

ON NONCOMMUTATIVE GENERALISATIONS OF BOOLEAN ALGEBRAS

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In honor of Jonathan Leech

ABSTRACT. Skew Boolean algebras (SBA) and Boolean-like algebras (n BA) are one-pointed and n -pointed noncommutative generalisation of Boolean algebras, respectively. We show that any n BA is a cluster of n isomorphic right-handed SBAs, axiomatised here as the variety of skew star algebras. The variety of skew star algebras is shown to be term equivalent to the variety of n BAs. We use SBAs in order to develop a general theory of multideals for n BAs. We also provide a representation theorem for right-handed SBAs in terms of n BAs of n -partitions.

1. INTRODUCTION

Boolean algebras are the main example of a well-behaved double-pointed variety - meaning a variety \mathcal{V} whose type includes two distinct constants $0, 1$ in every nontrivial $A \in \mathcal{V}$. Since there are other double-pointed varieties of algebras that have Boolean-like features, in [21, 14] the notion of Boolean-like algebra (of dimension 2) was introduced as a generalisation of Boolean algebras to a double-pointed but otherwise arbitrary similarity type. The idea behind this approach was that a Boolean-like algebra of dimension 2 is an algebra A such that every $a \in A$ is 2-central in the sense of Vaggione [25], meaning that $\theta(a, 0)$ and $\theta(a, 1)$ are complementary factor congruences of A . Central elements can be given an equational characterisation through the ternary operator q satisfying the fundamental properties of the if-then-else connective of computer science. Algebraic analogues of the if-then-else construction have been studied extensively in the literature; the best known of these realisations is the ternary discriminator function $t : A^3 \rightarrow A$ of general algebra [26], defined for all $a, b, c \in A$ by $t(a, b, c) = c$ if $a = b$ and a otherwise. Varieties generated by a class of algebras with a common discriminator term are called discriminator varieties and are the most successful generalisation of Boolean algebras to date ([6, Section IV.9]).

It turns out that some important properties of Boolean algebras are shared by n -pointed algebras whose elements satisfy all the equational conditions of n -central elements through an operator q of arity $n + 1$ satisfying the fundamental properties of a generalised if-then-else connective. These algebras, and the varieties they form, were termed *Boolean-like algebras of dimension n* (n BA, for short) in [5]. Varieties of n BAs have many remarkable properties in common with the variety of Boolean algebras. In particular, any variety of n BAs with compatible operations is generated by the n BAs of finite cardinality n . In the pure case (i.e., when the type includes just the generalised if-then-else q and the n constants $\mathbf{e}_1, \dots, \mathbf{e}_n$), there is (up to isomorphism) a unique n BA \mathbf{n} of cardinality n , so that any pure n BA is isomorphic to a subalgebra of \mathbf{n}^I , for a suitable set I . Another remarkable property of the 2-element Boolean algebra is the definability of all finite Boolean functions in terms of the connectives AND, OR,

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NOT. This property is inherited by the algebra \mathbf{n} : all finite functions on the universe of \mathbf{n} are term-definable, so that the variety of pure n BAs is primal. More generally, a variety of an arbitrary type with one generator is primal if and only if it is a variety of n BAs.

Lattices and boolean algebras have been generalised in other directions: in the last decades weakenings of lattices where the meet and join operations may fail to be commutative have attracted the attention of various researchers. A non-commutative generalisation of lattices, probably the most interesting and successful, is the concept of *skew lattice* [16] along with the related notion of *skew Boolean algebra* (SBA) (the interested reader is referred to [17, 3, 15] or [23] for a comprehensive account). Here, a SBA is a skew lattice with zero in the sense of Leech [16], structurally enriched with an implicative BCK-difference [12] operation. Roughly speaking, a SBA is a non-commutative analogue of a generalised Boolean algebra. The significance of SBAs is revealed by a result of Leech [15], stating that any right-handed SBA can be embedded into some SBA of partial functions. This result has been revisited and further explored in [1] and [13], showing that any SBA is dual to a sheaf over a locally-compact Boolean space.

SBAs are also closely related to discriminator varieties (see [3, 7] for the one-pointed case and [21] for the double-pointed one). Seminal results of Bignall and Leech [3] show that every algebra in a one-pointed discriminator variety can be presented, up to term equivalence, as a skew Boolean intersection algebra (SBIA) with compatible operations. SBIA's are closely related to the SBAs of Leech [15]. Every SBIA has a SBA term reduct, but not conversely.

The present paper explores the connection between skew Boolean algebras and Boolean-like algebras of dimension n . We prove that any n BA \mathbf{A} contains a symmetric \cap -skew cluster of right-handed SBIA's $S_1^\cap(\mathbf{A}), \dots, S_n^\cap(\mathbf{A})$, called its \cap -*skew reducts*. Interestingly, every permutation σ of the symmetric group S_n determines a bunch of isomorphisms

$$S_1^\cap(\mathbf{A}) \cong S_{\sigma_1}^\cap(\mathbf{A}) \quad \dots \quad S_n^\cap(\mathbf{A}) \cong S_{\sigma_n}^\cap(\mathbf{A})$$

which shows the inner symmetry of the n BAs. Every n BA has also a skew cluster $S_1(\mathbf{A}), \dots, S_n(\mathbf{A})$ of isomorphic right-handed SBAs, called its *skew reducts*, which are the skew Boolean algebra reducts of members of the \cap -skew cluster of \mathbf{A} . The skew reducts of a n BA are so deeply correlated that they allow to recover the full structure of the n BA. We introduce a new variety of algebras, called *skew star* algebras, equationally axiomatising a bunch of skew Boolean algebras and their relationships, and we prove that it is term equivalent to the variety of n BAs. We also provide a representation theorem for right-handed skew Boolean algebras in terms of n BAs of n -partitions. This result follows on combining Leech's example [15] showing that every right-handed skew Boolean algebra can be embedded in an algebra of partial functions with codomain $\{1, 2\}$ with the result given in [5] that every n BA is isomorphic to a n BA of n -partitions.

The notion of ideal plays an important role in order theory and universal algebra. Ideals, filters and congruences are interdefinable in Boolean algebras. In the case of n BAs, the couple ideal-filter is replaced by *multideals*, which are tuples (I_1, \dots, I_n) of disjoint skew Boolean ideals satisfying some compatibility conditions that extend in a conservative way those of the Boolean case. We show that there exists a bijective correspondence between multideals and congruences on n BAs, rephrasing the well known correspondence of the Boolean case. The proof of this result makes an essential use of the notion of a coordinate, originally defined in [5] and rephrased here in terms of the operations of the skew reducts. Any element x of a n BA \mathbf{A} univocally determines a n -tuple of elements of the canonical inner Boolean algebra B of \mathbf{A} , its coordinates, codifying x as a "linear combination". In the Boolean case, there is a bijective correspondence between maximal ideals and homomorphisms onto $\mathbf{2}$. In the last section of the paper we show that every multideal can be extended to an ultramultideal, and that there exists a bijective

correspondence between ultramultideals and homomorphisms onto \mathbf{n} . Moreover, ultramultideals are proved to be exactly the prime multideals.

2. PRELIMINARIES

The notation and terminology in this paper are pretty standard. For concepts, notations and results not covered hereafter, the reader is referred to [6, 20] for universal algebra, to [17, 15, 23] for skew Boolean algebras and to [21, 14, 5] for n BAs.

2.1. Algebras. If τ is an algebraic type, an algebra \mathbf{A} of type τ is called a τ -*algebra*, or simply an algebra when τ is clear from the context. An algebra is *trivial* if its carrier set is a singleton set.

Superscripts that mark the difference between operations and operation symbols will be dropped whenever the context is sufficient for a disambiguation.

$\text{Con}(\mathbf{A})$ is the lattice of all congruences on \mathbf{A} , whose bottom and top elements are, respectively, $\Delta = \{(a, a) : a \in A\}$ and $\nabla = A \times A$. Given $a, b \in A$, we write $\theta(a, b)$ for the smallest congruence θ such that $(a, b) \in \theta$.

We say that an algebra \mathbf{A} is: (i) *subdirectly irreducible* if the lattice $\text{Con}(\mathbf{A})$ has a unique atom; (ii) *simple* if $\text{Con}(\mathbf{A}) = \{\Delta, \nabla\}$; (iii) *directly indecomposable* if \mathbf{A} is not isomorphic to a direct product of two nontrivial algebras.

A class \mathcal{V} of τ -algebras is a *variety* (equational class) if it is closed under subalgebras, direct products and homomorphic images. If K is a class of τ -algebras, the variety $\mathcal{V}(K)$ generated by K is the smallest variety including K . If $K = \{\mathbf{A}\}$ we write $\mathcal{V}(\mathbf{A})$ for $\mathcal{V}(\{\mathbf{A}\})$.

Following Blok and Pigozzi [4], two elements a, b of an algebra \mathbf{A} are said to be *residually distinct* if they have distinct images in every non-trivial homomorphic image of \mathbf{A} .

We say that a variety \mathcal{V} is *n -pointed* iff it has at least n nullary operators that are distinct in any nontrivial member of \mathcal{V} . Boolean algebras are the main example of a double-pointed variety.

A one-pointed variety \mathcal{V} is *0-regular* if the congruences of algebras in \mathcal{V} are uniquely determined by their 0-classes. Fichtner [9] has shown that a one-pointed variety is 0-regular if and only if there exist binary terms $d_1(x, y), \dots, d_n(x, y)$ satisfying the following two conditions:

- $d_i(x, x) = 0$ for every $i = 1, \dots, n$;
- $d_1(x, y) = d_2(x, y) = \dots = d_n(x, y) = 0 \Rightarrow x = y$.

2.1.1. Notations. If A is a set and $X \subseteq A$, then \bar{X} denotes the set $A \setminus X$.

Let $\hat{n} = \{1, \dots, n\}$ and q be an operator of arity $n + 1$. If d_1, \dots, d_k is a partition of \hat{n} and $a, b_1, \dots, b_k \in A$, then

$$(1) \quad q(a, b_1/d_1, \dots, b_k/d_k)$$

denotes $q(a, c_1, \dots, c_n)$, where for all $1 \leq i \leq n$, $c_i = b_j$ iff $i \in d_j$. Notice that $q(a, b_1/d_1, \dots, b_k/d_k)$ is well-defined as d_1, \dots, d_k partition \hat{n} . If d_j is a singleton $\{i\}$, then we write b/i for b/d_j . If $d_i = \hat{n} \setminus d_r$ is the complement of d_r , then we may write b/\bar{d}_r for b/d_i . The notation (1) will be used extensively throughout the paper, mainly to define derived term operations in the context of n BAs.

2.2. Factor Congruences and Decomposition. Directly indecomposable algebras play an important role in the characterisation of the structure of a variety of algebras. For example, if the class of indecomposable algebras in a Church variety (see Section 3.1 and [21]) is universal, then any algebra in the variety is a weak Boolean product of directly indecomposable algebras. In this section we summarize the basic ingredients of factorisation: tuples of complementary factor congruences and decomposition operators (see [20]).

Definition 2.1. A sequence (ϕ_1, \dots, ϕ_n) of congruences on a τ -algebra \mathbf{A} is a n -tuple of complementary factor congruences exactly when:

- (1) $\bigcap_{1 \leq i \leq n} \phi_i = \Delta$;
- (2) $\forall (a_1, \dots, a_n) \in A^n$, there is $u \in A$ such that $a_i \phi_i u$, for all $1 \leq i \leq n$.

