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H-PRINCIPLES FOR THE INCOMPRESSIBLE EULER EQUATIONS

A. CHOFFRUT

Abstract. In [DLS12b], De Lellis and Székelyhidi construct Hölder continuous, dissipative (weak) solutions to the incompressible Euler equations in the torus $\mathbb{T}^3$. The construction consists in adding fast oscillations to the trivial solution. We extend this result by establishing optimal h-principles in two and three space dimensions. Specifically, we identify all subsolutions (defined in a suitable sense) which can be approximated in the H^{-1}-norm by exact solutions. Furthermore, we prove that the flows thus constructed on $\mathbb{T}^3$ are genuinely three-dimensional and are not trivially obtained from solutions on $\mathbb{T}^2$.

1. Introduction

1.1. Incompressible Euler equations and h-principle.

We consider the **(incompressible) Euler equations**

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p = 0, \qquad \operatorname{div} v = 0 \tag{1}$$

on the torus $\mathbb{T}^d$, $d = 2$ or 3. Here, v is the velocity vector field and the pressure p enforces the divergence-free condition. If (v, p) is a *classical* solution to (1), scalar multiplication with v and the chain rule give $\partial_t \frac{|v|^2}{2} + \operatorname{div}_x \left(\left(\frac{|v|^2}{2} + p \right) v \right) = 0$. Integrating in space shows that classical solutions to the incompressible Euler equations conserve the total kinetic energy:

$$\frac{d}{dt} \int_{\mathbb{T}^d} |v|^2(x,t)\, dx = 0$$

Anomalous dissipation. The existence of *weak* solutions violating the conservation of kinetic energy was first suggested in [Ons49] by Onsager, where indeed he conjectured the existence of Hölder continuous solutions in 3 space dimensions with any exponent smaller than $\frac{1}{3}$. Onsager also asserted that such solutions do not exist if we impose the Hölder continuity with exponent larger than $\frac{1}{3}$ and this part of his conjecture was proved in [Eyi94] and [CET94]. The considerations of Onsager are motivated by the Kolmogorov theory of isotropic 3-dimensional turbulence, where the phenomenon of anomalous dissipation in the Navier-Stokes equations is postulated. This assumption seems to be widely confirmed experimentally, whereas no such phenomenon is observed in 2 dimensions. Indeed, for $d = 2$ the conservation law for the enstrophy does prevent it for solutions which start from sufficiently smooth initial data. However, the considerations put forward by Onsager which

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pertain to the mathematical structure of the equations do not depend on the dimension and this independence appears clearly also in the proof of [CET94], which works for any $d \geq 2$.

The first proof of the existence of a weak solution violating the energy conservation was given in the groundbreaking work of Scheffer [Sch93], which showed the existence of a compactly supported nontrivial weak solution in $\mathbb{R}^2 \times \mathbb{R}$. A different construction of the existence of a compactly supported nontrivial weak solution in $\mathbb{T}^2 \times \mathbb{R}$ was then given by Shnirelman in [Shn97]. In both cases the solutions are only square summable as a function of both space and time variables. The first proof of the existence of a solution for which the total kinetic energy is a monotone decreasing function has been given by Shnirelman in [Shn00]. Shnirelman's example is in the energy space $L^\infty([0,\infty), L^2(\mathbb{R}^3))$.

In [DLS09, DLS10] these existence results were extended to solutions with bounded velocity and pressure and in any space dimensions. The same methods were also used to give quite severe counterexamples to the uniqueness of admissible solutions, both for incompressible and compressible Euler. Further developments in fluid dynamics inspired by these works appeared subsequently in [Chi12, CFG11, Shv11, Szé11, SW11, Wie11] and are surveyed in the note [DLS12c]. In [DLS12a, DLS12b], De Lellis and Székelyhidi devised a new iteration scheme , which produces continuous and even Hölder continuous solutions on $\mathbb{T}^3$. Furthermore, one may prescribe the **total kinetic energy** profile

$$\fint_{\mathbb{T}^d} |v(x,t)|^2 \, dx = e(t)$$

where $d = 3$, and $\fint_{\mathbb{T}^d} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d}$. (For notational convenience we omit the usual factor 1/2 and average over the domain.)

Solutions of class C^1 are therefore "rigid" compared to less regular solutions. In fact, the paper [DLS09] introduced a new point of view in the subject, highlighting connections to other counterintuitive solutions of (mainly geometric) systems of partial differential equations: in geometry these solutions are, according to Gromov, instances of the h-principle, the prime example of which is Nash's theorem on C^1 isometric embeddings [Nas54]. (See in [CDLS11] an earlier discussion on the striking similarities between Onsager's conjecture and the rigidity and flexibility properties of the isometric problem.) We recall that an embedding $u_0 \colon M^n \to \mathbb{R}^N$, $N > n$ is said to be (strictly) short if $\partial_i u_0 \cdot \partial_j u_0 < g_{ij}$ where g is a prescribed Riemannian metric. Nash (and Kuiper) proved that any strictly short embedding can be uniformly approximated by an isometric embedding $u \in C^1(M; \mathbb{R}^N)$, $\partial_i u \cdot \partial_j u = g_{ij}$, in the sense that $\|u_0 - u\|_{C^0}$ can be made arbitrarily small. For the isometric problem, the h-principle is the statement that u_0 can be deformed into u *via* a *homotopy* (hence the name). In the sequel we will leave this aspect of the h-principle aside and view the h-principle as a density statement.

The main idea in [Nas54] is to add fast oscillations in order to increase the metric induced by a short embedding u_0 and thereby reducing the defect $g_{ij} - \partial_i u_0 \cdot \partial_j u_0$. Thus, u_0 is

taken closer to the boundary of the set short embeddings, precisely made up of isometric embeddings. Nash's idea has been further developped by Kuiper, Gromov, and others, and falls nowadays under the name of **convex integration**, see [DLS12c, EM02, Gro86, Spr98].

If convex integration alone produces C^1 isometric embeddings, refinements can achieve $C^{1,\alpha}$ regularity for certain α depending on n and N, see [Bor65, Bor04, CDLS11] for precise statements and references therein. For the Euler equations, the natural space for convex integration is C^0. The method used in [DLS09] producing solutions in L^∞ was a weak form of convex integration. The iteration scheme of [DLS12a] is closer to the approach of [Nas54], see the introduction of [DLS12a] for a thorough discussion. Finally, [DLS12b], with the improved regularity for Euler, parallels [CDLS11] for the isometric problem.

In this article we establish h-principles for the Euler equations in 2 and 3 space dimensions, using the convex integration procedure developped in [DLS12a] and sophisticated in [DLS12b]. We shall first motivate our definition of subsolutions to the Euler equations, analogous to the short embeddings of Nash for the isometric problem, and the notion of the h-principle in use here. It is generally accepted that the onset of turbulence in incompressible fluids is due to the appearance of high-frequency oscillations in the velocity field [DLS12c, Fri95, Maj91]. For example, if (v_ν, p_ν, f_ν) is a sequence of approximate solutions,

$$\partial_t v_\nu + \operatorname{div}(v_\nu \otimes v_\nu) + \nabla p_\nu = f_\nu, \qquad \operatorname{div} v_\nu = 0,$$

with uniformly bounded kinetic energy, and converges weakly to $(v, \pi, 0)$, then in general

$$\partial_t v + \operatorname{div}(v \otimes v + R) + \nabla \pi = 0, \qquad \operatorname{div} v = 0 \tag{2}$$

in the weak sense, where R is a symmetric, positive semi-definite matrix, called the **Reynolds stress tensor**. It appears because the operation of taking weak limits does not commute with the nonlinear operator $\otimes$. A strategy to construct an exact solution to the Euler equations (1) is then to reintroduce the oscillations so as to eliminate R on average. A crucial point in the construction of [DLS12a] is therefore the ability to generate the tensor $R \geq 0$ with a fast oscillating perturbation W (see Lemma 7 and Section 3.5 for details): we seek a velocity field W solving the stationary Euler equations and satisfying

$$\fint_{\mathbb{T}^d} W \otimes W \, d\xi = R.$$

In three dimensions, this is done using Beltrami flows. However, these flows seem to be insufficient to capture all possible oscillatory behaviors in the Euler equations, see Proposition 5, where it is also shown that this problem does not exist in two dimensions.

Remark Beltrami flows are defined as those three-dimensional flows of the form $\operatorname{curl} v(x) = \lambda(x) v(x)$ for some scalar function $\lambda(x)$. These are stationary flows, see [MB02]. There is a connection with two-dimensional stationary flows, see Proposition 2.11 in [MB02]. For these flows however, the function λ is in general not constant. In the construction of [DLS12a],

on the other hand, the function $\lambda(x) = \lambda$ is constant. ■

With these general considerations being done, we now turn to precise definitions. It will be more convenient to work with an alternative form of (2). Letting $\mathring{R}$ be (minus) the trace-free part of R, $R = \frac{\operatorname{tr} R}{d}\operatorname{Id} - \mathring{R}$, then (2) becomes

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \operatorname{div} \mathring{R}, \qquad \operatorname{div} v = 0 \tag{3}$$

where $p = \pi + \frac{\operatorname{tr} R}{d}$. We shall refer to (3) as the **Euler-Reynolds system**. It is equivalent to (2) provided one fixes $\operatorname{tr} R$. (Indeed, if (v, R, π) solves (2), then so does $(v, R + f\operatorname{Id}, \pi - f)$ for any function f.)

We emphasize that the notion of short embedding for the isometric problem is relative to a prescribed metric g. In the context of ideal hydrodynamics, a natural quantity to prescribe is the kinetic energy $e(t)$. We shall say that $(v, \mathring{R}, p)$ is a **strict subsolution** to the Euler equations (relative to the kinetic energy $e(t)$) if $(v, \mathring{R}, p)$ solves the Euler-Reynolds system (in the classical sense), where $\mathring{R}$ is trace-free, and if

$$\frac{e - \fint_{\mathbb{T}^d} |v(x', t)|^2 \, dx'}{d}\operatorname{Id} - \mathring{R}(x, t) > 0, \qquad x \in \mathbb{T}^d, \quad t \in [0, T]. \tag{4}$$

This amounts to fixing $\operatorname{tr} R = e(t) - \fint_{\mathbb{T}^d} |v(x, t)|^2 \, dx$ and imposing that $R > 0$ in the (v, R, π) formulation. In particular we have

$$e > \fint_{\mathbb{T}^d} |v|^2.$$

As for the isometric problem, the boundary of the set of subsolutions consists of exact solutions to the Euler equations with prescribed kinetic energy.

The main focus [DLS12a, DLS12b] is the construction of *some* solutions with a certain amount of regularity, and thus used the particular (trivial) subsolution $(0, 0, 0)$. Their Geometric Lemma (Lemma 3.2 in [DLS12a]) was sufficient for this purpose. Here, we prove an optimal Geometric Lemma, see Lemma 7, and identify the largest class of subsolutions for which the convex integration scheme of [DLS12a] produces an exact solution to (1). A subsolution $(v, \mathring{R}, p)$ is **strong** if it satisfies the condition, stronger than (4), that

$$\frac{e(t) - \fint_{\mathbb{T}^d} |v_0(x', t)|^2 \, dx'}{d(d-1)}\operatorname{Id} + \mathring{R}_0(x, t) > 0, \qquad x \in \mathbb{T}^d, \quad t \in [0, T]. \tag{5}$$

(Equivalently, $\frac{e(t) - \fint_{\mathbb{T}^d} |v_0(x', t)|^2 \, dx'}{d}\operatorname{Id} - \mathring{R}_0(x, t) \in \mathcal{M}_d$, see Section 2.1 for definitions.)

We say that the **h-principle** holds for (1) if, given $\sigma > 0$ and a strict subsolution $(v_0, \mathring{R}_0, p_0)$ relative to $e(t)$, there exists an exact solution (v, p) with $\fint_{\mathbb{T}^d} |v(x, t)|^2 \, dx = e(t)$ and such that $\|v - v_0\|_{H^{-1}(\mathbb{T}^d)} < \sigma$. The h-principle holds **for strong subsolutions** if $(v_0, \mathring{R}_0, p_0)$ is required to be strong.

Remark Another possible notion of subsolutions is to fix a function $\overline{e} = \overline{e}(x, t)$ and impose the pointwise condition that $\frac{\overline{e}(x,t) - |v(x,t)|^2}{d}\operatorname{Id} - \mathring{R}(x, t) > 0$. This is the notion used

in [DLS09] in the context of L^∞-solutions. The two notions are different, and one does not imply the other. The pointwise notion seems ill suited for the construction in use here. Indeed, the pointwise control on the velocity field along the iteration seems insufficient.

