Edinburgh School of Economics

Discussion Paper Series

Number 230

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Date
October 2013

Published by
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http://edin.ac/16ja6A6

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Suppliers of Priors:
A Theory of Retailing Inspired by the Market for Chinese Antiquities*

by

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* We are grateful to colleagues in the School of Economics at the University of Edinburgh, and especially David Pugh, for their extremely useful comments.

Abstract

Adverse selection may thwart trade between an informed seller, who knows the probability p that an item of antiquity is genuine, and an uninformed buyer, who does not know p. The buyer might not be wholly uninformed, however. Suppose he can perform a simple inspection, a test of his own: the probability that an item passes the test is g if the item is genuine, but only $f < g$ if it is fake. Given that the buyer is no expert, his test may have little power: f may be close to g. Unfortunately, without much power, the buyer's test will not resolve the difficulty of adverse selection; gains from trade may remain unexploited.

But now consider a "store", where the seller groups a number of items, perhaps all with the same quality, the same probability p of being genuine. (We show that in equilibrium the seller *will* choose to group items in this manner.) Now the buyer can conduct his test across a large sample, perhaps all, of a group of items in the seller's store. He can thereby assess the overall quality of these items; he can invert the aggregate of his test results to uncover the underlying p; he can form a "prior". There is thus no longer asymmetric information between seller and buyer: gains from trade can be exploited. This is our theory of retailing: by grouping items together – setting up a store – a seller is able to supply buyers with priors, as well as the items themselves.

We show that the *weaker* the power of the buyer's test (the closer f is to g), the greater the seller's profit. So the seller has no incentive to assist the buyer – e.g., by performing her own tests on the items, or by cleaning them to reveal more about their true age.

The paper ends with an analysis of which sellers should specialise in which qualities. We show that quality will be low in busy locations and high in expensive locations.

1. Introduction

Antiquities are valuable if they are genuine. We take a simple-minded view, and say that the "quality" – and hence the value – of an item is proportional to the probability p that the item is genuine. But determining p is not easy. And the ease with which cheap and convincing fakes can be produced only serves to make worse the problem of a more informed seller, who might know p, trying to strike a deal with a less informed buyer, who might not. The difficulty of adverse selection can be so severe that it thwarts all trade.

Buyers may not be completely uninformed, however. Suppose that a buyer can perform a simple inspection, a test of his own. The probability that an item passes the buyer's test is g if the item is genuine, but only $f < g$ if it is a fake. Given that the buyer is no expert, his test may have little power: f may be close to g. Unfortunately, without much power, the buyer's test will not resolve the difficulty of adverse selection. A single item may remain unsold even when there are gains from trade.

But now consider a "store", where the seller groups a number of items, perhaps all with the same quality, the same probability p of being genuine. (We show that in equilibrium the seller *will* choose to group homogeneous items in this manner.) Now the buyer can conduct his test across a large sample, perhaps all, of a group of items in the seller's store. He can thereby assess the overall quality of these items; he can invert the aggregate of his test results to uncover the underlying p; he can form a "prior". There is thus no longer asymmetric information between seller and buyer: gains from trade can be exploited.

This, then, is our theory of retailing. By grouping (possibly homogeneous) items together – setting up a store – a seller is able to supply buyers with priors, as well as the items themselves.

In colloquial terms, when we visit a store to buy something, particularly a large store, we scan the items for sale to form a general assessment of quality. Armed with this belief, we then proceed to test out, and perhaps purchase, an item that, on aesthetic or practical grounds, either takes our fancy or meets our needs. Our belief is our security against being fleeced: against being sold a fake.

A central question is: will a seller *want* to group homogeneous items? Might she not, for example, seed the better items with others that she knows are cheap fakes? The upside of doing so would be to masquerade a fake as a genuine item, and thereby sell it at a profit. The downside would be to reduce the buyer's overall assessment of the store's quality.

We show that the seller's discounted expected profit is strictly convex in p. That is, she does *not* want to mix qualities, and, in particular, she does not want introduce fakes.

