Conservative numerical methods for nonlinear strings

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In this article, a class of numerical schemes for the simulation of nonlinear coupled longitudinal/transverse string vibration is presented. Though there are various ways of arriving at such schemes, special attention is paid here to energy conservation in nonlinear model systems and its transfer to an analogous discrete quantity in a difference scheme. Such exact numerical energy conservation can lead to simple global stability conditions, which can be otherwise difficult to ascertain for nonlinear difference schemes—in particular, such conditions may be arrived at without any reliance on frequency domain concepts (i.e., Fourier or Laplace transforms), which are of only moderate utility in the analysis of nonlinear systems. Implementation details are discussed and numerical results are presented. © *2005 Acoustical Society of America.* [DOI: 10.1121/1.2046787]

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I. INTRODUCTION

Work on nonlinear string vibration has been ongoing for more than a century, and the field has, in some respects, reached maturity. There are various partial differential equation (PDE) models which describe the large amplitude vibration of a nonlinear string, and in particular the coupling between longitudinal and transverse motion, to varying degrees of complexity. Though a complete summary is impossible in this short article, it is worth mentioning that there are two discernible families of such models: (1) general forms for which the nonlinearity is pointwise dependent on the string state,[1,2] and (2) simplified forms (referred to here as Kirchhoff-Carrier type models) for which the longitudinal motion is "averaged out" to yield a nonlinear system in the transverse motion alone, with the nonlinearity intervening in a distributed, but not spatially varying way.[3–6] It is to be noted that the formulation and study of such models is of fundamental importance, as the nonlinear string is perhaps the simplest distributed nonlinear system occurring in nature. The reader is referred to the text by Morse and Ingard[1] and the work of Vallette[7] for an overview. Nonlinear string models are intimately related to (and can often be viewed as special cases of) various models of beam vibration, and in particular, the so-called "geometrically-exact" formulation of Simo and Vu-Quoc.[8,9]

Though there do exist various established PDE models for nonlinear string dynamics, when it comes to numerical solution techniques, the situation is far less clear, and this aspect has been given relatively little attention in the literature. This is surprising, given the utility of numerical results as a means of comparing theory and measured data. Though there have been a few cursory descriptions of numerical schemes for nonlinear strings,[10–12] these are usually simple extensions of schemes for the linear wave equation; in particular, the important question of numerical stability is not addressed. (Two exceptions are the work of Furihata,[13] who employs a technique loosely related to that to be presented here to a simplified transverse-only nonlinear string model, and Rubin's work on numerical methods based on the theory of a Cosserat point;[14] in neither case is any numerical stability analysis carried out.)

In previous work, this author has described finite difference schemes for a Kirchhoff-Carrier string model in a single transverse polarization.[15–17] These schemes mirror not only the dynamics of the continuous string model, but also possess a conserved quantity analogous to an energy—this conservation property may then be used to find useful global stability conditions (i.e., stability which is independent of initial conditions), through analysis often referred to as the *energy method*,[18–21] which relies on concepts in functional analysis.[22] Stability verification for nonlinear difference schemes is otherwise difficult, and, in particular, cannot be arrived at definitively through Fourier or spectral analysis techniques (often referred to as von Neumann analysis in this context[23,24]). (The text by Gustaffson, Kreiss, and Oliger[18] discusses the ranges of applicability of spectral techniques in great detail.) Energy-based analysis and construction of numerical schemes has seen quite a bit of increased activity in recent years, and has been applied to various types of solid systems (see, e.g., Ref. 25).

Kirchhoff-Carrier type models, however, are but simplified representations of nonlinear string dynamics, and the numerical results in previous work by this author[15] exploit the spatially averaged character of the nonlinearity (the same can be said of extended digital waveguide type approaches to solutions of the so-called "tension-modulated" string[26–28] as well as quasimodal descriptions of nonlinear strings,[29,30] which lead to highly accurate energy conserving methods[17] of the spectral type[31,32]). Furthermore, as mentioned earlier, longitudinal motion is not explicitly modeled. In more realistic nonlinear string models, however, the pointwise nonlinear coupling between the transverse and longitudinal motion persists; such models are the subject of this article, and are related to so-called "geometrically exact" beam vibration formulations mentioned earlier. The case of planar motion is discussed here—an extension to full three-dimensional motion (see, e.g., Ref. 33) is immediate.

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The techniques earlier applied to Kirchhoff-Carrier models are here extended to the much more general case of pointwise nonlinear string equations of various types, which are given along with an energetic analysis in Sec. II. Several commonly encountered approximate forms of the nonlinear string equations are presented (see Table I), only one of which ($\mathbf{S}_4^*$) is suitable for energy-based stability analysis. In Sec. III, some basic facts regarding grid functions and finite difference operators are presented, followed by finite difference schemes of the interleaved type for the string system $\mathbf{S}_4^*$ described in Sec. II. A discussion of conservation properties ensures, and a particular scheme for which numerical stability may be guaranteed is presented (scheme $\mathbf{s}_4^{*,(c)}$, given in Table II). The section is concluded with a cursory look at boundary conditions and implementation issues. Numerical results follow in Sec. IV.

As a note, the author wishes to add that a latent goal of the present study is to develop a general methodology for the robust analysis and synthesis of sound for stringed musical instruments; such schemes are the only rigorous means of generating synthetic sound for those musical instruments whose timbre is inextricably linked to nonlinear effects (so-called "phantom partials" in piano tones[10,34] and pitch glides in plucked instruments such as the Finnish kantele[26,27] being two notable examples). Though this article is intended for a more general readership, the numerical examples are chosen so as to hint at the possibilities for musical sound synthesis.

II. MODEL EQUATIONS

A general model of nonlinear string dynamics, described by Morse and Ingard[1] and which can be related to the geometrically exact theory of beams,[35] is given by the following set of equations:

$$\rho\ddot{\xi} = EA\xi'' - (EA - T_0)\left(\frac{1+\xi'}{\sqrt{(1+\xi')^2+(\eta')^2}}\right)', \tag{1a}$$

$$\rho\ddot{\eta} = EA\eta'' - (EA - T_0)\left(\frac{\eta'}{\sqrt{(1+\xi')^2+(\eta')^2}}\right)'. \tag{1b}$$

Here, $\xi(x,t)$ and $\eta(x,t)$ describe, respectively, the longitudinal and transverse deviation of a point on the string as a function of time $t \geq 0$ and distance along the string $x \in [0,L]$. Such a point, located at Cartesian coordinates $(x,0)$ when the string is at rest, will have dynamic coordinates $(x+\xi, \eta)$. E, A, ρ, and T_0 are Young's modulus, cross-sectional area, linear mass density, and nominal tension for the string, all assumed constant here. Dots and primes indicate partial differentiation with respect to time and space, respectively. System (1) may be generalized further in various ways, including through the introduction of linear terms modeling loss and dispersion, as well as excitations, and to include motion in both transverse polarizations; all of the results on finite difference schemes which follow are affected only in a minor way by such generalizations. It may also be generalized to include more subtle higher order effects, as per the work of Narasimha;[2] such improved models may fall outside the range of the techniques presented here, and their consideration is left to a future work.

Two types of boundary conditions are be considered here. For analysis purposes, most useful are so-called periodic boundary conditions[18] of the form

$$\xi(0,t) = \xi(L,t), \quad \eta(0,t) = \eta(L,t) \tag{2}$$

As a more realistic case, conditions of the fixed type, i.e.,

$$\xi(0,t) = \xi(L,t) = 0, \quad \eta(0,t) = \eta(L,t) = 0 \tag{3}$$

are briefly examined as well.

System (1) requires the specification of four initial conditions, namely $\xi(x,0)$, $\dot{\xi}(x,0)$, $\eta(x,0)$, and $\dot{\eta}(x,0)$.

A. A transmission-line form

It is useful to rewrite system (1) as a system of four first-order equations. Defining the variables

$$p_\xi = \dot{\xi}, \quad q_\xi = \xi', \quad p_\eta = \dot{\eta}, \quad q_\eta = \eta' \tag{4}$$

it then follows that

$$\begin{aligned} \rho\dot{p}_\xi &= EAq_\xi' - (EA - T_0)\left(\frac{\partial\Psi}{\partial q_\xi}\right)', \\ \rho\dot{p}_\eta &= EAq_\eta' - (EA - T_0)\left(\frac{\partial\Psi}{\partial q_\eta}\right)', \end{aligned} \tag{S}$$

$$\dot{q}_\xi = p_\xi', \quad \dot{q}_\eta = p_\eta' \quad \text{(Auxiliary System)}, \tag{5}$$

where the quantity $\Psi(q_\xi, q_\eta)$ (representing a contribution to the potential energy density of the string due to the nonlinearity) is defined by

$$\Psi = \sqrt{(1+q_\xi)^2 + q_\eta^2} + a + a_\xi q_\xi + a_\eta q_\eta \tag{6}$$

for arbitrary constants a, a_ξ, and a_η [note that these constants have no effect on the solutions of system (1)]. System **S** (signifying "string"), accompanied by the *auxiliary system* (5), can be viewed as a pair of nonlinearly coupled transmission lines, one in the variables p_ξ, q_ξ, and the other in the variables p_η, q_η. The coupling occurs through the terms containing $\Psi(q_\xi, q_\eta)$.