Genuinely 3D flows. There is a trivial way to produce flows on $\mathbb{T}^3$ from flows on $\mathbb{T}^2$. As a consequence of precise estimates of our main result, Theorem 1, we show that the flows obtained for $d = 3$ are genuinely three-dimensional and do not coincide with those obtained for $d = 2$, see Corollary 3. In order to formulate our statement precisely, consider a solution (v, p) to the Euler equations (1) on $\mathbb{R}^3 \times [0, T]$ and denote with the same letters the corresponding solution on $\mathbb{R}^3 \times [0, T]$ with the obvious periodicity in space. We say that a solution is **not genuinely three-dimensional** if, after suitably changing coordinates in space, it takes the form

$$v(x,t) = (v_1(x_1, x_2, t), v_2(x_1, x_2, t), v_3) \tag{6}$$

where v_3 is a constant. Otherwise it is **genuinely three-dimensional**.

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1.2. **Statement of results.** In [DLS12a] and [DLS12b], solutions to the Euler equations with prescribed kinetic energy were constructed using convex integration starting from the trivial subsolution $(v_0, \mathring{R}_0, p_0) = (0, 0, 0)$ on $\mathbb{T}^3$. Since the building blocks are a certain class of Beltrami flows, which are inherently three-dimensional, it is not immediately clear whether the method should work in other space dimensions. In our main result, Theorem 1, we establish the largest set of subsolutions for which the h-principle holds, in dimensions two and three. It is based on a refined Geometric Lemma, see Proposition 5 and Lemma 7.

Theorem 1 (h-principle)**.** *Assume $d = 2$ or 3. Let $e(t)$, $t \in [0, T]$, be smooth, positive. Let $(v_0, \mathring{R}_0, p_0)$ be a strong subsolution. Let $0 < \theta < \frac{1}{10}$ and $\sigma > 0$. Then:*

(1) *there exists a vector field $v \in C^0(\mathbb{T}^d \times [0, T])$ and a function $p \in C^0(\mathbb{T}^d \times [0, T])$ which solve the Euler equations (1) (in the weak sense) and satisfy*

$$|v(x,t) - v(x',t)| \le C|x - x'|^\theta \qquad x, x' \in \mathbb{T}^d, \qquad t \in [0, T]$$

and

$$\sup_{t\in[0,T]} \|v(\cdot,t) - v_0(\cdot,t)\|_{H^{-1}(\mathbb{T}^d)} < \sigma; \tag{7}$$

(2) *the solution can be constructed so that, for all $t \in [0, T]$,*

$$\left| \fint_{\mathbb{T}^d} \Big(v(x,t) \otimes v(x,t) - v_0(x,t) \otimes v_0(x,t) + \mathring{R}_0(x,t) \Big)\, dx - \frac{e(t) - \fint_{\mathbb{T}^2} |v_0(x,t)|^2\, dx}{d} \mathrm{Id} \right| < \sigma. \tag{8}$$

Remark As in [DLS12b], the proof of Theorem 1 yields further regularity on both v and p. Namely, they are Hölder continuous in both x and t with

$$|v(x,t) - v(x',t')| \leq C \left(|x-x'|^\theta + |t-t'|^\theta \right)$$

and

$$|p(x,t) - p(x',t')| \leq C \left(|x-x'|^{2\theta} + |t-t'|^{2\theta} \right).$$

■

Corollary 2. *If $d = 2$, then the h-principle holds for strict subsolutions.*

Corollary 3 (Genuine 3D flows)**.** *Assume $d = 3$ in Theorem 1. Then the flows are genuinely three-dimensional provided σ is chosen sufficiently small.*

2. Proof of Theorem 1, part 1

2.1. **Notation.** *Spaces of symmetric matrices.* All matrices in this article will be symmetric, and thus the qualifier "symmetric" will often be omitted. We shall denote by

$$\mathcal{S}^{d\times d} = \left\{ M \in \mathbb{R}^{d\times d} \,:\, M^\top = M \right\}$$

the set of symmetric $d \times d$ matrices, by

$$\mathcal{S}^{d\times d}_{++} = \left\{ M \in \mathcal{S}^{d\times d} \,:\, M > 0 \right\}$$

the open convex cone of (symmetric) positive definite matrices, and by

$$\mathcal{S}^{d\times d}_{0} = \left\{ M \in \mathcal{S}^{d\times d} \,:\, \operatorname{tr} M = 0 \right\}$$

the closed linear subset of $\mathcal{S}^{d\times d}$ of trace-free matrices. We also introduce

$$\mathbb{M}_d := \left\{ \mathrm{Id} - b \otimes b \mid b \in \mathbb{S}^{d-1} \right\}$$

where $\mathbb{S}^{d-1} = \left\{ b \in \mathbb{R}^d \mid |b| = 1 \right\}$ denotes the $(d-1)$-dimensional sphere. Of interest will be the open subset

$$\mathcal{M}_d := \operatorname{int} \mathbb{M}_d^{\mathrm{conic}} \subset \mathcal{S}^{d\times d}_{++}$$

where $\mathbb{M}_d^{\mathrm{conic}}$ denotes the **conic hull** of $\mathbb{M}_d$, that is, the set of all matrices of the form

$$\sum_{i=1}^m \alpha_i \left(\mathrm{Id} - b_i \otimes b_i \right), \qquad \text{where} \quad \alpha_i > 0 \quad \text{and} \quad |b_i| = 1.$$

The norm on these spaces will be the operator norm.

$B_{r_o}(\mathrm{Id})$ and the parameter $\overline{r}_0$. For $r_0 > 0$, $B_{r_0}(\mathrm{Id})$ will always denote the open ball in $\mathcal{S}^{d\times d}$. By Proposition 5, we fix $\overline{r}_0 > 0$ sufficiently small so that

$$\overline{B_{2\overline{r}_0}(\mathrm{Id})} \subset \mathcal{M}_d. \tag{9}$$

Hölder norms. For a time-independent function $f = f(x)$, the sup-norm is denoted $\|f\|_0 = \sup_{\mathbb{T}^d} |f|$, and the Hölder seminorms are given by

$$[f]_m := \max_{|\gamma|=m} \|D^\gamma f\|_0, \qquad [f]_{m+\alpha} := \max_{|\gamma|=m} \sup_{x\neq y} \frac{|D^\gamma f(x) - D^\gamma f(y)|}{|x-y|^\alpha}$$

and the Hölder norms are given by

$$\|f\|_m := \sum_{j=0}^{m} [f]_j\,, \qquad \|f\|_{m+\alpha} := \|f\|_m + [f]_{m+\alpha}\,.$$

For a time-dependent function $f = f(x,t)$, and $r \geq 0$, $\|f\|_r$ will denote the "Hölder norm in space", that is

$$\|f\|_r = \sup_{t\in[0,T]} \|f(\cdot,t)\|_r\,.$$

while the Hölder norms in space and time will be denoted by $\|\cdot\|_{C^r}$.

Constants. We will follow [DLS12b] for the convention pertaining to the constants involved in the estimates of Section 4 and the Appendix.

- C: will denote universal constants.
- C_h: will denote constants in estimates concerning standard functional inequalities in Hölder spaces C^r. These constants depend only on the specific norm used and therefore only on the parameter $r \geq 0$.
- C_e: throughout the rest of the paper the prescribed energy will be assumed to be a fixed smooth function bounded below by a positive function. Several estimates depend on these bounds and the relate constants will be denoted C_e.
- C_v: in addition to the dependence on e, there will be estimates which depend also on $\|v\|_0$. See the constant A in Proposition 4.
- C_s, $C_{e,s}$, $C_{v,s}$: will denote constants which are typically involved in Schauder estimates for $C^{m+\alpha}$ norms of elliptic operators, when $m \in \mathbb{N}$ and $0 < \alpha < 1$. These constants not only depend on the specific norm used, but they also degenerate as $\alpha \downarrow 0$ and $\alpha \uparrow 1$. The ones denoted by $C_{e,s}$ and $C_{v,s}$ depend also, respectively, on e and e and $\|v\|_0$.

We emphasize that constants never depend on the parameters $\mu, \ell, \delta, \lambda$ and D, although they may depend on ε (see Section 4 and Appendix for definitions of these parameters).

2.2. **The iterative scheme: Proposition 4.** In order to motivate the main Proposition of this Section, we briefly sketch the strategy to construct exact solutions to the Euler equations (1). Given a strict subsolution $(v, \mathring{R}, p)$, *i.e.*

$$\partial_t v + \operatorname{div}(v\otimes v) + \nabla p = \operatorname{div}\mathring{R}, \qquad \operatorname{div} v = 0$$

and

$$\frac{e(t) - \fint_{\mathbb{T}^d} |v(x,t)|^2\,dx}{d}\,\mathrm{Id} - \mathring{R} > 0,$$

we construct a triple $(v_1, \mathring{R}_1, p_1)$ which is closer to being a solution, in the sense that the energy gap $e(t) - \fint_{\mathbb{T}^d} |v_1(x,t)|^2\,dx$ and the trace-free tensor $\mathring{R}_1$ are both smaller, while $\frac{e(t)-\fint_{\mathbb{T}^d}|v_1(x,t)|^2\,dx}{d}\,\mathrm{Id} - \mathring{R}_1$ remains positive definite. An iteration is needed since $e(t) - \fint_{\mathbb{T}^d} |v_1(x,t)|^2\,dx$ and $\mathring{R}_1$ cannot be made to vanish exactly. Yet the iteration converges because this can be done with arbitrary accuracy.

Proposition 4. *Suppose $d = 2$ or 3, and fix $\overline{r}_0 > 0$ as in (9). Let $K \subset \mathcal{M}_d$ be compact and contain $B_{2\overline{r}_0}(\mathrm{Id})$, and let $\mathcal{N}$ be an open neighborhood of K such that $\overline{\mathcal{N}} \subset \mathcal{M}_d$. Fix $e_0 \geq 0$, $\Delta_0 > 0$, and $\tilde{r}_0 = \tilde{r}_0(d, K, e_0, \Delta_0)$ as in Lemma 12.*

Fix now $r_0 \leq \min\{\overline{r}_0, \tilde{r}_0\}$ and set

$$\eta = \frac{\min \Delta_0}{4d} r_0. \tag{10}$$

Then, there exists $M = M(e_0, \Delta_0)$ with the following properties.

Let $\varepsilon > 0$ and $0 < \zeta \leq \frac{1}{2}$. Suppose $0 < \delta \leq 1$ and $(v, \mathring{R}, p)$ satisfy

$$\partial_t v + \operatorname{div}(v \otimes v) + \nabla p = \operatorname{div} \mathring{R}, \qquad \operatorname{div} v = 0,$$

$$\left| e_0(t) + \Delta_0(t)(1-\delta) - \fint_{\mathbb{T}^d} |v(x,t)|^2\,dx \right| \leq \zeta/2\delta\Delta_0(t), \tag{11}$$

and, posing $\overline{\delta} := \zeta\delta^{\frac{3}{2}}$, that

$$\mathrm{Id} - \frac{d}{e_0(t) + (1-\overline{\delta})\Delta_0(t) - \fint_{\mathbb{T}^d} |v(x,t)|^2\,dx} \mathring{R}(x,t) \in K, \qquad x \in \mathbb{T}^d, \quad t \in \mathbb{S}^1. \tag{12}$$

Set $D := \max\left\{1, \|v\|_1, \|\mathring{R}\|_1\right\}$.

Then there exists $(v_1, \mathring{R}_1, p_1)$ satisfying

$$\partial_t v_1 + \operatorname{div}(v_1 \otimes v_1) + \nabla p_1 = \operatorname{div} \mathring{R}_1, \qquad \operatorname{div} v_1 = 0$$

and such that

$$\left| e_0(t) + \Delta_0(t)(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v_1(x,t)|^2\,dx \right| \leq \zeta/2\overline{\delta}\Delta_0(t), \tag{13}$$

$$\|\mathring{R}_1\|_0 \leq \eta\overline{\delta}, \tag{14}$$

$$\|v_1 - v\|_0 \leq M\sqrt{\delta}, \tag{15}$$

$$\sup_t \|v_1(\cdot,t) - v(\cdot,t)\|_{H^{-1}(\mathbb{T}^d)} \leq r_0\delta^{\frac{1}{4}}, \tag{16}$$

$$\|p_1 - p\|_0 \leq M^2\delta, \tag{17}$$

$$\max\left\{1, \|v_1\|_1, \|\mathring{R}_1\|_1\right\} \leq A\delta^{\frac{3}{2}}\left(\frac{D}{\overline{\delta}^2}\right)^{1+\varepsilon} \tag{18}$$

where the constant A depends on $d, e, \varepsilon > 0$ and $\|v\|_0$, see (69).

