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Dynamics on strata of trigonal Jacobians and some integrable problems of rigid body motion

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DYNAMICS ON STRATA OF TRIGONAL JACOBIANS AND SOME INTEGRABLE PROBLEMS OF RIGID BODY MOTION

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ABSTRACT. We present an algebraic geometrical and analytical description of the Goryachev case of rigid body motion. It belongs to a family of systems sharing the same properties: although completely integrable, they are not algebraically integrable, their solution is not meromorphic in the complex time and involves dynamics on the strata of the Jacobian varieties of trigonal curves.

Although the strata of hyperelliptic Jacobians have already appeared in the literature in the context of some dynamical systems, the Goryachev case is the first example of an integrable system whose solution involves a more general curve. Several new features (and formulae) are encountered in the solution given in terms of sigma-functions of such a curve.

1. INTRODUCTION

Most of the known finite-dimensional integrable systems of classical mechanics and mathematical physics are also algebraically completely integrable: following [AvM989], their invariant tori can be extended to specific complex tori, Abelian varieties, and the complexified flow is a straight-line flow on them. As a direct consequence of this property, all the solutions of such systems are meromorphic functions of the complex time, and can be described explicitly in terms of theta-functions or generalized theta-functions. The property of meromorphicity led to the Kovalevskaya–Painlevé integrability test, which was effectively applied to detect several integrable cases. In some cases, such as the famous Neumann system describing the motion of a point on a sphere with a quadratic potential, or the Steklov–Lyapunov integrable case of the Kirchhoff equations, the complex tori are Jacobians of hyperelliptic curves (or possibly coverings of the Jacobians). In other, more complicated situations, for example the Frahm–Manakov top on $so(n)$ or the Kovalevskaya top, the complexified tori are not Jacobians but Abelian subvarieties thereof, with a non-principal polarization (Prym varieties)[1].

On the other hand, there are many other systems, including generalizations of the above ones, which preserve integrability but lose the meromorphicity of the complex solutions. In such systems the genus of the underlying algebraic curve (often the spectral curve) is greater than the dimension of the invariant tori, and the latter are certain non-Abelian subvarieties (strata) of Jacobians. The algebro-geometric properties of such systems and the nature of the singularities of their complex solutions have been described in [Van995, AF00, EPR03, FG07, EEKL993, EEKT994]. Until recently, however, the only examples known were of systems related to the strata of hyperelliptic Jacobians.

In present paper we consider the first example of a mechanical system whose complex invariant varieties are strata of Jacobians of a non-hyperelliptic curve, here a trigonal curve of genus 3 given by the equation $y^3 + p(x)y + q(x) = 0$. The latter appear in the reduction to quadratures of the integrable Goryachev case of the Kirchhoff equations [Gor912]: the quadratures involve 2 points on the genus 3 curve (which has no additional involution in general, so not allowing reduction to a Prym) and so the quadratures lead to an incomplete Abel map which cannot be inverted in terms of meromorphic functions. This means that, as in other non-algebraic integrable systems, the Goryachev case cannot be detected by the Kovalevskaya–Painlevé test. We emphasize that this example is not unique: it is in fact a member of a family of integrable Hamiltonian flows on the sphere S^2 that have extra cubic integrals and that were described recently in [Val10], see also [VT12, Yeh02]. In particular this family also includes the non-trivial case found by Dullin and Matveev in [DM04]. As was shown in [Tsi05, VT09], most of the systems of the family are reduced

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[1] These can be related to other Jacobians in the case of dimension 2.

to "trigonal" quadratures similar to those of the Goryachev system[2]. For concreteness, the present paper considers only the Goryachev system.

Our solution builds on the explicit description of Abelian functions of trigonal curves and (in the terminology of [BEL999]) the more general (n, s)-curves. There has been a resurgence of interest in this area with many new analytic results obtained [BEL00, BL05, EEMOP07, EE09, Nak10, EEO11] including the inversion of the Abel-Jacobi map on strata of Jacobians [BG06, MP08, MP11, NYo12]. Although we will need to extend this work in various ways these new studies are foundational to providing explicit solutions for non-algebraically completely integrable systems.

The paper is organized as follows. In Section 2 we reproduce the reduction to quadratures of the Goryachev system first made in [VT09] and interpret them as sums of two holomorphic differentials on a genus 3 trigonal curve $\mathcal{C}$, also indicating its canonical form. The original variables of the system are then expressed in terms of coordinates of two points on the curve. In Section 3, following [EEMOP07], we describe in detail the inversion of the complete Abel map on the (3,4)-trigonal curve. Here the main tools are the sigma-function of the curve and its logarithmic derivatives (Kleinian-Weierstrass functions), which are direct generalizations of the Weierstrass elliptic σ- and $\wp$-functions respectively (see [BEL997, Bak897]). We give a method of an effective calculation of the vector of Riemann constants for the trigonal Jacobians which allows us to calculate the sigma function explicitly by relating it to the corresponding theta-function. We also present an analytic description of the Wirtinger strata in the Jacobian of $\mathcal{C}$ as zeros of the sigma-function and of some of its derivatives. Section 4 describes the inversion of the incomplete Abel map. The formal explicit solution to the inversion problem is obtained from the formulae of the previous section (inversion of the complete map) by a certain limiting procedure. The resulting solution is given in terms of the sigma-function and its derivatives whose arguments are restricted to the 2-dimensional stratum in the Jacobian of $\mathcal{C}$. (Note that for the case of a cyclic trigonal curve of genus 4 similar results were obtained in [BG06].)

Then global analytical properties of the solutions as functions of the complex time are described. We show that these functions have an infinite number of branch points and are single-valued only on an infinite ramified covering of the complex time plane. Finally the local singularities of the complex solutions are described in Section 5 by using the expansion of the sigma-function near generic and special points of the curve $\mathcal{C}$. An Appendix contains some rather long and technical proofs of two theorems.

2. THE GORYACHEV INTEGRABLE CASE. SEPARATION OF VARIABLES AND REDUCTION TO QUADRATURES.

Recall that the classical Kirchhoff equations describing the motion of a rigid body in an ideal fluid have (in an appropriate coordinate frame) the form

$$
(1) \qquad
\begin{aligned}
\dot{J} &= J \times \frac{\partial H}{\partial J} + \gamma \times \frac{\partial H}{\partial \gamma}, \\
\dot{\gamma} &= \gamma \times \frac{\partial H}{\partial J},
\end{aligned}
$$

where $J = (J_1, J_2, J_3)^T$ and $\gamma = (\gamma_1, \gamma_2, \gamma_3)^T$ are the angular and linear momentum respectively and $H(J, \gamma)$ is the Hamiltonian, which is also a first integral. In addition to the Hamiltonian the equations always possess the integrals (Casimir functions)

$$C_1 = \langle J, \gamma \rangle, \quad C_2 = \langle \gamma, \gamma \rangle.$$

Apart from the well known integrable cases of Kirchhoff, Clebsch, Steklov and Lyapunov (and their gyroscopic generalizations) there are some further special cases of integrability cases where an additional integral exists only under the condition $C_1 = 0$. In the most classical case found by D. Goryachev [Gor912] and S. Chaplygin [Chap904] the extra integral is cubic in J. In this case Chaplygin himself [Chap904] gave a separation of variables and reduced the system to quadratures containing integrals on a hyperelliptic genus 2 curve. A detailed algebro-geometric description of the complex invariant manifolds was made in [BvM987]. Further rather exotic special cases of integrability also exist for which neither separation of variables nor explicit solution were known until recently. Here we concentrate on the Goryachev case [Gor916] which was reduced to quadratures in [VT09] by using a bi-Hamiltonian structure and the corresponding separating Darboux–Nijenhuis variables.

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[2]There are several exceptions in the family: one of them is the classical Goryachev–Chaplygin system [Chap904, Gor916], which is linearized on Jacobians of genus 2 hyperelliptic curve.

The Goryachev case, the focus of this paper, has Hamiltonian $H_1 = H$ and extra integral H_2 that take the form

$$H_1 = J_1^2 + J_2^2 + \frac{4}{3}J_3^2 + \frac{a\gamma_1 + b}{\gamma_3^{2/3}}, \quad a, b \text{ being arbitrary constants,}$$

(2)

$$H_2 = -\frac{2}{3}J_3(J_1^2 + J_2^2 + \frac{8}{9}J_3^2 + \frac{a\gamma_1 + b}{\gamma_3^{2/3}}) + a\gamma_3^{1/3} J_1.$$

The corresponding Kirchhoff equations are then

$$\dot{J}_1 = \frac{2}{3}J_2 J_3 - \frac{2}{3}\gamma_2(a\gamma_1 + b)\gamma_3^{-5/3},$$

$$\dot{J}_2 = -\frac{2}{3}J_1 J_3 + a\gamma_3^{1/3} + \frac{2}{3}\gamma_1(a\gamma_1 + b)\gamma_3^{-5/3},$$

$$\dot{J}_3 = -a\frac{\gamma_2}{\gamma_3^{2/3}},$$

(3)

$$\dot{\gamma}_1 = 2(\frac{4}{3}J_3\gamma_2 - J_2\gamma_3),$$

$$\dot{\gamma}_2 = 2(-\frac{4}{3}J_3\gamma_1 + J_1\gamma_3),$$

$$\dot{\gamma}_3 = 2(\gamma_1 J_2 - \gamma_2 J_1).$$

Without loss of generality one can set $C_2 = \langle \gamma, \gamma \rangle = 1$. Then, since $C_1 = 0$, equations (1) can be reduced to a Hamiltonian system on the cotangent bundle of the unit sphere $S^2 = \{\langle \gamma, \gamma \rangle = 1\}$ with coordinates and momenta

(4)
$$u = \gamma_3, \quad p_u = \frac{J_1\gamma_2 - J_2\gamma_1}{\gamma_1^2 + \gamma_2^2}, \quad \phi = \arctan(\gamma_1/\gamma_2), \quad p_\phi = -J_3.$$

In terms of these the original variables become

$$J_1 = \frac{u\sin\phi\, p_\phi + \cos\phi(1 - u^2)\, p_u}{\sqrt{1 - u^2}}, \quad J_2 = \frac{u\cos\phi\, p_\phi - \sin\phi(1 - u^2)\, p_u}{\sqrt{1 - u^2}},$$

(5)
$$\gamma_1 = \sqrt{1 - u^2}\sin\phi, \quad \gamma_2 = \sqrt{1 - u^2}\cos\phi, \quad \gamma_3 = u.$$

The paper [VT09] introduced separating variables q_1, q_2 as the roots of the polynomial

$$A(\lambda) = \lambda^2 + u^{1/3}\left(\frac{up_\phi}{1 - u^2} - \imath p_u\right)\lambda - \frac{\imath a e^{\imath\phi}}{\sqrt{1 - u^2}}, \quad \imath = \sqrt{-1}.$$

Observe this polynomial depends not only on the coordinates u, ϕ of S^2 but also on their momenta p_u, p_ϕ. Following [VT09], in the Darboux coordinates q_i, p_i such that

(6)
$$\{q_i, p_j\} = \delta_{ij}, \quad \{q_i, q_j\} = \{p_i, p_j\} = 0,$$

one has

$$p_\phi = \imath\, q_1 q_2 \frac{p_2 - p_1}{q_1 - q_2}, \quad u = \left[-\frac{2}{3}\imath\frac{q_1 p_1 - q_2 p_2}{q_1 - q_2}\right]^{3/2},$$

(7)
$$u^{1/3}\left(\frac{up_\phi}{1 - u^2} - \imath p_u\right) = -q_1 - q_2, \quad \frac{\imath a e^{\imath\phi}}{\sqrt{1 - u^2}} = -q_1 q_2.$$

Here one should stress that in the real case the coordinates q_1, p_1, q_2, p_2 are complex. In the sequel, by default, we consider all the variables as complex, leaving the analysis of real conditions to a separate study in the future. In particular, q_i, p_i will be regarded as Darboux coordinates on the complexified cotangent bundle $T^* S^2$.

Under the above substitution the two Hamiltonians take the Stäckel form

$$H_1 = (S^{-1})_{11}U(p_1, q_1) + (S^{-1})_{21}U(p_2, q_2),$$

(8)
$$H_2 = (S^{-1})_{12}U(p_1, q_1) + (S^{-1})_{22}U(p_2, q_2),$$

$$U(q, p) = \frac{3}{2\imath q^2 p}\left(-\frac{8}{27}\imath p^3 q^4 + \lambda^4 + \frac{1}{4}a^2 - b\lambda^2\right),$$

with the Stäckel matrix

$$S = \begin{pmatrix} 1 & 1 \\ 3\imath/(2p_1q_1) & 3\imath/(2p_2q_2) \end{pmatrix}.$$

Setting for convenience $\lambda_i = q_i$, $\mu_i = 2/3\,\imath q_i p_i$, we get

$$U(q, p) = \bar{U}(\lambda, \mu) = \frac{1}{\lambda\mu}(\lambda\mu^3 + \lambda^4 + a^2/4 - b\lambda^2), \quad S = \begin{pmatrix} 1 & 1 \\ -1/\mu_1 & -1/\mu_2 \end{pmatrix}$$

and then observe that the above relations are equivalent to

$$H_1 - H_2\frac{1}{\mu_1} = \bar{U}(\lambda_1, \mu_1), \quad H_1 - H_2\frac{1}{\mu_2} = \bar{U}(\lambda_2, \mu_2)$$

Then, upon fixing the values of the integrals, $H_1 = h_1$, $H_2 = h_2$, the pairs (λ_i, μ_i) are subject to the algebraic relation

$$\mu h_1 - h_2 = \mu^3 + \lambda^3 + \frac{a^2}{4\lambda} - b\lambda$$

or

(9)
$$F = \lambda^4 - b\lambda^2 + (\mu^3 - h_1\mu + h_2)\lambda + a^2/4 = 0.$$

As was also mentioned in [VT09], equation (9) defines an algebraic curve $\mathcal{C}' \subset \mathbb{C}^2(\lambda, \mu)$, which, for generic values of h_1, h_2 is smooth, has genus 3, and is non-hyperelliptic, i.e., cannot be transformed to the form $y^2 = P(x)$ by a birational change of coordinates. A basis of holomorphic differentials on $\mathcal{C}'$ is

(10)
$$\omega_1 = \frac{d\lambda}{\partial F/\partial\mu}, \quad \omega_2 = \frac{\lambda d\lambda}{\partial F/\partial\mu}, \quad \omega_3 = \frac{\mu\,d\lambda}{\partial F/\partial\mu}, \quad \frac{\partial F}{\partial\mu} = \lambda(3\mu^2 - h_1).$$