B. Series approximations and simplifications

Slightly more tractable forms, as presented in Table I, may be obtained through the use of Taylor series approximations about $q_\xi = q_\eta = 0$ (employing the choices $a = a_\xi = -1$ and $a_\eta = 0$) to the function $\Psi(q_\xi, q_\eta)$, given in the following to fourth order:

$$\Psi = \tfrac{1}{2}q_\eta^2 - \tfrac{1}{2}q_\xi q_\eta^2 + \tfrac{1}{2}q_\xi^2 q_\eta^2 - \tfrac{1}{8}q_\eta^4 + \cdots. \tag{7}$$

As discussed in the following, the energetic behavior of the resulting system depends critically on the type of approximation made. The standard series approximation, found, e.g., in Morse,[1] is given by truncating this series approximation to fourth order, yielding the system $\mathbf{S}_4$. Truncation of Ψ to third order is also sometimes employed,[10,11] giving the form $\mathbf{S}_3$, and truncation to second order uncouples the longitudinal and transverse motion completely (see form $\mathbf{S}_2$).

The approximation Ψ_4^*, employed by Anand,[5] and given by

TABLE I. Approximations to system **S**, as well as the associated expressions for kinetic and potential energy.

	Defining equations	(aux)	Kinetic energy	Potential energy
$\mathbf{S}_4$	$\rho\dot{p}_\xi = EAq'_\xi + \frac{EA-T_0}{2}(q_\eta^2[1-2q_\xi])'$ $\rho\dot{p}_\eta = T_0 q'_\eta + \frac{EA-T_0}{2}(q_\eta^3 + 2q_\eta q_\xi[1-q_\xi])'$	$\dot{q}_\xi = p'_\xi$ $\dot{q}_\eta = p'_\eta$	$\mathcal{T}_\mathbf{S} = \frac{\rho}{2}(\|p_\xi\|^2 + \|p_\eta\|^2)$	$\mathcal{V}_{\mathbf{S}_4} = \frac{T_0}{2}(\|q_\xi\|^2 + \|q_\eta\|^2) + \frac{EA-T_0}{8}(\|q_\eta^2 + 2q_\xi\|^2 - \|q_\eta q_\xi\|^2)$
$\mathbf{S}_4^*$	$\rho\dot{p}_\xi = EAq'_\xi + \frac{EA-T_0}{2}(q_\eta^2)'$ $\rho\dot{p}_\eta = T_0 q'_\eta + \frac{EA-T_0}{2}(q_\eta^3 + 2q_\eta q_\xi)'$			$\mathcal{V}_{\mathbf{S}_4^*} = \frac{T_0}{2}(\|q_\xi\|^2 + \|q_\eta\|^2) + \frac{EA-T_0}{8}\|q_\eta^2 + 2q_\xi\|^2$
$\mathbf{S}_3$	$\rho\dot{p}_\xi = EAq'_\xi + \frac{EA-T_0}{2}(q_\eta^2)'$ $\rho\dot{p}_\eta = T_0 q'_\eta + \frac{EA-T_0}{2}(2q_\eta q_\xi)'$			$\mathcal{V}_{\mathbf{S}_3} = \frac{EA}{2}\|q_\xi\|^2 + \frac{T_0}{2}\|q_\eta\|^2 + \frac{EA-T_0}{2}\langle q_\xi, q_\eta^2\rangle$
$\mathbf{S}_2$	$\rho\dot{p}_\xi = EAq'_\xi$ $\rho\dot{p}_\eta = T_0 q'_\eta$			$\mathcal{V}_{\mathbf{S}_2} = \frac{EA}{2}\|q_\xi\|^2 + \frac{T_0}{2}\|q_\eta\|^2$
$\mathbf{S}_{T,4}$	$\rho\dot{p}_\eta = T_0 q'_\eta + \frac{EA-T_0}{2}(q_\eta^3)'$	$\dot{q}_\eta = p'_\eta$	$\mathcal{T}_{\mathbf{S}_T} = \frac{\rho}{2}\|p_\eta\|^2$	$\mathcal{V}_{\mathbf{S}_{T,4}} = \frac{EA}{2}\|q_\eta\|^2 + \frac{EA-T_0}{8}\|q_\eta^2\|^2$

$$\Psi_4^*(q_\xi, q_\eta) = \tfrac{1}{2}q_\eta^2 - \tfrac{1}{2}q_\xi q_\eta^2 - \tfrac{1}{8}q_\eta^4 \tag{8}$$

is correct to third order, but lacks one of the fourth-order terms of Ψ, and leads to the system $\mathbf{S}_4^*$. One might well ask in what sense the use of system $\mathbf{S}_4^*$ is justified. One answer follows from the energetic properties to be discussed in the next section; another relates to the relative orders of magnitude of q_ξ and q_η in system **S**. As noted by Anand[5] and Morse,[1] q_ξ is of the same order of magnitude as q_η^2, and it is perhaps more natural, then, to use a homogeneous approximation, truncated to powers of, say, q_η. Given that the term $q_\xi^2 q_\eta^2$ is clearly of sixth order in q_η, it is then justified to neglect it with respect to the term in q_η^4.

As a useful nonlinear test problem for finite difference schemes, it is also worthwhile to consider the transverse-only system (i.e., assuming $p_\xi = q_\xi = 0$) under a fourth-order nonlinearity, given as $\mathbf{S}_{T,4}$ in Table I.

C. Energetic analysis

The procedure by which one may extract a conserved energy-like quantity from a system is similar to that which was applied to a Kirchhoff-Carrier system by this author in Ref. 15. A condensed treatment is provided here.

As a notational preliminary, it is worth recalling here the definition of the spatial L^2 inner product of two real-valued functions $f(x,t)$ and $g(x,t)$ over the interval $x \in [0, L]$, as well as the associated norm:

$$\langle f, g\rangle = \int_0^L fg\,dx, \quad \|f(\cdot,t)\| = \langle f, f\rangle^{1/2}. \tag{9}$$

Consider first system **S**. Multiplying the first and second equations by p_ξ and p_η, respectively, and integrating over the range $x \in [0, L]$ gives

$$\int_0^L \rho p_\xi \dot{p}_\xi dx = \int_0^L p_\xi \left(EAq_\xi - (EA - T_0)\frac{\partial \Psi}{\partial q_\xi}\right)' dx,$$

$$\int_0^L \rho p_\eta \dot{p}_\eta dx = \int_0^L p_\eta \left(EAq_\eta - (EA - T_0)\frac{\partial \Psi}{\partial q_\eta}\right)' dx.$$

Using integration by parts on the right-hand side of these equations, and either of the boundary conditions of Eq. (2) or Eq. (3), as well as the auxiliary equations (5) gives

$$\int_0^L \rho p_\xi \dot{p}_\xi dx = -\int_0^L \dot{q}_\xi \left(EAq_\xi - (EA - T_0)\frac{\partial \Psi}{\partial q_\xi}\right) dx,$$

$$\int_0^L \rho p_\eta \dot{p}_\eta dx = -\int_0^L \dot{q}_\eta \left(EAq_\eta - (EA - T_0)\frac{\partial \Psi}{\partial q_\eta}\right) dx.$$

Finally, applying the definition of the norm as per Eq. (9) and summing the resulting equations gives

$$\frac{d}{dt}\mathcal{H}_\mathbf{S} = 0 \Rightarrow \mathcal{H}_\mathbf{S} = \mathcal{T}_\mathbf{S} + \mathcal{V}_\mathbf{S} = \text{constant} \tag{10}$$

with

$$\mathcal{T}_\mathbf{S} = \frac{\rho}{2}(\|p_\xi\|^2 + \|p_\eta\|^2),$$

$$\mathcal{V}_\mathbf{S} = \frac{EA}{2}(\|q_\xi\|^2 + \|q_\eta\|^2) - (EA - T_0)\int_0^L \Psi dx.$$

Thus $\mathcal{H}_\mathbf{S}$ is a scalar conserved quantity of system **S**; in particular, it behaves as an energy ($\mathcal{H}$ signifies "Hamiltonian"), with a kinetic part $\mathcal{T}_\mathbf{S}$ dependent on p_ξ and p_η and a potential part $\mathcal{V}_\mathbf{S}$ dependent on q_ξ and q_η.

The systems $\mathbf{S}_4$, $\mathbf{S}_4^*$, $\mathbf{S}_3$, and $\mathbf{S}_2$ all result from series approximations to Ψ, and as such, also possess conserved quantities. The kinetic energy terms remain the same in all cases, and the potential energy is given by the above-presented expression for $\mathcal{V}_\mathbf{S}$, with Ψ replaced by the series truncated form. The resulting expressions are denoted by $\mathcal{V}_{\mathbf{S}_4}$, $\mathcal{V}_{\mathbf{S}_4^*}$, $\mathcal{V}_{\mathbf{S}_3}$, and $\mathcal{V}_{\mathbf{S}_2}$, respectively, and appear at the right in

TABLE II. Finite difference schemes of the interleaved type for the systems $\mathbf{S}_2$, $\mathbf{S}_4^*$, and $\mathbf{S}_{T,4}$ as given in Table I.

