Energy-conserving finite difference schemes for tension-modulated strings

ENERGY-CONSERVING FINITE DIFFERENCE SCHEMES FOR TENSION-MODULATED STRINGS

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ABSTRACT

The timbre of certain stringed instruments is strongly dependent on large-amplitude vibration, in which case linear models (such as the 1D wave equation), which are often used for sound synthesis purposes, are unsatisfactory. We discuss here a nonlinear generalization of the wave equation, sometimes called the Kirchhoff-Carrier Equation, which models large amplitude vibration through a modulation of string tension. In particular, we look at a finite difference scheme for the Kirchhoff-Carrier Equation which is both efficient, and has excellent stability properties (this is often difficult to ensure for nonlinear difference schemes). The key to this stability property is the close attention paid to the energetic behavior of the model and its analogue in the finite difference scheme; such a difference scheme is capable of discrete energy conservation to machine precision. Implementation details are discussed, and simulation results are presented.

1. INTRODUCTION

There has recently been an increase in interest in modelling of nonlinear systems in the context of realistic musical sound synthesis. One such system, the so-called tension-modulated string has been modelled using digital waveguides [1, 2, 3]. One problem that can be encountered in any numerical simulation (by digital waveguides or any other method) is that of maintaining stability. This problem is especially acute for highly nonlinear systems such as the string under large amplitude vibration conditions. In this paper, we return to the equation of motion describing such a string, sometimes called the Kirchhoff-Carrier Equation [4, 5, 6], and show how the energy method [7] may be applied to prove stability of a nonlinear finite difference scheme; in essence, if one is able to show that the model system enforces an energy conservation property, and also that the finite difference scheme inherits this property, then the scheme is numerically stable. This is indeed the case for the Kirchhoff-Carrier Equation. The energy method of stability verification is distinct from spectral analysis techniques (such as Von Neumann analysis [8]) which cannot be usefully applied to nonlinear systems.

In Section 2, we revisit simple finite difference schemes for the linear 1D wave equation with an emphasis on energetic behavior, and review the energy method of stability verification. In Section 3, we present the Kirchhoff-Carrier equation, as well as a difference scheme which possesses energetic properties which mirror those of the model system itself. Implementation details are discussed in Section 4, and some simulation results are given in Section 5.

2. THE WAVE EQUATION AND CENTERED FINITE DIFFERENCE SCHEMES

For a string undergoing transverse motion, the 1D wave equation is a crude first approximation. It can be written simply as

$$\rho \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

Here, $t \geq 0$ is a time variable, $x \in [0, L]$ is a space variable, and $u(x,t)$ is the transverse string displacement. ρ is the linear mass density of the string, and T_0 is the tension applied to the string; both are assumed constant. Given two initial conditions, $u(x,0)$ and $\frac{\partial u}{\partial t}(x,0)$, and appropriate boundary conditions, the solution to the wave equation exists and is unique for all future times $t > 0$. The wave equation can be complemented by additional linear terms which model loss and dispersion[9]. For large amplitude displacement, however, the wave equation no longer holds, and must be generalized. We will describe one such generalization, applicable to stiff strings, in Section 3.

For analysis purposes, and for the development of numerical methods (such as finite difference schemes) [8], it is useful to rewrite the wave equation as a first-order system, i.e.,

$$\frac{\partial p}{\partial t} = c_0 \frac{\partial q}{\partial x} \qquad \frac{\partial q}{\partial t} = c_0 \frac{\partial p}{\partial x} \qquad c_0 = \sqrt{\frac{T_0}{\rho}} \tag{2}$$

where we have introduced the new variables $p = \sqrt{\rho}\frac{\partial u}{\partial t}$, and $q = \sqrt{T_0}\frac{\partial u}{\partial x}$. Both have units of root energy density. This system is of the form of the transmission line, or telegrapher's equations [10].

