gives a good quotient morphism $W \to Y$, which is a trivial fibration outside a

subset Δ of pure codimension one. Now, we choose a log resolution $\widetilde{Y} \to Y$ of $(Y, \Delta \cup \operatorname{supp}_Y D)$ and denote the normalization of the closure of the graph of $X \dashrightarrow \widetilde{Y}$ in $X \times \widetilde{Y}$ by $\widetilde{X}$. The projections to X and $\widetilde{Y}$ induce the morphisms φ and $\widetilde{\pi}$, respectively.

The following lemma shows that it is enough to resolve to such a locally toric situation for checking the log canonicity of a T-pair (X, D), consisting of a T-variety and a T-invariant divisor.

Lemma 4.2. *The pair (X, D) is log canonical if and only if the discrepancies of ψ are at least -1, i.e. we have canonical divisors on X and $\widetilde{X}$, such that*

$$K_{\widetilde{X}} = \psi^*(K_X + D) + E$$

holds, with E being supported only on $\operatorname{supp} D$ and the exceptional divisor of ψ and the coefficients of E are at least -1.

Proof. One direction is obvious. Let θ be a log resolution of $(\widetilde{X}, -E)$. Then the discrepancies of θ with respect to $(\widetilde{X}, -E)$ are the same as the discrepancies of $\theta \circ \varphi$ with respect to (X, D). Hence, for the other direction we have to show $(\widetilde{X}, -E)$ is log canonical provided that the coefficients of E are at least -1. But this is well known (and can easily be checked) for toric pairs. Since, we are locally formally in a toric situation the claim follows. $\square$

In general we consider "twiddled" versions of our objects defined on X, i.e. we will denote the horizontal and vertical prime divisors of $\widetilde{X}$ by $\widetilde{\mathcal{H}}$ and $\widetilde{\mathcal{V}}$, respectively. The set of maximal orbits in $\widetilde{X}$ will be denoted by $\widetilde{X}^\circ$.

5. COMPARING THRESHOLDS

Remember, that we defined a boundary divisor $B = \sum_Z \frac{m_Z - 1}{m_Z} \cdot Z$ on Y, where m_Z is the maximal order of a generic stabilizer on a vertical divisor over Z. If there are no such divisors we set $m_Z = 0$ and obtain an $-\infty$-coefficient for the boundary at Z. Y° was defined as the open set $Y \setminus \bigcup_{\mathcal{V}_Z = \emptyset} Z$.

Lemma 5.1. *Let X be a symmetric T-variety and G the corresponding finite subgroup $G \subset \mathcal{N}(T)$. Then $(T \times G)$-invariant $\mathbf{Q}$-divisors D, with $Q_X \sim_{\mathbf{Q}} -K_X$ are exactly those of the form*

$$(2) \qquad Q_X = \pi^* Q_Y + \sum_{D \in \mathcal{H}} D + \sum_{D \in \mathcal{V}} (1 - \mu(D)) \cdot D$$

where Q_Y is an G-invariant divisor on Y with $Q_Y \sim_{\mathbf{Q}} -K_Y$.
  Moreover, Q_X is effective iff $Q_Y \geq B$.

Proof. Due to Proposition 3.1 the divisor Q_X defined in (2) is linearly equivalent to $-K_X$. It is obviously T-invariant. The summand $\pi^* Q_Y$ is a G-invariant, since the G-action on Y° is defined via π. Horizontal prime divisors are necessarily mapped to horizontal ones by any equivariant automorphism. Hence, $\sum_{E \in \mathcal{H}} E$ is G-invariant. Similarly, vertical prime divisors are mapped to vertical ones. Moreover, the order of a vertical prime divisor is an invariant of the T-action on X. Hence, $\sum_{D \in \mathcal{V}} (1 - \mu(D)) \cdot D$ is G-invariant, as well.

