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LINEAR MAHLER MEASURES AND DOUBLE L-VALUES OF MODULAR FORMS

EVGENY SHINDER, MASHA VLASENKO

ABSTRACT. We consider the Mahler measure of the polynomial $1 + x_1 + x_2 + x_3 + x_4$, which is the first case not yet evaluated explicitly. A conjecture due to F. Rodriguez-Villegas represents this Mahler measure as a special value at the point 4 of the L-function of a modular form of weight 3. We prove that this Mahler measure is equal to a linear combination of double L-values of certain meromorphic modular forms of weight 4.

1. INTRODUCTION

The logarithmic Mahler measure of a Laurent polynomial

$$P(x_1, \dots, x_n) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$$

is defined as

$$m(P) = \frac{1}{(2\pi i)^n} \int_{|x_1|=\dots=|x_n|=1} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}.$$

One can show that this integral is always convergent. For a monic polynomial in one variable $P \in \mathbb{C}[x]$ one can compute $m(P)$ by Jensen's formula

$$(1.1) \quad \frac{1}{2\pi i} \int_{|x|=1} \log |P(x)| \frac{dx}{x} = \sum_{\alpha: P(\alpha)=0} \max(0, \log |\alpha|),$$

but no explicit formula is known for polynomials in several variables. Let us consider the simplest case of linear forms, namely $m(1 + x_1 + \dots + x_n)$. In 1981 C. Smyth discovered ([Sm]) that

$$(1.2) \quad m(1 + x_1 + x_2) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

where $\chi_{-3}(n) = \left(\frac{-3}{n}\right)$, $L(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} = 1 - \frac{1}{2^s} + \frac{1}{4^s} - \frac{1}{5^s} + \dots$ and

$$(1.3) \quad m(1 + x_1 + x_2 + x_3) = \frac{7}{2\pi^2} \zeta(3).$$

These formulas can be proved by explicit integration. Later we will see another method due to F. Rodriguez-Villegas ([RV]) to obtain (1.2) and (1.3) with the help of modular forms. Already in the next case no explicit formula for $m(1 + x_1 + x_2 + x_3 + x_4)$ is known, and this is the subject of the present paper. One can find in [RTV] the numerical value

$$m(1 + x_1 + x_2 + x_3 + x_4) = 0.544412561752185\dots$$

and also there is the following conjectural formula.

Conjecture (F. Rodriguez-Villegas, [BLVD], see also [Zud1]):

$$m(1 + x_1 + x_2 + x_3 + x_4) \stackrel{?}{=} 6 \left(\frac{\sqrt{-15}}{2\pi i} \right)^5 L(f_{15}, 4)$$

where

$$f_{15} = \eta(3z)^3 \eta(5z)^3 + \eta(z)^3 \eta(15z)^3 = q + q^2 - 3q^3 - 3q^4 + \dots$$

is a CM modular form of weight 3, level 15 and Nebentypus $\left(\frac{-15}{\cdot}\right)$.

This modular form arises in [PTV] in the relation to the variety

$$\begin{cases} 1 + x_1 + x_2 + x_3 + x_4 = 0 \\ 1 + \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} = 0 \end{cases}$$

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which can be compactified to a $K3$ surface of Picard rank 20. Namely, C.Peters, J. Top, M. van der Vlugt show that if X is the minimal resolution of singularities of the above surface then the L-function of $H^2(X)$ has generic Euler factor

$$(1 - pT)^{16} \left(1 - \left(\frac{-3}{p}\right)pT\right)^4 \left(1 - A_p T + \left(\frac{-15}{p}\right)p^2 T^2\right)$$

where A_p is the p th coefficient in the q -expansion of f_{15} .

In order to state our results, consider the modular function $t(z)$ and modular form $f(z)$ of weight 2

$$(1.4) \quad \begin{aligned} t(z) &= -\left(\frac{\eta(2z)\eta(6z)}{\eta(z)\eta(3z)}\right)^6 = -q - 6q^2 - 21q^3 + \dots \\ f(z) &= \frac{(\eta(z)\eta(3z))^4}{(\eta(2z)\eta(6z))^2} = 1 - 4q + 4q^2 - 4q^3 + \dots \end{aligned}$$

for the group

$$\Gamma_0(6) + 3 = \Gamma_0(6) \cup \left\{ \sqrt{3} \begin{pmatrix} a & b/3 \\ 2c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}) \mid a, b, c, d \in \mathbb{Z} \right\}.$$

Throughout the paper we use the differential operator $D = \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}$. We need the following modular forms of weight 4

$$(1.5) \quad \begin{aligned} g_1 &= \frac{Dt}{t} f = 1 + 2q - 14q^2 + 38q^3 - 142q^4 + 252q^5 - 266q^6 + \dots \\ g_2 &= \frac{t}{1-t} g_1 = -q - 7q^2 - 6q^3 + 5q^4 + 120q^5 + 498q^6 + \dots \\ g_3 &= \frac{t(212t^2 + 251t - 13)}{(1-t)^3} g_1 = 13q + 316q^2 + 2328q^3 + \dots \end{aligned}$$

Here g_1 is indeed a modular form and one can write it as a linear combination of Eisenstein series (see (3.8)), while g_2 and g_3 have poles at the discrete set of points where $t(z) = 1$. Our main result is the following.

Theorem. *Consider the Chowla-Selberg period for the field $K = \mathbb{Q}(\sqrt{-15})$*

$$(1.6) \quad \Omega_{15} = \frac{1}{\sqrt{30}\pi} \left(\prod_{j=1}^{14} \Gamma\left(\frac{j}{15}\right)^{\chi_K(j)} \right)^{1/4}$$

and the two numbers

$$(1.7) \quad L(g_j, g_1, 3, 1) = (2\pi)^4 \int_0^\infty g_1(is) \int_s^\infty \int_{s_1}^\infty \int_{s_2}^\infty g_j(is_3) ds_3 ds_2 ds_1 ds$$

for $j = 2, 3$. One has

$$\begin{aligned} m(1 + x_1 + x_2 + x_3 + x_4) - \frac{4}{5} m(1 + x_1 + x_2 + x_3) \\ = \frac{3\sqrt{5}\Omega_{15}^2}{20\pi} L(g_3, g_1, 3, 1) - \frac{3\sqrt{5}}{10\pi^3\Omega_{15}^2} L(g_2, g_1, 3, 1). \end{aligned}$$

The reader will find this statement in a slightly different notation under the name of Corollary 6.3. First, let us explain why the integrals in (1.7) converge. For $z \in i\mathbb{R}_+$ both $t(z)$ and $f(z)$ are real-valued and one can easily check that $t(z) < 0$. Therefore g_2 and g_3 have no poles along the imaginary half-axis. When $s \rightarrow \infty$ we have $g_1(is) = O(1)$, $g_2(is) = O(e^{-2\pi s})$ and the same holds for g_3 because q -expansions of g_2 and g_3 start in degree 1, therefore the integrated integrals above are convergent at ∞ . Also one can show that all three functions $g_j(is)$ are $o(s)$ when $s \rightarrow 0$, hence they are globally bounded and there is no problem with convergence at $s = 0$. With the help of PARI/GP we find that numerically

$$\begin{aligned} L(g_2, g_1, 3, 1) &= -0.44662442\dots \\ L(g_3, g_1, 3, 1) &= 8.5383217\dots \end{aligned}$$

which agrees with the statement of the theorem.

These two numbers are periods whose geometric meaning is not clear at the moment. If for example g_2 were a holomorphic cusp form then the number defined in (1.7) would be indeed the value of the corresponding double L-function $L(g_2, g_1, s_2, s_1)$ at $s_2 = 3, s_1 = 1$, which is the motivation for our notation. We discuss double L-values of holomorphic modular forms in Section 6. But as soon as forms under consideration have poles in the upper half-plane the corresponding multiple integrals become path-dependent and there is no general theory of multiple L-values. Also we would like to remark that for the holomorphic modular form g_1 in our theorem one has $m(1 + x_1 + x_2 + x_3) = -\frac{1}{2}L(g_1, 1)$, the reader can find the proof of this statement in Section 3. Another observation is that the poles of g_2 and g_3 are located at the points from the same field $K = \mathbb{Q}(\sqrt{-15})$, namely at the images of $z = \frac{1}{8} + \frac{\sqrt{-15}}{24}$ under the group $\Gamma_0(6) + 3$.

The structure of the paper is as follows. Sections 2 and 3 follow the approach pioneered by Rodriguez-Villegas [RV]. In Section 2 we relate the Mahler measure of $1 + x_1 + \dots + x_n$ to the principal period of a pencil of Calabi-Yau varieties of dimension $n - 1$ given by

$$\left(1 + x_1 + \dots + x_n\right) \left(1 + \frac{1}{x_1} + \dots + \frac{1}{x_n}\right) = \lambda$$

and the corresponding Picard-Fuchs differential equation. One needs to do explicitly analytic continuation of its solutions from one singular point to another one in order to compute the Mahler measure. For $n = 2, 3$ the differential operators appear to have modular parametrization. This allows us to do necessary analytic continuation and derive (1.2) and (1.3) in Section 3.

When $n = 4$ the Picard-Fuchs differential operator is not modular. However one can apply Jensen's formula to reduce the number of variables: in Section 4 we observe that in fact $m(1 + x_1 + \dots + x_n)$ can be computed by analytic continuation of a solution of a non-homogeneous differential equation with the Picard-Fuchs differential operator corresponding to $m(1 + x_1 + \dots + x_{n-1})$. A non-homogeneous differential equation arises if one considers the generating function for the moments of a solution of a homogeneous differential equation along a path. Moreover, the differential operator depends only on the initial differential equation being independent of the particular solution and the path, while the right-hand side depends on this data (Proposition 4.1). In Section 6 we discuss modular interpretation of solutions to a non-homogeneous equation in the case when the differential operator has modular parametrization and show that double L-values of modular forms appear naturally in this context.

Though our main interest is the case $n = 4$, we keep applying our technique parallelly to the case $n = 3$ throughout the paper (Theorem 4.2 and Corollary 6.2). This leads to a linear relation (6.4) between a double L-value of two Eisenstein series of weight 3, and ordinary L-values $L(\chi_3, 2)$ and $\zeta(3)$. We give a direct proof of this relation in Section 7 using a method due to Mathew Rogers and Wadim Zudilin.

Our original interest in Mahler's measure came from the beautiful paper [RV] which has been inspiring us the whole time we were working on this project. We would like to thank our friends and colleagues Sergey Galkin, Vasily Golyshev, Anton Mellit, Maxim Smirnov and Wadim Zudilin for their interest in our work. Both authors are grateful to the Max-Planck-Institute für Mathematik in Bonn for providing wonderful working conditions where a significant part of this work has been done.