	Defining equations	(Auxiliary)
$\mathbf{s}_2$	$\rho\delta_{t+}p_{\xi,i}^{n-1}=EA\,\delta_{x-}q_{\xi,i+1/2}^{n-1/2}$ $\rho\delta_{t+}p_{\eta,i}^{n-1}=T_0\delta_{x-}q_{\eta,i+1/2}^{n-1/2}$	$\delta_{t+}q_{\xi,i+1/2}^{n-1/2}=\delta_{x+}p_{\xi,i}^{n}$ $\delta_{t+}q_{\eta,i+1/2}^{n-1/2}=\delta_{x+}p_{\eta,i}^{n}$
$\mathbf{s}_4^{*,(a)}$	$\rho\delta_{t+}p_{\xi,i}^{n-1}=EA\,\delta_{x-}q_{\xi,i+1/2}^{n-1/2}+\frac{EA-T_0}{2}\delta_{x-}(q_{\eta,i+1/2}^{n-1/2})^2$ $\rho\delta_{t+}p_{\eta,i}^{n-1}=T_0\delta_{x-}q_{\eta,i+1/2}^{n-1/2}+\frac{EA-T_0}{2}\delta_{x-}([q_{\eta,i+1/2}^{n-1/2}]^3+2q_{\xi,i+1/2}^{n-1/2}q_{\eta,i+1/2}^{n-1/2})$	
$\mathbf{s}_4^{*,(b)}$	$\rho\delta_{t+}p_{\xi,i}^{n-1}=EA\,\delta_{x-}q_{\xi,i+1/2}^{n-1/2}+\frac{EA-T_0}{2}\delta_{x-}(q_{\eta,i+1/2}^{n-1/2})^2$ $\rho\delta_{t+}p_{\eta,i}^{n-1}=T_0\delta_{x-}q_{\eta,i+1/2}^{n-1/2}+\frac{EA-T_0}{2}\delta_{x-}([[q_{\eta,i+1/2}^{n-1/2}]^2+2q_{\xi,i+1/2}^{n-1/2}]\mu_{t0}q_{\eta,i+1/2}^{n-1/2})$	
$\mathbf{s}_4^{*,(c)}$	$\rho\delta_{t+}p_{\xi,i}^{n-1}=EA\,\delta_{x-}q_{\xi,i+1/2}^{n-1/2}+\frac{EA-T_0}{2}\delta_{x-}(q_{\eta,i+1/2}^{n-1/2}\mu_{t0}q_{\eta,i+1/2}^{n-1/2})$ $\rho\delta_{t+}p_{\eta,i}^{n-1}=T_0\delta_{x-}q_{\eta,i+1/2}^{n-1/2}+\frac{EA-T_0}{2}\delta_{x-}([q_{\eta,i+1/2}^{n-1/2}]^2\mu_{t0}q_{\eta,i+1/2}^{n-1/2}+q_{\eta,i+1/2}^{n-1/2}\mu_{t+}\mu_{t-}q_{\xi,i+1/2}^{n-1/2})$	
$\mathbf{s}_{T,4}^{(a)}$	$\rho\delta_{t+}p_{\eta,i}^{n-1}=T_0\delta_{x-}q_{\eta,i+1/2}^{n-1/2}+\frac{EA-T_0}{2}\delta_{x-}(q_{\eta,i+1/2}^{n-1/2})^3$	$\delta_{t+}q_{\eta,i+1/2}^{n-1/2}=\delta_{x+}p_{\eta,i}^{n}$
$\mathbf{s}_{T,4}^{(b)}$	$\rho\delta_{t+}p_{\eta,i}^{n-1}=T_0\delta_{x-}q_{\eta,i+1/2}^{n-1/2}+\frac{EA-T_0}{2}\delta_{x-}([q_{\eta,i+1/2}^{n-1/2}]^2\mu_{t0}q_{\eta,i+1/2}^{n-1/2})$	

Table I. For the simplified nonlinear transverse-only system $\mathbf{S}_{T,4}$, the kinetic and potential energies are denoted by $\mathcal{T}_{\mathbf{S}_T}$ and $\mathcal{V}_{\mathbf{S}_{T,4}}$; explicit expressions for these are also given in Table I.

D. Bounds on solution size

As a prelude to an energetic analysis of finite difference schemes, it is useful to compare the above-mentioned approximations to the system $\mathbf{S}$, especially insofar as the conserved quantities mentioned earlier lead to global bounds on the size of the solution.

Consider system $\mathbf{S}_2$, which, it is to be recalled, is the linearized form of the string equation. From the expressions given in Table I, it is clearly true that $\mathcal{T}_{\mathbf{S}}\geqslant 0$ and $\mathcal{V}_{\mathbf{S}_2}\geqslant 0$, implying $\mathcal{H}_{\mathbf{S}_2}\geqslant 0$ and, furthermore,

$$\|p_\xi\|,\|p_\eta\|\leqslant\sqrt{\frac{2\mathcal{H}_{\mathbf{S}_2}}{\rho}},\tag{11}$$

$$\|q_\xi\|\leqslant\sqrt{\frac{2\mathcal{H}_{\mathbf{S}_2}}{EA}},\quad \|q_\eta\|\leqslant\sqrt{\frac{2\mathcal{H}_{\mathbf{S}_2}}{T_0}}.\tag{12}$$

As $\mathcal{H}_{\mathbf{S}_2}$ is constant and equal to its value at time $t=0$, there are thus global bounds on the size of the solution of system $\mathbf{S}_2$ at all future times.

In contrast, for the commonly encountered approximate systems $\mathbf{S}_3$ and $\mathbf{S}_4$, such global conditions do not exist, as the energy function is not necessarily positive for all possible choices of the string state; choosing, for instance, $q_\xi=\mathrm{sgn}(T_0-EA)q_\eta^2$, then $\mathcal{V}_{\mathbf{S}_4}$ and $\mathcal{V}_{\mathbf{S}_3}$ are negative and unbounded in the limit as q_η becomes large. Because the solutions themselves cannot be bounded, it is unreasonable to expect to find finite difference schemes which are provably numerically stable under general conditions; systems $\mathbf{S}_3$ and $\mathbf{S}_4$ are not considered henceforth in this article, though it should be emphasized that they do possess a conserved energy-like quantity which can be transferred to discrete time.

System $\mathbf{S}_4^*$, however, is well-behaved in this sense, under the restricted condition

$$EA\geqslant T_0.\tag{13}$$

This is indeed the case for many systems of practical (and also musical) interest. $\mathcal{T}_{\mathbf{S}}$ and $\mathcal{V}_{\mathbf{S}_4^*}$ are both non-negative, and it follows immediately that

$$\|p_\xi\|,\|p_\eta\|\leqslant\sqrt{\frac{2\mathcal{H}_{\mathbf{S}_4^*}}{\rho}},\quad \|q_\xi\|,\|q_\eta\|\leqslant\sqrt{\frac{2\mathcal{H}_{\mathbf{S}_4^*}}{T_0}}.\tag{14}$$

It is probably possible to tighten the above-mentioned bounds through further analysis.

For the simplified system $\mathbf{S}_{T,4}$, the energy $\mathcal{H}_{\mathbf{S}_{T,4}}=\mathcal{T}_{\mathbf{S}_T}+\mathcal{V}_{\mathbf{S}_{T,4}}$ is non-negative, again under condition (13), leading to bounds

$$\|p_\eta\|\leqslant\sqrt{\frac{2\mathcal{H}_{\mathbf{S}_{T,4}}}{\rho}},\quad \|q_\eta\|\leqslant\sqrt{\frac{2\mathcal{H}_{\mathbf{S}_{T,4}}}{EA}}.\tag{15}$$

III. FINITE DIFFERENCE SCHEMES

In the following, finite difference schemes for the various nonlinear string systems discussed in Sec. II are introduced. Stability conditions which are based on discrete energy conservation, when available, are then derived. A good tutorial treatment of the energy method for finite difference schemes is given by Vu-Quoc and Li.[36,37] The reader is reminded that frequency-domain concepts are nowhere employed in this method.

A. Preliminaries

In order to approximate a real-valued function such as $f(x,t)$ which appears as a dependent variable in a PDE, a first step is to introduce a grid function f_i^n, which serves as an approximation at the coordinates $x=ih_x$, $t=nh_t$, for i and n integer. Here h_x and h_t are the grid spacing and time step, respectively. In view of the use of such grid functions in initial boundary value problems such as the string, the restrictions $n \geqslant 0$ and $i=0,\ldots,N-1$, where $N=L/h_x$, are also imposed. As the difference schemes to be discussed here are of the *interleaved* or *finite-difference time domain* variety,[38,39] it is also helpful to define grid functions such as $g_{i+1/2}^{n+1/2}$, again for integer n and $i=0,\ldots,N-1$, representing an approximation to a continuously variable function $g(x,t)$ at time $t=(n+1/2)h_t$ and at location $x=(i+1/2)h_x$.

The forward time difference operator δ_{t+} and time-mean operators μ_{t+}, μ_{t-}, and μ_{t0} are defined by

$$\delta_{t+}f_i^n = \frac{1}{h_t}(f_i^{n+1} - f_i^n),$$

$$\mu_{t+}f_i^n = \tfrac{1}{2}(f_i^{n+1} + f_i^n) \quad \mu_{t-}f_i^n = \tfrac{1}{2}(f_i^n + f_i^{n-1}),$$

$$\mu_{t0}f_i^n = \tfrac{1}{2}(f_i^{n+1} + f_i^{n-1})$$

and forward and backward spatial difference operators δ_{x+} and δ_{x-} by

$$\delta_{x+}f_i^n = \frac{1}{h_x}(f_{i+1}^n - f_i^n), \quad \delta_{x-}f_i^n = \frac{1}{h_x}(f_i^n - f_{i-1}^n).$$

For periodic boundary conditions, the spatial indices of the grid function are to be taken modulo N. For instance, $\delta_{x+}f_{N-1}^n = (1/h_x)(f_0^n - f_{N-1}^n)$.