2.1. **The quadratures.** Let t_1, t_2 denote the time of the flows on T^*S^2 defined respectively by the Hamiltonians H_1 and H_2. To describe the evolution of $q_i = \lambda_i$ with H_2, we use the bracket (6) and the expressions (8) to obtain

$$\frac{d}{d\,t_2}\lambda_1 = \frac{\partial H_2}{\partial p_1} = \frac{\partial H_2}{\partial\mu_1}\frac{2}{3}\imath\lambda_1 = \left[\left(\frac{\mu_2}{\mu_1 - \mu_2} - \frac{\mu_1\mu_2}{(\mu_1 - \mu_2)^2}\right)(\bar{U}_1 - \bar{U}_2) + \frac{\mu_1\mu_2}{\mu_1 - \mu_2}\frac{\partial\bar{U}_1}{\partial\mu_1}\right]\frac{2}{3}\imath\lambda_1$$

$$= \frac{2}{3}\imath\lambda_1\frac{\mu_2}{\mu_1 - \mu_2}\left(\bar{U}_1 + \mu_1\frac{\partial\bar{U}_1}{\partial\mu_1} - \underbrace{\frac{1}{\mu_1 - \mu_2}(\mu_1\bar{U}_1 - \mu_2\bar{U}_2)}_{H_1(\lambda,\mu)}\right), \quad \bar{U}_i = \bar{U}(\lambda_i, \mu_i).$$

In view of the expression for F in (9) this is equivalent to

(11)
$$\frac{d\lambda_1}{d\,t_2} = \frac{2}{3}\imath\frac{\mu_2}{\mu_1 - \mu_2}\frac{\partial F(\lambda_1, \mu_1)}{\partial\mu_1}, \quad \text{and, similarly,} \quad \frac{d\lambda_2}{d\,t_2} = \frac{2}{3}\imath\frac{\mu_1}{\mu_2 - \mu_1}\frac{\partial F(\lambda_2, \mu_2)}{\partial\mu_2}.$$

Similarly one obtains for the flow with the quadratic Hamiltonian H_1

(12)
$$\frac{d\lambda_1}{d\,t_1} = -\frac{2}{3}\imath\frac{1}{\mu_1 - \mu_2}\frac{\partial F(\lambda_1, \mu_1)}{\partial\mu_1}, \quad \frac{d\lambda_2}{d\,t_1} = -\frac{2}{3}\imath\frac{1}{\mu_2 - \mu_1}\frac{\partial F(\lambda_2, \mu_2)}{\partial\mu_2},$$

and also

(13)
$$\frac{d\mu_1}{d\,t_1} = \frac{\partial\mu_1}{\partial\lambda_1}\frac{d\lambda_1}{d\,t_1} = \frac{2}{3}\imath\frac{1}{\mu_1 - \mu_2}\frac{\partial F(\lambda_1, \mu_1)}{\partial\lambda_1}, \quad \frac{d\mu_2}{d\,t_1} = \frac{2}{3}\imath\frac{1}{\mu_2 - \mu_1}\frac{\partial F(\lambda_2, \mu_2)}{\partial\lambda_2}.$$

Expressions (12), (11) yield the following quadratures in the differential form

$$\frac{d\lambda_1}{\partial F(\lambda_1,\mu_1)/\partial\mu_1} + \frac{d\lambda_2}{\partial F(\lambda_2,\mu_2)/\partial\mu_2} = -\frac{2}{3}\imath\, dt_2$$

(14)

$$\frac{\mu_1 d\lambda_2}{\partial F(\lambda_1,\mu_1)/\partial\mu_1} + \frac{\mu_2 d\lambda_2}{\partial F(\lambda_2,\mu_2)/\partial\mu_2} = -\frac{2}{3}\imath\, dt_1.$$

We will return to these later.

2.2. The original variables in terms of the separating ones.
From (7) and other formulae from the paper [VT09] one has

$$\gamma_3^2 = -\left(\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^3, \quad \gamma_3^{2/3} = -\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2},$$

$$J_3 = -p_\phi = -\frac{3}{2}\frac{\lambda_1\mu_2-\lambda_2\mu_1}{\lambda_1-\lambda_2},$$

$$\exp(\imath\phi) = \frac{2\imath}{a}\lambda_1\lambda_2\sqrt{1+\left(\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^3},$$

$$\gamma_2+\imath\gamma_1 = \frac{2\imath}{a}\lambda_1\lambda_2\left(1+\left(\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^3\right), \quad \gamma_2-\imath\gamma_1 = \frac{a}{2\imath\,\lambda_1\lambda_2},$$

(15)

$$J_1+\imath J_2 = -\imath\left(\imath p_u - \frac{u}{1-u^2}p_\phi\right)e^{-\imath\phi}\sqrt{1-u^2}$$

$$= -\frac{a}{2}\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}\left(-\frac{\mu_1-\mu_2}{\lambda_1-\lambda_2}\right)^{-1/2},$$

$$J_1-\imath J_2 = \frac{1}{\gamma_2-\imath\gamma_1}\left((J_1+\imath J_2)(\gamma_2+\imath\gamma_1)+2\imath J_3\gamma_3\right) \quad \text{(due to the condition } J\cdot\gamma=0)$$

$$= \frac{2\,\lambda_1\,\lambda_2\left(\left(1+\frac{(\mu_1-\mu_2)^3}{(\lambda_1-\lambda_2)^3}\right)(\lambda_1+\lambda_2)+\frac{3\,(\mu_1-\mu_2)^2\,(\lambda_1\,\mu_2-\mu_1\,\lambda_2)}{(\lambda_1-\lambda_2)^3}\right)}{a\sqrt{-\frac{\mu_1-\mu_2}{\lambda_1-x_2}}}$$

Substituting the above expressions into the integrals (2) and using the equation of the curve (9) for each pair (λ_i,μ_i) one may verify the identities $H_1 = h_1, H_2 = h_2$. Thus to solve for the original system (1), (2) it suffices to solve for the pairs (λ_i,μ_i).

2.3. Canonical form of the curve and of the quadratures.
Rather than using the variables (λ_i,μ_i) directly we now make one final birational transformation to bring the curve $\mathcal{C}'$ to a canonical form. This allows us to make connection with the literature on (n,s) curves and so permits us to solve for the motion in terms of the multi-dimensional σ-function.

By making the birational change

(16)
$$\lambda = \frac{1}{x}\sqrt{\frac{a}{2}}, \quad \mu = -\frac{y}{x}\sqrt{\frac{a}{2}}$$

the curve (9) can be transformed to the *canonical trigonal form* with respect to y

(17)
$$G(x,y) = y^3 - 2\frac{h_1}{a}x^2\,y - \left(x^4 + \frac{2\sqrt{2}h_2}{a^{3/2}}x^3 - 2\frac{b}{a}x^2 + 1\right) = 0.$$

We will refer to this curve as $\mathcal{C}$. In the terminology of [BEL00] (see also the next section) this is a (3,4)-curve having one infinite branch point ∞, where all the 3 sheets of the covering $\mathcal{C} \to \mathbb{P}^1 = \{x\}$ come together. Under the above transformation the holomorphic differentials $\omega_1, \omega_2, \omega_3$ in (10) take the following respective forms,

(18)
$$\Omega_2 = -\frac{x\,\mathrm{d}x}{\partial G/\partial y}, \quad \Omega_1 = \frac{\mathrm{d}x}{\partial G/\partial y}, \quad \Omega_3 = \frac{y\,\mathrm{d}x}{\partial G/\partial y}, \quad \frac{\partial G}{\partial y} = 3y^2 - \frac{2h_1}{a}x^2.$$

Then, in the new coordinates, the quadratures (14) take the form

$$\frac{x_1\,\mathrm{d}x_1}{\partial G(x_1,y_1)/\partial y_1} + \frac{x_2\mathrm{d}x_2}{\partial G(x_2,y_2)/\partial y_2} = -\frac{2}{3}\imath\,\mathrm{d}t_2,$$

(19)

$$\frac{y_1\mathrm{d}x_1}{\partial G(x_1,y_1)/\partial y_1} + \frac{y_2\mathrm{d}x_2}{\partial G(x_2,y_2)/\partial y_2} = -\frac{2}{3}\imath\,\mathrm{d}t_1.$$

We again stress that, although in the real case the curve (17) is real, the new separating variables x_1, x_2 are complex. The description of their behaviour in the real case could be an object of a separate study.

2.4. **Non-algebraic integrability.** We are now in the situation described in the introduction. The genus of the curve $\mathcal{C}$ is greater than that of the dimension of generic invariant tori of the system: here we have 2 separating variables while $\mathcal{C}$ is of genus 3. Such a situation occurs in many algebraically integrable systems, for example the Clebsch integrable case of the Kirchhoff equations or the Kovalevskaya top. In these examples however, although the genus of the underlying curve (the spectral curve of the corresponding Lax representation) is greater than the number of degrees of freedom, the relevant curves possess an additional involution which extends to the Jacobian variety. Then, as a rule, the complex invariant manifolds of the systems turn out to be 2-dimensional Abelian (Prym) subvarieties of the Jacobians, whose real part gives the invariant tori. This however is not the case for the Goryachev system on T^*S^2 we are considering. For generic h_1, h_2 the curve $\mathcal{C}$ has no further symmetries, and the differentials in (19) do not reduce to those of a genus 2 curve. As we shall see below, this pathological property means such systems are not algebraically completely integrable. In particular, their complex invariant manifolds are non-Abelian subvarieties of the Jacobian of $\mathcal{C}$ and the variables J_i, γ_i are not meromorphic functions of the complex times t_1, t_2.

In terms of the differentials (18) the quadratures (19) may be extended and written in the following integral form

(20)
$$\int_{\infty}^{P_1} (\Omega_1, \Omega_2, \Omega_3)^T + \int_{\infty}^{P_2} (\Omega_1, \Omega_2, \Omega_3)^T = \begin{cases} u_1 \\ u_2 = -2\imath t_2/3 + u_{20} \\ u_3 = -2\imath t_1/3 + u_{30} \end{cases}, \quad P_i = (x_i, y_i) \in \mathcal{C},$$

where u_{20}, u_{30} are constant phases and the coordinate u_1 is a transcendental function of u_2, u_3, whose properties will be described in the next sections. Thus (20) defines a map of the symmetric product $\mathcal{C} \times \mathcal{C}$ to a codimension one subvariety (stratum) of the Jacobian variety of $\mathcal{C}$. To invert the map, i.e., to express symmetric functions of the coordinates $(x_1, y_1), (x_2, y_2)$ and consequently the variables J_i, γ_i in terms of u_2, u_3, at least locally, we next recall some basic facts about the standard Jacobi inversion problem associated to trigonal curves.

3. JACOBI INVERSION PROBLEM FOR THE GENUS THREE TRIGONAL CURVE

The curve (17) belongs to a class of (n, s)-curves. These are smooth curves with $s > n \geq 2$ and $\gcd(n, s) = 1$ that have one point at infinity and whose affine part may be defined by an equation

$$y^n - x^s - \sum_{\alpha,\beta} \nu_{\alpha n + \beta s} x^\alpha y^\beta = 0, \qquad 0 \leq \alpha < s - 1, \quad 0 \leq \beta < n - 1.$$

The curves are of genus $g = (n-1)(s-1)/2$ and their properties and relation to integrable hierarchies of KP type are widely discussed in the literature.

Below we concentrate on the trigonal $(3,4)$-curve $\widetilde{\mathcal{C}}$, which we write in the canonical form

(21) $\qquad f(x, y) = y^3 + (\mu_2 x^2 + \mu_5 x + \mu_8)y - (x^4 + \mu_3 x^3 + \mu_6 x^2 + \mu_9 x + \mu_{12}) = 0,$

where μ_j are parameters. This is more general than our curves (17) and we will specialize in due course. The curve has one infinite point ∞, where all 3 sheets of the covering $\mathcal{C} \to \mathbb{P} = \{x\}$ come together. Let $\xi = x^{-1/3}$ be a local coordinate in a neighborhood of ∞. That is, $\xi(\infty) = 0$, and the coordinates in the vicinity of this point have expansions

(22) $\qquad x = \frac{1}{\xi^3}, \quad y = \frac{1}{\xi^4} - \frac{\mu_2}{3}\frac{1}{\xi^2} + \frac{\mu_3}{3}\frac{1}{\xi} + O(\xi).$

Choose the vector of holomorphic differentials $\boldsymbol{\Omega} = (\Omega_1, \Omega_2, \Omega_3)^T$,

$$(23) \qquad \boldsymbol{\Omega} = \frac{dx}{f_y(x,y)} \begin{pmatrix} 1 \\ x \\ y \end{pmatrix}.$$

Near ∞, they admit the expansions

$$(24) \qquad \begin{aligned} \Omega_1 &= -\xi^4 \left(1 + \frac{\mu_2}{3}\xi^2 - \frac{2\mu_3}{3}\xi^3 + O(\xi^5) \right) d\xi, \\ \Omega_2 &= -\xi \left(1 + \frac{\mu_2}{3}\xi^2 - \frac{2\mu_3}{3}\xi^3 + O(\xi^5) \right) d\xi, \\ \Omega_3 &= - \left(1 - \frac{\mu_3}{3}\xi^3 - \frac{\mu_2^2}{9}\xi^4 + O(\xi^6) \right) d\xi, \end{aligned}$$

so the orders of their zeros at ∞ decrease.

Next, choose a canonical basis of cycles of $H_1(\mathcal{C}, \mathbb{Z})$

$$(\mathfrak{a}, \mathfrak{b}) = (\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3; \mathfrak{b}_1, \mathfrak{b}_2, \mathfrak{b}_3), \quad \text{such that} \quad \mathfrak{a}_i \circ \mathfrak{a}_j = \mathfrak{b}_i \circ \mathfrak{b}_j = 0, \quad \mathfrak{a}_i \circ \mathfrak{b}_j = -\mathfrak{b}_i \circ \mathfrak{a}_j = 1$$

and introduce matrices of periods of the above differentials

$$(25) \qquad \mathcal{A} = \left(\oint_{\mathfrak{a}_j} \Omega_i \right)_{i,j=1,2,3}, \quad \mathcal{B} = \left(\oint_{\mathfrak{b}_j} \Omega_i \right)_{i,j=1,2,3}.$$

Throughout the whole paper we will use two normalization of periods: the first, given above, $(\mathcal{A}, \mathcal{B})$, where we have specified the differentials; the second utilizes the so called $\mathfrak{a}$-normalized differentials for which the matrix of periods takes the form $(1_3, \tau)$. Here τ is the Riemann period matrix

$$(26) \qquad \tau = \mathcal{A}^{-1}\mathcal{B}, \quad \tau^T = \tau, \quad \operatorname{Im}\tau > 0.$$

We denote the $\mathfrak{a}$-normalized holomorphic differentials by

$$(27) \qquad \overline{\boldsymbol{\Omega}} = \mathcal{A}^{-1}\boldsymbol{\Omega} \quad \Leftrightarrow \quad \oint_{\mathfrak{a}_j} \bar{\Omega}_i = \delta_{ij}, \quad i, j = 1, 2, 3.$$

The first normalization is used in the definition of the σ-function while the second in the definition of Riemann's theta-function $\theta(\boldsymbol{z}; \tau)$ associated with $\mathcal{C}$ and the Riemann period matrix τ given in (26); it is defined by the series

$$(28) \qquad \theta(\boldsymbol{v}; \tau) = \sum_{\boldsymbol{n} \in \mathbb{Z}^3} \exp\left\{ \imath \pi \boldsymbol{n}^T \cdot \tau \cdot \boldsymbol{n} + 2\imath\pi\boldsymbol{v}^T \cdot \boldsymbol{n} \right\}.$$

With these normalizations we define the Jacobi variety $\operatorname{Jac}(\mathcal{C})$ of the curve $\mathcal{C}$ to be the quotient $\operatorname{Jac}(\mathcal{C}) = \mathbb{C}^3(u_1, u_2, u_3)/\{\mathcal{A} \oplus \mathcal{B}\}$ and also $\overline{\operatorname{Jac}}(\mathcal{C}) = \mathbb{C}^3(v_1, v_2, v_3)/\{1_3 \oplus \tau\}$, where the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ are related by $\boldsymbol{v} = (v_1, v_2, v_3)^T = \mathcal{A}^{-1}\boldsymbol{u}$.