2. MAHLER MEASURES AND DIFFERENTIAL EQUATIONS

For a Laurent polynomial $P(x_1, \dots, x_n)$ the function

$$a(t) = \frac{1}{(2\pi i)^n} \int_{|x_1|=\dots=|x_n|=1} \frac{1}{1 - tP(x_1, \dots, x_n)} \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}$$

is well defined for small t since $|P|$ is bounded on the torus. We call $a(t)$ the principal period of P . It is the generating function for the sequence

$$(2.1) \quad a_m = \text{the constant term of } P(x_1, \dots, x_n)^m$$

since

$$\begin{aligned} a(t) &= \sum_{m=0}^{\infty} t^m \frac{1}{(2\pi i)^n} \int_{|x_1|=\dots=|x_n|=1} P(x_1, \dots, x_n)^m \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \\ &= \sum_{m=0}^{\infty} a_m t^m. \end{aligned}$$

Suppose that the polynomial P takes only nonnegative real values on the torus $\{|x_i| = 1\}$. Then the Mahler measure $m(P)$ can be computed as follows. For small real $t < 0$ one has

$$\begin{aligned} m\left(P - \frac{1}{t}\right) &= \frac{1}{(2\pi i)^n} \int_{|x_i|=1} \log\left(P(x_1, \dots, x_n) - \frac{1}{t}\right) \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \\ &= \frac{1}{(2\pi i)^n} \int_{|x_i|=1} \log\left(-\frac{1}{t}(1 - tP(x_1, \dots, x_n))\right) \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \\ &= -\log(-t) - \sum_{m=1}^{\infty} \frac{t^m}{m} \frac{1}{(2\pi i)^n} \int_{|x_i|=1} P(x_1, \dots, x_n)^m \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \\ &= -\log(-t) - \sum_{m=1}^{\infty} \frac{t^m}{m} a_m = -\left(t \frac{d}{dt}\right)^{-1} a(t). \end{aligned}$$

Though we did computation only for small real $t < 0$, the first integral here and the terminal expression are holomorphic in t and defined in some neighbourhood of real negative half-axis (apart from possibly finitely many punctures where $a(t)$ has singularities). Therefore

$$(2.2) \quad m(P) = -\operatorname{Re} \left(t \frac{d}{dt} \right)^{-1} a(t) \Big|_{t=\infty},$$

where the analytic continuation is done along $-\infty < t < 0$ and we added real part to be independent of the branch of $\log(t)$, e.g. we can now assume throughout the paper that

$$\left(t \frac{d}{dt} \right)^{-1} \sum_{m=0}^{\infty} a_m t^m = a_0 \log t + \sum_{m=1}^{\infty} \frac{a_m}{m} t^m.$$

On the other hand, it is known ([SB]) that the sequence (2.1) always satisfies a recursion, i.e. $a(t)$ is a solution to an ordinary differential equation

$$(2.3) \quad \mathcal{L}\left(t, t \frac{d}{dt}\right) a(t) = 0$$

where \mathcal{L} is a certain polynomial in two non-commuting variables. Finally we see that Mahler measure $m(P)$ can be computed by doing analytic continuation of a particular solutions to an ordinary differential equation which one constructs from the polynomial P .

Let us apply this strategy to the linear polynomials. Observe that

$$m(1 + x_1 + \dots + x_n) = \frac{1}{2} m(P_n)$$

where

$$(2.4) \quad P_n = \left(1 + x_1 + \dots + x_n\right) \left(1 + \frac{1}{x_1} + \dots + \frac{1}{x_n}\right)$$

takes nonnegative real values on the torus. Consider the sequence of the free coefficients of the powers of P_n :

$$\begin{aligned} n = 2 & \quad a_m : 1, 3, 15, 93, 639 \dots \\ n = 3 & \quad a_m : 1, 4, 28, 256, 2716 \dots \\ n = 4 & \quad a_m : 1, 5, 45, 545, 7885 \dots \end{aligned}$$

The corresponding differential equations

$$\mathcal{L}_n\left(t, t \frac{d}{dt}\right) a(t) = 0$$

are given by:

$$\mathcal{L}_2(t, \theta) = \theta^2 - t(10\theta^2 + 10\theta + 3) + 9t^2(\theta + 1)^2$$

$$\mathcal{L}_3(t, \theta) = \theta^3 - 2t(2\theta + 1)(5\theta^2 + 5\theta + 2) + 64t^2(\theta + 1)^3$$

$$\begin{aligned} \mathcal{L}_4(t, \theta) &= \theta^4 - t(35\theta^4 + 70\theta^3 + 63\theta^2 + 28\theta + 5) \\ &\quad + t^2(\theta + 1)^2(259\theta^2 + 518\theta + 285) - 225t^3(\theta + 1)^2(\theta + 2)^2 \end{aligned}$$

(see [Ver08] for the general form of the operator.) We use the notation $\theta = t \frac{d}{dt}$ to distinguish it from $D = q \frac{d}{dq}$.

In all three cases there is a unique analytic at $t = 0$ solution satisfying $a(t) = 1 + o(t)$.

The equations $P_2(x_1, x_2) = \lambda$ and $P_3(x_1, x_2, x_3) = \lambda$ describe families of elliptic curves and $K3$ -surfaces of rank 19 respectively. It is therefore natural that the differential equations \mathcal{L}_2 and \mathcal{L}_3 have modular parametrization, hence we can easily do analytic continuation of their solutions and compute the corresponding Mahler measures by formula (2.2). We do this in the next section.

Unfortunately, this method is not applicable in the case $m(1+x_1+x_2+x_3+x_4)$. The equation $P_4(x_1, x_2, x_3, x_4) = \lambda$ describes a family of Calabi-Yau threefolds, and hence we do not expect \mathcal{L}_4 to have modular parametrization. Indeed, one can check that the differential operator \mathcal{L}_4 is not a symmetric cube of any second order differential operator and therefore it does not admit a modular parametrization. Further in the paper we show that one can still use the operator \mathcal{L}_3 to compute $m(1+x_1+x_2+x_3+x_4)$, though the price paid is that we have to consider non-homogeneous differential equations.

3. MODULAR PARAMETRIZATIONS OF \mathcal{L}_2 AND \mathcal{L}_3

Recall that for an arbitrary modular function $t(z)$ and a modular form $f(z)$ of weight k on a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ one can construct an ordinary differential operator of order $k+1$ with algebraic coefficients

$$(3.1) \quad \sum_{i=0}^{k+1} c_i(t) \left(\frac{d}{dt} \right)^i, \quad c_i(t) \in \overline{\mathbb{C}(t)}$$

such that the functions

$$(3.2) \quad f(z), z f(z), \dots, z^k f(z)$$

span the kernel of the pull-back of $\mathcal{L}_{t,f}$ to the upper half-plane

$$\sum_{i=0}^{k+1} c_i(t(z)) \left(\frac{1}{t'(z)} \frac{d}{dz} \right)^i = \sum_{i=0}^{k+1} \tilde{c}_i(z) \left(\frac{d}{dz} \right)^i.$$

It follows that an operator with these properties is unique up to multiplication by algebraic functions of t on the left. On the other hand, the operator

$$\frac{1}{t'(z) \cdot f(z)} \left(\frac{d}{dz} \right)^{k+1} \frac{1}{f(z)}$$

obviously annihilates the local system (3.2) and it is a routine to check that if we rewrite it as

$$\frac{1}{t'(z) \cdot f(z)} \left(\frac{d}{dz} \right)^{k+1} \frac{1}{f(z)} = \sum_{i=0}^{k+1} g_i(z) \left(\frac{d}{dt} \right)^i$$

all coefficients $g_i(z)$ will be modular functions and hence can be written as some algebraic functions $g_i(z) = c_i(t(z))$. The reader could refer to [Zag], Proposition 21 for several constructions of the differential equation satisfied by a modular form.

Recall that $D = \frac{1}{2\pi i} \frac{d}{dz} = q \frac{d}{dq}$. In the view of the above said we make a choice and define the operator

$$(3.3) \quad \mathcal{L}_{t,f} = \frac{1}{Dt \cdot f} D^{k+1} \frac{1}{f}.$$

This choice corresponds to the leading coefficient in (3.1) being

$$c_{k+1}(t) = \frac{D^k t}{f^2}.$$

It is not hard to check that both \mathcal{L}_2 and \mathcal{L}_3 can be obtained from certain pairs of a modular function and modular form, namely the following ones (see [Ver96] for all the details in the case of \mathcal{L}_3 .)

Proposition 3.1. *With*

$$(3.4) \quad \begin{aligned} t &= \frac{\eta(6z)^8 \eta(z)^4}{\eta(3z)^4 \eta(2z)^8} = q - 4q^2 + 10q^3 + \dots \\ f &= \frac{\eta(2z)^6 \eta(3z)}{\eta(z)^3 \eta(6z)^2} = 1 + 3q + 3q^2 + 3q^3 + \dots \end{aligned}$$

one has

$$\mathcal{L}_{t,f} = \frac{1}{t} \mathcal{L}_2\left(t, t \frac{d}{dt}\right).$$

Proposition 3.2. *With $t(z)$ and $f(z)$ as in (1.4) one has*

$$\mathcal{L}_{t,f} = \frac{1}{t} \mathcal{L}_3\left(t, t \frac{d}{dt}\right).$$

Let us use these modular parametrizations to compute $m(P_2)$ and $m(P_3)$ by formula (2.2). With (3.4) we see that at $q = 0$ we have $t = 0$ and $f = 1$, hence f coincides with $a(t)$ near $t = 0$. Since t runs over the negative real axis when $z \in \frac{1}{2} + i\mathbb{R}_-$ we have by (2.2)

$$(3.5) \quad \begin{aligned} m(P_2) &= -\operatorname{Re} \left(t \frac{d}{dt} \right)^{-1} a(t) \Big|_{t=\infty} = -\operatorname{Re} \left(\frac{t}{Dt} D \right)^{-1} f \Big|_{z=\frac{1}{2}} \\ &= -\operatorname{Re} D^{-1} \left(\frac{Dt}{t} f \right) \Big|_{z=\frac{1}{2}}. \end{aligned}$$

Here and throughout the paper we make a particular choice for D^{-1} by letting it to act on q -series as

$$D^{-1} \sum_{n=0}^{\infty} a_n q^n = a_0 \log q + \sum_{n=1}^{\infty} \frac{a_n}{n} q^n,$$

which is in accordance with our choice for $(t \frac{d}{dt})^{-1}$, so that the above computation is correct. To evaluate the terminal expression in (3.5) let us recall the definition of L -function of a modular form.

For a modular form $g = \sum_{n=0}^{\infty} c_n q^n$ of weight k on a congruence subgroup of $\operatorname{SL}(2, \mathbb{Z})$ one has $c_n = O(n^{k-1})$ and the L -function of g is defined by

$$L(g, s) = \sum_{n=1}^{\infty} \frac{c_n}{n^s}$$

when $\operatorname{Re} s > k$. This function can be continued as a meromorphic function to the whole \mathbb{C} . Moreover, if g is a cusp form then $L(g, s)$ is holomorphic everywhere in \mathbb{C} .

Proposition 3.3. *Let g be a modular form of weight k with $c_0 = 0$ and $p < k$ be an arbitrary integer. If $L(g, s)$ has no poles with $\operatorname{Res} \geq p$ then one has*

$$\lim_{q \rightarrow 1} (D^{-p} g)(q) = L(g, p).$$

If $c_0 \neq 0$ the same holds when $p < 0$,

$$\lim_{q \rightarrow 1} g(q) = c_0 + L(g, 0)$$

and

$$\lim_{q \rightarrow 1} (D^{-1} g)(q) = \lim_{q \rightarrow 1} \left(c_0 \log q + \sum_{n=1}^{\infty} \frac{c_n}{n} q^n \right) = L(g, 1)$$

assuming that the branch of $\log q$ is taken so that $\lim_{q \rightarrow 1} \log q = 0$.