Another issue is testing. Should the seller assist the buyer – e.g., by performing her own tests on the items, or by cleaning them to reveal more about their true age? No. We show that the seller's payoff is an *increasing* function of f: the seller benefits from the low power of the buyer's test. Of course, there is a discontinuity at the limit: if f were to equal g, then the buyer would be unable to form a prior and the problem of adverse selection would typically kill the possibility of trade.

The paper ends with an analysis of which sellers should specialise in which qualities. We show that, in equilibrium, quality will be low in busy locations and high in expensive locations.

2. Background Environment

A buyer values an item of antiquity at $p_b v$ if he likes the item aesthetically, and zero if he doesn't, where p_b is his assessment of the probability that the item is genuine, the "quality" of the item. We normalise v to equal 1.

The buyer does not have the expertise to form a *direct* assessment of p_b. But he can perform a simple pass/fail test on the item, where

$$\text{probability of passing test} \mid \text{item is genuine} = g$$

$$\text{probability of passing test} \mid \text{item is fake} = f < g$$

We take g to equal 1. Because the buyer has little expertise, the power of the test may be low: f may be close to 1.

The seller *does* have a private assessment, $p > 0$, of the probability that the item is genuine, and her cost/opportunity cost of it is strictly increasing in p. For example, suppose the seller can buy fakes (that she knows are definitely not genuine) at zero cost, but she has a strictly positive opportunity cost of selling her current item now (she may be able to sell it at a later date). Then, because of the asymmetric information, the buyer will be exposed to an extreme Akerlof lemons problem (Akerlof, 1970): the seller always strictly prefers to substitute a fake item for her current one.

The basic premise of the paper is that retailing – selling clusters of items together – is a mechanism for overcoming the Akerlof lemons problem.

The seller has a cost $c > 0$ per unit measure of items on sale, per unit of time.

3. A Static Model

Suppose the seller has s items for sale, and they are all of the same quality, the same probability p that they are genuine.

Not knowing p directly, the buyer tests all the items (the outcomes of his tests are independent across items). He finds that

$$d = [p + f(1 - p)]s$$

items pass. From these results, he can infer the value of p:

$$p = \frac{(d/s) - f}{(1 - f)}$$

That is, the buyer *indirectly* learns p.

His (and the seller's) Bayesian update on the posterior probability that an item that has passed his test is genuine equals

$$p_b = \frac{p}{p + f(1 - p)}$$

which of course exceeds p because f < 1. The point is that this item is of better quality than p because the positive test result has raised the posterior probability above the prior – for both the buyer and the seller.

For simplicity (in this static model), assume that the buyer aesthetically likes *all* the items on offer in the seller's store. Then he will be willing to pay p_b for all d items that passed his test.

We suppose throughout that the seller has all the bargaining power. So she charges p_b per item:

$$price = \frac{p}{p + f(1 - p)}$$

Hence her profit equals

$$price\ x\ quantity - cost = ps - cs$$

One suggestive thought-experiment is to ask what happens if, instead, the seller doubles the size of her store (assume best case: the additional cost is cs), filling it up with fakes?

By the earlier logic, her overall profit then equals

$$2\,[\underbrace{(p/2)s}_{\text{buyer's assessment of average quality}} - cs] = ps - 2cs$$

which is *lower* than before. Fakes reduce profit, because their storage cost (c) exceeds their contribution to quality (zero).

On the other hand, had the seller mixed two different groups of items, each of measure s, one of quality $p_1 > c$ and the other of quality $p_2 > c$, then her profit from the "mixed" store would be

$$2\left[\frac{p_1 + p_2}{2}s - cs\right] = (p_1 s - cs) + (p_2 s - cs)$$

which is the *same* as if she had sold the groups separately in two different stores (or if she had maintained separate areas in the double-sized store). In other words, mixing qualities in this static model is neither a good nor a bad strategy.

However, this static model is extreme, insofar as the seller disposes of *all* her genuine items in one go (more precisely: her ex post assessment, of the probability that the items she hasn't sold are genuine, equals zero). Arguably, a better model slows down her rate of sales; inter alia, this endogenises her opportunity cost of selling an item on any given date. To this

end, we introduce time into the model, and no longer assume that the buyer aesthetically likes (or can afford to buy!) all the items on offer.