2.1. Energetic Analysis

Since the wave equation, (or the equivalent first-order system) is linear and shift-invariant, it is possible to perform a complete analysis and arrive at solutions using spectral techniques, such as Laplace and Fourier transforms [8, 7]. As we would like to be able to be able to generalize the analysis to the nonlinear case, we will make use of non-spectral techniques, and in particular, the energy method [7]. In the linear case, one can perform an energetic analysis as follows: multiply the first of (2) by the variable p, and the second by q, and add the two equations. This yields

$$\frac{\partial}{\partial t}\left(\frac{1}{2}p^2 + \frac{1}{2}q^2\right) = c_0 \frac{\partial (pq)}{\partial x}$$

Integrating over the range $x \in [0, L]$ gives

$$\frac{d}{dt}\int_0^L \left(\frac{1}{2}p^2 + \frac{1}{2}q^2\right) dx = c_0 \int_0^L \frac{\partial (pq)}{\partial x} dx = pq|_0^L$$

Under a simple choice of boundary condition such as $p = 0$ or $q = 0$ at both boundaries $x = 0$ and $x = L$, we can then write

$$\mathcal{E}(t) = \frac{1}{2}\|p\|^2 + \frac{1}{2}\|q\|^2 = \text{constant} \tag{3}$$

where we have used the notation $\|f\| = \left(\int_0^L f^2 dx\right)^{1/2}$ for square-integrable functions $f \in L^2(0, L)$. Here, $\mathcal{E}(t)$ has the interpretation of the total string energy; the term $\|p\|^2/2$ represents the kinetic energy, and the term in $\|q\|^2/2$ the potential energy. As the wave equation does not model effects of loss, this energy must remain constant (and equal to the initial energy present in the system). In a numerical setting, it is useful to have such a energy conservation property (i.e., a positive definite function of the state), as it can be used to ensure numerical stability, as we will show shortly.

Before proceeding to a difference scheme for the transmission line equations, we provide here some basic facts about grid functions [8] and difference operators.

2.2. Grid Functions

A real-valued grid function f_i^n, employed in a 1D finite difference scheme, is to be viewed as an approximation to a continuous time/space variable $f(x, t)$, at the coordinates $x = ih$, $t = nk$, for integer i and n; here h is the grid spacing, and k is the time-step (both assumed constant here). In this section, for simplicity we assume that the spatial domain is unbounded, i.e., $x \in [-\infty, \infty]$. We will consider boundary conditions in Section 4.

The forward time difference operator δ_t and time-averaging operator μ_t are defined by

$$\delta_t f_i^n = \frac{1}{k}(f_i^{n+1} - f_i^n) \qquad \mu_t f_i^n = \frac{1}{2}(f_i^{n+1} + f_i^n)$$

The identities

$$(\mu_t f_i^n)(\delta_t f_i^n) = \frac{1}{2}\delta_t (f_i^n)^2 \tag{4}$$

$$\mu_t \delta_t f_i^n = \frac{1}{2k}(f_i^{n+2} - f_i^n) \tag{5}$$

follow immediately from the definitions above. Forward and backward spatial difference operators δ_x and δ_x^{-1} are defined by

$$\delta_x f_i^n = \frac{1}{h}(f_{i+1}^n - f_i^n) \qquad \delta_x^{-1} f_i^n = \frac{1}{h}(f_i^n - f_{i-1}^n)$$

The operators δ_t, μ_t, δ_x, and δ_x^{-1} all commute.

For the subsequent energetic analysis of difference schemes, it is useful to define an inner product between two real-valued grid functions f_i^n and g_i^n by

$$\langle f^n, g^n \rangle = \sum_{i=-\infty}^{\infty} h f_i^n g_i^n$$

An L^2 norm, for square-summable sequences, then follows as

$$\|f^n\| = \langle f^n, f^n \rangle^{1/2}$$

(These definitions can be easily modified for problems defined over a bounded spatial domain, in which case the summation above is finite.)

The useful identity

$$\langle f^n, \delta_x^{-1} g^n \rangle = -\langle \delta_x f^n, g^n \rangle \tag{6}$$

is the discrete analogue of integration by parts.

