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Variations on a theme of Grothendieck

Or, I've a feeling we're not in Kansas any more

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The theme of the paper is that principal bundles over the complex projective line with reductive structure group can be reduced to a maximal torus, in a unique fashion modulo automorphisms and the action of the Weyl group. Grothendieck proved this in a remarkable paper [11] written, in quite a different style from that of his later works, during the year 1955, which he spent in Kansas. Harder [13] showed in 1968 that the statement remains valid over an arbitrary ground field, provided that the reductive group is split and the bundle is trivial over the generic point.

Our variations are extensions of this result, on the existence and uniqueness of such a reduction, to principal bundles over other base schemes or stacks:

1. a μ_n*-equivariant line*, that is, a line modulo the group μ_n of nth roots of unity;
2. a *football*, that is, an orbifold whose coarse moduli space is the projective line, with orbifold structure at two points;
3. a *gerbe* with structure group μ_n over a football;
4. a *torus-equivariant line*, that is, a line modulo an action of a split torus;
5. a *chain of lines* meeting in nodal singularities; and
6. a *torus-equivariant chain*, that is, a chain of lines modulo an action of a split torus.

We also prove that the automorphism group schemes of such bundles are smooth, affine, and connected.

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Theme: the projective line

Let k be an arbitrary field. All schemes and stacks throughout are over k.

Let X be a scheme or stack, and let G, H be linear algebraic groups, that is, smooth affine group schemes of finite type over k.

A *principal bundle* over X with *structure group* G is a G-scheme or G-stack $E \to X$ locally trivial in the étale topology. Isomorphism classes of principal G-bundles over X are naturally in bijection with the étale cohomology set $H^1(X, G)$.

If $\rho : G \to H$ is a homomorphism, the *extension of structure group* is the principal H-bundle $E_\rho := [(E \times H)/G]$. A *reduction of structure group* to a subgroup $H \subset G$ is a section of $[E/H]$, the fiber bundle associated to E with fiber G/H. The inverse image of this section in E is a principal H-bundle, which we denote E_H; its extension of structure group by the inclusion $H \subset G$ is naturally isomorphic to E. Two reductions of structure group of E to H may, of course, be interchanged by an automorphism of the G-bundle E; but, as is easily seen, this is the case if and only if the principal H-bundles are isomorphic.

Suppose henceforth that G is a split reductive group, which we take to be connected; let $T \subset G$ be a split maximal torus; and let $B \subset G$ be a Borel subgroup containing T. Let $\eta = \operatorname{Spec} k(t)$ be the generic point of $\mathbb{P}^1$, and define a G-bundle over $\mathbb{P}^1$ to be *rationally trivial* if its restriction to η is trivial. The set, denoted $\bar{H}^1(\mathbb{P}^1, G)$, of isomorphism classes of rationally trivial G-bundles is therefore nothing but $\ker H^1(\mathbb{P}^1, G) \to H^1(\eta, G)$. All T-bundles over $\mathbb{P}^1$ are rationally trivial by Theorem 90 [17, III 4.9], so $\bar{H}^1(\mathbb{P}^1, T) = H^1(\mathbb{P}^1, T)$. Furthermore, if k is algebraically closed of characteristic zero, a theorem of Steinberg [19, 1.9] implies that all G-bundles over $\mathbb{P}^1$ are rationally trivial, so $\bar{H}^1(\mathbb{P}^1, G) = H^1(\mathbb{P}^1, G)$ in this case.

(0.1) Theorem (Existence for lines). *Extension of structure group induces a surjection $H^1(\mathbb{P}^1, T) \to \bar{H}^1(\mathbb{P}^1, G)$; that is, every rationally trivial principal G-bundle over $\mathbb{P}^1$ admits a reduction to T.*

Proof following Harder [13].

Step A: *The case $G = PGL_2$.* Let F be a rationally trivial PGL_2-bundle over $\mathbb{P}^1$. The associated $\mathbb{P}^1$-bundle $[(F \times \mathbb{P}^1)/PGL_2]$ is trivial over the generic point. Let D be the closure of a generic section; then $\mathcal{O}(D)$ is a line bundle restricting to $\mathcal{O}(1)$ on each fiber. Its direct image under the projection to $\mathbb{P}^1$ is then a rank 2 vector bundle whose dual E satisfies $F = \mathbb{P}E$. It therefore suffices to show that every rank 2 vector bundle E over $\mathbb{P}^1$ is isomorphic to $\mathcal{O}(d) \oplus \mathcal{O}(e)$ for some $d \geq e$.

Any such bundle has a line subbundle $\mathcal{O}(d)$ of maximal degree. Indeed, being rationally trivial, E has a nonzero rational section. Let D be its divisor of zeroes and poles, and let $d = |D|$; there is then a nowhere vanishing section of $E(-d)$ yielding a short exact sequence of the form

$$0 \longrightarrow \mathcal{O}(d) \longrightarrow E \longrightarrow \mathcal{O}(e) \longrightarrow 0.$$

So a line subbundle exists. Furthermore, if $\mathcal{O}(c) \to E$ is any line subbundle, then either $c = d$ or there is a nonzero map $\mathcal{O}(c) \to \mathcal{O}(e)$, in which case $c \leq e$. Hence the set of possible c is bounded above; assume without loss of generality that d is maximal among

them.

We claim that then $d \geq e$. If not, then $H^0(\mathcal{O}(e-d-1)) \neq 0$, so there exists a nonzero map $\mathcal{O}(d+1) \to \mathcal{O}(e)$. Since in the long exact sequence

$$H^0(\mathrm{Hom}(\mathcal{O}(d+1), E)) \longrightarrow H^0(\mathrm{Hom}(\mathcal{O}(d+1), O(e))) \longrightarrow H^1(\mathrm{Hom}(\mathcal{O}(d+1), \mathcal{O}(d)))$$

the right-hand term vanishes, our nonzero map comes from a nonzero map $\mathcal{O}(d+1) \to E$. Let D be its divisor of vanishing; there is then a nowhere zero map $\mathcal{O}(d+1+|D|) \to E$, contradicting the maximality of d. This proves the claim.

The extension class of the short exact sequence lies in $H^1(\mathcal{O}(d-e))$, but by the claim this vanishes. The sequence therefore splits. This completes Step A.

Step B: *Reduction to B.* Rational triviality guarantees that a generic section of the associated G/B-bundle $[E/B] \to \mathbb{P}^1$ exists. Any cover of $\mathbb{P}^1$ by finitely many étale trivializations of E provides a faithfully flat and quasi-compact morphism to $\mathbb{P}^1$ by which the base change of $[E/B] \to \mathbb{P}^1$ is proper. Hence $[E/B] \to \mathbb{P}^1$ is itself proper [12, 2.7.1], so the generic section extends to a regular section by the valuative criterion. Hence E reduces to B.

Step C: *An adapted reduction to B.* We wish the corresponding B-bundle E_B to enjoy the property that for all positive roots $\alpha : B \to \mathbb{G}_m$, the associated line bundle E_α has degree ≥ 0. This is false in general, but the given reduction to B can be modified to give a new one where this property holds. We call such a reduction an *adapted* reduction (Harder calls it "reduziert").

To prove this, observe first that it suffices for this property to hold for the simple roots α_i, since any positive root is a nonnegative combination of them. Next consider the parabolic subgroup P_i whose Lie algebra $\mathfrak{p}_i$ is the sum of $\mathfrak{b}$ and the root space $\mathfrak{g}_{-\alpha_i}$. Then $P_i/B \cong \mathbb{P}^1$, so the P_i/B-bundle associated to E_B is a $\mathbb{P}^1$-bundle with structure group PGL_2. Its total space is naturally contained in that of the G/B-bundle above and contains the chosen section. But it also contains the section corresponding to $\mathcal{O}(d)$, the line bundle of greater degree, in the splitting of Step A. If this section is different, it gives a different reduction of E to B. For this new reduction, the line bundle associated to α_i is $\mathcal{O}(d-e)$, whose degree is nonnegative. But the line bundles associated to all other simple roots are unchanged from the previous reduction, since for $j \neq i$ the homomorphisms $\alpha_j : B \to \mathbb{G}_m$ factor through P_i, and the section of the associated G/P_i-bundle is unchanged. We may therefore modify the section of our G/B-bundle one simple root at a time until the desired property holds for all simple roots.

Step D: *Conclusion.* Thus we obtain an adapted reduction E_B to B. We seek a further reduction to T, that is, a section of the associated B/T-bundle. Recall now that the Borel is a semidirect product $B = T \ltimes U$, where U is the maximal unipotent subgroup. Hence B/T is canonically isomorphic to U as a variety; but the natural actions of B do not match, so the B/T-bundle and the U-bundle associated to our B-bundle are not the same. Rather, our B/T-bundle is a torsor for F, the bundle of groups associated to E_B via the conjugation action of B on U. The identity of course furnishes a section of F, so it suffices to show that all torsors for F are trivial.

This is where we use the adapted property established in Step C. As a group with B-action, U is filtered by subgroups whose successive quotients are isomorphic to the additive

group $\mathbb{G}_a$, with B-action given by a positive root α. Hence F is filtered by subbundles of groups whose successive quotients are line bundles of nonnegative degree. On $\mathbb{P}^1$, any such line bundle has trivial H^1, so from the long exact sequences associated to the filtration it follows that $H^1(F) = 0$ and hence that every torsor over F is trivial. $\square$

(0.2) Corollary. *Let $\Lambda := \mathrm{Hom}(\mathbb{G}_m, T)$ be the cocharacter lattice, W the Weyl group of G. Then there is a natural surjection from Λ/W to $\bar{H}^1(\mathbb{P}^1, G)$.*

Proof. Tensoring the usual isomorphism $\mathbb{Z} \to \mathrm{Pic}\, \mathbb{P}^1 = H^1(\mathbb{P}^1, \mathbb{G}_m)$ by the free $\mathbb{Z}$-module $\Lambda = \mathrm{Hom}(\mathbb{G}_m, T)$ yields an isomorphism $\Lambda \to H^1(\mathbb{P}^1, T)$. By Theorem 0.1, its composition with the extension of structure group $T \to G$ is a surjection $\Lambda \to H^1(\mathbb{P}^1, G)$. There is some redundancy, however. Since for any inner automorphism $\rho : G \to G$ there is a natural isomorphism of principal bundles $E_\rho \cong E$, and every Weyl group element $w : T \to T$ extends to an inner automorphism of G, it follows that λ and $w \cdot \lambda$ determine isomorphic G-bundles. The surjection above therefore descends to Λ/W. $\square$

(0.3) Theorem (Uniqueness for lines). *The latter surjection is also an injection: that is, for every rationally trivial principal G-bundle E over $\mathbb{P}^1$, the isomorphism class of the reduction to T is unique modulo the action of W.*

Proof following Grothendieck [11].

First, we prove the uniqueness for the case $G = GL_n$, that is, for vector bundles. This amounts to proving that if $\bigoplus \mathcal{O}(a_i) \cong \bigoplus \mathcal{O}(b_j)$, then the a_i and the b_j are the same up to permutation. Or equivalently, if both sequences are weakly increasing, then they are the same. Since only bundles of nonnegative degree have nonzero sections, in order to have maps in both directions nonvanishing on all summands, we must have $a_n = b_n$. Then the map from $\mathcal{O}(a_n)$ to $\bigoplus \mathcal{O}(b_j)$ must be a constant map to the summands of degree a_n; we may split off its image and proceed by induction on n to finish the GL_n case.

Next, we reduce the general case to the GL_n case. Suppose $\lambda, \lambda' \in \Lambda$ induce isomorphic G-bundles on $\mathbb{P}^1$. Then the vector bundles associated to any representation $G \to GL_n$ are also isomorphic and hence split into line bundles of the same degrees. Conjugate the representation so that it takes T to the diagonal matrices. Then its value on $\lambda(t)$ is $\mathrm{diag}(t^{a_1}, \cdots, t^{a_n})$ where the integers a_i are the splitting type, as above. Hence the character of any representation takes the same values on λ as on λ'.