Such an element u such that $a_i \phi_i u$ for every i is unique by Definition 2.1(1).

If (ϕ_1, \dots, ϕ_n) is a n -tuple of complementary factor congruences on \mathbf{A} , then the function $f : \mathbf{A} \rightarrow \prod_{i=1}^n \mathbf{A}/\phi_i$, defined by $f(a) = (a/\phi_1, \dots, a/\phi_n)$, is an isomorphism. Moreover, every factorisation of \mathbf{A} in n factors univocally determines a n -tuple of complementary factor congruences.

A pair (ϕ_1, ϕ_2) of congruences is a pair of complementary factor congruences if and only if $\phi_1 \cap \phi_2 = \Delta$ and $\phi_1 \circ \phi_2 = \nabla$. The pair (Δ, ∇) corresponds to the product $\mathbf{A} \cong \mathbf{A} \times \mathbf{1}$, where $\mathbf{1}$ is a trivial algebra; obviously $\mathbf{1} \cong \mathbf{A}/\nabla$ and $\mathbf{A} \cong \mathbf{A}/\Delta$.

A *factor congruence* is any congruence which belongs to a pair of complementary factor congruences. The set of factor congruences of \mathbf{A} is not, in general, a sublattice of $\text{Con}(\mathbf{A})$.

Notice that, if (ϕ_1, \dots, ϕ_n) is a n -tuple of complementary factor congruences, then ϕ_i is a factor congruence for each $1 \leq i \leq n$, because the pair $(\phi_i, \bigcap_{j \neq i} \phi_j)$ is a pair of complementary factor congruences.

It is possible to characterise n -tuple of complementary factor congruences in terms of certain algebra homomorphisms called *decomposition operators* (see [20, Def. 4.32] for additional details).

Definition 2.2. An n -ary decomposition operator on an algebra \mathbf{A} is a function $f : A^n \rightarrow A$ satisfying the following conditions:

- D1:** $f(x, x, \dots, x) = x$;
- D2:** $f(f(x_{11}, x_{12}, \dots, x_{1n}), \dots, f(x_{n1}, x_{n2}, \dots, x_{nn})) = f(x_{11}, \dots, x_{nn})$;
- D3:** f is an algebra homomorphism from \mathbf{A}^n to \mathbf{A} :
 $f(g(x_{11}, x_{12}, \dots, x_{1k}), \dots, g(x_{n1}, x_{n2}, \dots, x_{nk})) = g(f(x_{11}, \dots, x_{n1}), \dots, f(x_{1k}, \dots, x_{nk}))$,
for every $g \in \tau$ of arity k .

There is a bijective correspondence between n -tuples of complementary factor congruences and n -ary decomposition operators, and thus, between n -ary decomposition operators and factorisations of an algebra in n factors.

Theorem 2.1. Any n -ary decomposition operator $f : \mathbf{A}^n \rightarrow \mathbf{A}$ on an algebra \mathbf{A} induces a n -tuple of complementary factor congruences ϕ_1, \dots, ϕ_n , where each $\phi_i \subseteq A \times A$ is defined by:

$$a \phi_i b \text{ iff } f(a, \dots, a, b, a, \dots, a) = a \quad (b \text{ at position } i).$$

Conversely, any n -tuple ϕ_1, \dots, ϕ_n of complementary factor congruences induces a decomposition operator f on \mathbf{A} : $f(a_1, \dots, a_n) = u$ iff $a_i \phi_i u$ for all i .

We say that two functions $f : A^m \rightarrow A$ and $g : A^n \rightarrow A$ commute (see [20, Definition 4.34]) if

$$f(g(x_{11}, \dots, x_{1n}), \dots, g(x_{m1}, \dots, x_{mn})) = g(f(x_{11}, \dots, x_{m1}), \dots, f(x_{1n}, \dots, x_{mn})).$$

In this case, f is a homomorphism from $(A, g)^m$ into (A, g) and g is a homomorphism from $(A, f)^n$ into (A, f) .

The following proposition is [20, Exercise 4.38(15)].

Proposition 2.2. Let f and g be an m -ary and an n -ary decomposition operator of an algebra \mathbf{A} . Then $f(g(x_{11}, \dots, x_{1n}), \dots, g(x_{m1}, \dots, x_{mn}))$ is a decomposition operator of \mathbf{A} if and only if f and g commute.

The variables occurring in $f(g(x_{11}, \dots, x_{1n}), \dots, g(x_{m1}, \dots, x_{mn}))$ may be not all distinct, as explained in the following proposition.

Proposition 2.3. *If f is a n -ary decomposition operator and d_1, \dots, d_k ($k \geq 2$) is a partition of $\hat{n} = \{1, \dots, n\}$, then the map h , defined by*

$$h(y_1, \dots, y_k) = f(z_1, \dots, z_n), \text{ where for all } 1 \leq i \leq n, z_i = y_j \text{ iff } i \in d_j,$$

is a k -ary decomposition operator.

2.3. Factor Elements. The notion of decomposition operator and of factorisation can sometimes be internalised: some elements of the algebra, the so called factor elements, can embody all the information codified by a decomposition operator.

Let \mathbf{A} be a τ -algebra, where we distinguish a $(n+1)$ -ary term operation q .

Definition 2.3. We say that an element e of \mathbf{A} is a *factor element with respect to q* if the n -ary operation $f_e : A^n \rightarrow A$, defined by

$$f_e(a_1, \dots, a_n) = q^{\mathbf{A}}(e, a_1, \dots, a_n), \text{ for all } a_i \in A,$$

is a n -ary decomposition operator (that is, f_e satisfies identities (D1)-(D3) of Definition 2.2).

An element e of \mathbf{A} is a factor element if and only if the tuple of relations (ϕ_1, \dots, ϕ_n) , defined by $a \phi_i b$ iff $q(e, a, \dots, a, b, a, \dots, a) = a$ (b at position i), constitute a n -tuple of complementary factor congruences of \mathbf{A} .

By [7, Proposition 3.4] the set of factor elements is closed under the operation q : if $a, b_1, \dots, b_n \in A$ are factor elements, then $q(a, b_1, \dots, b_n)$ is also a factor element.

We notice that

- different factor elements may define the same tuple of complementary factor congruences.
- there may exist n -tuples of complementary factor congruences that do not correspond to any factor element.

In Section 3 we describe a class of algebras, called Church algebras of dimension n , where the $(n+1)$ -ary operator q induces a bijective correspondence between a suitable subset of factor elements, the so-called n -central elements, and the set of all n -ary decomposition operators.

2.4. Skew Boolean Algebras. We review here some basic definitions and results on *skew lattices* [16] and *skew Boolean algebras* [15].

Definition 2.4. A *skew lattice* is an algebra $\mathbf{A} = (A, \vee, \wedge)$ of type $(2, 2)$, where both \vee and \wedge are associative, idempotent binary operations, connected by the absorption law: $x \vee (x \wedge y) = x = x \wedge (x \vee y)$; and $(y \wedge x) \vee x = x = (y \vee x) \wedge x$.

The absorption condition is equivalent to the following pair of biconditionals: $x \vee y = y$ iff $x \wedge y = x$; and $x \vee y = x$ iff $x \wedge y = y$.

In any skew lattice we define the following relations:

- (1) $x \leq y$ iff $x \wedge y = x = y \wedge x$.
- (2) $x \preceq_{\mathcal{D}} y$ iff $x \wedge y \wedge x = x$.
- (3) $x \preceq_{\mathcal{L}} y$ iff $x \wedge y = x$.
- (4) $x \preceq_{\mathcal{R}} y$ iff $y \wedge x = x$.

The relation \leq is a partial ordering, while the relations $\preceq_{\mathcal{D}}, \preceq_{\mathcal{L}}, \preceq_{\mathcal{R}}$ are preorders. The equivalences \mathcal{D}, \mathcal{L} and \mathcal{R} , respectively induced by $\preceq_{\mathcal{D}}, \preceq_{\mathcal{L}}$ and $\preceq_{\mathcal{R}}$, are congruences.

A skew lattice is *left-handed* (*right-handed*) if $\mathcal{L} = \mathcal{D}$ ($\mathcal{R} = \mathcal{D}$). The following conditions are equivalent for a skew lattice \mathbf{A} : (a) \mathbf{A} is right-handed (left-handed); (b) for all $a, b \in A$, $a \wedge b \wedge a = b \wedge a$ ($a \wedge b \wedge a = a \wedge b$).

Observe that

- (i) The quotient \mathbf{A}/\mathcal{D} is the maximal lattice image of \mathbf{A} . This is the skew-lattice theoretic analogue [16, Theorem 1.7] of the well-known Clifford-McLean theorem for bands.
- (ii) The algebras \mathbf{A}/\mathcal{L} and \mathbf{A}/\mathcal{R} are the maximal right-handed and left-handed images of \mathbf{A} respectively.
- (iii) The skew lattice \mathbf{A} is the fibered product of its maximal right-handed image \mathbf{A}/\mathcal{L} with its maximal left-handed \mathbf{A}/\mathcal{R} over its maximal lattice image. This result is the skew-lattice theoretic analogue [16, Theorem 1.15] of the Kimura factorisation theorem for idempotent semigroups.

In a skew lattice elements commuting under \vee need not commute under \wedge and vice-versa. A skew lattice, satisfying $x \wedge y = y \wedge x$ if and only if $x \vee y = y \vee x$ for all x and y , is called *symmetric*. Symmetric skew lattices form a variety characterised by the following identities (See [24, Theorem SSL-6]):

$$x \vee y \vee (x \wedge y) = (y \wedge x) \vee y \vee x; \quad x \wedge y \wedge (x \vee y) = (y \vee x) \wedge y \wedge x.$$

The two most significant classes of examples, skew lattices of idempotents in rings (see, e.g., [16]) and skew Boolean algebras (see [15] and Definition 2.5 below), consist of symmetric skew lattices.

If we expand skew lattices by a subtraction operation and a constant 0, we get the following non-commutative variant of Boolean algebras (see [15]).

Definition 2.5. A *skew Boolean algebra* (SBA, for short) is an algebra $\mathbf{A} = (A, \vee, \wedge, \setminus, 0)$ of type $(2, 2, 2, 0)$ such that:

- (S1) its reduct (A, \vee, \wedge) is a skew lattice satisfying
 - Normality: $x \wedge y \wedge z \wedge x = x \wedge z \wedge y \wedge x$;
 - Symmetry: $x \wedge y = y \wedge x$ iff $x \vee y = y \vee x$;
 - Distributivity: $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$ and $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$;
- (S2) 0 is left and right absorbing w.r.t. skew lattice meet;
- (S3) the operation \setminus satisfies the identities
 - $(x \wedge y \wedge x) \vee (x \setminus y) = x = (x \setminus y) \vee (x \wedge y \wedge x)$;
 - $x \wedge y \wedge x \wedge (x \setminus y) = 0 = (x \setminus y) \wedge x \wedge y \wedge x$.

Every SBA is strongly distributive, i.e., it satisfies the identities $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ and $(y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x)$.

It can be seen that, for every $a \in A$, the natural partial order of the subalgebra $a \wedge A \wedge a = \{a \wedge b \wedge a : b \in A\} = \{b : b \leq a\}$ of \mathbf{A} is a Boolean lattice. Indeed, the algebra $(a \wedge A \wedge a, \vee, \wedge, 0, a, \neg)$, where $\neg b = a \setminus b$ for every $b \leq a$, is a Boolean algebra with minimum 0 and maximum a .