For the opposite direction consider a $(T \times G)$-invariant principal divisor $\operatorname{div}(f)$ on X. Then f is a semi-invariant rational function, i.e. $f(t.x) = u(t) \cdot f(x)$ holds for some character $u \in M$, i.e. $\deg(f) = u$. Moreover, we may assume that f is G-invariant, in particular the character u has to be G-invariant. Now, the symmetry condition on X implies that $u = 0$ and $f \in K(X)^T = \pi^* K(Y)$. Hence, $\operatorname{div}(f) = \pi^* \operatorname{div}(g)$ holds for some principal divisor $\operatorname{div}(g)$ on Y. Therefore, if some divisor D differs from Q_X in (2) only by a divisor having a principal multiple, then D is also of the desired form.

$Q_Y \geq B$ holds if and only if $Q_Y - B$ is effective on Y°. This is equivalent to the effectiveness of $\pi^*(Q_Y - B)$. By the pullback formula (1) one obtains

$$\pi^*(Q_Y - B) = \pi^* Q_Y + \sum_Z \sum_{D \in \mathcal{V}_Z} (1 - m_Z) \cdot D$$

$$\geq \pi^* Q_Y + \sum_Z \sum_{D \in \mathcal{V}_Z} (1 - \mu(D)) \cdot D$$

$$= \pi^* Q_Y + \sum_{D \in \mathcal{V}} (1 - \mu(D)) \cdot D$$

Hence, $Q_Y \geq B$ implies the effectiveness of Q_X. For one of the prime divisors $D \in \mathcal{V}_Z$ we have $\mu(D) = m_Z$. Therefore, we also obtain the opposite direction. $\qquad\square$

Proof of Theorem 1.2. At the beginning we introduce some notation.

- We consider the toroidal resolution and the maps from Proposition 4.1.
- As before we consider $G \times T$-invariant effective divisors $Q_X \sim_{\mathbf{Q}} -K_X$.
- We set

$$B_X = \sum_{D \in \mathcal{V}} (\mu(D) - 1) \cdot D, \qquad B_{\widetilde{X}} = \sum_{\widetilde{D} \in \widetilde{\mathcal{V}}} (\mu(\widetilde{D}) - 1) \cdot \widetilde{D}$$

- We may choose some representations of the canonical classes on $\widetilde{X}$ and X, respectively:

$$K_X = \pi^* K_Y + B_X - \sum_{D \in \mathcal{H}} D, \qquad K_{\widetilde{X}} = \widetilde{\pi}^* K_{\widetilde{Y}} + B_{\widetilde{X}} - \sum_{\widetilde{D} \in \widetilde{\mathcal{H}}} \widetilde{D}$$

As above we also choose a G-invariant effective divisor $Q_X \sim_{\mathbf{Q}} -K_X$. Then we have $Q_X = \pi^* Q_Y - B_X$, with $(Q_y - B) \in |-K_Y - B|_{\mathbf{Q}}^G$ by Lemma 5.1 and $\pi^* B \geq B_X$. If $\lambda \leq 1$ one obtains for $E_{\widetilde{X}} := K_{\widetilde{X}} - \varphi^*(K_X + \lambda Q_X)$ on $\widetilde{X}^\circ$

$$
\begin{aligned}
E_{\widetilde{X}} &= K_{\widetilde{X}} - \varphi^*(K_X + \lambda Q_X) \\
&= \widetilde{\pi}^* K_{\widetilde{Y}} + B_{\widetilde{X}} - \varphi^*(\pi^* K_Y + B_X + \lambda(\pi^* Q_Y - B_X)) \\
&= \widetilde{\pi}^* K_{\widetilde{Y}} + B_{\widetilde{X}} - \varphi^*(\pi^* K_Y + (1 - \lambda) B_X + \lambda \pi^* Q_Y) \\
&\geq \widetilde{\pi}^* K_{\widetilde{Y}} + B_{\widetilde{X}} - \varphi^*(\pi^* K_Y + (1 - \lambda) \pi^* B + \lambda \pi^* Q_Y) \\
&= \widetilde{\pi}^*(K_{\widetilde{Y}} - \psi^*(K_Y + B + \lambda(Q_Y - B))) + B_{\widetilde{X}}
\end{aligned}
$$

(3)