By passing to a field extension, it suffices to assume that k is algebraically closed, with nonzero elements that are not roots of unity. Then the algebra of regular class functions on G is generated by the characters of irreducible representations [19, 6.1]. Hence all such functions agree on λ and λ', so the compositions of λ and λ' with the projection to $G/G = T/W$ are equal. In particular, if $t_0 \in \mathbb{G}_m$ is a k-rational point that is not a root of unity, so that it generates a Zariski dense subset, then $\lambda(t_0)$ and $\lambda'(t_0)$ are conjugate by some constant $w \in W$. Hence the same is true of λ and λ' themselves. $\square$

Let us mention two further results, though they do not feature in Grothendieck's paper: on the existence and uniqueness of a Harder-Narasimhan reduction, and on the connectedness of the automorphism group.

(0.4) Theorem. *A rationally trivial G-bundle E over $\mathbb{P}^1$ admits an unique reduction to a parabolic subgroup P containing T which (a) is rigid in that its infinitesimal deformations are trivial and (b) is minimal among all such rigid reductions to parabolic subgroups. Any reduction of E to T is a further reduction of this one.*

Proof. Observe first that, as any parabolic P is the normalizer in G of its own Lie algebra $\mathfrak{p} \subset \mathfrak{g}$ [3, 11.16], G/P parametrizes the subalgebras of $\mathfrak{g}$ conjugate to $\mathfrak{p}$. Consequently, a reduction of E to P, being a section $\sigma : \mathbb{P}^1 \to E/P$, determines, and is determined by, a bundle $\operatorname{ad} E_P$ of subalgebras of $\operatorname{ad} E$ conjugate to $\mathfrak{p}$. The infinitesimal deformations of the reduction are sections of the normal bundle to $\sigma(\mathbb{P}^1)$ in E/P, namely $\operatorname{ad} E/\operatorname{ad} E_P$.

By Theorem 0.1 there exists a reduction of E to T. Then by Corollary 0.2, E is induced by a cocharacter λ, so there is a splitting

$$\operatorname{ad} E \cong \mathcal{O}^r \oplus \bigoplus_{\alpha \in \Phi} \mathcal{O}(\alpha \cdot \lambda)$$

corresponding to the splitting of $\mathfrak{g}$ into Cartan subalgebra and root spaces. If any of the summands of nonnegative degree is not contained in $\operatorname{ad} E_P$, then it will map nontrivially to $\operatorname{ad} E/\operatorname{ad} E_P$, giving the latter a nonzero section. Hence for the reduction to be rigid, $\operatorname{ad} E_P$ must contain all of the summands of nonnegative degree in the splitting above.

On the other hand, the summands of nonnegative degree span a bundle of parabolic subalgebras with the desired rigidity. It therefore determines the unique minimal rigid reduction. As $\operatorname{ad} E_P$ contains the bundle of Cartan subalgebras $\operatorname{ad} E_T \cong \mathcal{O}^r$, this reduction reduces further to T. $\square$

(0.5) Theorem (Connectedness for lines). *The automorphism group $\Gamma(\mathbb{P}^1, \operatorname{Ad} E)$ is smooth, affine, and connected, as is the kernel of the evaluation map $\Gamma(\mathbb{P}^1, \operatorname{Ad} E) \to G^\ell$ defined when E is trivialized at ℓ rational points.*

Proof. Let $E_P \subset E$ be the Harder-Narasimhan reduction of Theorem 0.4. Since it is unique and rigid, any family of automorphisms must preserve it, so $\Gamma(\operatorname{Ad} E) = \Gamma(\operatorname{Ad} E_P)$. Let L be the Levi factor of P containing T and let U be the maximal unipotent, so that $P = L \ltimes U$. The adjoint action of T preserves L and U, and even the root subgroups U_α that directly span U. Hence $\operatorname{Ad} E_P$ becomes a semidirect product of group bundles $Q \ltimes R$, where Q is an adjoint L-bundle, R is an adjoint U-bundle, and R is directly spanned by line bundles R_α. Hence as a scheme $\Gamma(\operatorname{Ad} E_P) \cong \Gamma(Q) \times \Gamma(R)$, and furthermore $\Gamma(R) \cong \prod_\alpha \Gamma(R_\alpha)$, where α runs over the root spaces in the Lie algebra of U.

Consider first $\Gamma(Q)$. Let $\lambda : \mathbb{G}_m \to T$ be the 1-parameter subgroup defining E, as usual, and note that for any root α of the Levi, $\lambda \cdot \alpha = 0$. Hence $\lambda(\mathbb{G}_m)$ is contained in the center of the Levi, so its adjoint action on L is trivial, and Q is trivial so that $\Gamma(Q) = L$, which is smooth, affine, and connected.

On the other hand, $\Gamma(R) \cong \prod_\alpha \Gamma(R_\alpha)$, and since R_α is a line bundle, $\Gamma(R_\alpha)$ is an affine space, so $\Gamma(R)$ is also an affine space, hence certainly smooth, affine, and connected.

As for the kernel of the evaluation map, it is a semidirect product of the corresponding kernels for $\Gamma(Q)$ and $\Gamma(R)$. From the above, the first kernel is clearly trivial, while the second one is clearly an affine space. $\square$

Variation 1: a μ_n-equivariant line

Let μ_n be the group of nth roots of unity, acting on the projective line $\mathbb{P}^1$ by $\lambda[x,y]=[\lambda x,y]$. This variation is concerned with G-bundles over the stack $[\mathbb{P}^1/\mu_n]$, or equivalently, with μ_n-equivariant G-bundles over $\mathbb{P}^1$.

A G-bundle over $[\mathbb{P}^1/\mu_n]$ is said to be *rationally trivial* if its restriction to the generic point Spec $k(t)$ of $[\mathbb{P}^1/\mu_n]$ is trivial.

(1.1) Counterexample where the bundle is not rationally trivial and does not reduce to T. If k is the field of real numbers and $G=PGL_2$, then the subgroup generated by $\left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ is isomorphic to μ_2 but lies in no split torus. For all split tori are conjugate over k [3, 20.9], so the eigenvalues of any matrix in a split torus are real. The bundle $[(\mathbb{P}^1\times G)/\mu_2]$ over $[\mathbb{P}^1/\mu_2]$ therefore does not reduce to the split torus T. In light of the following lemma, this bundle cannot be rationally trivial.

(1.2) Lemma. *A homomorphism $\phi:\mu_n\to PGL_2$ has image in a split torus if and only if the induced PGL_2-bundle $[(\mathbb{P}^1\times PGL_2)/\mu_n]$ over $[\mathbb{P}^1/\mu_n]$ is rationally trivial.*

Proof. For such a bundle to be rationally trivial means that there exists a rational map $f:\mathbb{G}_m \dashrightarrow PGL_2$ such that the following diagram commutes, where the top arrow is multiplication and the bottom arrow is given by $(\lambda,g)\mapsto\phi(\lambda)g$:

$$\begin{array}{ccc}
\mu_n\times\mathbb{G}_m & \longrightarrow & \mathbb{G}_m\\
\Big\downarrow{\scriptstyle \mathrm{id}\times f} & & \Big\downarrow{\scriptstyle f}\\
\mu_n\times PGL_2 & \longrightarrow & PGL_2.
\end{array}$$

If ϕ has image in a split torus, then it extends to a 1-parameter subgroup $f:\mathbb{G}_m\to PGL_2$, which has the desired property.

Conversely, suppose that the rational map f exists. Consider the completion of PGL_2 to the projective space $\mathbb{P}^3$ parametrizing all 2×2 matrices modulo scalars. By the valuative criterion, f extends to a morphism $\bar f:\mathbb{P}^1\to\mathbb{P}^3$, and the diagram

$$\begin{array}{ccc}
\mu_n\times\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1\\
\Big\downarrow{\scriptstyle \mathrm{id}\times \bar f} & & \Big\downarrow{\scriptstyle \bar f}\\
\mu_n\times\mathbb{P}^3 & \longrightarrow & \mathbb{P}^3
\end{array}$$

still commutes.

Every torus in PGL_2 fixes two points in $\mathbb{P}^1$ over some extension field of k. If one of these points is k-rational, then clearly so is the other. As the split tori are all conjugate over k [3, 20.9], this occurs if and only if the torus is split. Hence ϕ has image in a split torus if and only if it fixes some k-rational point in $\mathbb{P}^1$.

If ϕ is trivial, the statement is immediate. Otherwise, the action of μ_n on PGL_2 fixes no points, while that on the quadric surface $\mathbb{P}^3\setminus PGL_2\cong\mathbb{P}^1\times\mathbb{P}^1$ is induced from the action

of μ_n on the first factor via ϕ. Since the action of μ_n on $\mathbb{P}^1$ by $\lambda[x,y]=[\lambda x,y]$ fixes $[1,0]$ and $[0,1]$, the commutativity of the diagram implies they must map to k-rational points on the quadric whose first factors are fixed by the μ_n-action via ϕ. Hence ϕ has image in a split torus. $\square$

(1.3) Theorem (Existence for $\boldsymbol{\mu_n}$-equivariant lines). *Extension of structure group induces a surjection $H^1([\mathbb{P}^1/\mu_n],T)\to\bar{H}^1([\mathbb{P}^1/\mu_n],G)$; that is, every rationally trivial principal G-bundle over $[\mathbb{P}^1/\mu_n]$ admits a reduction to T.*

Proof. **Step A**: *The case $G=PGL_2$.* Let E be a rationally trivial PGL_2-bundle over $[\mathbb{P}^1/\mu_n]$. By Theorem 0.1, its pullback to $\mathbb{P}^1$ lifts to a GL_2-bundle which splits as $\mathcal{O}(d)\oplus\mathcal{O}(e)$ with $d\geq e$. If $d=e$, then E is trivial as a PGL_2-bundle over $\mathbb{P}^1$; since any split torus is conjugate to T, the result then follows from Lemma 1.2.

Otherwise, the section of $\pi:\mathbb{P}(\mathcal{O}(d)\oplus\mathcal{O}(e))\to\mathbb{P}^1$ corresponding to $\mathcal{O}(d)$ is the unique section with negative normal bundle and hence is preserved by the μ_n-action. It is therefore a μ_n-invariant effective divisor D restricting to $\mathcal{O}(1)$ on each fiber of π. Hence $\mathcal{O}(D)$ is an equivariant line bundle, that is, a line bundle over $[\mathbb{P}^1/\mu_n]$, so that $\tilde{E}=\pi_*\mathcal{O}(D)^*$ is an equivariant lifting of E to GL_2 preserving $\mathcal{O}(1)$. That is, over $[\mathbb{P}^1/\mu_n]$ there is an exact sequence

$$0\longrightarrow\mathcal{O}(d)\longrightarrow\tilde{E}\longrightarrow\mathcal{O}(e)\longrightarrow 0.$$

The extension class lies in $H^1([\mathbb{P}^1/\mu_n],\mathcal{O}(d-e))$. Since μ_n is a multiplicative group, its rational representations are completely reducible, so taking invariants is exact. Hence any line bundle L over $[\mathbb{P}^1/\mu_n]$ satisfies $H^1([\mathbb{P}^1/\mu_n],L)=H^1(\mathbb{P}^1,L)^{\mu_n}$. Consequently $H^1([\mathbb{P}^1/\mu_n],\mathcal{O}(d-e))=H^1(\mathbb{P}^1,\mathcal{O}(d-e))^{\mu_n}=0$, and hence $\tilde{E}$ splits over $[\mathbb{P}^1/\mu_n]$ as a sum $\mathcal{O}(d)\oplus\mathcal{O}(e)$. This reduces its structure group to the maximal torus $\tilde{T}$ of GL_2, determining a μ_n-invariant section of $\tilde{E}/\tilde{T}=E/T$, which in turn reduces E to T as a bundle over $[\mathbb{P}^1/\mu_n]$.

Steps B, C, and D then proceed exactly as in the proof of Theorem 0.1. $\square$

(1.4) Corollary. *Let $P=\operatorname{Pic}[\mathbb{P}^1/\mu_n]$; then the surjection above descends to a natural surjection $(P\otimes\Lambda)/W\to\bar{H}^1([\mathbb{P}^1/\mu_n],G)$.*

Proof. Similar to that of Corollary 0.2. $\square$

(1.5) Lemma. *Suppose that a diagonalizable group scheme S acts on a scheme X satisfying $\Gamma(\mathcal{O}_X^\times)=(\mathbb{G}_m)_k$, and suppose that* Pic X *is étale, generated by a finite number of line bundles on which the S-action is linearized. Then the short exact sequence*

$$1\longrightarrow\hat{S}\longrightarrow\operatorname{Pic}[X/S]\to\operatorname{Pic}X\to 1,$$

where $\hat{S}=\operatorname{Hom}(S,\mathbb{G}_m)$, is split by sending a line bundle L over X to its unique lifting to a line bundle over $[X/S]$ such that S acts trivially on the fiber over x.