Remark The conclusions imply that

$$\mathrm{Id} - \frac{d}{e_0(t) + (1-\overline{\overline{\delta}})\Delta_0(t) - \fint_{\mathbb{T}^d} |v_1(x,t)|^2\,dx}\mathring{R}_1(x,t) \in B_{r_0}(\mathrm{Id}) \subset K, \qquad x \in \mathbb{T}^d, \quad t \in \mathbb{S}^1 \tag{19}$$

where $\overline{\overline{\delta}} = \zeta\overline{\delta}^{\frac{3}{2}}$. Indeed,

$$e_0 + \Delta_0(1-\overline{\overline{\delta}}) - \fint_{\mathbb{T}^d} |v_1|^2 = \Delta_0(\overline{\delta} - \overline{\overline{\delta}}) + e_0 + \Delta_0(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v_1|^2 \geq \frac{\Delta_0\overline{\delta}}{4}$$

so that $\left\| \frac{d}{e_0+\Delta_0(1-\overline{\overline{\delta}})-\fint_{\mathbb{T}^d}|v_1(x,t)|^2\,dx}\mathring{R}_1 \right\| \leq r_0$. Therefore, an iteration can be carried out by repeated use of Proposition 4. ■

2.3. **Proof of Theorem 1, part 1.** Assume Proposition 4 is proved.

Periodicity in t. In this paragraph we show that we may assume, without loss of generality, that $v_0, \mathring{R}_0, p_0$, and e are periodic in t. (Although this is not necessary for the construction, this feature will prove to be convenient as mollification in space *and* time is used in the estimates, see Section 3.5). Let's then start with a strong subsolution $(v_0, \mathring{R}_0, p_0)$ defined for $x \in \mathbb{T}^d$ and $t \in [0,T]$ relative to $e(t)$ which is a smooth positive function defined for $t \in [0,T]$. It is standard that $v_0(x,t), p_0(x,t)$ can be extended to smooth functions for $x \in \mathbb{T}^d$ and $t \in \mathbb{R}$ which vanish for $t \geq \frac{3}{2}T$ and $t \leq -\frac{T}{2}$, and such that $\operatorname{div} v_0 = 0$ and $\fint_{\mathbb{T}^d} v_0(x,t)\,dx = 0$ for all $t \in \mathbb{R}$. See for instance the proof of Corollary 1.3.7, p. 138, Part II of [Ham82]. We may then repeat v_0 and p_0 periodically in t with period $2T$. We define

$$\mathring{R}_0 := \mathcal{R}\left(\partial_t v_0 + \operatorname{div}(v_0 \otimes v_0) + \nabla p_0\right)$$

for $t \notin [0,T]$, see Definition 9 for the operator $\mathcal{R}$. Since the argument of the right-hand side has average 0 over $\mathbb{T}^d$, the triple $(v_0, \mathring{R}_0, p_0)$ solves the Euler-Reynolds system and is periodic in t, see Lemma 10. Finally, it is clear that $e(t)$ can be extended to a smooth, positive, periodic functions for $t \in \mathbb{R}$ with period $2T$ such that

$$\frac{e(t) - \fint_{\mathbb{T}^d} |v_0(x',t)|^2\,dx'}{d}\,\mathrm{Id} - \mathring{R}_0(x,t) \in \mathcal{M}_d, \qquad x \in \mathbb{T}^d, \quad t \in \mathbb{R}.$$

Rescaling in t, we may assume that $v_0, \mathring{R}_0, p_0$, and e have period 2π.

Setting parameters. Set

$$e_0(t) := \fint_{\mathbb{T}^d} |v_0(x,t)|^2\,dx, \qquad \Delta_0(t) := e(t) - e_0(t).$$

We may then choose $0 < \zeta \leq \frac{1}{2}$ such that

$$\frac{(1-\zeta)\Delta_0(t)}{d}\mathrm{Id} - \mathring{R}_0(x,t) \in \mathcal{M}_d, \qquad x \in \mathbb{T}^d, \quad t \in \mathbb{R}. \tag{20}$$

Set

$$K := \left\{ \mathrm{Id} - \frac{d}{(1-\zeta)\Delta_0(t)} \mathring{R}_0(x,t) \,:\, x \in \mathbb{T}^d, t \in [0,T] \right\} \cup \overline{B_{2\overline{r}_0}(\mathrm{Id})},$$

where $\overline{r}_0$ is as in (9). Fix an open neighborhood $\mathcal{N}$ of K such that $\overline{\mathcal{N}} \subset \mathcal{M}_d$. Fix $\tilde{r}_0$ as in Lemma 12 and let $r_0 \leq \min\{\overline{r}_0, \tilde{r}_0\}$ to be specified later, see Section 6.

Define inductively

$$\delta_{n+1} = \zeta \delta_n^{\frac{3}{2}}, \qquad \delta_0 = 1.$$

Fix $\varepsilon > 0$ and $\sigma > 0$.

The iterates. Use Proposition 4 inductively to construct a sequence $(v_n, \mathring{R}_n, p_n)$ with $\delta = \delta_n$, $\overline{\delta} = \delta_{n+1}$. Since $(v_0, \mathring{R}_0, p_0)$ clearly satisfies the assumptions of Proposition 4, the remark following Proposition 4 shows that $\mathrm{Id} - \frac{d}{e_0 + \Delta_0(1-\delta_{n+1}) - \fint_{\mathbb{T}^d} |v_n|^2} \mathring{R}_n \in K$ for each n. The sequence satisfies

$$\left| e_0(t) + \Delta_0(t)(1-\delta_n) - \fint_{\mathbb{T}^d} |v_n(x,t)|^2 \, dx \right| \leq \zeta/2 \delta_n \Delta_0(t), \tag{21}$$

for $n \geq 0$, and for $n \geq 1$

$$\|\mathring{R}_n\|_0 \leq \eta \delta_n, \tag{22}$$

$$\|v_n - v_{n-1}\|_0 \leq M\sqrt{\delta}_{n-1}, \tag{23}$$

$$\sup_t \|v_n(\cdot,t) - v_{n-1}(\cdot,t)\|_{H^{-1}(\mathbb{T}^d)} \leq r_0 \delta_{n-1}^{\frac{1}{4}} \tag{24}$$

$$\|p_n - p_{n-1}\|_0 \leq M^2 \delta_{n-1}, \tag{25}$$

and

$$D_{n+1} := \max\left\{ \|v_{n+1}\|_{C^1}, \|\mathring{R}_{n+1}\|_{C^1} \right\} \leq A \delta_n^{\frac{3}{2}} \left(\frac{D_n}{\delta_{n+1}^2} \right)^{1+\varepsilon} \tag{26}$$

Convergence of δ_n and D_n. With $d_n = \ln(\zeta^2 \delta_n)$ we have $d_{n+1} = \frac{3}{2} d_n$ and so

$$\delta_n = \zeta^{-2} \zeta^{3(\frac{3}{2})^{n-1}} \qquad (n \geq 0).$$

Next, define $x_n := \delta_n^\gamma D_n$ where $\gamma > 0$ will be chosen later. Then, (26) gives

$$x_{n+1} \leq A \zeta^{-2(1+\varepsilon)+\gamma} \, \delta_n^{\gamma(\frac{1}{2}-\varepsilon) - 3(\frac{1}{2}+\varepsilon)} \, x_n^{1+\varepsilon}.$$

There is no loss in assuming that $\varepsilon < \frac{1}{2}$ (since we will take $\varepsilon \downarrow 0$). Let

$$\gamma > 3 \frac{1+2\varepsilon}{1-2\varepsilon}$$

and observe that $0 < \delta_n \leq 1$ so that $x_{n+1} \leq A \zeta^{-2(1+\varepsilon)+\gamma} \, x_n^{1+\varepsilon}$. Let $B := \left(A \zeta^{-2(1+\varepsilon)+\gamma} \right)^{-\frac{1}{\varepsilon}}$, pose $z_n = \ln(B x_n)$ One easily finds

$$D_{n+1} = \zeta^{2\gamma} B^{-1} \zeta^{-3\gamma(\frac{3}{2})^{n+1}} (B x_1)^{(1+\varepsilon)^n}.$$

Since $0 < \varepsilon < \frac{1}{2}$, $\gamma > 0$, and $\ln \zeta < 0$, the term in $(\frac{3}{2})^n$ will dominate that in $(1+\varepsilon)^n$. That is, for any $\gamma' > \gamma$, there exists $C' = C'(\zeta, \varepsilon, A, \gamma, \gamma')$ such that

$$D_{n+1} \leq C' \zeta^{-3\gamma'(\frac{3}{2})^n}.$$

Convergence in C^0 and (weak) solution to the Euler equations. Since δ_n vanishes very fast, and from (21), (22), (23), and (25), we conclude that (v_n, p_n) converges uniformly to a (weak) solution (v, p) to the Euler equations (1) with kinetic energy $e(t) = \fint_{\mathbb{T}^d} |v(x,t)|^2 \, dx$. In fact,

$$\|v_n - v_0\|_0 \leq M \sum_{j=0}^{\infty} \delta_n^{\frac{1}{2}} \leq CM \tag{27}$$

where C is some universal constant. In turn, the constant A in Proposition 4 can be taken to depend only on ε and e.

Convergence in C^θ. We have

$$\|v_{n+1} - v_n\|_0 \leq M\sqrt{\delta_n} \leq M\zeta^{-1}\zeta^{(\frac{3}{2})^n}$$

$$\|v_{n+1} - v_n\|_{C^1} \leq D_n + D_{n+1} \leq C'\zeta^{-3\gamma'(\frac{3}{2})^n}$$

and therefore by interpolation we find

$$\|v_{n+1} - v_n\|_{C^\theta} \leq \|v_{n+1} - v_n\|_{C^0}^{1-\theta} \|v_{n+1} - v_n\|_{C^1}^{\theta} \leq (M\zeta^{-1})^{1-\theta} (C')^\theta \, \zeta^{((1-\theta)-3\gamma'\theta)(\frac{3}{2})^n}.$$

The critical value for θ for which the right-hand side remains bounded is therefore $\frac{1}{1+3\gamma'}$. Since $\gamma' > \gamma > 3\frac{1+2\varepsilon}{1-2\varepsilon}$ are completely arbitrary, this means that any value

$$\theta < \frac{1}{1 + 9\frac{1+2\varepsilon}{1-2\varepsilon}} = \frac{1-2\varepsilon}{10+16\varepsilon}$$

is achievable. Letting $\varepsilon \downarrow 0$, any value $\theta < \frac{1}{10}$ is achievable.

H^{-1}-estimate. We have by (24)

$$\sup_t \|v(\cdot,t) - v_0(\cdot,t)\|_{H^{-1}(\mathbb{T}^d)} \leq r_0 \sum_{n=0}^{\infty} \delta_n^{\frac{1}{4}} \leq \sigma \tag{28}$$

by choosing r_0 sufficiently small. ■

2.4. **Proof of Corollary 2.** Corollary 2 will follow from

Proposition 5. *With the notation of Section 2.1, if $d \geq 2$, then*

$$\mathcal{M}_d \subset \mathcal{S}_{++}^{d\times d} \qquad \text{and} \qquad \mathrm{Id} \in \mathcal{M}_d.$$

Furthermore, $R \in \mathcal{S}_{++}^{d\times d}$ is in $\mathcal{M}_d$ if and only if

$$\frac{\operatorname{tr} R}{d-1} \mathrm{Id} - R > 0.$$

In particular, $\mathcal{M}_2 = \mathcal{S}^{2\times 2}_{++}$ *and* $\mathcal{M}_d \subsetneq \mathcal{S}^{d\times d}_{++}$ *for* $d \geq 3$.

Proof. It is obvious that $\mathcal{M}_d \subset \mathcal{S}^{d\times d}_{++}$ and one easily verifies that $\mathrm{Id} = \frac{1}{d-1}\sum_{i=1}^d (\mathrm{Id} - e_i \otimes e_i)$ where $\{e_1, \dots, e_d\}$ is the canonical basis for $\mathbb{R}^d$.