Note, that here we used that π is defined on X° and surjective. Hence, $\widetilde{\pi}^* \circ \psi^*$ and $\varphi^* \circ \pi^*$ coincide as homomorphisms $\mathrm{Div}_{\mathbf{Q}}(Y) \to \mathrm{Div}_{\mathbf{Q}}(\widetilde{X}^\circ)$. Now, assume, that $(Y, B + \lambda(Q_Y - B))$ is log canonical, then by definition all the coefficients of $E_{\widetilde{Y}} := K_{\widetilde{Y}} - \psi^*(K_Y + B + \lambda(Q_Y - B))$ are at least -1, but by our pullback formula (1) we infer that the coefficients of $\widetilde{\pi}^* E_Y + B_{\widetilde{X}}$ are ≥ -1 as well. If we replace X° by an open subset obtained by removing all vertical prime divisors D which are not of maximal order in $\mathcal{V}_{\pi(D)}$, then we even have equality in (3). Hence, the coefficients of $E_{\widetilde{X}}$ are ≥ -1 if and only if the same is true for $E_{\widetilde{Y}}$.

It remains to check the discrepancies at the horizontal divisors. Remember, that we have $Q_X \sim_{\mathbf{Q}} -K_X$, i.e. $(Q_X + K_X) = 1/m \cdot \mathrm{div}(f)$ for some torus invariant function $f \in K(X)^T$. We get

$$
\begin{aligned}
K_{\widetilde{X}} - \varphi^*(K_X + \lambda Q_X) &= K_{\widetilde{X}} - \varphi^*(K_X + Q_X + (\lambda - 1) Q_X) \\
&= K_{\widetilde{X}} - \varphi^*(1/m \cdot \mathrm{div}(f)) + \varphi^*((1 - \lambda) Q_X) \\
&\geq K_{\widetilde{X}} - 1/m \cdot \mathrm{div}(\varphi^* f)
\end{aligned}
$$

Note, that we assumed $\lambda \leq 1$, again. Now, $\varphi^* f$ is again a torus invariant function. Hence, $\mathrm{div}(\varphi^* f)$ is supported only on vertical divisors and for $K_{\widetilde{X}} = \widetilde{\pi}^* K_{\widetilde{Y}} + B_{\widetilde{X}} - \sum_{\widetilde{D} \in \widetilde{\mathcal{H}}} \widetilde{D}$ all the coefficients at horizontal divisors are -1.

It remains to show that we can indeed assume that $\lambda \leq 1$, i.e. $\mathbf{glct}(X, G) \leq 1$. First note, that the resolution $\widetilde{X} \to \widetilde{Y}$ is a fibration into complete toric varieties. Let us denote the general

fiber by X' and an arbitrary invariant prime divisor of X' by D. Now, there is a big open subset of the form $X' \times U \subset \widetilde{X}$ with $U \subset \widetilde{Y}$. Now, $D \subset X'$ gives rise to a horizontal prime divisor $H = \overline{D \times U} \subset \widetilde{X}$. In particular there exists a horizontal divisor H on $\widetilde{X}$. We choose $Q_X = K_X$. Now we have $K_{\widetilde{X}} - \varphi^*(K_X + \lambda Q_X) = K_{\widetilde{X}} + (1 - \lambda) \cdot \varphi^* K_X$. Consider a horizontal divisor H on $\widetilde{X}$. On the one hand then the coefficient of $K_{\widetilde{X}}$ at H is -1. On the other hand by the fact that X is log terminal, we know that the coefficient of $K_{\widetilde{X}} - \varphi^* K_X$ is bigger than -1. Hence, the coefficient of $\varphi^* K_X$ at H is positive and the coefficient of $K_{\widetilde{X}} + (1 - \lambda) \cdot \varphi^* K_X$ is ≥ -1 if and only if $\lambda \leq 1$. $\qquad\square$