Proof. The last hypothesis guarantees that the natural homomorphism $\operatorname{Pic}[X/S]\to\operatorname{Pic}X$ is surjective, and the first guarantees that its kernel, consisting of the group of linearizations

on $\mathcal{O}_X$, is isomorphic to $\hat{S}$. The étale hypothesis then guarantees that the splitting described is a morphism. $\square$

(1.6) Theorem (Uniqueness for μ_n-equivariant lines). *The latter surjection is also an injection: that is, for every rationally trivial principal G-bundle E over $[\mathbb{P}^1/\mu_n]$, the isomorphism class of the reduction to T is unique modulo the action of W.*

Proof. Similar to that of Theorem 0.3, but with the following additions.

For a character $\chi \in \hat{\mu}_n$ and an integer $d \in \mathbb{Z}$, denote by $\mathcal{O}(\chi, d)$ the lifting of $\mathcal{O}(d)$ to $[\mathbb{P}^1/\mu_n]$ such that μ_n acts on the fiber over $[1, 0]$ via χ. By Lemma 1.5, every line bundle over $[\mathbb{P}^1/\mu_n]$ is isomorphic to exactly one of these.

In the case $G = GL_n$, therefore, one has to prove that if $\bigoplus \mathcal{O}(\chi_i, a_i) = \bigoplus \mathcal{O}(\psi_j, b_j)$, then the (χ_i, a_i) and the (ψ_j, b_j) are the same up to permutation. This is accomplished by arguing as for Theorem 0.3 that there exist summands $\mathcal{O}(\psi_j, b_j)$ of degree a_n, then using Schur's lemma to find one such summand where $\psi_j = \chi_n$.

For general G, assume first that k has characteristic zero. There is a W-equivariant isomorphism $P \otimes \Lambda \cong \mathrm{Hom}(\hat{P}, T)$, so regard $\lambda, \lambda' \in P \otimes \Lambda$ as homomorphisms $\hat{P} \to T$. Since $P = \mathrm{Pic}\,[\mathbb{P}^1/H] \cong \mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$, its Cartier dual satisfies $\hat{P} \cong \mathbb{G}_m \times \mu_n$ and hence is a smooth group scheme having a k-rational point t_0 that generates a dense subgroup. Then follow the proof of Theorem 0.3.

When k has characteristic $p > 0$, there is the complication that $\hat{P} \cong \mathbb{G}_m \times \mu_n$ may not be reduced, so homomorphisms are not determined by their values on geometric points. Suppose, however, that $\lambda, \lambda' \in P \otimes \Lambda$ determine isomorphic G-bundles on $[\mathbb{P}^1/\mu_n]$. If $G \to GL_N$ is any representation over k, then the associated GL_N-bundles are also isomorphic and therefore correspond to the same element of $(P \otimes \Lambda_N)/W_N$, where Λ_N and W_N are the cocharacter lattice and Weyl group of GL_N.

Now recall that every pair (G, T) consisting of a reductive group and maximal torus over k is the base change of a Chevalley group $G_{\mathbb{Z}}$ over $\mathbb{Z}$ with maximal torus $T_{\mathbb{Z}}$, and that the cocharacter lattice $\Lambda_{\mathbb{Z}} = \mathrm{Hom}(\mathbb{G}_m, T_{\mathbb{Z}})$ is constant over Spec $\mathbb{Z}$ as a group with W-action. For any dominant weight β, let L_β be the corresponding line bundle over $G_{\mathbb{Z}}/B_{\mathbb{Z}}$, and let $V_\beta = H^0(G_{\mathbb{Z}}/B_{\mathbb{Z}}, L_\beta)$. Then V_β is a finitely generated, free abelian group acted on by $G_{\mathbb{Z}}$, and for any field K, the Borel-Weil representation satisfies $H^0(G_K/B_K, L_\beta) = V_\beta \otimes K$. Choosing an integer basis for V_β consisting of weight vectors determines a homomorphism $G_{\mathbb{Z}} \to (GL_N)_{\mathbb{Z}}$ taking $T_{\mathbb{Z}}$ to the diagonal matrices. The induced map $\Lambda \to \Lambda_N$ is therefore constant, hence independent of K.

For $K = \mathbb{C}$, all irreducible representations appear in this fashion. For $K = k$, our field of characteristic p, the Borel-Weil representation may be reducible; nevertheless, the elements of $(P \otimes \Lambda_N)/W_N$ corresponding to λ and λ' must be the same. The map $(P \otimes \Lambda)/W \to (P \otimes \Lambda_N)/W_N$ is the unique one making the diagram

$$\begin{array}{ccc} P \otimes \Lambda & \longrightarrow & P \otimes \Lambda_N \\ \downarrow & & \downarrow \\ (P \otimes \Lambda)/W & \longrightarrow & (P \otimes \Lambda_N)/W_N \end{array}$$

commute, and the top row is independent of K, so the bottom row is as well. Hence for $K = \mathbb{C}$ as well as $K = k$, the elements of $(P \otimes \Lambda_N)/W_N$ corresponding to λ and λ' are equal. By the characteristic zero case, the G-bundles over $\mathbb{C}$ corresponding to λ and λ' must be isomorphic, so there exists $w \in W$ such that $\lambda' = w\lambda$, as desired. $\square$

(1.7) Remark. There is in general no unique Harder-Narasimhan reduction in the sense of Theorem 0.4 for bundles on μ_n-equivariant lines. Even on $[\mathbb{P}^1/\mu_2]$, if one lets E be $\mathcal{O} \oplus \mathcal{O} \to \mathbb{P}^1$ where μ_2 acts with weights 1 and -1 on the two summands, then either summand gives a reduction to $B \subset GL_2$ satisfying all the conditions of Theorem 0.4. That is why the following proof uses the Harder-Narasimhan reduction on $\mathbb{P}^1$ itself rather than on $[\mathbb{P}^1/\mu_n]$.

(1.8) Theorem (Connectedness for $\boldsymbol{\mu_n}$-equivariant lines). *Suppose the derived subgroup of G is simply connected. Then the automorphism group $\Gamma([\mathbb{P}^1/\mu_n], \operatorname{Ad} E)$ is smooth, affine, and connected, as is the kernel of the evaluation map $\Gamma([\mathbb{P}^1/\mu_n], \operatorname{Ad} E) \to G^\ell$ defined when E is trivialized at ℓ rational points.*

Proof. Pull back E from $[\mathbb{P}^1/\mu_n]$ to $\mathbb{P}^1$ and let E_P be the Harder-Narasimhan reduction of Theorem 0.4. An automorphism of E over $[\mathbb{P}^1/\mu_n]$ is the same as an automorphism over $\mathbb{P}^1$ commuting with the μ_n-action. Since the Harder-Narasimhan reduction is unique and rigid, it must be preserved both by the automorphism and by the μ_n-action. Hence $\operatorname{Ad} E_P$ descends to $[\mathbb{P}^1/\mu_n]$, and there $\Gamma(\operatorname{Ad} E) = \Gamma(\operatorname{Ad} E_P)$.

As in the proof of Theorem 0.5, decompose $P = L \ltimes U$, a semidirect product of a Levi factor and a maximal unipotent, and let $\operatorname{Ad} E_P = Q \ltimes R$ be the corresponding decomposition. Since μ_n acts on E via a reduction to T by Theorem 1.3, it preserves this decomposition, as well as the splitting of R into a product of line bundles. So Q and R (and its splitting) descend to $[\mathbb{P}^1/\mu_n]$.

Consider first $\Gamma(Q)$. The action of μ_n on the trivial bundle Q over $\mathbb{P}^1$ must be by conjugation by some homomorphism $\mu_n \to L$. Hence $\Gamma(Q) = Z_L(\mu_n)$, the centralizer of the image of this homomorphism. As L is a torus centralizer in a reductive group G with simply connected derived group, it too has simply connected derived group: see Steinberg [21, 2.17]. But the centralizer of the image of μ_n in such a group is smooth [10, XI 5.2] and connected. If the image of μ_n is étale, as in characteristic zero, the connectedness is a well-known result of Steinberg [20, 8.1]. The necessary argument in the general case was kindly communicated to us by Brian Conrad and is set down in Theorem 7.2 of the Appendix.

As for $\Gamma(R)$, the argument in the proof of Theorem 0.5 goes through without change, showing that $\Gamma(R)$ is smooth, affine, and connected.

The kernel of the evaluation map is treated similarly, regardless of whether the rational points are fixed by μ_n. $\square$

(1.9) Remark. In the preceding proof, and throughout the paper, centralizers are taken in the scheme-theoretic sense. This should cause no confusion, in light of the aforementioned result [10, XI 5.2] implying that the centralizer, in a linear algebraic group, of a diagonalizable sub-group scheme is a smooth sub-group scheme, that is to say, a linear algebraic subgroup.

(1.10) Counterexample where G is not simply connected. Let $\mu_2 \to PGL_2$ be a nontrivial homomorphism extending to $\mathbb{G}_m \to PGL_2$. This determines a PGL_2-bundle over $[\mathbb{P}^1/\mu_2]$ whose pullback to $\mathbb{P}^1$ is trivial. By Lemma 1.2 it is rationally trivial. But its automorphism group is the centralizer $Z_{PGL_2}(\mu_2)$, which is disconnected.

Variation 2: a football

Now let a, b be positive integers. Let $\mathbb{P}^1_{a,b}$ be the smooth stack, tame in the sense of Abramovich-Olsson-Vistoli [1], whose coarse moduli space is $\mathbb{P}^1$, but with isotropy μ_a over $p_+ = [1,0]$, isotropy μ_b over $[0,1] = p_-$, and trivial isotropy elsewhere. We call such a stack a *football* (referring to an American football, not a soccer ball). It can be constructed by performing the root construction of Cadman [7] on $\mathbb{P}^1$, first to order a at p_+, then to order b at p_-. Note that the global quotient $[\mathbb{P}^1/\mu_{ab}] = \mathbb{P}^1_{ab,ab}$ admits a morphism to $\mathbb{P}^1_{a,b}$ inducing an isomorphism on coarse moduli spaces.

(2.1) Lemma. *The morphism above induces an equivalence between the category of all vector bundles over $\mathbb{P}^1_{a,b}$ and the category of those vector bundles over $\mathbb{P}^1_{ab,ab}$ on which the subgroups μ_b and μ_a of the isotropy act trivially at p_+ and p_-, respectively.*

Proof. Let $\phi : [\mathbb{A}^1/\mu_a] \to \mathbb{A}^1/\mu_a$ be the coarse moduli map. Alper proves [2, 4.5] that for any vector bundle E over the target, the natural adjunction map $E \to \phi_*\phi^*E$ is an isomorphism. He also proves [2, 10.3] that, for any vector bundle F over the source on which the isotropy group at the origin acts trivially, the natural adjunction map $\phi^*\phi_*F \to F$ is an isomorphism.

The standard action of μ_{ab} on $\mathbb{A}^1$ induces an action of $\mu_b \cong \mu_{ab}/\mu_a$ on $\mathbb{A}^1/\mu_a$. If a vector bundle E or F is linearized for this action, then so are its pullbacks and pushforwards, and the adjunction maps are compatible with these linearizations. Hence pullback and pushforward by the corresponding map $[\mathbb{A}^1/\mu_{ab}] \to [\mathbb{A}^1/\mu_b]$ induces an equivalence between the category of all vector bundles over $[\mathbb{A}^1/\mu_b]$ and the category of those vector bundles over $[\mathbb{A}^1/\mu_{ab}]$ on which the subgroup μ_a of the isotropy at the origin acts trivially.

