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A REMARK ON NORMAL FORMS AND THE "UPSIDE-DOWN" I-METHOD FOR PERIODIC NLS: GROWTH OF HIGHER SOBOLEV NORMS

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Abstract. We study growth of higher Sobolev norms of solutions to the one-dimensional periodic nonlinear Schrödinger equation (NLS). By a combination of the normal form reduction and the *upside-down* I-method, we establish

$$\|u(t)\|_{H^s} \lesssim (1+|t|)^{\alpha(s-1)+}$$

with $\alpha = 1$ for a general power nonlinearity. In the quintic case, we obtain the above estimate with $\alpha = 1/2$ via the space-time estimate due to Bourgain [4, 5]. In the cubic case, we concretely compute the terms arising in the first few steps of the normal form reduction and prove the above estimate with $\alpha = 4/9$. These results improve the previously known results (except for the quintic case.) In Appendix, we also show how Bourgain's idea in [4] on the normal form reduction for the quintic nonlinearity can be applied to other powers.

Contents

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1. Introduction

We consider the periodic defocusing nonlinear Schrödinger equation (NLS):

$$\begin{cases} iu_t - u_{xx} + |u|^{2p}u = 0 \\ u\big|_{t=0} = u_0 \in H^s(\mathbb{T}), \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R} \tag{1.1}$$

where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $p \in \mathbb{N}$, $s > 1$. NLS (1.1) is a Hamiltonian PDE with Hamiltonian:

$$H(u) = \frac{1}{2}\int_{\mathbb{T}} |u_x|^2 + \frac{1}{2p+2}\int_{\mathbb{T}} |u|^{2p+2}. \tag{1.2}$$

Indeed, (1.1) can be written as

$$u_t = i\frac{\partial H}{\partial \bar{u}}. \tag{1.3}$$

Recall that (1.1) also conserves the L^2-norm and the momentum $P(u) = i\int_{\mathbb{T}} u\overline{u}_x$. Moreover, the cubic NLS ($p = 1$) is known to be completely integrable [16] in the sense that it enjoys the Lax pair structure and so infinitely many conservation laws. For $p \geq 2$, the L^2-norm, the momentum, and the Hamiltonian are the only known conservation laws.

In [2], Bourgain proved local well-posedness of (1.1)

- in $L^2(\mathbb{T})$ for the cubic NLS ($p = 1$),
- in $H^s(\mathbb{T})$, $s > 0$, for the quintic NLS ($p = 2$),
- in $H^s(\mathbb{T})$, $s > \frac{1}{2} - \frac{1}{p}$, for $p \geq 3$.

Hence, (1.1) is globally well-posed in $H^1(\mathbb{T})$ for any $p \in \mathbb{N}$, since the conservation of the L^2-norm and the Hamiltonian yields an a priori global-in-time bound on the H^1-norm of solutions. However, except for the cubic case ($p = 1$), there is no a priori upperbound on the H^s-norm for $s > 1$.

In this paper, we study growth of higher Sobolev norms $\|u(t)\|_{H^s}$, $s > 1$, of solutions to (1.1). By iterating the local theory, we easily obtain an exponential bound

$$\|u(t)\|_{H^s} \leq C_1 e^{C_2|t|},$$

where C_1 and C_2 depend only on s, p, and u_0. This exponential bound is not satisfactory at all. Polynomial bounds were then obtained in Bourgain [3], Staffilani [14]. The basic idea is to establish an improved iteration bound:

$$\|u(t+\tau)\|_{H^s} \leq \|u(t)\|_{H^s} + C\|u(t)\|_{H^s}^{1-\delta}$$

for all $t \in \mathbb{R}$, with some $\delta = \delta(s,p) \in (0,1)$, where τ and C depend on s, p, and u_0. This in turn implies

$$\|u(t)\|_{H^s} \leq C(1+|t|)^{\frac{1}{\delta}}, \tag{1.4}$$

where $C = C(s,p,u_0)$. Fourier multiplier method was used in [3], and careful multilinear analysis was performed in [14]. (The only result in [3, 14] for the one-dimensional periodic NLS is for the (nonhomogeneous) cubic NLS with $\delta^{-1} = (s-1)+$ in [14].) Then, Sohinger [12] applied the *upside-down* I-method (see below) to study this problem and proved (1.4) with $\delta^{-1} = 2s+$ for $p \geq 2$ and with $\delta^{-1} = \frac{1}{2}s+$ for $p = 1$.[1]

In the appendix of [4], Bourgain applied the normal form reduction to the quintic NLS and obtained a growth bound; if u is a global solution to the quintic NLS (1.1) with $p = 2$, then we have

$$\|u(t)\|_{H^s} \lesssim_{s,p,u_0} (1+|t|)^{\frac{1}{2}(s-1)+} \tag{1.5}$$

[1] Note the presence of s in place of $s-1$ unlike other results. See Remark 1.3.

for $s > 1$.[2] His idea can be applied to other powers, which yields

Theorem 1.1. *Fix $s > 1$. Given $u_0 \in H^s(\mathbb{T})$, let u be the global solution to* (1.1) *with initial condition u_0.*

(a) *Let $p = 1, 2$. Then, the a priori bound* (1.5) *holds.*
(b) *Let $p \geq 3$. Then, the following a priori bound holds:*

$$\|u(t)\|_{H^s} \lesssim (1 + |t|)^{2(s-1)+}. \tag{1.6}$$

Note that both (1.5) and (1.6) provide slightly better estimates than those in [12]. For the cubic ($p = 1$) case, there are uniform bounds on Sobolev norms due to the complete integrability. Our interest in this article is to establish an a priori bound without using such a structure in an explicit manner.

Consider the Hamiltonian corresponding to (1.1) in the frequency space:[3]

$$\begin{aligned} H(q) = H(q, \bar{q}) &= \sum_n n^2 |q_n|^2 + \sum_{n_1 - n_2 + \cdots - n_{2p+2} = 0} q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2p+1}} \bar{q}_{n_{2p+2}} \\ &=: H_0(q) + H_1(q), \end{aligned} \tag{1.7}$$

where $q_n = \widehat{q}(n)$. Normal form reduction is a sequence of phase space transformations, transforming the nonlinear part $H_1(q)$ of the Hamiltonian into expressions involving only "nearly-resonant" monomials for the form

$$q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}}, \qquad r \geq p + 1, \tag{1.8}$$

where

$$n_1 - n_2 + \cdots + n_{2r-1} - n_{2r} = 0 \tag{1.9}$$

and

$$|n_1^2 - n_2^2 + \cdots + n_{2r-1}^2 - n_{2r}^2| < K \tag{1.10}$$

for some large $K > 0$, (plus a non-resonant error, which needs to be estimated in a suitable topology.) By choosing $K = T^{-\delta}$ for some small $\delta > 0$, Bourgain [4] applied the normal form reduction with the L^6-Strichartz estimate (see (3.23) and (3.24) below) and established (1.5) for $|t| \leq T$.[4] In Appendix, we briefly discuss how his idea can be applied to other powers.

In order to improve Theorem 1.1, we combine this normal form reduction with the *upside-down* I-method. For $s > 1$, let $\mathcal{D}$ be the Fourier multiplier operator given by the multiplier $m : \mathbb{Z} \to \mathbb{R}$, where

$$m(n) = \begin{cases} 1, & |n| \leq N \\ \big(\frac{|n|}{N}\big)^{s-1}, & |n| > N. \end{cases} \tag{1.11}$$

The operator $\mathcal{D}$ is basically a differentiation operator of order $s - 1$. Moreover, it satisfies

$$\|\mathcal{D}q\|_{H^1} \leq \|q\|_{H^s} \leq N^{s-1} \|\mathcal{D}q\|_{H^1}. \tag{1.12}$$

The upside down I-method first appeared in [6] (in the low regularity setting.) The growth of Sobolev norm is related to the low-to-high frequency cascade, and the (upside-down) I

[2] We use $A \lesssim$ B to denote an estimate of the form $A \leq CB$ for some $C > 0$. Similarly, we use $A \sim B$ to denote $A \lesssim B$ and $B \lesssim A$. In (1.5), the expression $\lesssim_{s,p,u_0}$ shows that the implicit constant C depends on s, p, and u_0. In the following, we omit such subscripts when there is no confusion.

[3] In the following, we often drop constants, when they do not play an important role.

[4] In (1.5), the implicit constant is independent of T, and hence the bound (1.5) holds for all $t \in \mathbb{R}$.

method is a suitable tool to study such a phenomenon. As a result, we obtain the following improvement.

Theorem 1.2. *Fix $s > 1$. Given $u_0 \in H^s(\mathbb{T})$, let u be the global solution to* (1.1) *with initial condition u_0.*

(a) *Let $p \geq 3$. Then, we have*

$$\|u(t)\|_{H^s} \lesssim (1+|t|)^{(s-1)+}. \tag{1.13}$$

(b) *Let $p = 2$. Then, the a priori bound* (1.5) *holds.*

(c) *Let $p = 1$. Then, the following a priori bound holds:*

$$\|u(t)\|_{H^s} \lesssim (1+|t|)^{\frac{4}{9}(s-1)+}. \tag{1.14}$$

Remark 1.3. In [12], Sohinger defined $\mathcal{D}$ to be a differentiation of order s and proved an estimate on $\|\mathcal{D}u(t)\|_{L^2}$, i.e. his argument is based on almost conservation of the L^2-norm. However, it seems that by using $\mathcal{D}$ as in (1.11) with almost conservation of the Hamiltonian ($\sim H^1$-norm), one can obtain the results in [12], but with $s-1$ in place of s.

Our argument is closely related to that by Bourgain in [5], where he combined the normal form reduction and the I-method to study global well-posedness of the defocusing quintic NLS on $\mathbb{T}$. There are two main steps in the proof of Theorem 1.2. First, we apply the normal form reduction to the Hamiltonian H in (1.7) and obtain a new Hamiltonian $\mathcal{H} = H \circ \Gamma$ with a certain symplectic transformation Γ so that the transformed Hamiltonian $\mathcal{H}$ is of the form

$$\mathcal{H}(q) = H_0(q) + \mathcal{N}(q),$$

where $\mathcal{N}$ consists of nearly-resonant terms (plus "small" error.) Our choice of the symplectic transformation Γ satisfies $\|\Gamma q\|_{L^2} = \|q\|_{L^2}$ and $\|\Gamma q\|_{H^1} \sim \|q\|_{H^1}$. Recall from [4] that $K = T^{-\delta}$ for Theorem 1.1. For Theorem 1.2, we choose $K = N^\delta$ for some small $\delta > 0$, and then choose N in terms of T as in the usual (upside-down) I-method.