Proof of the Theorem 2.4. In general we do not have $(\pi \circ \varphi)^* D = (\psi \circ \widetilde{\pi})^* D$ on $\widetilde{X}$, since functoriality does not hold for rational maps. Nevertheless, we have functoriality for rational functions, hence $(\pi \circ \varphi)^* D = (\psi \circ \widetilde{\pi})^* D$ holds if $D \sim_{\mathbf{Q}} 0$. Therefore, on $\widetilde{X}^\circ$ we have

$$\begin{aligned}
K_{\widetilde{X}} - \varphi^*(K_X + Q_X) &= K_{\widetilde{X}} - \varphi^*(\pi^*(K_Y + Q_Y)) \\
&= K_{\widetilde{X}} - \widetilde{\pi}^*(\psi^*(K_Y + Q_Y)) \\
&= \widetilde{\pi}^*(K_{\widetilde{Y}} - \psi^*(K_Y + Q_Y)) + B_{\widetilde{X}}.
\end{aligned}$$

By the pullback formula (1) the coefficients of this divisor are ≥ -1 if and only if this is true for the divisor $K_{\widetilde{Y}} - \psi^*(K_Y + Q_Y)$, but the latter is equivalent to the log canonicity of the pair (Y, Q_Y). Hence, G is valuable for X if and only if the same is true for (Y, B). $\qquad\square$

Proof of the Theorem 1.1. For the case of complexity one the quotient has to be $\mathbf{P}^1$ (it has to be a curve and the Fano property implies a finitely generated Picard group) and the quotient map is indeed defined on the whole set X° and automatically surjective. One easily checks (see e.g. Example 2.5) that for $B = \sum_P \frac{m_P - 1}{m_P}$ we have $\mathbf{glct}_G(\mathbf{P}^1, B) \geq 1$ exactly in the following case

    (i) $\#\operatorname{supp} B \geq 3$
    (ii) $\#\operatorname{supp} B = 2$ and G acts without fixed point on $\operatorname{supp} B$
    (iii) G acts without fixed point on $\mathbf{P}^1$

These are exactly the cases from Theorem 1.1. Now, the claim follows from Tian's theorem. $\qquad\square$

6. Examples and Kähler-Einstein metric on their deformations

6.1. **Hypersurfaces of bidegree** $(1, 2)$. We come back to the manifold of Example 3.3, i.e. the hypersurface $V(xu^2 + yv^2 + zw^2) \subset \mathbf{P}^2 \times \mathbf{P}^2$ of bidegree $(1, 2)$. By Theorem 1.1 this manifold admits a Kähler-Einstein metric. A semi-universal deformation of X is given by the family

$$\mathcal{X}_{\alpha, \beta, \gamma} = V(xu^2 + yv^2 + zw^2 + \alpha \cdot xvw + \beta \cdot yuw + \gamma \cdot zuv)$$

over $(\mathbf{C}^3, 0)$. Note, that the action of $(\mathbf{C}^*)^2$ on $X = \mathcal{X}_0$ naturally extends to an action on this family. Remember the weights for the action on X:

$$\begin{array}{cccccc}
x & y & z & u & v & w
\end{array}$$
$$\begin{pmatrix}
2 & 0 & 0 & -1 & 0 & 0 \\
0 & 2 & 0 & 0 & -1 & 0
\end{pmatrix}$$

Hence, the equation of the family has to be of weight 0 and the weights for the action on the base are given by

$$\begin{array}{ccc}
\alpha & \beta & \gamma
\end{array}$$
$$\begin{pmatrix}
-2 & 1 & 1 \\
1 & -2 & 1
\end{pmatrix}$$

A point (α, β, γ) is *polystable*, i.e. has a closed $(\mathbf{C}^*)^2$-orbit under this action, if and only if $\alpha\beta\gamma \neq 0$. This can be seen by checking the existence of the limits

$$\lim_{t \to 0} (t^{v_1}, t^{v_2}).(\alpha, \beta, \gamma)$$

for every one-parameter subgroup $t \mapsto (t^{v_1}, t^{v_2}) \in (\mathbf{C}^*)^2$. Alternatively one can use the explicit combinatorial criterion in terms of balanced families of weights, given in [RT12, Proposition 3.3.3.].