Notice that

- The normal axiom implies the commutativity of \wedge and \vee in the interval $a \wedge A \wedge a$.
- Axiom (S2) expresses that 0 is the minimum of the natural partial order on A .
- Axiom (S3) implies that, for every $b \in a \wedge A \wedge a$, the element $a \setminus b$ is the complement of b in the Boolean lattice $a \wedge A \wedge a$. We point out here that $a \setminus b$ is in fact a kind of relative complement that acts ‘locally’ on subalgebras of the form $a \wedge A \wedge a$.

An element m of a SBA \mathbf{A} is *maximal* if $a \preceq_{\mathcal{D}} m$ for every $a \in A$ (i.e., $a \wedge m \wedge a = a$, for every $a \in A$). When they exist, maximal elements form an equivalence class (module \mathcal{D}) called the *maximal class*. If \mathbf{A} is a SBA, then \mathbf{A}/\mathcal{D} , where \mathcal{D} is the Clifford-McLean congruence on \mathbf{A} , is a Boolean algebra iff \mathbf{A} has a maximal class. Skew Boolean algebras with a maximal class thus constitute a very specialised class of skew Boolean algebras. It is known that every skew Boolean algebra embeds into a skew Boolean algebra with a maximal class.

A nonempty subset I of a SBA \mathbf{A} closed under \vee is an *ideal* of \mathbf{A} (see [18, Section 4]) if it satisfies one of the following equivalent conditions:

- $a \in A, b \in I$ and $a \preceq b$ imply $a \in I$;
- $a \in A$ and $b \in I$ imply $a \wedge b, b \wedge a \in I$;
- $a \in A$ and $b \in I$ imply $a \wedge b \wedge a \in I$.

Given a congruence ϕ on a SBA, the equivalence class $0/\phi$ is an ideal. However, congruences on a SBA are not in general in 1-1 correspondence with ideals.

2.5. Skew Boolean algebras with intersection. Skew Boolean algebras such that every finite set of their universe has an infimum w.r.t. the underlying natural partial ordering of the algebra stand out for their significance. We denote the infimum of a and b w.r.t. the natural partial order by $a \cap b$ and refer to the operation \cap as *intersection* in order to distinguish it from the skew lattice meet \wedge . It turns out that SBAs with the additional operation \cap can be given an equational characterisation provided we include the operation \cap into the signature.

Definition 2.6. A *skew Boolean \cap -algebra* (SBIA, for short) is an algebra $\mathbf{A} = (A; \vee, \wedge, \cap, \setminus, 0)$ of type $(2, 2, 2, 2, 0)$ such that:

- (i) The reduct $(A; \vee, \wedge, \setminus, 0)$ is a SBA and the reduct $(A; \cap)$ is a meet semilattice;
- (ii) \mathbf{A} satisfies the identities $x \cap (x \wedge y \wedge x) = x \wedge y \wedge x$ and $x \wedge (x \cap y) = x \cap y = (x \cap y) \wedge x$.

The next theorem by Bignall and Leech [3], which we present in its simplest form, provides a powerful bridge between the theories of SBAs and pointed discriminator varieties.

Theorem 2.4. *The variety of type $(3, 0)$ generated by the class of all one-pointed discriminator algebras $(A; t, 0)$, where t is the discriminator function on A and 0 is a constant, is term equivalent to the variety of right handed SBIA's.*

2.6. A term equivalence result for skew Boolean algebras. In [7] Cvetko-Vah and the second author have introduced the variety of semicentral right Church algebras (SRCA) and have shown that the variety of right-handed SBAs is term equivalent to the variety of SRCAs. It is worth noticing that, in SRCAs, a single ternary operator q replaces all the binary operators of SBAs.

An algebra $\mathbf{A} = (A, q, 0)$ of type $(3, 0)$ is called a *right Church algebra* (RCA, for short) if it satisfies the identity $q(0, x, y) = y$.

Definition 2.7. Let $\mathbf{A} = (A, q, 0)$ be a RCA. An element $a \in A$ is called *semicentral* if it is a factor element (w.r.t. q) satisfying $q(a, a, 0) = a$.

Lemma 2.5. [7, Proposition 3.9] *Let $\mathbf{A} = (A, q, 0)$ be an RCA. Every semicentral element $e \in A$ determines a pair of complementary factor congruences:*

$$\phi_e = \{(a, b) : q(e, a, b) = a\} \text{ and } \bar{\phi}_e = \{(a, b) : q(e, a, b) = b\}$$

such that $\phi_e = \theta(e, 0)$, the least congruence of \mathbf{A} equating e and 0 .

Definition 2.8. An algebra $\mathbf{A} = (A, q, 0)$ of type $(3, 0)$ is called a *semicentral RCA* (SRCA, for short) if every element of A is semicentral.

To help the reader in understanding the term equivalence of SRCAs and right-handed SBAs, it is perhaps useful to provide an explicit axiomatisation of SRCAs. Such an axiomatisation is not long:

- (1) $q(0, x, y) = y$;
- (2) $q(w, w, 0) = w$;
- (3) $q(w, y, y) = y$;

- (4) $q(w, q(w, x, y), z) = q(w, x, z)$;
- (5) $q(w, x, q(w, y, z)) = q(w, x, z)$;
- (6) $q(w, q(y_1, y_2, y_3), q(z_1, z_2, z_3)) = q(q(w, y_1, z_1), q(w, y_2, z_2), q(w, y_3, z_3))$.

The last five identities equationally formalise that the element w is semicentral.

Theorem 2.6. [7] *The variety of right-handed SBAs is term equivalent to the variety of SRCAs.*

The proof is based on the following correspondence between the algebraic similarity types of SBAs and of SRCAs:

$$\begin{aligned} q(x, y, z) &\rightsquigarrow (x \wedge y) \vee (z \setminus x) \\ x \vee y &\rightsquigarrow q(x, x, y) \\ x \wedge y &\rightsquigarrow q(x, y, 0) \\ y \setminus x &\rightsquigarrow q(x, 0, y). \end{aligned}$$

The natural partial order and preorders of a SRCA are the partial order \leq and the preorders $\preceq_{\mathcal{D}}, \preceq_{\mathcal{L}}, \preceq_{\mathcal{R}}$ of its corresponding SBA.

Example 1. (see [8, 7]) Let $\mathcal{F}(X, Y)$ be the set of all partial functions from X into Y . The algebra $\mathbf{F} = (\mathcal{F}(X, Y), q, 0)$ is a SRCA, where

- $0 = \emptyset$ is the empty function;
- For all functions $f : F \rightarrow Y$, $g : G \rightarrow Y$ and $h : H \rightarrow Y$ ($F, G, H \subseteq X$),

$$q(f, g, h) = g|_{G \cap F} \cup h|_{H \cap \overline{F}}.$$

By Theorem 2.6 \mathbf{F} is term equivalent to the right-handed SBA of universe $\mathcal{F}(X, Y)$, whose operations are defined as follows:

$$f \wedge g = g|_{G \cap F}; \quad f \vee g = f \cup g|_{G \cap \overline{F}}; \quad g \setminus f = g|_{G \cap \overline{F}}.$$

3. BOOLEAN-LIKE ALGEBRAS OF FINITE DIMENSION

Some important properties of Boolean algebras are shared by n -pointed algebras whose elements satisfy all the equational conditions of n -central elements through an operator q of arity $n + 1$ satisfying the fundamental properties of a generalised if-then-else connective. These algebras, and the varieties they form, were termed Boolean-like algebras of dimension n in [5].

3.1. Church algebras of finite dimension. In this section we recall from [5] the notion of a Church algebra of dimension n . These algebras have n nullary operations $\mathbf{e}_1, \dots, \mathbf{e}_n$ ($n \geq 2$) and an operation q of arity $n + 1$ (a sort of “generalised if-then-else”) satisfying the identities $q(\mathbf{e}_i, x_1, \dots, x_n) = x_i$. The operator q induces, through the so-called n -central elements, a decomposition of the algebra into n factors.

Definition 3.1. Algebras of type τ , equipped with at least n nullary operations $\mathbf{e}_1, \dots, \mathbf{e}_n$ ($n \geq 2$) and a term operation q of arity $n + 1$ satisfying $q(\mathbf{e}_i, x_1, \dots, x_n) = x_i$, are called *Church algebras of dimension n* (n CA, for short); n CAs admitting only the $(n + 1)$ -ary q operator and the n constants $\mathbf{e}_1, \dots, \mathbf{e}_n$ are called *pure n CAs*.

If \mathbf{A} is an n CA, then $\mathbf{A}_0 = (A, q, \mathbf{e}_1, \dots, \mathbf{e}_n)$ is the *pure reduct* of \mathbf{A} .

Church algebras of dimension 2 were introduced as Church algebras in [19] and studied in [21]. Examples of Church algebras of dimension 2 are Boolean algebras (with $q(x, y, z) = (x \wedge y) \vee (\neg x \wedge z)$) or rings with unit (with $q(x, y, z) = xy + z - xz$). Next, we present some examples of Church algebra having dimension greater than 2.

Example 2. (*Semimodules*) Let R be a semiring and V be an R -semimodule generated by a finite set $E = \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Then we define an operation q of arity $n + 1$ as follows (for all $\mathbf{v} = \sum_{j=1}^n v_j \mathbf{e}_j$ and $\mathbf{w}^i = \sum_{j=1}^n w_j^i \mathbf{e}_j$):

$$q(\mathbf{v}, \mathbf{w}^1, \dots, \mathbf{w}^n) = \sum_{i=1}^n v_i \mathbf{w}^i.$$

Under this definition, V becomes a n CA. As a concrete example, if B is a Boolean algebra, B^n is a semimodule (over the Boolean ring B) with the following operations: $(a_1, \dots, a_n) + (b_1, \dots, b_n) = (a_1 \vee b_1, \dots, a_n \vee b_n)$ and $b(a_1, \dots, a_n) = (b \wedge a_1, \dots, b \wedge a_n)$. B^n is also called a Boolean vector space (see [10, 11]).

Example 3. (*n -Sets*) Let I be a set. A n -subset of I is a sequence (Y_1, \dots, Y_n) of subsets Y_i of I . We denote by $\text{Set}_n(I)$ the family of all n -subsets of I . $\text{Set}_n(I)$ becomes a pure n CA if we define an $(n + 1)$ -ary operator q and n constants $\mathbf{e}_1, \dots, \mathbf{e}_n$ as follows, for all n -subsets $\mathbf{y}^i = (Y_1^i, \dots, Y_n^i)$:

$$q(\mathbf{y}^0, \mathbf{y}^1, \dots, \mathbf{y}^n) = \left(\bigcup_{i=1}^n Y_i^0 \cap Y_1^i, \dots, \bigcup_{i=1}^n Y_i^0 \cap Y_n^i \right); \quad \mathbf{e}_1 = (I, \emptyset, \dots, \emptyset), \dots, \mathbf{e}_n = (\emptyset, \dots, \emptyset, I).$$

In [25], Vaggione introduced the notion of *central element* to study algebras whose complementary factor congruences can be replaced by certain elements of their universes. Central elements coincide with central idempotents in rings with unit and with members of the centre in ortholattices.

