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Citation for published version:

Rossi, R The Newsvendor problem: analysis of the cost structure under normally distributed demand.

Link:

[Link to publication record in Edinburgh Research Explorer](#)

Document Version:

Preprint (usually an early version)

Publisher Rights Statement:

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The Newsvendor problem

analysis of the cost structure under normally distributed demand

Roberto Rossi
University of Edinburgh Business School
Edinburgh, UK

Abstract

Well-known derivation of a closed form solution for the expected total cost expression of the Newsvendor problem.

The Newsvendor problem

We consider the Newsvendor problem, this is a classical inventory control problem in which we consider a single item and a single stocking location. The aim is to control stock over a single period planning horizon under a single opportunity to replenish stocks at the beginning of the horizon. Without loss of generality, we assume the initial inventory to be zero. Consider an order quantity Q and a normally distributed demand d with mean μ and standard deviation σ . Let ϕ_d and Φ_d denote the probability density function and the cumulative distribution function of d , respectively; furthermore, we shall denote as ϕ and Φ the probability density function and the cumulative distribution function of a standard normally distributed random variable, respectively. The newsvendor faces "overage" costs of o dollars per unit left, if the end of period inventory is positive. Conversely, the newsvendor faces "underage" costs of u dollars per unit short, if the end of period inventory is negative. The aim is to find the optimal order quantity Q that minimizes expected total cost, $C(Q)$, composed by expected total overage and underage costs:

$$C(Q) = o \int_{-\infty}^Q (Q - t) \phi_d(t) dt + u \int_Q^{\infty} (t - Q) \phi_d(t) dt$$

■ Example

$$\mu = 5;$$

$$\sigma = 2;$$

$$o = 1;$$

$$u = 4;$$

$$\text{NMinimize} \left[o \int_{-\infty}^q (q - t) \text{PDF}[\text{NormalDistribution}[\mu, \sigma], t] dt + \right. \\ \left. u \int_q^{\infty} (t - q) \text{PDF}[\text{NormalDistribution}[\mu, \sigma], t] dt, \{q\} \right]$$

$$\{2.79962, \{q \rightarrow 6.68324\}\}$$

A well-known result in inventory theory is the following

$$C(Q) = (o + u) \sigma \left(\phi\left(\frac{Q - \mu}{\sigma}\right) - \left(1 - \Phi\left(\frac{Q - \mu}{\sigma}\right)\right) \frac{Q - \mu}{\sigma} \right) + o(Q - \mu)$$

or alternatively

$$C(Q) = (o + u) \sigma \left(\Phi\left(\frac{Q-\mu}{\sigma}\right) \frac{Q-\mu}{\sigma} + \phi\left(\frac{Q-\mu}{\sigma}\right) \right) - u(Q - \mu)$$

■ Example

$$Q = 6.683242467145828 ;$$

$$\begin{aligned} & N \left[(o + u) \sigma \left(\text{PDF} \left[\text{NormalDistribution} [0, 1], \frac{Q - \mu}{\sigma} \right] - \right. \right. \\ & \quad \left. \left. \left(1 - \text{CDF} \left[\text{NormalDistribution} [0, 1], \frac{Q - \mu}{\sigma} \right] \right) \frac{Q - \mu}{\sigma} \right) + o (Q - \mu) \right] \\ & N \left[(o + u) \sigma \left(\text{CDF} \left[\text{NormalDistribution} [0, 1], \frac{Q - \mu}{\sigma} \right] \frac{Q - \mu}{\sigma} + \text{PDF} \left[\text{NormalDistribution} [0, 1], \frac{Q - \mu}{\sigma} \right] \right) - \right. \\ & \quad \left. u (Q - \mu) \right] \end{aligned}$$

$$2.79962$$

$$2.79962$$

In the rest of this note we shall analytically derive the second of these expressions, the first can be derived in a similar fashion.

Consider the expected total cost

$$C(Q) = o \int_{-\infty}^Q (Q - t) \phi_d(t) dt + u \int_Q^{\infty} (t - Q) \phi_d(t) dt$$

and separate the overage component and the underage component. The overage is

$$o \int_{-\infty}^Q (Q - t) \phi_d(t) dt$$

the underage is

$$u \int_Q^{\infty} (t - Q) \phi_d(t) dt$$

the underage can be rewritten as

$$u \int_{-\infty}^Q (t - Q) \phi_d(t) dt - u(Q - \mu)$$

note that to derive the other expression discussed above, we must rewrite the overage cost in a similar fashion, rather than the underage cost, as here discussed.