Suppose that $R \in \mathcal{S}^{d\times d}_{++}$ is of the form $R = \sum_i a_i (\mathrm{Id} - b_i \otimes b_i)$, where $a_i > 0$ and $|b_i| = 1$. Then, $\sum_i a_i = \frac{\operatorname{tr} R}{d-1}$ and hence

$$0 < \sum_i a_i b_i \otimes b_i = \frac{\operatorname{tr} R}{d-1}\,\mathrm{Id} - R.$$

Conversely, suppose $R \in \mathcal{S}^{d\times d}_{++}$ satisfies $\frac{\operatorname{tr} R}{d-1}\,\mathrm{Id} - R > 0$. R is diagonalizable and all its eigenvalues satisfy $\lambda_i < \frac{\operatorname{tr} R}{d-1}$. It is then easy to verify that, with $a_i := \frac{\operatorname{tr} R}{d-1} - \lambda_i > 0$, we have after diagonalization

$$R = \sum_{i=1}^d a_i (\mathrm{Id} - e_i \otimes e_i)$$

where $\{e_1, \dots, e_d\}$ is the canonical basis of $\mathbb{R}^d$.

Finally, note that $\sum_{i=1}^d \lambda_i = \operatorname{tr} R$ and $\lambda_i \geq 0$. If $d = 2$, then $\lambda_i \leq \operatorname{tr} R$ for $i = 1, 2$. Otherwise, if $d \geq 3$, the condition $\frac{\operatorname{tr} R}{d-1}\,\mathrm{Id} - R > 0$ can be violated. ■

Proof of Corollary 2: It is easy to see that $R = \frac{e - \fint |v|^2}{d}\,\mathrm{Id} - \mathring{R}$ satisfies (5) if and only if $R \in \mathcal{M}_d$. Thus, Proposition 5 implies Corollary 2. ■

2.5. **Proof of Corollary 3.** For sufficiently small σ we have from the bound (8)

$$|v_3| > \frac{1}{2} \fint_{\mathbb{T}^d} \left(|v_{0,33}(x,t)|^2 + \frac{e(t) - \fint_{\mathbb{T}^d} |v_0(x',t)|^2}{d} - \mathring{R}_{0,33}(x,t) \right) dx > \sigma$$

for sufficiently small σ whereas from (7) we would have $|v_3| < \sigma$. Thus, the solution cannot be of the form (6) if σ is chosen sufficiently small.

The analogous conclusion holds as well for the case $d = 2$: the flows constructed in Theorem 1 are genuinely two-dimensional, that is, they are not parallel flows. However, this conclusion can be arrived at by more elementary means. Indeed, it is classical that such flows are necessarily stationary, and this is not possible if $e(t)$ is chosen non-constant. ■

3. Construction of the iterates

3.1. **Linear spaces of stationary flows.** An essential ingredient in the construction introduced in [DLS12a] is a *linear* set of functions (W, Q) (in the ξ-variable) satisfying the stationary Euler equations. The existence of such spaces seems to hold for different reasons for $d = 2$ and $d = 3$, and we consider these cases separately.

Dimension 3. For $k \in \mathbb{Z}^3$, we let

$$b_k(\xi) := B_k e^{ik\cdot\xi}, \qquad \psi_k(\xi) := D_k e^{ik\cdot\xi} \tag{29}$$

where $B_k \in \mathbb{C}^3$ satisfies $|B_k| = \frac{1}{\sqrt{2}}$, $k \cdot B_k = 0$, and $\overline{B_k} = B_{-k}$, and $D_k = i\frac{k\times B_k}{|k|^2}$ so that

$$b_k = \mathrm{curl}_\xi\, \psi_k, \qquad \mathrm{div}_\xi b_k = 0, \qquad \overline{b_k} = b_{-k}. \tag{30}$$

Here, the operator $\mathrm{curl}_\xi = \nabla_\xi \times \cdot$ is defined as usual. The ψ_k are **vector potentials** for the vector fields b_k. Concerning the analysis in this paper, they will play the same role as the stream functions in $d = 2$ dimensions introduced in (31).

Dimension 2. For $k \in \mathbb{Z}^2$, we let

$$b_k(\xi) := i\frac{k^\perp}{|k|}e^{ik\cdot\xi}, \qquad \psi_k(\xi) = \frac{e^{ik\cdot\xi}}{|k|} \tag{31}$$

so that

$$b_k(\xi) = \mathrm{curl}_\xi\, \psi_k(\xi), \qquad \mathrm{div}_\xi\, b_k = 0, \qquad \overline{b_k} = b_{-k} \tag{32}$$

where this time $\mathrm{curl}_\xi = \nabla_\xi^\perp = (-\partial_{\xi^2}, \partial_{\xi^1})$ denotes the rotated gradient. From the analytic point of view, the **stream function** ψ_k is the analogue of the vector potential $D_k e^{ik\cdot\xi}$ defined in (29) for the case $d = 3$.

Lemma 6. *Let $\nu \geq 1$ and $d = 2$ or 3. For $k \in \mathbb{Z}^d$ such that $|k|^2 = \nu$, let $a_k \in \mathbb{C}$ such that $\overline{a_k} = a_{-k}$. Then*

$$W(\xi) = \sum_{|k|^2=\nu} a_k b_k(\xi), \qquad Q := \begin{cases} -\frac{|W|^2}{2} + \fint \frac{|W|^2}{2}\, d\xi & (d=3) \\ -\frac{|W|^2}{2} + \nu\frac{\Psi^2}{2} & (d=2) \end{cases}, \tag{33}$$

where $\Psi(\xi) = \sum_{|k|^2=\nu} a_k \psi_k(\xi)$, are $\mathbb{R}$-valued and satisfy

$$\mathrm{div}_\xi\, (W \otimes W) + \nabla_\xi\, Q = 0, \quad \mathrm{div}_\xi\, W = 0. \tag{34}$$

Furthermore,

$$\fint_{\mathbb{T}^d} W \otimes W\, d\xi = \sum_{|k|^2=\nu} |a_k|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right). \tag{35}$$

Proof. If $d = 3$, this is Lemma 3.1 of [DLS12a]. (A constant is added in our definition so that $\fint_{\mathbb{T}^3} Q\, d\xi = 0$.) Suppose $d = 2$. By direct computation one finds $\Delta_\xi \psi_k = -|k|^2\psi_k$, and hence that $\Delta_\xi \Psi = -\nu\Psi$. Recall the identities

$$\mathrm{div}_\xi (W \otimes W) = \frac{1}{2}\nabla_\xi |W|^2 + (\mathrm{curl}_\xi W)W^\perp$$

where $\mathrm{curl}_\xi W = \partial_{\xi^1} W^2 - \partial_{\xi^2} W^1 = \Delta_\xi \Psi$ and $W^\perp = (-W^2, W^1)$. Then,

$$\mathrm{div}_\xi (W \otimes W) = \nabla_\xi \frac{|W|^2}{2} - \nu\Psi\nabla_\xi\Psi$$

as desired.

As for the average, write

$$\fint_{\mathbb{T}^2} W \otimes W(\xi)\, d\xi = \sum_{j,k} a_k \overline{a_j} \fint_{\mathbb{T}^d} e^{i(k-j)\cdot\xi}\, d\xi \frac{j^\perp}{|j|} \otimes \frac{k^\perp}{|k|} = \sum_{|k|^2=\nu} |a_k|^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right)$$

where the last identity follows from $\frac{k^\perp}{|k|} \otimes \frac{k^\perp}{|k|} = \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right)$ by direct calculation. ■

3.2. **The Geometric Lemma.** The next Lemma is a quantified examination of the range of positive definite matrices that the flows from Lemma 6 are able to generate.

Lemma 7 (Geometric Lemma)**.** *Suppose $d \geq 2$ and $N \geq 1$. Let $K \subset \mathcal{M}_d$ be compact, and $\mathcal{N}$ an open neighborhood of K such that $\overline{\mathcal{N}} \subset \mathcal{M}_d$. Then, there exist $\nu \geq 1$, pairwise disjoint subsets*

$$\Lambda_j \subset \{k \in \mathbb{Z}^d \ : \ |k|^2 = \nu\} \qquad j \in \{1, \dots, N\}$$

and smooth positive functions

$$\gamma_k^{(j)} \in C^\infty(\mathcal{N}), \qquad j \in \{1, \dots, N\}, \quad k \in \Lambda_j$$

such that

(1) *$k \in \Lambda_j$ implies $-k \in \Lambda_j$ and $\gamma_k^{(j)} = \gamma_{-k}^{(j)}$;*
(2) *for each $R \in \mathcal{N}$ and $j = 1, 2, \dots, N$ we have*

$$R = \sum_{k \in \Lambda_j} \left(\gamma_k^{(j)}(R)\right)^2 \left(\mathrm{Id} - \frac{k}{|k|} \otimes \frac{k}{|k|}\right). \tag{36}$$

Proof. Each $R \in \overline{\mathcal{N}} \subset \mathcal{M}_d$ is in the interior of a simplex $\Sigma(R)$ with vertices of the form

$$A_i(R) = \kappa_i(R)\left(\mathrm{Id} - b_i(R) \otimes b_i(R)\right), \qquad i = 1, \dots, d+1$$

where $|b_i(R)| = 1$ and $\kappa_i(R) > 0$. $\Sigma(R)$ can be chosen with pairwise distinct vertices and hence R is of the form

$$R = \sum_{i=1}^{d+1} c_i(R)\left(\mathrm{Id} - b_i(R) \otimes b_i(R)\right)$$

where $c_i(R) > 0$.

Since $\overline{\mathcal{N}}$ is compact, we may extract a finite subcover $\{\Sigma_l\}_{l=1}^L$ where $\Sigma_l := \Sigma(R_l)$. Observe now that since R_l is in the interior of Σ_l, it is also in the interior of any simplex with vertices slightly perturbed. Recall now that $\mathbb{Q}^d \cap \mathbb{S}^{d-1}$ is dense in $\mathbb{S}^{d-1}$ (the proof in [DLS12a] using stereographic projection holds in any dimension). Then, by taking $\nu \in \mathbb{N}$ sufficiently large, there exist $k_{i,l}^{(j)} \in \mathbb{Z}^d$, $i = 1, \dots, d+1$, $l = 1, \dots, L$, $j = 1, \dots, N$, all distinct, satisfying $|k_{i,l}^{(j)}|^2 = \nu$, and such that, for each $l = 1, \dots, L$, R_l is in the interior of $\Sigma_l^{(j)}$ for $j = 1, \dots, N$,

where $\Sigma_l^{(j)}$ is a simplex whose vertices are multiples of $\left(\mathrm{Id} - \frac{k_{i,l}^{(j)}}{|k_{i,l}^{(j)}|} \otimes \frac{k_{i,l}^{(j)}}{|k_{i,l}^{(j)}|}\right)$. We then write

$$R_l = \sum_{i=1}^{d+1} c_{i,l}^{(j)} \left(\mathrm{Id} - \frac{k_{i,l}^{(j)}}{|k_{i,l}^{(j)}|} \otimes \frac{k_{i,l}^{(j)}}{|k_{i,l}^{(j)}|}\right), \qquad l = 1, \dots, L, \quad j = 1, \dots, N$$

where $c_{i,l}^{(j)} > 0$, $l = 1, \dots, L$, $i = 1, \dots, d+1$.