For a positive divisor $\mathcal{D} = P_1 + P_2 + P_3$, $P_i = (x_i, y_i) \in \mathcal{C}$, we consider *the Abel map* with a base point P_0, $\mathfrak{A} : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \longrightarrow \operatorname{Jac}(\mathcal{C})$, given by

$$(29) \qquad \mathfrak{A} : \mathcal{D} \longrightarrow \int_{P_0}^{P_1} \boldsymbol{\Omega} + \int_{P_0}^{P_2} \boldsymbol{\Omega} + \int_{P_0}^{P_3} \boldsymbol{\Omega} = \boldsymbol{u}, \quad \text{or, equivalently,} \quad \int_{P_0}^{P_1} \overline{\boldsymbol{\Omega}} + \int_{P_0}^{P_2} \overline{\boldsymbol{\Omega}} + \int_{P_0}^{P_3} \overline{\boldsymbol{\Omega}} = \boldsymbol{v},$$

Henceforth we assume that $P_0 = (\infty, \infty)$. The Jacobi inversion problem refers to the inversion of Abel map. For the $g = 3$ case being considered this is solved in terms of Riemann's θ-functions as follows.

Theorem 3.1. (Riemann) *Let $\boldsymbol{e}$ be a vector such that the function $F(P) = \theta(\int_\infty^P \overline{\boldsymbol{\Omega}} - \boldsymbol{e}; \tau)$ does not vanish identically. Then $F(P)$ has exactly 3 zeros on $\mathcal{C}$, P_1, P_2, P_3, and these provide a solution of the Jacobi inversion problem*

$$(30) \qquad \int_{3\infty}^{P_1+P_2+P_3} \overline{\boldsymbol{\Omega}} = \boldsymbol{e} - \overline{\boldsymbol{K}}_\infty,$$

where $\overline{\boldsymbol{K}}_\infty = (\overline{K}_1, \overline{K}_1, \overline{K}_3)$ *is the vector of Riemann constants with base point* ∞ *and whose components are given by*

$$(31) \qquad \overline{K}_r = \frac{1}{2}(2\pi\imath + \tau_{rr}) - \frac{1}{2\pi\imath} \sum_{l \neq r}^{3} \left(\oint_{\mathfrak{a}_l} \overline{\Omega}_l(P) \int_\infty^P \overline{\Omega}_r \right), \qquad r = 1, 2, 3.$$

The corresponding divisor $\mathcal{D} = P_1 + P_2 + P_3$ *is non-special, and in the vicinity of* $\mathcal{D}$ *the map* $\mathfrak{A}$ *is uniquely invertible.*

The rank of the Abel map is maximal on non-special divisors and decreases on special subvarieties of the Jacobian, the *Wirtinger strata*. Here $W^{(0)} = \overline{\boldsymbol{K}}_\infty \subset W^{(1)} \subset W^{(2)} \subset \mathrm{Jac}(\mathcal{C})$ where

$$(32) \qquad \begin{aligned} W^{(1)}: \quad & \boldsymbol{v} \in \mathrm{Jac}(X), \quad \boldsymbol{v} = \int_\infty^P \overline{\boldsymbol{\Omega}} + \overline{\boldsymbol{K}}_\infty, \quad \forall P \in \mathcal{C}, \\ W^{(2)}: \quad & \boldsymbol{v} \in \mathrm{Jac}(X), \quad \boldsymbol{v} = \int_\infty^{P_1} \overline{\boldsymbol{\Omega}} + \int_\infty^{P_2} \overline{\boldsymbol{\Omega}} + \overline{\boldsymbol{K}}_\infty, \quad \forall (P_1, P_2) \in \mathcal{C} \times \mathcal{C}. \end{aligned}$$

The equation $\theta(\boldsymbol{v}; \tau) = 0$ defines a codimension one subvariety $\Theta \in \mathrm{Jac}(\mathcal{C})$ (with singularities for $g > 2$) called the *theta-divisor*, which coincides with the stratum $W^{(2)}$. This is equivalent to the fact that for any points $P_1, P_2 \in \mathcal{C}$,

$$(33) \qquad \theta\left(\int_\infty^{P_1} \overline{\boldsymbol{\Omega}} + \int_\infty^{P_2} \overline{\boldsymbol{\Omega}} + \overline{\boldsymbol{K}}_\infty; \tau \right) \equiv 0.$$

Note that a consequence of Riemann's theorem is that the vector $\overline{\boldsymbol{K}}_\infty$ for which (33) holds is unique.

We also introduce characteristics, represented by real 2×3 matrices

$$[\varepsilon] = \begin{pmatrix} \boldsymbol{\varepsilon}^T \\ \boldsymbol{\varepsilon}'^T \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ \varepsilon_1' & \varepsilon_2' & \varepsilon_3' \end{pmatrix},$$

so that any vector $\boldsymbol{u} \in \mathrm{Jac}(\mathcal{C})$ can be written in the form $\boldsymbol{u} = \mathcal{A}\boldsymbol{\varepsilon}' + \mathcal{B}\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}' \in \mathbb{R}^3$. We denote the characteristic of $\boldsymbol{u}$ by $[\boldsymbol{u}]$. In what follows we concentrate on rational characteristics, in particular half-integer ones, for which $\varepsilon_i, \varepsilon_j' = \frac{1}{2}$ or 0. We shall also need the Riemann theta-functions with characteristics $[\varepsilon]$ given by

$$\theta\begin{bmatrix} \boldsymbol{\varepsilon}^T \\ \boldsymbol{\varepsilon}'^T \end{bmatrix}(\boldsymbol{v}; \tau) = \exp\left\{ \imath\pi(\boldsymbol{\varepsilon}^T \tau \boldsymbol{\varepsilon} + 2\boldsymbol{\varepsilon}^T(\boldsymbol{v} + \boldsymbol{\varepsilon}')) \right\} \theta(\boldsymbol{v} + \tau\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}'; \tau)$$

$$= \sum_{\boldsymbol{n} \in \mathbb{Z}^3} \exp\left\{ \imath\pi(\boldsymbol{n} + \boldsymbol{\varepsilon})^T \tau(\boldsymbol{n} + \boldsymbol{\varepsilon}) + 2\imath\pi(\boldsymbol{n} + \boldsymbol{\varepsilon})^T(\boldsymbol{v} + \boldsymbol{\varepsilon}') \right\},$$

3.1. **Calculation of** $\overline{\boldsymbol{K}}_\infty$. The vector of Riemann constant $\overline{\boldsymbol{K}}_\infty$ given by (31) includes Abelian integrals over $\mathfrak{a}$-cycles, which are difficult to calculate directly. It is known that if a curve has a point P_* such that the canonical divisor is linearly equivalent to $2(g-1)P_*$ then the vector of Riemann constants with base point P_* is a half-period [Fay973, FK980]. Such is the case for a hyperelliptic curve. For all (n, s)-curves Nakayashiki [Nak10] observed that these admit a holomorphic differential that vanishes to order $2g - 2$ at the point $P_0 = (\infty, \infty)$ and so we have the following.

Proposition 3.2. *The vector* $\overline{\boldsymbol{K}}_\infty$ *of the trigonal curve* $\mathcal{C}$ *is a half-period in* $\mathrm{Jac}(\mathcal{C})$.

We may further restrict the choice of $\overline{\boldsymbol{K}}_\infty$. There are 64 half-periods in $\mathrm{Jac}(\mathcal{C})$, 28 odd and 36 even ones. Up to exponential terms the σ-function on $\mathcal{C}$ is the θ-function shifted by $\overline{\boldsymbol{K}}_\infty$ (see below). Thus the leading terms of the expansion of the θ-function at $\overline{\boldsymbol{K}}_\infty$ coincides with those of the corresponding σ-function. For $(3, 4)$ curves it was shown in [BEL999] that the σ-function begins with an odd order Schur function and it follows then that $\overline{\boldsymbol{K}}_\infty$ is an odd half-period. Of course the explicit expression for $\overline{\boldsymbol{K}}_\infty$ depends on the choice of homology basis on $\mathcal{C}$ and so to proceed further we must first fix the homology basis. We may specify which half-period then corresponds to $\overline{\boldsymbol{K}}_\infty$ by first calculating the period matrix and then checking the condition (33) by computing the expansion of θ-function near *all* odd half-integer characteristics. For example, for the curve $\mathcal{C}$ given by

$$y^3 - 4yx^2 - x^4 - 3x^3 - 4x^2 - x + 5 = 0$$

and in the Tretkoff–Tretkoff basis of cycles on it given by `algcurves` package of `Maple` one obtains

$$(34) \qquad [\overline{\boldsymbol{K}}_\infty] = \frac{1}{2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

We note in passing that we already encounter here one of the differences with Jacobians of hyperelliptic curves: for a genus three *hyperelliptic* curve the vector of Riemann constants is given by an *even singular* half-integer characteristic whereas the characteristic (34) is non-singular and odd.

3.2. **The σ-functions.** Apart from the theta-functions, in many cases it is more convenient to use the σ-function of $\boldsymbol{u}$. To describe it, we first introduce the basis of 3 *meromorphic* differentials $\boldsymbol{\Upsilon} = (\Upsilon_1, \Upsilon_2, \Upsilon_3)^T$ having a unique pole at the infinity of $\mathcal{C}$ and specified by the pairing conditions

$$(35) \qquad \operatorname{Res}_{P=\infty} \; \Omega_i(P) \int_{P_0}^{P} \Upsilon_j = \delta_{ij}, \qquad i, j = 1, 2, 3.$$

Then, following [EEMOP07],

$$(36) \qquad \boldsymbol{\Upsilon} = \frac{\mathrm{d}x}{f_y(x,y)} \begin{pmatrix} \Upsilon_1 \\ -2xy + \mu_1 x^2 \\ -x^2 \end{pmatrix},$$

where

$$\Upsilon_1 = -(5x^2 + (\mu_1\mu_2 - 3\mu_3)x + \mu_2\mu_4 + \mu_6)y + mu_2 y^2 + 3\mu_1 x^3$$
$$- (\mu_2^2 + 2\mu_3\mu_1 - 2\mu_4)x^2 - (\mu_5\mu_2 + \mu_6\mu_1 + \mu_3\mu_4)x + \frac{3}{4}\mu_1 f_x(x,y) - \left(\frac{1}{3}\mu_2 - \frac{1}{4}\mu_1^2\right) f_y(x,y).$$

The matrices of periods of the differentials

$$(37) \qquad \mathcal{S} = -\left(\oint_{a_k} \mathrm{d}\Upsilon_i\right)_{i,k=1,2,3}, \qquad \mathcal{T} = -\left(\oint_{b_k} \mathrm{d}\Upsilon_i\right)_{i,k=1,2,3},$$

satisfy the generalized Legendre relation

$$(38) \qquad \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{S} & \mathcal{T} \end{pmatrix} J \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{S} & \mathcal{T} \end{pmatrix}^T = -2\pi\iota J, \qquad J = \begin{pmatrix} 0_3 & -1_3 \\ 1_3 & 0_3 \end{pmatrix}.$$

Let us also introduce the normalized second period matrix $\varkappa = \mathcal{S}\mathcal{A}^{-1}$, which is necessarily symmetric.

The fundamental σ-function of the curve $\mathcal{C}$ is defined by the formula (see e.g., [BEL997, EEL00])

$$(39) \qquad \sigma(\boldsymbol{u}) = C \, \theta[\boldsymbol{K}](\mathcal{A}^{-1}\boldsymbol{u}) \exp\left\{\frac{1}{2}\boldsymbol{u}^T \varkappa \boldsymbol{u}\right\},$$

where $\theta[\boldsymbol{K}](\boldsymbol{v})$ is the theta-function with the characteristic corresponding to the vector of Riemann constants (for a chosen base point and homology basis), and C is a constant depending on the period matrix $\mathcal{A}$ and the coefficients of the curve. The constant C provides the modular invariance of (39), just as in the case of the Weierstrass elliptic σ-function. An explicit expression for C is given in [EEMOP07] and it is not necessary for our exposition (we will deal only with ratios of the sigma-function derivatives). According to the definition, the fundamental σ-function is normalized in such the way that its expansion at $\boldsymbol{u} \sim \boldsymbol{0}$ starts with the Schur-Weierstrass polynomial (see [BEL999] for details). It follows that $\sigma(\boldsymbol{u})$ is just the theta-function of $\mathcal{C}$ whose rescaled argument is shifted by the vector $\overline{\boldsymbol{K}}_\infty$ and multiplied by a quadratic exponent of $\boldsymbol{u}$. Thus, the knowledge of the corresponding characteristic $[\boldsymbol{K}]$ calculated above is important in the explicit description of $\sigma(\boldsymbol{u})$.

The function $\sigma(\boldsymbol{u})$ is quasi-periodic just as is $\theta(\boldsymbol{v})$: when $\boldsymbol{u}$ is shifted by a period $\mathcal{A}\boldsymbol{n} + \mathcal{B}\boldsymbol{m}$, $\boldsymbol{n}, \boldsymbol{m} \in \mathbb{Z}^3$ then $\sigma(\boldsymbol{u})$ is multiplied by the exponent

$$(40) \qquad \sigma(\boldsymbol{u} + \mathcal{A}\boldsymbol{n} + \mathcal{B}\boldsymbol{m}) = \sigma(\boldsymbol{u})\exp\left\{(\mathcal{S}\boldsymbol{n} + \mathcal{T}\boldsymbol{m})^T \left(\boldsymbol{u} + \frac{1}{2}\mathcal{A}\boldsymbol{n} + \frac{1}{2}\mathcal{B}\boldsymbol{m}\right)\right\}$$

For the case of the curve (21), at the origin $\boldsymbol{u} = 0$, the σ-function admits the following expansion (see [EEMOP07])

$$(41) \qquad \sigma(\boldsymbol{u}) = u_1 - u_2^2 u_3 + \frac{1}{20}u_3^5 - \frac{\mu_2}{168}u_3^7 + \frac{\mu_2}{6}u_3^3 u_2^2 + \frac{\mu_3}{40}u_3^6 u_2 - \frac{\mu_3}{2}u_3^2 u_2^3 + \text{higher order terms.}$$

It follows from property (33) and the definition (39) that for any points $P_1, P_2 \in \mathcal{C}$

$$(42) \qquad \sigma\left(\int_\infty^{P_1} \boldsymbol{\Omega} + \int_\infty^{P_2} \boldsymbol{\Omega}\right) \equiv 0.$$

That is, in $\boldsymbol{u}$-coordinates on $\mathrm{Jac}(\mathcal{C})$, the stratum $W^{(2)}$ is given by the condition $\sigma(\boldsymbol{u}) = 0$.