2.3. A Finite Difference Scheme for the Transmission-Line Equations

The transmission-line equations (2) are amenable to simple interleaved finite difference schemes of the Yee variety [11, 12]. The variables p and q are approximated at alternating spatial locations and time-steps. We thus define grid functions p_i^n and $q_{i+\frac{1}{2}}^{n+\frac{1}{2}}$, both for integer i and n. p is calculated at coordinates x and t at even multiples of $h/2$ and $k/2$, respectively, and q at odd multiples.

A difference scheme corresponding to (2) is then

$$\delta_t p_i^{n-1} = c_0 \delta_x^{-1} q_{i+\frac{1}{2}}^{n-\frac{1}{2}} \tag{7a}$$

$$\delta_t q_{i+\frac{1}{2}}^{n-\frac{1}{2}} = c_0 \delta_x p_i^n \tag{7b}$$

which can be rewritten as

$$p_i^n = p_i^{n-1} + \lambda \left(q_{i+\frac{1}{2}}^{n-\frac{1}{2}} - q_{i-\frac{1}{2}}^{n-\frac{1}{2}} \right) \tag{8a}$$

$$q_{i+\frac{1}{2}}^{n+\frac{1}{2}} = q_{i+\frac{1}{2}}^{n-\frac{1}{2}} + \lambda \left(p_{i+1}^n - p_i^n \right) \tag{8b}$$

We have introduced here the important parameter $\lambda = c_0 k/h$, sometimes called the Courant number, which plays an important role in the stability analysis to follow. This scheme is consistent with system (2), and second order accurate, by virtue of the interleaving of grid quantities.

2.4. The Energy Method and Numerical Stability

Analysis of the system (7) can be carried out as follows. First, multiply (7a) by $\mu_t p_i^{n-1}$, to get

$$(\mu_t p_i^{n-1})(\delta_t p_i^{n-1}) = c_0 (\mu_t p_i^{n-1})(\delta_x^{-1} q_{i+\frac{1}{2}}^{n-\frac{1}{2}})$$

and by applying identity (4), we arrive at

$$\frac{1}{2}\delta_t (p_i^{n-1})^2 = c_0 (\mu_t p_i^{n-1})(\delta_x^{-1} q_{i+\frac{1}{2}}^{n-\frac{1}{2}})$$

We then sum over i and multiply by h, to get

$$\frac{1}{2}\delta_t \|p^{n-1}\|^2 = c_0 \langle \mu_t p^{n-1}, \delta_x^{-1} q^{n-\frac{1}{2}} \rangle$$

and, continuing,

$$\begin{aligned} \frac{1}{2}\delta_t \|p^{n-1}\|^2 &= -c_0 \langle \delta_x \mu_t p^{n-1}, q^{n-\frac{1}{2}} \rangle \\ &= -c_0 \langle \mu_t \delta_x p^{n-1}, q^{n-\frac{1}{2}} \rangle \\ &= -\langle \mu_t \delta_t q^{n-\frac{3}{2}}, q^{n-\frac{1}{2}} \rangle \\ &= -\frac{1}{2k}\langle q^{n+\frac{1}{2}} - q^{n-\frac{3}{2}}, q^{n-\frac{1}{2}} \rangle \end{aligned}$$

where the above steps follow from identity (6), commutativity of the operators μ_t and δ_x, (7b), and identity (5) respectively. It is easy to conclude that

$$\frac{1}{2}\delta_t \|p^{n-1}\|^2 = -\frac{1}{2}\delta_t \langle q^{n-\frac{1}{2}}, q^{n-\frac{3}{2}} \rangle \tag{9}$$

and then, from (8b), that

$$\frac{1}{2}\delta_t \|p^{n-1}\|^2 = -\frac{1}{2}\delta_t \left(\|q^{n-\frac{1}{2}}\|^2 - \lambda h \langle q^{n-\frac{1}{2}}, \delta_x p^{n-1} \rangle \right) \tag{10}$$

From this last equation we may define the quantity $\mathcal{E}^n$ by

$$\mathcal{E}^n = \frac{1}{2}\left(\|p^n\|^2 + \|q^{n+\frac{1}{2}}\|^2 - \lambda h\langle q^{n+\frac{1}{2}}, \delta_x p^n\rangle\right) \qquad (11)$$

which, from (10), remains constant as the recursion progresses. $\mathcal{E}^n$ is very nearly the discrete-time equivalent of the continuous-time energy as defined in (3). It remains to find the conditions under which $\mathcal{E}^n$ is positive definite (so that it can be used as a numerical stability guarantee).