For each $l = 1, \dots, L$, $j = 1, \dots, N$ there exist positive functions $\alpha_{i,l}^{(j)} \in C^\infty(\Sigma_l)$, $i = 1, \dots, d+1$, such that

$$R = \sum_{i=1}^{d+1} \alpha_{i,l}^{(j)}(R) \left(\mathrm{Id} - \frac{k_{i,l}^{(j)}}{|k_{i,l}^{(j)}|} \otimes \frac{k_{i,l}^{(j)}}{|k_{i,l}^{(j)}|}\right) \qquad \text{for} \quad R \in \Sigma_l^{(j)}.$$

(Indeed, $R \in \Sigma_l^{(j)}$ is the unique convex combination of the vertices of $\Sigma_l^{(j)}$, and the coefficients are algebraic expressions of R.) For each $j = 1, \dots, N$, let now $\{\eta_l^{(j)}\}_{l=1}^L$ be a C^∞ partition of unity subordinate to the cover $\{\Sigma_l^{(j)}\}_{l=1}^L$. Then for any $R \in \overline{\mathcal{N}}$ we have

$$R = \sum_{l=1}^{L} \sum_{i=1}^{d+1} \eta_l^{(j)}(R) \alpha_{i,l}^{(j)}(R) \left(\mathrm{Id} - \frac{k_{i,l}^{(j)}}{|k_{i,l}^{(j)}|} \otimes \frac{k_{i,l}^{(j)}}{|k_{i,l}^{(j)}|}\right), \qquad j = 1, \dots, N.$$

Set now

$$\Gamma_j := \left\{k_{i,l}^{(j)} \in \mathbb{Z}^d \,:\, i = 1, \dots, d+1, l = 1, \dots, N\right\}, \qquad j = 1, \dots, N$$

and $a_k := \sqrt{\eta_l^{(j)} \alpha_{i,l}^{(j)}}$ for $k = k_{i,l}^{(j)}$. Next let

$$\Lambda_j := \Gamma_j \cup (-\Gamma_j), \qquad j = 1, \dots, N$$

(once again by density, we can arrange for the Λ_j's to be pairwise disjoint) and $a_k = 0$ if $k \notin \Gamma_j$. Finally, taking

$$\gamma_k(R) := \frac{\sqrt{2}}{2} \sqrt{a_k(R) + a_{-k}(R)}, \qquad k \in \Lambda_j$$

finishes the proof. ■

3.3. **The operator $\mathcal{R} = \mathrm{div}^{-1}$.**

Definition 8 (The Leray projector)**.** *Let $d \geq 2$. For a vector field $v \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$, set*

$$\mathcal{Q}v := \nabla\phi + \fint_{\mathbb{T}^d} v$$

where $\phi \in C^\infty(\mathbb{T}^d)$ is the solution to $\Delta\phi = \mathrm{div}\, v$ in $\mathbb{T}^d$ subject to $\fint_{\mathbb{T}^d} \phi = 0$. We denote by $\mathcal{P} := I - \mathcal{Q}$ the Leray projector onto divergence-free vector fields with zero average.

The operator $\mathcal{R}$ was introduced in [DLS12a] for $d = 3$. Its generalization for any $d \geq 2$ is given by the following

Definition 9 (The operator $\mathcal{R}$)**.** *Let $d \geq 2$. For any smooth vector field $v \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$, we define $\mathcal{R}v$ to be the matrix-valued periodic function*

$$\mathcal{R}v = \frac{d-2}{2(d-1)}\left(\nabla\mathcal{P}u + (\nabla\mathcal{P}u)^\top\right) + \frac{d}{2(d-1)}\left(\nabla u + (\nabla u)^\top\right) + \frac{1}{1-d}(\mathrm{div} u)\mathrm{Id} \tag{37}$$

where $u \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ is the solution to

$$\Delta u = v - \fint_{\mathbb{T}^d} v \quad \text{in } \mathbb{T}^d, \qquad \text{subject to} \quad \fint_{\mathbb{T}^d} u = 0.$$

By direct verification one obtains

Lemma 10 ($\mathcal{R} = \mathrm{div}^{-1}$)**.** *Let $d \geq 2$. For any $v \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ we have*

(1) *$\mathcal{R}v(x)$ is a symmetric trace-free matrix for each $x \in \mathbb{T}^d$;*
(2) $\mathrm{div}\,\mathcal{R}v = v - \fint_{\mathbb{T}^d} v$.

3.4. **Further technical preliminary.** The following is proved in [DLS12a] for $d = 3$ and the proof is valid as it is for any number of space dimensions. Denote $\mathcal{C}_1, \ldots, \mathcal{C}_{2^d}$ the equivalence classes of $\mathbb{Z}^d/\sim$ where $k \sim l$ if $k - l \in (2\mathbb{Z})^d$.

Proposition 11 (Partition of the space of velocities)**.** *Let $d \geq 2$ and $\mu \in \mathbb{N}$. There exists a partition of the space of velocities, namely $\mathbb{R}$-valued functions $\alpha_l(v)$ for $l \in \mathbb{Z}^d$ satisfying*

$$\sum_{l\in\mathbb{Z}^d} (\alpha_l(v))^2 \equiv 1 \tag{38}$$

such that, setting $\phi_k^{(j)}(v,\tau) = \sum_{l\in\mathcal{C}_j} \alpha_l(\mu v) e^{-i(k\cdot\frac{l}{\mu})\tau}$, for $j = 1, \ldots, 2^d$, and $k \in \mathbb{Z}^d$, then we have $\overline{\phi_k^{(j)}} = \phi_{-k}^{(j)}$ and

$$|\phi_k^{(j)}(v,\tau)|^2 = \sum_{l\in\mathcal{C}_j} \alpha_l^2(v). \tag{39}$$

3.5. **The maps w_o, v_1, p_1, and $\mathring{R}_1$.** Let $e_0, \Delta_0, \zeta, (v, \mathring{R}, p), r_0, K, \mathcal{N}, \delta$ be as in Proposition 4.

Mollifications. Let $\chi \in C_c^\infty(\mathbb{R}^d \times \mathbb{R})$ be a smooth standard nonnegative radial kernel supported in $[-\pi, \pi]^{d+1}$ and denote by

$$\chi_\ell(x,t) := \frac{1}{\ell^{d+1}}\chi(\frac{x}{\ell}, \frac{t}{\ell})$$

the corresponding family of mollifiers ($0 < \ell < 1$). We define

$$v_\ell(x,t) := \int_{\mathbb{T}^d\times\mathbb{S}^1} v(x-y, t-s)\chi_\ell(y,s)\,dy\,ds, \tag{40}$$

$$\mathring{R}_\ell(x,t) := \int_{\mathbb{T}^d\times\mathbb{S}^1} \mathring{R}(x-y, t-s)\chi_\ell(y,s)\,dy\,ds$$

and set

$$\rho_\ell(t) := \frac{1}{d(2\pi)^d}\left(e_0(t) + \Delta_0(t)(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v_\ell(x,t)|^2\,dx\right) \tag{41}$$

and

$$R_\ell(x,t) := \rho_\ell(t)\mathrm{Id} - \mathring{R}_\ell(x,t), \qquad x \in \mathbb{T}^d, \quad t \in \mathbb{S}^1. \tag{42}$$

The oscillation term w_o. Provided $\frac{R_\ell}{\rho_\ell} \in \mathcal{N}$, see Lemma 12, we may define

$$w_o(x,t) := W(x,t;\lambda t, \lambda x) \tag{43}$$

where (the b_k's are defined in (31) and (29))

$$\begin{aligned} W(y,s;\tau,\xi) &:= \sum_{|k|^2=\nu} a_k(y,s;\tau) b_k(\xi) \\ &:= \sqrt{\rho_\ell(s)} \sum_{j=1}^{2^d} \sum_{k\in\Lambda_j} \gamma_k\left(\frac{R_\ell(y,s)}{\rho_\ell(s)}\right) \phi^{(j)}_{k,\mu}(v_\ell(y,s),\tau) b_k(\xi). \end{aligned} \tag{44}$$

and where for $k \in \Lambda_j$,

$$a_k(y,s;\tau) = \sqrt{\rho_\ell(s)}\gamma_k\left(\frac{R_\ell(y,s)}{\rho_\ell(y,s)}\right)\phi^{(j)}_{k,\mu}(v_\ell(y,s);\tau) \tag{45}$$

The corresponding stream function ($d=2$) and vector potential ($d=3$) are both formally defined by

$$\psi_o(x,t) := \Psi(x,t;\lambda t, \lambda x) \tag{46}$$

where (see again (31) and (29))

$$\Psi(y,s;\tau,\xi) := \sum_{|k|^2=\nu} a_k(y,s;\tau)\psi_k(\xi).$$

The velocity field v_1. It is defined by

$$v_1 := v + w := v + w_o + w_c, \qquad w_c := -\mathcal{Q} w_o$$

where w_o is given in (43) and $\mathcal{Q}$ is the Leray projector of Definition 8. Note that $\operatorname{div} v_1 = 0$.

The pressure p_1. It is defined by

$$p_1 := p + q = p + \tilde{p} - 2\frac{(v - v_\ell)\cdot w}{d} \tag{47}$$

where

$$\tilde{q}(x,t) := Q(x,t;\lambda t, \lambda x) \tag{48}$$

and

$$Q(y,s;\tau,\xi) := \sum_{1\le|k|\le 2\nu} \tilde{a}_k(y,s;\tau)e^{ik\cdot\xi} := \begin{cases} -\frac{|W|^2}{2} + \fint \frac{|W|^2}{2}\,d\xi & (d=3) \\ -\frac{|W|^2}{2} + \nu\frac{\psi_o^2}{2} & (d=2) \end{cases} \tag{49}$$

and where v_ℓ is given in (40), w_o in (43), W in (44), and in case $d = 2$, ν is given by Geometric Lemma 7.

The tensor $\mathring{R}_1$. We define

$$\begin{aligned}\mathring{R}_1 \quad := \quad & \mathring{R} - \mathring{R}_\ell \\ & + w \otimes (v - v_\ell) + (v - v_\ell) \otimes w - 2\frac{(v - v_\ell)\cdot w}{d}\mathrm{Id} \\ & + \mathcal{R}\left[\mathrm{div}\,(w_o \otimes w_o + \mathring{R}_\ell + \tilde{q}\mathrm{Id})\right. \\ & + \mathcal{R}\,\partial_t w_c \\ & + \mathcal{R}\,\mathrm{div}\,((v_\ell + w)\otimes w_c + w_c \otimes (v_\ell + w) - w_c \otimes w_c) \\ & + \mathcal{R}\,\mathrm{div}\,(w_o \otimes v_\ell) \\ & + \mathcal{R}\,\left[\partial_t w_o + \mathrm{div}\,(v_\ell \otimes w_o)\right] \quad \left(= \mathcal{R}\left[\partial_t w_o + v_\ell \cdot \nabla w_o\right]\right).\end{aligned} \tag{50}$$

One easily verifies (see § 3.5 in [DLS12b] for details) that $\mathring{R}_1 \in \mathcal{S}_0^{d\times d}$ and that

$$\mathrm{div}\,\mathring{R}_1 = \partial_t v_1 + \mathrm{div}\,(v_1 \otimes v_1) + \nabla p_1.$$

4. Proof of Proposition 4

4.1. **Conditions on the parameters.** Let $e_0, \Delta_0, \overline{r}_0, \varepsilon, \zeta, K, \mathcal{N}$ be as in Proposition 4. Set $e := e_0 + \Delta_0$ and $\omega := \frac{\varepsilon}{2+\varepsilon}$ so that

$$1 + \varepsilon = \frac{1+\omega}{1-\omega}.$$

We assume $D \geq 1$ and $\delta \leq 1$ are given.

The estimates in the following Section as well as those established in [DLS12b], see Propostions 16, 17, and 18 in the Appendix, are derived under the assumptions on the parameters λ, μ and ℓ that they satisfy

$$\lambda, \mu, \frac{\lambda}{\mu} \in \mathbb{N}$$

and

$$\mu \geq \delta^{-1} \geq 1, \qquad \ell^{-1} \geq \frac{D}{\eta\delta} \geq 1, \qquad \lambda \geq \max\left\{(\mu D)^{1+\omega}, \ell^{-(1+\omega)}\right\}. \tag{51}$$

4.2. w_o **is well defined.** The following estimates are standard:

$$\|v_\ell\|_r \leq C(r)D\ell^{-r} \qquad (r \geq 1), \tag{52}$$

$$\|v - v_\ell\|_0 + \|\mathring{R}_\ell - \mathring{R}\|_0 \leq CD\ell, \tag{53}$$

$$\|\mathring{R}_\ell\|_0 \leq \|\mathring{R}\|_0. \tag{54}$$

As a consequence, writing $\left||v_\ell|^2 - |v|^2\right| \leq |v - v_\ell|^2 + 2|v||v - v_\ell|$, and using $D\ell \leq \eta\delta \leq \frac{\Delta_0}{4d} r_0$ from (51) and (10) we obtain

$$\fint_{\mathbb{T}^d} \left||v_\ell(x,t)|^2 - |v(x,t)|^2\right| \, dx \leq CD\ell \left(\max e^{\frac{1}{2}} + 1\right) \leq C\eta\delta \left(\max e^{\frac{1}{2}} + 1\right). \tag{55}$$