All the discrete operators defined earlier are pairwise commutative. (The symbols δ and μ are intended as mnemonics for "difference" and "mean," respectively.)

The identities

$$(\mu_{t+}f_i^n)(\delta_{t+}f_i^n) = \delta_{t+}\left(\tfrac{1}{2}(f_i^n)^2\right), \tag{16}$$

$$\mu_{t+}\delta_{t+}f_i^n = \frac{1}{2h_t}(f_i^{n+2} - f_i^n), \tag{17}$$

$$(\mu_{t0}f_i^n)(\mu_{t+}\delta_{t+}f_i^{n-1}) = \mu_{t+}\delta_{t+}\left(\tfrac{1}{2}(f_i^{n-1})^2\right) \tag{18}$$

follow immediately from the above-presented definitions.

It is useful to define an inner product at time step n between two real-valued grid functions f_i^n and g_i^n (and the associated norm) by

$$\langle f^n, g^n\rangle = \sum_{i=0}^{N-1} h_x f_i^n g_i^n, \quad \|f^n\| = \langle f^n, f^n\rangle^{1/2}.$$

(The grid function g_i^n above may be replaced by a grid function $g_{i+1/2}^{n+1/2}$, interleaved with respect to f_i^n without affecting the above-presented definition.) It then follows that

$$\langle f^n, r^n, g^n\rangle = \langle f^n r^n, g^n\rangle \tag{19}$$

for any three grid functions f, g, and r.

Recall also the triangle inequality,

$$\|f^n + g^n\| \leqslant \|f^n\| + \|g^n\| \tag{20}$$

which implies, in particular, that

$$\|\delta_{x-}f^n\| \leqslant \frac{2}{h_x}\|f^n\|. \tag{21}$$

The useful identity

$$\langle f^n, \delta_{x-}g^n\rangle = -\langle \delta_{x+}f^n, g^n\rangle \tag{22}$$

holds for periodic boundary conditions and is the discrete analogue of integration by parts.

B. Interleaved schemes

In developing difference schemes, it is perhaps simplest to begin from the uncoupled linear system $\mathbf{S}_2$. Due to this lack of coupling, it is permissible to examine the two systems, one in p_ξ and q_ξ, the other in p_η and q_η, in isolation. A centered finite difference scheme, here called $\mathbf{s}_2$, can then be written as shown in the first row of Table II. Here, the grid functions $p_{\xi,i}^n$ and $p_{\eta,i}^n$, approximations to $p_\xi(x,t)$ and $p_\eta(x,t)$ are interleaved in time and space with respect to the grid functions $q_{\xi,i+1/2}^{n-1/2}$ and $q_{\eta,i+1/2}^{n-1/2}$, which are approximations to $q_\xi(x,t)$ and $q_\eta(x,t)$.

To see this interleaving property clearly, it is helpful to rewrite system $\mathbf{s}_2$ in update form. For the uncoupled system in $p_{\eta,i}^n$ and $q_{\eta,i+1/2}^{n-1/2}$,

$$p_{\eta,i}^n = p_{\eta,i}^{n-1} + \frac{T_0}{\rho}\alpha(q_{\eta,i+1/2}^{n-1/2} - q_{\eta,i-1/2}^{n-1/2}),$$

$$q_{\eta,i+1/2}^{n+1/2} = q_{\eta,i+1/2}^{n-1/2} + \alpha(p_{\eta,i+1}^n - p_{\eta,i}^n),$$

where the important parameter α is defined by

$$\alpha = h_t/h_x. \tag{23}$$

If the updates are performed in the order in which they are presented above, the scheme is fully explicit. The other subsystem in $p_{\xi,i}^n$ and $q_{\xi,i+1/2}^{n-1/2}$ is of the same form.

As a first example of a nonlinear difference scheme, consider the simplified transverse-only nonlinear system $\mathbf{S}_{T,4}$, which depends only on p_η and q_η. There are plainly many ways to approximate the nonlinearity. Perhaps the most straightforward choice is scheme $\mathbf{s}_{T,4}^{(a)}$, given in Table II. This finite difference scheme, like system $\mathbf{s}_2$, is again interleaved and can be updated explicitly; unfortunately, it does not possess a simple conserved quantity analogous to $\mathcal{H}_{\mathbf{S}_{T,4}}$, and its stability properties are difficult to ascertain. Another choice of difference scheme is given by $\mathbf{s}_{T,4}^{(b)}$. Many other choices are possible, but as discussed presently, scheme $\mathbf{s}_{T,4}^{(b)}$ possesses a simple conserved energy which allows for a convenient global stability condition. It should be noted, however, that scheme $\mathbf{s}_{T,4}^{(b)}$ is not fully explicit, i.e., it requires the solution of a sparse linear system at each time step. More details are provided in Sec. III F.

Returning now to the full coupled system of interest, $\mathbf{S}_4^*$, in Table II are presented, for the sake of comparison, three distinct schemes, all of the interleaved type: $\mathbf{s}_4^{*,(a)}$, $\mathbf{s}_4^{*,(b)}$ and $\mathbf{s}_4^{*,(c)}$. Scheme $\mathbf{s}_4^{*,(a)}$ is explicit, and schemes $\mathbf{s}_4^{*,(b)}$ and $\mathbf{s}_4^{*,(c)}$ are implicit. Both $\mathbf{s}_4^{*,(b)}$ and $\mathbf{s}_4^{*,(c)}$ possess conserved energy-like

quantities, but as will be seen shortly, only for scheme $\mathbf{s}_4^{*,(c)}$ may the energy function be constrained to be positive, and thus provably stable.

By virtue of the centering of difference operators, all the schemes shown in Table II are second-order accurate[23] in both h_x and h_t.

C. Conserved quantities

A good place to embark on a study of the conservation properties of the algorithms given in Table II is certainly the linear scheme $\mathbf{s}_2$; as mentioned earlier, it is composed of two uncoupled systems, one in $p_{\eta,i}^n, q_{\eta,i+1/2}^{n-1/2}$, and the other in $p_{\xi,i}^n, q_{\xi,i+1/2}^{n-1/2}$. Considering only the transverse subsystem, take the inner product of the second equation of system $\mathbf{s}_2$ with $\mu_{t+}p_{\eta,i}^{n-1}$ to get

$$\rho\langle\mu_{t+}p_\eta^{n-1},\delta_{t+}p_\eta^{n-1}\rangle = T_0\langle\mu_{t+}p_\eta^{n-1},\delta_{x-}q_\eta^{n-1/2}\rangle.$$

Using identity (16), this may be rewritten as

$$0=\delta_{t+}\left(\frac{\rho}{2}\|p_\eta^{n-1}\|^2\right)-T_0\langle\mu_{t+}p_\eta^{n-1},\delta_{x-}q_\eta^{n-1/2}\rangle.$$

Using summation by parts Eq. (22), and commutativity of the operators δ_{x+} and μ_{t+} gives

$$0=\delta_{t+}\left(\frac{\rho}{2}\|p_\eta^{n-1}\|^2\right)+T_0\langle\mu_{t+}\delta_{x+}p_\eta^{n-1},q_\eta^{n-1/2}\rangle$$

and finally, using the transverse auxiliary equation, and identity (17),

$$0=\delta_{t+}\left(\frac{\rho}{2}\|p_\eta^{n-1}\|^2+\frac{T_0}{2}\langle q_\eta^{n-1/2},q_\eta^{n-3/2}\rangle\right). \tag{24}$$

By symmetry, one may obtain, for the longitudinal subsystem,

$$0=\delta_{t+}\left(\frac{\rho}{2}\|p_\xi^{n-1}\|^2+\frac{EA}{2}\langle q_\xi^{n-1/2},q_\xi^{n-3/2}\rangle\right) \tag{25}$$

and combining Eqs. (24) and (25), finally, one arrives at

$$\delta_{t+}\mathcal{H}_{\mathbf{s}_2}^{n-1}=0\Rightarrow\mathcal{H}_{\mathbf{s}_2}^n=\mathcal{H}_{\mathbf{s}_2}^0=\text{constant} \tag{26}$$

for the scalar function $\mathcal{H}_{\mathbf{s}_2}^n$ defined by

$$\mathcal{H}_{\mathbf{s}_2}^n=\mathcal{T}_{\mathbf{s}}^n+\mathcal{V}_{\mathbf{s}_2}^n$$

with

$$\mathcal{T}_{\mathbf{s}}=\frac{\rho}{2}(\|p_\xi^n\|^2+\|p_\eta^n\|^2),$$

$$\mathcal{V}_{\mathbf{s}_2}=\frac{EA}{2}\langle q_\xi^{n+1/2},q_\xi^{n-1/2}\rangle+\frac{T_0}{2}\langle q_\eta^{n+1/2},q_\eta^{n-1/2}\rangle.$$

The quantity $\mathcal{H}_{\mathbf{s}_2}^n$ is clearly analogous to $\mathcal{H}_{\mathbf{S}_2}$ for the model system $\mathbf{S}_2$; it is a conserved quantity of difference scheme $\mathbf{s}_2$, but is not necessarily positive. The determination of conditions under which $\mathcal{H}_{\mathbf{s}_2}^n$ is positive is related to stability conditions for scheme $\mathbf{s}_2$. Such stability conditions will be derived in the next section.