4. The Dynamic Model

Time is continuous, $t \geq 0$. Let the common interest rate be $r \geq 0$. *Per item*, suppose that the flow of buyers that like it aesthetically equals b.

At time t, let the seller's stock be s_t, and have quality p_t: i.e., the seller assesses that *all* the items in s_t have a probability p_t of being genuine. Behind this lies a conjecture: the seller does not mix qualities.

Consider a buyer visiting the seller's store between t and $t + \Delta t$. The number of genuine items in the seller's stock equals

$$p_t s_t$$

The number of fake items in the seller's stock equals

$$(1 - p_t)s_t$$

The number of the seller's items that fail the buyer's test equals

$$(1 - f)(1 - p_t)s_t$$

The number of the seller's items that pass the buyer's test equals

$$p_t s_t + f(1 - p_t)s_t$$

– a fraction $b\Delta t$ of which the buyer likes aesthetically. Hence

$$s_{t+\Delta t} = s_t - (b\Delta t)[p_t s_t + f(1-p_t)s_t]$$

Taking the limit $\Delta t \to 0$,

$$\frac{ds_t}{dt} = -b[p_t s_t + f(1-p_t)s_t] \qquad (1)$$

The updated fraction of genuine items in the seller's store at time $t + \Delta t$ equals

$$p_{t+\Delta t} = \frac{p_t s_t - (b\Delta t)p_t s_t}{s_t - (b\Delta t)[p_t s_t + f(1-p_t)s_t]}$$

Taking the limit $\Delta t \to 0$,

$$\frac{dp_t}{dt} = -b(1-f)p_t(1-p_t) \qquad (2)$$

Equations (1) and (2) are nested first-order differential equations in s_t and p_t. Note that p_t falls through time, because the genuine items are purchased disproportionately often.

An item that passes the buyer's test at time t thus has a posterior quality

$$\frac{p_t}{p_t + f(1-p_t)}$$

And, given that the seller has all the bargaining power, at time t she posts a price equal to this posterior quality.

The buyer purchases $(b\Delta t)[p_t s_t + f(1-p_t)s_t]$ items at this price, and so the seller's gross flow revenue, price x quality, equals

$$bp_t s_t$$

With a flow storage cost of c, this means that the seller's net profit flow equals

$$(bp_t - c)\, s_t$$

The seller's initial stock is s_0, with quality p_0. To ensure profitability, we assume

$$c/b < p_0$$ – requiring $c < b$.

It is optimal for the seller to jettison her remaining stock at time t^*, where t^* solves

$$p_{t^*} = c/b$$

Hence the seller's maximised discounted total profit at $t = 0$ is given by

$$\Pi = \int_0^{t^*} (bp_t - c) s_t e^{-rt} dt$$

where s_t and p_t solve (1) and (2).

Claim

$$\Pi = \frac{bs_0}{b+r}\left\{p_0 - \frac{c}{b} - (1 - p_0)\frac{c(1-f)}{bf+r}(1-\theta)\right\}$$

where $$\theta \equiv \left[\left(\frac{1-p_0}{p_0}\right)\left(\frac{c}{b-c}\right)\right]^{\left(1+\frac{r}{bf}\right)\left(\frac{f}{1-f}\right)}$$

Proof: See Appendix.

From the Claim, note that Π is proportional to s_0: the seller operates in an environment with constant returns to scale (constant flow storage cost *per item*; constant arrival flow of buyers *per item*). Also note that if $c = 0$ (so that $t^* = \infty$), then

$$\Pi = \frac{b}{b+r}p_0 s_0$$

which is both proportional to p_0 and independent of f. But these properties do *not* hold for $c > 0$.

We now use the expression for Π to prove the following three propositions.

Proposition 1. Π is increasing and strictly convex in p_0.

Proof: See Appendix.

This proposition confirms our conjecture at the start of this Section: the seller strictly prefers not to mix qualities. But this conclusion should be tempered somewhat. Were the seller to mix qualities but then adopt the strategy of removing lower qualities once the

threshold c/b is reached, then discounted profit would be as great as if no mixing had occurred. However, such a strategy would entail the seller keeping an eye on each subgroup in order to jettison it at the optimal time (when the quality of the subgroup drops to c/b). Even then, mixing confers no benefit. From the seller's perspective, it is simpler to keep the qualities separate.