Since a similar statement holds for $[\mathbb{A}^1/\mu_{ab}] \to [\mathbb{A}^1/\mu_a]$, and the assertion is local on the base, the result follows. $\square$

(2.2) Proposition. *The Picard group* $\operatorname{Pic} \mathbb{P}^1_{a,b}$ *is the free abelian group generated by $\mathcal{O}(p_+)$ and $\mathcal{O}(p_-)$, modulo the subgroup generated by $\mathcal{O}(ap_+ - bp_-)$.*

Proof. Straightforward using Lemma 2.1 for line bundles and then applying Lemma 1.5 to $[\mathbb{P}^1/\mu_{ab}]$. $\square$

Define the *degree* to be the map $\deg : \operatorname{Pic} \mathbb{P}^1_{a,b} \to \mathbb{Q}$ given by $\deg \mathcal{O}(ip_+ + jp_-) = i/a + j/b$.

(2.3) Proposition. *We have $h^0(\mathbb{P}^1_{a,b}, \mathcal{O}(ip_+ + jp_-)) = \max(\lfloor i/a \rfloor + \lfloor j/b \rfloor + 1, 0)$. In particular,* $\deg < 0$ *implies $h^0 = 0$. There exists a section nonvanishing at p_+ (resp. p_-) if and only if $i|a$ (resp. $j|b$). The canonical bundle is $\kappa_{\mathbb{P}^1_{a,b}} \cong \mathcal{O}(p_+ + p_-)$, so by Serre duality $h^1(\mathbb{P}^1_{a,b}, \mathcal{O}(ip_+ + jp_-)) = h^0(\mathbb{P}^1_{a,b}, \mathcal{O}((1-i)p_+ + (1-j)p_-))$. In particular,* $\deg \geq 0$ *implies $h^1 = 0$.*

Proof. Again, straightforward from Lemma 2.1. □

A principal G-bundle over $\mathbb{P}^1_{a,b}$ is said to be *rationally trivial* if its restriction to the generic point Spec $k(t)$ of $\mathbb{P}^1_{a,b}$ is trivial.

(2.4) Theorem (Existence for footballs). *Extension of structure group induces a surjection $H^1(\mathbb{P}^1_{a,b}, T) \to \bar{H}^1(\mathbb{P}^1_{a,b}, G)$; that is, every rationally trivial principal G-bundle over $\mathbb{P}^1_{a,b}$ admits a reduction to T.*

Proof. **Step A**: *The case $G = PGL_2$.* Arguing exactly as in the proof of Theorem 0.1 reduces the PGL_2 case to the GL_2 case.

For $G = GL_2$, by Lemma 2.1 it suffices to consider the case of $\mathbb{P}^1_{n,n} = [\mathbb{P}^1/\mu_n]$ for $n = ab$; but this is covered by Theorem 1.3.

Steps B, C, and D then proceed exactly as in the proof of Theorem 0.1, using Lemma 2.3 for the vanishing of H^1 in Step D. □

(2.5) Counterexample where there are three orbifold points on the line. Let $X = \mathbb{P}^1_{2,2,2}$, that is, the stack obtained from $\mathbb{P}^1$ by performing the root construction to order 2 at each of 3 rational points. As shown by Borne [5, 3.13], there is an equivalence of categories between vector bundles on X and parabolic bundles on $\mathbb{P}^1$ with 3 marked points having half-integral weights. So let $E = \mathcal{O} \otimes k^2$ be the trivial bundle on $\mathbb{P}^1$, endowed with a parabolic structure at the 3 marked points consisting of 3 distinct flags in k^2, each with weights 0 and $1/2$. Clearly this does not split as a sum of parabolic line bundles, so the corresponding vector bundle on X does not split either.

(2.6) Corollary. *Let $P = \operatorname{Pic} \mathbb{P}^1_{a,b}$; then the surjection above descends to a natural surjection $(P \otimes \Lambda)/W \to \bar{H}^1(\mathbb{P}^1_{a,b}, G)$.*

Proof. Similar to that of Corollary 0.2. □

(2.7) Theorem (Uniqueness for footballs). *The latter surjection is also an injection: that is, for every rationally trivial principal G-bundle E over $\mathbb{P}^1_{a,b}$, the isomorphism class of the reduction to T is unique modulo the action of W.*

Proof. The case $G = GL_n$ is an immediate consequence of Theorem 1.6 and Lemma 2.1. The general case follows from this case exactly as in the proof of Theorem 1.6. □

(2.8) Theorem (Connectedness for footballs). *Suppose the derived subgroup of G is simply connected. Then the automorphism group $\Gamma(\mathbb{P}^1_{a,b}, \operatorname{Ad} E)$ is smooth, affine, and connected, as is the kernel of the evaluation map $\Gamma(\mathbb{P}^1_{a,b}, \operatorname{Ad} E) \to G^\ell$ defined when E is trivialized at ℓ rational points.*

Proof. By Theorem 2.4, E reduces to T. Hence the splitting of $\mathfrak{g}$ as a T-representation into Cartan subalgebra and root spaces induces a splitting

$$\operatorname{ad} E \cong \mathcal{O}^r \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

As seen in the proof of Theorem 0.4, a reduction E_P of E to a parabolic P is determined by a bundle $\operatorname{ad} E_P$ of subalgebras of $\operatorname{ad} E$ conjugate to $\mathfrak{p}$. As was done there, let $\operatorname{ad} E_P$ be the bundle consisting of line bundles of degree ≥ 0 in the splitting above. In light of Remark 1.7, this may not be the smallest such rigid reduction; nevertheless, it follows from Proposition 2.3 that any automorphism of $\operatorname{ad} E$ preserves $\operatorname{ad} E_P$, and hence that $\Gamma(\operatorname{Ad} E) = \Gamma(\operatorname{Ad} E_P)$, since the normalizer of $\mathfrak{p} \subset \mathfrak{g}$ in G is P. Since T preserves the semidirect product decomposition $P = L \ltimes U$ of the corresponding parabolic, we have a decomposition $\operatorname{Ad} E_P = Q \ltimes R$. Hence as a scheme $\Gamma(\operatorname{Ad} E_P) = \Gamma(Q) \times \Gamma(R)$.

Consider first $\Gamma(Q)$. By Proposition 2.2, $\operatorname{Pic} \mathbb{P}^1_{a,b} \cong \mathbb{Z} \times \mathbb{Z}/d\mathbb{Z}$, where $d = (a, b)$ is the greatest common divisor, and the factor $\mathbb{Z}/d\mathbb{Z}$ corresponds to the bundles of degree zero. Pullback by the μ_d-cover $\pi : \mathbb{P}^1_{a/d,b/d} \to \mathbb{P}^1_{a,b}$ kills this $\mathbb{Z}/d\mathbb{Z}$ factor. Hence $\pi^*\mathfrak{q}$ is trivial. Since π^*Q is an adjoint bundle for the connected group L, this implies that π^*Q is also trivial. Hence $\Gamma(\pi^*Q) \cong L$, which, being a torus centralizer in G, has simply connected derived group [21, 2.17]. Then $\Gamma(Q)$ is the invariant part under the μ_d-action, namely the centralizer $Z_L(\mu_d)$. This is smooth [10, XI 5.2] and, of course, affine. By Theorem 7.2 from the Appendix, it is connected.

As for $\Gamma(R)$, observe that since E reduces to T, and the T-action on U preserves the root subgroups that directly span it, the bundle R splits as a product of line bundles, as before. Hence the argument in the proof of Theorem 0.5 goes through without change.

The kernel of the evaluation map is treated similarly, regardless of whether any of the rational points coincide with p_+ or p_-. $\square$

Variation 3: a gerbe over a football

We now extend the results of Variation 2 to bundles over a more general base space: a rationally trivial μ_n-gerbe J over a football, whose isotropy groups at p_+ and p_- are μ_{na} and μ_{nb}, respectively. Here, by a *μ_n-gerbe* over the football, we mean a stack over the football, locally isomorphic to a product with $B\mu_n$ in, say, the fppf topology on the football. Before we proceed, here are two counterexamples indicating why the hypothesis on the isotropy is necessary.

(3.1) Counterexample where the isotropy hypothesis does not hold. Over any field with two distinct roots $\pm i$ of -1, let $X = [\mathbb{P}^1/\mu_2] \times B\mu_2$. This is a trivial gerbe over $[\mathbb{P}^1/\mu_2] = \mathbb{P}^1_{2,2}$ with structure group μ_2. The quaternion group $\{\pm 1, \pm i, \pm j, \pm k\}$, acting on a 2-dimensional vector space with basis $\{1, j\}$, defines an irreducible 2-dimensional projective representation of μ_2^2, hence a PGL_2-bundle over X which does not reduce to a maximal torus.

(3.2) Counterexample where the band is nontrivial. Over any field with two distinct roots $\pm i$ of -1, let $Y = [\mathbb{P}^1/D_4]$ where the dihedral group D_4 acts via its quotient $\mu_2 = \mathbb{Z}/2\mathbb{Z}$. This is a μ_4-gerbe over the football $[\mathbb{P}^1/\mu_2]$ with nontrivial band. Any irreducible representation of D_4 of dimension 2 induces a vector bundle over Y, which does not split into a sum of line bundles; this may be seen by restricting it to one of the fixed points.

On the other hand, our hypothesis on the isotropy guarantees better behavior.

(3.3) Proposition. *Any μ_n-gerbe J over a football $\mathbb{P}^1_{a,b}$ with abelian isotropy at p_+ and p_- must have trivial band.*

Proof. Since $\operatorname{Aut}\mu_n = \operatorname{Aut}\hat{\mu}_n = \operatorname{Aut}\mathbb{Z}/n\mathbb{Z} = (\mathbb{Z}/n\mathbb{Z})^\times$, the isomorphism class of the band lies in $H^1(\mathbb{P}^1_{a,b}, (\mathbb{Z}/n\mathbb{Z})^\times)$. By hypothesis, the isotropy groups at $p_\pm$ are central extensions of μ_a and μ_b by μ_n. Hence the gerbes obtained by restricting J to $B\mu_a$ and $B\mu_b$ have trivial band. It therefore suffices to show that the restriction map $H^1(\mathbb{P}^1_{a,b}, (\mathbb{Z}/n\mathbb{Z})^\times) \to H^1(B\mu_a, (\mathbb{Z}/n\mathbb{Z})^\times) \times H^1(B\mu_b, (\mathbb{Z}/n\mathbb{Z})^\times)$ is injective. Since $(\mathbb{Z}/n\mathbb{Z})^\times$ is a finite abelian group, it suffices to prove the following lemma. $\square$

(3.4) Lemma. *The restriction $H^1(\mathbb{P}^1_{a,b}, \mathbb{Z}/q\mathbb{Z}) \to H^1(B\mu_a, \mathbb{Z}/q\mathbb{Z}) \times H^1(B\mu_b, \mathbb{Z}/q\mathbb{Z})$ is injective for any prime power q.*

Proof. When q is not a power of the characteristic, this is easy, for then

$$0 \longrightarrow \mathbb{Z}/q\mathbb{Z} \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1.$$

We therefore have a diagram

$$\begin{array}{ccc} H^1(\mathbb{P}^1_{a,b}, \mathbb{Z}/q\mathbb{Z}) & \longrightarrow & H^1(\mathbb{P}^1_{a,b}, \mathbb{G}_m) \\ \downarrow & & \downarrow \\ H^1(B\mu_a \cup B\mu_b, \mathbb{Z}/q\mathbb{Z}) & \longrightarrow & H^1(B\mu_a \cup B\mu_b, \mathbb{G}_m) \end{array}$$

whose horizontal arrows are injective. By Theorem 90 for stacks, the spaces on the right-hand side are Picard groups. By Proposition 2.2 the right-hand map can be identified with the obvious map $\mathbb{Z}^2/\langle(a,-b)\rangle \to \mathbb{Z}/a\mathbb{Z}\times\mathbb{Z}/b\mathbb{Z}$. This is not injective, but it becomes injective when restricted to the kernel of multiplication by q, which is the image of the horizontal map. The left-hand vertical map is therefore injective.