As a first example of energetic analysis applied to a nonlinear difference scheme, consider schemes for the transverse-only system $\mathbf{S}_{T,4}$. Two such schemes are presented in Table II, $\mathbf{s}_{T,4}^{(a)}$ and $\mathbf{s}_{T,4}^{(b)}$. The first of these is purely explicit, and certainly the simpler form. Unfortunately, it does not possess a conserved energy-like quantity, and thus stability conditions are not immediately forthcoming. The implicit scheme, $\mathbf{s}_{T,4}^{(b)}$, on the other hand, does. In general, implicit schemes for nonlinear equations are very problematic, in that existence and uniqueness are not easy to show (though, interestingly, through energy-based analysis, one may often show that if a solution *does* exist, it will be stable[13]). As is shown in Sec. III F, the schemes to be discussed here do not cause these difficulties, as the implicit character of the schemes intervenes in an essentially linear way.

Beginning from $\mathbf{s}_{T,4}^{(b)}$, again take an inner product with $\mu_{t+}p_i^{n-1}$, to get

$$\begin{aligned}0=&\rho\langle\mu_{t+}p_\eta^{n-1},\delta_{t+}p_\eta^{n-1}\rangle-T_0\langle\mu_{t+}p_\eta^{n-1},\delta_{x-}q_\eta^{n-1/2}\rangle\\&-\frac{EA-T_0}{2}\langle\mu_{t+}p_\eta^{n-1},\delta_{x-}([q_\eta^{n-1/2}]^2\mu_{t0}q_\eta^{n-1/2})\rangle.\end{aligned}$$

Applying the same steps as for system $\mathbf{s}_2$, one obtains

$$\begin{aligned}0=&\delta_{t+}\left(\frac{\rho}{2}\|p_\eta^{n-1}\|^2+\frac{T_0}{2}\langle q_\eta^{n-1/2},q_\eta^{n-3/2}\rangle\right)\\&+\frac{EA-T_0}{2}\langle\mu_{t+}\delta_{t+}q_\eta^{n-3/2},(q_\eta^{n-1/2})^2\mu_{t0}q_\eta^{n-1/2}\rangle\\=&\delta_{t+}\left(\frac{\rho}{2}\|p_\eta^{n-1}\|^2+\frac{T_0}{2}\langle q_\eta^{n-1/2},q_\eta^{n-3/2}\rangle\right)\\&+\frac{EA-T_0}{2}\langle(\mu_{t0}q_\eta^{n-1/2})(\mu_{t+}\delta_{t+}q_\eta^{n-3/2}),(q_\eta^{n-1/2})^2\rangle\\=&\delta_{t+}\left(\frac{\rho}{2}\|p_\eta^{n-1}\|^2+\frac{T_0}{2}\langle q_\eta^{n-1/2},q_\eta^{n-3/2}\rangle\right)\\&+\frac{EA-T_0}{4}\langle\mu_{t+}\delta_{t+}(q_\eta^{n-3/2})^2,(q_\eta^{n-1/2})^2\rangle=\delta_{t+}\left(\frac{\rho}{2}\|p_\eta^{n-1}\|^2\right.\\&\left.+\frac{T_0}{2}\langle q_\eta^{n-1/2},q_\eta^{n-3/2}\rangle+\frac{EA-T_0}{8}\langle(q_\eta^{n-3/2})^2,(q_\eta^{n-1/2})^2\rangle\right)\end{aligned}$$

where, in the final three steps above, the identities (19), (18), and (17), respectively, have been used. Clearly, then,

$$\mathcal{H}_{\mathbf{s}_{T,4}^{(b)}}^n=\mathcal{T}_{\mathbf{s}_{T,4}}^n+\mathcal{V}_{\mathbf{s}_{T,4}^{(b)}}^n=\text{constant}, \tag{27}$$

where $\mathcal{T}_{\mathbf{s}_{T,4}}^n$ and $\mathcal{V}_{\mathbf{s}_{T,4}^{(b)}}^n$ are as given in Table III.

Finally, consider the three schemes for the full coupled transverse/longitudinal system, as discussed in the previous section, and as given in Table II. Scheme $\mathbf{s}_4^{*,(a)}$ is explicit, but again, does not possess a conserved energy. Schemes $\mathbf{s}_4^{*,(b)}$ and $\mathbf{s}_4^{*,(c)}$ are implicit, and conservative; manipulations similar to those performed earlier may be applied in order to arrive at kinetic and potential energies. These are presented without further comment for both schemes in Table III.

TABLE III. Discrete kinetic and potential energies for the schemes given in Table II; their sum will be conserved. The closed circle (●) indicates that the scheme is not conservative.

	Kinetic energy	Potential energy
$\mathbf{s}_2$	$\mathcal{T}_{\mathbf{s}}^n = \frac{\rho}{2}(\|p_\xi^n\|^2 + \|p_\eta^n\|^2)$	$\mathcal{V}_{\mathbf{s}_2}^n = \frac{EA}{2}\langle q_\xi^{n+1/2}, q_\xi^{n-1/2}\rangle + \frac{T_0}{2}\langle q_\eta^{n+1/2}, q_\eta^{n-1/2}\rangle$
$\mathbf{s}_4^{*,(a)}$	●	●
$\mathbf{s}_4^{*,(b)}$	$\mathcal{T}_{\mathbf{s}}^n = \frac{\rho}{2}(\|p_\xi^n\|^2 + \|p_\eta^n\|^2)$	$\mathcal{V}_{\mathbf{s}_4^{*,(b)}}^n = \frac{T_0}{2}(\langle q_\xi^{n+1/2}, q_\xi^{n-1/2}\rangle + \langle q_\eta^{n+1/2}, q_\eta^{n-1/2}\rangle) + \frac{EA-T_0}{8}\langle (q_\eta^{n+1/2})^2 + 2q_\xi^{n+1/2}, (q_\eta^{n-1/2})^2 + 2q_\xi^{n-1/2}\rangle$
$\mathbf{s}_4^{*(c)}$	$\mathcal{T}_{\mathbf{s}}^n = \frac{\rho}{2}(\|p_\xi^n\|^2 + \|p_\eta^n\|^2)$	$\mathcal{V}_{\mathbf{s}_4^{*,(c)}}^n = \frac{EA}{2}\langle q_\xi^{n+1/2}, q_\xi^{n-1/2}\rangle + \frac{T_0}{2}\langle q_\eta^{n+1/2}, q_\eta^{n-1/2}\rangle + \frac{EA-T_0}{8}(\|q_\eta^{n+1/2}q_\eta^{n-1/2} + 2\mu_{t+}q_\xi^{n-1/2}\|^2 - 4\|\mu_{t+}q_\xi^{n-1/2}\|^2)$
$\mathbf{s}_{T,4}^{(a)}$	●	●
$\mathbf{s}_{T,4}^{(b)}$	$\mathcal{T}_{\mathbf{s}_4}^n = \frac{\rho}{2}\|p_\eta^n\|^2$	$\mathcal{V}_{\mathbf{s}_{T,4}^{(b)}}^n = \frac{T_0}{2}\langle q_\eta^{n+1/2}, q_\eta^{n-1/2}\rangle + \frac{EA-T_0}{8}\langle (q_\eta^{n+1/2})^2, (q_\eta^{n-1/2})^2\rangle$

D. Numerical stability

Given the conserved forms shown in Table III, it is relatively straightforward to arrive at bounds on the solution, provided that discrete energy may be shown to be positive. Beginning from the conserved energy for scheme $\mathbf{s}_2$, first note that

$$\langle q_\xi^{n+1/2}, q_\xi^{n-1/2}\rangle = \|\mu_{t+}q_\xi^{n-1/2}\|^2 - \frac{h_t^2}{4}\|\delta_{x+}p_\xi^n\|^2,$$

$$\langle q_\eta^{n+1/2}, q_\eta^{n-1/2}\rangle = \|\mu_{t+}q_\eta^{n-1/2}\|^2 - \frac{h_t^2}{4}\|\delta_{x+}p_\eta^n\|^2.$$

The total conserved energy for the scheme can then be rewritten as

$$\mathcal{H}_{\mathbf{s}_2}^n = \frac{1}{2}\left(\rho\|p_\xi^n\|^2 - \frac{EAh_t^2}{4}\|\delta_{x+}p_\xi^n\|^2 + EA\|\mu_{t+}q_\xi^{n-1/2}\|^2\right) + \frac{1}{2}\left(\rho\|p_\eta^n\|^2 - \frac{T_0h_t^2}{4}\|\delta_{x+}p_\eta^n\|^2 + T_0\|\mu_{t+}q_\eta^{n-1/2}\|^2\right).$$