Now, we have the following statement

Propositio 6.1. *For a sufficiently small neighborhood $0 \in \mathcal{B} \subset \mathbf{C}^3$ the manifold $\mathcal{X}_{\alpha,\beta,\gamma}$ admits a Kähler-Einstein metric if and only if $\alpha\beta\gamma \neq 0$.*

To prove this behavior we are using the following theorem implicitly given in [RT12]:

Theorem 6.2. *Assume that a simply connected smooth T-variety X admits a Kähler metric of constant scalar curvature (cscK metric) in the class $c(L)$ for an ample line bundle L. Consider the semi-universal deformation $\mathcal{X} \to \mathcal{B}$, where $0 \in \mathcal{B} \subset H^1(X, \mathcal{T}_X)$ is a sufficiently small neighborhood of 0. If $H^2(X, \mathcal{O}) = 0$ then L lifts to a line bundle $\mathcal{L}$ on $\mathcal{X}^1$.*

If in this situation $\mathrm{Aut}^\circ(X) = T$ and $H^2(X, \mathcal{T}_X) = 0$ holds, then the manifold $\mathcal{X}_t$ admits a cscK metric in the class $c(\mathcal{L}_t)$ if and only if $t \in \mathcal{B}$ is a polystable point of the T-action on $H^1(X, \mathcal{T}_X)$.

This corresponds to Theorem 1.2.3. in [RT12]. There it is stated for the toric case, but the proof uses only the facts that toric varieties are simply connected and that $H^2(X, \mathcal{O}_X) = 0$ holds for every toric variety. Note, that the original condition of Theorem 1.2.3 is that the complexification of the group of Hamiltonian isometries $H^{\mathbf{C}}$ is an algebraic torus. As it is pointed out later on in their paper, in our situation H is a maximal compact subgroup of $\mathrm{Aut}^\circ(X)$. Now, the condition on the Hamiltonian isometries follows from ours: $\mathrm{Aut}^\circ(X) = T$.

Proof of Proposition 6.1. By Kodaira vanishing we have conditions $H^2(X, \mathcal{O}_X) = 0$ and $H^2(X, \mathcal{T}_X) = 0$ fulfilled. Moreover, since X is Fano it is also simply connected. Since a Kähler-Einstein metric is a cscK metric in the anti-canonical class we may apply Theorem 6.2 and the claim follows. □

Remark. This is a similar situation as for the Mukai-Umemura threefold, where one finds a symmetric element (here admitting an Sl$_2$-action) in the family V_{22} of Fano threefolds which is Kähler-Einstein. In the neighborhood of this threefold the Sl$_2$-symmetry is lost and one finds nearby-elements not being Kähler-Einstein. See [Tia97] and [Don08] for a detailed discussion of this situation.

6.2. **Torus invariant blowup of the quadric threefold.** Consider th quadric $Q^3 = V(u_0^2 + u_1v_1 + u_2v_2) \subset \mathbf{P}^4$. We have an action of the 2-torus given by the weights

$$
\begin{array}{ccccc}
u_0 & u_1 & u_2 & v_1 & v_2
\end{array}
$$
$$
\begin{pmatrix}
0 & 1 & 0 & -1 & 0 \\
0 & 0 & 1 & 0 & -1
\end{pmatrix}
$$

We get the rational quotient map

$$\pi : Q \dashrightarrow \mathbf{P}^1, \quad (u_0 : u_1 : u_2 : v_1 : v_2) \mapsto (u_1v_1 : u_2v_2).$$

There is an equivariant involution $\sigma : Q \to Q$ induced by the change of variables $u_i \leftrightarrows v_i$ for $i = 1, 2$. By looking at the weights of u_i and v_i we see, that this corresponds to the involution $w \leftrightarrows -w$ on M. Hence, Q is symmetric.

Let us study the invariant prime divisors on Q. There are no horizontal prime divisors. Over 0 and ∞, respectively, we find pairs of divisors cut out by $[u_i = 0]$ and $[v_i = 0]$ and having order 1. Moreover, we there is also a vertical prime divisor of order 2 over the point -1. Over every other point $(a : b) \in \mathbf{P}^1$ there is exactly one vertical prime divisor, having order 1. Quadric hypersurfaces are Kähler-Einstein. Nevertheless, we cannot prove it by using Theorem 1.1. We are not interested in Q itself, but in the blowup of Q in the two conics C_1 an C_2 cut out by $[u_i = v_i = 0]$ for $i = 1, 2$. These conics are torus invariant and invariant by the involution.