The reason we do not consider $p > 1$ in the latter case is that it is not that clear how to define D^{-p} when $c_0 \neq 0$. Note also that one always has $L(g, p) = 0$ when $p < 0$. Indeed, the function $\Lambda(g, s) = \Gamma(s)/(2\pi)^s L(g, s)$ satisfies a functional equation when s goes to $k - s$ and it is obviously holomorphic when $\operatorname{Re} s > k$, hence also when $\operatorname{Re} s < 0$. Since $\Gamma(s)$ has poles at nonpositive integers $L(g, s)$ has zeros at all integers $s < 0$. If g is a cusp form then also $L(g, 0) = 0$ by the same reason since $\Lambda(g, s)$ is holomorphic in the entire complex plane.

Proof of Proposition 3.3. First let $c_0 = 0$ or $c_0 \neq 0$ but $p < 0$. When $\operatorname{Re} w > k - p$ one has

$$\begin{aligned} \frac{\Gamma(w)}{(2\pi)^w} L(g, p+w) &= \frac{\Gamma(w)}{(2\pi)^w} \sum_{n=1}^{\infty} \frac{c_n}{n^{p+w}} = \sum_{n=1}^{\infty} \frac{c_n}{n^p} \int_0^{\infty} t^{w-1} e^{-2\pi n t} dt \\ &= \int_0^{\infty} t^{w-1} (D^{-p} g)(it) dt. \end{aligned}$$

Since $\Gamma(w)$ is small when the imaginary part of w is large and $L(g, p+w)$ is uniformly bounded we can apply the inverse Mellin transform. Namely, with any real $c > k - p$ one has

$$\begin{aligned} (D^{-p}g)(it) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(w)}{(2\pi t)^w} L(g, p+w) dw \\ &= L(g, p) + \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \frac{\Gamma(w)}{(2\pi t)^w} L(g, p+w) dw \end{aligned}$$

where we moved the path of integration and $0 < \varepsilon < 1$ is chosen sufficiently small so that $\mathcal{L}(g, s)$ still has no poles with $\operatorname{Re} s \geq p - \varepsilon$. The last integral is $O(t^\varepsilon)$ and obviously vanishes when $t \rightarrow 0$.

If $c_0 \neq 0$ the calculation above would remain correct if we substitute $D^{-p}(q)$ by $g(q) - c_0$ and $D^{-1}(q) - c_0 \log q$ when $p = 0$ and $p = 1$ correspondingly. \square

Going back to (3.5), consider a modular form of weight 3 given by

$$\begin{aligned} (3.6) \quad g(z) &= \left(\frac{Dt}{t}f\right)\left(z + \frac{1}{2}\right) = 1 + q - 5q^2 + q^3 + 11q^4 - 24q^5 + \dots \\ &= E_{3, \chi_{-3}}(z) - 2E_{3, \chi_{-3}}(2z) - 8E_{3, \chi_{-3}}(4z) \end{aligned}$$

where $E_{3, \chi_{-3}} \in M_3(\Gamma_0(3), \chi_{-3})$ is the Eisenstein series

$$(3.7) \quad E_{3, \chi_{-3}} = -\frac{1}{9} + \sum_{n \geq 1} \sum_{d|n} \chi_{-3}(d) d^2 q^n.$$

Then $L(E_{3, \chi_{-3}}, s) = \zeta(s) L(\chi_{-3}, s-2)$ and

$$L(g, s) = \left(1 - \frac{2}{2^s} - \frac{8}{4^s}\right) \zeta(s) L(\chi_{-3}, s-2)$$

is holomorphic in the entire complex plane. Since Fourier coefficients of the form g are real $L(g, s)$ takes real values at real arguments s . Combining (3.5) with Proposition 3.3 we get

$$m(P_2) = -L(g, 1) = 2 \lim_{s \rightarrow 1} \zeta(s) L(\chi_{-3}, s-2) = \frac{3\sqrt{3}}{2\pi} L(\chi_{-3}, 2).$$

Recall that $m(1 + x_1 + x_2) = \frac{1}{2}m(P_2)$, hence we have just reproved (1.2).

Analogously, with (1.4) we have that $t(z)$ assumes all negative real values along the imaginary half-axis and $t = \infty$ at $z = 0$, hence

$$m(P_3) = -\operatorname{Re} D^{-1} \left(\frac{Dt}{t}f\right) \Big|_{q=1}.$$

We consider

$$\begin{aligned} (3.8) \quad g &= \frac{Dt}{t}f = 1 + 2q - 14q^2 + 38q^3 - 142q^4 + 252q^5 - 266q^6 + \dots \\ &= 2E_4(z) - 32E_4(2z) - 18E_4(3z) + 288E_4(6z) \end{aligned}$$

with the Eisenstein series

$$(3.9) \quad E_4 = \frac{1}{240} + \sum_{n \geq 1} \sum_{d|n} d^3 q^n.$$

The function $L(E_4, s) = \zeta(s)\zeta(s-3)$ has the only pole at $s = 4$, and one can easily see that

$$L(g, s) = \left(2 - \frac{32}{2^s} - \frac{18}{3^s} + \frac{288}{6^s}\right) \zeta(s)\zeta(s-3)$$

is holomorphic in the entire complex plane because the factor in the brackets vanishes at $s = 4$. Finally,

$$m(P_3) = -L(g, 1) = -\left(2 - \frac{32}{2} - \frac{18}{2} + \frac{288}{6}\right) \lim_{s \rightarrow 1} \zeta(s)\zeta(s-3) = \frac{7\zeta(3)}{\pi^2}.$$

This again reproves (1.3) because $m(1 + x_1 + x_2 + x_3) = \frac{1}{2}m(P_3)$.

4. COMPUTATION OF $m(P_n)$ VIA \mathcal{L}_{n-1}

Observe that in general the Mahler measure of a Laurent polynomial P which takes nonnegative real values on the torus $\{|x_1| = \dots = |x_n| = 1\}$ can be written as

$$m(P) = \int_{\lambda_{\min}}^{\lambda_{\max}} \log(\lambda) a^*(\lambda) d\lambda$$

where

$$\begin{aligned} \lambda_{\min} &= \min_{|x_1|=\dots=|x_n|=1} P(x_1, \dots, x_n), \\ \lambda_{\max} &= \max_{|x_1|=\dots=|x_n|=1} P(x_1, \dots, x_n), \end{aligned}$$

and $a^*(\lambda)$ is equal to the integral over the variety

$$\{P(x_1, \dots, x_n) = \lambda\} \cap \{|x_1| = \dots = |x_n| = 1\}$$

of the the $(n-1)$ -form ω_λ defined as the residue

$$\frac{1}{(2\pi i)^n} \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dx_n}{x_n} = \omega_\lambda \wedge d\lambda,$$

i.e.

$$a^*(\lambda) = \int_{\{P=\lambda\} \cap \{|x_i|=1\}} \omega_\lambda.$$

Along the same lines as we did in Section 2, one can recover $m(P)$ from the generating function of the “moments” of $a^*(\lambda)$

$$a_m = \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda^m a^*(\lambda) d\lambda, \quad a(t) = \sum_{m=0}^{\infty} a_m t^m$$

by formula (2.2). Indeed, the moment a_m is exactly the constant term of P^m , so this $a(t)$ is identical with the one in the previous section. Also one can see it directly by repeating the old trick in our new notation: for $-\infty < t < 0$

$$\begin{aligned} & \int_{\lambda_{\min}}^{\lambda_{\max}} \log\left(\lambda - \frac{1}{t}\right) a^*(\lambda) d\lambda \\ &= -\log(-t) - \sum_{m=1}^{\infty} \frac{t^m}{m} \int_{\lambda_{\min}}^{\lambda_{\max}} \lambda^m a^*(\lambda) d\lambda \\ &= -\log(-t) - \sum_{m=1}^{\infty} \frac{t^m}{m} a_m = -\left(t \frac{d}{dt}\right)^{-1} a(t), \end{aligned}$$

hence

$$m(P) = \int_{\lambda_{\min}}^{\lambda_{\max}} \log(\lambda) a^*(\lambda) d\lambda = -\operatorname{Re} \left(t \frac{d}{dt}\right)^{-1} a(t) \Big|_{t=-\infty}$$

where we again assume that the analytic continuation of $a(t)$ is done along the real negative halfaxis. In Section 2 we mentioned without a proof that $a(t)$ satisfies a differential equation (2.3). Now this fact follows from the proposition below because $a^*(\lambda)$ is a period for the 1-parametric family of varieties $P(x_1, \dots, x_n) = \lambda$ and it satisfies a certain differential equation

$$\tilde{\mathcal{L}}\left(\lambda, \lambda \frac{d}{d\lambda}\right) a^*(\lambda) = 0,$$

namely the Picard-Fuchs differential equation for this family. Proposition 4.1 states that moments of a solution of a differential equation satisfy another differential equation determined by the initial one, though in general they are solutions of this equation with a right-hand side. This right-hand side is a simple rational function which depends on the path of integration and on the choice of the solution along this path. In the above situation with $a^*(\lambda)$ along the real line from λ_{\min} to λ_{\max} the right-hand side exceptionally appears to vanish. But later we will need a general case as well.

For a Laurent series $\sum_n a_n t^n$ we introduce the notations

$$\begin{aligned} \left[\sum_n a_n t^n \right]_+ &= \sum_{n \geq 0} a_n t^n \\ \left[\sum_n a_n t^n \right]_- &= \sum_{n < 0} a_n t^n \end{aligned}$$

for its parts with nonnegative and negative powers correspondingly. One then has

$$\sum_n a_n t^n = \left[\sum_n a_n t^n \right]_- + \left[\sum_n a_n t^n \right]_+.$$

Proposition 4.1. *For a polynomial differential operator*

$$\tilde{\mathcal{L}}(\lambda, \theta) = \sum_{i=0}^M \sum_{j=0}^N c_{ij} \lambda^i \theta^j, \quad \theta = \lambda \frac{d}{d\lambda}$$

consider a solution $F(\lambda)$ of $\tilde{\mathcal{L}}F = 0$ along some path between $\lambda = \alpha$ and $\lambda = \beta$. Then for the generating function of its moments

$$b(t) = \sum_{n=0}^{\infty} t^n \int_{\alpha}^{\beta} \lambda^n F(\lambda) d\lambda$$

one has

$$\tilde{\mathcal{L}}\left(\frac{1}{t}, -\theta_t - 1\right) b(t) = h(t)$$

where the right-hand $h(t)$ is a rational function which can have poles at most at $t = 0, \frac{1}{\alpha}, \frac{1}{\beta}$ and is defined as follows. Let

$$\tilde{\mathcal{L}}^{(k)}(\lambda, \theta) = \sum_{i=0}^M \sum_{j=k}^N c_{ij} \lambda^i \theta^{j-k}$$

and for given λ consider a rational function of t

$$H_{\lambda}(t) = \lambda \sum_{j=0}^{N-1} (\theta^j F)(\lambda) \left[\tilde{\mathcal{L}}^{(j+1)}\left(\frac{1}{t}, -\theta_t - 1\right) \frac{1}{1 - \lambda t} \right]_+.$$