Using inequality (21), one then has

$$\mathcal{H}_{\mathbf{s}_2}^n \geqslant \frac{1}{2}(\rho - EA\alpha^2)\|p_\xi^n\|^2 + \frac{EA}{2}\|\mu_{t+}q_\xi^{n-1/2}\|^2 + \frac{1}{2}(\rho - T_0\alpha^2)\|p_\eta^n\|^2 + \frac{T_0}{2}\|\mu_{t+}q_\eta^{n-1/2}\|^2.$$

Under the conditions

$$\alpha < \sqrt{\rho/EA}, \quad \alpha < \sqrt{\rho/T_0} \tag{28}$$

the energy is strictly positive, and, furthermore, the following bounds may be obtained:

$$\|p_\xi^n\| \leqslant \sqrt{\frac{2\mathcal{H}_{\mathbf{s}_2}^0}{\rho - EA\alpha^2}}, \quad \|p_\eta^n\| \leqslant \sqrt{\frac{2\mathcal{H}_{\mathbf{s}_2}^0}{\rho - T_0\alpha^2}}, \tag{29}$$

$$\|\mu_{t+}q_\xi^{n-1/2}\| \leqslant \sqrt{\frac{2\mathcal{H}_{\mathbf{s}_2}^0}{EA}}, \quad \|\mu_{t+}q_\eta^{n-1/2}\| \leqslant \sqrt{\frac{2\mathcal{H}_{\mathbf{s}_2}^0}{T_0}}. \tag{30}$$

It is simple to translate the above mentioned bounds on $\|\mu_{t+}q_\xi^{n-1/2}\|$ into a more direct bound on $\|q_\xi^{n-1/2}\|$, by using the fact that $q_{\xi,i+1/2}^{n-1/2} = \mu_{t+}q_{\xi,i+1/2}^{n-1/2} + (h_t/2)\delta_{x+}p_{\xi,i}^n$, and an application of the triangle inequality (21). A similar statement holds for $\|\mu_{t+}q_\eta^{n-1/2}\|$.

Similar stability bounds can be course be obtained in this linear case by using von Neumann (Fourier) methods;[23] Vu-Quoc and Li[37] applied such analysis to energy-conserving schemes for the Klein-Gordon equation under linear conditions.

Consider now the conserved energy for the nonlinear transverse-only scheme $\mathbf{s}_{T,4}^{(b)}$, again given in Table III. It can be expressed as

$$\mathcal{H}_{\mathbf{s}_{T,4}^{(b)}}^n = \frac{1}{2}\left(p\|p_\eta^n\|^2 - \frac{T_0h_t^2}{4}\|\delta_{x+}p_\eta^n\|^2 + T_0\|\mu_{t+}q_\eta^{n-1/2}\|^2\right) + \frac{EA-T_0}{8}\langle (q_\eta^{n+1/2})^2, (q_\eta^{n-1/2})^2\rangle \geqslant \frac{1}{2}(\rho - T_0\alpha^2)\|p_\eta^n\|^2 + \frac{T_0}{2}\|\mu_{t+}q_\eta^{n-1/2}\|^2 + \frac{EA-T_0}{8}\langle (q_\eta^{n+1/2})^2, (q_\eta^{n-1/2})^2\rangle.$$

Under condition (13) and the second of Eq. (28), all terms are positive, leading to similar bounds

$$\|p_\eta^n\| \leq \sqrt{\frac{2\mathcal{H}^0_{\mathbf{s}^{(b)}_{T,4}}}{\rho - T_0\alpha^2}}, \quad \|\mu_{t+}q_\eta^{n-1/2}\| \leq \sqrt{\frac{2\mathcal{H}^0_{\mathbf{s}^{(b)}_{T,4}}}{T_0}}.$$

The coupled scheme $\mathbf{s}_4^{*,(c)}$ can be dealt with similarly. Rewriting the expression for conserved energy as earlier, and applying similar manipulations, one arrives at

$$\begin{aligned}\mathcal{H}^n_{\mathbf{s}_4^{*,(c)}} \geq{}& \frac{1}{2}(\rho - EA\alpha^2)\|p_\xi^n\|^2 + \frac{T_0}{2}\|\mu_{t+}q_\xi^{n-1/2}\|^2 \\ &+ \frac{1}{2}(\rho - T_0\alpha^2)\|p_\eta^n\|^2 + \frac{T_0}{2}\|\mu_{t+}q_\eta^{n-1/2}\|^2 \\ &+ \frac{EA - T_0}{8}\|q_\eta^{n+1/2}q_\eta^{n-1/2} + 2\mu_{t+}q_\xi^{n-1/2}\|^2,\end{aligned}$$

which is positive under conditions (28) and (13). Bounds on the solution size follow as before:

$$\|p_\xi^n\| \leq \sqrt{\frac{2\mathcal{H}^0_{\mathbf{s}_4^{*,(c)}}}{\rho - EA\alpha^2}}, \quad \|p_\eta^n\| \leq \sqrt{\frac{2\mathcal{H}^0_{\mathbf{s}_4^{*,(c)}}}{\rho - T_0\alpha^2}},$$

$$\|\mu_{t+}q_\xi^{n-1/2}\|, \|\mu_{t+}q_\eta^{n-1/2}\| \leq \sqrt{\frac{2\mathcal{H}^0_{\mathbf{s}_4^{*,(c)}}}{T_0}}.$$

In contrast, scheme $\mathbf{s}_4^{*,(b)}$ does *not* allow such bounds on the solution size. From the form of the potential energy $\mathcal{V}_{\mathbf{s}_4^{*,(b)}}$ given in Table III, it is clear that the last term, of highest order in the state variables, is not necessarily positive, in contrast with scheme $\mathbf{s}_4^{*,(c)}$. In fact, for certain choices of the state variables, it can become unbounded and negative. Thus conservative behavior in a difference scheme is not sufficient for stability. Further comments on this topic appear in Sec. IV.

The simplicity of the nonlinear stability results presented here are a direct consequence of the series-approximated form of the nonlinearity, in which case various algebraic symmetries may be exploited. The stability analysis here is thus less general than in a case for which the form of the nonlinearity is not as simple (e.g., that carried out by Li and Vu-Quoc[36] for the nonlinear Klein-Gordon equation, with a nonlinear term of unspecified form, which is much more involved).

E. Boundary conditions

In all the above-presented analysis, periodic boundary conditions of the form of Eq. (2) have been assumed; simple fixed conditions, as given in Eq. (3) change virtually none of this analysis. These are simply incorporated into any of the schemes presented in Table II, which are used as written for $i=1,\ldots,N-1$. At $i=0$, one may simply set $p^n_{\xi,0}=p^n_{\eta,0}=0$, and the computations of $\delta_{x+}p^n_{\xi,N-1}$ and $\delta_{x+}p^n_{\eta,N-1}$ may still be performed using periodicity, under this constraint. A fuller discussion of the distinction between periodic and fixed boundary conditions in the case of energetic analysis of difference schemes for strings appears in previous work by this author.[15]

F. Implementation details

As mentioned earlier, the schemes of most interest, namely $\mathbf{s}^{(b)}_{T,4}$ and $\mathbf{s}^{*,(c)}_4$ for which numerical stability bounds can be shown, are implicit. As such, it is worth discussing a computer implementation in detail. Fixed boundary conditions, as discussed in the previous section, are assumed here.

The implicit character of scheme $\mathbf{s}^{(b)}_{T,4}$ results directly from the inclusion of the term $\mu_{t0}q^{n-1/2}_{\eta,i+1/2}$. Noting that

$$\mu_{t0}q^{n-1/2}_{\eta,i+1/2} = q^{n-1/2}_{\eta,i+1/2} + \frac{h_t^2}{2}\delta_{x+}\delta_{t+}p^{n-1}_{\eta,i}$$

scheme $\mathbf{s}^{(b)}_{T,4}$ may be rewritten as

$$\begin{aligned}\rho\delta_{t+}p^{n-1}_{\eta,i} = T_0\delta_{x-}q^{n-1/2}_{\eta,i+1/2} + \frac{EA - T_0}{2}\delta_{x-}(q^{n-1/2}_{\eta,i+1/2})^3 \\ + \frac{h_t^2(EA - T_0)}{4}\delta_{x-}((q^{n-1/2}_{\eta,i+1/2})^2\delta_{x+}\delta_{t+}p^{n-1}_{\eta,i}).\end{aligned} \quad (31)$$

Introducing vectors $\mathbf{p}^n_\eta$ and $\mathbf{q}^{n-1/2}_\eta$, defined as

$$\mathbf{p}^n_\eta = [p^n_{\eta,1}, \ldots, p^n_{\eta,N-1}]^T, \quad \mathbf{q}^{n-1/2}_\eta = [q^{n-1/2}_{\eta,1/2}, \ldots, q^{n-1/2}_{\eta,N-1/2}]^T \quad (32)$$

system $\mathbf{s}^{(b)}_{T,4}$ may be rewritten, in vector-matrix form, as

$$\mathbf{p}^n_\eta = \mathbf{p}^{n-1}_\eta + (\mathbf{A}^{n-1/2})^{-1}\mathbf{D}_-\mathbf{b}^{n-1/2}, \quad (33a)$$

$$\mathbf{q}^{n+1/2}_\eta = \mathbf{q}^{n-1/2}_\eta + \mathbf{D}_+\mathbf{p}^n_\eta, \quad (33b)$$

where

$$\mathbf{A}^{n-1/2} = \mathbf{I}_{N-1} - \mathbf{D}_-(\mathbf{\Lambda}^{n-1/2})^2\mathbf{D}_+,$$

$$\mathbf{b}^{n-1/2} = \left(\frac{T_0}{\rho}\mathbf{I}_N + 2(\mathbf{\Lambda}^{n-1/2})^2\right)\mathbf{q}^{n-1/2}_\eta,$$

$$\mathbf{\Lambda}^{n-1/2} = \frac{\beta}{2}\,\mathrm{diag}(\mathbf{q}^{n-1/2}_\eta),$$

$$\beta = \sqrt{\frac{EA - T_0}{\rho}} \quad (34)$$

and the $N\times(N-1)$ matrix $\mathbf{D}_+$, which incorporates the fixed boundary condition constraints, is defined by

$$\mathbf{D}_+ = \alpha\begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & & -1 \end{bmatrix}. \quad (35)$$