After performing the normal form reduction, we apply the upside-down I-method to the transformed Hamiltonian $\mathcal{H}$. Suppose that $q(t)$ satisfies the Hamiltonian flow of $\mathcal{H}$, i.e.

$$q_t = i\frac{\partial \mathcal{H}}{\partial q}.$$

Then, differentiating in time as in [5], we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{H}(\mathcal{D}q) &= \frac{\partial \mathcal{H}}{\partial q}(\mathcal{D}q)\cdot \mathcal{D}q_t + \frac{\partial \mathcal{H}}{\partial \bar{q}}(\mathcal{D}q)\cdot \overline{\mathcal{D}q_t} \\
&= i\sum_n m(n)^2 n^2 \bigg(\bar{q}_n \frac{\partial \mathcal{N}}{\partial \bar{q}_n}(q) - q_n \frac{\partial \mathcal{N}}{\partial q_n}(q)\bigg) && (1.15)\\
&\quad + i\sum_n m(n) n^2 \bigg(q_n \frac{\partial \mathcal{N}}{\partial q_n}(\mathcal{D}q) - \bar{q}_n \frac{\partial \mathcal{N}}{\partial \bar{q}_n}(\mathcal{D}q)\bigg) && (1.16)\\
&\quad + i\sum_n m(n) \bigg(\frac{\partial \mathcal{N}}{\partial q_n}(\mathcal{D}q)\frac{\partial \mathcal{N}}{\partial \bar{q}_n}(q) - \frac{\partial \mathcal{N}}{\partial q_n}(q)\frac{\partial \mathcal{N}}{\partial \bar{q}_n}(\mathcal{D}q)\bigg). && (1.17)
\end{aligned}$$

As noted in [5], we have (1.15) + (1.16) = 0 and (1.17) = 0 if $\operatorname{supp} q \subset [-N, N]$. Hence, we assume that

$$\max(|n_1|, \dots, |n_{2r}|) > N \tag{1.18}$$

for the monomials of the form (1.8). Then, we prove Theorem 1.2 (a) by estimating the contributions from (1.15)–(1.17). When $p \leq 2$, we obtain an improvement from the space-time estimate by Bourgain [4, 5]. See (3.24) below. Finally, for the cubic nonlinearity ($p = 1$), we concretely compute the terms arising in the first few steps of the normal form reduction, and show that these terms (as well as the higher order terms) satisfy better estimates. A further improvement may be achieved by computing more terms in the normal form reduction. However, the actual computation becomes very cumbersome and we do not pursue this direction any further in this article. See Grébert-Kappeler-Pöschel [10] for the normal form theory of the defocusing cubic NLS, based on the integrability of the equation.

For the non-periodic cubic NLS, Sohinger [13] used the a priori bound on the H^k-norm, $k \in \mathbb{N}$, and obtained

$$\|u(t)\|_{H^s} \lesssim (1+|t|)^{\{s\}+},$$

where $\{s\}$ denotes the fractional part of $s > 1$. Note that such uniform bounds on the H^k-norms are results of integrability of the equation. See [8, 15]. One could try to prove a similar result in the periodic case. However, we do not pursue this direction in this article, since our focus is to present an analytical method without using the complete integrability in an explicit manner.

This paper is organized as follows. In Section 2, we briefly review the theory of the normal form reduction, and apply it in the NLS context. In Section 3, we apply the upside-down I-method to the transformed Hamiltonian and prove Theorem 1.2 (a) and (b). In Section 4, we focus on the cubic NLS. By explicitly computing the first few steps of the normal form reduction, we establish improved estimates in applying the upside-down I-method, and prove Theorem 1.2 (c). In Appendix, we discuss Theorem 1.1 and show how to apply Bourgain's idea [4] for general powers.

2. Normal Form Reduction

2.1. **Introduction.** The normal form reduction involves in eliminating non-resonant parts of the Hamiltonians by introducing suitable symplectic transformations. Our goal is to repeat this procedure so that the transformed Hamiltonian $\mathcal{H}$ consists of the quadratic part H_0, the resonant part $\mathcal{N}_0$, and the error $\mathcal{N}_r$. In the following, we briefly review the basic procedure of the normal form reduction. Also, see Kuksin-Pöschel [11], Bourgain [4, 5], Grébert [9].

Let

$$\widetilde{H}(q,\bar{q}) = \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} c(\bar{n}) q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}} \tag{2.1}$$

be (a part of) a Hamiltonian obtained at some stage of this process. Assume that $c(\bar{n}) := c(n_1, \cdots, n_{2r}) \in \mathbb{R}$ and that $c(\bar{n})$ is invariant[5] (modulo $\pm$ signs) under the permutation $n_{2k-1} \leftrightarrow n_{2k}$, $k = 1, \ldots, r$. Divide $\widetilde{H}$ into the resonant[6] part $\widetilde{H}_0$ and non-resonant part $\widetilde{H}_1$, i.e. $\widetilde{H}_0$ (and $\widetilde{H}_1$) is the restriction of $\widetilde{H}$ on $|D(\bar{n})| \leq K$ (and $|D(\bar{n})| > K$, respectively), where $D(\bar{n})$ is defined by

$$D(\bar{n}) := n_1^2 - n_2^2 + \cdots + n_{2r-1}^2 - n_{2r}^2. \tag{2.2}$$

[5]This is satisfied by the initial Hamiltonian (1.7), and thus is automatically satisfied by all the Hamiltonians appearing in the process.

[6]Strictly speaking, $\widetilde{H}_0$ is only "nearly resonant". However, we refer to it as the "resonant" part for simplicity.

We now introduce a symplectic transformation $\Gamma = \Gamma_F$, called the Lie transform, to eliminate $\widetilde{H}_1$. Define a Hamiltonian F (= "$D^{-1}\widetilde{H}_1$") by

$$F(q, \bar{q}) = \sum_{\substack{n_1 - n_2 + \cdots - n_{2r} = 0 \\ |D(\bar{n})| > K}} \frac{c(\bar{n})}{D(\bar{n})} q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}}. \tag{2.3}$$

Then, it is not difficult to see that F satisfies the following homological equation:

$$\{H_0, F\} = -\widetilde{H}_1, \tag{2.4}$$

where $H_0(q) = \sum_n n^2 |q_n|^2$ as in (1.7) and the Poisson bracket $\{\cdot, \cdot\}$ is defined by

$$\{H_1, H_2\} = i \sum_n \left[\frac{\partial H_1}{\partial q_n} \frac{\partial H_2}{\partial \bar{q}_n} - \frac{\partial H_1}{\partial \bar{q}_n} \frac{\partial H_2}{\partial q_n} \right]. \tag{2.5}$$

Consider a Hamiltonian flow associated to the Hamiltonian F:

$$q_t = i \frac{\partial F}{\partial \bar{q}}. \tag{2.6}$$

Let Γ_t denote the flow map generated by F at time t. Then, we define the Lie transform $\Gamma (= \Gamma_F)$ to be the time-1 map Γ_1.[7] As we see below, the non-resonant part $\widetilde{H}_1$ is eliminated under Γ.

Recall the following lemma. See [9, Lemma 2.8].

Lemma 2.1. *Let Γ_t be as above. Then, for a smooth function G, we have*

$$\frac{d}{dt}(G \circ \Gamma_t) = \{G, F\} \circ \Gamma_t.$$

Proof. By Chain Rule, we have

$$\begin{aligned} \frac{d}{dt}(G \circ \Gamma_t) &= \frac{\partial G}{\partial q}(q(t)) \cdot q_t + \frac{\partial G}{\partial \bar{q}}(q(t)) \cdot \bar{q}_t \\ &= i \frac{\partial G}{\partial q} \cdot \frac{\partial F}{\partial \bar{q}} - i \frac{\partial G}{\partial \bar{q}} \cdot \frac{\partial F}{\partial q} = \{G(q(t)), F(q(t))\} \end{aligned}$$

since $\overline{\partial F / \partial \bar{q}} = \partial F / \partial q$. $\square$

By the Taylor series expansion of $G \circ \Gamma_t$ centered at $t = 0$, we obtain

$$G \circ \Gamma = \sum_{k=0}^{\infty} \frac{1}{k!} \{G, F\}^{(k)}, \tag{2.7}$$

where $\{G, F\}^{(k)}$ denotes the k-fold Poisson bracket of G with F, i.e.

$$\{G, F\}^{(k)} := \{\cdots \{G, \underbrace{F\}, F\}, \cdots, F\}}_{k \text{ times}}$$

and $\{G, F\}^{(0)} = G$.

[7]Here, we simply assume that the flow exists up to time $t = 1$. See Subsection 2.3.

Suppose that we start with a Hamiltonian $H = H_0 + \widetilde{H}$, where H_0 is as in (1.7) and $\widetilde{H}$ is as in (2.1). From (2.7) and (2.4), the transformed Hamiltonian $H' = H \circ \Gamma$ is given by

$$\begin{aligned} H' = H \circ \Gamma &= H_0 \circ \Gamma + \widetilde{H}_0 \circ \Gamma + \widetilde{H}_1 \circ \Gamma \\ &= H_0 + \widetilde{H}_0 + \widetilde{H}_1 + \{H_0, F\} + \{\widetilde{H}_0, F\} + \{\widetilde{H}_1, F\} + \text{h.o.t.} \\ &= H_0 + \widetilde{H}_0 + \{\widetilde{H}_0, F\} + \{\widetilde{H}_1, F\} + \text{h.o.t.}, \end{aligned}$$

where "h.o.t." stands for higher order terms. Hence, we have eliminated the non-resonant part $\widetilde{H}_1$ by the Lie transform Γ. Then, we define the resonant part $\widetilde{H}_0'$ and the non-resonant part $\widetilde{H}_1'$ of the new Hamiltonian H' by

$$\begin{aligned} \widetilde{H}_0' &:= \widetilde{H}_0 + \text{resonant part of } \{\widetilde{H}_0, F\} + \{\widetilde{H}_1, F\} + \text{h.o.t.} \\ \widetilde{H}_1' &:= \text{non-resonant part of } \{\widetilde{H}_0, F\} + \{\widetilde{H}_1, F\} + \text{h.o.t.} \end{aligned}$$

Note that at each step, the lowest degree among the monomials in the non-resonant part increases at least by two since $\deg F \geq 4$.

Lastly, we discuss the regularity of the Lie transform Γ. It follows from Sobolev embedding that Γ acts boundedly on bounded subsets of $H^s(\mathbb{T})$, $s > \frac{1}{2}$. See [11]. Indeed, for F as in (2.3), by Hölder inequality and Sobolev embedding, we have

$$\begin{aligned} \bigg\| \frac{\partial F}{\partial \bar{q}} \bigg\|_{H^s} &\lesssim \sup_{\|p\|_{L^2}=1} K^{-1} \sum_{n_1 - n_2 + \cdots + n_{2r-1} - n = 0} |c(\bar{n})| |q_{n_1}| \cdots |q_{n_{2r-1}}| \cdot \langle n \rangle^s |p_n| \\ &\lesssim \sup_{\|p\|_{L^2}=1} \|p\|_{L^2} \|q\|_{H^s} \|q\|^{2r-2}_{H^{\frac{1}{2}+}} \leq \|q\|^{2r-1}_{H^s}, \end{aligned}$$

where we used the fact that $\langle n \rangle \lesssim \max(\langle n_1 \rangle, \dots, \langle n_{2r-1} \rangle)$ in the second inequality. This is sufficient for our purpose since we take the phase space to be $H^1(\mathbb{T})$ for $\mathcal{D}q$ (and $H^s(\mathbb{T})$, $s > 1$, for our initial data q.) See [4, 5] for the boundedness of Γ in $H^\varepsilon(\mathbb{T})$, $\varepsilon > 0$, for the quintic case.

2.2. **Normal form reduction.** In this subsection, we actually implement the normal form reduction to the Hamiltonian H in (1.7) corresponding to NLS (1.1). Our goal is the following; by a finite[8] sequence of Lie transforms, we transform H into a Hamiltonian of the form

$$\mathcal{H}(q) = H_0(q) + \mathcal{N}_0(q) + \mathcal{N}_r(q), \tag{2.8}$$

where H_0 is the quadratic part, $\mathcal{N}_0$ is the resonant part $\mathcal{N}_0$, and $\mathcal{N}_r$ is "small" error. We assume that $q = \{q_n\}_{n \in \mathbb{Z}}$ satisfies the following L^2- and H^1-bounds:

$$\|q\|_{L^2} \leq C_1, \tag{2.9}$$

$$\|q\|_{H^1} \leq C_2. \tag{2.10}$$

In Section 3, we use the result of this section with $\mathcal{D}q \in H^1$ (for given $q \in H^s$, $s > 1$) as the phase space element in place of q in (2.9) and (2.10).