Theorem 3.1. [5] *If \mathbf{A} is a n CA of type τ and $c \in A$, then the following conditions are equivalent:*

- (1) c is a factor element (w.r.t. q) satisfying the identity $q(c, \mathbf{e}_1, \dots, \mathbf{e}_n) = c$;
- (2) the sequence of congruences $(\theta(c, \mathbf{e}_1), \dots, \theta(c, \mathbf{e}_n))$ is a n -tuple of complementary factor congruences of \mathbf{A} ;
- (3) for all $a_1, \dots, a_n \in A$, $q(c, a_1, \dots, a_n)$ is the unique element such that

$$a_i \theta(c, \mathbf{e}_i) q(c, a_1, \dots, a_n),$$

for all $1 \leq i \leq n$;

- (4) The function f_c , defined by $f_c(a_1, \dots, a_n) = q(c, a_1, \dots, a_n)$ for all $a_1, \dots, a_n \in A$, is a n -ary decomposition operator on \mathbf{A} such that $f_c(\mathbf{e}_1, \dots, \mathbf{e}_n) = c$.

Definition 3.2. If \mathbf{A} is a n CA, then $c \in A$ is called n -central if it satisfies one of the equivalent conditions of Theorem 3.1. A n -central element c is *nontrivial* if $c \notin \{\mathbf{e}_1, \dots, \mathbf{e}_n\}$.

Every n -central element $c \in A$ induces a decomposition of \mathbf{A} as a direct product of the algebras $\mathbf{A}/\theta(c, \mathbf{e}_i)$, for $i \leq n$.

The set of all n -central elements of a n CA \mathbf{A} is a subalgebra of the pure reduct of \mathbf{A} . We denote by $\mathbf{Ce}_n(\mathbf{A})$ the algebra $(\mathbf{Ce}_n(\mathbf{A}), q, \mathbf{e}_1, \dots, \mathbf{e}_n)$ of all n -central elements of an n CA \mathbf{A} .

Factorisations of arbitrary algebras in n factors may be studied in terms of n -central elements of suitable n CAs of functions, as explained in the following example.

Example 4. Let \mathbf{A} be an arbitrary algebra of type τ and F be a set of functions from A^n into A , which includes the projections \mathbf{e}_i^F and all constant functions f_b ($b \in A$):

- (1) $\mathbf{e}_i^F(a_1, \dots, a_n) = a_i$, for every $a_1, \dots, a_n \in A$;
- (2) $f_b(a_1, \dots, a_n) = b$, for every $a_1, \dots, a_n \in A$;

and it is closed under the following operations (for all $f, h_i, g_j \in F$ and all $a_1, \dots, a_n \in A$):

- (3) $q^{\mathbf{F}}(f, g_1, \dots, g_n)(a_1, \dots, a_n) = f(g_1(a_1, \dots, a_n), \dots, g_n(a_1, \dots, a_n))$.
(4) $\sigma^{\mathbf{F}}(h_1, \dots, h_k)(a_1, \dots, a_n) = \sigma^{\mathbf{A}}(h_1(a_1, \dots, a_n), \dots, h_k(a_1, \dots, a_n))$, for every $\sigma \in \tau$ of arity k .

The algebra $\mathbf{F} = (F, \sigma^{\mathbf{F}}, q^{\mathbf{F}}, \mathbf{e}_1^{\mathbf{F}}, \dots, \mathbf{e}_n^{\mathbf{F}})_{\sigma \in \tau}$ is a n CA. It is possible to prove that a function $f \in F$ is a n -central element of \mathbf{F} if and only if f is a n -ary decomposition operator on the algebra \mathbf{A} commuting (see Section 2.2) with every function $g \in F$. The reader may consult [22] for the case $n = 2$.

3.2. Boolean-like algebras. Boolean algebras are Church algebras of dimension 2 all of whose elements are 2-central. It turns out that, among the n -dimensional Church algebras, those algebras all of whose elements are n -central inherit many of the remarkable properties that distinguish Boolean algebras. We now recall from [5] the notion of Boolean-like algebras of dimension n , the main subject of study of this paper.

In [5] n BAs are studied in the general case of an arbitrary similarity type. Here, we restrict ourselves to consider the *pure* case, where q is the unique operator of the algebra.

Definition 3.3. A pure n CA $\mathbf{A} = (A, q, \mathbf{e}_1, \dots, \mathbf{e}_n)$ is called a *Boolean-like algebra of dimension n* (n BA, for short) if every element of A is n -central.

The class of all n BAs is a variety axiomatised by the following identities:

- (B0) $q(\mathbf{e}_i, x_1, \dots, x_n) = x_i$ ($i = 1, \dots, n$).
(B1) $q(y, x, \dots, x) = x$.
(B2) $q(y, q(y, x_{11}, x_{12}, \dots, x_{1n}), \dots, q(y, x_{n1}, x_{n2}, \dots, x_{nn})) = q(y, x_{11}, \dots, x_{nn})$.
(B3) $q(y, q(x_{10}, \dots, x_{1n}), \dots, q(x_{n0}, \dots, x_{nn})) = q(q(y, x_{10}, \dots, x_{n0}), \dots, q(y, x_{1n}, \dots, x_{nn}))$.
(B4) $q(y, \mathbf{e}_1, \dots, \mathbf{e}_n) = y$.

In the following lemma we show that every nontrivial n BA has at least n elements.

Lemma 3.2. *The constants \mathbf{e}_i ($1 \leq i \leq n$) are pairwise residually distinct in every nontrivial n BA.*

Proof. Let \mathbf{A} be a non trivial n BA such that $\mathbf{e}_k = \mathbf{e}_j$ for some $k \neq j$. If $a_1, \dots, a_n \in A$ with $a_k \neq a_j$, then $a_k = q(\mathbf{e}_k, a_1, \dots, a_n) = q(\mathbf{e}_j, a_1, \dots, a_n) = a_j$, providing a contradiction. \square

Boolean-like algebras of dimension 2 were introduced in [21] with the name ‘‘Boolean-like algebras’’. *Inter alia*, it was shown in that paper that the variety of Boolean-like algebras of dimension 2 is term-equivalent to the variety of Boolean algebras.

Example 5. The algebra $\mathbf{C}\mathbf{e}_n(\mathbf{A})$ of all n -central elements of a n CA \mathbf{A} of type τ is a canonical example of n BA (see the remark after Definition 3.2).

Example 6. The algebra $\mathbf{n} = (\{\mathbf{e}_1, \dots, \mathbf{e}_n\}, q^{\mathbf{n}}, \mathbf{e}_1^{\mathbf{n}}, \dots, \mathbf{e}_n^{\mathbf{n}})$, where $q^{\mathbf{n}}(\mathbf{e}_i, x_1, \dots, x_n) = x_i$ for every $i \leq n$, is a n BA.

Example 7. (*n -Partitions*) Let I be a set. An *n -partition* of I is a n -subset (Y^1, \dots, Y^n) of I such that $\bigcup_{i=1}^n Y^i = I$ and $Y^i \cap Y^j = \emptyset$ for all $i \neq j$. The set of n -partitions of I is closed under the q -operator defined in Example 3 and constitutes the algebra of all n -central elements of the pure n CA $\text{Set}_n(I)$ of all n -subsets of I . Notice that the algebra of n -partitions of I , denoted by $\text{Par}_n(I)$, can be proved isomorphic to the n BA \mathbf{n}^I (the Cartesian product of I copies of the algebra \mathbf{n}).

The variety BA of Boolean algebras is semisimple as every $\mathbf{A} \in \text{BA}$ is subdirectly embeddable into a power of the 2-element Boolean algebra, which is the only subdirectly irreducible member of BA. This property finds an analogue in the structure theory of n BAs.

Theorem 3.3. [5] *The algebra \mathbf{n} is the unique subdirectly irreducible n BA and it generates the variety of n BAs.*

The next corollary shows that, for any $n \geq 2$, the n BA \mathbf{n} plays a role analogous to the Boolean algebra $\mathbf{2}$ of truth values.

Corollary 3.4. *Every n BA \mathbf{A} is isomorphic to a subdirect power of \mathbf{n}^I , for some set I .*

A subalgebra of the n BA $\text{Par}_n(I)$ of the n -partitions on a set I , defined in Example 7, is called a *field of n -partitions on I* . The Stone representation theorem for n BAs follows.

Corollary 3.5. *Any n BA is isomorphic to a field of n -partitions on a suitable set I .*

One of the most remarkable properties of the 2-element Boolean algebra, called *primality* in universal algebra [6, Sec. 7 in Chap. IV], is the definability of all finite Boolean functions in terms of the connectives AND, OR, NOT. This property is inherited by n BAs. An algebra of cardinality n is primal if and only if it admits the n BA \mathbf{n} as subreduct.

Definition 3.4. Let \mathbf{A} be a nontrivial algebra. \mathbf{A} is *primal* if it is of finite cardinality and, for every function $f : A^k \rightarrow A$ ($k \geq 0$), there is a k -ary term t such that for all $a_1, \dots, a_k \in A$, $f(a_1, \dots, a_k) = t^{\mathbf{A}}(a_1, \dots, a_k)$.

A variety \mathcal{V} is primal if $\mathcal{V} = \mathcal{V}(\mathbf{A})$ for a primal algebra \mathbf{A} .

Theorem 3.6. [5]

- (i) *The variety $n\text{BA} = \mathcal{V}(\mathbf{n})$ is primal;*
- (ii) *Let \mathbf{A} be a finite algebra of cardinality n . Then \mathbf{A} is primal if and only if it admits the algebra \mathbf{n} as subreduct.*

We would like to point out here that when an algebra \mathbf{A} is primal, the choice of fundamental operations is a matter of taste and convenience (since any set of functionally complete operations would serve), and hence is typically driven by applications.

4. SKEW BOOLEAN ALGEBRAS AND n BAS

In this section we prove that any n BA \mathbf{A} contains a symmetric \cap -skew cluster of right-handed SBAs $S_1^\cap(\mathbf{A}), \dots, S_n^\cap(\mathbf{A})$. The algebra $S_i^\cap(\mathbf{A})$, called the \cap -skew i -reduct of \mathbf{A} , has e_i as a bottom element, and the other constants $e_1, \dots, e_{i-1}, e_{i+1}, \dots, e_n$ as maximal elements. Rather interestingly, every permutation σ of the symmetric group S_n determines a bunch of isomorphisms

$$S_1^\cap(\mathbf{A}) \cong S_{\sigma_1}^\cap(\mathbf{A}) \quad \dots \quad S_n^\cap(\mathbf{A}) \cong S_{\sigma_n}^\cap(\mathbf{A})$$

which shows the inner symmetry of the n BAs. Every n BA has also a skew cluster $S_1(\mathbf{A}), \dots, S_n(\mathbf{A})$ of isomorphic right-handed SBAs, which are the skew Boolean algebra reducts of members of the \cap -skew cluster of \mathbf{A} . We conclude the section with a general representation theorem for right-handed SBAs in terms of n BAs of n -partitions.

4.1. The skew reducts of a n BA. In [7] it is shown that the variety of SBAs is term equivalent to the variety of SRCAs (see Section 2.6), whose type contains only a ternary operator and a nullary operator. Here we use the $n+1$ -ary operator q of a n BA \mathbf{A} to define ternary operators t_1, \dots, t_n such that the reducts (A, t_i, e_i) are isomorphic SRCAs. Their term equivalent SBAs are all isomorphic reducts of \mathbf{A} , too. We also show that these isomorphic SBAs are in their turn reducts of isomorphic SBAs.

For every $i \in \hat{n}$, we denote by \bar{i} the set $\hat{n} \setminus \{i\}$.