Clearly, we have, from the Cauchy-Schwartz and triangle inequalities [13], that

$$\langle q^{n+\frac{1}{2}}, \delta_x p^n\rangle \leq \|q^{n+\frac{1}{2}}\|\|\delta_x p^n\| \leq \frac{2}{h}\|q^{n+\frac{1}{2}}\|\|p^n\|$$

which implies that

$$\begin{aligned}\mathcal{E}^n &\geq \frac{1}{2}\left(\|p^n\|^2 + \|q^{n+\frac{1}{2}}\|^2 - 2\lambda\|q^{n+\frac{1}{2}}\|\|p^n\|\right)\\ &= \frac{1}{2}\left((\|p^n\| - \|q^{n+\frac{1}{2}}\|)^2 + (2-2\lambda)\|q^{n+\frac{1}{2}}\|\|p^n\|\right)\end{aligned}$$

Finally, we can then say that

$$\mathcal{E}^n \geq 0 \qquad \text{if} \qquad \lambda \leq 1$$

and clearly, for the more strict condition $\lambda < 1$, $\mathcal{E}^n = 0$ only when p and q vanish identically. This is the familiar CFL condition on λ for explicit numerical schemes for hyperbolic systems [8, 7].

3. THE KIRCHHOFF-CARRIER EQUATION

A nonlinear generalization of the wave equation (1) to model large-amplitude vibration in stiff strings was first put forth by Kirchhoff in 1883 [4], and subsequently rediscovered by Carrier [5]. It can be written as

$$\rho\frac{\partial^2 u}{\partial t^2} = \left(T_0 + \frac{EA}{2L}\int_0^L \left(\frac{\partial u}{\partial x}\right)^2 dx\right)\frac{\partial^2 u}{\partial x^2} \qquad (12)$$

where E is Young's modulus, and A is the string cross-sectional area. Essentially, the tension in the string consists of the applied tension T_0 plus an additional contribution due to the significant change in length of the string under large-amplitude conditions.

Like the wave equation, (12) can be written as a first order system

$$\frac{\partial p}{\partial t} = c_0\left(1 + B\int_0^L q^2 dx\right)\frac{\partial q}{\partial x} \qquad (13a)$$

$$\frac{\partial q}{\partial t} = c_0\frac{\partial p}{\partial x} \qquad (13b)$$

where $B = EA/2LT_0^2$, and the variables p and q are defined as before. A simple energy analysis similar to that applied to the wave equation yields a conservation law

$$E_{KC}(t) = \frac{1}{2}\|p\|^2 + \frac{1}{2}(1 + \frac{B}{2}\|q\|^2)\|q\|^2 = \text{constant}$$

under the assumption of $p = 0$ or $q = 0$ at both boundary points. This is again a positive definite function of the state variables p and q. It is thus desirable to design a difference scheme for which a discrete version of this energy is preserved through each step in the recursion.

3.1. An Energy-Conserving Scheme

System (13) can also be approximated using an interleaved difference scheme like (7), but now of the form

$$\delta_t p_i^{n-1} = c_0 g^{n-\frac{1}{2}}\delta_x^{-1} q_{i+\frac{1}{2}}^{n-\frac{1}{2}} \qquad (14a)$$

$$\delta_t q_{i+\frac{1}{2}}^{n-\frac{1}{2}} = c_0\delta_x p_i^n \qquad (14b)$$

We againconsider the problem defined on an infinite spatial domain; we return to a discussion of boundary conditions in he next section. Here, the function $g^{n-\frac{1}{2}}$ is some second-order accurate approximation to the quantity $1 + B\int_{-\infty}^{\infty} q^2 dx$; notice, in particular, that it is not a grid function, as it depends only on time. We leave its exact form unspecified for the moment, and will return to it shortly.