On the other hand, if $q = \operatorname{char} k$, we have the Artin-Schreier sequence

$$0 \longrightarrow \mathbb{Z}/q\mathbb{Z} \longrightarrow \mathbb{G}_a \xrightarrow{F-1} \mathbb{G}_a \longrightarrow 0.$$

From its long exact sequence, together with $H^1(\mathbb{P}^1_{a,b}, \mathcal{O}) = 0$ from Proposition 2.3, we deduce that $H^1(\mathbb{P}^1_{a,b}, \mathbb{Z}/q\mathbb{Z}) = 0$. More generally, if q is a power of $\operatorname{char} k$, then we use an analogue of the Artin-Schreier sequence, namely

$$0 \longrightarrow \mathbb{Z}/q\mathbb{Z} \longrightarrow W_i \xrightarrow{F-1} W_i \longrightarrow 0$$

where W_i is the sheaf of Witt vectors [18]. Since these also satisfy

$$0 \longrightarrow \mathcal{O} \longrightarrow W_i \longrightarrow W_{i-1} \longrightarrow 0,$$

the vanishing of $H^1(\mathbb{P}^1_{a,b}, \mathbb{Z}/q\mathbb{Z})$ again follows from that of $H^1(\mathbb{P}^1_{a,b}, \mathcal{O})$. $\square$

However, the isotropy hypothesis is not enough. The gerbe J is said to be *rationally trivial* if its restriction to the generic point $\operatorname{Spec} k(t)$ of $\mathbb{P}^1_{a,b}$ is trivial. (By Tsen's theorem this is automatic if k is algebraically closed.) Something like this is necessary, as the following example shows.

(3.5) Counterexample where the gerbe is not rationally trivial. Since the absolute Galois group of $\mathbb{R}$ is $\mathbb{Z}/2\mathbb{Z}$, we have $H^2(\text{Spec } \mathbb{R}, \mathbb{Z}/2\mathbb{Z}) = H^2(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Hence there is a nontrivial $\mu_2 = \mathbb{Z}/2\mathbb{Z}$-gerbe over Spec $\mathbb{R}$. Indeed, it is the quotient $[\text{Spec } \mathbb{C}/\mu_4]$, where the generator of μ_4 acts by complex conjugation. The quaternions, with the generator of μ_4 acting by multiplication by j, define a GL_2-bundle over this stack which is irreducible and hence does not reduce to T. The same is true of its pullback under $\mathbb{P}^1_{a,b} \to \text{Spec } \mathbb{R}$.

(3.6) Proposition. *Any rationally trivial μ_n-gerbe over a football $\mathbb{P}^1_{a,b}$ with trivial band is the gerbe of nth roots of a line bundle.*

Proof. The short exact sequence

$$1 \longrightarrow \mu_n \longrightarrow \mathbb{G}_m \longrightarrow \mathbb{G}_m \longrightarrow 1$$

gives long exact sequences (say in fppf cohomology)

$$\begin{array}{ccccccc} H^1(\mathbb{P}^1_{a,b}, \mathbb{G}_m) & \longrightarrow & H^1(\mathbb{P}^1_{a,b}, \mathbb{G}_m) & \longrightarrow & H^2(\mathbb{P}^1_{a,b}, \mu_n) & \longrightarrow & H^2(\mathbb{P}^1_{a,b}, \mathbb{G}_m) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(\eta, \mathbb{G}_m) & \longrightarrow & H^1(\eta, \mathbb{G}_m) & \longrightarrow & H^2(\eta, \mu_n) & \longrightarrow & H^2(\eta, \mathbb{G}_m), \end{array}$$

where η denotes the generic point. The images of the middle horizontal arrows are the root gerbes. According to Lieblich [15, 3.1.3.3], the right-hand vertical arrow is injective. $\square$

(3.7) Proposition. *Let J be the gerbe of nth roots of a line bundle L over $\mathbb{P}^1_{a,b}$. The Picard group of J is generated over* Pic $\mathbb{P}^1_{a,b}$ *by the tautological line bundle ξ modulo the single relation $\xi^n \cong L$. In particular, there is a well-defined degree* Pic $J \to \mathbb{Q}$ *extending the degree on* Pic $\mathbb{P}^1_{a,b}$. *The cohomology of a line bundle M is $H^i(J, M) \cong H^i(\mathbb{P}^1_{a,b}, N)$ if M is the pullback of a line bundle N over $\mathbb{P}^1_{a,b}$, and 0 otherwise. In particular,* deg ≥ 0 *implies $H^1 = 0$, whereas* deg ≤ 0 *implies $H^0 = 0$ except for a trivial bundle.*

Proof. The first statement is proved by Cadman [7, 3.1.2]. The second follows from applying the Leray sequence to the projection $J \to \mathbb{P}^1_{a,b}$. $\square$

A G-bundle over the rationally trivial gerbe J is said to be *rationally trivial* if, on the generic point Spec $k(t) \times B\mu_n$ of J, it is a trivial G-bundle with μ_n acting by a homomorphism into a split torus of $G_{k(t)}$. The last condition may seem ad hoc (and is non-vacuous even if k is algebraically closed), but something of this sort is necessary to avoid counterexamples similar to Counterexample 1.1, only with μ_2 acting trivially on $\mathbb{P}^1$.

(3.8) Theorem (Existence for gerbes). *Let J be a rationally trivial μ_n-gerbe over $\mathbb{P}^1_{a,b}$, having isotropy μ_{na} and μ_{nb} over p_+ and p_-, respectively. Extension of structure group induces a surjection $H^1(J, T) \to \bar{H}^1(J, G)$; that is, every rationally trivial principal G-bundle over J admits a reduction to T.*

Proof.

Step A: *The case* $G = PGL_2$. Suppose $E \to J$ is a rationally trivial principal PGL_2-bundle. Let $B\mu_2 \to J' \to J$ be the gerbe of liftings of E to SL_2. The pullback of E to J' tautologically lifts to an SL_2-bundle E'.

But J' is also a gerbe over the same football, whose structure group, say H, is an extension of μ_n by μ_2. As $\operatorname{Aut}\mu_2 = \operatorname{Aut}\hat{\mu}_2 = \operatorname{Aut}\mathbb{Z}/2\mathbb{Z} = 1$, all such extensions are central. Since $\operatorname{Hom}(\mu_n, \mu_2) = \operatorname{Hom}(\hat{\mu}_2, \hat{\mu}_n)$ is either $\mathbb{Z}/2\mathbb{Z}$ or trivial, the commutator map $\mu_n \to \operatorname{Hom}(\mu_n, \mu_2)$ factors through the group of components of μ_n, which is cyclic. But since every central extension of a cyclic group is abelian, this implies that H is abelian, and hence is either μ_{2n} or $\mu_2 \times \mu_n$. By the same token, the isotropy groups of J' over $p_\pm$ are also abelian. If $H \cong \mu_{2n}$, then J' has trivial band by Proposition 3.3.

If $H \cong \mu_2 \times \mu_n$, then the band of J' is a bundle over $\mathbb{P}^1_{a,b}$ of groups isomorphic to H, with structure group $\operatorname{Aut} H$. In fact, as J' is a gerbe over a gerbe, this structure group reduces to the subgroup preserving μ_2. The band surjects onto the band of J, which is the trivial μ_n-bundle, and its kernel is a bundle of groups isomorphic to μ_2, necessarily trivial as well since $\operatorname{Aut}\mu_2 = 1$. Hence the structure group of the band further reduces to $\operatorname{Hom}(\mu_n, \mu_2)$, which is $\mathbb{Z}/2\mathbb{Z}$ or trivial. Then by Lemma 3.4, J' again has trivial band.

Its structure group H therefore globally acts on E', regarded as a rank 2 vector bundle. If there are two distinct characters, this splits E' globally as a sum of line bundles, reducing it to a maximal torus $T' \subset GL_2$.

If there is only one character, then let L be a line bundle over J' on which H acts with the same character. This exists, as by Proposition 3.6 J' is a root gerbe (or a tensor product of two root gerbes if $H \cong \mu_2 \times \mu_n$), and root gerbes admit tautological line bundles on which μ_n acts by a generator of $\hat{\mu}_n$. Then the rank 2 vector bundle $E' \otimes L^{-1}$ is acted on trivially by H.

We want to conclude that $E' \otimes L^{-1}$ descends to $\mathbb{P}^1_{a,b}$. As in the proof of Lemma 2.1, this follows from an argument like that of Alper [2, 10.3]. Over $\mathbb{P}^1_{a,b}$ as proved in Theorem 2.4, it reduces to a maximal torus $T' \subset GL_2$. Hence so does E' over J'.

Since $E'/T' = E/T$ over J', it follows that E reduces to T over J'. But since E is pulled back from J, the isotropy group μ_2 acts trivially on E and hence on E/T over J'. By descent for affine morphisms [23, 4.3.1], the section of E/T over J' therefore descends to J. Hence E reduces to T over J.

Step B is a direct consequence of the rational triviality of E over J as we defined it. Since the generic isotropy μ_n maps into a split torus of $G_{k(t)}$ over the generic point $\eta = \operatorname{Spec} k(t)$ of $\mathbb{P}^1_{a,b}$, and since all maximal split tori are conjugate to $T_{k(t)}$ [3, 20.9], there is a rational section of E/B preserved by the generic isotropy, that is, a section of E/B over the generic point $\operatorname{Spec} k(t) \times B\mu_n$ of J. By the valuative criterion, this extends to a regular section of E/B over all of J.

Steps C and D then proceed exactly as in the proof of Theorem 0.1, using Proposition 3.7 for the vanishing of H^1 in Step D. $\square$

(3.9) Corollary. *Let* $P = \operatorname{Pic} J$*; then the surjection above descends to a natural surjection* $(P \otimes \Lambda)/W \to \bar{H}^1(J, G)$.

Proof. Similar to that of Corollary 0.2. □

(3.10) Theorem (Uniqueness for gerbes). *The latter surjection is also an injection: that is, for every rationally trivial principal G-bundle E over J, the isomorphism class of the reduction to T is unique modulo the action of W.*

Proof. Similar to that of Theorem 1.6.

In the case $G = GL_n$, one has to prove that if a vector bundle splits in two ways as a sum of line bundles, then the isomorphism classes of the line bundles are the same up to permutation. This is accomplished as in the proof of Theorem 0.3, using the last statement of Proposition 3.7 to show that a line bundle of maximal degree in the first sum maps isomorphically to a line bundle of maximal degree in the second sum, and then proceeding by induction.

To reduce the general case to the GL_n case in characteristic zero, as in the proof of Theorem 1.6 it suffices to show that the Cartier dual $\hat{P}$ of $\operatorname{Pic} J$ has a dense cyclic subgroup. As $\operatorname{Pic} J$ is abelian and finitely generated by Proposition 3.7, this amounts to showing that its torsion part is cyclic. It follows from Proposition 2.2 that the restriction $\operatorname{Tor}\operatorname{Pic} \mathbb{P}^1_{a,b} \to \operatorname{Pic} B\mu_a$ is injective. By Proposition 3.7 there is a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Tor}\operatorname{Pic} \mathbb{P}^1_{a,b} & \longrightarrow & \operatorname{Tor}\operatorname{Pic} J & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & & \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & \operatorname{Pic} B\mu_a & \longrightarrow & \operatorname{Pic} B\mu_{na} & \longrightarrow & \mathbb{Z}/n\mathbb{Z} & \longrightarrow & 0, \end{array} \tag{3.11}$$

where we have used the isotropy hypothesis that the restriction of J to p_+ is isomorphic to $B\mu_{na}$. Hence $\operatorname{Tor}\operatorname{Pic} J$ is a subgroup of a cyclic group, hence cyclic.

The proof in the characteristic $p > 0$ case proceeds exactly as in the proof of Theorem 1.6. □

(3.12) Theorem (Connectedness for gerbes). *Let J be as in Theorem 3.8, and suppose the derived subgroup of G is simply connected. Then the automorphism group $\Gamma(J, \operatorname{Ad} E)$ is smooth, affine, and connected, as is the kernel of the evaluation map $\Gamma(J, \operatorname{Ad} E) \to G^\ell$ defined when E is trivialized at ℓ rational points.*

Proof. The proof runs parallel to that of Theorem 2.8, except for one detail. In order to show the connectedness of $\Gamma(Q)$, it suffices to provide a reduced μ_d-cover $\pi : X \to J$ killing the torsion in $\operatorname{Pic} J$. That it be reduced is necessary to ensure that $\Gamma(\pi^* Q)$ consists only of L.