Then

$$h(t) = \left[\tilde{\mathcal{L}}\left(\frac{1}{t}, -\theta_t - 1\right) b(t) \right]_- - H_{\beta}(t) + H_{\alpha}(t).$$

Proof. Integration by parts yields

$$\int_{\alpha}^{\beta} \lambda^i (\theta^j F)(\lambda) d\lambda = \sum_{s=0}^{j-1} \lambda^{i+1} (-i-1)^s \theta^{j-1-s} F(\lambda) \Big|_{\lambda=\alpha}^{\lambda=\beta} + (-i-1)^j \int_{\alpha}^{\beta} \lambda^i F(\lambda) d\lambda,$$

and we apply this formula to every term below to get

$$\begin{aligned}
0 &= \sum_{n=0}^{\infty} t^n \int_{\alpha}^{\beta} \lambda^n \left(\tilde{\mathcal{L}}(\lambda, \theta_{\lambda}) F \right) (\lambda) d\lambda = \sum_{ij} c_{ij} \sum_{n=0}^{\infty} t^n \int_{\alpha}^{\beta} \lambda^{n+i} (\theta^j F) (\lambda) d\lambda \\
&= \sum_{n=0}^{\infty} t^n \sum_{i,j} c_{ij} \left[\sum_{s=0}^{j-1} \lambda^{n+i+1} (-n-i-1)^s \theta^{j-1-s} F(\lambda) \Big|_{\lambda=\alpha}^{\lambda=\beta} + (-n-i-1)^j \int_{\alpha}^{\beta} \lambda^{n+i} F(\lambda) d\lambda \right] \\
&= \sum_{k=0}^{N-1} \theta^k F(\lambda) \sum_{j \geq k+1} c_{ij} \frac{\lambda}{t^i} \sum_{n=0}^{\infty} \lambda^{n+i} t^{n+i} (-n-i-1)^{j-1-k} \Big|_{\lambda=\alpha}^{\lambda=\beta} \\
&\quad + \sum_{i,j} c_{ij} \frac{1}{t^i} \sum_{n=0}^{\infty} (-n-i-1)^j t^{n+i} \int_{\alpha}^{\beta} \lambda^{n+i} F(\lambda) d\lambda \\
&= \sum_{k=0}^{N-1} \theta^k F(\lambda) \left[\tilde{\mathcal{L}}^{(k+1)} \left(\frac{1}{t}, -\theta_t - 1 \right) \frac{1}{1-\lambda t} \Big|_{\lambda=\alpha}^{\lambda=\beta} + \left[\tilde{\mathcal{L}} \left(\frac{1}{t}, -\theta_t - 1 \right) b(t) \right]_+ \right] \\
&= H_{\beta}(t) - H_{\alpha}(t) + \left[\tilde{\mathcal{L}} \left(\frac{1}{t}, -\theta_t - 1 \right) b(t) \right]_+.
\end{aligned}$$

□

Now we introduce another idea which will allow to apply the above proposition to our case of linear polynomials. We consider $P = P_n$ as defined in (2.4), and let $\tilde{\mathcal{L}}_n$ be the corresponding Picard-Fuchs differential operator. It can be easily recovered from \mathcal{L}_n since (up to a simple multiplier) the operators $\mathcal{L}_n(t, \theta_t)$ and $\tilde{\mathcal{L}}_n\left(\frac{1}{t}, -\theta_t - 1\right)$ must be equal. For example, with

$$\begin{aligned}
\tilde{\mathcal{L}}_2 &= 9\theta^2 - \lambda(10\theta^2 + 10\theta + 3) + \lambda^2(\theta + 1)^2 \\
\tilde{\mathcal{L}}_3 &= 64\theta^3 - 2\lambda(2\theta + 1)(5\theta^2 + 5\theta + 2) + \lambda^2(\theta + 1)^3
\end{aligned}$$

one can easily check that

$$(4.1) \quad \tilde{\mathcal{L}}_2 \left(\frac{1}{t}, -\theta - 1 \right) = \frac{1}{9t^2} \tilde{\mathcal{L}}_2(9t, \theta) = \frac{1}{t^2} \mathcal{L}_2(t, \theta).$$

and

$$(4.2) \quad \tilde{\mathcal{L}}_3 \left(\frac{1}{t}, -\theta - 1 \right) = -\frac{1}{64t^2} \tilde{\mathcal{L}}_3(64t, \theta) = -\frac{1}{t^2} \mathcal{L}_3(t, \theta).$$

Let $a^*(\lambda)$ be the solution of $\tilde{\mathcal{L}}_n a^* = 0$ such that

$$(4.3) \quad m(P_n) = \int_0^{(n+1)^2} \log(\lambda) a^*(\lambda) d\lambda.$$

Applying Jensen's formula (1.1) in the variable x_{n+1} gives

$$\begin{aligned}
\frac{1}{2} m(P_{n+1}) &= \frac{1}{(2\pi i)^{n+1}} \int_{|x_i|=1} \log |1 + x_1 + \dots + x_{n+1}| \frac{dx_1}{x_1} \dots \frac{dx_{n+1}}{x_{n+1}} \\
&= \frac{1}{(2\pi i)^n} \int_{|x_i|=1, |1+x_1+\dots+x_n|>1} \log |1 + x_1 + \dots + x_n| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}
\end{aligned}$$

or

$$(4.4) \quad m(P_{n+1}) = \int_1^{(n+1)^2} \log(\lambda) a^*(\lambda) d\lambda.$$

Observe that (4.3) and (4.4) differ only by the lower limit of integration. This approach yields us to state that for every n there is a simple rational function $h_n(t)$ and an analytic solution $b_n(t)$ of

$$\mathcal{L}_n \left(t, t \frac{d}{dt} \right) b_n(t) = h_n(t)$$

such that

$$m(P_{n+1}) = -\operatorname{Re} \left(t \frac{d}{dt} \right)^{-1} b_n(t) \Big|_{t=\infty}.$$

Below we give exact statements for $n = 2$ and $n = 3$. We will formulate our results using solutions with rational coefficients rather than their transcendental linear combinations. This makes sense from the number-theoretical point of view and will be used later.

Theorem 4.2. *Take $\mathcal{L}_2(t, \theta) = \theta^2 - t(10\theta^2 + 10\theta + 3) + 9t^2(\theta + 1)^2$ and consider the following analytic at $t = 0$ solutions of*

$$\begin{aligned} \mathcal{L}_2\left(t, t \frac{d}{dt}\right) \phi(t) &= 0 & \phi(t) &= 1 + 3t + \dots \\ \mathcal{L}_2\left(t, t \frac{d}{dt}\right) \psi(t) &= \frac{t}{1-t} & \psi(t) &= t + \dots \end{aligned}$$

Then

$$m(P_3) = -\operatorname{Re} \left(t \frac{d}{dt} \right)^{-1} \left[\frac{3}{4} \phi(t) + \frac{6}{\pi^2} \psi(t) \right] \Big|_{t=\infty}.$$

Though all Mahler measures in this theorem were already computed before, we will use our result to relate them to double L-values of modular forms later. Next theorem already deals with the interesting case $n = 4$.

Theorem 4.3. *Let Ω_{15} be the Chowla-Selberg period for the field $K = Q(\sqrt{-15})$ as in (1.6) and $b(t)$ be the unique analytic at $t = 0$ solution of the non-homogeneous differential equation*

$$\mathcal{L}_3\left(t, t \frac{d}{dt}\right) b(t) = -\frac{3\sqrt{5}\Omega_{15}^2}{10\pi} \frac{t(212t^2 + 251t - 13)}{(1-t)^3} + \frac{3\sqrt{5}}{5\pi^3\Omega_{15}^2} \frac{t}{1-t}.$$

satisfying $b(t) = \frac{4}{5} + O(t)$. Then

$$m(P_4) = -\operatorname{Re} \left(t \frac{d}{dt} \right)^{-1} b(t) \Big|_{t=\infty}.$$

5. PROOFS

Recall from Section 2 that the function

$$a(t) = \frac{1}{(2\pi i)^n} \int_{|x_1|=\dots=|x_n|=1} \frac{1}{1-tP_n(x_1, \dots, x_n)} \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n}$$

is the unique analytic at $t = 0$ solution of $\mathcal{L}_n a = 0$ satisfying $a(0) = 1$. Let us write $a(t)$ as

$$a(t) = \int_0^{(n+1)^2} \frac{a^*(\lambda)}{1-t\lambda} d\lambda,$$

where $a^*(\lambda)$ is a solution of $\tilde{\mathcal{L}}_n a^* = 0$. Let us also introduce

$$\begin{aligned} b(t) &= \int_1^{(n+1)^2} \frac{a^*(\lambda)}{1-t\lambda} d\lambda = b_0 + b_1 t + \dots \\ c(t) &= \int_0^1 \frac{a^*(\lambda)}{1-t\lambda} d\lambda = c_0 + c_1 t + \dots \end{aligned}$$

In Section 4 we showed that

$$m(P_{n+1}) = -\operatorname{Re} \left(t \frac{d}{dt} \right)^{-1} b(t) \Big|_{t=\infty}.$$

In this section for $n = 2, 3$ we will compute the coefficient b_0 and prove that

$$h(t) := \mathcal{L}_n b(t) = -\mathcal{L}_n c(t)$$

has the form as in Theorems 4.2 and 4.3 correspondingly. The proofs of Theorems 4.2 and 4.3 go along the same lines. First, we identify the solution $a^*(\lambda)$ for $\lambda \in [0, 1]$ in terms of the Frobenius basis in the space of solutions near $\lambda = 0$. In order to do this we use asymptotics of $a(t)$ when t is large, and we use the modular parametrization of the differential equation to find this asymptotics. As soon as we know $a^*(\lambda)$ explicitly, we apply Proposition 4.1 to finish the proof.

We note that the differential operator $\tilde{\mathcal{L}}_2$ has singularities at $\lambda = 0, 1$ which are regular singular points of maximal unipotent monodromy, whereas the operator $\tilde{\mathcal{L}}_3$ has a regular singular point of maximal unipotent monodromy at

$\lambda = 0$ and is nonsingular at $\lambda = 1$. The period Ω_{15} appears in Theorem 4.3 because the point $\lambda = 1$ corresponds under the modular parametrization of $\widetilde{\mathcal{L}}_3$ to a CM point of conductor 15.

It hopefully will not confuse the reader that we use the same notation

$$a(t), a^*(\lambda), b(t), c(t)$$

for the integrals corresponding to the case $n = 2$ in Lemmas 5.1 – 5.3 and to the case $n = 3$ in Lemmas 5.4 – 5.7.