In the following definition we use the $(n+1)$ -ary operator q of n BAs to introduce some term operations needed to define the above described reducts of n BAs.

Definition 4.1. Let $\mathbf{A} = (A, q, \mathbf{e}_1, \dots, \mathbf{e}_n)$ be a n BA. Given $1 \leq i \leq n$, we define the following term operations:

- $t_i(x, y, z) = q(x, y/\bar{i}, z/i)$;
- $x \wedge_i y = t_i(x, y, \mathbf{e}_i)$;
- $x \vee_i y = t_i(x, x, y)$;
- $x \setminus_i y = t_i(y, \mathbf{e}_i, x)$;
- $d_i(x, y) = q(x, t_1(y, x \vee_i y, \mathbf{e}_i), t_2(y, x \vee_i y, \mathbf{e}_i), \dots, t_n(y, x \vee_i y, \mathbf{e}_i))$;
- $x \cap_i y = q(x, t_1(y, \mathbf{e}_i, x), t_2(y, \mathbf{e}_i, x), \dots, t_n(y, \mathbf{e}_i, x))$.

We now define three reducts of a n BA \mathbf{A} , for each $1 \leq i \leq n$.

Definition 4.2. Let \mathbf{A} be a n BA. We define the following three reducts of \mathbf{A} :

- (i) The right Church i -reduct $R_i(\mathbf{A}) = (A, t_i, \mathbf{e}_i)$.
- (ii) The skew i -reduct $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathbf{e}_i)$.
- (iii) The \cap -skew i -reduct $S_i^\cap(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathbf{e}_i, \cap_i)$.

In the remaining part of this subsection we will prove that $R_i(\mathbf{A})$ is a SRCA, $S_i^\cap(\mathbf{A})$ is a \mathbf{e}_i -regular (w.r.t. d_i) right-handed SBIA, and $S_i(\mathbf{A})$ is a right-handed SBA.

In the following lemmas we prove some properties of the term operations introduced in Definition 4.1.

Lemma 4.1. *The term operations of Definition 4.1 satisfy the following conditions when they are interpreted in the generator \mathbf{n} of the variety n BA:*

$$t_i(a, b, c) = \begin{cases} c & \text{if } a = \mathbf{e}_i \\ b & \text{if } a \neq \mathbf{e}_i \end{cases}; \quad a \wedge_i b = \begin{cases} \mathbf{e}_i & \text{if } a = \mathbf{e}_i \\ b & \text{if } a \neq \mathbf{e}_i \end{cases}; \quad a \vee_i b = \begin{cases} b & \text{if } a = \mathbf{e}_i \\ a & \text{if } a \neq \mathbf{e}_i \end{cases};$$

$$a \setminus_i b = \begin{cases} b & \text{if } a = \mathbf{e}_i \\ \mathbf{e}_i & \text{if } a \neq \mathbf{e}_i \end{cases}; \quad d_i(a, b) = \begin{cases} \mathbf{e}_i & \text{if } a = b \\ a \vee_i b & \text{if } a \neq b \end{cases}; \quad a \cap_i b = \begin{cases} a & \text{if } a = b \\ \mathbf{e}_i & \text{if } a \neq b \end{cases}.$$

Proof. The proof is trivial for $t_i, \wedge_i, \vee_i, \setminus_i$. We now prove the relation for $d_i(a, b)$. We distinguish three cases.

- $(a = b)$: $d_i(a, a) = q(a, t_1(a, a \vee_i a, \mathbf{e}_i), \dots, t_n(a, a \vee_i a, \mathbf{e}_i)) \stackrel{(B2)}{=} q(a, \mathbf{e}_i, \dots, \mathbf{e}_i) \stackrel{(B1)}{=} \mathbf{e}_i$.
- $(a = \mathbf{e}_k \text{ and } a \neq b)$: $d_i(\mathbf{e}_k, b) = q(\mathbf{e}_k, t_1(b, \mathbf{e}_k \vee_i b, \mathbf{e}_i), \dots, t_n(b, \mathbf{e}_k \vee_i b, \mathbf{e}_i)) = t_k(b, \mathbf{e}_k \vee_i b, \mathbf{e}_i) \stackrel{(b \neq \mathbf{e}_k)}{=} \mathbf{e}_k \vee_i b = a \vee_i b$.

By definition of \cap_i it is trivial to prove $a \cap_i a = a$. If $a = \mathbf{e}_k \neq b$, then we have: $a \cap_i b = q(a, t_1(b, \mathbf{e}_i, a), t_2(b, \mathbf{e}_i, a), \dots, t_n(b, \mathbf{e}_i, a)) \stackrel{(a = \mathbf{e}_k)}{=} t_k(b, \mathbf{e}_i, a) \stackrel{(b \neq \mathbf{e}_k)}{=} \mathbf{e}_i$. \square

Lemma 4.2. *The following identities hold in every n BA:*

- (1) $t_i(\mathbf{e}_i, x, y) = y$ and $t_i(\mathbf{e}_j, x, y) = x$, for every $j \neq i$;
- (2) $q(x, y_1, \dots, y_n) = t_1(x, t_2(x, t_3(x, \dots, t_n(x, z, y_n) \dots, y_3), y_2), y_1)$
 $= t_1(x, t_2(x, t_3(x, \dots, t_{n-1}(x, y_n, y_{n-1}) \dots, y_3), y_2), y_1)$.
- (3) \wedge_i and \vee_i are idempotents;
- (4) $d_i(x, x) = \mathbf{e}_i$ and $x \cap_i x = x$;
- (5) $t_i(d_i(x, y), x, y) = x$ and $t_i(x \cap_i y, y, x) = x$;
- (6) $(A, \cap_i, \mathbf{e}_i)$ is a meet semilattice with bottom \mathbf{e}_i ;
- (7) $x \cap_i (x \wedge_i y \wedge_i x) = x \wedge_i y \wedge_i x$ and $x \wedge_i (x \cap_i y) = x \cap_i y = (x \cap_i y) \wedge_i x$;
- (8) $d_i(x, y) = (x \vee_i y) \setminus_i (x \cap_i y)$;
- (9) $x \cap_i y = (x \wedge_i y) \setminus_i d_i(x, y)$.

Proof. The identities are checked in the generator \mathbf{n} of the variety $n\text{BA}$.

(2) First we have: $t_1(\mathbf{e}_i, t_2(\mathbf{e}_i, t_3(\mathbf{e}_i, \dots t_n(\mathbf{e}_i, c, b_n) \dots, b_3), b_2), b_1) = t_2(\mathbf{e}_i, t_3(\mathbf{e}_i, \dots t_n(\mathbf{e}_i, c, b_n) \dots, b_3), b_2) = \dots = t_i(\mathbf{e}_i, \dots t_n(\mathbf{e}_i, c, b_n) \dots, b_i) = b_i = q(\mathbf{e}_i, b_1, \dots, b_n)$. If $i \neq n$, a similar computation gives $t_1(\mathbf{e}_i, t_2(\mathbf{e}_i, t_3(\mathbf{e}_i, \dots t_{n-1}(\mathbf{e}_i, b_n, b_{n-1}) \dots, b_3), b_2), b_1) = b_i$. If $i = n$, then $t_1(\mathbf{e}_n, t_2(\mathbf{e}_n, t_3(\mathbf{e}_n, \dots t_{n-1}(\mathbf{e}_n, b_n, b_{n-1}) \dots, b_3), b_2), b_1) = \dots = t_{n-1}(\mathbf{e}_n, b_n, b_{n-1}) = b_n$.

(3)-(4) Trivial by Lemma 4.1.

(5) If $a = b$, then the conclusion is trivial by (4). Let now $a \neq b$.

- $t_i(a \cap_i b, b, a) =_{L.4.1} t_i(\mathbf{e}_i, b, a) = a$.
- If $a = \mathbf{e}_i$ then $t_i(d_i(\mathbf{e}_i, b), \mathbf{e}_i, b) =_{L.4.1} t_i(b, \mathbf{e}_i, b) =_{(b \neq \mathbf{e}_i)} \mathbf{e}_i = a$.
- If $a \neq \mathbf{e}_i$, then $t_i(d_i(a, b), a, b) =_{L.4.1} t_i(a, a, b) =_{(a \neq \mathbf{e}_i)} a$.

$$(8) (a \vee_i b) \setminus_i (a \cap_i b) = t_i(a \cap_i b, \mathbf{e}_i, a \vee_i b) = \begin{pmatrix} t_i(a, \mathbf{e}_i, a) = \mathbf{e}_i & \text{if } a = b \\ t_i(\mathbf{e}_i, \mathbf{e}_i, a \vee_i b) = a \vee_i b & \text{if } a \neq b \end{pmatrix} = d_i(a, y).$$

(9) First we have: $(a \wedge_i a) \setminus_i d_i(a, a) = t_i(\mathbf{e}_i, \mathbf{e}_i, a) = a = a \cap_i a$.

If $a \neq b$, then $(a \wedge_i a) \setminus_i d_i(a, b) = t_i(d_i(a, b), \mathbf{e}_i, a \wedge_i b) = t_i(a \vee_i b, \mathbf{e}_i, a \wedge_i b) = \mathbf{e}_i = a \cap_i b$, because by Lemma 4.1, $a \vee_i b \neq \mathbf{e}_i$ if $a \neq b$.

(6) and (7) can be similarly checked in the generator \mathbf{n} of the variety $n\text{BA}$ by using Lemma 4.1. \square

Lemma 4.3. *Let \mathbf{A} be a $n\text{BA}$, and $a, b \in A$. Then we have:*

- (i) $t_i(a, -, -)$ is a 2-ary decomposition operator on \mathbf{A} .
- (ii) $d_i(a, b) = \mathbf{e}_i \Rightarrow a = b$.

Proof. (i) The binary operator $t_i(a, -, -)$ is a decomposition operator, because it is obtained by the n -ary decomposition operator $q(a, -, \dots, -)$ equating some of its coordinates (see [20] and Proposition 2.3).

(ii) Let $d_i(a, b) = \mathbf{e}_i$. Then $a =_{L.4.2(5)} t_i(d_i(a, b), a, b) = t_i(\mathbf{e}_i, a, b) = b$. \square

We now characterise the reducts $R_i(\mathbf{A})$, $S_i(\mathbf{A})$ and $S_i^\cap(\mathbf{A})$ of a $n\text{BA}$ \mathbf{A} (see Definition 4.2).

Proposition 4.4. *Let \mathbf{A} be a $n\text{BA}$. Then the following conditions hold:*

- (i) *The right Church i -reduct $R_i(\mathbf{A}) = (A, t_i, \mathbf{e}_i)$ of \mathbf{A} is a SRCA;*
- (ii) *The skew i -reduct $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathbf{e}_i)$ of \mathbf{A} is a right-handed SBA.*

Proof. By Lemma 4.2(3) and Lemma 4.3(i) every element of A is a factor element (w.r.t. t_i) that is \wedge_i -idempotent. Then every element of A is semicentral, so that $R_i(\mathbf{A})$ is a SRCA. By Theorem 2.6 the skew i -reduct $S_i(\mathbf{A})$ is a right-handed SBA. \square

Hereafter, we denote by $\preceq_{\mathcal{D}}^i$, $\preceq_{\mathcal{L}}^i$, $\preceq_{\mathcal{R}}^i$ and \leq^i the natural preorders and order of the SBA $S_i(\mathbf{A})$ (see Section 2.4). Since $S_i(\mathbf{A})$ is right-handed $\preceq_{\mathcal{D}}^i$ and $\preceq_{\mathcal{R}}^i$ coincide.