Lemma 12 (w_o is well defined)**.** *Let $d, e_0, \Delta_0, K, \mathcal{N}, \overline{r}_0$ as in Proposition 4. Let $0 < \delta \leq 1$, $\overline{\delta} \leq \zeta\delta$, $0 < \zeta \leq \frac{1}{2}$, and $D\ell \leq \delta \frac{\min \Delta_0}{4d} \overline{r}_0$. Then there exists $\tilde{r}_0$ depending on d, K, e_0, and Δ_0 from Proposition 4, such that the following holds. If $r_0 \leq \tilde{r}_0$, $(v, \mathring{R}, p)$ satisfies (11) and (12), and if ρ_ℓ and R_ℓ are defined as in (41) and (42) respectively, then $\frac{R_\ell}{\rho_\ell} \in \overline{\mathcal{N}}$.*

Proof. By assumption $\mathrm{Id} - \frac{d}{e_0(t) + \Delta_0(t)(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v(x,t)|^2 \, dx} \mathring{R} \in K$. In order to prove that $\mathrm{Id} - \frac{d}{e_0(t) + \Delta_0(t)(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v_\ell(x,t)|^2 \, dx} \mathring{R}_\ell \in \mathcal{N}$, we shall prove that

$$\left\| \frac{d\mathring{R}}{e_0(t) + \Delta_0(t)(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v|^2} - \frac{d\mathring{R}_\ell}{e_0(t) + \Delta_0(t)(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v_\ell|^2} \right\|$$

is less than $\mathrm{dist}\,(K, \partial\mathcal{N}) := \inf\{|A - B| \, : \, A \in K, B \in \partial\mathcal{N}\}$. By assumptions on $\overline{\delta}$ and ζ,

$$e_0 + \Delta_0(1 - \overline{\delta}) - \fint_{\mathbb{T}^d} |v|^2 \geq \Delta_0\delta/4$$

and thus

$$\begin{aligned}
& d(2\pi)^d \rho_\ell(t) \\
= \; & e_0(t) + \Delta_0(t)(1 - \overline{\delta}) - \fint_{\mathbb{T}^d} |v_\ell(x,t)|^2 \, dx \\
= \; & e_0(t) + \Delta_0(t)(1 - \overline{\delta}) - \fint_{\mathbb{T}^d} |v(x,t)|^2 \, dx - \fint_{\mathbb{T}^d} \left(|v_\ell(x,t)|^2 - |v(x,t)|^2\right) \, dx \\
\geq \; & \Delta_0(t)\delta/4 - \fint_{\mathbb{T}^d} \left||v_\ell(x,t)|^2 - |v(x,t)|^2\right| \, dx.
\end{aligned} \tag{56}$$

By (55), making $\overline{r}_0$ smaller if necessary depending on d, Δ_0 and $e = e_0 + \Delta_0$, we have

$$d(2\pi)^d \rho_\ell(t) \geq \Delta_0(t)\delta/8.$$

Since $(v, \mathring{R}, p)$ satisfies (12), there exists a constant $C = C(K)$ such that

$$\left\| \frac{\mathring{R}}{e_0 + \Delta(1 - \overline{\delta}) - \fint_{\mathbb{T}^d} |v|^2} \right\| \leq C(K).$$

Using the above, (53), (55), and again $D\ell \leq \frac{\min \Delta_0}{d}\overline{r}_0$, we obtain

$$\begin{aligned}
& \left| \frac{\mathring{R}}{e_0 + \Delta_0(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v|^2} - \frac{\mathring{R}_\ell}{e_0 + \Delta_0(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v_\ell|^2} \right| \\
\leq \quad & \left| \frac{\mathring{R} - \mathring{R}_\ell}{e_0 + \Delta_0(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v|^2} \right| \\
& + |\mathring{R}_\ell| \left| \frac{1}{e_0 + \Delta(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v|^2} - \frac{1}{e_0 + \Delta(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v_\ell|^2} \right| \\
\leq \quad & \left| \frac{\mathring{R} - \mathring{R}_\ell}{e_0 + \Delta_0(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v|^2} \right| \\
& + \frac{|\mathring{R}|}{e_0 + \Delta_0(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v|^2} \, \frac{\fint_{\mathbb{T}^d} \left| |v_\ell|^2 - |v|^2 \right|}{e_0 + \Delta_0(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v_\ell|^2} \\
\leq \quad & \frac{4CD\ell}{\Delta_0 \delta} + C(K) \frac{4C\eta\delta(\max e^{\frac{1}{2}} + 1)}{\Delta_0 \delta} \\
\leq \quad & \frac{\overline{r}_0}{d} \left(C + C(K)(\max e^{\frac{1}{2}} + 1) \right) \qquad (57)
\end{aligned}$$

Therefore, the right-hand side is sufficiently small so that $\frac{R_\ell}{\rho_\ell} \in \mathcal{N}$, provided $\overline{r}_0 \leq \tilde{r}_0$ where $\tilde{r}_0$ is chosen sufficiently small depending on d, K, e, Δ_0. ■

4.3. **Proof of Proposition 4.** *Setting some parameters.* In the next paragraphs, we will use estimates from [DLS12b], see Propositions 16, 17, and 18 in the Appendix. These estimates are derived under the conditions listed in (51) on the parameters ℓ, λ, μ, D, and ε (*via* ω). We shall now *set* the parameters ℓ, μ, λ in terms of D, δ, ε so that these conditions are satisfied. Set

$$\alpha = \frac{\omega}{2(1+\omega)}.$$

In particular, both ω and α depend only on ε and $\alpha \in (0, \frac{\omega}{1+\omega})$ so that Propositions 17 and 18 are applicable. Note then that the constants $C_{v,s}$ become constants C_v. Also,

$$\alpha - \frac{1}{2} = -\frac{1}{2(1+\omega)} \quad < \quad 2\alpha - \frac{1}{2} = -\frac{1-\omega}{2(1+\omega)} \quad < \quad 0.$$

Recall that $\overline{\delta} = \zeta\delta^{\frac{3}{2}}$ and choose

$$\ell = \frac{1}{L_v} \frac{\overline{\delta}}{D} \qquad (58)$$

where $L_v \geq 1$ will be chosen sufficiently large, see Section 6. We shall impose

$$\mu^2 D = \lambda = \Lambda_v \left(\frac{D\delta}{\overline{\delta}^2} \right)^{\frac{1}{1-4\alpha}} = \Lambda_v \left(\frac{D\delta}{\overline{\delta}^2} \right)^{\frac{1+\omega}{1-\omega}} = \Lambda_v \left(\frac{D\delta}{\overline{\delta}^2} \right)^{1+\varepsilon} \qquad (59)$$

where $\Lambda_v \geq 1$ will be chosen sufficiently large, see Section 6. (We note that in principle we should require that $\lambda, \mu, \lambda/\mu \in \mathbb{N}$, but this can be arranged easily, up to universal constants.)

Now we verify that the conditions (51) on the parameters are satisfied with the above choices (58) and (59). Noting that $\overline{\delta} \leq \delta$, then $\ell^{-1} \geq \frac{D}{\eta\delta}$ is satisfied with

$$L_v \geq \eta^{-1}. \tag{60}$$

Next, (59) and $\Lambda_v \geq 1$ imply

$$\mu = \sqrt{\frac{\mu}{D}} = \Lambda_v^{\frac{1}{2}} \left(\frac{D}{\zeta^2\delta^2}\right)^{\frac{1+\varepsilon}{2}} D^{-\frac{1}{2}} = \frac{\Lambda_v^{\frac{1}{2}} D^{\frac{\varepsilon}{2}}}{\zeta^{1+\varepsilon}} \delta^{-(1+\varepsilon)} \geq \delta^{-1}$$

since $\zeta \leq \frac{1}{2}$, and $D \geq 1$. Also,

$$\frac{\lambda}{(\mu D)^{1+\omega}} = \lambda^{\frac{1-\omega}{2}} D^{\frac{1+\omega}{2}} = \Lambda_v^{\frac{1-\omega}{2}} \left(\frac{D}{\zeta^2\delta^2}\right)^{\frac{1+\omega}{2}} D^{\frac{1+\omega}{2}} = \frac{\Lambda_v^{\frac{1-\omega}{2}}}{\zeta^{1+\omega}} D^{1+\omega} \delta^{-(1+\omega)} \geq 1$$

since $0 < \omega < 1$, $\delta \leq 1$, $\Lambda_v \geq 1$, and $D \geq 1$. Also,

$$= \Lambda_v \left(\frac{D}{\zeta^2\delta^2}\right)^{\frac{1+\omega}{1-\omega}} \left(\frac{\zeta\delta^{\frac{3}{2}}}{L_v D}\right)^{\frac{1+\omega}{1-\omega}} = \frac{\Lambda_v}{\zeta^{\frac{(1+\omega)^2}{1-\omega}} L_v^{1+\omega}} \geq \frac{\Lambda_v}{\zeta^{\frac{(1+\omega)^2}{1-\omega}} L_v^{1+\omega}}$$

so that we require

$$\frac{\Lambda_v}{\zeta^{\frac{(1+\omega)^2}{1-\omega}} L_v^{1+\omega}} \geq 1 \tag{61}$$

In conclusion, the requirements (51) are satisfied provided $L_v \geq 1$ and $\Lambda_v \geq 1$ satisfy (60), and (61). Note that ζ shall be chosen first, then L_v, and finally Λ_v. (Further requirements will be imposed on ζ, L_v and Λ_v, see Section 6.)

Estimates on the energy. From Proposition 17, and with $\alpha = \frac{\omega}{1+\omega}$,

$$\begin{aligned}
&\left| e_0(t) + \Delta_0(1-\overline{\delta}) - \fint_{\mathbb{T}^d} |v_1(x,t)|^2 \, dx \right| \\
\leq\ & C_e D\ell + C_v \sqrt{\delta} D^{\frac{1}{2}} \lambda^{\alpha - \frac{1}{2}} \\
\leq\ & \frac{C_e}{L_v}\overline{\delta} + \frac{C_v}{\Lambda_v^{\frac{1}{2(1+\omega)}}} D^{\frac{1}{2} - \frac{1}{2(1-\omega)}} \delta^{\frac{1}{2} - \frac{1}{2(1-\omega)}} \overline{\delta}^{\frac{1}{1-\omega}} \\
\leq\ & \frac{C_e}{L_v}\overline{\delta} + \frac{C_v}{\Lambda_v^{\frac{1}{2(1+\omega)}}} \overline{\delta}
\end{aligned}$$

where simplifications follow since $D \geq 1$, $\frac{1}{2} - \frac{1}{2(1-\omega)} = -\frac{\omega}{2(1-\omega)} < 0$ for $0 < \omega < 1$, and

$$\delta^{\frac{1}{2} - \frac{1}{2(1-\omega)}} \overline{\delta}^{\frac{1}{1-\omega}} = \delta^{-\frac{\omega}{2(1-\omega)}} \overline{\delta}^{\frac{\omega}{1-\omega}} \overline{\delta} = \left(\frac{\overline{\delta}}{\delta^{\frac{1}{2}}}\right)^{\frac{\omega}{1-\omega}} \overline{\delta} \leq \overline{\delta}$$

since $\overline{\delta} \leq \sqrt{\delta}$. We can achieve (13) provided

$$\frac{C_e}{L_v} + \frac{C_v}{\Lambda_v^{\frac{1}{2(1+\omega)}}} \leq \zeta/2 \min_t \Delta_0(t). \tag{62}$$

Using $\frac{\alpha-\frac{1}{2}}{1-4\alpha} = -\frac{1}{2(1-\omega)}$ and $\delta^{\frac{1}{2}-\frac{1}{2(1-\omega)}}\overline{\delta}^{\frac{1}{1-\omega}} \leq \overline{\delta}$, established above, the second estimate in Proposition 17 becomes

$$\begin{aligned}
& \left| \int_{\mathbb{T}^d} (v_1 \otimes v_1 - v \otimes v - R_\ell)\, dx \right| \\
\leq\ & C_{v,e}\sqrt{\delta} D^{\frac{1}{2}} \lambda^{\alpha-\frac{1}{2}} + C_e \delta D^{\frac{1}{2}} \lambda^{-\frac{1}{2}} \\
\leq\ & C_{v,e} \Lambda_v^{-\frac{1}{2(1+\omega)}} D^{\frac{1}{2}-\frac{1}{2(1-\omega)}} \delta^{\frac{1}{2}-\frac{1}{2(1-\omega)}} \overline{\delta}^{\frac{1}{1-\omega}} + C_e \Lambda_v^{-\frac{1}{2}} \delta^{\frac{1}{2}} \overline{\delta} \\
\leq\ & \left(\frac{C_{v,e}}{\Lambda_v^{\frac{1}{2(1+\omega)}}} + \frac{C_e}{\Lambda_v^{\frac{1}{2}}} \right) \overline{\delta}.
\end{aligned}$$

Making $\Lambda_v \geq 1$ sufficiently large, so that

$$\frac{C_{v,e}}{\Lambda_v^{\frac{1}{2(1+\omega)}}} + \frac{C_e}{\Lambda_v^{\frac{1}{2}}} \leq r_0 \tag{63}$$

we can achieve

$$\left| \fint_{\mathbb{T}^d} (v_1 \otimes v_1 - v \otimes v - R_\ell)\, dx \right| \leq r_0 \overline{\delta}. \tag{64}$$

C^0-estimate on $\mathring{R}_1$. We have

$$\|\mathring{R}_1\|_0 \leq C_v \left(D\ell + \sqrt{\delta} D^{\frac{1}{2}} \lambda^{2\alpha-\frac{1}{2}} + \sqrt{\delta} D^{\frac{1}{2}} \lambda^{\alpha-\frac{1}{2}} \right)$$

using the fact that $\lambda \geq 1$ and thus we should keep the least negative of $\alpha - \frac{1}{2} < 2\alpha - \frac{1}{2} < 0$. Note also that we have used that $\overline{\delta} \leq \sqrt{\delta}$. Now (14) obtains provided

$$\frac{C_v}{L_v} + \frac{C_v}{\Lambda_v^{\frac{1}{2(1+\varepsilon)}}} \leq \eta. \tag{65}$$

C^0-estimate on $v_1 - v$. From Proposition 16,

$$\|v_1 - v\|_0 = \|w\|_0 \leq C_e \sqrt{\delta} \leq \frac{M}{2}\sqrt{\delta} \tag{66}$$

by making M sufficiently large. This is (15).