---

[1]see e.g. [Ser06, 3.3.11]

Hence, the resulting variety is again a symmetric T-variety. Moreover the quotient map is simply the composition of π with the blowup morphism (there is no choice, the quotient has to be $\mathbf{P}^1$ and the rational map has to coincide with π outside the exceptional divisors). It follows that the exceptional prime divisors are vertical prime divisors over 0 and ∞, respectively. Both have order 2, since locally we are blowing up $\mathbf{A}^3 = \operatorname{Spec} \mathbf{C}[x, y, z]$ in the ideal (y, z). Here, we may assume that $x = {}^{u_0}/_{v_2}$, $y = {}^{u_1}/_{v_2}$ and $z = {}^{v_1}/_{v_2}$ with weights

$$\begin{matrix} x & y & z \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \end{matrix} \cdot$$

Locally the blowup is given as $\{(x, y, z \,;\, u : v) \in \mathbf{A}^3 \times \mathbf{P}^1 \mid yu - zv = 0\}$ and the extended torus action by the additional weights $\deg(u) = \binom{-1}{-1}$ and $\deg(v) = \binom{0}{-2}$. Now, the stabilizer at a generic point $(x, 0, 0 \,;\, u : v)$ is $\{(1, 1), (-1, -1)\} \subset T$.

By Theorem 1.1 we obtain the following

Propositio 6.3. $\mathrm{Bl}_{C_1, C_2} Q$ *is Kähler-Einstein.*

Again we are interested in the deformation theory of the manifold. The semi-universal deformation of $X = \mathrm{Bl}_{C_1, C_2} Q^3$ is given by blowing up the family

$$\mathcal{X} = V(u_0^2 + u_1 v_1 + u_2 v_2 + \alpha \cdot u_1 v_2 + \beta \cdot u_2 v_1 + \gamma \cdot u_1 u_2 + \delta \cdot v_1 v_2)$$

in the subvarieties cut out by $[v_i = u_i = 0]$, $i = 1, 2$. The 2-torus acts on the base by the following weights

$$\begin{matrix} \alpha & \beta & \gamma & \delta \\ \begin{pmatrix} -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \end{matrix}$$

The polystable points are exactly those points $(\alpha, \beta, \gamma, \delta)$, such that $\alpha\beta \neq 0$ or $\gamma\delta \neq 0$. As in the previous section we can apply Theorem 6.2 to obtain deformations with Kähler-Einstein metric as well as those without such metrics.

Remark. Both constructions, that of blowups of quadrics and that of hypersurfaces of bidegree $(1, 2)$ generalize well to higher dimensions. Here, the quotients to consider are projective spaces of higher dimensions.

References

[AH06]   Klaus Altmann and Jürgen Hausen. Polyhedral divisors and algebraic torus actions. *Math. Ann.*, 334(3):557–607, 2006.

[AHS08] Klaus Altmann, Jürgen Hausen, and Hendrik Süß. Gluing affine torus actions via divisorial fans. *Transformation Groups*, 13(2):215–242, 2008.

[AIP+11] Klaus Altmann, Nathan O. Ilten, Lars Petersen, Hendrik Süß, and Robert Vollmert. The Geometry of T-Varieties. In Piotr Pragacz, editor, *Contributions to Algebraic Geometry*, Impanga Lect Notes, 2011, arXiv:1102.5760.

[BB96]   Frédéric Bien and Michel Brion. Automorphisms and local rigidity of regular varieties. *Compos. Math.*, 104(1):1–26, 1996.

[BHK12] H. Bäker, J. Hausen, and S. Keicher. On Chow quotients of torus actions. *ArXiv e-prints*, March 2012, arXiv:1203.3759.

[BS99]   Victor V. Batyrev and Elena Selivanova. Einstein-Kähler metrics on symmetric toric Fano manifolds. *J. Reine Angew. Math.*, 512:225–236, 1999.

[CDS12] X.-X. Chen, S. Donaldson, and S. Sun. Kahler-Einstein metrics and stability. *ArXiv e-prints*, October 2012, 1210.7494.