3.3. Inversion of the Abel map in terms of the sigma function.

We next introduce the Kleinian multi-index symbols

$$\wp_{i,j}(\boldsymbol{u}) = -\frac{\partial^2}{\partial u_i \partial u_j} \ln \sigma(\boldsymbol{u}), \qquad i,j = 1,2,3,$$

$$(43) \qquad \wp_{i,j,k}(\boldsymbol{u}) = -\frac{\partial^2}{\partial u_i \partial u_j \partial u_k} \ln \sigma(\boldsymbol{u}), \qquad i,j,k = 1,2,3$$

$$\vdots$$

These are multiply periodic (Abelian) functions

$$(44) \qquad \wp_{\mathcal{J}}(\boldsymbol{u} + \mathcal{A}\boldsymbol{n} + \mathcal{B}\boldsymbol{m}) = \wp_{\mathcal{J}}(\boldsymbol{u}),$$

where $\mathcal{J}$ is arbitrary multi-index with more than one entry. It will also be convenient to denote throughout

$$\sigma_i(\boldsymbol{u}) = \frac{\partial}{\partial u_i} \sigma(\boldsymbol{u})$$

though these are not Abelian functions.

Then, as was first shown in [BEL00, EEL00] (though in a somewhat hidden fashion), the problem of inversion of the map (29) is reduced to solving the following two equations with respect to x and y,

$$(45) \qquad \wp_{33}(\boldsymbol{u})y + \wp_{23}(\boldsymbol{u})x + \wp_{13}(\boldsymbol{u}) = x^2,$$

$$(46) \qquad (\wp_{23}(\boldsymbol{u}) - \wp_{333}(\boldsymbol{u}))y + (\wp_{22}(\boldsymbol{u}) - \wp_{233}(\boldsymbol{u}))x + \wp_{12}(\boldsymbol{u}) - \wp_{133}(\boldsymbol{u}) = 2xy,$$

the solutions of which give the coordinates of the points P_1, P_2, P_3 in (29). In contrast to the inversion problem for hyperelliptic case, equations (45) and (46) both contain the variables x and y. By elimination of y, one obtains a cubic equation with respect to x, whereas elimination of x yields a cubic equation for y. In the first case we get (the new expression, though a simple consequence of the previous)

$$(47) \qquad \begin{aligned} \mathcal{P}(x; \boldsymbol{u}) &= 2x^3 - (3\wp_{23}(\boldsymbol{u}) - \wp_{333}(\boldsymbol{u}))x^2 \\ &\quad - (\wp_{33}(\boldsymbol{u})\wp_{22}(\boldsymbol{u}) + \wp_{23}(\boldsymbol{u})\wp_{333}(\boldsymbol{u}) - \wp_{33}(\boldsymbol{u})\wp_{233}(\boldsymbol{u}) - \wp_{23}(\boldsymbol{u})^2 + 2\wp_{13}(\boldsymbol{u}))x \\ &\quad - \wp_{33}(\boldsymbol{u})(\wp_{12}(\boldsymbol{u}) - \wp_{133}(\boldsymbol{u})) + \wp_{13}(\boldsymbol{u})(\wp_{23}(\boldsymbol{u}) - \wp_{333}(\boldsymbol{u})) \\ &= 2(x - x_1)(x - x_2)(x - x_3). \end{aligned}$$

These formulae enable one to express the elementary symmetric functions of x_i as Abelian functions of $\boldsymbol{u}$. We shall use some of them in the next section.

3.4. The inverse trace formula.

Let $q_i = (0, y_i)$, $i = 1,2,3$ be the three points on the curve $\mathcal{C}$ over $x = 0$ and $\boldsymbol{r}_i = \int_\infty^{q_i} \boldsymbol{\Omega}$ be their images in $\mathrm{Jac}(\mathcal{C})$. By applying the classical method of residues one can also obtain the following "inverse trace formula":

$$(48) \qquad \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \sum_{i=1}^3 \partial_{\mathcal{V}_i} \log \sigma(\boldsymbol{r}_i - \boldsymbol{u}) + k, \quad \mathcal{V}_i = \left(\frac{1}{f_y(0, y_i)}, 0, \frac{y_i}{f_y(0, y_i)}\right)^T.$$

Here $\mathcal{V}_i$ is the tangent vector to $W^{(1)} \subset \mathrm{Jac}(\mathcal{C})$ at the point $\boldsymbol{r}_i$, and k is a constant depending on the curve $\mathcal{C}$ only.

Indeed, consider the single-valued function $F(P) = \theta \left(\int_\infty^P \overline{\Omega} - \boldsymbol{v} - \overline{\boldsymbol{K}}_\infty \right)$ defined on a simply connected dissection $\widetilde{\mathcal{C}}$ of $\mathcal{C}$ with boundary $\partial\widetilde{\mathcal{C}}$. Then, for a meromorphic function $f(P)$, $P \in \mathcal{C}$ with the poles $Q_1, \ldots, Q_s$ on $\mathcal{C}$, the residue formula gives (see, e.g., [Dub81],[BBEIM94])

$$(49) \qquad f(P_1) + f(P_2) + f(P_3) = \frac{1}{2\pi\imath} \oint_{\partial\widetilde{\mathcal{C}}} f(P) d\log F(P) - \sum_{k=1}^{s} \mathrm{Res}_{Q_s} f(P) d\log F(P).$$

Set here $f(P) = 1/x$ and observe that this function has simple poles precisely at $Q_i = q_i$, $i = 1, 2, 3$ and that the expression is independent of the choice of local parameter. Then, using the expansions of $F(P)$ in the neighborhood of q_i and the relation (39) between $\theta(\boldsymbol{v})$ and $\sigma(\boldsymbol{u})$, one arrives at (48). Alternatively, one can derive a σ-function analogue of formula (49).

Notice that $\mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3 = 0$, which, in view of the quasi-periodic property of $\sigma(\boldsymbol{u})$, ensures that the right hand side of (48) remains unchanged when $\boldsymbol{u}$ changes by a period vector of $\mathrm{Jac}(\mathcal{C})$. The constant k can be calculated explicitly by letting $x_1, x_2, x_3 \to \infty$ in (48) and evaluating the right hand side (see also formula (51) below).

For the curve relevant to the Goryachev case under consideration we have simplifications.

Proposition 3.3. *In the special case $\mu_8 = 0, \mu_{12} = 1$ in the equation of the trigonal curve (21) one has*

$$(50) \qquad q_i = (0, \rho^i), \quad \rho = \exp(2\pi\imath/3), \quad \mathcal{V}_i = \frac{1}{3}\left(\frac{1}{\rho^{2i}}, 0, \frac{1}{\rho^i}\right) = \frac{1}{3}(\rho^i, 0, \rho^{2i}).$$

In this case formula (48) takes the form

$$(51) \qquad \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} = \frac{1}{3}\sum_{i=1}^{3}\left(\rho^i\partial_1 \log\sigma(\boldsymbol{r}_i - \boldsymbol{u}) + \rho^{2i}\partial_3\log\sigma(\boldsymbol{r}_i - \boldsymbol{u})\right) + k,$$

$$(52) \qquad k = -\frac{1}{3}\left(\frac{\sigma_{23}(\boldsymbol{r}_3)}{\sigma_2(\boldsymbol{r}_3)} + \rho^2\frac{\sigma_{23}(\boldsymbol{r}_1)}{\sigma_2(\boldsymbol{r}_1)} + \rho^4\frac{\sigma_{23}(\boldsymbol{r}_2)}{\sigma_2(\boldsymbol{r}_2)}\right), \quad \boldsymbol{r}_i = \int_\infty^{\rho_i}\boldsymbol{\Omega}.$$

Proof of Proposition 3.3. The formula (51) follows from (48) under the conditions (50). To calculate the constant k, let the points $P_1, P_2, P_3 \in \mathcal{C}$ to tend to ∞ in such a way that under the Abel map (30) one has $u_1 \equiv u_3 \equiv 0$, and the coordinate u_2 tends to zero. Then the left hand of (51) tends to zero, while the right hand side becomes the sum of k and of the terms

$$\frac{\rho^{2i}\sigma_1(\boldsymbol{r}_i) + \rho^i\sigma_3(\boldsymbol{r}_i)}{3\sigma(\boldsymbol{r}_i)}.$$

Now as $\sigma_1(\boldsymbol{r}_i) = \sigma_3(\boldsymbol{r}_i) = \sigma(\boldsymbol{r}_i) = 0$ this expression in indeterminate, but applying l'Hopital's rule to each term we obtain

$$(53) \qquad \lim_{u_2\to 0}\frac{\rho^{2i}\sigma_1(\boldsymbol{r}_i + u_2\boldsymbol{e}_2) + \rho^i\sigma_3(\boldsymbol{r}_i + u_2\boldsymbol{e}_2)}{3\sigma(\boldsymbol{r}_3 + u_2\boldsymbol{e}_2)} = \lim_{u_2\to 0}\frac{\rho^{2i}\sigma_{12}(\boldsymbol{r}_i + u_2\boldsymbol{e}_2) + \rho^i\sigma_{23}(\boldsymbol{r}_i + u_2\boldsymbol{e}_2)}{3\sigma_2(\boldsymbol{r}_i + u_2\boldsymbol{e}_2)}$$

$$= \frac{\rho^{2i}\sigma_{12}(\boldsymbol{r}_i) + \rho^i\sigma_{23}(\boldsymbol{r}_i)}{3\sigma_2(\boldsymbol{r}_i)},$$

where $\boldsymbol{e}_2 = (0, 1, 0)^T$. The latter fractions are well determined. Finally, as we will prove in Proposition 4.2 below, $\sigma_{12}(\boldsymbol{r}_i) = 0$ for any $i = 1, 2, 3$, and so (53) gives (52). $\square$

4. Solving the inversion problem on the stratum $W^{(2)}$ and sigma-function solutions of the Goryachev system

We now identify the trigonal curve (17) that appears in the quadratures for the Goryachev system and the curve $\widetilde{\mathcal{C}}$ in (21) by setting

$$(54)$$

$$\mu_1 = \mu_4 = 0, \quad \mu_2 = -2h_1/a, \quad \mu_5 = \mu_8 = 0, \quad \mu_3 = 4h_2\left(\frac{2}{a}\right)^{2/3}, \quad \mu_6 = -2b/a, \quad \mu_9 = 0, \quad \mu_{12} = 1.$$

Notice again that the map $\mathcal{C} \times \mathcal{C} \to \mathrm{Jac}(\mathcal{C})$ in (20) contains only 2 points on $\mathcal{C}$ and cannot be identified with the full Abel map (29). According to the previous observations, (20) maps the symmetric product $\mathcal{C} \times \mathcal{C}$ to the stratum $W^{(2)} \subset \mathrm{Jac}(\mathcal{C})$ given analytically by the condition $\sigma(\boldsymbol{u}) = 0$.

4.1. The inversion of the Abel map on the stratum $W^{(2)}$.

One can extend this map by adding a third *fixed* point on $\mathcal{C}$, in particular, one of the three points $q_i = (0, \rho^i) \in \mathcal{C}$:

$$\int_{2\infty}^{P_1+P_2} \Omega + \int_{\infty}^{q_i} \Omega = \boldsymbol{u} + \boldsymbol{r}_i, \qquad \boldsymbol{r}_i = \int_{\infty}^{q_i} \Omega,$$

$\boldsymbol{u}$ being the right hand side of (20). Then, by using (47), one obtains the following formal complex expressions for the symmetric functions of x_1, x_2 (which hold for any $i = 1, 2, 3$ on the right hand side):

$$(55) \qquad x_1 + x_2 = 3\wp_{23}(\boldsymbol{u} + \boldsymbol{r}_i) - \wp_{333}(\boldsymbol{u} + \boldsymbol{r}_i),$$

$$x_1 x_2 = \wp_{33}(\boldsymbol{u} + \boldsymbol{r}_i)\wp_{22}(\boldsymbol{u} + \boldsymbol{r}_i) + \wp_{23}(\boldsymbol{u} + \boldsymbol{r}_i)\wp_{333}(\boldsymbol{u} + \boldsymbol{r}_i)$$

$$(56) \qquad - \wp_{33}(\boldsymbol{u} + \boldsymbol{r}_i)\wp_{233}(\boldsymbol{u} + \boldsymbol{r}_i) - \wp_{23}^2(\boldsymbol{u} + \boldsymbol{r}_i) + 2\wp_{13}(\boldsymbol{u} + \boldsymbol{r}_i),$$

where, according to (20), $u_3 = -2\imath t_1/3$, $u_2 = -2\imath t_2/3$, and the coordinate u_1 is defined (but not uniquely!) from the transcendental condition $\sigma(\boldsymbol{u}) = 0$.