Proposition 4.5. *The elements $\mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_n$ are maximal elements of $S_i(\mathbf{A})$.*

Proof. We show the maximality of the elements $\mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_n$ with respect to the natural preorder $\preceq_{\mathcal{D}}^i$ of the SBA $S_i(\mathbf{A})$, defined by $a \preceq_{\mathcal{D}}^i b$ iff $a \wedge_i b \wedge_i a = a$. If $k \neq i$ and $a \in A$, then $a \wedge_i \mathbf{e}_k \wedge_i a = a \wedge_i a = a$, because $\mathbf{e}_k \wedge_i a = t_i(\mathbf{e}_k, a, \mathbf{e}_i) = a$ by Lemma 4.2(1). \square

By Proposition 4.5 the skew i -reduct $S_i(\mathbf{A})$ has a maximal class M with $\mathbf{e}_1, \dots, \mathbf{e}_{i-1}, \mathbf{e}_{i+1}, \dots, \mathbf{e}_n \in M$. Then the algebra $S_i(\mathbf{A})/\mathcal{D}_i$ is a Boolean algebra.

Proposition 4.6. *The \cap -skew i -reduct $S_i^\cap(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathbf{e}_i, \cap_i)$ of \mathbf{A} is a \mathbf{e}_i -regular right-handed SBIA.*

Proof. By Proposition 4.4 the skew i -reduct $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathbf{e}_i)$ is a right-handed SBA. By Lemma 4.2(6,7) and by Definition 2.6 the \cap -skew i -reduct $S_i^\cap(\mathbf{A})$ is a right-handed SBIA. By Lemma 4.2(8) we have that $d_i(x, y) = (x \vee_i y) \setminus_i (x \cap_i y)$ is a term operation in the type of SBIA's. Then the \mathbf{e}_i -regularity w.r.t. d_i follows from Lemma 4.2(4) and Lemma 4.3(ii) (see Section 2.1 for the definition of regularity). \square

Remark 1. Skew Boolean algebras, whose underlying natural partial ordering is a meet semilattice, cannot be equationally axiomatised in the type of SBAs. Therefore, skew Boolean \cap -algebras of Definition 2.6 are equationally axiomatised in the type of SBAs enriched with a binary operator \cap of intersection. Rather interestingly, if \mathbf{A} is a n BA the term operation \cap_i is definable in terms of $\wedge_1, \vee_1, \setminus_1, \mathbf{e}_1, \dots, \wedge_n, \vee_n, \setminus_n, \mathbf{e}_n$. This follows from Definition 4.1, Lemma 4.2(2) and Theorem 2.6. Then the following question is natural. Let $\mathbf{A} = (A, \wedge, \vee, \setminus, 0)$ be a SBA, whose underlying natural partial ordering is a meet semilattice $(A, \cap, 0)$ with bottom. Does it exist a bunch of SBAs $\mathbf{A}_1 = (A, \wedge_1, \vee_1, \setminus_1, 0_1), \dots, \mathbf{A}_k = (A, \wedge_k, \vee_k, \setminus_k, 0_k)$ such that the meet operation \cap is definable in terms of the skew Boolean operations of $\mathbf{A}_1, \dots, \mathbf{A}_k$? A further analysis of this question will be given in Section 5.

4.2. A bunch of isomorphisms. It turns out that all the \cap -skew reducts of a n BA \mathbf{A} are isomorphic. In order to prove this, we study the action of the symmetric group S_n on \mathbf{A} . The first part of this section is rather technical.

Let \mathbf{A} be a n BA. For every permutation σ of the symmetric group S_n and $a, b_1, \dots, b_n \in A$, we define a sequence u_s ($1 \leq s \leq n+1$) parametrised by another permutation τ :

$$u_{n+1} = b_{\tau n}; \quad u_s = t_{\tau s}(a, u_{s+1}, b_{\sigma \tau s}) \quad (1 \leq s \leq n).$$

In the following lemma we prove that u_1 is independent of the permutation τ .

Notice that $u_n = q(a, b_{\tau n}/\overline{\tau n}, b_{\sigma \tau n}/\tau n)$ and $u_s = q(a, u_{s+1}/\overline{\tau s}, b_{\sigma \tau s}/\tau s)$.

Lemma 4.7. *For every $1 \leq s \leq n$ we have:*

$$u_s = q(a, b_{\tau n}/\{\tau 1, \tau 2, \dots, \tau(s-1)\}, b_{\sigma \tau s}/\tau s, b_{\sigma \tau(s+1)}/\tau(s+1), \dots, b_{\sigma \tau n}/\tau n).$$

Then $u_1 = q(a, b_{\sigma \tau 1}/\tau 1, b_{\sigma \tau 2}/\tau 2, \dots, b_{\sigma \tau n}/\tau n) = q(a, b_{\sigma 1}, b_{\sigma 2}, \dots, b_{\sigma n})$.

Proof. Assume that

$$u_{s+1} = q(a, b_{\tau n}/\{\tau 1, \tau 2, \dots, \tau s\}, b_{\sigma \tau(s+1)}/\tau(s+1), \dots, b_{\sigma \tau n}/\tau n).$$

Then we have:

$$\begin{aligned} u_s &= t_{\tau s}(a, u_{s+1}, b_{\sigma \tau s}) \\ &= q(a, u_{s+1}/\overline{\tau s}, b_{\sigma \tau s}/\tau s) \\ &\stackrel{(B2)}{=} q(a, b_{\tau n}/\{\tau 1, \tau 2, \dots, \tau(s-1)\}, b_{\sigma \tau s}/\tau s, b_{\sigma \tau(s+1)}/\tau(s+1), \dots, b_{\sigma \tau n}/\tau n). \end{aligned}$$

\square

We define

$$a^\sigma = q(a, \mathbf{e}_{\sigma 1}, \mathbf{e}_{\sigma 2}, \dots, \mathbf{e}_{\sigma n}).$$

The transposition (ij) exchanges i and j : $(ij)(i) = j$ and $(ij)(j) = i$.

Lemma 4.8. *The following conditions hold in every n BA, for all permutations σ, τ and indices $i \neq j$:*

(1) *The 2-ary decomposition operators $t_i(x, -, -)$ and $t_j(x, -, -)$ commute:*

$$t_i(x, t_j(x, y, z), t_j(x, u, w)) = t_j(x, t_i(x, y, u), t_i(x, z, w)) = q(x, y/\overline{\{i, j\}}, z/j, u/i);$$

(2) $t_i(x, t_j(x, y, z), u) = t_j(x, t_i(x, y, u), z) = q(x, y/\overline{\{i, j\}}, z/j, u/i);$

(3) $q(x, y_{\sigma 1}, \dots, y_{\sigma n}) = t_{\tau 1}(x, t_{\tau 2}(x, t_{\tau 3}(x, \dots, t_{\tau n}(x, y_{\tau n}, y_{\sigma \tau n}) \dots, y_{\sigma \tau 3})), y_{\sigma \tau 2}), y_{\sigma \tau 1});$

- (4) $x^\sigma = t_{\tau_1}(x, t_{\tau_2}(x, t_{\tau_3}(x, \dots, t_{\tau_n}(x, \mathbf{e}_{\tau n}, \mathbf{e}_{\sigma \tau n}) \dots, \mathbf{e}_{\sigma \tau 3})), \mathbf{e}_{\sigma \tau 2}), \mathbf{e}_{\sigma \tau 1});$
(5) $q(x^\sigma, y_1, \dots, y_n) = q(x, y_{\sigma 1}, \dots, y_{\sigma n});$
(6) $x^{(ij)} = t_i(x, t_j(x, x, \mathbf{e}_i), \mathbf{e}_j) = t_j(x, t_i(x, x, \mathbf{e}_j), \mathbf{e}_i);$
(7) $x^{\tau \circ \sigma} = (x^\sigma)^\tau;$
(8) $q(x, y_1, \dots, y_n)^\sigma = q(x, (y_1)^\sigma, \dots, (y_n)^\sigma).$

Proof. Let \mathbf{A} be a n BA and $a, b, c, d, e, b_1, \dots, b_n$ be elements of A .

(1)

$$\begin{aligned} t_i(a, t_j(a, b, c), t_j(a, d, e)) &= q(a, t_j(a, b, c)/\bar{i}, t_j(a, d, e)/i) \\ &= q(a, q(a, b/\bar{j}, c/j)/\bar{i}, q(a, d/\bar{j}, e/j)/i) \\ &\stackrel{(B2)}{=} q(a, b/\{i, j\}, c/j, d/i). \end{aligned}$$

By symmetry we also get $t_j(a, t_i(a, b, d), t_i(a, c, e)) = q(a, b/\{i, j\}, c/j, d/i)$.

(2) $t_i(a, t_j(a, b, c), d) = t_i(a, t_j(a, b, c), t_j(a, d, d)) \stackrel{(1)}{=} q(a, b/\{i, j\}, c/j, d/i)$, and similarly $t_j(a, t_i(a, b, d), c) = t_j(a, t_i(a, b, d), t_i(a, c, c)) = q(a, b/\{i, j\}, c/j, d/i)$.

(3) By Lemma 4.7 $u_1 = q(a, b_{\sigma 1}, \dots, b_{\sigma n})$. Then the conclusion follows from the unfolding of the definition of u_1 :

$$u_1 = t_{\tau_1}(a, u_2, b_{\sigma \tau 1}) = t_{\tau_1}(a, t_{\tau_2}(a, u_3, b_{\sigma \tau 2}), b_{\sigma \tau 1}) = \dots$$

(4) follows from (3) by putting $y_{\tau i} = \mathbf{e}_{\tau i}$.

(5) $q(a^\sigma, b_1, \dots, b_n) = q(q(a, \mathbf{e}_{\sigma 1}, \dots, \mathbf{e}_{\sigma n}), b_1, \dots, b_n) \stackrel{(B3)}{=} q(a, b_{\sigma 1}, \dots, b_{\sigma n})$.