Proof of Theorem 4.2. Here $a(t) = \phi(t)$. From Proposition 4.1 and Lemmas 5.2 and 5.3 below it follows that

$$\widetilde{\mathcal{L}}_2\left(\frac{1}{t}, -\theta - 1\right)c(t) = -\frac{6}{\pi^2}\left(\frac{1}{1-t} + \frac{1}{t}\right) = -\frac{6}{\pi^2}\frac{1}{t(1-t)}.$$

According to (4.1) we then have

$$\mathcal{L}_2(t, \theta)c(t) = t^2\widetilde{\mathcal{L}}_2\left(\frac{1}{t}, -\theta - 1\right)c(t) = -\frac{6}{\pi^2}\frac{t}{1-t}$$

and then

$$\mathcal{L}_2(t, \theta)b(t) = -\mathcal{L}_2(t, \theta)c(t) = \frac{6}{\pi^2}\frac{t}{1-t}.$$

From Lemma 5.3 we have $b_0 = 1 - c_0 = \frac{3}{4}$ and therefore $b(t) = \frac{3}{4}\phi(t) + \frac{6}{\pi^2}\psi(t)$. To finish the proof it remains to verify the three lemmas that follow below. □

Lemma 5.1.

$$\begin{aligned} a^*(\lambda) &= \frac{1}{\sqrt{3}\pi} + O(\lambda)_{\lambda \rightarrow 0} \\ a^*(\lambda) &= -\frac{3}{4\pi^2}\log(1-\lambda) + O(1)_{\lambda \rightarrow 1^-} \end{aligned}$$

Proof. Using the modular parametrization (3.4) we find that when $t \rightarrow -\infty$ along negative real axis (this corresponds to z going down the ray $\frac{1}{2} + i\mathbb{R}_+$)

$$(5.1) \quad t a(t) = \frac{1}{\sqrt{3}\pi}\log\left(-\frac{1}{t}\right) + O(1)$$

On the other hand

$$(5.2) \quad t a(t) = \int_0^9 \frac{a^*(\lambda)}{1/t - \lambda} d\lambda = -\int_0^9 \frac{a^*(\lambda)}{s + \lambda} d\lambda \Big|_{s=-\frac{1}{t}}.$$

Let us write $a^*(\lambda) = \alpha_0\phi_0(\lambda) + \alpha_1\phi_1(\lambda)$ in terms of the Frobenius basis

$$\begin{aligned} \phi_0(\lambda) &= 1 + O(\lambda) \\ \phi_1(\lambda) &= \log(\lambda)\phi_0(\lambda) + O(\lambda) \end{aligned}$$

of solutions near $\lambda = 0$. One can easily check that for any $\varepsilon > 0$

$$\begin{aligned} \int_0^\varepsilon \frac{d\lambda}{s + \lambda} &\sim -\log s + O(1) \\ \int_0^\varepsilon \frac{\log \lambda d\lambda}{s + \lambda} &\sim -\frac{1}{2}(\log s)^2 + O(1) \end{aligned}$$

and comparing (5.1) and (5.2) we see that $\alpha_1 = 0$ and $\alpha_0 = \frac{1}{\sqrt{3}\pi}$.

Now let

$$\begin{aligned} \kappa_0(\lambda) &= 1 + O(\lambda - 1) \\ \kappa_1(\lambda) &= \log(\lambda - 1)\kappa_0(\lambda) + O(\lambda - 1) \end{aligned}$$

be the Frobenius basis at $\lambda = 1$ and $a^*(\lambda) = \alpha_0\kappa_0(\lambda) + \alpha_1\kappa_1(\lambda)$ when $\lambda \rightarrow 1^-$. Using our modular parametrization we find that

$$\alpha_1 = \lim_{\lambda \rightarrow 1^-} \frac{a^*(\lambda)}{\log(1-\lambda)} = \lim_{s \rightarrow 0^+} \frac{1}{\sqrt{3}\pi} \frac{f(z)}{\log(1-9t(z))} \Big|_{z=is} = -\frac{3}{4\pi^2}.$$

□

Lemma 5.2. *In the notation of Proposition 4.1 applied to $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_2$, $\alpha = 0$, $\beta = 1$, $F(\lambda) = a^*(\lambda)$ we have:*

$$\begin{aligned} H_0(t) &= 0 \\ H_1(t) &= \frac{6}{\pi^2(1-t)} \end{aligned}$$

Proof. In the course of the proof we rely on the asymptotics given in Lemma 5.1. Since $\lambda = 0, 1$ are singular points we are going to compute $H_0(t)$ and $H_1(t)$ as the corresponding limits of $H_\lambda(t)$. We have

$$\begin{aligned} \left[\tilde{\mathcal{L}}_2^{(2)}\left(\frac{1}{t}, -\theta - 1\right) \frac{1}{1-\lambda t} \right]_+ &= \left[\left(9 - \frac{10}{t} + \frac{1}{t^2}\right) \frac{1}{1-\lambda t} \right]_+ \\ &= \frac{9t^2 - 10t + 1}{t^2(1-\lambda t)} - \frac{1}{t^2} - \frac{\lambda - 10}{t} = \frac{(\lambda - 1)(\lambda - 9)}{1 - \lambda t} \end{aligned}$$

and

$$\begin{aligned} \left[\tilde{\mathcal{L}}_2^{(1)}\left(\frac{1}{t}, -\theta - 1\right) \frac{1}{1-\lambda t} \right]_+ &= \left[\left(\left(9 - \frac{10}{t} + \frac{1}{t^2}\right)(-\theta - 1) + \left(-\frac{10}{t} + 2t^2\right) \right) \frac{1}{1-\lambda t} \right]_+ \\ &= -\left(9 - \frac{10}{t} + \frac{1}{t^2}\right) \frac{1}{(1-\lambda t)^2} + \left(-\frac{10}{t} + 2t^2\right) \frac{1}{1-\lambda t} - \frac{1}{t^2} = -\frac{(\lambda - 1)(\lambda - 9)}{(1-\lambda t)^2} \end{aligned}$$

These functions have finite limits when $\lambda \rightarrow 0$ and now we see that $H_0(t) = 0$ because $a^*(\lambda)$ is analytic at $\lambda = 0$ and therefore $\lim_{\lambda \rightarrow 0} \lambda \theta^j a^*(\lambda) = 0$ for any $j \geq 0$.

Since $\lim_{\lambda \rightarrow 1^-} (\lambda - 1) a^*(\lambda) = 0$ and $\lim_{\lambda \rightarrow 1^-} (\lambda - 1) \theta a^*(\lambda) = -\frac{3}{4\pi^2}$ we find that $H_1(t) = \frac{6}{\pi^2(1-t)}$. \square

Lemma 5.3. *The first coefficients c_0 in the power series expansion of $c(t)$ is equal to $\frac{1}{4}$ and we have*

$$\left[\tilde{\mathcal{L}}_2\left(\frac{1}{t}, -\theta - 1\right) c(t) \right]_- = -\frac{6}{\pi^2 t}.$$

Proof. It is easy to compute that

$$\left[\tilde{\mathcal{L}}_2\left(\frac{1}{t}, -\theta - 1\right) c(t) \right]_- = \frac{-3c_0 + c_1}{t}.$$

Using the modular parametrization (3.4) (with modular t and f from (3.4) and $\lambda = 9t$ one has $\mathcal{L}_{\lambda, f} = \frac{1}{9\lambda} \tilde{\mathcal{L}}_2(\lambda, \lambda \frac{d}{d\lambda})$) we compute that

$$\begin{aligned} c_0 &= \int_0^1 a^*(\lambda) d\lambda = \frac{9}{\sqrt{3}\pi} \int_{i\infty}^0 f(z) t'(z) dz \\ &= \frac{18}{\sqrt{3}} \int_0^\infty f(z)^3 t(z) (1 - 9t(z))(1 - t(z)) \Big|_{z=is} ds = \frac{1}{4} \end{aligned}$$

and

$$\begin{aligned} c_1 - 3c_0 &= \int_0^1 (\lambda - 3) a^*(\lambda) d\lambda \\ &= \frac{18}{\sqrt{3}} \int_0^\infty (9t(z) - 3) f(z)^3 t(z) (1 - 9t(z))(1 - t(z)) \Big|_{z=is} ds \\ &= -\frac{6}{\pi^2}. \end{aligned}$$

\square

Proof of Theorem 4.3. We let $\Omega = \Omega_{15}$. According to Proposition 4.1 and Lemmas 5.5 and 5.6 below we compute that

$$\begin{aligned} \tilde{\mathcal{L}}_3\left(\frac{1}{t}, -\theta - 1\right) c(t) &= \frac{3\sqrt{5}\Omega^2}{10\pi} \left(\frac{13}{t} - \frac{-13t^2 + 251t + 212}{(1-t)^3} \right) + \frac{3\sqrt{5}}{5\pi^3\Omega^2} \left(\frac{1}{t} + \frac{1}{1-t} \right) \\ &= \frac{3\sqrt{5}\Omega^2}{10\pi} \frac{-212t^2 - 251t + 13}{t(1-t)^3} + \frac{3\sqrt{5}}{5\pi^3\Omega^2} \frac{1}{t(1-t)}. \end{aligned}$$

Now it follows from (4.2) that we have

$$\begin{aligned} -h(t) = \mathcal{L}_3(t, \theta)c(t) &= -t^2 \widetilde{\mathcal{L}}_3\left(\frac{1}{t}, -\theta - 1\right)c(t) \\ &= \frac{3\sqrt{5}\Omega^2}{10\pi} \frac{t(212t^2 + 251t - 13)}{(1-t)^3} - \frac{3\sqrt{5}}{5\pi^3\Omega^2} \frac{t}{1-t}. \end{aligned}$$

Therefore the function

$$b(t) = \int_1^{16} \frac{a^*(\lambda)}{1-t\lambda} d\lambda$$

satisfies $\mathcal{L}_3 b = h(t)$ and its power series expansion at $t = 0$ starts with $b_0 = a_0 - c_0 = \frac{4}{5}$. The proof will be finished after verifying the three lemmas below. \square

Lemma 5.4. *In terms of the Frobenius basis*

$$\begin{aligned} \phi_0(\lambda) &= 1 + O(\lambda) \\ \phi_1(\lambda) &= \log(\lambda)\phi_0(\lambda) + O(\lambda) = \log(\lambda) + o(1) \\ \phi_2(\lambda) &= \log(\lambda)^2\phi_0(\lambda) + O(\lambda) = \log(\lambda)^2 + o(1) \end{aligned}$$

of solutions near $\lambda = 0$ we have

$$a^*(\lambda) = -\frac{3}{8\pi^2}(\phi_1(\lambda) - 6\log 2 \phi_0(\lambda)).$$

At $\lambda = 1$ we have

$$\begin{aligned} a^*(1) &= 0.1649669005300320\dots = \frac{3\sqrt{5}}{2\pi}\Omega^2 \quad (\Omega = \Omega_{15}) \\ \theta a^*(1) &= -0.032993380106006\dots = -\frac{3\sqrt{5}}{10\pi}\Omega^2 \\ \theta^2 a^*(1) &= 0.00330836512971504\dots = \frac{\sqrt{5}}{150}\left(\frac{13\Omega^2}{\pi} - \frac{2}{\pi^3\Omega^2}\right). \end{aligned}$$

Proof. With the help of modular parametrization (1.4) we find that when $t \rightarrow -\infty$ along negative real axis (this corresponds to z going down to 0 along the imaginary axis)

$$ta(t) + \frac{3}{16\pi^2} \log\left(-\frac{1}{64t}\right)^2 \rightarrow 0$$

On the other hand

$$ta(t) = \int_0^{16} \frac{a^*(\lambda)}{1/t - \lambda} d\lambda = -\int_0^{16} \frac{a^*(\lambda)}{s + \lambda} d\lambda \Big|_{s=-\frac{1}{t}}$$

and since for any $\varepsilon > 0$ one has when $s \rightarrow 0$

$$\begin{aligned} \int_0^\varepsilon \frac{d\lambda}{s + \lambda} &\sim -\log s + O(1) \\ \int_0^\varepsilon \frac{\log \lambda d\lambda}{s + \lambda} &\sim -\frac{1}{2}(\log s)^2 + O(1) \\ \int_0^\varepsilon \frac{(\log \lambda)^2 d\lambda}{s + \lambda} &\sim -\frac{1}{3}(\log s)^3 + O(\log(s)^2) \end{aligned}$$

we find that $\alpha_2 = 0$, $\alpha_1 = -\frac{3}{8\pi^2}$ and $\alpha_0 = \frac{9}{4\pi^2} \log 2$.