First, we need to define the "norm" $\| \cdot \|$ to measure a size of a (homogeneous) Hamiltonian. Given a homogeneous multilinear expression

$$\mathcal{N}(q, \bar{q}) = \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} c(\bar{n}) q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}}, \tag{2.11}$$

[8] We repeat the process only finitely many times. In particular, the degree $2r$ of monomials is finite.

define the "size" $\|\mathcal{N}\|$ of $\mathcal{N}$ by

$$\|\mathcal{N}\| = \sup_{*} \sum_n |c(\bar{n})||q^{(1)}_{n_1}||q^{(2)}_{n_2}| \cdots |q^{(2r)}_{n_{2r}}| \tag{2.12}$$

where the supremum is taken over factors $q^{(j)}$, $1 \leq j \leq 2r$ such that

- all factors satisfy (2.9)
- all except at most two factors also satisfy (2.10).

i.e. the supremum is taken over all the factors, allowing two *exceptional* ones. See [5] for a similar definition of a norm on homogeneous multilinear expressions. Like (3.6) in [5], we obtain the following proposition on closure of the Poisson bracket under this norm.

Proposition 2.2. *Let H_1 and H_2 be Hamiltonians of the form* (2.11). *Then, we have*

$$\|\{H_1, H_2\}\| \lesssim \|H_1\|\|H_2\|. \tag{2.13}$$

We need the following lemma to prove Proposition 2.2.

Lemma 2.3. *Let H be a Hamiltonian of the form* (2.11).
(a) *If all the factors of $\partial H/\partial \bar{q}$ satisfy both* (2.9) *and* (2.10), *then we have*

$$\left\| \frac{\partial H}{\partial \bar{q}} \right\|_{H^1} \leq \|H\|. \tag{2.14}$$

(b) *If all the factors of $\partial H/\partial \bar{q}$ satisfy* (2.9) *and all except at most one satisfy* (2.10), *then we have*

$$\left\| \frac{\partial H}{\partial \bar{q}} \right\|_{L^2} \leq \|H\|. \tag{2.15}$$

Proof. (a) Without loss of generality, assume $|n| \lesssim |n_1|$ since $n = n_1 - n_2 + \cdots + n_{2r-1}$. By duality, we have

$$\begin{aligned} \text{LHS of (2.14)} &\lesssim \sup_{\|p\|_{L^2}=1} \sum_{n_1 - \cdots + n_{2r-1} - n = 0} |c(\bar{n})|\big(|n_1||q_{n_1}|\big)|\bar{q}_{n_2}| \cdots |q_{n_{2r-1}}||\bar{p}_n| \\ &\leq \|H\|, \end{aligned}$$

since $\big\||n_1|q_{n_1}\big\|_{l^2}, \|\bar{p}_n\|_{l^2} \lesssim 1$, i.e. all the factors satisfy (2.9) and all, except for $|n_1|q_{n_1}$ and $\bar{p}_n$, satisfy (2.10). Part (b) follows in a similar manner. □

Proof of Proposition 2.2. It suffices to prove

$$\left\| \sum_n \frac{\partial H_1}{\partial q_n} \frac{\partial H_2}{\partial \bar{q}_n} \right\| \lesssim \|H_1\|\|H_2\|. \tag{2.16}$$

There are three cases, depending on the location of the two *exceptional* factors.

First, suppose that both appear in $\partial H_1/\partial q_n$. By duality, we have

$$\left\| \frac{1}{\langle n \rangle} \frac{\partial H_1}{\partial q_n} \right\|_{l^2_n} \leq \sup_{\|p\|_{L^2}=1} \sum_{n_1 - \cdots + n_{2r-1} - n = 0} |c_1(\bar{n})||q_{n_1}||\bar{q}_{n_2}| \cdots |q_{n_{2r-1}}|\big(\langle n \rangle^{-1}|\bar{p}_n|\big) \leq \|H_1\|,$$

(where $\langle n \rangle := 1 + |n|$), since $\langle n \rangle^{-1}\bar{p}_n$ with $\|p\|_{L^2} = 1$ satisfies both (2.9) and (2.10). Hence, from Lemma 2.3 (a), we have

$$\left| \sum_n \frac{\partial H_1}{\partial q_n} \frac{\partial H_2}{\partial \bar{q}_n} \right| \leq \left\| \frac{1}{\langle n \rangle} \frac{\partial H_1}{\partial q_n} \right\|_{l^2_n} \left\| \frac{\partial H_2}{\partial \bar{q}} \right\|_{H^1} \leq \|H_1\|\|H_2\|.$$

The same argument holds when both exceptional factors appear in $\partial H_2/\partial \bar{q}_n$. Finally, suppose that exactly one exceptional factor appears in each of $\partial H_1/\partial q_n$ and $\partial H_2/\partial \bar{q}_n$. Then, (2.16) follows from Cauchy-Schwarz inequality and Lemma 2.3 (b). □

Now, we inductively iterate the steps of the normal form reduction, assuming (2.9) and (2.10). For fixed N (to be chosen in terms of T in the next section), we set $K = N^\delta$ for some small $\delta > 0$. (Recall that we have $K = T^\delta$ in [4].) *Assume* that at some stage of the process, the Hamiltonian is of the form

$$\mathcal{H}(q) = \sum_n n^2 |q_n|^2 + \mathcal{N}_0(q) + \mathcal{N}_1(q) + \mathcal{N}_r(q), \tag{2.17}$$

where the monomials in the resonant part $\mathcal{N}_0$ satisfy

$$|D(\bar{n})| \leq N^\delta \tag{2.18}$$

for some small $\delta > 0$, (where $D(\bar{n})$ is as in (2.2)), the monomials in the non-resonant part $\mathcal{N}_1$ satisfy

$$|D(\bar{n})| > N^\delta, \tag{2.19}$$

and the remainder part $\mathcal{N}_r$ satisfies

$$\|\mathcal{N}_r\| < N^{-C} \tag{2.20}$$

for some large $C > 0$. Moreover, we have

$$\|\mathcal{N}_0\|, \ \|\mathcal{N}_1\| \lesssim 1. \tag{2.21}$$

Clearly, the initial Hamiltonian in (1.7) satisfies the above conditions. Note that there is no remainder part at this stage, i.e. $\mathcal{N}_r = 0$. By Sobolev embedding along with (2.9) and (2.10), we have

$$\begin{aligned} |H_1(q)| &= \Big| \sum_{n_1 - n_2 + \cdots - n_{2p+2} = 0} q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2p+1}} \bar{q}_{n_{2p+2}} \Big| \\ &\leq \|q\|_{L^2}^2 \|q\|_{L^\infty}^{2p} \leq \|q\|_{L^2}^2 \|q\|_{H^{\frac{1}{2}+}}^{2p} \lesssim 1. \end{aligned}$$

i.e. $\|H_1\| \lesssim 1$. Hence, the resonant and non-resonant parts of H_1 satisfy (2.21).

Assume (2.17). Suppose that (the collection of monomials of the lowest degree in) the non-resonant part $\mathcal{N}_1$ is given by

$$\sum_{\substack{n_1 - n_2 + \cdots - n_{2r} = 0 \\ |D(\bar{n})| > N^\delta}} c(\bar{n}) q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}}.$$

As in the previous subsection, define F by

$$F(q, \bar{q}) = \sum_{\substack{n_1 - n_2 + \cdots - n_{2r} = 0 \\ |D(\bar{n})| > N^\delta}} \frac{c(\bar{n})}{D(\bar{n})} q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}} \tag{2.22}$$

so that we have

$$\{F, H_0\} = \mathcal{N}_1. \tag{2.23}$$

Let Γ be the Lie transform associated to F. Then, by (2.7), we have

$$\begin{aligned}\mathcal{H}' = \mathcal{H}\circ\Gamma &= H_0 + \mathcal{N}_0 + \mathcal{N}_1 \\ &\quad + \{H_0, F\} + \{\mathcal{N}_0, F\} + \{\mathcal{N}_1, F\} + \text{h.o.t.} \\ &\quad + \mathcal{N}_r\circ\Gamma \\ &= H_0 + \mathcal{N}_0 + \{\mathcal{N}_0, F\} + \{\mathcal{N}_1, F\} + \text{h.o.t.} + \mathcal{N}_r\circ\Gamma.\end{aligned}$$

From (2.19) and (2.21), we have

$$\|F\| \leq N^{-\delta}\|\mathcal{N}_1\| \lesssim N^{-\delta}. \tag{2.24}$$

From Proposition 2.2, (2.20), (2.24), and (2.7), we see that $\mathcal{N}_r\circ\Gamma$ satisfies (2.20). It also follows that the higher order terms with sufficiently high degrees satisfy (2.20).

Let $\mathfrak{N}$ denote the sum of $\{\mathcal{N}_0, F\}$, $\{\mathcal{N}_1, F\}$, and the remaining part of the higher order terms, i.e.

$$\mathfrak{N} = \sum_{k=1}^{M}\frac{1}{k!}\{\mathcal{N}_0,\ F\}^{(k)} + \sum_{k=1}^{M}\frac{1}{k!}\{\mathcal{N}_1,\ F\}^{(k)} + \sum_{k=2}^{M}\frac{1}{k!}\{H_0,\ F\}^{(k)}$$

for some $M\in\mathbb{N}$. From Proposition 2.2, (2.22), and (2.21), we have

$$\|\{\mathcal{N}_0, F\}\| \lesssim N^{-\delta}\|\mathcal{N}_0\|\|\mathcal{N}_1\| \lesssim N^{-\delta}\|\mathcal{N}_1\|.$$

Similarly, we have $\|\{\mathcal{N}_0, F\}^{(k)}\| \lesssim N^{-k\delta}\|\mathcal{N}_1\|$. Then, from (2.24) and (2.23), we have

$$\|\mathfrak{N}\| \lesssim \|\mathcal{N}_1\|\bigg\{\sum_{k=1}^{M}\frac{1}{k!}N^{-k\delta} + \sum_{k=2}^{M}\frac{1}{k!}N^{-(k-1)\delta}\bigg\} \lesssim N^{-\delta}\|\mathcal{N}_1\|.$$

Now, according to (2.18) and (2.19), divide $\mathfrak{N}$ into its resonant part $\mathfrak{N}_0$ and its non-resonant part $\mathfrak{N}_1$. Hence, we can write the new Hamiltonian $\mathcal{H}'$ as

$$\mathcal{H}' = H_0 + \mathcal{N}_0' + \mathcal{N}_1' + \mathcal{N}_r'$$

where $\mathcal{N}_0' := \mathcal{N}_0 + \mathfrak{N}_0$ satisfies (2.21), $\mathcal{N}_1' := \mathfrak{N}_1$ satisfies

$$\|\mathcal{N}_1'\| \lesssim N^{-\delta}\|\mathcal{N}_1\|, \tag{2.25}$$

and $\mathcal{N}_r'$ satisfies (2.20). In view of (2.25), we can hide the non-resonant part into the remainder part, by iterating the process sufficiently many times.

Therefore, by a *finite* sequence of Lie transforms, we have obtained a new Hamiltonian $\mathcal{H}$ of the form

$$\mathcal{H}(q) = \sum_n n^2|q_n|^2 + \mathcal{N}_0(q) + \mathcal{N}_r(q), \tag{2.26}$$

where $\|\mathcal{N}_0\| \lesssim 1$ and $\|\mathcal{N}_r\| \lesssim N^{-C}$.