Proposition 2. Π is strictly increasing in f.

Proof: See Appendix.

This is perhaps a more intriguing result: the seller strictly prefers the buyer's test to have less power. An implication is that the seller won't choose to assist the buyer – e.g., by performing her own tests on the items, or by cleaning them to reveal more about their true age.

Of course, there is a discontinuity at $f = 1$. If the buyer's test were useless, then he could not learn the average quality of the seller's items, and there would be no rationale for grouping items together, for retailing.

Proposition 3. There exists a threshold function, $\underline{f}(b, c, r, p_0)$ say, with $0 \leq \underline{f}(b, c, r, p_0) < 1$, such that whenever $\underline{f}(b, c, r, p_0) < f < 1$, $\partial\Pi/\partial p_0$ is increasing in c and decreasing in b.

Proof: See Appendix.

A corollary of Proposition 3 is that higher c or lower b retailers specialise in higher p_0 items. That is, high quality items are sold in high rental outlets (high c); low quality items are sold in busy locations (high b). These conclusions appear to accord with reality.

Appendix

Claim

The solution to

$$\frac{dp}{dt} = -b(1-f)p(t)\big(1-p(t)\big), p(0) = p_0$$

is

$$p(t) = \frac{1}{1+\left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}}$$

Proof

The equation

$$\frac{dp}{dt} = -b(1-f)p(t)\big(1-p(t)\big), p(0) = p_0 \ (1)$$

is a first-order, second degree ordinary differential equation (ODE). One can always transform ODEs of this type into a linear ODE which can be solved in closed form.

First re-write equation 1 in standard form

$$\frac{dp}{dt} + b(1-f)p(t) = b(1-f)p(t)^2$$

and then divide through by $p(t)^2$ to obtain the following.

$$p(t)^{-2}\frac{dp}{dt} + b(1-f)p(t)^{-1} = b(1-f) \ (2)$$

Now define a new variable $z(t)$ as $z(t) = p(t)^{-1}$ and note that

$$\frac{dz}{dt} = \frac{dz}{dp}\frac{dp}{dt} = -p(t)^{-2}\frac{dp}{dt}$$

This result can be used to transform equation 2 into the following non-homogenous, linear ODE.

$$\frac{dz}{dt} - b(1-f)z(t) = -b(1-f), z(0) = p(0)^{-1} = {p_0}^{-1} \ (3)$$

Standard method for solving linear ODEs of this type is to express the solution as the sum of the complementary function, $z_c(t)$, and the particular integral, $z_p(t)$.

$$z(t) = z_c(t) + z_p(t)$$

The complementary function, $z_c(t)$, is just the general solution to the homogenous form of equation 3.

$$z_c(t) = Ae^{b(1-f)t}$$

The particular integral is simply any particular solution to equation 3. Try the simplest possible solution: $z(t) = k$, where k is some constant. If $z(t) = k$, then $\frac{dz}{dt} = 0$. Setting $\frac{dz}{dt} = 0$ in equation 3 and solving for $z(t)$ yields the desired expression for the particular integral.

$$z_p(t) = 1$$

Combining the complementary function and the particular integral yields the general form for the solution to equation 3.

$$z(t) = 1 + Ae^{b(1-f)t} \quad (4)$$

To get the definite solution, we first use the initial condition to solve for the constant *A*

$$A = z(0) - 1 = \frac{1}{p_0} - 1 = \frac{1 - p_0}{p_0}$$

and then substitute this result into equation 4 in order to obtain the final, definite solution to equation 3.

$$z(t) = 1 + \left(\frac{1 - p_0}{p_0}\right) e^{b(1-f)t} \quad (5)$$

To obtain the final, definite solution to equation 1, we now substitute

$$z(t) = p(t)^{-1}$$

into equation 5 and solve for $p(t)$.

$$p(t) = \frac{1}{1 + \left(\frac{1 - p_0}{p_0}\right) e^{b(1-f)t}} \quad (6)$$

Claim

The solution to

$$\frac{ds}{dt} = -b[s(t)p(t) + fs(t)\big(1 - p(t)\big)], s(0) = s_0$$

is

$$s(t) = s_0 p_0 \left(\frac{1}{p(t)}\right) e^{-bt}$$

Proof

Begin by dividing through by $s(t)$ to obtain the following.