We indicate how to find the values $\theta^j a^*(1)$ for $j = 0, 1, 2$. Again we use modular parametrization (1.4), namely with modular t and f from (1.4) and $\lambda = 64t$ one has $\mathcal{L}_{\lambda, f} = \frac{1}{64\lambda} \widetilde{\mathcal{L}}_3(\lambda, \lambda \frac{d}{d\lambda})$. This $\lambda(z)$ takes real values from the interval $(0, 1]$ for $z = \frac{1}{2} + is$ and $s \in (+\infty, \frac{\sqrt{15}}{6}]$ (let us denote $\tau = \frac{1}{2} + \frac{\sqrt{-15}}{6}$ so that $\lambda(\tau) = 1$). Using asymptotics at ∞ one can check that on the vertical half-line from τ to ∞

$$a^*(\lambda(z)) = -\frac{3}{8\pi^2} \cdot 2\pi i \left(z - \frac{1}{2}\right) f(z).$$

Now the problem is reduced to computing the values of modular forms and their derivatives at a CM-point of conductor 15, this leading to expressions involving Ω and π , see [Zag], Propositions 26, 27 and Corollary of Proposition 27.

□

Lemma 5.5. *In the notation of Proposition 4.1 applied to $\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_3$, $\alpha = 0$, $\beta = 1$, $F(\lambda) = a^*(\lambda)$ we have:*

$$\begin{aligned} H_0(t) &= 0 \\ H_1(t) &= \frac{3\Omega^2\sqrt{5}-13t^2+251t+212}{10\pi} \frac{1}{(1-t)^3} - \frac{3\sqrt{5}}{5\pi^3\Omega^2} \frac{1}{1-t} \end{aligned}$$

Proof. One easily checks that $\lim_{\lambda \rightarrow 0} \lambda \theta^j a^*(\lambda) = 0$, whence $H_0(t) = 0$. In order to compute $H_1(t)$ we need

$$\begin{aligned} \left[\tilde{\mathcal{L}}_3^{(3)} \left(\frac{1}{t}, -\theta_t - 1 \right) \frac{1}{1-t} \right]_+ &= \frac{45}{1-t} \\ \left[\tilde{\mathcal{L}}_3^{(2)} \left(\frac{1}{t}, -\theta_t - 1 \right) \frac{1}{1-t} \right]_+ &= \frac{9(t-6)}{(1-t)^2} \\ \left[\tilde{\mathcal{L}}_3^{(1)} \left(\frac{1}{t}, -\theta_t - 1 \right) \frac{1}{1-t} \right]_+ &= \frac{29+68t-7t^2}{(1-t)^3}, \end{aligned}$$

and then the formula for $H_1(t)$ follows after a simple computation with values of $\theta^j a^*(1)$ provided by Lemma 5.4. □

Lemma 5.6. *The first coefficient c_0 in the power series expansion of $c(t)$ is equal to $\frac{1}{5}$ and we have*

$$\left[\tilde{\mathcal{L}}_3 \left(\frac{1}{t}, -\theta - 1 \right) c(t) \right]_- = \left(\frac{39\sqrt{5}}{10\pi} \Omega^2 + \frac{3\sqrt{5}}{5\pi^3\Omega^2} \right) \frac{1}{t}.$$

Proof. It is easy to compute that

$$\begin{aligned} \left[\tilde{\mathcal{L}}_3 \left(\frac{1}{t}, -\theta - 1 \right) c(t) \right]_- &= \frac{4c_0 - c_1}{t}. \\ c_0 &= \int_0^1 a^*(\lambda) d\lambda = -\frac{3}{8\pi^2} \cdot 2\pi i \int_{i\infty}^\tau \left(z - \frac{1}{2} \right) f(z) \lambda'(z) dz \\ &\quad \left(\text{here we use that } \lambda = 64t \text{ and } \left(q \frac{dt}{dq} \right)^2 / f^2 = t^2(1-4t)(1-16t) \right) \\ &= -\frac{3}{8\pi^2} \cdot (2\pi i)^2 \cdot 64 \int_{i\infty}^\tau \left(z - \frac{1}{2} \right) f(z)^2 t(z) \sqrt{(1-4t(z))(1-16t(z))} dz \\ &= 96 \int_{\frac{\sqrt{45}}{6}}^\infty s f(z)^2 t(z) \sqrt{(1-4t(z))(1-16t(z))} \Big|_{z=\frac{1}{2}+is} ds = \frac{1}{5} \end{aligned}$$

and

$$\begin{aligned} c_1 - 4c_0 &= \int_0^1 (\lambda - 4) a^*(\lambda) d\lambda \\ &= 96 \int_{\frac{\sqrt{45}}{6}}^\infty s f(z)^2 (64t(z) - 4) t(z) \sqrt{(1-4t(z))(1-16t(z))} \Big|_{z=\frac{1}{2}+is} ds \\ &= -0.708951451918989714\dots = -7a^*(1) - 9\theta a^*(1) + 45\theta^2 a^*(1) \\ &= -\frac{39\sqrt{5}}{10\pi} \Omega^2 - \frac{3\sqrt{5}}{5\pi^3\Omega^2} \end{aligned}$$

□

6. DOUBLE L-VALUES OF MODULAR FORMS

In Theorem 4.2 we are led to evaluation of

$$\left(t \frac{d}{dt} \right)^{-1} \psi(t) \Big|_{t=\infty}$$

where ψ is the unique analytic at $t = 0$ solution of

$$\mathcal{L}_2 \left(t, t \frac{d}{dt} \right) \psi(t) = \frac{t}{1-t}$$

which satisfies the condition $\psi(t) = t + o(t)$. The same happens in Theorem 4.3. Putting this situation into a more general context, consider a solution of a non-homogeneous differential equation of order $k + 1$ which has a modular parametrization, i.e.

$$\mathcal{L}_{t,f} \psi = h(t),$$

where t is a modular function, f is a modular form of weight k , $\mathcal{L}_{t,f}$ is defined by (3.3) and $h(t)$ is a function of t which will be just rational in our cases. We can consider $\psi = \psi(t(z))$ as a function in the upper half-plane and we rewrite the above differential equation as

$$\frac{1}{Dt \cdot f} D^{k+1} \frac{\psi}{f} = h(t),$$

or

$$D^{k+1} \frac{\psi}{f} = h(t) \cdot Dt \cdot f.$$

Therefore $\frac{\psi}{f}$ is an Eichler integral of the modular form $h(t) \cdot Dt \cdot f$ of weight $k + 2$. (This conclusion is precisely the statement of Lemma 1 in [Yan].) Let us assume in addition that the modular function t takes values 0 and ∞ at $q = 0$ and $q = 1$ correspondingly. Then

$$\left(t \frac{d}{dt}\right)^{-1} \psi(t) \Big|_{t=\infty} = D^{-1} \left(\frac{Dt}{t} \psi\right) \Big|_{q=1} = D^{-1} (g_2 D^{-k-1} g_1) \Big|_{q=1}$$

where

$$g_1 = h(t) \cdot Dt \cdot f, \quad g_2 = \frac{Dt}{t} f$$

are two modular forms of weight $k + 2$. According to the proposition below the above expression appears to be the double L-value of these two forms. Let us give the definition.

Let $g_1 = \sum_{n \geq 0} a_n q^n$ and $g_2 = \sum_{m \geq 0} b_m q^m$ be two modular forms of weight k on a congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$, and let in addition $a_0 = 0$. Their double L-function (denoted by L^\bullet in [Sr]) is defined for $\mathrm{Re}(s_1 + s_2) > 2k$, $\mathrm{Re} s_2 > k$ by

$$L(g_1, g_2, s_1, s_2) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{a_n b_m}{n^{s_1} (n+m)^{s_2}}.$$

The question of the analytic continuation simultaneously in the two variables s_1, s_2 is rather tricky and we do not want to consider it here. But it appears that for any fixed integer $s_1 = p$ the function $L(g_1, g_2, p, s_2)$ is well-defined for s_2 with sufficiently large real part, and we can easily prove analytic continuation in this variable when $p > 0$. In order to do this one writes ([Sr])

$$(6.1) \quad L(g_1, g_2, p, s_2) = \frac{(2\pi)^{p+s_2}}{\Gamma(p)\Gamma(s_2)} \sum_{m=0}^{p-1} \Lambda(g_1, g_2, p-m, s_2+m)$$

where

$$\Lambda(g_1, g_2, s_1, s_2) = \int_0^\infty t^{s_2-1} g_2(it) \int_t^\infty v^{s_1-1} g_1(iv) dv dt.$$

Now observe that these integrals are well defined for all s_1, s_2 . Indeed, it follows from the estimates

$$\begin{aligned} \int_t^\infty v^{s_1-1} g_1(iv) dv &= O(t^{s_1-1} e^{-2\pi t}), \quad t \rightarrow \infty \\ &= O(t^{s_1-k} e^{-\frac{2\pi}{t}}), \quad t \rightarrow 0 \end{aligned}$$

since $g_1(it) = O\left(t^{-k} e^{-\frac{2\pi}{t}}\right)$ when $t \rightarrow 0$. Therefore formula (6.1) gives analytic continuation of $L(g_1, g_2, p, s_2)$ in the variable s_2 with integer $p > 1$. Moreover, this function is holomorphic in the entire complex plane because $1/\Gamma(s_2)$ is holomorphic, and we can speak of “double L-values” $L(g_1, g_2, p_1, p_2)$ with integers p_1, p_2 whenever $p_1 > 0$. Notice also that $L(g_1, g_2, p_1, p_2) = 0$ if $p_2 \leq 0$ as one can see from (6.1) since $\Gamma(s_2)$ has poles at nonpositive integers.