On the other hand, letting $x_3 \to \infty$ in the inverse trace formula (51), we find

$$\frac{x_1 + x_2}{x_1 x_2} = \frac{1}{3} \sum_{i=1}^{3} \left(\rho^i \partial_1 \log \sigma(\boldsymbol{r}_i - \boldsymbol{u}) + \rho^{2i} \partial_3 \log \sigma(\boldsymbol{r}_i - \boldsymbol{u}) \right) + k \bigg|_{\sigma(\boldsymbol{u})=0},$$

$$k = -\frac{1}{3} \sum_{i=1}^{3} \frac{\rho^{2i} \sigma_{2,3}(\boldsymbol{r}_i)}{\sigma_2(\boldsymbol{r}_i)}.$$

Upon combining the above with (55), one obtains the following alternative to (56)

$$(57) \qquad x_1 x_2 = \frac{3\wp_{23}(\boldsymbol{u} + \boldsymbol{r}_i) - \wp_{333}(\boldsymbol{u} + \boldsymbol{r}_i)}{\sum_{i=1}^{3} \partial_{\mathcal{V}_i} \log \sigma(\boldsymbol{u} - \boldsymbol{r}_i) + k} \bigg|_{\sigma(\boldsymbol{u})=0}, \qquad \mathcal{V}_i = \frac{1}{3}(\rho^i, 0, \rho^{2i}).$$

There is another way of writing the formal solution to the inversion of (20). (For the case of hyperelliptic curves this was proposed in [Gra990, Jor992] and also used in [EPR03].) Namely, consider again the full Abel map (29) with the three points $P_1 = (x_1, y_1), P_2 = (x_2, y_2), P_3 = (x_3, y_3)$ and observe that

$$x_1 + x_2 = \lim_{x_3 \to \infty} \frac{x_1 x_2 + x_2 x_3 + x_3 x_1}{x_1 + x_2 + x_3}.$$

Then, in view of (47),

$$x_1 + x_2 = \lim_{P_3 \to \infty} \frac{\wp_{33}(\boldsymbol{u})\wp_{22}(\boldsymbol{u}) + \wp_{23}(\boldsymbol{u})\wp_{333}(\boldsymbol{u}) - \wp_{33}(\boldsymbol{u})\wp_{233}(\boldsymbol{u}) - \wp_{23}(\boldsymbol{u})^2 + 2\wp_{13}(\boldsymbol{u})}{3\wp_{23}(\boldsymbol{u}) - \wp_{333}(\boldsymbol{u})}.$$

Using the definition of $\wp_{ij}$ in (43) and taking the limit (for which $\sigma(\boldsymbol{u}) = 0$) one obtains

$$(58) \qquad x_1 + x_2 = \frac{1}{2}\phi(\boldsymbol{u})\frac{\sigma_3(\boldsymbol{u})\sigma_{2,3}(\boldsymbol{u}) - \sigma_2(\boldsymbol{u})\sigma_{3,3}(\boldsymbol{u})}{\sigma_3^3(\boldsymbol{u})} + \frac{1}{2}\frac{\sigma_2(\boldsymbol{u})\phi_3(\boldsymbol{u}) - \sigma_3(\boldsymbol{u})\phi_2(\boldsymbol{u})}{\sigma_3^3(\boldsymbol{u})},$$

where we set

$$(59) \qquad \phi(\boldsymbol{u}) = \sigma_2(\boldsymbol{u}) - \sigma_{3,3}(\boldsymbol{u}), \quad \phi_j(\boldsymbol{u}) = \sigma_{j,2}(\boldsymbol{u}) - \sigma_{j,3,3}(\boldsymbol{u}), \quad j = 1, 2, 3, \quad \boldsymbol{u} \in W^{(2)},$$

and $\boldsymbol{u}$ is given by the right hand side of (20). In a similar way we find

$$(60) \qquad x_1 x_2 = \frac{1}{2}\phi(\boldsymbol{u})\frac{\sigma_1(\boldsymbol{u})\sigma_{3,3}(\boldsymbol{u}) - \sigma_3(\boldsymbol{u})\sigma_{1,3}(\boldsymbol{u})}{\sigma_3^3(\boldsymbol{u})} + \frac{1}{2}\frac{\sigma_3(\boldsymbol{u})\phi_1(\boldsymbol{u}) - \sigma_1(\boldsymbol{u})\phi_3(\boldsymbol{u})}{\sigma_3^3(\boldsymbol{u})}, \qquad \boldsymbol{u} \in W^{(2)}.$$

Next, using the relation (45) for the pairs $(x_1, y_1), (x_2, y_2)$, we find

$$(61) \qquad y_j = (x_j^2 - \wp_{23}(\boldsymbol{u})x_i - \wp_{13}(\boldsymbol{u}))/\wp_{33}(\boldsymbol{u}),$$

and, therefore,

$$\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} = \frac{x_1 x_2 - \wp_{13}(\boldsymbol{u})}{\wp_{33}(\boldsymbol{u})}, \qquad \frac{y_1 - y_2}{x_2 - x_1} = \frac{x_1 + x_2 - \wp_{23}(\boldsymbol{u})}{\wp_{33}(\boldsymbol{u})},$$

where $\boldsymbol{u}$ is again given by the right hand side of (20). In view of (43) and the condition $\sigma(\boldsymbol{u}) = 0$, this gives the following new compact expressions

$$(62) \qquad \frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} = \frac{\sigma_1(\boldsymbol{u})}{\sigma_3(\boldsymbol{u})}\bigg|_{\sigma(\boldsymbol{u})=0}, \qquad \frac{y_1 - y_2}{x_2 - x_1} = \frac{\sigma_2(\boldsymbol{u})}{\sigma_3(\boldsymbol{u})}\bigg|_{\sigma(\boldsymbol{u})=0}.$$

The above formulae lead to the following analytic description of the Wirtinger strata defined in (32).

Proposition 4.1. (1) *The strata* $W^{(0)}, W^{(1)}, W^{(2)}$ *are given by the conditions*

$$(63) \qquad \begin{aligned} W^{(0)}: & \quad \sigma(\boldsymbol{u}) = \sigma_3(\boldsymbol{u}) = \sigma_2(\boldsymbol{u}) = 0, \\ W^{(1)}: & \quad \{\boldsymbol{u}| \ \sigma(\boldsymbol{u}) = 0, \ \sigma_3(\boldsymbol{u}) = 0\}, \\ W^{(2)}: & \quad \{\boldsymbol{u}| \ \sigma(\boldsymbol{u}) = 0\} \end{aligned}$$

(2) *The coordinates* x, y *of the curve* $\mathcal{C} = W^{(1)} \subset \mathrm{Jac}(\mathcal{C})$ *admit the parameterization*

$$(64) \qquad x_1 = -\frac{\sigma_1(\boldsymbol{u})}{\sigma_2(\boldsymbol{u})}, \quad y_1 = \frac{\sigma_1(\boldsymbol{u})\sigma_{2,3}(\boldsymbol{u}) - \sigma_2(\boldsymbol{u})\sigma_{1,3}(\boldsymbol{u})}{\sigma_2^2(\boldsymbol{u})}, \quad \boldsymbol{u} \in W^{(1)}.$$

Proof. The description of $W^{(2)}$ was already given by (42). To pass to the stratum $W^{(1)}$ one should let $x_2 \to \infty$. Then both sides of (58) and (60) tend to infinity, which happens if and only if $\sigma(\boldsymbol{u}) = 0$ and $\sigma_3(\boldsymbol{u}) = 0$.

Next, in view of (58), (60), and the condition $\sigma_3(\boldsymbol{u}) = 0$,

$$\lim_{x_2 \to \infty} \frac{x_1 x_2}{x_1 + x_2} = x_1 = -\frac{\sigma_1(\boldsymbol{u})}{\sigma_2(\boldsymbol{u})}, \quad \boldsymbol{u} = \int_\infty^{P_1} \boldsymbol{\Omega},$$

which gives the first expression in (64). Letting here $x_1 \to \infty$ gives $\sigma_2(\boldsymbol{u}) = 0$ for $\boldsymbol{u} = 0$. The latter also follows directly from the expansion (41). The second formula (64) is obtained by the appropriate limit from (61). $\square$

We shall also use the following

Proposition 4.2. *Let* $\mathbf{w} = \boldsymbol{u} - \boldsymbol{r}_i = (w_1, w_2, w_3)$ *and set* $\sigma_{j,\dots,k}^{(i)} = \sigma_{j,\dots,k}(\boldsymbol{r}_i)$. *Then* $\sigma_1^{(i)} = \sigma_3^{(i)} = 0$ *and the first few coefficients of the expansion near* $\boldsymbol{r}_i \in W^{(1)}$, $i = 1, 2, 3$

$$\sigma(\boldsymbol{r}_i + \mathbf{w}) = \sigma_2^{(i)} w_2 + \frac{1}{2}\sigma_{1,1}^{(i)} w_1{}^2 + \sigma_{1,3}^{(i)} w_1 w_3 + \frac{1}{2}\sigma_{3,3}^{(i)} w_3^2 + \frac{1}{2}\sigma_{2,2}^{(i)} w_2^2 + \sigma_{1,2}^{(i)} w_1 w_2 + \sigma_{2,3}^{(i)} w_2 w_3$$

$$(65) \qquad + \frac{1}{6}\sigma_{1,1,1}^{(i)} w_1{}^3 + \frac{1}{2}\sigma_{1,1,2}^{(i)} w_2 w_1{}^2 + \frac{1}{2}\sigma_{1,1,3}^{(i)} w_3 w_1{}^2 + \frac{1}{2}\sigma_{1,2,2}^{(i)} w_1 w_2{}^2 + \sigma_{1,2,3}^{(i)} w_3 w_1 w_2$$

$$+ \frac{1}{2}\sigma_{1,3,3}^{(i)} w_3^2 w_1 + \frac{1}{6}\sigma_{2,2,2}^{(i)} w_2^3 + \frac{1}{2}\sigma_{2,2,3}^{(i)} w_3 w_2^2 + \frac{1}{2}\sigma_{2,3,3}^{(i)} w_3^2 w_2 + \frac{1}{6}\sigma_{3,3,3}^{(i)} w_3^3 + O(w^4)$$

are related as follows

$$(66) \qquad \sigma_{3,3}^{(i)} = \sigma_2^{(i)}, \quad \sigma_{1,3}^{(i)} = -\rho^i \sigma_2^{(i)}, \quad \sigma_{1,1}^{(i)} = -\frac{2}{\rho^i}\sigma_2^{(i)}, \quad \sigma_{3,3,3}^{(i)} = 3\sigma_{2,3}^{(i)},$$

$$(67) \qquad \sigma_{1,2}^{(i)} = \sigma_{1,1,1}^{(i)} = 0, \quad \sigma_{1,1,3}^{(i)} = \sigma_{1,3,3}^{(i)} = -2\rho^i \sigma_{2,3}^{(i)}.$$

Here, as above, $\rho = \exp(2\pi\imath/3)$.

The proof uses the expansions of the coordinate y near the points $q_i = (0, \rho^i) \in \mathcal{C}$,

$$(68) \qquad \begin{aligned} y &= \rho^i + \frac{1}{3}(\mu_2(\rho^i + 1) + \mu_6\rho^i)x^2 + \frac{\mu_3}{3}\rho^i x^3 + O(x^4) \\ &= \rho^i + \frac{1}{3}(-\mu_2\rho^{2i} + \mu_6\rho^i)x^2 + \frac{\mu_3}{3}\rho^i x^3 + O(x^4). \end{aligned}$$

and the corresponding expansions of the holomorphic differentials (18). Thus, for example, near $(0, \rho)$ one has

$$\Omega_1 = \left(\frac{\rho}{3} - \frac{1}{9} \frac{2\rho^2 \mu_6 - \mu_2}{\rho} x^2 - \frac{2}{9} \rho \mu_3 x^3 + \frac{1}{27} \frac{(3\rho^2 \mu_6 - 2\mu_2)\mu_6}{\rho} x^4 + O(x^5) \right) dx,$$

(69)
$$\Omega_2 = \left(\frac{\rho}{3} x - \frac{1}{9} \frac{2\rho^2 \mu_6 - \mu_2}{\rho} x^3 - \frac{2}{9} \rho \mu_3 x^4 + \frac{1}{27} \frac{(3\rho^2 \mu_6 - 2\mu_2)\mu_6}{\rho} x^5 + O(x^6) \right) dx,$$

$$\Omega_3 = \left(\frac{\rho^2}{3} - \frac{1}{9} \rho^2 \mu_6 x^2 - \frac{1}{9} \mu_3 \rho^2 x^3 + \frac{1}{27} \frac{\rho^4 \mu_6{}^2 + \rho^2 \mu_6 \mu_2 - \mu_2{}^2}{\rho^2} x^4 + O(x^5) \right) dx$$

and the expansions near $(0, \rho^2), (0, 1)$ are obtained by replacing ρ above by ρ^2 and 1 respectively.

The remainder of the proof is technical and is presented in the Appendix.

4.2. **Analytic properties of the inversion on** $W^{(2)}$. One should stress that formulae (58)-(62) provide only a local analytic solution to the inversion of the incomplete Abel map (20). This is because $\sigma(\boldsymbol{u}) = 0$ is a transcendental equation and the argument u_1 is an infinitely-valued complex function of u_2, u_3. This fact also admits a geometric description. Namely, let $\mathcal{W}^{(2)} \subset \mathbb{C}^3(u_1, u_2, u_3)$ be the universal covering of the stratum $W^{(2)} \subset \mathrm{Jac}(\mathcal{C})$ and consider the projection $\pi : \mathcal{W}^{(2)} \to \mathbb{C}^2(u_2, u_3)$.

Proposition 4.3. *Assume that* $\mathrm{Jac}(\mathcal{C})$ *has no Abelian subvarieties. Then, under the projection* π, *the variety* $\mathcal{W}^{(2)}$ *is an infinitely-sheeted covering of* $\mathbb{C}^2(u_2, u_3)$ *ramified along the subvariety* $\bar{W} \subset \mathcal{W}^{(2)}$ *defined by the conditions* $\{\sigma(u) = 0, \ \sigma_1(u) = 0\}$.

Moreover, let $\{\boldsymbol{u}^*\} \subset \mathcal{W}^{(2)}$ *be the equivalence class corresponding to any point* $\boldsymbol{u}^* \in W^{(2)}$. *Then the projection* $\pi\{\boldsymbol{u}^*\}$ *forms a dense set on* $\mathbb{C}^2(u_2, u_3)$.

Note that in the case of hyperelliptic curves of genus 2, when the codimension one stratum $W^{(1)}$ coincides with the curve itself, a similar description was made by Jacobi in connection with the inversion of a single hyperelliptic integral (see [Mark992]), whereas for hyperelliptic curves of any genus and strata of any codimension a similar theorem was proven in [AF00].

Proof of Proposition 4.3. The proof follows the same lines as that in [AF00]. Namely, let $\mathfrak{v}_1, \ldots, \mathfrak{v}_6 \in \mathbb{C}^2(u_2, u_3)$ be the π-projections of six independent period vectors of $\mathrm{Jac}(\mathcal{C})$. For any point $\boldsymbol{u}^* \in W^{(2)}$, the projections of its equivalence class on $\mathcal{W}^{(2)}$ has the form

$$\left\{ \pi(\boldsymbol{u}^*) + \sum_{j=1}^{6} m_j \mathfrak{v}_j \mid m_j \in \mathbb{Z} \right\}.$$

Under the condition of the proposition the periods themselves are not commensurable and consequently the integer coefficients m_j can always be chosen in such a way that the above sum will give a point in any small neighborhood of any point of $\mathbb{C}^2(u_2, u_3)$ fixed *a priori*. Next, since $\mathcal{W}^{(2)}$ is defined by the transcendental equation $\sigma(\boldsymbol{u}) = 0$, for any point $(u_2^*, u_3^*) \in \mathbb{C}^2$ there exists an infinite number of solutions u_1. In other words, the covering $\pi : \mathcal{W}^{(2)} \to \mathbb{C}^2(u_2, u_3)$ has an infinite number of sheets. Finally, by the implicit function theorem, π is ramified over the points satisfying $\sigma_1(u) = 0$. This establishes the proposition. $\square$

4.3. **Sigma-function solutions of the Goryachev system.** We conclude the section with the formal σ-function solutions for the original variables of the Goryachev system. First, applying the transformation (16) to (15), one gets

$$\gamma_3^2 = \left(\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \right)^3, \quad J_3 = \frac{3}{2} \sqrt{\frac{a}{2}} \frac{y_1 - y_2}{x_2 - x_1},$$

$$\gamma_2 + \imath\gamma_1 = \frac{\imath}{x_1 x_2} \left(1 - \left(\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \right)^3 \right), \quad \gamma_2 - \imath\gamma_1 = -\imath\, x_1 x_2,$$

(70)
$$J_1 + \imath J_2 = -\sqrt{\frac{a}{2}} (x_1 + x_2) \left(\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} \right)^{-1/2},$$

$$J_1 - \imath J_2 = \frac{1}{\gamma_2 - \imath\gamma_1} \left((J_1 + \imath J_2)(\gamma_2 + \imath\gamma_1) + 2\imath J_3 \gamma_3 \right).$$