[CDS13] X. Chen, S. Donaldson, and S. Sun. Kahler-Einstein metrics on Fano manifolds, III: limits as cone angle approaches 2\pi\ and completion of the main proof. *ArXiv e-prints*, February 2013, 1302.0282.

[CM07]   Alastair Craw and Diane Maclagan. Fiber fans and toric quotients. *Discrete Comput. Geom.*, 37(2):251–266, 2007.

[CP12]   O. V. Chuvashova and N. A. Pechenkin. Quotients of an affine variety by an action of a torus. *ArXiv e-prints*, February 2012, arXiv:1202.5760.

[CS08]   Ivan. Cheltsov and Constantin Shramov. Log canonical thresholds of smooth Fano threefolds. *Russ. Math. Surv.*, 63(5):859–958, 2008.

[DK01]   Jean-Pierre Demailly and János Kollár. Semi-continuity of complex singularity exponents and Kähler-Einstein metrics on Fano orbifolds. *Ann. Sci. Éc. Norm. Supér. (4)*, 34(4):525–556, 2001.

[Don08] Simon K. Donaldson. Kähler geometry on toric manifolds, and some other manifolds with large symmetry. Somerville, MA: International Press; Beijing: Higher Education Press, 2008.

[Fut83] A. Futaki. An obstruction to the existence of Einstein Kähler metrics. *Invent. Math.*, 73:437–443, 1983.

[Ilt12] Nathan Ilten. Versal deformations and local hilbert schemes. *Journal of Software for Algebra and Geometry*, 4:12–16, 2012. Package available at: `http://people.cs.uchicago.edu/~nilten/m2.html`.

[Kap93] Mikhail M. Kapranov. Chow quotients of Grassmannians. I. Gelfand, Sergej (ed.) et al., I. M. Gelfand seminar. Part 2: Papers of the Gelfand seminar in functional analysis held at Moscow University, Russia, September 1993. Providence, RI: American Mathematical Society. Adv. Sov. Math. 16(2), 29-110., 1993.

[KSZ91] M.M. Kapranov, Bernd Sturmfels, and Andrei V. Zelevinsky. Quotients of toric varieties. *Math. Ann.*, 290(4):643–655, 1991.

[LS10]   Alvaro Liendo and Hendrik Süß. Normal singularities with torus actions. *Tohoku Mathematical Journal*, May 2010. to be published.

[Mab87] Toshiki Mabuchi. Einstein-Kähler forms, Futaki invariants and convex geometry on toric Fano varieties. *Osaka J. Math.*, 24:705–737, 1987.

[MU83]   Shigeru Mukai and Hiroshi Umemura. Minimal rational threefolds. Algebraic geometry, Proc. Jap.-Fr. Conf., Tokyo and Kyoto 1982, Lect. Notes Math. 1016, 490-518 (1983)., 1983.

[PS11]   Lars Petersen and Hendrik Süß. Torus invariant divisors. *Israel Journal of Mathematics*, 182:481–505, 2011.

[RT11]   Julius Ross and Richard Thomas. Weighted projective embeddings, stability of orbifolds, and constant scarla curvature kär metrics. *J. Differ. Geom.*, 88(1):109–159, 2011.

[RT12]   Y. Rollin and C. Tipler. Deformations of extremal toric manifolds. *ArXiv e-prints*, January 2012, arXiv:1201.4137.

[Ser06] Edoardo Sernesi. *Deformations of algebraic schemes.* Berlin: Springer, 2006.

[Tia87] Gang Tian. On Kähler-Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. *Invent. Math.*, 89:225–246, 1987.

[Tia97] Gang Tian. Kähler-Einstein metrics with positive scalar curvature. *Invent. Math.*, 130(1):1–37, 1997.

[Tia12] G. Tian. K-stability and Kähler-Einstein metrics. *ArXiv e-prints*, November 2012, 1211.4669.

[WZ04]   Xu-Jia Wang and Xiaohua Zhu. Kähler–Ricci solitons on toric manifolds with positive first Chern class. *Adv. Math.*, 188(1):87–103, 2004.

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