2.3. **L^2- and H^1-bounds under the Lie transform.** Before proceeding with the upside-down I-method, let us discuss how the conditions (2.9) and (2.10) are affected under the Lie transform.

Given F as in (2.22), we define the Lie transform Γ to be the time-1 map of (2.6). Denoting by Γ_t the flow map of (2.6) at time t, we have

$$\Gamma_t q = q(t) = q(0) + i\int_0^t \frac{\partial F}{\partial\bar{q}}(q(t'))dt', \tag{2.27}$$

where $q = q(0)$ and $\Gamma_t q = q(t)$. Let $M(q) = \|q\|_{L^2}^2 = \sum_n |q_n|^2$. Then, by Lemma 2.1, we have

$$\frac{d}{dt} M(q(t)) = \{M(q(t)), F(q(t))\} = 0.$$

Hence, we have $\|\Gamma_t q\|_{L^2} = \|q\|_{L^2}$. In particular, we obtain

$$\|\Gamma q\|_{L^2} = \|q\|_{L^2}. \tag{2.28}$$

Now, take the H^1-norm in (2.27). From Lemma 2.3 (a), (2.12), and (2.19), we have

$$\begin{aligned}\|\Gamma_t q\|_{H^1} &\leq \|q\|_{H^1} + t \sup_{t' \in [0,t]} \left\| \frac{\partial F}{\partial \bar{q}}(q(t')) \right\|_{H^1} \leq \|q\|_{H^1} + Ct \sup_{t' \in [0,t]} \|F(q(t'))\| \\ &\leq \|q\|_{H^1} + C' t N^{-\delta} \sup_{t' \in [0,t]} \|q(t')\|_{L^2}^2 \|q(t')\|_{L^\infty}^{2r-2}.\end{aligned}$$

By taking the supremum over $t \in [0,1]$ and by Sobolev embedding along with interpolation on the L^2- and H^1-norms, we have

$$\sup_{t \in [0,1]} \|\Gamma_t q\|_{H^1} \leq \|q\|_{H^1} + C' N^{-\delta} \sup_{t \in [0,1]} \|\Gamma_t q\|_{H^1}^{r-1+}$$

Now, choose $N = N(\gamma)$ large enough such that if $X_t \leq 4\gamma$, then

$$X_t \leq \gamma + C'' N^{-\delta} X_t^{r-1+} \quad \text{implies} \quad X_t \leq 2\gamma. \tag{2.29}$$

For our purpose, let $\gamma = \|q\|_{H^1}$. By the local theory of (2.6), there exists a time $[0, \varepsilon_0]$ such that $X_t := \|\Gamma_t q\|_{H^1} \leq 2\gamma$ for $t \in [0, \varepsilon_0]$. In particular, we have $\|\Gamma_{\varepsilon_0} q\|_{H^1} \leq 2\gamma$. By the local theory again, there exists $\varepsilon > 0$ such that $X_t := \|\Gamma_t q\|_{H^1} \leq 4\gamma$ for $t \in [0, \varepsilon_0 + \varepsilon]$. By (2.29), we have $\|\Gamma_t q\|_{H^1} \leq 4\gamma$ for $t \in [0, \varepsilon_0 + \varepsilon]$. By iterating the argument with a fixed size of ε, we obtain $\|\Gamma_t q\|_{H^1} \leq 2\|q\|_{H^1}$ for $t \in [0,1]$. By inverting the time, we obtain

$$\|\Gamma q\|_{H^1} \sim \|q\|_{H^1} \sim \|\Gamma^{-1} q\|_{H^1}. \tag{2.30}$$

From (2.28) and (2.30), we see that the conditions (2.9) and (2.10) are preserved under the Lie transform.

3. Upside-down I-method

In this section, we estimate the terms (1.15), (1.16), and (1.17) appearing in $\frac{d}{dt}\mathcal{H}(\mathcal{D}q)$, where $\mathcal{H}$ is the Hamiltonian of the form (2.26) obtained in the previous section. The analysis is very similar to that in [5]. We estimate $|\frac{d}{dt}\mathcal{H}(\mathcal{D}q)|$ in terms of a negative power of N and then prove Theorem 1.2.

3.1. Estimates on (1.15), (1.16), and (1.17).

In the following, we assume that $\mathcal{N}$ is of the form (2.11). Then, we can rewrite (1.15) and (1.16) as follows.

$$(1.15) = - \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} c(\bar{n}) R(\bar{n}) q_{n_1} \bar{q}_{n_2} \cdots \bar{q}_{n_{2r}} \tag{3.1}$$

$$(1.16) = \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} c(\bar{n}) D(\bar{n}) \mathcal{D}q_{n_1} \overline{\mathcal{D}q}_{n_2} \cdots \overline{\mathcal{D}q}_{n_{2r}}, \tag{3.2}$$

where $D(\bar{n})$ is as in (2.2) and $R(\bar{n})$ is defined by

$$R(\bar{n}) = m(n_1)^2 n_1^2 - m(n_2)^2 n_2^2 + \cdots - m(n_{2r})^2 n_{2r}^2. \tag{3.3}$$

Recall that we assume (1.18):

$$n_1^* := \max(|n_1|, \ldots, |n_{2r}|) > N.$$

We use n_j^* to denote the j-th largest frequency in the absolute value. Then, we have $n_2^* \gtrsim N$ since $n_1 - n_2 + \cdots - n_{2r} = 0$.

In the following, we assume that $\mathcal{D}q$ satisfies both (2.9) and (2.10). Let $\mathbb{P}_{\geq N}$ be the Dirichlet projection onto the frequencies $\{|n| \geq N\}$. Then, we have

$$\|\mathbb{P}_{\geq N}\mathcal{D}q\|_{L^2} \lesssim N^{-1}\|\mathcal{D}q\|_{H^1} \lesssim N^{-1} \tag{3.4}$$

• **Estimate on** (1.17)**:** Let $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ be of the form (2.11) with frequencies $\{n_j\}_{j=1}^{2r}$ and $\{\widetilde{n}_j\}_{j=1}^{2\widetilde{r}}$. In the following, we estimate

$$\sum_n m(n)\bigg(\frac{\partial\mathcal{N}}{\partial q_n}(\mathcal{D}q)\frac{\partial\widetilde{\mathcal{N}}}{\partial\bar{q}_n}(q) - \frac{\partial\mathcal{N}}{\partial q_n}(q)\frac{\partial\widetilde{\mathcal{N}}}{\partial\bar{q}_n}(\mathcal{D}q)\bigg), \tag{3.5}$$

where

$$n = n_2 - n_3 + \cdots + n_{2r} = \widetilde{n}_1 - \widetilde{n}_2 + \cdots + \widetilde{n}_{2\widetilde{r}-1}. \tag{3.6}$$

If $\max(n_1^*, \widetilde{n}_1^*) \leq N$, then we have (3.5) $= 0$.[9] Hence, without loss of generality, assume $n_1^* > N$. We consider only the first term in (3.5) since the second term can be estimated in a similar manner. Now, we consider two cases: (a) $|n| \gtrsim N$, (b) $|n| \ll N$.

○ Case (a): Suppose $|n| \gtrsim N$. This implies $\widetilde{n}_1^* \gtrsim |n| \gtrsim N$. By Cauchy-Schwarz inequality, we have

$$\bigg|\sum_n m(n)\frac{\partial\mathcal{N}}{\partial q_n}(\mathcal{D}q)\frac{\partial\widetilde{\mathcal{N}}}{\partial\bar{q}_n}(q)\bigg| \leq \bigg\|\frac{\partial\mathcal{N}}{\partial q_n}(\mathcal{D}q)\bigg\|_{l^2_n}\bigg\|m(n)\frac{\partial\widetilde{\mathcal{N}}}{\partial\bar{q}_n}(q)\bigg\|_{l^2_n}. \tag{3.7}$$

First, let us consider the first factor. By (3.4), we have

$$\|Nm(n_1^*)q_{n_1^*}\|_{l^2_n} \lesssim 1.$$

By duality, we have

$$\bigg\|\frac{\partial\mathcal{N}}{\partial q_n}(\mathcal{D}q)\bigg\|_{l^2_n} = \sup_{\|p\|_{L^2}=1}\sum_{n-n_2+\cdots-n_{2r}=0} c(\bar{n})\cdot p_n\cdot\overline{\mathcal{D}q}_{n_2}\cdots\overline{\mathcal{D}q}_{n_{2r}} \lesssim N^{-1}\|\mathcal{N}\|, \tag{3.8}$$

where p_n and $N\mathcal{D}q_{n_1^*}$ are the exceptional factors.

Next, consider the second factor in (3.7). By the monotonicity of $m(\cdot)$ and (3.4), we have

$$\|Nm(n)q_{\widetilde{n}_1^*}\|_{l^2_{\widetilde{n}_1^*}} \lesssim \|Nm(\widetilde{n}_1^*)q_{\widetilde{n}_1^*}\|_{l^2_{\widetilde{n}_1^*}} \lesssim 1.$$

By duality, we have

$$\begin{aligned}\bigg\|m(n)\frac{\partial\widetilde{\mathcal{N}}}{\partial\bar{q}_n}(q)\bigg\|_{l^2_n} &= \sup_{\|p\|_{L^2}=1}\sum_{\widetilde{n}_1-\widetilde{n}_2+\cdots+\widetilde{n}_{2\widetilde{r}-1}-n=0} c(\bar{n})\cdot p_n\cdot m(n)q_{\widetilde{n}_1}\bar{q}_{\widetilde{n}_2}\cdots q_{\widetilde{n}_{2\widetilde{r}-1}}\\ &\lesssim N^{-1}\|\widetilde{\mathcal{N}}\|,\end{aligned} \tag{3.9}$$

where p_n and $Nm(n)q_{\widetilde{n}_1^*}$ are the exceptional factors.

○ Case (b): Suppose $|n| \ll N$. From (3.6), we have $n_2^* \gtrsim N$. Thus, we have

$$\|Nm(n_1^*)q_{n_1^*}\|_{l^2}, \|Nm(n_2^*)q_{n_2^*}\|_{l^2} \lesssim 1.$$

[9]Here, we abuse notation and set $n_1^* = \max(|n_2|, |n_3|, \ldots, |n_{2r}|)$ and $\widetilde{n}_1^* = \max(|\widetilde{n}_1|, |\widetilde{n}_2|, \ldots, |\widetilde{n}_{2\widetilde{r}-1}|)$.

By Cauchy-Schwarz inequality, we have

$$\Big|\sum_n m(n)\frac{\partial \mathcal{N}}{\partial q_n}(\mathcal{D}q)\frac{\partial \widetilde{\mathcal{N}}}{\partial \bar{q}_n}(q)\Big| \le \Big\|\frac{1}{\langle n\rangle}\frac{\partial \mathcal{N}}{\partial q_n}(\mathcal{D}q)\Big\|_{l^2_n}\Big\|\langle n\rangle m(n)\frac{\partial \widetilde{\mathcal{N}}}{\partial \bar{q}_n}(q)\Big\|_{l^2_n}. \tag{3.10}$$

By duality, we have

$$\begin{aligned}\Big\|\frac{1}{\langle n\rangle}\frac{\partial \mathcal{N}}{\partial q_n}(\mathcal{D}q)\Big\|_{l^2_n} &= \sup_{\|p\|_{L^2}=1}\sum_{n-n_2+\cdots-n_{2r}=0} c(\bar{n})\cdot\langle n\rangle^{-1}p_n\cdot\overline{\mathcal{D}q}_{n_2}\cdots\overline{\mathcal{D}q}_{n_{2r}} \\ &\lesssim N^{-2}\|\mathcal{N}\|, \end{aligned}\tag{3.11}$$

where $N\mathcal{D}q_{n_1^*}$ and $N\mathcal{D}q_{n_2^*}$ are the exceptional factors. By Lemma 2.3 (a) and the monotonicity of $m(\cdot)$, the second factor in (3.10) is bounded by $\|\widetilde{\mathcal{N}}\|$.