Then, upon comparing the above with the sigma-function expressions (58), (60), (62), one obtains after simplification

$$\gamma_3^{2/3} = \left.\frac{\sigma_1(\boldsymbol{u})}{\sigma_3(\boldsymbol{u})}\right|_{\sigma(\boldsymbol{u})=0}, \quad J_3 = \left.\frac{3}{2}\sqrt{\frac{a}{2}}\frac{\sigma_2(\boldsymbol{u})}{\sigma_3(\boldsymbol{u})}\right|_{\sigma(\boldsymbol{u})=0},$$

$$\gamma_2 - \imath\gamma_1 = \left.-\frac{\imath}{2}\frac{\Phi_2(\boldsymbol{u})}{\sigma_3^3(\boldsymbol{u})}\right|_{\sigma(\boldsymbol{u})=0},$$

(71)
$$\gamma_2 + \imath\gamma_1 = \left.2\imath\frac{\sigma_3^3(\boldsymbol{u}) - \sigma_1^3(\boldsymbol{u})}{\Phi_2(\boldsymbol{u})}\right|_{\sigma(\boldsymbol{u})=0},$$

$$J_1 + \imath J_2 = \left.-\sqrt{\frac{a}{2}}\frac{1}{2}\frac{\Phi_1(\boldsymbol{u})}{\sqrt{\sigma_1(\boldsymbol{u})\sigma_3(\boldsymbol{u})}\,\sigma_3^2(\boldsymbol{u})}\right|_{\sigma(\boldsymbol{u})=0},$$

$$J_1 - \imath J_2 = \left.\sqrt{\frac{a}{2}}\frac{\sqrt{\sigma_3(\boldsymbol{u})}}{\sqrt{\sigma_1(\boldsymbol{u})}}\frac{\Phi_1(\boldsymbol{u})(\sigma_3^3(\boldsymbol{u}) - \sigma_1^3(\boldsymbol{u})) + 3\Phi_2(\boldsymbol{u})\sigma_1^2(\boldsymbol{u})\sigma_2(\boldsymbol{u}))}{\Phi_2^2(\boldsymbol{u})}\right|_{\sigma(\boldsymbol{u})=0},$$

where we set

$$\Phi_1(\boldsymbol{u}) = \phi(\boldsymbol{u})\left(\sigma_3(\boldsymbol{u})\sigma_{2,3}(\boldsymbol{u}) - \sigma_2(\boldsymbol{u})\sigma_{3,3}(\boldsymbol{u})\right) + \sigma_2(\boldsymbol{u})\phi_3(\boldsymbol{u}) - \sigma_3(\boldsymbol{u})\phi_2(\boldsymbol{u}),$$
$$\Phi_2(\boldsymbol{u}) = \phi(\boldsymbol{u})\left(\sigma_1(\boldsymbol{u})\sigma_{3,3}(\boldsymbol{u}) - \sigma_3(\boldsymbol{u})\sigma_{1,3}(\boldsymbol{u})\right) + \sigma_3(\boldsymbol{u})\phi_1(\boldsymbol{u}) - \sigma_1(\boldsymbol{u})\phi_3(\boldsymbol{u}).$$

Here $\phi(\boldsymbol{u})$ and its derivatives were defined in (59), and the components u_2, u_3 of $\mathbf{u}$ are linear functions of t_1, t_2 as described in (20). These expressions give the solution of the Goryachev system for a general flow under the Hamiltonians H_1, H_2.

5. Expansions of the functions x_i, y_i and the Painlevé analysis of the system.

Apart from the formal σ-function solutions (71) it is important to know the complex singularities of the functions $J_i(t), \gamma_i(t)$: their poles, order of branching, etc. Here we shall make connection with the Painlevé analysis of [AvM989]. For simplicity we concentrate on the complex flow generated by the quadratic Hamiltonian H_1 (time $t = t_1$). As we have seen, this flow lies on the stratum $W^{(2)} \subset \mathrm{Jac}(\mathcal{C})$, on which the coordinate u_1 is a transcendental function of u_2, u_3 and for which u_2 is constant along the flow.

It follows from (71) that most of the variables have poles when the u_3-flow (i.e. the t_1-flow) on $W^{(2)}$ crosses the substratum $W^{(1)}$, on which $\sigma_3(\boldsymbol{u}) = 0$. Also, the right hand sides of (71) may have branching only when the coordinate u_1 (as a solution of $\sigma(u_1, u_2, u_3) = 0$) ceases to be a locally meromorphic function of u_2, u_3 (and, therefore, of t_1). As observed above, the implicit function theorem means this occurs along the 1-dimensional analytic subvariety $\mathcal{Z} = \{\sigma(\boldsymbol{u}) = 0, \sigma_1(\boldsymbol{u}) = 0\} \subset W^{(2)}$. (The solutions for $J_2 \pm \imath J_1$ in (71) have additional branching along $W^{(1)}$ due to presence of square roots.)

To describe the local behavior of $u_1 = u_1(u_2, u_3)$ in detail, choose a point $\boldsymbol{u}_0 \in W^{(1)}$. Then $\sigma(\boldsymbol{u}_0) = \sigma_3(\boldsymbol{u}_0) = 0$. Let δu_i be the increments of the coordinates u_i such that $\boldsymbol{u}_0 + \delta\boldsymbol{u} \in W^{(2)}$. Then the following expansion holds

(72)
$$\sigma(\boldsymbol{u}_0 + \delta\boldsymbol{u}) = \sigma_1(\boldsymbol{u}_0)\,\delta u_1 + \sigma_2(\boldsymbol{u}_0)\,\delta u_2 + \sum_{1 \leq i,j \leq 3}\frac{\sigma_{i,j}(\boldsymbol{u}_0)}{2}\,\delta u_i\delta u_j + \cdots = 0\,.$$

For a generic $\boldsymbol{u}_0 \in W^{(1)}$ and u_3-flow ($\delta u_2 = 0$) this implies

(73)
$$\delta u_1 = \varkappa\,(\delta u_3)^2 + O\left((\delta u_3)^3\right), \qquad \varkappa = -\frac{\sigma_{33}(\boldsymbol{u}_0)}{2\sigma_1(\boldsymbol{u}_0)}.$$

The above expansion does not hold for the points on $W^{(1)} = \mathcal{C}$ with $\sigma_1(\boldsymbol{u}) = 0$, i.e., at the points of $\mathcal{Z} \cap W^{(1)}$.

Proposition 5.1. *The subvariety $\mathcal{Z} = \{\sigma(\boldsymbol{u}) = 0, \sigma_1(\boldsymbol{u}) = 0\}$ has precisely 3 common points with $W^{(1)}$:* $\mathcal{Z} \cap W^{(1)} = \{\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3\}$.

Proof. In view of (64), along the stratum $W^{(1)}$ one has $\sigma_1(\boldsymbol{u}) = 0$ if and only if $x_1 = 0$, which corresponds to the points $\boldsymbol{r}_1, \boldsymbol{r}_2, \boldsymbol{r}_3$. $\square$

Next, it follows from Proposition 4.2, that near $\boldsymbol{u} = \boldsymbol{r}_i$, for the u_3-flow ($\delta u_2 = 0$), one has

$$(74) \qquad \sigma(\boldsymbol{r}_i + \delta\boldsymbol{u}) = \sigma_2^{(i)}\left(-\frac{1}{\rho^i}(\delta u_1)^2 - \rho^i \delta u_1\,\delta u_3 + \frac{1}{2}(\delta u_3)^2\right) + O\left(\delta u_1(\delta u_3)^2\right) + O\left((\delta u_3)^3\right) = 0,$$

where, as above, $\rho = \exp(2\pi\imath/3)$. Then the increment δu_1, as a function of δu_3, has 2 local branches

$$(75) \qquad\qquad\qquad \delta u_1 = \frac{1}{2}\rho^{2i}\left(-1 \pm \sqrt{3}\right)\delta u_3 + O\left((\delta u_3)^2\right).$$

Indeed, substituting this into the expansion (74), up to cubic terms, we get

$$\sigma(\boldsymbol{r}_i + \delta\boldsymbol{u}) = \pm\frac{\sigma_2^{(i)}}{2}(\rho^{3i} - 1)(\delta u_3)^2,$$

which is zero for any $i = 1, 2, 3$.

Note also that $\boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 \equiv 0$ in $\mathrm{Jac}(\mathcal{C})$ being the image of the divisor of x under the Abel map. Due to expansion (24) of the differentials Ω_j near $\infty \in \mathcal{C}$, a tangent vector to $W^{(1)} \subset W^{(2)}$ at the origin $u = 0$, $T_0 W^{(1)}$, is $(0,0,1)^T$. Hence the u_3-flow is tangent to $W^{(1)}$ at the origin. Next, in view of (69), $T_{\boldsymbol{r}_i} W^{(1)} = (\rho^i, 0, \rho^{2i})^T$, which means that the flow is also tangent to the projection of $W^{(1)}$ onto the (u_2, u_3)-plane at $\boldsymbol{r}_i$. All these observations are depicted in Figure 5.1.
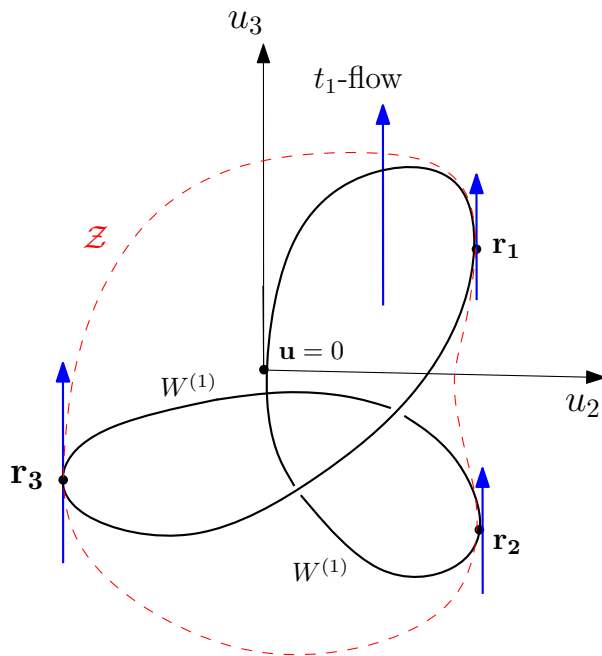

FIGURE 1. A sketch of the stratum $W^{(1)} \subset W^{(2)}$ (solid line) and $\mathcal{Z}$ (dashed line) in the projection onto the (u_2, u_3)-plane, and the u_3-flow on $W^{(2)}$.

5.1. **Expansions of the solutions near $W^{(1)}$ along the u_3-flow.** The order of poles of the variables J_i, γ_i, as functions of t_1 (or u_3) depend on the nature of the intersection (for example, transversal or tangential) of the u_3-flow with $W^{(1)} \subset W^{(2)}$. The expansions of the solutions (71) in powers of u_3 may be found by using the corresponding expansions of the sigma-function and its derivatives near a point $\boldsymbol{u}_0 \in W^{(1)}$, as well as the expansions (73), (75). It is more convenient however to find the expansions of the coordinates $(x_1, y_1), (x_2, y_2)$ of the points $P_1, P_2 \in S$ under the incomplete Abel map (19) and then use the formulae (70).

To do this, first note that, according to the definition of $W^{(1)}$, when $\boldsymbol{u}$ belongs to $W^{(1)} \setminus \{0\}$, one of the points P_i on $\mathcal{C}$, say P_1, coincides with ∞, whereas P_2 remains finite. Now let $\Gamma(t)$, $|t - t_0| < 1$ be a complex analytic arc in $W^{(2)}$ such that $\Gamma(t_0) \cap W^{(1)} = \boldsymbol{u}_0$, and the projection of the arc onto $\mathbb{C}^2 = (u_2, u_3)$ is a

segment of a straight line $u_2 =$const. Let $D(t) = \{P_1(t), P_2(t)\}$ be a divisor on $\mathcal{C}$ such that its Abel image gives $\Gamma(t)$. Then $P_1(t_0) = \infty, P_2(t_0) = (x_0, y_0)$.

Theorem 5.2. *For a generic $\boldsymbol{u}_0 = (u_{10}, u_{20}, u_{30}) \in W^{(1)}$, near $t = t_0$ the coordinates of the points P_1, P_2 admit the expansion*

$$
\begin{aligned}
x_1 &= \frac{1}{(u_3 - u_{30})^3} - \frac{3y_0}{2x_0} \frac{1}{(u_3 - u_{30})^2} + O\left((u_3 - u_{30})^{-1}\right) \\
&= \frac{27}{8}\imath \frac{1}{(t - t_0)^3} + \frac{27}{8}\frac{y_0}{x_0} \frac{1}{(t - t_0)^2} + O\left((t - t_0)^{-1}\right), \\
y_1 &= \frac{1}{(u_3 - u_{30})^4} + \frac{2y_0}{x_0} \frac{1}{(u_3 - u_{30})^3} + O((u_3 - u_{30})^{-2}) \\
&= \frac{81}{16(t - t_0)^4} + \frac{27\imath y_0}{4x_0} \frac{81}{16(t - t_0)^3} + O((t - t_0)^{-2}), \\
x_2 &= x_0 - \frac{1}{2\phi_{20}}(u_3 - u_{30})^2 + O((u_3 - u_{30})^3), \\
y_2 &= y_0 + O((u_3 - u_{30})^2).
\end{aligned}
\tag{76}
$$

Next, near each point $\boldsymbol{u}_0 = \boldsymbol{r}_i$, $i = 1, 2, 3$, at which $x_{20} = 0$, there are two expansions:

$$
\begin{aligned}
x_1 &= \frac{1}{g^3} \frac{1}{(u_3 - u_{30})^3} - \frac{3\rho^i}{2g^3} \frac{1}{(u_3 - u_{30})^2} + O((u_3 - u_{30})^{-1}), \\
y_1 &= \frac{1}{g^4} \frac{1}{(u_3 - u_{30})^4} - \frac{2\rho^i}{g^4} \frac{1}{(u_3 - u_{30})^3} + O((u_3 - u_{30})^{-2}), \\
x_2 &= \frac{3}{2}(-1 \pm \sqrt{3})\rho^i (u_3 - u_{30}) - \frac{3}{\rho^i}g (u_3 - u_{30})^2 + O((u_3 - u_{30})^3), \\
y_2 &= \rho^i + \frac{3}{4}(\mu_6 - \mu_3\rho^i)(-1 \pm \sqrt{3})^2 (u_3 - u_{30})^2 + O((u_3 - u_{30})^3),
\end{aligned}
\tag{77}
$$

where $g = (3 \mp \sqrt{3})/2$.

The proof is given in Appendix.