By Proposition 3.7, $\operatorname{Pic} J$ is a finitely generated abelian group of rank 1, so by (3.11), $\operatorname{Pic} J \cong Z \times \mathbb{Z}/d\mathbb{Z}$ for some $d \in \mathbb{Z}$. Let L be a line bundle over J whose isomorphism class generates the second factor, choose a nowhere zero section of $L^{\otimes d}$, and let X be the inverse image of this section under the dth power map $L \to L^{\otimes d}$.

The pullback by $X \to J$ tautologically kills L. On the other hand, X is easily seen to be reduced, as follows. By Proposition 3.6, the part of J over $\mathbb{P}^1_{a,b} \setminus p_-$ is $[\mathbb{A}^1/\mu_{na}]$ for some linear action of μ_{na} on $\mathbb{A}^1$, and then by Proposition 3.7, over this open set the line bundle L

is induced by a character of μ_{na}, which by (3.11) has order exactly d. Hence the part of X over this open set is isomorphic to $[\mathbb{A}^1/\mu_{na/d}]$, which is clearly reduced. A similar argument applies to the open set $\mathbb{P}^1_{a,b} \setminus p_+$. $\square$

Variation 4: a chain of lines

Now let n be a positive integer. Let C be a nodal chain of n projective lines, where $[0,1]$ on the first line coincides with $[1,0]$ on the second line, and so on. Denote $[1,0]$ on the first line by p_+ and $[0,1]$ on the last line by p_-. Let $\mathcal{O}(1)$ over $\mathbb{P}^1$ have the standard trivialization at $[0,1]$ and $[1,0]$ and, given $d = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, define $\mathcal{O}(d)$ to be the line bundle over C obtained by gluing together the line bundles $\mathcal{O}(d_i)$ on the ith line preserving (the tensor powers of) these trivializations. This induces an isomorphism $\mathbb{Z}^n \to \operatorname{Pic} C$.

Tensoring with Λ induces a further isomorphism $\Lambda^n \to H^1(C,T)$, since $H^1(C,T) \cong \operatorname{Pic} C \otimes \Lambda$ canonically. For $\lambda \in \Lambda^n$, denote by $E(\lambda)$ the G-bundle associated to the image of λ under the latter isomorphism. Note that $E(\lambda)$ too acquires a standard trivialization at $p_\pm$ and at each of the nodes. Thus one may define homomorphisms $V_\pm : \operatorname{Aut} E(\lambda) \to G$ evaluating an automorphism at $p_\pm$ in terms of the standard trivializations.

(4.1) Lemma. (a) *The image of $V_\pm$ is a parabolic subgroup $P_\pm$ of G, the Lie algebra of whose Levi factor is the direct sum of $\mathfrak{t}$ and those root spaces $\mathfrak{g}_\alpha$ for which all $\alpha \cdot \lambda_i = 0$.* (b) *Let X be a nonempty set of rational points of C and $\operatorname{Aut}(E(\lambda), X)$ the subgroup of automorphisms trivial over X. Then $V_\pm(\operatorname{Aut}(E(\lambda), X))$ is a smooth unipotent subgroup of $P_\pm$ directly spanned by its root subgroups.*

Proof. Without loss of generality consider V_+. As shown by Brion [6, 4.2], $\operatorname{Aut} E(\lambda)$ is a group scheme, locally of finite type, with Lie algebra $H^0(\operatorname{ad} E(\lambda))$. Let us first determine the Lie algebra of the image, which is the image of the Lie algebra. This amounts to finding the image of the evaluation $H^0(\operatorname{ad} E(\lambda)) \to \mathfrak{g}$ at p_+.

Since the Lie algebra of G splits as a $\mathfrak{t}$-representation into root spaces

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha,$$

the vector bundle $\operatorname{ad} E(\lambda)$ splits under the reduction of structure group to T as

$$\operatorname{ad} E(\lambda) = (\mathcal{O} \otimes \mathfrak{t}) \oplus \bigoplus_{\alpha \in \Phi} L_\alpha,$$

where $L_\alpha \cong \mathcal{O}(\alpha \cdot \lambda_1, \ldots, \alpha \cdot \lambda_n)$.

Clearly the line bundle $\mathcal{O}(d_1, \ldots, d_n)$ has a section nonvanishing at p_+ if and only if $(0, \ldots, 0) \preccurlyeq (d_1, \ldots, d_n)$ in the lexicographic ordering. Only finitely many line bundles $\mathcal{O}(\alpha \cdot \lambda_1, \ldots, \alpha \cdot \lambda_n)$ appear in the sum above. So choose rational numbers $x_1 >> x_2 >> \cdots >> x_n > 0$; then the lexicographic inequality holds for these bundles if and only if $0 \le \sum x_i(\alpha \cdot \lambda_i) = \alpha \cdot \sum x_i \lambda_i$. Furthermore, equality holds in the latter inequality if and

only if all $\alpha \cdot \lambda_i = 0$. The image of dV_+ is therefore

$$\mathfrak{t} \oplus \bigoplus_{\alpha \cdot (\sum x_i \lambda_i) \geq 0} \mathfrak{g}_\alpha. \tag{4.2}$$

This is a parabolic subalgebra $\mathfrak{p}_+$, and its Levi factor is

$$\mathfrak{t} \oplus \bigoplus_{\alpha \cdot (\sum x_i \lambda_i) = 0} \mathfrak{g}_\alpha = \mathfrak{t} \oplus \bigoplus_{\alpha \cdot \lambda_i = 0} \mathfrak{g}_\alpha.$$

In characteristic zero, where all group schemes over a field are smooth, this immediately implies that the image of V_+ contains the parabolic subgroup P_+ whose Lie algebra is $\mathfrak{p}_+$.

In general, a little more care is needed, since *a priori* the image might not be smooth. So note instead that, as the Levi factor consists of those roots α where all $\alpha \cdot \lambda_i = 0$, the semidirect product decomposition $P = L \ltimes U$ determines a decomposition $\operatorname{ad} E_P \cong Q \ltimes R$ as in Theorem 0.5 where Q is a trivial L-bundle over C and R is a unipotent bundle. The evaluation $\Gamma(Q) \to L$ is then clearly surjective; to see that the evaluation $\Gamma(R) \to U$ is surjective as well, observe that the splitting of U (as a scheme) into a product of root subgroups U_α determines a similar splitting of R, say $R = \prod_\alpha R_\alpha$, and also that $\Gamma(R_\alpha) \to U_\alpha$ is surjective as mentioned before.

Hence, in any characteristic, the image of V_+ is a group subscheme of G containing P_+, whose Lie algebra coincides with that of P_+. It is therefore smooth with identity component P_+; but it must then be connected [3, 11.16], hence coincides with P_+. This proves (a).

As for (b), by cutting the chain at the nodes in X we may assume without loss of generality that X contains no nodes, hence defines a Cartier divisor. The statement then follows by a similar argument, splitting the vector bundle $\operatorname{ad}(E(\lambda))(-X)$ as before, then observing that the intersection of $\operatorname{Aut}(E(\lambda), X)$ with Q is trivial, while its intersection with R is again a product of line bundles corresponding to the roots. $\square$

Say that a G-bundle over C is *rationally trivial* if it is trivial over the generic point of each line in C, and let $\bar{H}^1(C, G)$ denote the set of isomorphism classes of rationally trivial G-bundles, as before.

(4.3) Theorem (Existence for chains). *Extension of structure group induces a surjection $H^1(C, T) \to \bar{H}^1(C, G)$; that is, every rationally trivial principal G-bundle over C admits a reduction to T.*

For $k = \mathbb{C}$, this has been proved by Teodorescu [22].

Proof by induction on n, the case $n = 1$ being Theorem 0.1.

For $n > 0$, we will prove that the structure group may be reduced to $N(T)$ in such a way that it reduces further to T on each irreducible component of C. This suffices, as the associated principal $N(T)/T$-bundle is then trivial on each component and hence clearly has a section.

Let E be a rationally trivial principal G-bundle over C. Express C as a union of two strictly shorter chains C_+ and C_- intersecting in a single point x. By the induction hypothesis, the restrictions of E to $C_\pm$ reduce to T and hence are isomorphic to bundles

of the form $E(\lambda_\pm)$ described earlier. Therefore E may be obtained from $E(\lambda_\pm)$, equipped with their standard trivializations at x, by using some gluing parameter $g \in G$ to identify their fibers. It suffices to show that this parameter may be moved into $N(T)$ by acting by automorphisms of $E(\lambda_\pm)$. Since the Bruhat decomposition says $G = BWB$, or rather $G = BN(T)B$, it now suffices to observe that the images of the evaluation homomorphisms $V_\pm : \operatorname{Aut} E(\lambda_\pm) \to G$ both contain a Borel containing T, as an immediate consequence of Lemma 4.1. $\square$

(4.4) Corollary. *This descends to a natural surjection $\Lambda^n/W \to H^1(C, G)$.*

Proof. Since $\mathbb{Z}^n \cong \operatorname{Pic} C = H^1(C, \mathbb{G}_m)$, tensoring by Λ yields $\Lambda^n \cong H^1(C, T)$ as W-modules. If $\lambda' = w\lambda \in \Lambda^n$, then the T-bundles corresponding to λ' and λ are related by extension of structure group by $w : T \to T$. Since this extends to an inner automorphism of G, the associated G-bundles $E(\lambda')$ and $E(\lambda)$ are isomorphic. $\square$

(4.5) Counterexample where the base curve is not a chain but still has arithmetic genus zero. Let X be a nodal curve consisting of 4 projective lines configured like the letter E. Let V be a vector bundle over X constructed by gluing 3 copies of $\mathcal{O} \oplus \mathcal{O}(1)$ (over each of the horizontal lines) to a trivial bundle $\mathcal{O} \otimes k^2$ (over the vertical line) so that the fibers of $\mathcal{O}(1)$ over the nodes are glued to 3 distinct lines in k^2. Then V does not split as a sum of line bundles. For the only splittings of $\mathcal{O} \otimes k^2$ arise from splittings of k^2, whereas any splitting of $\mathcal{O} \oplus \mathcal{O}(1)$ must include the given $\mathcal{O}(1)$ summand by the last statement of Theorem 0.4.

For another such example, let Y be the curve consisting of 3 projective lines meeting pairwise nodally in a single point with 3-dimensional Zariski tangent space, like the 3 axes in $\mathbb{A}^3$. Let $\pi : X \to Y$ be the obvious morphism. Then $\pi_* V$ is a vector bundle over Y, which may be constructed analogously to V, and which again does not split, for the same reasons.

(4.6) Theorem (Uniqueness for chains). *The latter surjection is also an injection: that is, for every rationally trivial principal G-bundle E over C, the isomorphism class of the reduction to T is unique modulo the action of W.*

Proof by induction on n, the case $n = 1$ being Theorem 0.3.

Let λ and $\lambda' \in \Lambda^n$ be n-tuples determining isomorphic G-bundles $E(\lambda) \cong E(\lambda')$ over C. It suffices to find $w \in W$ such that $\lambda' = w\lambda$.

Express C as a union of two shorter chains $C_+ \cup C_-$ intersecting at a single node x, the first chain having m lines. Write $\lambda = (\lambda_+, \lambda_-)$ for $\lambda_+ \in \Lambda^m$ and $\lambda_- \in \Lambda^{n-m}$, and similarly for λ'. Then $E(\lambda_\pm)$ are the restrictions of $E(\lambda)$ to $C_\pm$, equipped with trivializations at x, and similarly for $E(\lambda')$. Certainly $E(\lambda_\pm) \cong E(\lambda'_\pm)$, so by the induction hypothesis, there exist $w_\pm \in W$ such that $\lambda'_\pm = w_\pm \lambda_\pm$.

From bundles $E_\pm$ over $C_\pm$ trivialized at x, a bundle over C may be constructed by identifying the trivialized fibers using a gluing parameter, that is, a group element $g \in G$; denote this bundle $E_+ \vee_g E_-$. Then we have isomorphisms

$$E(\lambda_+) \vee_e E(\lambda_-) \cong E(\lambda) \cong E(\lambda') \cong E(\lambda'_+) \vee_e E(\lambda'_-) = E(w_+\lambda_+) \vee_e E(w_-\lambda_-).$$

On the other hand, extension of structure group by the homomorphisms $w_\pm : T \to T$ induces isomorphisms $E(w_\pm\lambda_\pm) \cong E(\lambda_\pm)$. These may act nontrivially on the fiber at x, so the gluing parameter must be adjusted to obtain

$$E(w_+\lambda_+) \vee_e E(w_-\lambda_-) \cong E(\lambda_+) \vee_{w_-^{-1}w_+} E(\lambda_-),$$

where we abuse notation by using w_+ and w_- to denote any of their representatives in $N(T)$. Concatenating these two chains of isomorphisms, we conclude that there exist automorphisms $\phi_+ \in \operatorname{Aut} E(\lambda_+)$ and $\phi_- \in \operatorname{Aut} E(\lambda_-)$ whose evaluations $v_\pm = V_\pm(\phi_\pm)$ at x satisfy $v_-^{-1}v_+ = w_-^{-1}w_+ \in N(T)$.