(6)

$$\begin{aligned} t_i(a, t_j(a, a, \mathbf{e}_i), \mathbf{e}_j) &= t_j(a, t_i(a, a, \mathbf{e}_j), \mathbf{e}_i) && \text{by (1)} \\ &= q(a, a/\{i, j\}, \mathbf{e}_i/j, \mathbf{e}_j/i) && \text{by (1)} \\ &= q(a, q(a, \mathbf{e}_1, \dots, \mathbf{e}_n)/\{i, j\}, \mathbf{e}_i/j, \mathbf{e}_j/i) && \text{by (B4)} \\ &= a^{(ij)} && \text{by (B2)} \end{aligned}$$

(8)

$$\begin{aligned} q(a, b_1, \dots, b_n)^\sigma &= q(q(a, b_1, \dots, b_n), \mathbf{e}_{\sigma 1}, \mathbf{e}_{\sigma 2}, \dots, \mathbf{e}_{\sigma n}) \\ &\stackrel{(B3)}{=} q(a, q(b_1, \mathbf{e}_{\sigma 1}, \mathbf{e}_{\sigma 2}, \dots, \mathbf{e}_{\sigma n}), \dots, q(b_n, \mathbf{e}_{\sigma 1}, \mathbf{e}_{\sigma 2}, \dots, \mathbf{e}_{\sigma n})) \\ &= q(a, (b_1)^\sigma, \dots, (b_n)^\sigma). \end{aligned}$$

□

Theorem 4.9. *For every transposition $(rk) \in S_n$, the map $x \mapsto x^{(rk)}$ defines an isomorphism from $S_r^\cap(\mathbf{A})$ onto $S_k^\cap(\mathbf{A})$.*

Proof. Let $\sigma = (rk)$ in this proof. The map $x \mapsto x^\sigma$ is bijective, because $(a^\sigma)^\sigma \stackrel{L.4.8(7)}{=} a^{\sigma \circ \sigma} = a^{\text{id}} = q(a, \mathbf{e}_1, \dots, \mathbf{e}_n) \stackrel{(B4)}{=} a$, for every $a \in A$. We now prove that $x \mapsto x^\sigma$ is a homomorphism of SBIA's. We recall from Definition 4.1 that the operations $\wedge_r, \vee_r, \setminus_r$ are defined in terms of t_r . Then to get the conclusion it is sufficient to prove the following equalities, for all $a, b, c \in A$: $t_r(a, b, c)^\sigma = t_k(a^\sigma, b^\sigma, c^\sigma)$, $(a \cap_r b)^\sigma = a^\sigma \cap_k b^\sigma$ and $\mathbf{e}_r^\sigma = \mathbf{e}_k$:

- $t_r(a, b, c)^\sigma \stackrel{L.4.8(8)}{=} t_r(a, b^\sigma, c^\sigma) = t_r((a^\sigma)^\sigma, b^\sigma, c^\sigma) = q((a^\sigma)^\sigma, b^\sigma/\bar{r}, c^\sigma/r) \stackrel{L.4.8(5)}{=} q(a^\sigma, b^\sigma/\bar{k}, c^\sigma/k) = t_k(a^\sigma, b^\sigma, c^\sigma)$.
- $(\mathbf{e}_r)^\sigma = \mathbf{e}_{\sigma r} = \mathbf{e}_k$.

$$\begin{aligned}
(a \cap_r b)^\sigma &= q(a, t_1(b, \mathbf{e}_r, a), \dots, t_n(b, \mathbf{e}_r, a))^\sigma \\
&= q(a, t_1(b, \mathbf{e}_r, a)^\sigma, \dots, t_n(b, \mathbf{e}_r, a)^\sigma) \\
&= q(a, t_1(b, \mathbf{e}_r^\sigma, a^\sigma), \dots, t_n(b, \mathbf{e}_r^\sigma, a^\sigma)) \\
&= q(a, t_1(b, \mathbf{e}_k, a^\sigma), \dots, t_n(b, \mathbf{e}_k, a^\sigma)) \\
&= q((a^\sigma)^\sigma, t_1((b^\sigma)^\sigma, \mathbf{e}_k, a^\sigma), \dots, t_n((b^\sigma)^\sigma, \mathbf{e}_k, a^\sigma)) \\
&= q((a^\sigma)^\sigma, \dots, t_r((b^\sigma)^\sigma, \mathbf{e}_k, a^\sigma), \dots, t_k((b^\sigma)^\sigma, \mathbf{e}_k, a^\sigma), \dots) \\
&= q(a^\sigma, \dots, t_k((b^\sigma)^\sigma, \mathbf{e}_k, a^\sigma), \dots, t_r((b^\sigma)^\sigma, \mathbf{e}_k, a^\sigma), \dots) \\
&= q(a^\sigma, \dots, t_r(b^\sigma, \mathbf{e}_k, a^\sigma), \dots, t_k(b^\sigma, \mathbf{e}_k, a^\sigma), \dots) \\
&= a^\sigma \cap_k b^\sigma.
\end{aligned}$$

□

4.3. A general representation theorem for right-handed SBAs. In this section we show that, for every $n \geq 3$, there is a representation of an arbitrary right-handed SBA within the skew i -reduct of a suitable n BA of n -partitions (described in Example 7). The theorem also provides a new proof that every SBA can be embedded into a SBA with a maximal class (see Proposition 4.5).

Theorem 4.10. *Let $n \geq 3$. Then every right-handed SBA can be embedded into the skew i -reduct $S_i(\mathbf{A})$ of a suitable n BA \mathbf{A} of n -partitions.*

Proof. (a) By [15, Corollary 1.14] every right-handed SBA can be embedded into an algebra of partial functions with codomain the set $\{1, 2\}$ (see Example 1), where $0 = \emptyset$ is the empty function, $f \wedge g = g|_{G \cap F}$, $f \vee g = f \cup g|_{G \cup F}$ and $g \setminus f = g|_{G \setminus F}$ (with F, G and H the domains of the functions f, g, h , respectively).

(b) By Corollary 3.5 every n BA is isomorphic to a n BA of n -partitions of a suitable set I (see Examples 3 and 7). If $P = (P_1, \dots, P_n)$ and $Q = (Q_1, \dots, Q_n)$ are n -partitions of I , then

$$(2) \quad P \wedge_i Q = t_i(P, Q, \mathbf{e}_i) = q(P, Q/\bar{i}, \mathbf{e}_i/i) = (\bar{P}_i \cap Q_1, \dots, P_i \cup (\bar{P}_i \cap Q_i), \dots, \bar{P}_i \cap Q_n).$$

The other operations can be similarly defined.

(c) We define an injective function $*$ between the set of partial functions from a set I into $\{1, 2\}$ and the set of n -partitions of I . If $f : I \rightarrow \{1, 2\}$ is a partial function, then $f^* = (P_1, \dots, P_n)$ is the following n -partition of I : $P_1 = f^{-1}(1)$, $P_2 = f^{-1}(2)$, $P_i = I \setminus \text{dom}(f)$ and $P_k = \emptyset$ for any $k \neq 1, 2, i$.

(d) The map $*$ preserves the meet. Let $f : F \rightarrow \{1, 2\}$ and $g : G \rightarrow \{1, 2\}$ ($F, G \subseteq I$) be functions. Then we derive $(f \wedge g)^* = f^* \wedge_i g^*$ as follows:

$$\begin{aligned}
f^* &= (f^{-1}(1), f^{-1}(2), \emptyset, \dots, \emptyset, \bar{F}, \emptyset, \dots, \emptyset); \quad g^* = (g^{-1}(1), g^{-1}(2), \emptyset, \dots, \emptyset, \bar{G}, \emptyset, \dots, \emptyset) \\
(f \wedge g)^* &= (g|_{G \cap F})^* \\
&= (F \cap g^{-1}(1), F \cap g^{-1}(2), \emptyset, \dots, \emptyset, \bar{G} \cup \bar{F}, \emptyset, \dots, \emptyset) \\
&= (F \cap g^{-1}(1), F \cap g^{-1}(2), \emptyset, \dots, \bar{F} \cup (F \cap \bar{G}), \emptyset, \dots, \emptyset) \\
&= f^* \wedge_i g^*.
\end{aligned}$$

Similarly for the other operations. □

5. SKEW STAR ALGEBRAS

The skew reducts of a n BA are so deeply related that they allow to recover the full structure of the n BA. It is worthy to introduce a new variety of algebras, called *skew star* algebras, equationally axiomatising n isomorphic SBAs and their relationships. In the main result of this section we prove that the variety of skew star algebras is term equivalent to the variety of n BA.

By Lemma 4.2(2) we have that the identity

$$q(x, y_1, \dots, y_n) = t_1(x, t_2(x, t_3(x, \dots t_{n-1}(x, y_n, y_{n-1}) \dots, y_3), y_2), y_1)$$

holds in every n BA. It follows that

$$t_n(x, y, z) = t_1(x, t_2(x, t_3(x, \dots t_{n-1}(x, z, y) \dots, y), y), y),$$

so that t_n is term definable by the remaining t_i ($1 \leq i \leq n-1$). This is one of the reasons for introducing $n-1$ (and not n) ternary operators in the following definition. Another reason is technical simplification.

Definition 5.1. An algebra $\mathbf{B} = (B, t_1, \dots, t_{n-1}, 0_1, \dots, 0_n)$, where t_i is ternary and 0_j is a nullary operator, is called a *skew star algebra* if the following conditions hold, for every $1 \leq i, k \leq n-1$ and $1 \leq j \leq n$:

- (N0) $(B, t_i, 0_i)$ is a SRCA.
- (N1) $t_i(0_j, y, z) = y$ ($i \neq j$).
- (N2) $t_1(x, t_2(x, t_3(x, (\dots t_{n-1}(x, 0_n, 0_{n-1}) \dots), 0_3), 0_2), 0_1) = x$.
- (N3) $t_i(x, t_k(x, y, z), t_k(x, u, w)) = t_k(x, t_i(x, y, u), t_i(x, z, w))$ ($i \neq k$).
- (N4) $t_i(x, t_k(x, y, z), u) = t_k(x, t_i(x, y, u), z)$ ($i \neq k$).
- (N5) $t_i(x, -, -)$ is a homomorphism of the algebra $(B, t_k, 0_k) \times (B, t_k, 0_k)$ into $(B, t_k, 0_k)$:

$$t_i(x, t_k(y_1, y_2, y_3), t_k(z_1, z_2, z_3)) = t_k(t_i(x, y_1, z_1), t_i(x, y_2, z_2), t_i(x, y_3, z_3)).$$

Skew star algebras constitute a variety of algebras.

The identities characterising skew star algebras deserve some explanation. Let $\mathbf{B}_i = (B, t_i, 0_i)$ ($i = 1, \dots, n-1$) be a family of SRCAs having the same universe B and such that $0_1, \dots, 0_{n-1}$ are distinct elements of B . Let 0_n be another element of B distinct from $0_1, \dots, 0_{n-1}$. Let $\mathbf{B} = (B, t_1, \dots, t_{n-1}, 0_1, \dots, 0_n)$ be the algebra collecting the basic operations of the algebras \mathbf{B}_i and the constant 0_n . Roughly speaking, the structure of an n BA on \mathbf{B} with respect to the term operation q_t , defined by

$$(3) \quad q_t(x, y_1, \dots, y_n) := t_1(x, t_2(x, t_3(x, \dots t_{n-1}(x, y_n, y_{n-1}) \dots, y_3), y_2), y_1),$$

can be recovered from the cluster of SRCAs \mathbf{B}_i if (N1)-(N5) hold:

- (i) (N1) implies that \mathbf{B} is a n CA with respect to the operation q_t .
- (ii) Since \mathbf{B}_i is a SRCA, then, for every $b \in B$, the function $t_i(b, -, -)$ satisfies conditions (D1) and (D2) of Definition 2.2. Then, axiom (N5) implies that, for every $b \in B$, the binary functions $t_1(b, -, -), \dots, t_{n-1}(b, -, -)$ are 2-ary decomposition operators of the n CA \mathbf{B} .
- (iii) (N3) means that the decomposition operators $t_1(b, -, -), \dots, t_{n-1}(b, -, -)$ are pairwise commuting. Hence, by Proposition 2.2 and by Proposition 2.3 the n -ary operator $q_t(b, -, \dots, -)$ (see (3) above) is a n -ary decomposition operator of the n CA \mathbf{B} .
- (iv) (N2) implies that the factor element b satisfies the identity $q_t(b, 0_1, \dots, 0_n) = b$. Then b is a n -central element of the n CA \mathbf{B} , for every $b \in B$. We conclude that axioms (N1), (N2), (N3) and (N5) collectively imply that \mathbf{B} is a n BA (w.r.t. q_t).
- (v) Axiom (N4) is used to recover the ternary operations t_i ($1 \leq i \leq n-1$) from q_t , i.e., $t_i(b, y, z) = q_t(b, y/\bar{i}, z/i)$.