$$\frac{1}{s(t)} \frac{ds}{dt} = -b\big[p(t) + f\big(1 - p(t)\big)\big] \quad (7)$$

Integrating both sides of equation 7 yields:

$$\int \frac{1}{s(t)} ds = \int -b\big[p(t) + f\big(1 - p(t)\big)\big] dt$$

$$\ln s(t) = \int -b\big[p(t) + f\big(1 - p(t)\big)\big] dt + C$$

$$s(t) = Ae^{\int -b\left[p(t)+f\left(1-p(t)\right)\right]dt} \quad (8)$$

Thus in order to find an expression for $s(t)$ we need to evaluate the following integral.

$$\int -b\big[p(t) + f\big(1 - p(t)\big)\big] dt$$

Substituting for $p(t)$ using equation (6) yields the following.

$$\int -b\left(\frac{1 + f\left(\frac{1 - p_0}{p_0}\right) e^{b(1-f)t}}{1 + \left(\frac{1 - p_0}{p_0}\right) e^{b(1-f)t}}\right) dt \quad (9)$$

Define a new variable $u(t)$ as follows.

$$u(t) = 1 + \left(\frac{1 - p_0}{p_0}\right) e^{b(1-f)t} \quad (10)$$

This implies that

$$du = b(1 - f)\left(\frac{1 - p_0}{p_0}\right) e^{b(1-f)t} dt$$

$$du - b\left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}dt = -bf\left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}dt$$

$$du - b - b\left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}dt = -b - bf\left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}dt$$

$$du - b\left(1 + \left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}\right)dt = -b\left(1 + f\left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}\right)dt$$

which reduces to

$$du(t) - bu(t)dt = -b\left(1 + f\left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}\right)dt \quad (11)$$

Equations 10 and 11 can be used to rewrite the integral in 9 as follows.

$$\int -b\left(\frac{1 + f\left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}}{1 + \left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}}\right)dt = \int\left(\frac{du(t) - bu(t)dt}{u(t)}\right)$$

$$= \int\left(\frac{du(t) - bu(t)dt}{u(t)}\right)$$

$$= \int\frac{du(t)}{u(t)} - b\int dt$$

$$= \ln u(t) - bt + C \quad (12)$$

Substituting this result into equation 8 yields the following general solution for $s(t)$.

$$s(t) = A\left(1 + \left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t}\right)e^{-bt} \quad (13)$$

To solve for A we need to make use of the initial condition $s(0) = s_0$:

$$s_0 = A\left(1 + \left(\frac{1 - p_0}{p_0}\right)\right)$$

which implies that $A = s_0 p_0$. Therefore the definite solution is

$$s(t) = s_0 p_0 \left(\frac{1}{p(t)}\right) e^{-bt} \quad (14)$$

Claim

$$\Pi = \frac{bs_0}{b+r}\left\{p_0 - \frac{c}{b} - (1 - p_0)\frac{c(1-f)}{bf+r}(1-\theta)\right\}$$

where $\theta \equiv \left[\left(\frac{1-p_0}{p_0}\right)\left(\frac{c}{b-c}\right)\right]^{\left(1+\frac{r}{bf}\right)\left(\frac{f}{1-f}\right)}$

Proof

Begin by solving for t^* (i.e., the point in time at which a seller will jettison her entire remaining stock). From our work so far we know that

$$p(t^*) = \frac{1}{1 + \left(\frac{1-p_0}{p_0}\right)e^{b(1-f)t^*}} = \frac{c}{b}$$

Solving for t^* yields:

$$t^* = \ln\left(\left(\frac{b-c}{c}\right)\left(\frac{p_0}{1-p_0}\right)\right)^{\frac{1}{b(1-f)}} \quad (15)$$

Next, substitute equations (6) and (14) into the expression for profits and simplify to obtain:

$$\Pi = \int_0^{t^*} (b-c)s_0p_0\, e^{-(b+r)t} - cs_0(1-p_0)e^{-(r+bf)t}dt \quad (16)$$