On the other hand, $v_-v_+^{-1} \in P_-P_+$, where $P_\pm$ are the parabolic subgroups of Lemma 4.1. If $W_\pm \subset W$ are the Weyl groups of their Levi factors and $\bar{W} = W_-\backslash W/W_+$, then the Bruhat decomposition is a disjoint union $G = P_-\bar{W}P_+$ [3, 21.16], meaning that the inverse image under the quotient $\pi : N(T) \to W$ induces a bijection between the orbits of $W_- \times W_+$ acting on W and those of $P_- \times P_+$ acting on G. Consequently, $P_-P_+ \cap N(T) = \pi^{-1}(W_-)\pi^{-1}(W_+)$, so there exist $u_\pm \in \pi^{-1}(W_\pm)$ such that $u_-^{-1}u_+ = v_-^{-1}v_+$.

Let $w = w_-u_-^{-1} = w_+u_+^{-1}$. Since every element of $W_\pm$ fixes $\lambda_\pm$, we have $w\lambda_- = w_-u_-^{-1}\lambda_- = w_-\lambda_- = \lambda'_-$ and $w\lambda_+ = w_+u_+^{-1}\lambda_+ = w_+\lambda_+ = \lambda'_+$, so $w\lambda = \lambda'$, as desired. $\square$

(4.7) Theorem (Connectedness for chains). *The automorphism group $\Gamma(C, \operatorname{Ad} E)$ is smooth, affine, and connected, as is the kernel of the evaluation map $\Gamma(C, \operatorname{Ad} E) \to G^\ell$ defined when E is trivialized at ℓ rational points.*

Proof by induction on n, the case $n = 1$ being Theorem 0.5.

Let $X = \{x_1, \ldots, x_\ell\}$ be the set of rational points. Assume the statement for chains shorter than C. Let E be a rationally trivial G-bundle over C. The kernel of the evaluation map is $\operatorname{Aut}(E, X)$, the group of automorphisms of E trivial over X. Without loss of generality, we may assume that X contains no nodes, for if it does, the automorphism group simply splits as a product of groups of the same form on shorter chains.

Express C as a union of two shorter chains $C_+ \cup C_-$, glued along a node x. Let $X_+ = X \cap C_+$ and $E_+ = E|_{C_+}$, and similarly for X_- and E_-. Trivialize E at x, so that automorphisms of that fiber take values in G. Let G_+ be the image of $\operatorname{Aut}(E_+, X_+)$ in G under evaluation at x, and similarly for G_-. Then $\operatorname{Aut}(E, X)$ becomes the fibered product over G of $\operatorname{Aut}(E_+, X_+)$ and $\operatorname{Aut}(E_-, X_-)$. As such, it lies in the short exact sequence

$$1 \longrightarrow \operatorname{Aut}(E_+, X_+ \cup \{x\}) \times \operatorname{Aut}(E_-, X_- \cup \{x\}) \longrightarrow \operatorname{Aut}(E, X) \longrightarrow G_+ \cap G_- \longrightarrow 1.$$

The left-hand term is smooth, affine, and connected by the induction hypothesis, so it suffices to show that $G_+ \cap G_-$ is smooth, affine, and connected. (That extensions of smooth affine groups are smooth and affine is clear, as the quotient map is a torsor, locally trivial in the fppf topology [10, VIA 3.2], and these properties are preserved by fppf base change [23, 1.15].)

Now apply Lemma 4.1. If X is empty, it says that $G_\pm$ are parabolic subgroups, whose intersection is smooth, affine, and connected [10, XXVI 4.1.1]. On the other hand, if X is nonempty, it says that either G_+ or G_- is a smooth unipotent subgroup directly spanned

by its root subgroups. Furthermore, the other one contains the root subgroup for every root space in its Lie algebra. Therefore the intersection is directly spanned by these root subgroups and hence is smooth, affine, and connected as well. $\square$

Variation 5: a torus-equivariant line

Let S be a split torus acting on $\mathbb{P}^1$ so that $p_+ = [1,0]$ and $p_- = [0,1]$ are fixed. This variation is concerned with G-bundles over the stack $[\mathbb{P}^1/S]$, or equivalently, with S-equivariant G-bundles over $\mathbb{P}^1$.

A G-bundle over $[\mathbb{P}^1/S]$ is said to be *rationally trivial* if its pullback to the generic point Spec $k(t)$ of $\mathbb{P}^1$ is trivial.

(5.1) Theorem (Existence for torus-equivariant lines). *Extension of structure group induces a surjection $H^1([\mathbb{P}^1/S], T) \to \bar{H}^1([\mathbb{P}^1/S], G)$; that is, every rationally trivial principal G-bundle over $[\mathbb{P}^1/S]$ admits a reduction to T.*

For $G = GL_n$ and $k = \mathbb{C}$, this has been proved by Kumar [14].

Proof. **Step A**: The case $G = PGL_2$. This is proved exactly as for Theorem 1.3, except that, in the case where the pullback of E to $\mathbb{P}^1$ is trivial, the result follows immediately, since the image of the split torus S in PGL_2 must be split [3, 8.4], hence either trivial or conjugate to T [3, 20.9].

Step B: *Reduction to B.* Let E be a rationally trivial G-bundle over $[\mathbb{P}^1/S]$. Then S fixes the generic point η and hence acts on E_η, which may be trivialized as E is rationally trivial. This gives a homomorphism $\phi : S(t) \to G(t)$ over the residue field $k(t)$ of η. Any split torus acting on a projective variety fixes a rational point [3, 15.2], so $S(t)$ fixes a $k(t)$-rational point on $G(t)/B(t)$. Since $B(t)$ is split solvable, $H^1(k(t), B(t)) = 1$, there is a $k(t)$-rational point $g(t) \in G(t)$ lying over the fixed point on $G(t)/B(t)$ [3, 15.7]. Then $\phi(S(t)) \subset g(t)B(t)g^{-1}(t)$. That is to say, S preserves a rational family of Borel subgroups parametrized by t and defined over an open subset of $\mathbb{P}^1$. It therefore preserves a rational section of $[E/B]$ over $\mathbb{P}^1$. By the valuative criterion, this extends to a regular section.

Steps C and D then proceed as in the proof of Theorem 0.1. In Step D, note that since S is a multiplicative group, its rational representations are completely reducible, so that taking invariants is exact. Hence for any line bundle L over $[\mathbb{P}^1/S]$, $H^1([\mathbb{P}^1/S], L) = H^1(\mathbb{P}^1, L)^S$. So if L has positive degree, $H^1([\mathbb{P}^1/\mu_n], L) = 0$ as required. $\square$

(5.2) Corollary. *Let $P = \operatorname{Pic}[\mathbb{P}^1/S]$; then the surjection above descends to a natural surjection $(P \otimes \Lambda)/W \to \bar{H}^1([\mathbb{P}^1/S], G)$.*

Proof. Similar to that of Corollary 0.2. $\square$

(5.3) Theorem (Uniqueness for torus-equivariant lines). *The latter surjection is also an injection: that is, for every rationally trivial principal G-bundle E over $[\mathbb{P}^1/S]$, the isomorphism class of the reduction to T is unique modulo the action of W.*

Proof. Similar to that of Theorem 1.6, though without the complication in characteristic p. In the present case the Cartier dual satisfies $\hat{P} \cong \mathbb{G}_m^{r+1}$, where r is the rank of S; hence, after replacing k by a field extension if necessary, one may find a k-rational point $t_0 \in \hat{P}$ generating a dense subgroup. $\square$

(5.4) Theorem (Connectedness for torus-equivariant lines). *The automorphism group $\Gamma([\mathbb{P}^1/S], \operatorname{Ad} E)$ is smooth, affine, and connected, as is the kernel of the evaluation map $\Gamma([\mathbb{P}^1/S], \operatorname{Ad} E) \to G^\ell$ defined when E is trivialized at ℓ rational points.*

Proof. Parallel to that of Theorem 0.5. As the Harder-Narasimhan filtration of Theorem 0.4 is unique and rigid, the S-action must preserve it. In the ensuing decomposition $\Gamma(\operatorname{Ad} E) = \Gamma(Q) \times \Gamma(R)$, the first factor is the centralizer of a torus in L, which is smooth [10, XI 5.3] and connected [3, 11.12], while the second factor is handled as before. $\square$

Variation 6: a torus-equivariant chain of lines

Let C be a chain of n lines with endpoints $p_\pm$ as in Variation 4. Let S be a split torus acting on C so that $p_\pm$ are fixed. This determines a homomorphism $S \to \operatorname{Aut}(C, p_\pm) \cong \mathbb{G}_m^n$, not necessarily an immersion. This variation is concerned with G-bundles over the stack $[C/S]$, or equivalently, with S-equivariant G-bundles over C.

Given $d \in \mathbb{Z}^n$, a line bundle $\mathcal{O}(d)$ over C of multidegree d was constructed in Variation 4, yielding an isomorphism $\mathbb{Z}^n \to \operatorname{Pic} C$. Furthermore, given any fixed k-rational point x of the S-action on C, applying Lemma 1.5 to $[C/S]$ splits the natural exact sequence

$$1 \longrightarrow \hat{S} \longrightarrow \operatorname{Pic}[C/S] \longrightarrow \operatorname{Pic} C \longrightarrow 1, \tag{6.1}$$

yielding an isomorphism $\hat{S} \times \mathbb{Z}^n \to \operatorname{Pic}[C/S]$.

Tensoring with $\Lambda = \operatorname{Hom}(\mathbb{G}_m, T)$ induces a further isomorphism $\operatorname{Hom}(S, T) \times \Lambda^n \to H^1([C/S], T)$. Once a fixed point p has been specified, denote by $E(\chi, \lambda)$ the G-bundle associated to the image of $(\chi, \lambda) \in \operatorname{Hom}(S, T) \times \Lambda^n$ under the latter isomorphism. It inherits from $E(\lambda)$ a standard trivialization at $p_\pm$ and at each of the nodes, over each of which S acts on the trivialized fiber by a homomorphism $S \to T$. Let $\chi_\pm : S \to T$ denote these homomorphisms at the points $p_\pm$, and let $S_\pm = \chi_\pm(S)$.

Any automorphism in $\operatorname{Aut} E(\chi, \lambda)$ must, by definition, commute with the S-action. The two homomorphisms $V_\pm : \operatorname{Aut} E(\chi, \lambda) \to G$ evaluating an automorphism at the points $p_\pm$ must therefore take values in $Z(S_\pm)$, the centralizers of the split tori $S_\pm \subset G$. Note that a root $\alpha \in \operatorname{Hom}(T, \mathbb{G}_m)$ of G is also a root of $Z(S_\pm)$ if and only if $\alpha(S_\pm) = 1$, which is the linear condition that α lie in the annihilator in $\mathfrak{t}^*$ of $\mathfrak{s}_\pm$.

(6.2) Lemma. (a) *The image of $V_\pm$ is a parabolic subgroup $P_\pm$ of $Z(S_\pm)$, the Lie algebra of whose Levi factor is the direct sum of $\mathfrak{t}$ and those root spaces $\mathfrak{g}_\alpha$ of $Z(S_\pm)$ for which all $\alpha \cdot \lambda_i = 0$.* (b) *Let X be a nonempty set of rational points of C and $\operatorname{Aut}(E(\lambda), X)$ the subgroup of automorphisms trivial over X. Then $V_\pm(\operatorname{Aut}(E(\lambda), X))$ is a smooth unipotent subgroup of $P_\pm$ directly spanned by its root subgroups.*

Proof. The centralizer $Z(S_\pm)$ is smooth [10, XI 5.3], reductive, and connected [3, 11.12]. The proof is therefore entirely parallel to that of Lemma 4.1.