Theorem 5.2 implies that when the u_3-flow crosses the substratum $W^{(1)}$ at a generic point $\boldsymbol{u}_0$, we have

$$
\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} = \frac{x_0}{u_3 - u_{30}} + O(1), \quad \frac{y_1 - y_2}{x_2 - x_1} = -\frac{1}{u_3 - u_{30}} + O(u_3 - u_{30}),
$$

$$
x_1 + x_2 = \frac{1}{(u_3 - u_{30})^3} + O((u_3 - u_{30})^{-2}), \quad x_1 x_2 = \frac{x_0}{(u_3 - u_{30})^3} + O((u_3 - u_{30})^{-2}).
\tag{78}
$$

In the case of crossing $W^{(1)}$ at the points $\boldsymbol{r}_i \in W^{(1)}$, one has instead

$$
\frac{x_2 y_1 - x_1 y_2}{x_2 - x_1} = O(1) + O((u_3 - u_{30})), \quad \frac{y_1 - y_2}{x_2 - x_1} = -\frac{1}{g(u_3 - u_{30})} + O(1),
$$

$$
x_1 + x_2 = \frac{1}{g^3(u_3 - u_{30})^3} + O(u_3 - u_{30}), \quad x_1 x_2 = \frac{3\rho^i(-1 \pm \sqrt{3})}{2g^3(u_3 - u_{30})^2} + O(1).
\tag{79}
$$

Note that the orders of the above expansions are compatible with those predictable from the sigma-function solutions (62), (58), (60).

We stress that in all the cases the above symmetric functions of x_i, y_i, as functions of u_3 or t_1, do not have poles with branching. However, they have finite branching along the intersection with $\mathcal{Z} \subset W^{(1)}$.

Using (71) and the above expansions, one can estimate the leading terms of formal series solutions for the variables $J_1 \pm \imath J_2, \gamma_2 \pm \imath \gamma_1, J_3, \gamma_3$ near the poles. In the generic case one has

$$
\begin{aligned}
\gamma_3 &= O\left(1/(t - t_0)^{3/2}\right), \quad J_3 = O\left(1/(t - t_0)\right), \\
\gamma_2 + \imath\gamma_1 &= O(1), \quad \gamma_2 - \imath\gamma_1 = O\left(1/(t - t_0)^3\right), \\
J_1 + \imath J_2 &= O\left(1/(t - t_0)^{5/2}\right), \quad J_1 - \imath J_2 = O\left((t - t_0)^{1/2}\right),
\end{aligned}
\tag{80}
$$

and when the u_3-flow crosses the substratum $W^{(1)}$ at $\boldsymbol{r}_i$,

$$\gamma_3 = O(1), \quad J_3 = O\left(1/(t-t_0)\right),$$
(81)
$$\gamma_2 + \imath\gamma_1 = O(t - t_0), \quad \gamma_2 - \imath\gamma_1 = O\left(1/(t-t_0)\right),$$
$$J_1 + \imath J_2 = O\left(1/(t-t_0)\right), \quad J_1 - \imath J_2 = O(t - t_0).$$

These expansions correspond precisely with the leading behavior of formal series solutions of the Goryachev system obtained directly by Kovalevskaya–Painlevé analysis. To observe this, first rewrite the Goryachev system (3) with the Hamiltonian H_1 in the form

$$\frac{d}{dt_1}(\gamma_2 - \imath\gamma_1) = -2\imath\frac{4}{3}J_3(\gamma_2 - \imath\gamma_1) + 2\gamma_3(J_1 + \imath J_2),$$

$$\frac{d}{dt_1}(\gamma_2 + \imath\gamma_1) = 2\imath\frac{4}{3}J_3(\gamma_2 + \imath\gamma_1) + 2\gamma_3(J_1 - \imath J_2),$$

$$\frac{d}{dt_1}(J_1 + \imath J_2) = -\frac{2}{3}J_3\imath(J_1 + \imath J_2) - \frac{2}{3}\frac{a\gamma_1 + b}{\gamma_3^{5/3}}(\gamma_2 - \imath\gamma_1) + \imath\, a\gamma_3^{1/3},$$

(82)
$$\frac{d}{dt_1}(J_1 - \imath J_2) = \frac{2}{3}J_3\imath(J_1 - \imath J_2) - \frac{2}{3}\frac{a\gamma_1 + b}{\gamma_3^{5/3}}(\gamma_2 + \imath\gamma_1) - \imath\, a\gamma_3^{1/3},$$

$$\frac{d}{dt_1}J_3 = -a\frac{\gamma_2}{\gamma_3^{2/3}},$$

$$\frac{d}{dt_1}\gamma_3 = -(J_1 + \imath J_2)(\gamma_2 + \imath\gamma_1) - (J_1 - \imath J_2)(\gamma_2 - \imath\gamma_1).$$

Then the corresponding formal Puiseaux (or Laurent) series solutions of (82) are of 2 kinds. According to terminology of [AvM989], those depending on the maximal number of free parameters (here three, for example, the constants of motion h_1, h_2 and a local coordinate on $W^{(1)}$) represent the *principle balances* of the solutions. Their leading behavior coincides with (80). The series solutions depending on 2 or less free parameters (called *secondary balances*) correspond to (81).

Appendix.

Proof of Proposition 4.2. Let, as above, $q_i = (0, \rho^i) \in \mathcal{C}$, $\boldsymbol{r}_i = \int_\infty^{q_i} \boldsymbol{\Omega}$. Since $\boldsymbol{r}_i \in W^{(1)}$ and $x(q_i) = 0$, Proposition 4.1 implies that $\sigma_1(\boldsymbol{r}_i) = \sigma_3(\boldsymbol{r}_i) = 0$. Let ξ be the local coordinate on $\mathcal{C}$ near ∞ and x be such coordinate near $q_i = (0, \rho^i) \in \mathcal{C}$. Introduce the functions

$$\boldsymbol{U}(\xi) = (U_1, U_2, U_3) = \int_\infty^P \boldsymbol{\Omega}, \quad \text{and} \quad \boldsymbol{w}(x) = (w_1, w_2, w_3) = \int_{q_i}^P \boldsymbol{\Omega},$$

In view of the expansions (24), we obtain

$$U_1 = -\frac{1}{5}\xi^5 - \frac{1}{21}\mu_2\xi^7 + \cdots,$$

(83)
$$U_2 = -\frac{1}{2}\xi^2 - \frac{1}{12}\mu_2\xi^4 + \frac{2}{15}\mu_3\xi^5 + \cdots,$$

$$U_3 = -\xi + \frac{1}{12}\mu_3\xi^4 + \frac{1}{45}\mu_2^2\xi^5 + \cdots,$$

and, in view of (69), for $i = 1$ we have

$$w_1 = \frac{\rho}{3}x - \frac{1}{27}\frac{2\rho^2\mu_6 - \mu_2}{\rho}x^3 - \frac{1}{18}\rho\mu_3 x^4 + \frac{1}{135}\frac{(3\rho^2\mu_6 - 2\mu_2)\mu_6}{\rho}x^5 + \mathrm{O}(x^6),$$

$$w_2 = \frac{\rho}{6}x^2 - \frac{1}{36}\frac{2\rho^2\mu_6 - \mu_2}{\rho}x^4 - \frac{2}{45}\rho\mu_3 x^5 + \mathrm{O}(x^6),$$

$$w_3 = \frac{\rho^2}{3}x - \frac{1}{27}\rho^2\mu_6 x^3 - \frac{1}{36}\mu_3\rho^2 x^4 + \frac{1}{135}\frac{\rho^4\mu_6^2 + \rho^2\mu_6\mu_2 - \mu_2^2}{\rho^2}x^5 + \mathrm{O}(x^6).$$

The expansions for $i = 2, 3$ are obtained from the above by replacing ρ by ρ^2 and $\rho^3 = 1$ respectively.

Now note that for any $x, \xi \in \mathbb{C}$ we have $\boldsymbol{r}_i + \boldsymbol{w}(x) \in W^{(1)}$ and $\boldsymbol{r}_i + \boldsymbol{w}(x) + \boldsymbol{U}(\xi) \in W^{(2)}$. Hence, in view of Proposition 4.1,

$$\Sigma(x, \xi) := \sigma(\boldsymbol{r}_i + \boldsymbol{w}(x) + \boldsymbol{U}(\xi)) \equiv 0, \quad \Sigma_3(x) = \sigma_3(\boldsymbol{r}_i + \boldsymbol{w}(x)) \equiv 0.$$

Substituting the above two expansions into the sigma-expansions (65) for $i = 1$, we obtain[3]

$$\Sigma(x, \xi) = \left(\frac{1}{6}\, \sigma_2\, \rho + \frac{1}{18}\, \sigma_{3,3}\, \rho^4 + \frac{1}{18}\, \sigma_{1,1}\, \rho^2 + \frac{1}{9}\, \sigma_{1,3}\, \rho^3 \right) x^2$$

$$- \frac{1}{3}(\sigma_{3,3}\, \rho^2 + \sigma_{1,3}\, \rho)\, \xi\, x + \left(-\frac{1}{2}\, \sigma_2 + \frac{1}{2}\, \sigma_{3,3} \right) \xi^2$$

$$+ \left(\frac{1}{54}\, \sigma_{1,3,3}\, \rho^5 + \frac{1}{18}\, \sigma_{1,2}\, \rho^2 + \frac{1}{162}\, \sigma_{3,3,3}\, \rho^6 + \frac{1}{54}\, \sigma_{1,1,3}\, \rho^4 + \frac{1}{18}\, \sigma_{2,3}\, \rho^3 + \frac{1}{162}\, \sigma_{1,1,1}\, \rho^3 \right) x^3$$

$$- \left(\frac{1}{18}\, \sigma_{1,1,3}\, \rho^2 + \frac{1}{9}\, \sigma_{1,3,3}\, \rho^3 + \frac{1}{6}\, \sigma_{2,3}\, \rho + \frac{1}{18}\, \sigma_{3,3,3}\, \rho^4 \right) \xi\, x^2$$

$$+ \left(-\frac{1}{6}\, \sigma_{2,3}\, \rho^2 + \frac{1}{6}\, \sigma_{1,3,3}\, \rho - \frac{1}{6}\, \sigma_{1,2}\, \rho + \frac{1}{6}\, \sigma_{3,3,3}\, \rho^2 \right) \xi^2\, x + \left(\frac{1}{2}\, \sigma_{2,3} - \frac{1}{6}\, \sigma_{3,3,3} \right) \xi^3 + O(\xi^3 x)$$

and

$$\Sigma_3(x) = \left(\frac{1}{3}\, \sigma_{1,3}\, \rho + \frac{1}{3}\, \sigma_{3,3}\, \rho^2 \right) x + \left(\frac{1}{18}\, \sigma_{3,3,3}\, \rho^4 + \frac{1}{18}\, \sigma_{1,1,3}\, \rho^2 + \frac{1}{9}\, \sigma_{1,3,3}\, \rho^3 + \frac{1}{6}\, \sigma_{2,3}\, \rho \right) x^2 + O(x^3).$$

Then, upon equating the coefficients of the expansions $\Sigma(x, \xi), \Sigma_3(x)$ to zero, we get a system of equations for the coefficients $\sigma_{i,\ldots,k}^{(\alpha)}$. (Note that the indicated coefficients of $\Sigma_3(x)$ are also coefficients of $\Sigma(x, \xi)$, so they, in fact, do no bring new conditions.) Solving it, we obtain the first group of relations (66).

To find the relations (67), we proceed in the same manner as in (53) and consider the limit

$$Q_i = \lim_{\mathbf{u} \to \mathbf{0}} \frac{\sigma_1(\boldsymbol{r}_i + \mathbf{u}) + \rho^i \sigma_i(\boldsymbol{r}_i + \mathbf{u})}{\sigma(\boldsymbol{r}_i + \mathbf{u})}.$$

Applying l'Hopital's rule, we get

$$Q_i = \lim_{u_3 \to 0} \frac{\sigma_1(\boldsymbol{r}_i + u_3 \mathbf{e_3}) + \rho^i\, \sigma_i(\boldsymbol{r}_i + u_3 \mathbf{e_3})}{\sigma(\boldsymbol{r}_i + u_3 \mathbf{e_3})} = \lim_{u_3 \to 0} \frac{\sigma_{13}(\boldsymbol{r}_i + u_3 \mathbf{e_3}) + \rho^i\, \sigma_{33}(\boldsymbol{r}_i + u_3 \mathbf{e_3})}{\sigma_3(\boldsymbol{r}_i + u_3 \mathbf{e_3})}$$

$$= \frac{\sigma_{133}(\boldsymbol{r}_i) + \rho^i\, \sigma_{333}(\boldsymbol{r}_i)}{\sigma_{33}(\boldsymbol{r}_i)}, \qquad \mathbf{e_3} = (0, 0, 1)^T.$$

On the other hand,

$$Q_i = \lim_{u_2 \to 0} \frac{\sigma_1(\boldsymbol{r}_i + u_2 \mathbf{e_2}) + \rho^i\, \sigma_3(\boldsymbol{r}_i + u_2 \mathbf{e_2})}{\sigma(\boldsymbol{r}_i + u_2 \mathbf{e_2})} = \frac{\sigma_{12}(\boldsymbol{r}_i) + \rho^i\, \sigma_{23}(\boldsymbol{r}_i)}{\sigma_2(\boldsymbol{r}_i)}, \qquad \mathbf{e_2} = (0, 1, 0)^T,$$

$$Q_i = \lim_{u_1 \to 0} \frac{\sigma_1(\boldsymbol{r}_i + u_1 \mathbf{e_1}) + \rho^i\, \sigma_3(\boldsymbol{r}_i + u_1 \mathbf{e_1})}{\sigma(\boldsymbol{r}_i + u_1 \mathbf{e_1})} = \lim_{u_1 \to 0} \frac{\sigma_{11}(\boldsymbol{r}_i + u_1 \mathbf{e_1}) + \rho^i\, \sigma_{13}(\boldsymbol{r}_i + u_1 \mathbf{e_1})}{\sigma_1(\boldsymbol{r}_i + u_1 \mathbf{e_1})}$$

$$= \frac{\sigma_{111}(\boldsymbol{r}_i) + \rho^i\, \sigma_{113}(\boldsymbol{r}_i)}{\sigma_{11}(\boldsymbol{r}_i)}, \qquad \mathbf{e_1} = (1, 0, 0)^T.$$

Comparing the above limits and using (66), we get

$$\sigma_{133}(\boldsymbol{r}_3) + 3\rho^i \sigma_{23}(\boldsymbol{r}_3) = \sigma_{12}(\boldsymbol{r}_3) + \rho^i \sigma_{23}(\boldsymbol{r}_3),$$

$$-\rho^i \frac{\sigma_{111}(\boldsymbol{r}_3) + \rho^i \sigma_{113}(\boldsymbol{r}_3)}{2\sigma_2(\boldsymbol{r}_3)} = \frac{\sigma_{12}(\boldsymbol{r}_3) + \rho^i \sigma_{23}(\boldsymbol{r}_3)}{\sigma_2(\boldsymbol{r}_3)}.$$

These relations are compatible with a vanishing of the leading coefficients of the expansions $\Sigma(x, \xi), \Sigma_3(x)$ if and only if relations (67) hold. This proves the proposition.

---

[3] Here we omit the index (i) in the σ-derivatives.