From (3.7)–(3.11), we obtain

$$|(1.17)| \lesssim N^{-2}\|\mathcal{N}\|^2. \tag{3.12}$$

Lastly, by writing $\mathcal{N} = \mathcal{N}_0 + \mathcal{N}_r$ and $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_0 + \widetilde{\mathcal{N}}_r$, if $\mathcal{N}_r$ or $\widetilde{\mathcal{N}}_r$ appears in either the first or the second factor, then we have $|(1.17)| \lesssim N^{-C}$.

• **Estimate on** (3.1)**:** Let $\eta(n^2) = m(n)^2n^2$. i.e. we have

$$\eta(u) = \begin{cases} u, & u \le N^2 \\ N^{2(1-s)}u^s, & u \ge N^2.\end{cases}$$

In particular, we have $\eta'(u) \lesssim \eta(u)/u$.

◦ $\mathcal{N}_0$-contribution: Assume $n_1^* = |n_1|$. Then, without loss of generality, we can assume $n_2^* \sim |n_2|$ since $|D(\bar{n})| \le N^\delta < (n_1^*)^\delta$ for small $\delta > 0$. Thus, we have

$$n_1^2 = n_2^2 + O((n_3^*)^2 + N^\delta).$$

By Mean Value Theorem, we have

$$|\eta(n_1^2) - \eta(n_2^2)| \lesssim m(n_1)^2 O((n_3^*)^2 + N^\delta).$$

Thus, we have

$$\begin{aligned}|R(\bar{n})| &\lesssim m(n_1)^2O((n_3^*)^2 + N^\delta) + m(n_3^*)^2(n_3^*)^2 \\ &\lesssim m(n_1)^2O((n_3^*)^2 + N^\delta),\end{aligned}$$

where $R(\bar{n})$ is defined in (3.3). Now, we consider two cases, depending on the size of n_3^*: (a) $n_3^* \lesssim N^{\frac{\delta}{2}}$, (b) $n_3^* \gg N^{\frac{\delta}{2}}$.

◦ Case (a): Suppose $n_3^* \lesssim N^{\frac{\delta}{2}}$. In this case, we have $|R(\bar{n})| \lesssim m(n_1)m(n_2)N^\delta$. Hence, from (3.4), we have

$$\begin{aligned}|(3.1)| &\lesssim N^\delta \sum_{n_1-n_2+\cdots-n_{2r}=0}|c(\bar{n})|\cdot m(n_1)q_{n_1}\cdot m(n_2)\bar{q}_{n_2}\cdot q_{n_3}\cdots\bar{q}_{n_{2r}} \\ &\lesssim N^{-2+\delta}\|\mathcal{N}_0\|.\end{aligned}\tag{3.13}$$

◦ Case (b): Suppose $n_3^* \gg N^{\frac{\delta}{2}}$. In this case, we have $n_4^* \sim n_3^*$ since $|D(\bar{n})| \ll (n_3^*)^2$. Otherwise, if $n_4^* \ll n_3^*$, then we would have

$$(1+o(1))(n_3^*)^2 = |(n_1-n_2)(n_1+n_2)| = (1+o(1))n_3^*(n_1^*+n_2^*) \tag{3.14}$$

since n_1 and n_2 have the same sign in view of $|n_1 - n_2| = (1 + o(1))n_3^*$. Then, it follows from (3.14) that $n_3^* = 0$, which in turn implies $n_1 = n_2$ and $n_3^* = \cdots = n_{2r}^* = 0$. In this case, we have (3.1) $= 0$ since $R(\bar{n}) = 0$.

Thus, we have $|R(\bar{n})| \lesssim m(n_1)m(n_2)n_3^* n_4^*$. By Hölder inequality and Sobolev embedding on the physical side, we have

$$\begin{aligned} |(3.1)| &\lesssim \sum_{n_1 - n_2 + \cdots - n_{2r} = 0} |c(\bar{n})| \cdot m(n_1) q_{n_1} \cdot m(n_2) \bar{q}_{n_2} \cdot n_3^* n_4^* q_{n_3} \cdots \bar{q}_{n_{2r}} \\ &\lesssim \|\mathbb{P}_{\gtrsim N} \mathcal{D} q\|_{H^{\frac{1}{2}+}}^2 \big\| |\partial_x| \, q \big\|_{L^2}^2 \|q\|_{H^{\frac{1}{2}+}}^{2r-4} \\ &\lesssim N^{-1+} \|\mathcal{D} q\|_{H^1}^{2r} \end{aligned} \tag{3.15}$$

since $\|\mathbb{P}_{\gtrsim N}\mathcal{D}q\|_{H^{\frac{1}{2}+}} \lesssim N^{-\frac{1}{2}+}\|\mathcal{D}q\|_{H^1}$.

From (3.13) and (3.15), we obtain

$$|(3.1)| \lesssim N^{-1+}. \tag{3.16}$$

$\circ$ $\mathcal{N}_r$-contribution: In this case, we use $|R(\bar{n})| \lesssim m(n_1^*)^2(n_1^*)^2$. Then, proceeding with $\|\mathcal{D}q\|_{H^1} \lesssim 1$, we have

$$|(3.1)| \lesssim \|\mathcal{N}_r\| < N^{-C}. \tag{3.17}$$

$\bullet$ **Estimate on** (3.2)**:**

$\circ$ $\mathcal{N}_0$-contribution: By proceeding with $|D(\bar{n})| \le N^\delta$ and (3.4) as before, we obtain

$$|(3.2)| \lesssim N^{-2+\delta}\|\mathcal{N}_0\|. \tag{3.18}$$

$\circ$ $\mathcal{N}_r$-contribution: In this case, we have $|D(\bar{n})| \lesssim (n_1^*)^2$. Hence, we obtain (3.17).

3.2. **Proof of Theorem 1.2 (a).** Now, we are ready to put all the estimates together and prove Theorem 1.2 (a). Given $u_0 \in H^s$ with $s > 1$, let

$$\mathcal{D}q_0 = \Gamma^{-1}\mathcal{D}u_0.$$

Then, from (1.12) and (2.30), we have

$$\|\mathcal{D}q_0\|_{H^1} \sim \|\mathcal{D}u_0\|_{H^1} \le \|u_0\|_{H^s} \sim_{s,u_0} 1.$$

From (3.12), (3.16), and (3.18), we have

$$\left| \frac{d}{dt}\mathcal{H}(\mathcal{D}q)(t) \right| \lesssim N^{-1+} \tag{3.19}$$

assuming

$$\|\mathcal{D}q(t)\|_{H^1} \lesssim 1. \tag{3.20}$$

Now, fix $T > 0$. Suppose that (3.20) holds true for $|t| \le T$. Then, from (3.19), we have

$$\|\mathcal{D}q(t)\|_{H^1}^2 \sim \mathcal{H}(\mathcal{D}q(t)) \le \mathcal{H}(\mathcal{D}q(0)) + CTN^{-1+}, \qquad |t| \le T. \tag{3.21}$$

By choosing $N \sim T^{1+}$, we indeed have

$$\|\mathcal{D}q(t)\|_{H^1} \lesssim 1, \qquad |t| \le T. \tag{3.22}$$

Note that we performed the upside-down I-method on the transformed coordinates. By (1.12), (2.30), and (3.22), we obtain

$$\|u(t)\|_{H^s} \lesssim N^{s-1}\|\mathcal{D}u(t)\|_{H^1} \sim N^{s-1}\|\mathcal{D}q(t)\|_{H^1} \lesssim T^{(s-1)+}, \qquad |t| \le T.$$

Therefore, we conclude that

$$\|u(t)\|_{H^s} \lesssim (1+|t|)^{(s-1)+}.$$

This completes the proof of Theorem 1.2 (a).

3.3. **Improvement for $p \leq 2$: Theorem 1.2 (b).** In this subsection, we briefly discuss how to improve the result when $p = 1, 2$. The basic idea is to use the estimate due to Bourgain. In [4, 5], Bourgain studied the quintic NLS, where he used space-time estimates to obtain purely spatial estimates. For this purpose, the L^6-Strichartz estimate [2]:

$$\|e^{-it\Delta}\phi\|_{L^6(\mathbb{T}^2)} \lesssim C_N \|\phi\|_{L^2}, \quad \operatorname{supp} \widehat{\phi} \subset [-N, N] \tag{3.23}$$

plays a crucial role, where $C_N = \exp\big(C\frac{\log N}{\log\log N}\big) \ll N^{0+}$. Then, one inductively proves estimates for Hamiltonians with higher order nonlinearity, which appear in the process of the normal form reduction. In the end, one obtains [5, (5.13)]:[10]

$$\max_{a\in\mathbb{Z}} \Bigg| \sum_{\substack{n_1-n_2+\cdots-n_{2r}=0\\ D(\bar{n})=a}} |c(\bar{n})||q^{(1)}_{n_1}||q^{(2)}_{n_2}|\cdots|q^{(2r)}_{n_{2r}}| \Bigg| \lesssim (n_1^*)^{0+} \prod_{j=1}^{2r} \|q^{(j)}_{n_j}\|_{L^2}. \tag{3.24}$$

For the cubic case ($p = 1$), one can basically repeat the same computation to establish (1.5), thanks to the L^4-Strichartz estimate [17]:

$$\|e^{-it\Delta}\phi\|_{L^4(\mathbb{T}^2)} \lesssim \|\phi\|_{L^2}. \tag{3.25}$$

Unlike (3.23), there is no derivative loss in (3.25). However, one has a small derivative loss in the inductive steps, causing the + sign in (1.5). See (A.22) in [4]. As a conclusion, the estimate (3.24) holds when $p = 1, 2$.

Theorem 1.2 (b) follows once we show

$$\left|\frac{d}{dt}\mathcal{H}(\mathcal{D}q)(t)\right| \lesssim N^{-2+}. \tag{3.26}$$

In view of (3.12), (3.13), and (3.18) with $\delta = 0+$, we only need to improve Case (b) in Estimate on (3.1). By applying (3.24) and (3.4) in (3.15) (in place of Hölder inequality and Sobolev embedding), we have

$$\begin{aligned} |(3.1)| &\lesssim N^\delta \max_{|a|\leq N^\delta} \sum_{\substack{n_1-n_2+\cdots-n_{2r}=0\\ D(\bar{n})=a}} |c(\bar{n})| \cdot m(n_1)q_{n_1} \cdot m(n_2)\bar{q}_{n_2} \cdot n_3^* \, n_4^* \, q_{n_3} \cdots \bar{q}_{n_{2r}} \\ &\lesssim N^\delta \|\mathbb{P}_{\gtrsim N}\mathcal{D}q\|^2_{H^{0+}} \big\||\partial_x|\, q\big\|^2_{L^2} \|q\|^{2r-4}_{L^2} \\ &\lesssim N^{-2+\delta+}\|\mathcal{D}q\|^{2r}_{H^1}. \end{aligned}$$

Hence, (3.26) follows and this proves Theorem 1.2 (b).