Let us now adopt this latter expression in $C(Q)$

$$C(Q) = o \int_{-\infty}^Q (Q - t) \phi_d(t) dt + u \int_{-\infty}^Q (t - Q) \phi_d(t) dt - u(Q - \mu)$$

rewrite

$$C(Q) = (o + u) \int_{-\infty}^Q (Q - t) \phi_d(t) dt - u(Q - \mu)$$

By noting that

$$\int_{-\infty}^Q (t - Q) \phi_d(t) dt = \int_{-\infty}^Q \Phi_d(t) dt$$

we rewrite

$$C(Q) = (o + u) \int_{-\infty}^Q \Phi_d(t) dt - u(Q - \mu)$$

We now standardize the above expression by using the standard normal distribution

$$C(Q) = (o + u) \sigma \int_{-\infty}^{\frac{Q-\mu}{\sigma}} \Phi(t) dt - u(Q - \mu)$$

integrate by parts the expression

$$C(Q) = (o + u) \sigma \left(\Phi\left(\frac{Q-\mu}{\sigma}\right) \frac{Q-\mu}{\sigma} - \int_{-\infty}^{\frac{Q-\mu}{\sigma}} t \phi(t) dt \right) - u(Q - \mu)$$

rewrite

$$C(Q) = (o + u) \sigma \left(\Phi\left(\frac{Q-\mu}{\sigma}\right) \frac{Q-\mu}{\sigma} - \int_{-\infty}^{\frac{Q-\mu}{\sigma}} t \phi(t) dt \right) - u(Q - \mu)$$

$$C(Q) = (o + u) \sigma \left(\Phi\left(\frac{Q-\mu}{\sigma}\right) \frac{Q-\mu}{\sigma} + \phi\left(\frac{Q-\mu}{\sigma}\right) \right) - u(Q - \mu)$$

qed.

■ Example

```

Q = 6.683242467145828 ;
μ = 5 ;
σ = 2 ;
o = 1 ;
u = 4 ;
Print ["C(Q)="];
N[o ∫-∞Q (Q - t) PDF[NormalDistribution[μ, σ], t] dt +
  u ∫Q∞ (t - Q) PDF[NormalDistribution[μ, σ], t] dt]
Print ["Overage="];
N[o ∫-∞Q (Q - t) PDF[NormalDistribution[μ, σ], t] dt]
Print ["Underage="];
N[u ∫Q∞ (t - Q) PDF[NormalDistribution[μ, σ], t] dt]
Print ["Underage="];
N[u ∫-∞Q (Q - t) PDF[NormalDistribution[μ, σ], t] dt - u (Q - μ)]
Print ["C(Q)="];
N[o ∫-∞Q (Q - t) PDF[NormalDistribution[μ, σ], t] dt +
  u ∫-∞Q (Q - t) PDF[NormalDistribution[μ, σ], t] dt - u (Q - μ)]
N[o ∫-∞Q (Q - t) PDF[NormalDistribution[μ, σ], t] dt +
  u ∫-∞Q (Q - t) PDF[NormalDistribution[μ, σ], t] dt - u (Q - μ)]
N[(o + u) ∫-∞Q (Q - t) PDF[NormalDistribution[μ, σ], t] dt - u (Q - μ)]
N[(o + u) ∫-∞Q CDF[NormalDistribution[μ, σ], t] dt - u (Q - μ)]
N[(o + u) σ ∫-∞ $\frac{Q-\mu}{\sigma}$  CDF[NormalDistribution[0, 1], t] dt - u (Q - μ)]
N[(o + u) σ (CDF[NormalDistribution[0, 1],  $\frac{Q-\mu}{\sigma}$ ]  $\frac{Q-\mu}{\sigma}$  -

```

$$\int_{-\infty}^{\frac{Q-\mu}{\sigma}} t \text{PDF}[\text{NormalDistribution}[0, 1], t] dt - u(Q - \mu)$$

$$N\left[(o + u) \sigma \left(\text{CDF}[\text{NormalDistribution}[0, 1], \frac{Q - \mu}{\sigma}] \frac{Q - \mu}{\sigma} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{Q-\mu}{\sigma}} t e^{-\frac{t^2}{2}} dt \right) - u(Q - \mu) \right]$$

$$N\left[(o + u) \sigma \left(\text{CDF}[\text{NormalDistribution}[0, 1], \frac{Q - \mu}{\sigma}] \frac{Q - \mu}{\sigma} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(\frac{Q-\mu}{\sigma})^2}{2}} \right) - u(Q - \mu) \right]$$

$$N\left[(o + u) \sigma \left(\text{CDF}[\text{NormalDistribution}[0, 1], \frac{Q - \mu}{\sigma}] \frac{Q - \mu}{\sigma} + \text{PDF}[\text{NormalDistribution}[0, 1], \frac{Q - \mu}{\sigma}] \right) - u(Q - \mu) \right]$$

$$N\left[(o + u) \sigma \left(\text{PDF}[\text{NormalDistribution}[0, 1], \frac{Q - \mu}{\sigma}] - \left(1 - \text{CDF}[\text{NormalDistribution}[0, 1], \frac{Q - \mu}{\sigma}]\right) \frac{Q - \mu}{\sigma} \right) + o(Q - \mu) \right]$$

C(Q) =

2.79962

Overage =

1.90652

Underage =

0.893101

Underage =

0.893101

C(Q) =

2.79962

2.79962

2.79962

2.79962

2.79962

2.79962

2.79962

2.79962

2.79962

2.79962