The energetic analysis of this system is identical to that carried out in Section 2.4, except for the extra factor of $g^{n-\frac{1}{2}}$, up until the form given in (9). Now, however, we have

$$\frac{1}{2}\delta_t\|p^{n-1}\|^2 = -\frac{1}{2}g^{n-\frac{1}{2}}\delta_t\langle q^{n-\frac{1}{2}}, q^{n-\frac{3}{2}}\rangle$$

and cannot proceed further to a conservation law until the form of $g^{n-\frac{1}{2}}$ is specified. One possible choice for such an approximation is

$$g^{n-\frac{1}{2}} = 1 + B\mu_t\langle q^{n-\frac{1}{2}}, q^{n-\frac{3}{2}}\rangle$$

and we arrive, through a further application of identity (4), at

$$\frac{1}{2}\delta_t\|p^{n-1}\|^2 = -\frac{1}{2}\delta_t\langle q^{n-\frac{1}{2}}, q^{n-\frac{3}{2}}\rangle - \frac{B}{4}\delta_t\langle q^{n-\frac{1}{2}}, q^{n-\frac{3}{2}}\rangle^2$$

from which we can derive an energy $\mathcal{E}_{KC}^n$

$$\mathcal{E}_{KC}^n = \mathcal{E}^n + \frac{B}{4}\langle q^{n-\frac{1}{2}}, q^{n-\frac{3}{2}}\rangle^2 = \text{constant}$$

Notice that it is defined in terms of the energy function (11) for the difference scheme for the transmission line equations. The important point is that it is *larger* than this energy, and thus $\mathcal{E}_{KC}^n$ will be positive definite for $\lambda < 1$. This is now only a sufficient condition for stability; the nonlinear multiplier in the difference scheme apparently has a stabilizing effect on the difference scheme. A necessary condition will be derived in a later work.

The above stabilizing effect is probably related to the fact that, due to our choice of $g^{n-\frac{1}{2}}$, the difference system (14) is now *implicit*[8]. As we will show in the following section, this implicit character does not engender a huge increase in computational requirements; this is primarily due to the fact that the nonlinearity is not spatially-varying (i.e., in the Kirchhoff-Carrier model the tension is averaged over the entire string).

4. IMPLEMENTATION DETAILS

For a finite length string, defined over the interval $x \in [0, L]$, we will choose the grid spacing such that $M = L/h$ is an integer. Since the difference scheme (14) is interleaved, we should also align our grid functions accordingly. Since we will take fixed boundary conditions $p(0,t) = p(L,t) = 0$, it is natural to choose the grid function p_i^n to lie on the boundary itself. In this case, p_i^n need only be updated at the $M-1$ locations $i = 1, \ldots, M-1$.

The grid function $q_{i+\frac{1}{2}}^{n+\frac{1}{2}}$ will necessarily be updated at the M locations $i = 0, \ldots, M-1$. It is useful to define the vectors $\mathbf{p}^n$ and $\mathbf{q}^{n+\frac{1}{2}}$ by

$$\mathbf{p}^n = [p_1^n, \ldots, p_{M-1}^n]^T \qquad \mathbf{q}^{n+\frac{1}{2}} = [q_{\frac{1}{2}}^{n+\frac{1}{2}}, \ldots, q_{M-\frac{1}{2}}^{n+\frac{1}{2}}]^T$$