The only subtlety is in characterizing the sections of the line bundles L_α over $[C/S]$. For such a bundle to have a section nonvanishing at p_+, clearly S must act trivially on L_α at p_+ (which is the condition that α be a root of $Z(S_+)$). Therefore L_α must be the unique lift of $\mathcal{O}(\alpha\cdot\lambda_1,\ldots,\alpha\cdot\lambda_n)$ on which S acts trivially over p_+. A moment's reflection shows that, despite the action of S on C, the unique lift of $\mathcal{O}(d_1,\ldots,d_n)$ on which S acts trivially over p_+ has a section over $[C/S]$ nonvanishing at p_+ if and only if $(0,\ldots,0)\preccurlyeq(d_1,\ldots,d_n)$ in the lexicographic ordering. That is to say, for line bundles on which S acts trivially at p_+, the presence of the S-action causes no further complications. $\square$

(6.3) Lemma. *Let G, T, and S be as before. Any two homomorphisms $S\to T$ which are conjugate by a k-rational $g\in G$ are also conjugate by a k-rational $w\in N(T)$.*

Proof. Without loss of generality assume that the homomorphisms are immersions. Let their images be $S_\pm\subset T$, with $gS_-g^{-1}=S_+$. Then $gTg^{-1}\supset gS_-g^{-1}=S_+$. Hence the centralizer $Z(S_+)\supset T\cup gTg^{-1}$, so these two maximal tori are conjugate in the reductive group $Z(S_+)$ by a k-rational $h\in Z(S_+)$, say $hTh^{-1}=gTg^{-1}$ [3, 20.9]. Let $w=h^{-1}g$. Then the two homomorphisms are conjugate by w, as the same is true of g while h^{-1} fixes S_+; but also $T=wTw^{-1}$, so $w\in N(T)$. $\square$

A G-bundle over $[C/S]$ is said to be *rationally trivial* if its pullback to C is rationally trivial in the sense of Variation 4.

(6.4) Theorem (Existence for torus-equivariant chains). *Extension of structure group induces a surjection $H^1([C/S],T)\to\bar{H}^1([C/S],G)$; that is, every rationally trivial principal G-bundle over $[C/S]$ admits a reduction to T.*

Proof by induction on n, the case $n=1$ being Theorem 5.1.

For $n>0$, let E be a rationally trivial principal G-bundle over $[C/S]$, where C is a chain of n lines. Express C as a union of two strictly shorter chains C_+ and C_- intersecting in a single point x. (By the way, even if S acts effectively on C, it may not act effectively on $C_\pm$, which is why we consider this more general case.) By the induction hypothesis, the restrictions $E_\pm$ of E to $C_\pm$ may be reduced to T and endowed with their standard trivializations over the fixed points of S. The actions of S on the fibers of $E_\pm$ over x are via two homomorphisms $\chi_\pm: S\to T\subset G$.

These homomorphisms are, of course, conjugate by a k-rational element $g\in G$, namely the relevant gluing parameter. By Lemma 6.3, any two such homomorphisms are, in fact, conjugate by a k-rational element $w\in N(T)$. In light of this, it clearly suffices to assume that the standard trivializations of $E_\pm$ over x have the same S-action, say by $\chi_+=\chi_-=\chi: S\to T$. For the reduction of E_- to T may be replaced with its extension of structure group by the homomorphism $w: T\to T$, which induces an isomorphic G-bundle. The gluing parameter $g\in G$ with respect to these trivializations commutes with the S-action, hence it belongs to the centralizer $Z(\chi(S))$.

The remainder of the proof exactly follows that of Theorem 4.3, but using the Bruhat

decomposition of the reductive group $Z(\chi(S))$, together with Lemma 6.2. $\square$

(6.5) Corollary. *Let $P = \operatorname{Pic}[C/S]$; then the surjection above descends to a natural surjection $(P \otimes \Lambda)/W \to \bar{H}^1([C/S], G)$.*

Proof. Similar to that of Corollary 0.2. $\square$

(6.6) Theorem (Uniqueness for torus-equivariant chains). *The latter surjection is also an injection: that is, for every rationally trivial principal G-bundle E over $[C/S]$, the isomorphism class of the reduction to T is unique modulo the action of W.*

Proof by induction on n, the case $n = 1$ being Theorem 5.3.

Express C as a union of two shorter chains $C_+ \cup C_-$ intersecting at a single node x, the first chain having m lines. Use the point x to split the exact sequence 6.1 and thereby determine an isomorphism $\operatorname{Hom}(S, T) \times \Lambda^n \to H^1([C/S], T)$. For $(\chi, \lambda) \in \operatorname{Hom}(S, T) \times \Lambda^n$, let $E(\chi, \lambda)$ be the corresponding G-bundle.

Now suppose that $E(\chi, \lambda) \cong E(\chi', \lambda')$. It suffices to find $w \in W$ such that $(\chi', \lambda') = w(\chi, \lambda)$. Restricting both bundles to x shows that χ and χ' are conjugate by some element of G, and hence by some element of W according to Lemma 6.3. It therefore suffices to assume that $\chi' = \chi$.

Write $\lambda = (\lambda_+, \lambda_-)$ for $\lambda_+ \in \Lambda^m$ and $\lambda_- \in \Lambda^{n-m}$, and similarly for λ'. Then $E(\chi, \lambda_\pm)$ are the restrictions of $E(\chi, \lambda)$ to $C_\pm$, equipped with given trivializations at x, and similarly for $E(\chi, \lambda')$. Certainly $E(\chi, \lambda_\pm) \cong E(\chi, \lambda'_\pm)$, so by the induction hypothesis, there exist $w_\pm \in W$ such that $(\chi, \lambda'_\pm) = w_\pm(\chi, \lambda_\pm)$. In particular, $\chi = w_\pm \chi$, so $w_\pm$ belong to $W_{Z(\chi(S))}$, the Weyl group of the centralizer $Z(\chi(S))$.

The remainder of the proof exactly follows that of Theorem 4.6, but using the Bruhat decomposition of the centralizer $Z(\chi(S))$, which is smooth [10, XI 5.3], reductive, and connected [3, 11.12, 13.17], together with Lemma 6.2. $\square$

(6.7) Theorem (Connectedness for torus-equivariant chains). *The automorphism group $\Gamma([C/S], \operatorname{Ad} E)$ is smooth, affine, and connected, as is the kernel of the evaluation map $\Gamma([C/S], \operatorname{Ad} E) \to G^\ell$ defined when E is trivialized at ℓ rational points.*

Proof. Parallel to that of Theorem 4.7, using Lemma 6.2. When E is trivialized at the node x, the S-action on the fiber determines a homomorphism $\phi : S \to G$, and G_+ and G_- are then subgroups of the centralizer $Z_G(\phi(S))$, which is smooth [10, XI 5.3], reductive, and connected [3, 11.12, 13.17]; it plays the role of G in the ensuing argument. $\square$

This connectedness theorem, in the case where the torus S is 1- or 2-dimensional, and where the evaluation is at the two endpoints p_+ and p_- of the chain, plays a vital part in another paper of the authors [16].

There is every reason to expect that the existence, uniqueness, and connectedness theorems will continue to hold, under suitable hypotheses, over a common generalization of all the foregoing variations: say, a torus-equivariant μ_n-gerbe over a chain of footballs. You are invited, dear reader, to prove this as an exercise.

Appendix: connectedness of centralizers in positive characteristic

We thank Brian Conrad for communicating the following statements and their proofs. As in the body of the paper, let G be a connected reductive group over a field k. (It is elsewhere assumed that G is split, but this is not necessary here.) Centralizers are taken in the scheme-theoretic sense, as always.

(7.1) Lemma. *For any integer $n \geq 1$, the image of any k-homomorphism $\mu_n \to G$ is contained in a k-torus of G.*

Proof. Without loss of generality assume that the homomorphism is a closed immersion, so that $\mu_n \subset G$.

It is equivalent to find a maximal k-torus T of G centralizing μ_n, as such a T is its own centralizer in G. Since the centralizer $Z_G(\mu_n)$ is smooth [10, XI 5.3], its maximal k-tori are maximal in G if and only if the same holds true after a field extension. Therefore we may assume that k is algebraically closed.

A theorem of Steinberg [20, 8.1] then asserts that any semisimple rational point of G is contained in a torus. This implies the lemma whenever the characteristic of k does not divide n, for then μ_n is étale and generated by a semisimple rational point of G.

If the characteristic p does divide n, then decompose $\mu_n \cong E \times \mu_{p^r}$ for some r, where E is étale. By Steinberg's theorem, E is contained in a torus in the centralizer $Z_G(E)$, which is reductive and contains μ_{p^r}. It is therefore in the identity component $H = Z_G(E)^0$, and indeed in its center. Replacing G by the quotient H/E, we may therefore assume that $n = p^r$.

Proceed by induction on r. If $r = 1$, choose an embedding $G \subset GL(V)$ as a closed subgroup; since all algebraic representations of μ_p are completely reducible, the tangent vector X to μ_p at the identity acts semisimply on $k[GL(V)]$, so $X \in \mathfrak{g}$ is a semisimple element [3, 4.3(2), 4.4(4)]. It is therefore tangent to some torus $T \subset G$ [3, 11.8]. The p-Lie functor defines a map $\mathrm{Hom}(\mu_p, T) \to \mathrm{Hom}_{p\text{-Lie}}(\mathrm{Lie}(\mu_p), \mathfrak{t})$, which is bijective because μ_p has vanishing relative Frobenius morphism [10, VIIA 7.2, 7.4]. But the same is true with T replaced by G. Hence the inclusion $\mu_p \subset G$ factors through T.

If $r > 1$, there is a subgroup $M \subset \mu_{p^r}$ with $M \cong \mu_p$. The centralizer $Z_H(M)$ is smooth with reductive identity component $Z_H(M)^0$ [9, A.8.12]. Being central, M lies in a maximal torus of $Z_H(M)^0$, so the inverse image of some (and hence every) maximal torus in $Z_H(M)^0/M$ is a maximal torus in $Z_H(M)^0$. The induction hypothesis then applies to $\mu_{p^{r-1}} \cong \mu_{p^r}/M \subset Z_H(M)^0/M$, and we are done. $\square$

(7.2) Theorem. *Suppose the derived subgroup of G is simply connected. Then the centralizer of the image of any homomorphism $\mu_n \to G$ is connected.*

Proof. Again without loss of generality assume that the homomorphism is a closed immersion, and also that k is algebraically closed.

Let $M = \mu_n$. If the characteristic of k does not divide n, so that M is étale, then the statement is again a theorem of Steinberg [21, 2.15].

To deduce the general case from this case, first observe that, by Lemma 7.1, the subgroup

$M \subset G$ lies in a maximal torus T.

Now recall that every pair (G, T) consisting of a reductive group and maximal torus over k is the base change of a Chevalley group $G_{\mathbb{Z}}$ over $\mathbb{Z}$ with maximal torus $T_{\mathbb{Z}}$. Every geometric fiber of $G_{\mathbb{Z}}$ has simply connected derived group, for this property is equivalent to the quotient of the cocharacter lattice by the coroot lattice being torsion-free [21, 2.4], and these lattices are constant over $\mathbb{Z}$.

The category of diagonalizable groups over any connected base scheme is equivalent, under Cartier duality, to the category of abelian groups. Both M and T are diagonalizable, so the inclusion $M \subset T$ is dual to a surjection $\mathbb{Z}^r \to \mathbb{Z}/n\mathbb{Z}$. Therefore M is the base change of a multiplicative subgroup $M_{\mathbb{Z}} \subset T_{\mathbb{Z}}$ dual to the same surjection.

The centralizer $Z_{G_{\mathbb{Z}}}(M_{\mathbb{Z}})$ is then a group subscheme of $G_{\mathbb{Z}}$, smooth over $\mathbb{Z}$ [10, XI 5.3]. Its geometric fibers are the centralizers of linearly reductive groups and hence have reductive identity components [9, A.8.12].

The generic fiber is connected by the theorem of Steinberg mentioned above. But the locus of connected fibers, for any smooth affine group scheme whose fibers have reductive identity components, is always closed [8, 3.1.12]. Hence the fiber $Z_G(M)$ over k is connected. $\square$

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