Taking the integral yields:

$$\Pi = \left[-\left(\frac{(b-c)s_0p_0}{b+r}\right)e^{-(b+r)t} + (1-p_0)\left(\frac{cs_0}{r+bf}\right)e^{-(r+bf)t}\right]_0^{t^*} \quad (17)$$

Evaluating the integral yields:

$$\Pi = \left(\frac{(b-c)s_0p_0}{b+r}\right) - \left(\frac{(b-c)s_0p_0}{b+r}\right)e^{-(b+r)t^*} + (1-p_0)\left(\frac{cs_0}{r+bf}\right)e^{-(r+bf)t^*} - (1-p_0)\left(\frac{cs_0}{r+bf}\right)$$

$$= \left(\frac{(b-c)s_0p_0}{b+r}\right)\left[1 - e^{-(b+r)t^*}\right] - (1-p_0)\left(\frac{cs_0}{r+bf}\right)\left[1 - e^{-(r+bf)t^*}\right] \quad (18)$$

In order to simplify further, it is useful to define a new variable θ as follows:

$$\theta = \left[\left(\frac{c}{b-c}\right)\left(\frac{1-p_0}{p_0}\right)\right]^{\left(1+\frac{r}{bf}\right)\left(\frac{f}{1-f}\right)} \quad (19)$$

Using our definition of θ equation 18 can be re-written as

$$= \left(\frac{(b-c)s_0p_0}{b+r}\right)\left(1 - \theta\left(\frac{c}{b-c}\right)\left(\frac{1-p_0}{p_0}\right)\right) - (1-p_0)\left(\frac{cs_0}{r+bf}\right)(1-\theta)$$

$$= \left(\frac{(b-c)s_0p_0}{b+r}\right)\left(1 - \left(\frac{c}{b-c}\right)\left(\frac{1-p_0}{p_0}\right) + \left(\frac{c}{b-c}\right)\left(\frac{1-p_0}{p_0}\right)(1-\theta)\right) - (1-p_0)\left(\frac{cs_0}{r+bf}\right)(1-\theta)$$

$$= \left(\frac{bs_0}{b+r}\right)\left[p_0 - \frac{c}{b} - (1-p_0)\left(\frac{c(1-f)}{r+bf}\right)(1-\theta)\right] \quad (20)$$

Lemma 1

Although θ is strictly decreasing in p_0, whether or not θ is strictly concave (convex) depends on p_0.

Proof

First, we need to show that $\frac{d\theta}{dp_0} < 0$.

$$\frac{d\theta}{dp_0} = -\left(\frac{1}{p_0}\right)\left(\frac{1}{1-p_0}\right)\left(1+\frac{r}{bf}\right)\left(\frac{f}{1-f}\right)\theta < 0$$

Next, we need to show under what conditions θ will be strictly concave (convex) in p_0. Differentiating again with respect to p_0 yields:

$$\frac{d^2\theta}{dp_0^2} = \left(\frac{1}{p_0}\right)\left(\frac{1}{1-p_0}\right)\left(1+\frac{r}{bf}\right)\left(\frac{f}{1-f}\right)\left[\left(\frac{1-2p_0}{p_0(1-p_0)}\right)\theta - \frac{d\theta}{dp_0}\right]$$

It follows that θ will be strictly concave (convex) if and only if

$$\left(\frac{1-2p_0}{p_0(1-p_0)}\right)\theta < (>)\frac{d\theta}{dp_0}$$

Substituting for $\frac{d\theta}{dp_0}$ and re-arranging shows that θ will be strictly concave (convex) if and only if the initial quality satisfies

$$p_0 > (<)\frac{1}{2} + \frac{1}{2}\left(1+\frac{r}{bf}\right)\left(\frac{f}{1-f}\right)$$

Lemma 2

θ is strictly decreasing in *f*.

Proof

We need to show that $\frac{d\theta}{df} < 0$. Differentiating with respect to *f* yields:

$$\frac{d\theta}{df} = \theta\left(\frac{b+r}{b(1-f)^2}\right)\ln\left(\left(\frac{1-p_0}{p_0}\right)\left(\frac{c}{b-c}\right)\right) < 0$$

which is negative because, by assumption, $p_0 > \frac{c}{b}$ which insures that

$$\ln\left(\left(\frac{1-p_0}{p_0}\right)\left(\frac{c}{b-c}\right)\right) < 0$$

Proposition 1

Π is increasing and convex in p_0.