Defining the vectors $\mathbf{a}^{n-\frac{1}{2}}$ and $\mathbf{b}^{n-1}$ and the scalar $\gamma^{n-\frac{1}{2}}$ by

$$\begin{aligned}
\mathbf{a}^{n-\frac{1}{2}} &= \lambda\sqrt{\frac{B}{2}}h^{\frac{3}{2}}[\delta_x q_{\frac{1}{2}}^{n-\frac{1}{2}}, \ldots, \delta_x q_{M-\frac{1}{2}}^{n-\frac{1}{2}}] \\
\gamma^{n-\frac{1}{2}} &= \sqrt{\frac{2}{Bh}}\left(1 + Bh(\mathbf{q}^{n-\frac{1}{2}})^T\mathbf{q}^{n-\frac{1}{2}}\right) + (\mathbf{a}^{n-\frac{1}{2}})^T\mathbf{p}^{n-1} \\
\mathbf{b}^{n-1} &= \mathbf{p}^{n-1} + \gamma^{n-\frac{1}{2}}\mathbf{a}^{n-\frac{1}{2}}
\end{aligned}$$

then (14a) can be written, in explicit form, through an application of the matrix inversion lemma [13] as

$$\mathbf{p}^n = \left(\mathbf{I}_{M-1} - \frac{\mathbf{a}^{n-\frac{1}{2}}(\mathbf{a}^{n-\frac{1}{2}})^T}{1 + \mathbf{a}^{n-\frac{1}{2}}(\mathbf{a}^{n-\frac{1}{2}})}\right)\mathbf{b}^{n-1}$$

where $\mathbf{I}_{M-1}$ is the $M-1 \times M-1$ identity matrix. Once $\mathbf{p}^n$ has been updated, $\mathbf{q}^{n+\frac{1}{2}}$ may be calculated explicitly using (14b) (using, in this case, boundary values $p_0^n = p_M^n = 0$).

5. NUMERICAL EXPERIMENTS

The Kirchhoff-Carrier model is lossless, and thus not a complete model for sound synthesis purposes. The difference scheme we have proposed is merely a first step towards such a synthesis algorithm. It does, however, possess a useful stability property, which is dependent, as in the linear case, only on the value of the Courant number. During a simulation, the discrete energy function $\mathcal{E}_{KC}$ remains invariant to machine precision. We simulated a string of length 0.65 m, made of steel (of linear density $\rho = 6 \times 10^{-4}$ kg/m and with Young's Modulus $E = 2 \times 10^{11}\text{N/m}^2$), of cross-sectional area $A = 3.6 \times 10^{-8}\text{m}^2$, and under tension $T_0 = 120\text{N}$[1]. At a sample rate of 44 100 Hz, and with an initial string profile of a raised cosine of height 5 cm, width 13 cm, centered on the string, $\mathcal{E}_{KC} = 25.169736984700$ J, which remains constant, to 12 decimal places, throughout the 1 s duration of the simulation.

On the other hand, though perfectly lossless, this difference scheme suffers, as is to be expected, from severe parasitic oscillations; this is often the case for difference schemes of the centered variety [8, 7]. The usual cure is some form of artificial dissipation (also called artificial viscosity); we will examine this technique, and its consequences in terms of the conservation property of the scheme in a subsequent work.

6. CONCLUSIONS

We have shown how perfectly energy-conserving difference schemes for a nonlinear string equation may be derived, and how this property leads to a numerical stability condition. Such robustness of a difference scheme is extremely useful in the context of musical sound synthesis, where the algorithm must be able to handle a large variety of possible excitations; here, stability depends only on the Courant number, not on any initial conditions applied. On the other hand, this scheme suffers from severe parasitic oscillations, so it is not yet ready to be used for synthesis purposes. We remark that digital waveguides [1, 2, 3] have been proposed as a means of simulating a tension-modulated string; here, the solution is modelled in terms of travelling waves of variable speed (leading to variable-length digital delay lines), so parasitic oscillations are presumably not an issue. The Kirchhoff-Carrier Equation, however, does not admit such travelling-wave solutions, so the digital waveguide technique must be considered as an approximation this system. On the other hand, the Kirchhoff-Carrier Equation itself is an approximation; there are obviously many issues which need clarification, not least of which is the relative perceptual significances of such approximations in sound synthesis applications.

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