Proposition 6.1. *Let g_1, g_2 be two modular forms of weight k on a congruence subgroup, g_1 vanishing at ∞ . Then for any integers $0 < p_1 \leq k$ and $p_2 < k$ one has*

$$(6.2) \quad \lim_{q \rightarrow 1} D^{-p_2} (g_2 \cdot D^{-p_1} g_1)(q) = L(g_1, g_2, p_1, p_2)$$

Proof. Whenever $\operatorname{Re}(p_2 + w) > k$ one has

$$\begin{aligned} \frac{\Gamma(w)}{(2\pi)^w} L(g_1, g_2, p_1, p_2 + w) &= \frac{\Gamma(w)}{(2\pi)^w} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{a_n b_m}{n^{p_1} (n+m)^{p_2+w}} \\ &= \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{a_n b_m}{n^{p_1} (n+m)^{p_2}} \int_0^{\infty} t^{w-1} e^{-2\pi t(n+m)} \\ &= \int_0^{\infty} t^{w-1} D^{-p_2} (g_2 \cdot D^{-p_1} g_1)(it) dt. \end{aligned}$$

By Mellin's inversion theorem with an arbitrary real $c > k - p_2$ one has

$$\begin{aligned} D^{-p_2} (g_2 \cdot D^{-p_1} g_1)(it) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(w)}{(2\pi t)^w} L(g_1, g_2, p_1, p_2 + w) dw \\ &= L(g_1, g_2, p_1, p_2) + \frac{1}{2\pi i} \int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} \frac{\Gamma(w)}{(2\pi t)^w} L(g_1, g_2, p_1, p_2 + w) dw \end{aligned}$$

with any $0 < \varepsilon < 1$ and we moved the path using the fact that $L(g_1, g_2, p_1, p_2 + w)$ is everywhere holomorphic in w . The last integral obviously vanishes when $t \rightarrow 0$, and (6.2) follows. \square

In the case of Theorem 4.2 we use the modular parametrization (3.4) and $h(t) = \frac{1}{1-t}$. Since $t = \infty$ at $z = \frac{1}{2}$ we consider the shifted forms

$$\begin{aligned} (6.3) \quad g_1(z) &= \left(\frac{Dt}{t} f \right) \left(z + \frac{1}{2} \right) = 1 + q - 5q^2 + q^3 + 11q^4 - 24q^5 + \dots \\ &= E_{3, \chi_{-3}}(z) - 2E_{3, \chi_{-3}}(2z) - 8E_{3, \chi_{-3}}(4z) \\ g_2(z) &= \left(\frac{Dt}{1-t} f \right) \left(z + \frac{1}{2} \right) = -q - 4q^2 - q^3 + 16q^4 + 24q^5 - 4q^6 + \dots \\ &= -E_{3, \chi_{-3}}(z) - 7E_{3, \chi_{-3}}(2z) + 8E_{3, \chi_{-3}}(4z) \end{aligned}$$

where $E_{3, \chi}$ is the Eisenstein series defined in (3.7). The form g_1 already appeared in (3.6) and we had that $m(P_2) = -L(g_1, 1)$. Using Proposition 6.1 we now rewrite the statement of Theorem 4.2 as follows.

Corollary 6.2. *With the modular forms g_1, g_2 of weight 3 defined in (6.3) one has*

$$m(P_3) - \frac{3}{4}m(P_2) = -\frac{6}{\pi^2}L(g_2, g_1, 2, 1).$$

Plugging in the values of $m(P_2)$ and $m(P_3)$ which we compute from (1.2), (1.3) into the formula given in Corollary 6.2 we obtain the following relation between double and ordinary L-values:

$$(6.4) \quad L(g_2, g_1, 2, 1) = \frac{3\sqrt{3}\pi}{2^4}L(\chi_{-3}, 2) - \frac{7}{6}\zeta(3).$$

We give a straightforward proof of this relation in the next section.

For the Theorem 4.3 we use the modular parametrization (1.4) and we have to consider two solutions with $h(t) = \frac{1}{1-t}$ and $h(t) = \frac{212t^2 + 251t - 13}{(1-t)^3}$. Also we have $t = 0$ at $z = i\infty$ and $t = \infty$ at $z = 0$. According to our strategy, we define the modular forms of weight 4

$$\begin{aligned} g_1 &= \frac{Dt}{t} f = 1 + 2q - 14q^2 + 38q^3 - 142q^4 + 252q^5 - 266q^6 + \dots \\ g_2 &= \frac{Dt}{1-t} f = -q - 7q^2 - 6q^3 + 5q^4 + 120q^5 + 498q^6 + \dots \\ g_3 &= \frac{212t^2 + 251t - 13}{(1-t)^3} Dt \cdot f = 13q + 316q^2 + 2328q^3 + \dots \end{aligned}$$

(observe that they are the same ones as in (1.5)). Here g_1 is a holomorphic modular form, it already appeared in (3.8) and there we had $m(P_3) = -L(g_1, 1)$. The forms g_2 and g_3 are meromorphic with the poles at the points where $t = 1$. Using the fact that t has no poles on imaginary half-axis we defined the corresponding double L-values $L(g_2, g_1, 3, 1)$, $L(g_3, g_1, 3, 1)$ in (1.7) in Section 1.

Corollary 6.3. *With the double L -values defined in (1.7) one has*

$$m(P_4) - \frac{4}{5}m(P_3) = \frac{3\sqrt{5}\Omega_{15}^2}{10\pi}L(g_3, g_1, 3, 1) - \frac{3\sqrt{5}}{5\pi^3\Omega_{15}^2}L(g_2, g_1, 3, 1).$$

Proof. Due to Theorem 4.3 we have that $m(P_4) = -\operatorname{Re} \left(t \frac{d}{dt} \right)^{-1} b(t) \Big|_{t=\infty}$. We know from Proposition 3.2 that this differential equation has modular parametrization by $t(z)$ and $f(z)$. Therefore

$$b(t(z)) = \frac{4}{5}f(z) + f(z)D^{-3} \left(-\frac{3\sqrt{5}\Omega_{15}^2}{10\pi}g_3(z) + \frac{3\sqrt{5}}{5\pi^3\Omega_{15}^2}g_2(z) \right).$$

We use the path in the upper halfplane from $z = i\infty$ to $z = 0$ along imaginary half-axis, exactly where one has $-\infty < t(z) < 0$. As it was explained in Section 1, all three forms are holomorphic along this path. For $g_j(z)$ with both $j = 2, 3$ we then have

$$D^{-1}(g_1 \cdot D^{-3}g_j)(iv) = (2\pi)^4 \int_v^\infty g_1(is) \int_s^\infty \int_{s_1}^\infty \int_{s_2}^\infty g_j(is_3) ds_3 ds_2 ds_1 ds$$

and therefore the numbers (1.7) are the limiting values at $v = 0$. \square

7. EXPLICIT COMPUTATION OF A DOUBLE L -VALUE IN FORMULA (6.4)

In this section we show how to compute the iterated integral

$$(7.1) \quad \begin{aligned} \int_0^{i\infty} g_1(z) \cdot D^{-2}g_2(z)dz &= \frac{1}{2\pi i} \int_1^0 g_1(q) \cdot D^{-2}g_2(q) \frac{dq}{q} \\ &= -\frac{1}{2\pi i} D^{-1}(g_1 \cdot D^{-2}g_2) \Big|_{q=1} \end{aligned}$$

for two modular forms g_1, g_2 of weight 3 defined in (6.3) which leads to an alternative proof of formula (6.4).

We use a powerful method due to Mathew Rogers and Wadim Zudilin [RZ1], [RZ2], [Zud2], [Zud3] of computing double L -values of Eisenstein-like series. We are grateful to Wadim for explaining to us his method and its applicability in this situation. Unfortunately, the more complicated and exciting L -values from Corollary 6.3 do not seem to be computable in the same way due to lack of Eisenstein-like representation for the forms g_1, g_2, g_3 referred to in Corollary 6.3.

We briefly describe the method as follows: the Atkin-Lehner involution $z \rightarrow -\frac{1}{12z}$ is applied to g_1 , the resulting modular form being denoted by $\hat{g}_1(z)$:

$$g_1(z) = \operatorname{const} \cdot \hat{g}_1\left(-\frac{1}{12z}\right)z^{-3},$$

so that the integral (7.1) will take the following form:

$$\operatorname{const} \cdot \int_0^{i\infty} \hat{g}_1\left(-\frac{1}{12z}\right)D^{-2}g_2(z)z^{-3}dz.$$

We then expand the integral as a quadruple sum, make a variable change and collapse the sum back in order to get

$$\operatorname{const} \cdot \int_0^{i\infty} f_1(u)(\operatorname{const} + f_2\left(-\frac{1}{12u}\right))u du,$$

where f_1 and f_2 are Eisenstein series of weight 1. We apply the Atkin-Lehner involution again:

$$f_2\left(-\frac{1}{12u}\right) = \operatorname{const} \cdot \hat{f}_2(u)u,$$

this time rewriting the integral as

$$\operatorname{const} \cdot \int_0^{i\infty} f_1(u)(\operatorname{const} + \hat{f}_2(u)u)u du = \operatorname{const} \cdot L(f_1, 2) + \operatorname{const} \cdot L(f_1\hat{f}_2, 3),$$

with f_1 an Eisenstein series of weight 1 and character χ_{-3} yielding $\zeta(2) \cdot L(\chi_{-3}, 2)$ and $f_1\hat{f}_2$ an Eisenstein series for $SL_2(\mathbb{Z})$ of weight 2 yielding $\zeta(2) \cdot \zeta(3)$.

Note that no regularization is necessary in our integral, since g_2 vanishes at $z = \infty$ and g_1 vanishes at $z = 0$.

In the course of the computation we will need to apply Atkin-Lehner involution to Eisenstein series. For that we express Eisenstein series as linear combinations of eta-products. Let $N \geq 1$. Consider an eta-product

$$f(z) = \prod_j \eta(d_j \cdot z)^{k_j}$$

where k_j are integers and d_j non-negative integers dividing N . Let $d'_j = N/d_j$ and

$$\hat{f}(z) = \prod_j \eta(d'_j \cdot z)^{k_j}.$$

We have

$$(7.2) \quad f\left(-\frac{1}{Nz}\right) = (-i)^w z^w \prod_j d_j^{k_j/2} \hat{f}(z)$$

for $w = \frac{1}{2} \sum_j k_j$ (the weight). This follows from the basic transformation formula:

$$\eta\left(-\frac{1}{z}\right) = \sqrt{-iz} \eta(z).$$

We use the following two Eisenstein series of weight 3

$$E_{3,\chi_{-3}}(z) = -\frac{1}{9} \frac{\eta(z)^9}{\eta(3z)^3} = -\frac{1}{9} + \sum_{n,m \geq 1} \chi_{-3}(n) n^2 q^{nm} = -\frac{1}{9} + q - 3q^2 + q^3 + \dots$$

$$\tilde{E}_{3,\chi_{-3}}(z) = \frac{\eta(3z)^9}{\eta(z)^3} = \sum_{n,m \geq 1} \chi_{-3}(m) n^2 q^{nm} = q + 3q^2 + 9q^3 + 13q^4 + \dots$$

$$g_1(z) = [1, -2, -8] \cdot [E_{3,\chi_{-3}}(z), E_{3,\chi_{-3}}(2z), E_{3,\chi_{-3}}(4z)]$$

$$g_2(z) = [-1, -7, 8] \cdot [E_{3,\chi_{-3}}(z), E_{3,\chi_{-3}}(2z), E_{3,\chi_{-3}}(4z)].$$