Proof

First, we need to show that $\frac{d\Pi}{dp_0} > 0$.

$$\frac{d\Pi}{dp_0} = \left(\frac{bs_0}{b+r}\right)\left(1 + \left(\frac{c(1-f)}{bf+r}\right)\left[(1-\theta) + (1-p_0)\frac{d\theta}{dp_0}\right]\right)$$

From this expression it is clear that $\frac{d\Pi}{dp_0} > 0$ if an only if

$$1 + \left(\frac{c(1-f)}{bf+r}\right)\left[(1-\theta) + (1-p_0)\frac{d\theta}{dp_0}\right] > 0$$

After substituting for $\frac{d\theta}{dp_0}$ using the result from Lemma 1, and re-arranging the above inequality simplifies to

$$p_0 + p_0(1-\theta)\left(\frac{c(1-f)}{bf+r}\right) > \left(\frac{c}{b}\right)\theta$$

This inequality must hold because, by assumption, $p_0 > \left(\frac{c}{b}\right)$ and

$$p_0 + p_0(1-\theta)\left(\frac{c(1-f)}{bf+r}\right) > p_0 > \frac{c}{b} > \left(\frac{c}{b}\right)\theta$$

Next we need to show that $\frac{d^2\Pi}{dp_0^2} > 0$. Differentiating again with respect to p_0 yields:

$$\frac{d^2\Pi}{dp_0^2} = \left(\frac{bs_0}{b+r}\right)\left(\frac{c(1-f)}{bf+r}\right)\left[(1-p_0)\frac{d^2\theta}{dp_0^2} - \frac{d\theta}{dp_0}\right]$$

which will be strictly positive if and only if

$$(1-p_0)\frac{d^2\theta}{dp_0^2} > \frac{d\theta}{dp_0}$$

Using the results from Lemma 1 we can re-write this inequality as

$$\frac{d\theta}{dp_0} - \left(\frac{1-2p_0}{p_0(1-p_0)}\right)\theta < \left(\frac{1}{1-p_0}\right)\theta$$

A bit of re-arranging yields

$$\frac{d\theta}{dp_0} < \left(\frac{1}{p_0}\right)\theta$$

which implies that the inequality holds as

$$\frac{d\theta}{dp_0} < 0 < \left(\frac{1}{p_0}\right)\theta$$

Proposition 2

Π is strictly increasing in *f*.

Proof

We need to show that $\frac{d\Pi}{df} > 0$.

$$\frac{d\Pi}{df} = -\left(\frac{bs_0}{b+r}\right)(1-p_0)\frac{d}{df}\left(\left(\frac{c(1-f)}{bf+r}\right)(1-\theta)\right)$$

$$= \left(\frac{bs_0}{b+r}\right)(1-p_0)\left(\left(\frac{c(b+r)}{(bf+r)^2}\right)(1-\theta) + \left(\frac{c(1-f)}{bf+r}\right)\frac{d\theta}{df}\right)$$

Thus we find that $\frac{d\Pi}{df} > 0$ if and only if

$$\left(\frac{b+r}{bf+r}\right)(1-\theta) + (1-f)\frac{d\theta}{df} > 0$$

Substituting for $\frac{d\theta}{df}$ using Lemma 2 and simplifying the left-hand side of the above inequality yields

$$\left(\frac{b+r}{bf+r}\right)[(1-\theta) + \theta \ln \theta]$$

which is strictly positive if and only if

$$(1-\theta) + \theta \ln \theta > 0$$

To confirm that this inequality holds for all $0 < \theta < 1$, it is sufficient to note that

$$(1-\theta) + \theta \ln \theta$$

obtains a unique global minimum of zero at $\theta = 1$.

Proposition 3: There exists a threshold function, $\underline{f}(b, c, r, p_0)$ say, with $0 \leq \underline{f}(b, c, r, p_0) < 1$ such that whenever $\underline{f}(b, c, p_0) < f < 1$, $\frac{d\Pi}{dp_0}$ is increasing in c and decreasing in b.