C^0-estimate on $p_1 - p$. The pressure p_1 has been defined in (47) as $p_1 = p + \tilde{q} - 2\frac{(v-v_\ell)\cdot w}{d}$ where $\tilde{q}$ is given in (48). Making M larger than previously if necessary (depending on ν in

the case $d = 2$), we have $\|p_1 - p\|_0 \leq \frac{M^2}{4}\delta + \|v - v_\ell\|_0 \|w\|_0$. But from $\|v - v_\ell\|_0 \leq CD\ell \leq C\overline{\delta}$, we get $CD\ell C_e\sqrt{\delta} \leq CC_e\overline{\delta}\sqrt{\delta} \leq \frac{M^2}{4}\delta$. Increasing M if necessary, we get (17):

$$\|p_1 - p\|_0 \leq \frac{M^2}{4}\delta + C_e\delta \leq \frac{M^2}{2}\delta. \tag{67}$$

C^1-estimates. Since $\lambda \geq 1$ and $\alpha - \frac{1}{2} < 2\alpha - \frac{1}{2} < 0$, we have from Proposition 18

$$\|\mathring{R}_1\|_{C^1} \leq C_v\lambda \left\{ \sqrt{\delta}\frac{\overline{\delta}}{L_v} + \sqrt{\delta}D^{\frac{1}{2}}\lambda^{2\alpha-\frac{1}{2}} + \sqrt{\delta}D^{\frac{1}{2}}\lambda^{\alpha-\frac{1}{2}} \right\} \leq \lambda\overline{\delta}\left(\frac{C_v}{L_v} + \frac{C_v}{\Lambda^{\frac{1-\omega}{2(1+\omega)}}} \right)$$

and therefore $\|\mathring{R}_1\|_1 \leq \lambda\overline{\delta}$ provided

$$\frac{C_v}{L_v} + \frac{C_v}{\Lambda_v^{\frac{1-\omega}{2(1+\omega)}}} < 1. \tag{68}$$

From Proposition 16, $\|v_1\|_{C^1} \leq \|v\|_{C^1} + \|w\|_{C^1} \leq D + C_{e,v}\sqrt{\delta}\lambda$ so that, with $\overline{\delta} \leq \sqrt{\delta}$,

$$\max\left\{ \|v_1\|_{C^1}, \|\mathring{R}_1\|_{C^1} \right\} \leq D + C_{e,v}\sqrt{\delta}\Lambda_v \left(\frac{D\delta}{\overline{\delta}^2}\right)^{1+\varepsilon} \leq 2C_{e,v}\Lambda_v\delta^{\frac{3}{2}} \left(\frac{D}{\overline{\delta}^2}\right)^{1+\varepsilon}$$

since $D \geq 1$ and $\delta^{\frac{3}{2}} \geq \overline{\delta}^2$. Now set $A := 2C_{e,v}\Lambda_v$. From (27), we conclude

$$A := 2C_e\Lambda_v. \tag{69}$$

Estimate on $\|v_1(\cdot,t) - v(\cdot,t)\|_{H^{-1}(\mathbb{T}^d)}$. By construction we have $v_1 - v = w = w_o + w_c$ and we will estimate $\|w_o\|_{H^{-1}(\mathbb{T}^d)}$ and $\|w_c\|_{H^{-1}(\mathbb{T}^d)}$ separately.

Let f be any test vector field. By definition (43) of w_o, and according to estimates from Propositions 14 and 15 in the Appendix, we have

$$\begin{aligned} \left| \int_{\mathbb{T}^d} w_o \cdot f\, dx \right| &\leq C \sum_{|k|^2=\nu} \frac{\|a_k\|_1 \|f\|_1}{\lambda} \\ &\leq \frac{C_e}{\Lambda_v^{\frac{1}{2}}} D^{-\omega}\delta^{-\frac{\omega}{1-\omega}}\overline{\delta}^{\frac{1+\omega}{1-\omega}}\|f\|_1 \\ &\leq \frac{C_e}{\Lambda_v^{\frac{1}{2}}} \left(\frac{\overline{\delta}}{\delta}\right)^{\frac{\omega}{1-\omega}} \overline{\delta}^{\frac{1}{1-\omega}}\|f\|_1 \\ &\leq \frac{C_e}{\Lambda_v^{\frac{1}{2}}}\overline{\delta}\|f\|_1 \end{aligned} \tag{70}$$

Next, observe from (30)=(32) and (43) that

$$w_o(x,t) = \frac{1}{\lambda}\mathrm{curl}\left(\sum_k a_k(x,t;\lambda t)\psi_k(\lambda x) \right) - \frac{1}{\lambda}\sum_k \psi_k(\lambda x) \cdot \mathrm{curl}\, a_k(x,t;\lambda t)$$

and thus $w_c(x,t) = -\mathcal{Q}w_o(x,t) = \frac{1}{\lambda}\mathcal{Q}\left(\sum_{|k|^2=\nu}\psi_k(\lambda x)\cdot \mathrm{curl}\, a_k(x,t,\lambda t)\right) = \frac{1}{\lambda}\mathcal{Q}\, u_c$. (Recall the interpretation of the curl operator in dimension $d = 2$ from Section 3.1.) The function

u_c is of the form $u_c(x,t) = \sum_{|k|^2=\nu} \tilde{c}_k(x,t;\lambda t)e^{i\lambda k\cdot x}$ where the coefficients $\tilde{c}_k$ satisfy the same estimates as the coefficients ∇a_k, see Proposition 15 in the Appendix. Then, with $0<\gamma<1$ to be specified later, we find

$$\begin{aligned}
\left|\int_{\mathbb{T}^d} w_c\cdot f\,dx\right| &\leq \frac{1}{\lambda}\left|\int_{\mathbb{T}^d} u_c\cdot \mathcal{Q}f\,dx\right| \\
&\leq C\frac{1}{\lambda}\sum_{|k|^2=\nu}\frac{\|\tilde{c}_k\|_\gamma\|\mathcal{Q}f\|_\gamma}{\lambda^\gamma} \\
&\leq \|\mathcal{Q}f\|_\gamma\frac{1}{\lambda^{1+\gamma}}C_e\sqrt{\delta}(\mu^{1+\gamma}D^{1+\gamma}+\mu D\ell^{-\gamma}) \\
&\leq \|f\|_1 C_e\left\{\lambda^{-\frac{1}{2}(1+\gamma)}\sqrt{\delta}D^{\frac{1}{2}(1+\gamma)}+\lambda^{-\frac{1}{2}-\gamma}\sqrt{\delta}D^{\frac{1}{2}+\gamma}\overline{\delta}^{-\gamma}\right\} \\
&\leq \|f\|_1 C_e\Bigg\{\Lambda_v^{-\frac{1}{2}(1+\gamma)}D^{\frac{1}{2}(1+\gamma)(1-\frac{1+\omega}{1-\omega})}\delta^{\frac{1}{2}-\frac{1}{2}(1+\gamma)\frac{1+\omega}{1-\omega}}\overline{\delta}^{(1+\gamma)\frac{1+\omega}{1-\omega}} \\
&\qquad +\Lambda_v^{-\frac{1}{2}-\gamma}D^{(\frac{1}{2}+\gamma)(1-\frac{1+\omega}{1-\omega})}\delta^{-(\frac{1}{2}+\gamma)\frac{1+\omega}{1-\omega}+\frac{1}{2}}\overline{\delta}^{(\frac{1}{2}+\gamma)\frac{1+\omega}{1-\omega}-\gamma}\Bigg\} \\
&\leq \|f\|_1 C_e\left\{\Lambda_v^{-\frac{1}{2}(1+\gamma)}\overline{\delta}+\lambda^{-\frac{1}{2}-\gamma}\overline{\delta}^{\frac{1}{2}-\gamma}\right\}
\end{aligned}$$

where we have used that $\frac{1}{2}(1+\gamma)\frac{1+\omega}{1-\omega}-\frac{1}{2}>0$ for $0<\omega<1$ and $0<\gamma<1$, $D\geq 1$, and $\overline{\delta}\leq\delta$. Fix $\gamma=\frac{1}{4}$ so that $\left|\int_{\mathbb{T}^d} w_c\cdot f\,dx\right| \leq \|f\|_1\frac{C_e}{\Lambda_v^{\frac{5}{8}}}\overline{\delta}^{\frac{1}{4}}$. From this and (70) we conclude $\left|\int_{\mathbb{T}^d} w\cdot f\,dx\right| \leq \frac{C_e}{\Lambda_v^{\frac{1}{2}}}\overline{\delta}^{\frac{1}{4}}\|f\|_1$ which implies that $\|w\|_{H^{-1}(\mathbb{T}^d)} \leq \frac{C_e}{\Lambda_v^{\frac{1}{2}}}\overline{\delta}^{\frac{1}{4}}$. The bound (16) is satisfied provided

$$\frac{C_e}{\Lambda_v^{\frac{1}{2}}} < r_0. \tag{71}$$

5. Proof of Theorem 1, part 2

We consider again the sequence $(v_n, \mathring{R}_n, p_n)$ and the limit v from Section 2.3. We denote with some abuse $R_{n,\ell} = \rho_{n,\ell}\mathrm{Id} - \mathring{R}_{n,\ell}$ and $v_{n,\ell}$ the corresponding quantities (since actually $\ell=\ell_n$). Write

$$\begin{aligned}
v\otimes v - v_0\otimes v_0 + \mathring{R}_0 - \frac{1}{d}\Delta_0\mathrm{Id} &= \sum_{n=0}^{\infty}(v_{n+1}\otimes v_{n+1} - v_n\otimes v_n - R_{n,\ell}) \\
&\quad + \sum_{n=0}^{\infty}\left(\rho_{n,\ell} - \frac{1}{d}\Delta_0(\delta_n-\delta_{n+1})\right)\mathrm{Id} \\
&\quad - \sum_{n=1}^{\infty}\mathring{R}_{n,\ell} + \mathring{R}_0 - \mathring{R}_{0,\ell}.
\end{aligned}$$

From (64),

$$\left|\fint_{\mathbb{T}^d} (v_{n+1}\otimes v_{n+1} - v_n\otimes v_n - R_{n,\ell})\, dx\right| \leq r_0\delta_{n+1} \qquad (n\geq 0)$$

and from (22) and (54),

$$\left|\fint_{\mathbb{T}^d} \mathring{R}_{n,\ell}\, dx\right| \leq \eta\delta_{n-1} \qquad (n\geq 1).$$

As for the remaining term,

$$\begin{aligned} d(2\pi)^d\rho_{n,\ell} &= e_0 + \Delta_0(1-\delta_{n+1}) - \fint_{\mathbb{T}^d} |v_{n,\ell}|^2\, dx \\ &= \Delta_0(\delta_n - \delta_{n+1}) + \fint_{\mathbb{T}^d} \left(|v_n|^2 - |v_{n,\ell}|^2\right)\, dx \\ &\quad + e_0 + \Delta_0(1-\delta_n) - \fint_{\mathbb{T}^d} |v_n|^2\, dx \end{aligned}$$