The matrix $\mathbf{D}_-$ is defined by $\mathbf{D}_- = -\mathbf{D}_+^T$, and $\mathbf{I}_M$ is the $M\times M$ identity matrix. Clearly, in order to solve Eq. (33a), it is not necessary to invert the matrix $\mathbf{A}^{n-1/2}$, but merely to solve a sparse linear system (note that $\mathbf{A}^{n-1/2}$ is tridiagonal), which can be done in $O(N-1)$ operations, at each time step, which is comparable to the cost for an explicit scheme.

Difference scheme $\mathbf{s}^{*,(c)}_4$ may be treated similarly. Defining the vectors $\mathbf{p}^n_\xi$ and $\mathbf{q}^{n-1/2}_\xi$ in analogy with Eq. (32), it may be written as

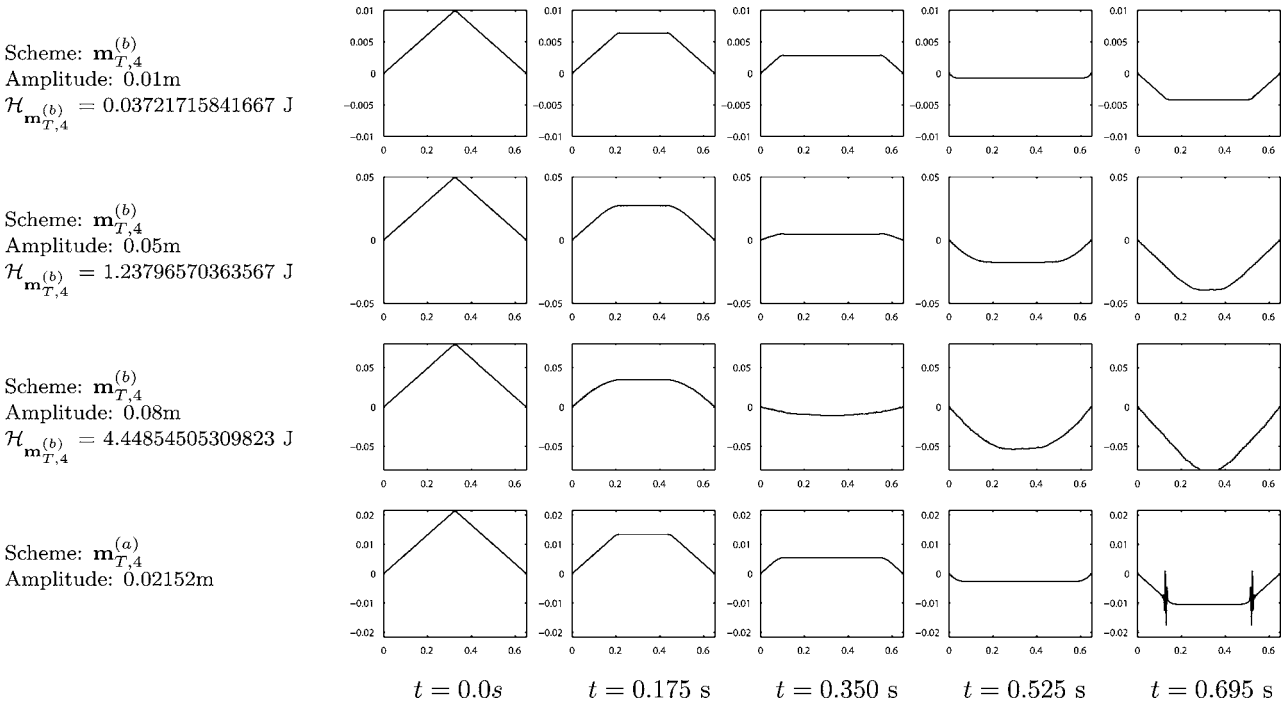

FIG. 1. Snapshots of the time evolution of the profile of a plucked string, according to the test model $\mathbf{S}_{T,4}$. The string is assumed made of steel (with $E=2.1\times10^{11}$ N/m^2 and density 7850 kg/m^3), under tension $T_0=120$ N, of length $L=0.65$ m, and of cross-sectional area $A=3.6\times10^{-8}$ m^2. The string is plucked at the center, according to the amplitudes given in the left-most column. For the first three simulations (shown in the first three rows), the energy-conserving scheme $\mathbf{s}_{T,4}^{(b)}$ is used; the conserved energy, in joules, is also given for each simulation, in the left-most column. For the fourth simulation, scheme $\mathbf{s}_{T,4}^{(a)}$ is used. In all cases, a value of $\alpha=0.85\sqrt{\rho/T_0}$ and a sample rate of 200 kHz are used. Plot units are in m.

$$\begin{bmatrix} \mathbf{p}_\xi^n \\ \mathbf{p}_\eta^n \end{bmatrix} = \begin{bmatrix} \mathbf{p}_\xi^{n-1} \\ \mathbf{p}_\eta^{n-1} \end{bmatrix} + (\mathbf{A}^{n-1/2})^{-1} \begin{bmatrix} \mathbf{D}_- & \cdot \\ \cdot & \mathbf{D}_- \end{bmatrix} \mathbf{b}^{n-1/2},$$

where

$$\mathbf{A}^{n-1/2} = \begin{bmatrix} \mathbf{I}_{N-1} & -\frac{\sqrt{2}}{\beta}\mathbf{D}_-\mathbf{\Lambda}^{n-1/2}\mathbf{D}_+ \\ -\frac{\sqrt{2}}{\beta}\mathbf{D}_-\mathbf{\Lambda}^{n-1/2}\mathbf{D}_+ & \mathbf{I}_{N-1} - \mathbf{D}_-(\mathbf{\Lambda}^{n-1/2})^2\mathbf{D}_+ \end{bmatrix},$$

$$\mathbf{b}^{n-1/2} = \begin{bmatrix} \frac{EA}{\rho} & \beta\mathbf{\Lambda}^{n-1/2} \\ 2\beta\mathbf{\Lambda}^{n-1/2} & 2(\mathbf{\Lambda}^{n-1/2})^2 \end{bmatrix} \begin{bmatrix} \mathbf{q}_\xi^{n-1/2} \\ \mathbf{q}_\eta^{n-1/2} \end{bmatrix}$$

and $\mathbf{\Lambda}^{n-1/2}$ and $\mathbf{D}_-$ are as defined in Eqs. (34) and (35).

It is important to note that these algorithms preserve energy exactly in exact machine arithmetic; in a finite precision machine, it will necessarily fluctuate, due to round-off error. In this respect, an important concern will be the means of solution of the linear systems mentioned earlier. Typically, this must be done using an iterative method of some sort, perhaps a variant of the conjugate gradient method;[23] if the matrix defining the linear system is poorly conditioned, convergence may be slow and, worse still, unconverged results can lead to large energy fluctuations and finally instability. More comments on this topic are to be found in Sec. IV.

Initialization of any of the above-mentioned interleaved algorithms is complicated, marginally, by the fact that the grid variables are not computed simultaneously. In general, grid variables p and q must be set at time steps 0 and 1/2, respectively. Given initial conditions $p_\eta(x,0)$ and $q_\eta(x,0)$, one may clearly set $p_{\eta,i}^0 = p_\eta(ih_x,0)$; $q_{\eta,i+1/2}^{1/2}$ requires slightly more care—from the auxiliary equation, it may be set according to a one-sided difference formula as $q_{\eta,i+1/2}^{1/2} = q_\eta(ih_x,0) + (h_t/2)\delta_{x+}p_{\eta,i}^0$. The initialization of $p_{\xi,i}^0$ and $q_{\xi,i+1/2}^{1/2}$ is similar. Other initialization strategies are also possible.