Proof

Set $r = 0$ and re-write the expressions for Π and θ as

$$\widehat{\Pi} = s_0\left[1 - y - x - xy\left(\frac{1}{\varphi}\right)(1 - \hat{\theta})\right]$$

and

$$\hat{\theta} = \left[\left(\frac{x}{1-x}\right)\left(\frac{y}{1-y}\right)\right]^{\varphi}$$

where $x = \left(\frac{c}{b}\right), y = 1 - p_0$, and $\varphi = \left(\frac{f}{1-f}\right)$.

Differentiating $\hat{\theta}$ with respect to x and y yields:

$$\frac{d\hat{\theta}}{dx} = \varphi\frac{1}{x}\left(\frac{1}{1-x}\right)\hat{\theta}$$
$$\frac{d\hat{\theta}}{dy} = \varphi\frac{1}{y}\left(\frac{1}{1-y}\right)\hat{\theta}$$
$$\frac{d^2\hat{\theta}}{dxdy} = \varphi^2\frac{1}{xy}\left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right)\hat{\theta}$$

Differentiating $\widehat{\Pi}$ with respect to x and y yields:

$$\frac{d\widehat{\Pi}}{dx} = s_0\left[-1 - \left[y\left(\frac{1}{\varphi}\right)(1 - \hat{\theta}) - xy\left(\frac{1}{\varphi}\right)\frac{d\hat{\theta}}{dx}\right]\right]$$

$$= -s_0 - s_0 y\left(\frac{1}{\varphi}\right) + s_0 y\left(\frac{1}{\varphi}\right)\hat{\theta} + s_0 xy\left(\frac{1}{\varphi}\right)\frac{d\hat{\theta}}{dx}$$

$$\frac{d^2\hat{\Pi}}{dxdy} = -s_0\left(\frac{1}{\varphi}\right) + s_0\left(\frac{1}{\varphi}\right)\hat{\theta} + s_0 y\left(\frac{1}{\varphi}\right)\frac{d\hat{\theta}}{dy} + s_0 x\left(\frac{1}{\varphi}\right)\frac{d\hat{\theta}}{dx} + s_0 xy\left(\frac{1}{\varphi}\right)\frac{d^2\hat{\theta}}{dxdy}$$

$$= -s_0\left(\frac{1}{\varphi}\right) + s_0\left(\frac{1}{\varphi}\right)\hat{\theta} + s_0\left(\frac{1}{1-y}\right)\hat{\theta} + s_0\left(\frac{1}{1-x}\right)\hat{\theta} + s_0\varphi\left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right)\hat{\theta}$$

$$= s_0\left[\left(\frac{1}{\varphi} + \left(\frac{1}{1-x}\right) + \left(\frac{1}{1-y}\right) + \varphi\left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right)\right)\hat{\theta} - \frac{1}{\varphi}\right]$$

In order for $\frac{d^2\hat{\Pi}}{p_0 dc} > 0$ and $\frac{d^2\hat{\Pi}}{dp_0 db} < 0$, we need $\frac{d^2\hat{\Pi}}{dxdy} < 0$. A necessary and sufficient condition for the desired inequalities to hold is that

$$\frac{1}{\varphi} > \left(\frac{1}{\varphi} + \left(\frac{1}{1-x}\right) + \left(\frac{1}{1-y}\right) + \varphi\left(\frac{1}{1-x}\right)\left(\frac{1}{1-y}\right)\right)\hat{\theta}$$

$$\hat{\theta} < \frac{(1-x)(1-y)}{(1-x+\varphi)(1-y+\varphi)}$$

If we define

$$\Gamma(f) = \frac{(1-x)(1-y)}{(1-x+\varphi)(1-y+\varphi)} - \hat{\theta}(f)$$

then there exists a threshold value $0 \leq \underline{f}(b, c, r, p_0) < 1$ such that sets $\Gamma\left(\underline{f}\right) = 0$ and for values of $\underline{f}(b, c, r, p_0) < f < 1$, we have that $\frac{d^2\hat{\Pi}}{dxdy} < 0$ which implies that $\frac{d\Pi}{dp_0}$ is increasing in c and decreasing in b.

Reference

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