But from (21) and from (55) we find

$$\left|d(2\pi)^d\rho_{n,\ell} - \Delta_0(\delta_n - \delta_{n+1})\right| \leq \zeta/2\Delta_0\delta_n + C\eta\delta_n(\max e^{\frac{1}{2}} + 1).$$

Also, by definition (58), $(D\ell)_0 = \frac{\delta_1}{L_v} \leq \zeta$ and thus from (53) we find

$$\|\mathring{R}_0 - \mathring{R}_{0,\ell}\| \leq C\zeta.$$

With all the above we conclude

$$\begin{aligned} &\left|\fint_{\mathbb{T}^d} \left(v\otimes v - v_0\otimes v_0 + \mathring{R}_0 - \frac{1}{d}\Delta_0 \mathrm{Id}\right)\, dx\right| \\ \leq\ & 2\left(r_0 + \zeta/2\Delta_0 + C\frac{\Delta_0}{d} r_0(\max e^{\frac{1}{2}} + 1)\right) + C\zeta. \end{aligned} \tag{72}$$

Thus, making r_0 and ζ sufficiently small depending σ, e and Δ_0, we can achieve

$$\left|\fint_{\mathbb{T}^d} \left(v\otimes v - v_0\otimes v_0 + \mathring{R}_0 - \frac{1}{d}\Delta_0 \mathrm{Id}\right)\, dx\right| < \sigma.$$

■

6. Fixing the parameters M, ζ, r_0, L_v, and Λ_v

We list the requirements on the parameters M, ζ, r_0, L_v, and Λ_v:

- M: (66), (67);
- ζ: (20), (72);
- r_0: (28), (55), (56), (57), (72);
- L_v: (60), (61), (62), (63), (65), (68);
- Λ_v: (61), (62), (63), (65) (68), (71).

Recall that $d, e, \Delta_0, (v_0, \mathring{R}_0, p_0)$, and ε (hence ω) are given. We set the parameters M, ζ, r_0, L_v and Λ_v in this order as follows:

(1) set M larger than a constant C_e so that it satisfies (66), (67) (specifically, the constant depends on e and $\sup_{k,y,\tau} \|a_k(\cdot, y; \tau)\|_1$);
(2) set ζ sufficiently small depending on $(v_0, \mathring{R}_0, p_0)$ and σ so that it satisfies (20), (72);
(3) set r_0 sufficiently small depending on d, σ, Δ_0 and e so that it satisfies (28), (55), (56), (57), (72);
(4) set $L_v \geq 1$ sufficiently large depending on r_0, Δ_0, and ε so that it satisfies (60), (62), (63), (65), (68);
(5) finally set Λ_v sufficiently large, depending on $\zeta, \Delta_0, \varepsilon, r_0$, and L_v, so that it satisfies (61), (62), (63), (65) (68), (71).

Appendix A. Estimates from [DLS12b]

The following Propositions have been proved in [DLS12b] and [DLS12a].

Proposition 13 (Schauder estimates for elliptic operators)**.** *Let $d \geq 2$. For any $\alpha \in (0,1)$ and any $m \in \mathbb{N}$ there exists a constant $C_s(d, m, \alpha)$ so that the following estimates hold.*

$$\begin{aligned}
\|\mathcal{Q}v\|_{m+\alpha} &\leq C_s(d,m,\alpha)\|v\|_{m+\alpha} \\
\|\mathcal{P}v\|_{m+\alpha} &\leq C_s(d,m,\alpha)\|v\|_{m+\alpha} \\
\|\mathcal{R}v\|_{m+1+\alpha} &\leq C_s(d,m,\alpha)\|v\|_{m+\alpha} \\
\|\mathcal{R}\,\mathrm{div}\, A\|_{m+\alpha} &\leq C_s(d,m,\alpha)\|A\|_{m+\alpha} \\
\|\mathcal{R}\mathcal{Q}\mathrm{div}\, A\|_{m+\alpha} &\leq C_s(d,m,\alpha)\|A\|_{m+\alpha}
\end{aligned}$$

Proof. This is Proposition 4.3 of [DLS12b], valid as it is for any $d \geq 2$ provided $\mathcal{R}$ is defined according to Definition 9. ■

Proposition 14 (Stationary phase lemma)**.** *Let $d \geq 1$. For $k \in \mathbb{Z}^d$ and $\lambda \geq 1$,*

(1) *For any function $a \in C^\infty(\mathbb{T}^d)$ and $m \in \mathbb{N}$ we have*

$$\left| \int_{\mathbb{T}^d} a(x) e^{i\lambda k \cdot x}\, dx \right| \leq \frac{[a]_m}{\lambda^m}.$$

(2) *Let $k \in \mathbb{Z}^d \setminus \{0\}$. For any vector field $F \in C^\infty(\mathbb{T}^d, \mathbb{R}^d)$ let $F_\lambda(x) := F(x)e^{i\lambda k \cdot x}$. Then,*

$$\begin{aligned}
\|\mathcal{R}\, F_\lambda\|_\alpha &\leq \frac{C_s}{\lambda^{1-\alpha}}\|F\|_0 + \frac{C_s}{\lambda^{m-\alpha}}\,[F]_m\,\frac{C_s}{\lambda^m}\,[F]_{m+\alpha} \\
\|\mathcal{R}\,\mathcal{Q}\, F_\lambda\|_\alpha &\leq \frac{C_s}{\lambda^{1-\alpha}}\|F\|_0 + \frac{C_s}{\lambda^{m-\alpha}}\,[F]_m\,\frac{C_s}{\lambda^m}\,[F]_{m+\alpha}
\end{aligned}$$

where $C_s = C_s(d, m, \alpha)$ (i.e. they do not depend on λ nor $k \neq 0$).

Proof. This is Proposition 4.4 of [DLS12b], valid as it is for any $d \geq 1$. ■

Of the estimates from Proposition 5.1 from [DLS12b], we will only recall the one which is explicitly used here (in the estimate of $\|w_c\|_{H^{-1}(\mathbb{T}^d)}$).

Proposition 15 (Estimates on the coefficients)**.** *Let $a_k \in C^\infty(\mathbb{T}^d \times \mathbb{S}^1 \times \mathbb{R})$ be given by (45). For any $r \geq 1$,*

$$\|a_k(\cdot, s; \tau)\|_r \leq C_e \sqrt{\delta} \left(\mu^r D^r + \mu D \ell^{1-r}\right).$$

Proposition 16 (Estimates on w_o, w_c, and w)**.** *Assuming (51) and $r \geq 0$, we have*

$$\|w_o\|_r \leq C_e \sqrt{\delta} \lambda^r$$
$$\|\partial_t w_o\|_r \leq C_v \sqrt{\delta} \lambda^{r+1}$$

and for $r > 0$, $r \notin \mathbb{N}$,

$$\|w_c\|_r \leq C_{e,s} \sqrt{\delta} D \mu \lambda^{r-1}$$
$$\|\partial_t w_c\|_r \leq C_{v,s} \sqrt{\delta} D \mu \lambda^r.$$

In particular

$$\|w\|_0 \leq C_e \sqrt{\delta}$$
$$\|w\|_1 \leq C_{e,v} \sqrt{\delta} \lambda.$$

Proof. This is Proposition 6.1 of [DLS12b]. ■

Proposition 17 (Estimates on the energy)**.** *For any $\alpha \in (0, \frac{\omega}{1+\omega})$ there is a constant $C_{v,s}$ depending only on d, α, e, and $\|v\|_0$, such that, under the assumptions (51), we have*

$$\left| e_0(t) + \Delta_0(t)(1 - \overline{\delta}) - \fint_{\mathbb{T}^d} |v_1(x,t)|^2 \, dx \right| \leq C_e D \ell + C_{v,s} \sqrt{\delta} \mu D \lambda^{\alpha - 1}$$

and

$$\left| \fint_{\mathbb{T}^d} \left(v_1 \otimes v_1 - v \otimes v - R_\ell\right) \, dx \right| \leq C_{v,e} \sqrt{\delta} D \mu \lambda^{\alpha-1} + C_e \delta D \mu \lambda^{-1}.$$

Proof. The first estimate is Proposition 7.1 of [DLS12b]. The second estimate holds since since the first and second term of

$$\int_{\mathbb{T}^d} \left(v_1 \otimes v_1 - v \otimes v - R_\ell\right) \, dx = \int_{\mathbb{T}^d} \left(v_1 \otimes v_1 - v \otimes v - w_o \otimes w_o\right) \, dx + \int_{\mathbb{T}^d} \left(w_o \otimes w_o - R_\ell\right) \, dx.$$

are estimated exactly as $\int_{\mathbb{T}^d} \left(|v_1|^2 - |v|^2 - |w_o|^2\right) \, dx$ and $\int_{\mathbb{T}^d} \left(|w_o|^2 - \operatorname{tr} R_\ell\right) \, dx$ in the proof of Proposition 7.1 of [DLS12b], respectively. ■

Proposition 18 (Estimates on $\mathring{R}_1$)**.** *For every $\alpha \in (0, \frac{\omega}{1+\omega})$ there is a constant $C_{v,s}$ depending only on d, α, ω, e, and $\|v\|_0$, such that, under the assumptions (51), we have*

$$\|\mathring{R}_1\|_0 \leq C_{v,s} \left(D\ell + \sqrt{\delta} D \mu \lambda^{2\alpha - 1} + \sqrt{\delta} \mu^{-1} \lambda^\alpha\right)$$
$$\|\mathring{R}_1\|_1 \leq C_{v,s} \lambda \left(\sqrt{\delta} D \ell + \sqrt{\delta} D \mu \lambda^{2\alpha - 1} + \sqrt{\delta} \mu^{-1} \lambda^\alpha\right)$$

Proof. This is Proposition 8.1 of [DLS12b]. For convenience for the reader, we recall briefly how the "zero-mode" of $w_o \otimes w_o$ cancels with R. From definition (44) of W we have

$$W \otimes W(y,s;\tau,\xi) = U_o(y,s) + \sum_{1\le|k|\le 2\nu} U_k(y,s;\tau)e^{ik\cdot\xi}$$

for some coefficients U_k. The "zero-mode" $U_0(y,s)$ is precisely $R_\ell(y,s)$ since

$$\begin{aligned}
\fint_{\mathbb{T}^d} W\otimes W\, d\xi &= \rho_\ell \sum_{j=1}^{2^d}\sum_{k\in\Lambda_j}\left(\gamma_k\left(\frac{R_\ell}{\rho_\ell}\right)\right)^2 |\phi_k^{(j)}(v,\tau)|^2\left(\mathrm{Id}-\frac{k}{|k|}\otimes\frac{k}{|k|}\right)\\
&= \rho_\ell \sum_{j=1}^{2^d}\sum_{k\in\Lambda_j}\sum_{l\in\mathcal{C}_j}\left(\gamma_k\left(\frac{R_\ell}{\rho_\ell}\right)\right)^2 \alpha_l^2(v)\left(\mathrm{Id}-\frac{k}{|k|}\otimes\frac{k}{|k|}\right)\\
&= R_\ell \sum_{j=1}^{2^d}\sum_{l\in\mathcal{C}_j}\alpha_l^2(v)\\
&= R_\ell.
\end{aligned}$$

The crucial identities are (35), (36), (38), and (39). Thus,

$$\begin{aligned}
\mathrm{div}\,(w_o\otimes w_0 + \mathring{R}_\ell + \tilde{q}\mathrm{Id}) &= \mathrm{div}_y\,(W\otimes W - R_\ell) + \nabla_y Q + \lambda \mathrm{div}_\xi\,(W\otimes W + Q\mathrm{Id})\\
&= \sum_{1\le|k|\le 2\nu}(\mathrm{div}_y\, U_k + \nabla_y \tilde{a}_k \mathrm{Id})\, e^{i\lambda k\cdot x}
\end{aligned}$$

where a cancelation occurs since (W,Q) is a stationary solution (in the ξ-variable) to the Euler equations. Here we have used that $\rho_\ell = \rho_\ell(t)$, and the $\tilde{a}_k$'s are the coefficients of Q, see (49). In the end, $\mathrm{div}\,(w_o\otimes w_o + R_\ell + \tilde{q}\mathrm{Id})$ is oscillatory.

The other terms in $\mathrm{div}\,\mathring{R}_1$ are linear in w and hence are also oscillatory. ■

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