IV. NUMERICAL EXPERIMENTS

As a test of the various schemes presented in this article, several simulation results are here discussed. First, consider the test system $\mathbf{S}_{T,4}$, and the associated difference schemes $\mathbf{s}_{T,4}^{(a)}$ and $\mathbf{s}_{T,4}^{(b)}$. For a string of parameters as given in Fig. 1, simulation results are given under triangular initial displacement or "center-plucked" conditions, of various amplitudes. For the energy-conserving scheme, the discrete energy is conserved to machine accuracy (values given adjacent in the figure); recall that this scheme is stable for any initial conditions, provided $EA \geqslant T_0$ (which is true in this case), and for $\alpha < \sqrt{\rho/T_0}$. The value $\alpha=0.85\sqrt{\rho/T_0}$ has been chosen for all simulations presented in the figure. The scheme is implicit, and the stabilized biconjugate gradient method has been used in order to solve the linear system which arises, as discussed in the previous section. In the first case, for an initial displacement amplitude of 0.01 m, the problem is essentially linear, but for higher amplitudes of 0.05 and 0.08 m, nonlinear effects may be observed, in particular the increase in the propagation speed, in a gross sense. For the nonconservative, explicit scheme $\mathbf{s}_{T,4}^{(a)}$, an instability develops (bottom right panel of Fig. 1), which quickly diverges—this behavior occurs even under the relatively small amplitude of

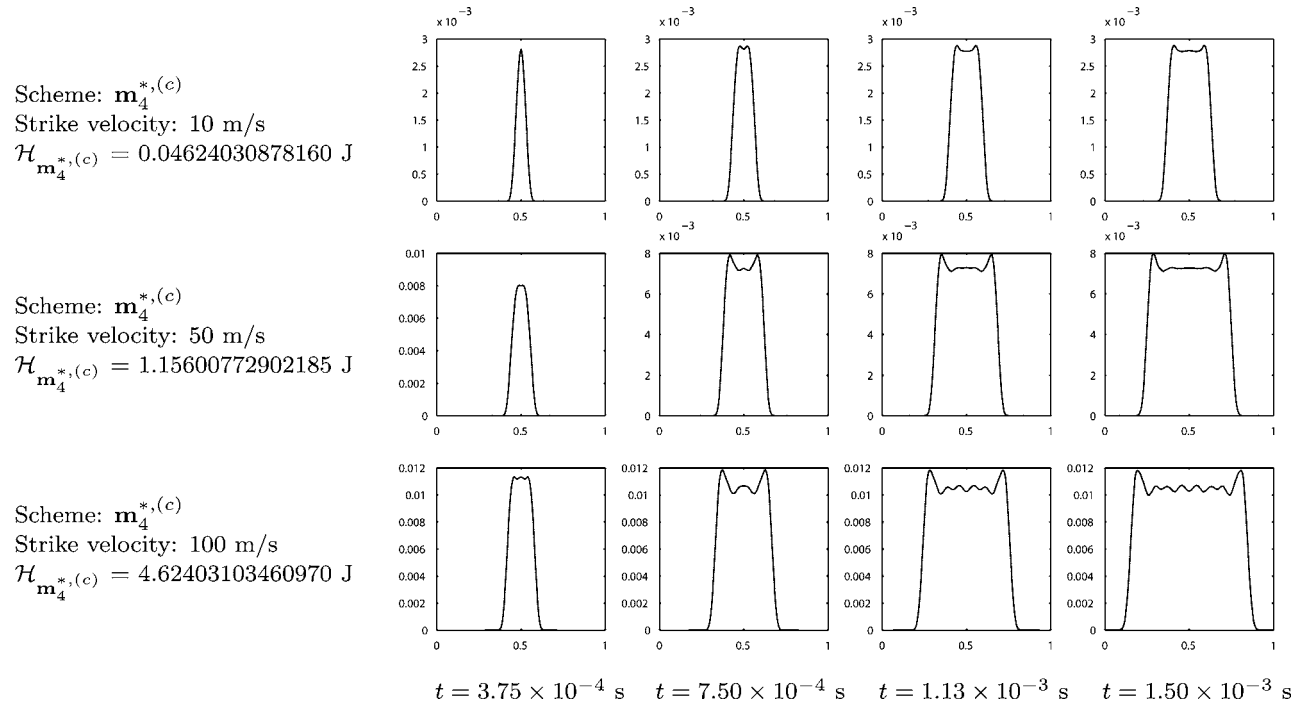

FIG. 2. Snapshots of the time evolution of the profile of a plucked string, according to the test model $\mathbf{S}_4^*$. The string is assumed made of steel (with $E = 2.1 \times 10^{11}$ N/m^2 and density 7850 kg/m^3), under tension $T_0 = 120$ N, of length $L = 1$ m, and of cross-sectional area $A = 3.14 \times 10^{-6}$ m^2. The string is struck at the center, according to a raised-cosine velocity distribution, of width 0.1 m, and of peak velocity 10, 50, and 100 m/s (in the three rows, respectively). For all three simulations, the energy-conserving scheme $\mathbf{s}_4^{*,(c)}$ is used; the conserved energy, in joules, is also given for each simulation, in the left-most column. In all cases, a value of $\alpha = 0.9\sqrt{\rho/EA}$ and a sample rate of 1 MHz are used. Plot units are in m.

0.021 52 m. Though this instability may be weakened through the use of lower values of α (at the expense of reduced efficiency), conditions under which it will not arise are not forthcoming.

In order to examine the results of application of scheme $\mathbf{s}_4^{*,(c)}$ for the coupled longitudinal/transverse system M_4^*, and for the sake of variety, the results of several initial velocity, or striking simulations are presented in Fig. 2, where string and striking parameters are as given in the accompanying caption. Again, energy is conserved to machine accuracy, and the scheme is globally stable, even for large strike amplitudes as shown. Note, again, the gross change in wave speed as strike velocity is increased, as well as the distinctive nonlinear distortion for high velocities—this is not a result of any kind of numerical dispersion of spurious oscillations in the numerical scheme (which was run at 1 MHz).

In analogy with the results presented in Fig. 1, it would be possible to demonstrate instability in the simple explicit scheme $\mathbf{s}_4^{*,(a)}$, even under relatively low striking velocities. More interesting, however, is the case of scheme $\mathbf{s}_4^{*,(b)}$, which is conservative, but not globally stable. In general, this scheme performs better than the explicit scheme (i.e., over a wider range of velocities), but suffers from another weakness, namely poor conditioning of the linear system which must be solved. This is an important issue which cannot be explored in any depth in this short article, but in essence, if the linear system is not adequately solved at each time step, the conservation guarantee fails, eventually leading to instabilities of the form seen when using, e.g., scheme $\mathbf{s}_4^{*,(a)}$. Both types of instability take the form of explosive oscillatory behavior, similar in character to those which occur in the last row of panels in Fig. 1; for this reason, simulation results are not presented here. As for the previous transverse-only system, decreasing α has an ameliorating effect on any stability concerns, for either of these two schemes.

V. CONCLUSION

Many interrelated issues have been raised here regarding model and difference scheme choice for nonlinear string vibration simulation. It is worth stepping back to view the main considerations, which all stem from an insistence on energy conservation, and rely in no way on frequency domain techniques. As far as model choice goes, in order to be able to arrive at difference schemes which are provably stable under an energetic criterion, the following constraints must be obeyed:

(1) The model should possess a conserved energy-like quantity.
(2) The conserved quantity should be a positive function of the state variables.

The first follows from good modeling of the system itself (though in certain models it is violated[10]); the second condition is also a very natural one, though it is not satisfied by several series-approximated forms which appear in the literature (system $\mathbf{S}_4$ in particular). As discussed in this article, form $\mathbf{S}_4^*$ does satisfy the second condition above, and is thus a more suitable candidate for the creation of a stable scheme. (A related question is: How should series approximations to nonlinearities be carried out, so as to respect the second constraint listed above? As was discussed earlier, an examination

of the relative orders of magnitude of the dependent variables is a key to this issue, but further amplification is necessary.)

If the above-mentioned conditions are observed, and a difference scheme is constructed, then the following constraints are also to be observed:

(1) The difference scheme should possess a conserved energy-like quantity.
(2) The conserved quantity should be a positive function of the discrete grid functions.

This second pair of constraints does not follow immediately for a given difference scheme, even if it is constructed from a model system which obeys the first set. In particular, often the most obvious choice of scheme (such as the explicit schemes discussed given in Table II) satisfies neither; a scheme which satisfies both of the above-mentioned conditions $\mathbf{s}_4^{*,(c)}$ has been presented. It is interesting that even if the first of these conditions is satisfied, *and* the continuous model problem itself satisfied the first set of conditions mentioned earlier, this is *still* insufficient for global stability; scheme $\mathbf{s}_4^{*,(b)}$ is an example of such a scheme. As is often the case for Lyapunov-type stability analysis (the present case of the nonlinear string being one example), it is not at all clear that there is a systematic framework for *constructing* a scheme which is energy conserving; trial and error, informed by experience and intuition appear to be the only tools available to the algorithm designer, as exemplified by other energy-based methods for related systems which appear in the literature.[13,36] Given that the number of discretization possibilities for nonlinear systems is vast, more work is clearly necessary, and an investigation of the related symplectic integration techniques[40] for Hamiltonian-type systems may be of some value.

In the present case of the string, one might ask whether difference schemes for the model system **S** may be approached directly using energy-based methods. Certain results in the literature point to an affirmative answer,[13] but the resulting schemes are either (a) explicit, and without a positivity guarantee on the discrete energy, or (b) implicit, and lacking any guarantee of existence or uniqueness of solutions.[21] For the series-truncated forms discussed here, it has been shown here that relatively simple, computationally efficient implicit globally stable schemes are available, and that existence/uniqueness issues do not arise, as the solution may always be arrived at through the solution of a linear system. This is a distinct advantage, and one that would appear to have wide applicability not merely to string vibration problems, but to other nonlinear systems of similar form which appear throughout solid mechanics (simulation of nonlinear plate vibration being a prime example).

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