We apply (7.2):

$$E_{3,\chi_{-3}}(z) \Big|_{-\frac{1}{12z}} = E_{3,\chi_{-3}}\left(-\frac{1}{12z}\right) = -2^6 3^{5/2} i z^3 \tilde{E}_{3,\chi_{-3}}(4z)$$

$$E_{3,\chi_{-3}}(2z) \Big|_{-\frac{1}{12z}} = E_{3,\chi_{-3}}\left(-\frac{1}{6z}\right) = -2^3 3^{5/2} i z^3 \tilde{E}_{3,\chi_{-3}}(2z)$$

$$E_{3,\chi_{-3}}(4z) \Big|_{-\frac{1}{12z}} = E_{3,\chi_{-3}}\left(-\frac{1}{3z}\right) = -3^{5/2} i z^3 \tilde{E}_{3,\chi_{-3}}(z)$$

and hence

$$[a, b, c] \cdot [E_{3,\chi_{-3}}(z), E_{3,\chi_{-3}}(2z), E_{3,\chi_{-3}}(4z)] \left(-\frac{1}{12z}\right) = -3^{5/2} i z^3 [c, 2^3 b, 2^6 a] \cdot [\tilde{E}_{3,\chi_{-3}}(z), \tilde{E}_{3,\chi_{-3}}(2z), \tilde{E}_{3,\chi_{-3}}(4z)].$$

In particular for $g_1(z) = [1, -2, -8] \cdot [E_{3,\chi_{-3}}(z), E_{3,\chi_{-3}}(2z), E_{3,\chi_{-3}}(4z)]$ we have

$$\begin{aligned} g_1\left(-\frac{1}{12z}\right) &= 8 \cdot 3^{5/2} i z^3 \hat{g}_1(z) \\ \hat{g}_1(z) &= [1, 2, -8] \cdot [\tilde{E}_{3,\chi_{-3}}, \tilde{E}_{3,\chi_{-3}}^{(1)}, \tilde{E}_{3,\chi_{-3}}^{(2)}](z), \\ &= q + 5q^2 + 9q^3 + 11q^4 + 24q^5 + \dots \end{aligned}$$

or equivalently

$$(7.3) \quad g_1(z) = -\frac{iz^{-3}}{2^3 3^{1/2}} \hat{g}_1\left(-\frac{1}{12z}\right)$$

Formula (7.3) allows us to rewrite the iterated integral (7.1) as

$$(7.4) \quad -\frac{i}{2^3 3^{1/2}} \int_0^{i\infty} \hat{g}_1\left(-\frac{1}{12z}\right) D^{-2} g_2(z) z^{-3} dz$$

We now make use of quadruple sums. For that we write our form \hat{g}_1 and g_2 as:

$$\begin{aligned}\hat{g}_1(z) &= \sum_{m_1, n_1 \geq 1} a_1(m_1)b_1(n_1)n_1^2 q^{m_1 n_1} = q + 5q^2 + 9q^3 + 11q^4 + 24q^5 + \dots \\ g_2(z) &= \sum_{m_2, n_2 \geq 1} a_2(m_2)b_2(n_2)n_2^2 q^{m_2 n_2} = -q - 4q^2 - q^3 + 16q^4 + 24q^5 + \dots \\ a_1(m) &= \chi_{-3}(m) \\ b_1(n) &= 1 + \frac{1}{2}[n \text{ even}] - \frac{1}{2}[n \text{ divisible by } 4] \\ a_2(m) &= -1 - 7[m \text{ even}] + 8[m \text{ divisible by } 4] \\ b_2(n) &= \chi_{-3}(n)\end{aligned}$$

We obtain a quadruple sum by plugging in the expansions

$$\begin{aligned}\hat{g}_1\left(-\frac{1}{12z}\right) &= \sum_{m_1, n_1 \geq 1} a_1(m_1)b_1(n_1)n_1^2 \exp\left(-\frac{2\pi i n_1 m_1}{12z}\right) \\ D^{-2}g_2(z) &= \sum_{m_2, n_2 \geq 1} a_2(m_2)b_2(n_2)\frac{1}{m_2^2} \exp(2\pi i m_2 n_2 z)\end{aligned}$$

into (7.4):

$$\begin{aligned}& -\frac{i}{2^3 3^{1/2}} \int_0^{i\infty} \hat{g}_1\left(-\frac{1}{12z}\right) D^{-2}g_2(z) z^{-3} dz \\ &= -\frac{i}{2^3 3^{1/2}} \sum_{m_1, n_1, m_2, n_2} a_1(m_1)b_1(n_1)a_2(m_2)b_2(n_2)\frac{n_1^2}{m_2^2} \int_0^{i\infty} \exp\left(2\pi i\left(-\frac{m_1 n_1}{12z} + m_2 n_2 z\right)\right) z^{-3} dz.\end{aligned}$$

Then we change variables. First we let $w = -\frac{1}{12z}$ and obtain:

$$-\frac{12^2 i}{2^3 3^{1/2}} \sum_{m_1, n_1, m_2, n_2} a_1(m_1)b_1(n_1)a_2(m_2)b_2(n_2)\frac{n_1^2}{m_2^2} \int_0^{i\infty} \exp\left(2\pi i\left(m_1 n_1 w - \frac{m_2 n_2}{12w}\right)\right) w dw$$

and then $u = \frac{n_1 w}{m_2}$:

$$-2 \cdot 3^{3/2} i \sum_{m_1, n_1, m_2, n_2} a_1(m_1)b_1(n_1)a_2(m_2)b_2(n_2) \int_0^{i\infty} \exp\left(2\pi i\left(m_1 m_2 u - \frac{n_1 n_2}{12u}\right)\right) u du$$

which we collapse to

$$(7.5) \quad -2 \cdot 3^{3/2} i \int_0^{i\infty} f_1(u)\left(f_2\left(-\frac{1}{12u}\right) - \frac{1}{6}\right) u du$$

where

$$\begin{aligned}f_1(z) &= \sum_{m_1, m_2 \geq 1} a_1(m_1)a_2(m_2)q^{m_1 m_2} \\ f_2(z) &= \frac{1}{6} + \sum_{n_1, n_2 \geq 1} b_1(n_1)b_2(n_2)q^{n_1 n_2}\end{aligned}$$

(the term $\frac{1}{6}$ makes $f_2(z)$ a modular form.)

Let

$$E_1(z) = \frac{1}{6} + \sum_{m, n \geq 1} \chi_{-3}(m)q^{nm},$$

then

$$\sum_{m, n \geq 1} \chi_{-3}(m)[n \text{ divisible by } l]q^{nm} = E_1(lz) - \frac{1}{6},$$

so that

$$\begin{aligned} f_1(z) &= -E_1(z) - 7E_1(2z) + 8E_1(4z) = -q - 7q^2 - q^3 + 7q^4 - 7q^6 - 2q^7 + \dots \\ f_2(z) &= E_1(z) + \frac{1}{2}E_1(2z) - \frac{1}{2}E_1(4z) = \frac{1}{6} + q + 1/2q^2 + q^3 + 1/2q^4 + 1/2q^6 + 2q^7 + \dots \end{aligned}$$

In order to evaluate (7.5) we apply Atkin-Lehner involution to f_2 . We first express f_2 as a combination of eta-products:

$$f_2(z) = \frac{1}{2} \frac{\eta(4z)^2 \eta(12z)^2}{\eta(2z) \eta(6z)} + \frac{1}{6} \frac{\eta(2z)^6 \eta(3z)}{\eta(z)^3 \eta(6z)^2}$$

and then use (7.2):

$$\begin{aligned} f_2\left(-\frac{1}{12z}\right) &= \frac{1}{2} \left(\frac{3^2}{6 \cdot 2}\right)^{1/2} (-iz) \frac{\eta(3z)^2 \eta(z)^2}{\eta(6z) \eta(2z)} + \frac{1}{6} \left(\frac{6^6 4}{12^3 2^2}\right)^{1/2} (-iz) \frac{\eta(6z)^6 \eta(4z)}{\eta(12z)^3 \eta(2z)^2} \\ &= -\frac{3^{1/2}}{2} iz \hat{f}_2(z), \\ \hat{f}_2(z) &= \frac{1}{2} \frac{\eta(3z)^2 \eta(z)^2}{\eta(6z) \eta(2z)} + \frac{\eta(6z)^6 \eta(4z)}{\eta(12z)^3 \eta(2z)^2} = -E_1(z) + 2E_1(2z) + 8E_1(4z) \\ &= 3/2 - q + 2q^2 - q^3 + 7q^4 + 2q^6 + \dots \end{aligned}$$

We continue rewriting the integral (7.5) using

$$\begin{aligned} \int_0^{i\infty} f(z) z^{k-1} dz &= \frac{(k-1)!}{(-2\pi i)^k} L(f, k) \\ \dots &= 3^{1/2} i \int_0^{i\infty} f_1(u) u du - 3^2 \int_0^{i\infty} f_1(u) \hat{f}_2(u) u^2 du \\ &= \frac{3^{1/2} i}{(-2\pi i)^2} L(f_1, 2) - \frac{3^2 \cdot 2}{(-2\pi i)^3} L(f_1(u) \hat{f}_2(u), 3) \\ &= -\frac{3^{1/2} i}{4\pi^2} L(f_1, 2) + \frac{3^2 i}{2^2 \pi^3} L(f_1(u) \hat{f}_2(u), 3) \end{aligned}$$

We evaluate the L -values that have popped up:

$$\begin{aligned} f_1(u) &= -E_1(z) - 7E_1(2z) + 8E_1(4z) \\ L(f_1(u), s) &= (-1 - 7 \cdot 2^{-s} + 8 \cdot 4^{-s}) \zeta(s) L(\chi_{-3}, s) \\ L(f_1(u), 2) &= (-1 - \frac{7}{2^2} + \frac{8}{4^2}) \zeta(2) L(\chi_{-3}, 2) = -\frac{3}{8} \pi^2 L(\chi_{-3}, 2) \end{aligned}$$

and

$$\begin{aligned} f_1(u) \hat{f}_2(u) &= -\frac{3}{2} G_2(z) - 5G_2(2z) + \frac{19}{2} G_2(3z) + 24G_2(4z) - 35G_2(6z) + 8G_2(12z) \\ L(f_1(u) \hat{f}_2(u), s) &= \left(-\frac{3}{2} - 5 \cdot 2^{-s} + \frac{19}{2} \cdot 3^{-s} + 24 \cdot 4^{-s} - 35 \cdot 6^{-s} + 8 \cdot 12^{-s}\right) \zeta(s) \zeta(s-1) \\ L(f_1(u) \hat{f}_2(u), 3) &= \left(-\frac{3}{2} - \frac{5}{2^3} + \frac{19}{2 \cdot 3^3} + \frac{24}{4^3} - \frac{35}{6^3} + \frac{8}{12^3}\right) \zeta(3) \zeta(2) \\ &= -\frac{14}{9} \zeta(3) \zeta(2) = -\frac{7}{27} \pi^2 \zeta(3) \end{aligned}$$

Eventually we finish the computation of the integrals in (7.1):

$$\int_0^{i\infty} g_1(z) \cdot D^{-2} g_2(z) dz = \frac{3^{3/2} i}{2^5} L(\chi_{-3}, 2) - \frac{7i}{12\pi} \zeta(3),$$

and multiplying by $-2\pi i$ we get

$$D^{-1}(g_1(q) D^{-2} g_2(q)) \Big|_{q=1} = \frac{3\sqrt{3}\pi}{2^4} L(\chi_{-3}, 2) - \frac{7}{6} \zeta(3)$$

which is the same as (6.4).

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