4. Cubic Case: Theorem 1.2 (c)

In this section, we consider the cubic case ($p = 1$) and prove Theorem 1.2 (c). First, we explicitly compute first few terms appearing in the normal form reduction in Subsection 4.1. See also Erdoğan-Zharnitsky [7]. Then, we establish improved estimates on those terms and prove Theorem 1.2 (c) in Subsection 4.2.

[10]One can indeed obtain this estimate with $(n_3^*)^{0+}$ in place of $(n_1^*)^{0+}$, but it is not useful for our purpose.

4.1. **Normal form reduction: cubic NLS.** Let H be as in (1.7) with $p = 1$. i.e. we have

$$H(q) = \sum_n n^2 |q_n|^2 + \sum_{n_1 - n_2 + n_3 - n_4 = 0} q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4} =: H_0(q) + H_1(q).$$

Now, divide H_1 into the resonant part $\mathcal{R}$ and the non-resonant part $\mathcal{N}$, depending on $D_1(\bar{n}) = 0$ or $\neq 0$, where $D_1(\bar{n})$ is defined by

$$D_1(\bar{n}) := n_1^2 - n_2^2 + n_3^2 - n_4^2 = -2(n_1 - n_2)(n_3 - n_2).$$

We further split $\mathcal{R}$ into two parts:

$$\mathcal{R} = \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ D_1(\bar{n}) = 0}} q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4} = 2 \sum_{n_1} \sum_{n_3} |q_{n_1}|^2 |q_{n_3}|^2 - \sum_n |q_n|^4 =: \mathcal{R}_1 + \mathcal{R}_2. \tag{4.1}$$

By the conservation of the L^2-norm, we have

$$\mathcal{R}_1 = 2\mu \sum_n |q_n|^2,$$

where $\mu = (2\pi)^{-1} \int |u|^2 dx$. By a direct computation, one easily sees that $\{\mathcal{R}_1, F\} = 0$ for smooth F of the form (2.3).

As the first step of the normal form reduction, define F_1 such that $\{H_0, F_1\} = -\mathcal{N}$. i.e.

$$F_1 = \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ n_2 \neq n_1, n_3}} \frac{q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4}}{-2(n_1 - n_2)(n_3 - n_2)}. \tag{4.2}$$

Let Γ_1 be the Lie transform associated with F_1. Then, we have

$$H' := H \circ \Gamma_1 = H_0 + \mathcal{R}_1 + \mathcal{R}_2 + \{\mathcal{R}_2, F_1\} + \tfrac{1}{2}\{\mathcal{N}, F_1\} + \text{h.o.t.} \tag{4.3}$$

Here, we used the fact that $\{\mathcal{R}_1, F_1\} = 0$ and $\{\mathcal{N}, F\} + \frac{1}{2}\{\{H_0, F_1\}, F_1\} = \frac{1}{2}\{\mathcal{N}, F\}$. From (4.1) and (4.2), we have

$$\{\mathcal{R}_2, F_1\} = 2i(\mathcal{I}_0 - \overline{\mathcal{I}_0}), \tag{4.4}$$

where $\mathcal{I}_0$ is given by

$$\begin{aligned} \mathcal{I}_0 &= \sum_{\substack{n_1 - n_2 + n_3 - n_4 = 0 \\ n_2 \neq n_1, n_3}} \frac{q_{n_1} \bar{q}_{n_2} q_{n_3} |q_{n_4}|^2 \bar{q}_{n_4}}{(n_1 - n_2)(n_3 - n_2)} \\ &= \sum_{\substack{n_1 - n_2 + n_3 - n_4 + n_5 - n_6 = 0 \\ n_2 \neq n_1, n_3 \\ n_4 = n_5 = n_6}} \frac{q_{n_1} \bar{q}_{n_2} q_{n_3} \bar{q}_{n_4} q_{n_5} \bar{q}_{n_6}}{(n_1 - n_2)(n_3 - n_2)}. \end{aligned} \tag{4.5}$$

Next, we introduce two more transformations to eliminate the "non-resonant" parts of $\{\mathcal{R}_2, F_1\}$ and $\frac{1}{2}\{\mathcal{N}, F_1\}$. First, we divide them into the resonant parts (with (r)) and the non-resonant parts (with (nr)),

$$\{\mathcal{R}_2, F_1\} = \{\mathcal{R}_2, F_1\}^{(\mathrm{r})} + \{\mathcal{R}_2, F_1\}^{(\mathrm{nr})}$$
$$\{\mathcal{N}, F_1\} = \{\mathcal{N}, F_1\}^{(\mathrm{r})} + \{\mathcal{N}, F_1\}^{(\mathrm{nr})},$$

depending on

$$|D_2(\bar{n})| \leq N^\beta \quad \text{or} \quad |D_2(\bar{n})| > N^\beta \tag{4.6}$$

for some $\beta > 0$ (to be chosen later), where $D_2(\bar{n})$ is defined by

$$D_2(\bar{n}) := n_1^2 - n_2^2 + n_3^2 - n_4^2 + n_5^2 - n_6^2.$$

Now, define F_2 and F_3 such that

$$\{H_0, F_2\} = \tfrac{1}{2}\{\mathcal{N}, F_1\}^{(\mathrm{nr})}$$
$$\{H_0, F_3\} = \{\mathcal{R}_2, F_1\}^{(\mathrm{nr})}$$

i.e. we have

$$F_2 \sim \text{“}D_2^{-1}\{\mathcal{N}, F_1\}^{(\mathrm{nr})}\text{”} \quad \text{and} \quad F_3 \sim \text{“}D_2^{-1}\{\mathcal{R}_2, F_1\}^{(\mathrm{nr})}\text{”}. \tag{4.7}$$

Let Γ_2 and Γ_3 be the Lie transforms associated with F_2 and F_3. Then, from (4.3), we have

$$H'' := H \circ \Gamma_1 \circ \Gamma_2 \circ \Gamma_3 = H_0 + \mathcal{R}_1 + \mathcal{R}_2 + \{\mathcal{R}_2, F_1\}^{(\mathrm{r})} + \tfrac{1}{2}\{\mathcal{N}, F_1\}^{(\mathrm{r})} + \text{h.o.t.} \tag{4.8}$$

From (4.4) and (4.2), we have

$$\{\mathcal{R}_2, F_1\}^{(\mathrm{r})} = 2i(\mathcal{I}_1 - \overline{\mathcal{I}_1}), \tag{4.9}$$
$$\tfrac{1}{2}\{\mathcal{N}, F_1\}^{(\mathrm{r})} = 2i(\mathcal{I}_2 - \overline{\mathcal{I}_2}), \tag{4.10}$$

where $\mathcal{I}_1$ is the resonant part (i.e. $|D_2(\bar{n})| \leq N^\beta$) of $\mathcal{I}_0$ defined in (4.5) and $\mathcal{I}_2$ is given by

$$\mathcal{I}_2 = \sum_{\substack{n_1 - n_2 + n_3 - n_4 + n_5 - n_6 = 0 \\ n_2 \neq n_1, n_3 \\ n_5 \neq n_4, n_6 \\ |D_2(\bar{n})| \leq N^\beta}} \frac{q_{n_1}\bar{q}_{n_2}q_{n_3}\bar{q}_{n_4}q_{n_5}\bar{q}_{n_6}}{(n_1 - n_2)(n_3 - n_2)}. \tag{4.11}$$

In the next subsection, we estimate the terms $\mathcal{R}_1$, $\mathcal{R}_2$, $\{\mathcal{R}_2, F_1\}^{(\mathrm{r})}$, and $\frac{1}{2}\{\mathcal{N}, F_1\}^{(\mathrm{r})}$ appearing in (4.8). Also, note that all the higher order terms in (4.8) are Poisson-bracketed with F_2 or F_3 at least once. i.e. they have an extra decay of $|D_2|^{-1} < N^{-\beta}$ from (4.6) and (4.7).

After this point, we perform the (usual) normal form reductions (as in Section 2) on the higher order terms in (4.8). In particular, we use (2.18) and (2.19) with $\delta = 0+$ to distinguish the resonant and non-resonant terms. In the process, we construct Hamiltonians F of the form (2.22) to eliminate the non-resonant parts of the higher order terms in (4.8). For such F, it follows from the observation in the previous paragraph that $c(\bar{n})$ in (2.22) is small, i.e. $|c(\bar{n})| < N^{-\beta}$. After a finite number of iterations, (4.8) is reduced to

$$\mathcal{H} = \widetilde{H}_0 + \mathcal{R}_2 + \{\mathcal{R}_2, F_1\}^{(\mathrm{r})} + \tfrac{1}{2}\{\mathcal{N}, F_1\}^{(\mathrm{r})} + \underbrace{\mathcal{N}_0 + \mathcal{N}_r}_{= \text{h.o.t.}} =: \widetilde{H}_0 + \widetilde{\mathcal{N}}, \tag{4.12}$$

where $\widetilde{H}_0$ is the new quadratic part defined by

$$\widetilde{H}_0 := H_0 + \mathcal{R}_1 = \sum_n (n^2 + 2\mu)|q_n|^2$$

and the higher order terms have an extra factor of $N^{-\beta}$. (Compare this with (2.26).)

4.2. **Improved estimates.** In this subsection, we prove Theorem 1.2 (c) by establishing improved estimates for all the terms in (4.12). Differentiating (4.12) in time, we obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(\mathcal{D}q) &= \frac{\partial \mathcal{H}}{\partial q}(\mathcal{D}q)\cdot \mathcal{D}q_t + \frac{\partial \mathcal{H}}{\partial \bar{q}}(\mathcal{D}q)\cdot \overline{\mathcal{D}q_t}\\ &= i\sum_n m(n)^2(n^2+2\mu)\bigg(\bar{q}_n\frac{\partial \widetilde{\mathcal{N}}}{\partial \bar{q}_n}(q) - q_n \frac{\partial \widetilde{\mathcal{N}}}{\partial q_n}(q)\bigg) \qquad (4.13)\\ &\quad + i\sum_n m(n)(n^2+2\mu)\bigg(q_n\frac{\partial \widetilde{\mathcal{N}}}{\partial q_n}(\mathcal{D}q) - \bar{q}_n \frac{\partial \widetilde{\mathcal{N}}}{\partial \bar{q}_n}(\mathcal{D}q)\bigg) \qquad (4.14)\\ &\quad + i\sum_n m(n)\bigg(\frac{\partial \widetilde{\mathcal{N}}}{\partial q_n}(\mathcal{D}q)\frac{\partial \widetilde{\mathcal{N}}}{\partial \bar{q}_n}(q) - \frac{\partial \widetilde{\mathcal{N}}}{\partial q_n}(q)\frac{\partial \widetilde{\mathcal{N}}}{\partial \bar{q}_n}(\mathcal{D}q)\bigg). \qquad (4.15)\end{aligned}$$

In the following, we simply use $\langle n\rangle$ for $\langle n\rangle_\mu := (n^2+2\mu)^{\frac{1}{2}}$ since μ is a fixed constant thanks to the L^2-conservation.

First, note that Theorem 1.2 (c) follows once we prove

$$\bigg|\frac{d}{dt}\mathcal{H}(\mathcal{D}q)(t)\bigg| \le |(4.13)| + |(4.14)| + |(4.15)| \lesssim N^{-\frac{9}{4}+}. \qquad (4.16)$$

Also, note that the terms (4.13)–(4.15) are basically the same as (1.15)–(1.17). Thus, by comparing (3.26) and (4.16), it suffices to show that there is an additional decay of $N^{-\frac{1}{4}+}$ in this case.