Proof of Theorem 5.2. It is sufficient to study the expansion of the map (19) near the point $\{\infty, (x_0, y_0)\} \in \mathcal{C} \times \mathcal{C}$. Again let $\xi = x^{-1/3}$ be a local coordinate of $P_1 \in \mathcal{C}$ near ∞ and $\kappa = x - x_0$ be such a coordinate of P_2 near (x_0, y_0), $x_0 \neq 0$. The expansions of the differentials Ω_i near (x_0, y_0) are

$$\Omega_2 = \frac{x}{\partial G / \partial y(x, y)} dx = \big(\varphi_{20} + \varphi_{21}\kappa + O(\kappa^2)\big) d\kappa,$$

$$\Omega_3 = \frac{y}{\partial G / \partial y(x, y)} dx = \big(\varphi_{30} + \varphi_{31}\kappa + O(\kappa^2)\big) d\kappa,$$

where $\varphi_{20}, \ldots, \varphi_{31}$ are certain nonzero constants depending on x_0 and the coefficients of the curve. Note that $\frac{\phi_{30}}{\phi_{20}} = \frac{y_0}{x_0}$. In view of this and of the expansions of Ω_i in (24), the differential of the map (19) reads

$$(84) \qquad \begin{pmatrix} \xi + \frac{\mu_2}{3}\xi^3 + O(\xi^4) & \varphi_{20} + \varphi_{21}\kappa + O(\kappa^2) \\ 1 - \frac{\mu_3}{3}\xi^3 + O(\xi^4) & \varphi_{30} + \varphi_{31}\kappa + O(\kappa^2) \end{pmatrix} \begin{pmatrix} d\xi \\ d\kappa \end{pmatrix} = \begin{pmatrix} du_2 \\ du_3 \end{pmatrix}.$$

Taking into account that $\varphi_{20}, \varphi_{30}$ are non-zero, we may invert the above matrix expansion yielding (up to quadratic terms)

$$(85) \qquad \begin{pmatrix} d\xi \\ d\kappa \end{pmatrix} = \begin{pmatrix} -\dfrac{\phi_{30}}{\phi_{20}} - \dfrac{\phi_{30}^2}{\phi_{20}^2}\xi - \Phi\kappa & 1 + \dfrac{\phi_{30}}{\phi_{20}}\xi \\ \dfrac{1}{\phi_{20}} + \dfrac{\phi_{30}}{\phi_{20}^2}\xi - \dfrac{\phi_{21}}{\phi_{20}^2}\kappa & -\dfrac{1}{\phi_{20}}\xi \end{pmatrix} \begin{pmatrix} du_2 \\ du_3 \end{pmatrix}, \qquad \Phi = \frac{\phi_{31}\phi_{20} - \phi_{21}\phi_{30}}{\phi_{20}^2}.$$

For the t_1-flow we have $du_3 = \frac{2}{3}\imath\, dt$ and $du_2 = 0$. Then (85) gives a system of 2 ODEs with respect to $u_3 - u_{30}$ (or $t - t_0$). Applying the condition $\xi(0) = 0, \kappa(0) = 0$, we find initial terms of the series solutions

$$\xi = u_3 - u_{30} + \frac{1}{2}\frac{y_0}{x_0}(u_3 - u_{30})^2 + O((u_3 - u_{30})^3) = \frac{2}{3}\imath(t - t_0) - \frac{1}{2}\frac{y_0}{x_0}\frac{2^2}{3^2}(t - t_0)^2 + \cdots,$$

$$\kappa = -\frac{1}{2\phi_{20}}(u_3 - u_{30})^2 + O((u_3 - u_{30})^3) = \frac{1}{2\phi_{20}}\frac{2^2}{3^2}(t - t_0)^2 + \cdots.$$

Substituting this into the expansion (22) and taking into account (54), we get the expansions (76).

The above argument fails to work only when $P_2(t_0) = q_i = (0, \rho^i)$ and $\phi_{20}(q_i) = 0$, i.e., when the differential relation (84) cannot be locally inverted. In this case we will use the expansions of the differentials Ω_i near q_i given by (69) and replace (84) by the differential of the Abel map (20) taking the part with Ω_1, Ω_3 and u_1, u_3. Let x be a local coordinate on $\mathcal{C}$ near $q_i = (0, \rho^i)$. For $\alpha = 1$ the expansion of the differential is

$$(86) \qquad \begin{pmatrix} \xi^4 + \frac{\mu_2}{3}\xi^6 + O(\xi^7) & \frac{\rho}{3} - \frac{1}{9\rho}(2\rho^2\mu_6 - \mu_2)x^2 + O(x^3) \\ 1 - \frac{\mu_3}{3}\xi^3 + O(\xi^4) & \frac{\rho^2}{3} - \frac{\mu_6\rho^2}{9}x^2 + O(x^3) \end{pmatrix} \begin{pmatrix} d\xi \\ dx \end{pmatrix} = \begin{pmatrix} du_1 \\ du_3 \end{pmatrix}.$$

The latter is invertible and, up to cubic terms in ξ, x, gives the expansions

$$(87) \qquad \begin{pmatrix} d\xi \\ dx \end{pmatrix} = \begin{pmatrix} -\rho - \rho^2\,\xi + \frac{1}{3\rho^2}\left(-\mu_6 + \rho\,\mu_2\right)x^2 - \xi^2 & 1 + \xi\,\rho + \rho^2\,\xi^2 \\ \frac{3}{\rho} + 3\,\xi + (2\,\mu_6\,\rho^2 - \mu_2)\,x^2 + 3\,\rho\,\xi^2 & -\frac{3\,\xi}{\rho} - 3\,\xi^2 \end{pmatrix} \begin{pmatrix} du_1 \\ du_3 \end{pmatrix}.$$

According to (75), for $i = 1$ one has $du_1 = \nu du_3$, $\nu = \frac{1}{2}(-1 \pm \sqrt{3})\rho^2$. Then the above expansions lead to the following system of 2 ODEs with respect to $u_3 - u_{30}$

$$\frac{d\xi}{du_3} = g(1 + \rho\,\xi + \rho^2\,\xi^2) + \frac{\nu}{3\rho^2}(\rho\mu_2 - \mu_6)x^2 + O(\xi^3)$$

$$\frac{dx}{du_3} = \frac{3\nu}{\rho} - \frac{3g}{\rho}\xi - 3g\xi^2 + (2\rho^2\mu_6 - \mu_2)\nu\,x^2 + O(\xi x^2),$$

where

$$g = 1 - \rho\nu = 1 - \frac{1}{2}(-1 \pm \sqrt{3}) = \frac{3 \mp \sqrt{3}}{2}.$$

Applying again the conditions $\xi(0) = 0, x(0) = 0$, we find the series solutions with initial terms

$$\xi = g(u_3 - u_{30}) + \frac{\rho\, g}{2}(u_3 - u_{30})^2 + O\left((u_3 - u_{30})^3\right),$$

$$x = \frac{3\nu}{\rho}(u_3 - u_{30}) - \frac{3g}{\rho}(u_3 - u_{30})^2 + O\left((u_3 - u_{30})^3\right),$$

which, in view of the expansions (22), (68), give (77). The expansions for $i = 2, 3$ are obtained by replacing ρ by ρ^2 and by $\rho^3 = 1$ respectively. $\square$

Acknowledgments

References

[AvM989] M. Adler, and P. van Moerbeke, *The complex geometry of the Kowalevski–Painlevé analysis.* Invent. Math. **97** (1989), 3–51.

[AF00] S. Abenda and Yu. Fedorov, *On the weak Kowalevski-Painlevé property for hyperelliptically separable systems*, Acta Appl.Math. **60** (2000), 137-178.

[Bak897] H. F. Baker, *Abel's theorem and the allied theory of theta functions*, Cambridge Univ. Press, Cambridge, 1897, Reprinted in 1995.

[Bak907] H. F. Baker, *Multiple periodic functions*, Cambridge Univ. Press, Cambridge, 1907. Reprinted in 2008.

[BG06] S. Baldwin and J. Gibbons, *Genus 4 trigonal reduction of the Benney equations*, J. Phys. A 39 (2006), no. 14, 3607-3639.

[BvM987] C. Bechlivanidis, and P. van Moerbeke, *The Goryachev-Chaplygin top and the Toda lattice*, Comm. Math. Phys. **110** (1987), no. 2, 317–324.

[BBEIM94] E. D. Belokolos, and A.I Bobenko, and V.Z.Enol'skii, and A.R.Its, and V. B Matveev *Algebro-Geometrical Approach to Nonlinear Integrable Equations*, Springer-Verlag, 1994.

[BEL997] V. M. Buchstaber and V. Z. Enolskii and D. V. Leykin, *Kleinian functions, hyperelliptic Jacobians and applications*, Reviews in Mathematics and Math. Physics, I. M. Krichever, S. P. Novikov Editors, **10** (1997) part 2, Gordon and Breach, London, 3–120.

[BEL999] V. M. Buchstaber and V. Z. Enolskii and D. V. Leykin, *Rational analogues of Abelian functions*, Funk. Anal. Appl., **33** (1999) no. 2, 1-15.

[BEL00] V. M. Buchstaber, V. Z. Enolskii, and D. V. Leykin, *Uniformisation of Jacobi varieties of trigonal curves and nonlinear differential equations*, Func. Anal. Appl. **34** (2000), no. 3, 159–171.

[BL05] V. M. Buchstaber, D. V. Leykin, *Addition laws on Jacobian varieties of plane algebraic curves*, Proceedings of the Steklov Math. Inst., **251** (2005) no. 4, 49–120.

[Chap904] S. A. Chaplygin, *A new partial solution of the problem of rotation of a heavy rigid body about a fixed point*, Trudy Otd. Fiz. Nauk Mosk. Obshch. Lyub. Estest. **12** (1904) 1–4.

[Dub81] B. A. Dubrovin, *Theta-functions and non-linear equations.* (Russian) With an appendix by I.M.Krichever. Uspekhi Mat. Nauk **36**, no. 2(218) (1981) 11–80.

[DM04] H. R. Dullin and V. S. Matveev, *A new integrable system on the sphere*, Math. Res. Lett. **11** (2004) 715–722.

[EEKL993] J. C. Eilbeck, and V. Z. Enolskii, and V. Z. Kuznetsov, and D. V. Leykin, *Linear r-matrix algebra for systems separable in parabolic coordinates*, Phys. Lett. A **180** (1993), no. 3, 208–214.

[EEO11] J. C. Eilbeck, and M. England, and Yo. Ônishi, *Abelian functions associated with genus three algebraic curves*, LMS J. Comput.Math. **14**, (2011) 291-326.

[EEKT994] J. C. Eilbeck, and V. Z. Enolskii V.Z., and V. B. Kuznetsov, and A. V. Tsiganov A.V., *Linear r-matrix algebra for classical separable systems*, J.Phys. A: Math. Gen., **27**, (1994) 567-578.

[EEL00] J. C. Eilbeck, V. Z. Enolskii, and D. V. Leykin, *On the Kleinian construction of Abelian functions of canonical algebraic curves*, Proceedings of the Conference SIDE III: Symmetries of Integrable Differences Equations , Saubadia, May 1998, CRM Proceedings and Lecture Notes 25, 2000, pp. 121–138.

[EE09] M. England and J. C. Eilbeck, *Abelian functions associated with a cyclic tetragonal curve of genus six.* J.Phys.A: Math.Theor. **42** (2009) 095210, 27pp.

[EEMOP07] J. C. Eilbeck and V. Z. Enolskii, and Sh. Matsutani and Yo. Ônishi and E. Previato, *Abelian Functions for Trigonal Curve of Genus Three*, Int. Math. Res. Notes, **2007**, (2007): rnm 140-68, arXiv: math.AG/0610019.

[EPR03] V. Z. Enolskii, M. Pronine, and P. Richter, *Double pendulum and theta-divisor*, J.Nonlin.Sci., **13** (2003) 157-174.

[Gor912] D. N. Goryachev, *On a motion of a heavy rigid body about a fixed point in the case of $A = B = 4C$*. Mat. Sbonik Kruzhka Lyub. Mat. Nauk **21** (1912) 431–438.

[Gor916] D. N. Goryachev, *New cases of integrability of Euler's dynamical equations.* Warshav. Univ. Izv. **3** (1916) 1–15.

[FK980] H. M. Farkas and I. Kra, *Riemann Surfaces* , Springer, New York, 1980.

[Fay973] J. D. Fay, *Theta functions on Riemann surfaces*, Lectures Notes in Mathematics (Berlin), **352**, Springer, 1973.

[FG07]   Yu.N.Fedorov and D. Gómes-Ulate, *Dynamical systems on infinitely sheeted Riemann surfaces*, Physica D **227** (2007) 120-134.

[Gra990] D. Grant, *Formal groups in genus two*, J. reine angew. Math. **411** (1990) 96-121.

[Jor992] J. Jorgenson, *On directional derivatives of the theta function along its divisor*, Israel J.Math. **77** (1992), 274–284.

[MP08]   S. Matsutani, and E. Previato, *Jacobi inversion on strata of the Jacobian of the C_{rs} curve $y^r = f(x)$*, J. Math. Soc. Jpn. **60** (2008) 1009-1044;

[MP11]   S. Matsutani and E. Previato, *Jacobi inversion on strata of the Jacobian of the C_{rs} curve $y^r = f(x)$ II* J. Math. Soc. Jpn. *in press* Preprint (2011) arXiv: 1006.1090 [math.AG].

[Nak10]   A. Nakayashiki, *Algebraic Expression of Sigma Functions of (n, s) Curves*, Asian J.Math. **14**:2 (2010), 174-211.

[NYo12]   A. Nakayashiki, and K. Yori, *Derivatives of Schur, Tau and Sigma Functions on Abel-Jacobi Images*, Dedicated to Michio Jimbo on his sixtieth birthday. Preprint 2012.

[Mark992] A.I.Markushevich, *Introduction to the classical theory of Abelian functions* (1992), AMS, Providence.

[Tsi05]   A. V. Tsiganov, *On a family of integrable systems on S^2 with a cubic integral of motion.* J. Phys. A: Math. Gen., **38** (2005) 921-927.

[VT09]   A.V. Vershilov, A. V. Tsiganov, *On bi-Hamiltonian geometry of some integrable systems on the sphere with cubic integral of motion.* J. Phys. A, **42** (2009), no. 10, 105203, 12 pp.

[VT12]   A.V. Vershilov, A.V. Tsiganov, On One Integrable System With a Cubic First Integral. *Lett. Math. Phys.*, **101** (2012), no.2, 143–156.

[Van995] P. Vanhaecke, *Stratification of hyperelliptic Jacobians and the Sato Grassmannian* Acta Appl. Math., **40** (1995), 143–172.

[Val10]   G. Valent, *On a class of integrable systems with a cubic first integral,* Commun. Math. Phys. **299** (2010), 631–649.

[Yeh02]   H.M. Yehia *On certain two-dimensional conservative mechanical systems with a cubic second integral* J. Phys. A: Math. Gen. **35** (2002) 9469–9487

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