As mentioned at the end of the last subsection, all the higher order terms in (4.12) have an extra decay of $|D_2|^{-1} < N^{-\beta}$. Hence, by repeating the argument in Subsection 3.3 with this extra decay of $N^{-\beta}$, we have

$$|(4.13)| + |(4.14)| + |(4.15)| \lesssim N^{-2-\beta+}. \qquad (4.17)$$

Moreover, if either of $\mathcal{N}$ or $\widetilde{\mathcal{N}}$ in (3.5), say $\mathcal{N}$, is one of the higher order terms, then, we also gain an extra decay of $N^{-\beta}$ from $\mathcal{N}$, and thus (4.17) holds.

Therefore, we only consider the contributions from $\mathcal{R}_2$, $\mathcal{I}_1$ for $\{\mathcal{R}_2, F_1\}^{(\mathrm{r})}$, and $\mathcal{I}_2$ for $\frac{1}{2}\{\mathcal{N}, F_1\}^{(\mathrm{r})}$ in the following. Recall that the main idea in Subsections 3.1 and 3.3 is to identify large frequencies and apply (3.4) to gain a negative power of N. In particular, it follows from (3.24) and (3.4) that for each large frequency $\gtrsim N$, we basically gain a power of N^{-1}.

First, let us use (3.24) to establish preliminary estimates on $\mathcal{R}_2$, $\mathcal{I}_1$, and $\mathcal{I}_2$, assuming (1.18):

$$n_1^* := \max(|n_1|, \dots, |n_{2r}|) > N.$$

• **(i) On $\mathcal{R}_2$:** By writing $\mathcal{R}_2$ in the form (2.11), we have

$$\mathcal{R}_2(q) = -\sum_n |q_n|^4 = -\sum_{\substack{n_1-n_2+n_3-n_4=0\\ n_1=n_2=n_3=n_4}} q_{n_1}\bar{q}_{n_2}q_{n_3}\bar{q}_{n_4}. \qquad (4.18)$$

By (1.18), we have $|n_j| > N$, $j = 1, \cdots, 4$. Then, from (3.24) and (3.4), we have

$$|\mathcal{R}_2(q)| \lesssim \|\mathbb{P}_{\geq N} q\|^4_{H^{0+}} \le N^{-4+}\|q\|^4_{H^1}. \qquad (4.19)$$

• **(ii) On $\mathcal{I}_1$:** From (4.5), we have

$$\mathcal{I}_1(q) = \sum_{\substack{n_1-n_2+n_3-n_4+n_5-n_6=0\\ n_2\neq n_1,n_3\\ n_4=n_5=n_6\\ |D_2(\bar n)|\leq N^\beta}} \frac{q_{n_1}\bar q_{n_2} q_{n_3}\bar q_{n_4} q_{n_5}\bar q_{n_6}}{(n_1-n_2)(n_3-n_2)}. \tag{4.20}$$

If $|n_4| \gtrsim N$, then we have at least three large frequencies ($\gtrsim N$.) Thus, from (3.24) and (3.4), we have

$$|\mathcal{I}_1(q)| \lesssim N^\beta \Big(\prod_{j=1}^3 \|q_{n_j}\|_{l^2}\Big)\|\mathbb{P}_{\geq N} q\|^3_{H^{0+}} \leq N^{-3+\beta+}\|q\|^6_{H^1}. \tag{4.21}$$

If $|n_4| \ll N$, then there are at least two frequencies among n_1, n_2, n_3 of size $\gtrsim N$. If $\min(|n_1|, |n_2|, |n_3|) \gtrsim N$, then we have $|\mathcal{I}_1(q)| \leq N^{-3+\beta+}\|q_n\|^6_{H^1}$ as in (4.21). If $\min(|n_1|, |n_2|, |n_3|) \ll N$, then we have

$$|(n_1-n_2)(n_3-n_2)| \gtrsim N.$$

Hence, we have

$$|\mathcal{I}_1(q)| \lesssim N^{-1+\beta}\|\mathbb{P}_{\gtrsim N} q\|^2_{H^{0+}}\|q_n\|^4_{l^2} \leq N^{-3+\beta+}\|q\|^6_{H^1}. \tag{4.22}$$

• **(iii) On $\mathcal{I}_2$:** We have $n_1^* \geq n_2^* \gtrsim N$. If $n_3^* \gtrsim N$, then we have

$$|\mathcal{I}_2(q)| \lesssim N^\beta\|\mathbb{P}_{\gtrsim N} q\|^3_{H^{0+}}\|q_n\|^3_{l^2} \leq N^{-3+\beta+}\|q\|^6_{H^1}. \tag{4.23}$$

Hence, suppose $n_3^* \ll N$ in the following.

◦ Case (iii.1): Suppose $\max(|n_1|, |n_2|, |n_3|) \gtrsim N$. Then we have $|(n_1-n_2)(n_3-n_2)| \gtrsim N$. Hence, $|\mathcal{I}_2(q)| \lesssim N^{-3+\beta+}\|q\|^6_{H^1}$ as before.

◦ Case (iii.2): Suppose $\max(|n_1|, |n_2|, |n_3|) \ll N$. Let $\beta < 1$. Then, $|D_2(\bar n)| \leq N^\beta$ implies $|n_5| \geq n_2^*$. Otherwise, i.e., if $|n_5| \leq n_3^* \ll N$, then we would have $|n_4|, |n_6| \gtrsim N$, and thus

$$-n_4^2 - n_6^2 = D(\bar n) + o(N^2) = o(N^2).$$

This is clearly a contradiction. Hence, we have $|n_5| \geq n_2^*$. Without loss of generality assume $|n_4| \geq |n_6|$. i.e. $\{|n_4|, |n_5|\} = \{n_1^*, n_2^*\}$.

⋄ Subcase (iii.2.a): Suppose $n_3^* \lesssim N^{\frac12-}$. Then, write n_4 as $n_4 = n_5 + m$, where $m = O(N^{\frac12-})$ and $m \neq 0$. (Recall $n_4 \neq n_5$.) Then, we have

$$|D_2(\bar n)| = |n_4^2 - n_5^2 + O(N^{1-})| = |2mn_5 + O(N^{1-})| \gtrsim |mn_5| \gtrsim N.$$

This contradicts with $|D_2(\bar n)| \leq N^\beta \ll N$.

⋄ Subcase (iii.2.b): Suppose $n_3^* \gg N^{\frac12-}$. Then, we have $|D_2(\bar n)| \leq N^\beta \leq N^{1-} \ll (n_3^*)^2$. This in turn implies $n_4^* \sim n_3^*$ as in Case (b) of Estimate on (1.15) in Subsection 3.1. Thus, we have

$$|\mathcal{I}_2(q)| \lesssim N^\beta\|\mathbb{P}_{\gtrsim N} q\|^2_{H^{0+}}\|\mathbb{P}_{\gtrsim N^{\frac12-}} q\|^2_{L^2}\|q\|^2_{L^2} \leq N^{-3+\beta+}\|q\|^6_{H^1}. \tag{4.24}$$

Therefore, we have $|\mathcal{I}_2(q)| \lesssim N^{-3+\beta+}\|q\|^6_{H^1}$ as long as $\beta < 1$.

In the following, we estimate the contributions from $\mathcal{R}_2$, $\mathcal{I}_1$, and $\mathcal{I}_2$ for (4.13), (4.14), and (4.15), assuming $\beta < 1$.

• **Estimate on** (4.14)**:** Since $\sum_{j=1}^{2r}(-1)^{j+1}(n_j^2+2\mu) = D(\bar{n})$, we can rewrite (4.14) in the form (3.2). First, note that there is no contribution from $\mathcal{R}_2$ since $D_1(\bar{n}) = 0$. From (4.19)–(4.24), we have

$$|\mathcal{I}_1|,\ |\mathcal{I}_2| \lesssim N^{\beta} N^{-3+\beta+} = N^{-3+2\beta+}.$$

Therefore, we have

$$|(4.14)| \lesssim N^{-3+2\beta+}. \tag{4.25}$$

• **Estimate on** (4.13)**:** First, we rewrite (4.13) as before.

$$(4.13) = -\sum_{n_1-n_2+\cdots-n_{2r}=0} c(\bar{n})\widetilde{R}(\bar{n}) q_{n_1}\bar{q}_{n_2}\cdots\bar{q}_{n_{2r}}$$

where $\widetilde{R}(\bar{n})$ is defined by

$$\widetilde{R}(\bar{n}) = m(n_1)^2\langle n_1\rangle^2 - m(n_2)^2\langle n_2\rangle^2 + \cdots - m(n_{2r})^2\langle n_{2r}\rangle^2. \tag{4.26}$$

Once again, there is no contribution from $\mathcal{R}_2$ since $\widetilde{R}(\bar{n}) = 0$ when $n_1 = \cdots = n_4$.

In the following, we estimate the contribution from $\mathcal{I}_1$ and $\mathcal{I}_2$ on (4.13). By repeating the computation in Estimate on (1.15) in Subsection 3.1, we have

$$|\widetilde{R}(\bar{n})| \lesssim m(n_1^*)^2 O((n_3^*)^2 + N^{\beta}).$$

∘ Case (a): Suppose $n_3^* \lesssim N^{\frac{\beta}{2}}$. In this case, we have $|\widetilde{R}(\bar{n})| \lesssim m(n_1^*)m(n_2^*)N^{\beta}$. Hence, from (4.19)–(4.24), the contribution from $\mathcal{I}_1$ and $\mathcal{I}_2$ can be estimated as

$$|(4.13)| \lesssim N^{-3+2\beta+}. \tag{4.27}$$

∘ Case (b): Suppose $n_3^* \gg N^{\frac{\beta}{2}}$. In this case, we have $n_4^* \sim n_3^*$ as in Subsection 3.1. Hence, we have $|\widetilde{R}(\bar{n})| \lesssim m(n_1^*)m(n_2^*)n_3^*n_4^*$.

First, we estimate the contribution from $\mathcal{I}_1$.

⋄ Subcase (b.1): Suppose $|n_4| \gtrsim N$. This implies that $\max(n_1,n_2,n_3) \geq n_4^* \sim n_3^* \gtrsim N$. If $n_5^* \ll N$, then we have $\mathrm{med}(n_1,n_2,n_3) = n_5^* \ll N$ and thus $|(n_1-n_2)(n_3-n_2)| \gtrsim N$. Hence, we have

$$\begin{aligned} |(4.13)| &\lesssim N^{-1+\beta}\|\mathbb{P}_{\gtrsim N}\mathcal{D}q\|_{H^{0+}}^2\Big(\prod_{j=3}^{4}\|n_j^* q_{n_j^*}\|_{l^2}\Big)\|q\|_{L^2}^2 \leq N^{-3+\beta+}\|\mathcal{D}q\|_{H^1}^6 \\ &\lesssim N^{-3+\beta+}. \end{aligned} \tag{4.28}$$

Otherwise, i.e. if $n_5^* \gtrsim N$, then we have

$$|(4.13)| \lesssim N^{\beta}\|\mathbb{P}_{\geq N}\mathcal{D}q\|_{H^{0+}}^2\Big(\prod_{j=3}^{4}\|n_j^* q_{n_j^*}\|_{l^2}\Big)\|\mathbb{P}_{\geq N}q\|_{H^{0+}}\|q\|_{L^2} \lesssim N^{-3+\beta+}. \tag{4.29}$$

⋄ Subcase (b.2): Suppose $|n_4| \ll N$. This implies $n_3^* \sim n_4^* \ll N$. Hence, we have $|(n_1-n_2)(n_3-n_2)| \gtrsim N$ and (4.28) holds in this case.

Next, we estimate the contribution from $\mathcal{I}_2$.

⋄ Subcase (b.3): Suppose $n_3^* \gtrsim N$. We have $n_4^* \gtrsim N$ since $n_4^* \sim n_3^*$. Then, as in Subcase (b.1), we obtain (4.28) or (4.29), depending on the size of n_5^*.

$\diamond$ Subcase (b.4): Suppose $n_3^* \ll N$. If $\max(n_1, n_2, n_3) \gtrsim N$, then we have $|(n_1 - n_2)(n_3 - n_2)| \gtrsim N$. Hence, $|(4.13)| \lesssim N^{-3+\beta+}$ as in (4.28).

Now, suppose $n_3^* \ll N$ and $\max(n_1, n_2, n_3) \ll N$. Then, as in Subcase (iii.2.a) for the preliminary estimate on $\mathcal{I}_2$, the case $n_3^* \lesssim N^{\frac{1}{2}-}$ can not occur. Hence, we have $n_3^* \gg N^{\frac{1}{2}-}$.

$\diamond$ Subsubcase (b.4.i): Suppose $n_5^* \ll n_3^*$. It follows from $\max(n_1, n_2, n_3) \ll N \lesssim n_2^*$ that either (a) two frequencies of $|n_1|, |n_2|, |n_3|$ are $O(n_3^*)$, and the other one is $o(n_3^*)$, or (b) one frequency of $|n_1|, |n_2|, |n_3|$ is $O(n_3^*)$, and the other two are $o(n_3^*)$. In either case, we have

$$|(n_1 - n_2)(n_3 - n_2)| \gtrsim O(n_3^*) \gg N^{\frac{1}{2}-}.$$

Hence, we have

$$|(4.13)| \lesssim N^{-\frac{1}{2}+\beta+} \|\mathbb{P}_{\gtrsim N} \mathcal{D} q\|_{H^{0+}}^2 \Big(\prod_{j=3}^4 \|n_j^* q_{n_j^*}\|_{l^2} \Big) \|q\|_{L^2}^2 \lesssim N^{-\frac{5}{2}+\beta+}. \tag{4.30}$$

$\diamond$ Subsubcase (b.4.ii): Suppose $n_5^* \sim n_3^*$. In this case, we have

$$|(4.13)| \lesssim N^{\beta} \|\mathbb{P}_{\geq N} \mathcal{D} q\|_{H^{0+}}^2 \Big(\prod_{j=3}^4 \|n_j^* q_{n_j^*}\|_{l^2} \Big) \|\mathbb{P}_{\geq N^{\frac{1}{2}-}} q\|_{H^{0+}} \|q\|_{L^2} \lesssim N^{-\frac{5}{2}+\beta+}. \tag{4.31}$$

From (4.27)–(4.31), we conclude

$$|(4.13)| \lesssim N^{-\frac{5}{2}+\beta+}. \tag{4.32}$$

• **Estimate on** (4.15)**:** We follow the argument in Estimate on (1.17) in Subsection 3.1. It suffices to estimate $\sum_n m(n) \frac{\partial \mathcal{N}_1}{\partial q_n}(\mathcal{D}q) \frac{\partial \mathcal{N}_2}{\partial \bar{q}_n}(q)$. where $\mathcal{N}_1$ and $\mathcal{N}_2$ are either $\mathcal{R}_2$, $\mathcal{I}_1$, or $\mathcal{I}_2$ with frequencies $\{n_j\}_{j=1}^{2r}$ and $\{\widetilde{n}_j\}_{j=1}^{2\widetilde{r}}$.

$\circ$ Case (a): Suppose $|n| \gtrsim N$. In this case, we have $\widetilde{n}_1^* \gtrsim |n| \gtrsim N$.

If $\mathcal{R}_2$ appears in one of the factors, say $\mathcal{N}_1 = \mathcal{R}_2$, then, by duality with (3.24) (note $D_1(\bar{n}) = 0$), we have

$$\begin{aligned} \Big\| \frac{\partial \mathcal{R}_2}{\partial q_n}(\mathcal{D}q) \Big\|_{l_n^2} &= \sup_{\|p\|_{L^2}=1} \sum_{\substack{n-n_2+n_3-n_4=0 \\ n_2=n_3=n_4=n}} p_n \cdot \overline{\mathcal{D}q}_{n_2} \mathcal{D}q_{n_3} \overline{\mathcal{D}q}_{n_4} \\ &\lesssim \|\mathbb{P}_{\geq N} q\|_{H^{0+}}^3 \lesssim N^{-3+}. \end{aligned}$$

By Cauchy-Schwarz inequality with (3.9), we have

$$\begin{aligned} \Big| \sum_n m(n) \frac{\partial \mathcal{R}_2}{\partial q_n}(\mathcal{D}q) \frac{\partial \mathcal{N}_2}{\partial \bar{q}_n}(q) \Big| &\leq \Big\| \frac{\partial \mathcal{R}_2}{\partial q_n}(\mathcal{D}q) \Big\|_{l_n^2} \Big\| m(n) \frac{\partial \mathcal{N}_2}{\partial \bar{q}_n}(q) \Big\|_{l_n^2} \\ &\lesssim N^{-3+} N^{-1} \|\mathcal{N}_2\| \lesssim N^{-4+}. \end{aligned} \tag{4.33}$$

Hence, we assume that both $\mathcal{N}_1$ and $\mathcal{N}_2$ are either $\mathcal{I}_1$ or $\mathcal{I}_2$. Then, by applying Cauchy-Schwarz inequality, duality, and the preliminary estimates on $\mathcal{I}_1$ or $\mathcal{I}_2$ in (ii)–(iii) on each factor, we obtain

$$|(4.15)| \lesssim N^{-2+\beta+} N^{-2+\beta+} = N^{-4+2\beta+}. \tag{4.34}$$

Note that we gain only $N^{-2+\beta+}$ from each factor, instead of $N^{-3+\beta+}$ as in (ii)–(iii). This is due to the fact that a duality variable p is only in L^2 and thus we can not gain an extra power of N through (3.4).

∘ Case (b): Suppose $|n| \ll N$. Then, we have either $n_1^*, n_2^* \gtrsim N$ or $\widetilde{n}_1^*, \widetilde{n}_2^* \gtrsim N$.[11] Suppose $n_1^*, n_2^* \gtrsim N$. Then, we can use the preliminary estimates on $\mathcal{R}_2$, $\mathcal{I}_1$, or $\mathcal{I}_2$ in (i)–(iii) for the first factor (after duality) and use Lemma 2.3 (a) for the second factor:

$$|(4.15)| \lesssim \bigg\| \frac{1}{\langle n \rangle} \frac{\partial \mathcal{N}_1}{\partial q_n}(\mathcal{D}q) \bigg\|_{l^2_n} \bigg\| \langle n \rangle m(n) \frac{\partial \mathcal{N}_2}{\partial \bar{q}_n}(q) \bigg\|_{l^2_n} \lesssim N^{-3+\beta+}. \tag{4.35}$$

From (4.34) and (4.35), we conclude

$$|(4.15)| \lesssim \max(N^{-4+2\beta+}, N^{-3+\beta+}) = N^{-3+\beta+}. \tag{4.36}$$

for $\beta < 1$.

Putting together all the estimates from (4.17), (4.25),(4.32), and (4.36), we have

$$|(4.13)| + |(4.14)| + |(4.15)| \lesssim \max(N^{-2-\beta+}, N^{-3+2\beta+}, N^{-\frac{5}{2}+\beta+})$$

By choosing $\beta = \frac{1}{4}$, (4.16) follows. This completes the proof of Theorem 1.2 (c).

Appendix A. On Theorem 1.1

In [4], Bourgain presented details for the quintic nonlinearity ($p = 2$.) After the normal form reduction, (3.24) was enough to conclude the result. For the cubic case ($p = 1$), Theorem 1.1 (a) follows once we note that (3.24) still holds in this case, as discussed in subsection 3.3.

For $p \geq 3$, there is no Strichartz estimate available in the periodic setting, and thus we need to rely on Sobolev inequality. However, we can still perform the normal form reduction as in Section 2 (with $K = T^\delta$) since both (2.9) and (2.10) are satisfied for all $t \in \mathbb{R}$ thanks to the L^2-conservation and the conservation of the defocusing Hamiltonian. Hence, we can proceed as in [4].

Let $I_s(q) = \|q\|^2_{H^s} = \sum_n |n|^{2s}|q_n|^2$. Then, after the normal form reduction, we have (see (A.29) in [4])

$$\partial_t I_s \lesssim \bigg| \sum_{\substack{n_1 - n_2 + \cdots - n_{2r} = 0 \\ |D(\bar{n})| < K}} c(\bar{n}) D_s(\bar{n}) q_{n_1} \bar{q}_{n_2} \cdots q_{n_{2r-1}} \bar{q}_{n_{2r}} \bigg| \tag{A.1}$$

where $D_s(\bar{n}) := \sum_j (-1)^j |n_j|^{2s}$. By Lemma on p.1355 in [4], we have

$$|D_s(\bar{n})| \lesssim (n_1^*)^{2(s-1)}(n_3^* n_4^* + K).$$

(Note a typo in the statement (A.32) in [4].) Assume that $n_j^* = |n_j|$, $j = 1, \dots, 4$. Moreover, assume $K \leq |n_3||n_4|$. Then, we have

$$|(\mathrm{A}.1)| \lesssim \sum_{\substack{n_1 - n_2 + \cdots - n_{2r} = 0 \\ |D(\bar{n})| < K}} |c(\bar{n})| \bigg(\prod_{j=1}^{2} |n_j|^{(s-1)} |q_{n_j}| \bigg) \bigg(\prod_{j=3}^{4} |n_j||q_{n_j}| \bigg) \bigg(\prod_{j=5}^{2r} |q_{n_j}| \bigg). \tag{A.2}$$

For the quintic case in [4], it is at this point (see (A.37)–(A.38) in [4]) that the space-time estimate [4, (A.18)] was used. As mentioned above, (A.18) in [4] follows from from the L^6-Strichartz estimate (3.23). For $p \geq 3$, we do not have such an estimate. Thus, we simply proceed by Hölder inequality and Sobolev embedding on the physical side, and obtain

$$|(\mathrm{A}.2)| \lesssim \|q\|^2_{H^{s-\frac{1}{2}+}} \|q\|^2_{H^1} \|q\|^{2r-4}_{H^{\frac{1}{2}+}} \lesssim I_s^{1-\theta} \|q\|^{2r-2+2\theta}_{H^1}, \tag{A.3}$$

[11] Recall a slight abuse of notation for n_j^* and $\widetilde{n}_j^*$. See Estimate on (1.17) in Subsection 3.1.

where in the last step we used interpolation: $\|q\|_{H^{s-\frac{1}{2}+}} \leq \|q\|_{H^s}^{1-\theta}\|q\|_{H^1}^{\theta}$ with

$$\theta = \frac{1}{2(s-1)+}. \tag{A.4}$$

If $|n_3||n_4| \leq K = T^\delta$, then we obtain (A.3) with an extra factor of $K = T^\delta$. In view of the uniform bound on the H^1-norm on solutions, we obtain

$$\partial_t I_s \lesssim T^\delta I_s^{1-\theta} \quad \Longrightarrow \quad \partial_t(I_s^\theta) \lesssim T^\delta.$$

Hence, we obtain $I_s(t) \lesssim T^{\frac{1+\delta}{\theta}} = T^{2(s-1)+}$ for $|t| \leq T$ (with $\delta = 0+$.) This proves Theorem 1.1 (b).

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