We now are going to prove that the variety of skew star algebras and of n BAs are term equivalent. Consider the following correspondence between the algebraic similarity types of n BAs and of skew star algebras.

- Beginning on the n BA side: $t_i(x, y, z) := q(x, y/\bar{i}, z/i)$ ($1 \leq i \leq n-1$) and $0_j := e_j$ ($1 \leq j \leq n$).
- Beginning on the skew star algebra side:

$$q_t(x, y_1, \dots, y_n) := t_1(x, t_2(x, t_3(x, \dots, t_{n-1}(x, y_n, y_{n-1}) \dots, y_3), y_2), y_1); \quad e_j := 0_j.$$

If \mathbf{B} is a skew star algebra, then $\mathbf{B}^\bullet = (B; q_t, e_1, \dots, e_n)$ denotes the corresponding algebra in the similarity type of n BAs. Similarly, if \mathbf{A} is a n BA, then $\mathbf{A}^* = (A; t_1, \dots, t_{n-1}, 0_1, \dots, 0_n)$ denotes the corresponding algebra in the similarity type of skew star algebras.

It is not difficult to prove the following theorem.

Theorem 5.1. *The above correspondences define a term equivalence between the varieties of n BAs and of skew star algebras. More precisely,*

- (i) *If \mathbf{A} is a n BA, then \mathbf{A}^* is a skew star algebra;*
- (ii) *If \mathbf{B} is a skew star algebra, then \mathbf{B}^\bullet is a n BA;*
- (iii) $(\mathbf{A}^*)^\bullet = \mathbf{A}$;
- (iv) $(\mathbf{B}^\bullet)^* = \mathbf{B}$.

Proof. (i) (N0) derives from Proposition 4.4, while (N1) from Lemma 4.2(1). (N2) follows from $t_1(x, t_2(x, t_3(x, \dots, t_n(x, 0_{n-1}, 0_n) \dots, 0_3), 0_2), 0_1) =_{(B2)} q(x, 0_1, \dots, 0_n) = x$. (N3) is a consequence of Lemma 4.8(1). For (N4) we apply Lemma 4.8(2). (N5) follows from (B3).

(ii) (B0) derives from (N0) and (N1). By (N0) and (N5), $t_i(x, -, -)$ ($1 \leq i \leq n$) is a decomposition operator on \mathbf{B} . Then, for every $b \in B$, $q_t(b, -, \dots, -)$ is a n -ary decomposition operator on \mathbf{B}^\bullet (i.e., (B1)-(B3) hold), because commuting decomposition operators are closed under composition (see [20], Proposition 2.2 and Proposition 2.3). (B4) is a consequence of (N2).

(iii) Let \mathbf{A} be a n BA. Since $t_i(x, y, z) = q(x, y/\bar{i}, z/i)$, then by (B2) we have $q_t(x, y_1, \dots, y_n) = q(x, t_2(x, \dots)/\bar{1}, y_1/1) = q(x, y_1, t_2(x, \dots), \dots, t_2(x, \dots)) = q(x, y_1, y_2, t_3(x, \dots), \dots, t_3(x, \dots)) = \dots = q(x, y_1, \dots, y_n)$.

(iv) Let $\mathbf{B} = (B, t_1, \dots, t_{n-1}, 0_1, \dots, 0_n)$ be a skew star algebra. The conclusion $(\mathbf{B}^\bullet)^* = \mathbf{B}$ follows because by (N4) we obtain that $t_i(x, y, z) = q_t(x, y/\bar{i}, z/i)$ for every $1 \leq i \leq n-1$. \square

6. MULTIDEALS

The notion of ideal plays an important role in order theory and universal algebra. Ideals, filters and congruences are interdefinable in Boolean algebras. For every Boolean ideal I , we have that $a \in I$ if and only if $\neg a \in \neg I$ if and only if $a\theta_I 0$ if and only if $\neg a\theta_I 1$. In the case of n BAs, the couple $(I, \neg I)$ is replaced by a n -tuple (I_1, \dots, I_n) satisfying some compatibility conditions that extend in a conservative way those of the Boolean case.

Definition 6.1. Let \mathbf{A} be a n BA. A *multideal* is a n -partition (I_1, \dots, I_n) of a subset I of A such that

- (m1) $e_k \in I_k$;
- (m2) $a \in I_r, b \in I_k$ and $c_1, \dots, c_n \in A$ imply $q(a, c_1, \dots, c_{r-1}, b, c_{r+1}, \dots, c_n) \in I_k$;
- (m3) $a \in A$ and $c_1, \dots, c_n \in I_k$ imply $q(a, c_1, \dots, c_n) \in I_k$.

The set I is called the *carrier* of the multideal. A *ultramultideal* of \mathbf{A} is a multideal whose carrier is A .

The following Lemma, whose proof is straightforward, shows the appropriateness of the notion of multideal. In Section 7 we show that there exists a bijective correspondence between multideals and congruences.

Lemma 6.1. *If θ is a proper congruence on a n BA \mathbf{A} , then $I(\theta) = (\mathbf{e}_1/\theta, \dots, \mathbf{e}_n/\theta)$ is a multideal of \mathbf{A} .*

Multideals extend to the n -ary case the fundamental notions of Boolean ideal and filter, as shown in the following Proposition.

Recall from [21] that a 2BA $\mathbf{A} = (A, q, \mathbf{e}_1, \mathbf{e}_2)$ is term equivalent to the Boolean algebra $\mathbf{A}^* = (A, \wedge, \vee, \neg, 0, 1)$, where $0 = \mathbf{e}_2$, $1 = \mathbf{e}_1$, $x \wedge y = q(x, y, 0)$, $x \vee y = q(x, 1, y)$, $\neg x = q(x, 0, 1)$.

Proposition 6.2. *Let \mathbf{A} be a 2BA, and $I_1, I_2 \subseteq A$. Then (I_1, I_2) is a multideal of \mathbf{A} if and only if I_2 is an ideal of \mathbf{A}^* , and $I_1 = \neg I_2$ is the filter associated to I_2 in \mathbf{A}^* .*

Proof. If (I_1, I_2) is a multideal, then $0 = \mathbf{e}_2 \in I_2$ and $1 = \mathbf{e}_1 \in I_1$. Moreover, if $a, b \in I_2$ then $a \vee b = q(a, 1, b) \in I_2$ by (m2), and if $a \in I_2$ and $b \in A$, then $b \wedge a = q(b, a, 0) \in I_2$ by (m3). It follows that I_2 is a Boolean ideal. By (m2) and the definition of \neg we have $I_1 \supseteq \{\neg a \mid a \in I_2\}$ and $I_2 \supseteq \{\neg a \mid a \in I_1\}$. Then from $\neg \neg a = a$ it follows that $I_1 = \{\neg a \mid a \in I_2\}$.

Conversely, if I_2 is a Boolean ideal of \mathbf{A}^* and $I_1 = \neg I_2$, then the condition (m1) is clearly satisfied. Concerning (m2), it is worth noticing that $q(a, c, b) = (a \wedge c) \vee (\neg a \wedge b)$. Then if $a \in I_2, b \in I_1, c \in A$ (for instance, the other 3 cases being similar to this one), we have that $\neg a \in I_1$, so that $\neg a \wedge b \in I_1$ and we conclude that $(a \wedge c) \vee (\neg a \wedge b) = q(a, c, b) \in I_1$. Concerning (m3), if $a, b \in I_2$ and $c \in A$ then $c \wedge a, \neg c \wedge b \in I_2$, hence $(c \wedge a) \vee (\neg c \wedge b) = q(c, a, b) \in I_2$. If $a, b \in I_1$ and $c \in A$, then $(c \wedge a) \vee (\neg c \wedge b) \geq (c \wedge (a \wedge b)) \vee (\neg c \wedge (a \wedge b)) = (c \vee \neg c) \wedge (a \wedge b) = a \wedge b \in I_1$, so that $(c \wedge a) \vee (\neg c \wedge b) = q(c, a, b) \in I_1$. \square

In the n -ary case, multideals of \mathbf{A} may be characterised as n -tuples of ideals in the skew i -reducts $S_i(\mathbf{A})$ of \mathbf{A} , satisfying the conditions expressed in the following Proposition.

Proposition 6.3. *Let \mathbf{A} be a n BA and (I_1, \dots, I_n) be a n -partition of a subset I of A . Then (I_1, \dots, I_n) is a multideal if and only if the following conditions are satisfied:*

- (I1) $\mathbf{e}_r \in I_r$;
- (I2) $a \in I_r, b \in I_k$ and $c \in A$ imply $t_r(a, c, b) \in I_k$.
- (I3) $a, b \in I_r$ and $c \in A$ imply $t_k(c, a, b) \in I_r$, for all k .

Proof. Showing that a multideal satisfies I1, I2 and I3 is straightforward. A n -partition satisfying I1, I2 and I3, trivially verifies (m1). Concerning (m2), let us suppose that $a \in I_r, b \in I_k$ and $c_1, \dots, c_n \in A$. In order to show that $q(a, c_1, \dots, c_{r-1}, b, c_{r+1}, \dots, c_n) \in I_k$, we apply Lemma 4.8(4): $q(x, c_{\sigma 1}, \dots, c_{\sigma n}) = t_{\tau 1}(x, t_{\tau 2}(x, t_{\tau 3}(x, \dots, t_{\tau n}(x, c_{\tau n}, c_{\sigma \tau n}) \dots, c_{\sigma \tau 3})), c_{\sigma \tau 2}), c_{\sigma \tau 1})$ in the case $\sigma = Id, \tau = (1r)$, and we get

$$q(a, c_1, \dots, c_{r-1}, b, c_{r+1}, \dots, c_n) = t_r(a, t_2(a, \dots, c_2), b) \in I_k, \text{ by I2.}$$

Concerning (m3), let $a_1, \dots, a_n \in I_k$ and $b \in A$. By Lemma 4.2(2) we have

$$q(b, a_1, \dots, a_n) = t_1(b, t_2(b, t_3(b, \dots, t_{n-1}(b, a_n, a_{n-1}) \dots, a_3), a_2), a_1).$$

By applying I3 n times, we conclude that $q(b, a_1, \dots, a_n) \in I_k$, since $t_{n-1}(b, a_n, a_{n-1}) \in I_k$, hence $t_{n-2}(b, t_{n-1}(b, a_n, a_{n-1}), a_{n-2}) \in I_k$, and so on. \square

By using the characterisation of Proposition 6.3 it is easy to see that the components of a multideals are ideals of the SBA corresponding to their index.

Recall from Section 2.4 the notion of an ideal of a SBA.

Corollary 6.4. *If (I_1, \dots, I_n) is a multideal of a n BA \mathbf{A} and $1 \leq i \leq n$, then I_i is an ideal of the skew i -reduct $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathbf{e}_i)$.*

Proof. Since $S_i(\mathbf{A})$ is right-handed, a non empty set $K \subseteq A$ is an ideal of $S_i(\mathbf{A})$ if and only if, for all $a, b \in K$ and $c \in A$, $a \vee_i b \in K$ and $a \wedge_i c \in K$ (see Section 2.4). Given $a, b \in I_i$ and $c \in A$, we have $a \vee_i b = t_i(a, a, b) \in I_i$ and $a \wedge_i c = t_i(a, c, \mathbf{e}_i) \in I_i$, by using in both cases condition (I2) of Proposition 6.3 (notice that $\mathbf{e}_i \in I_i$, by (I1)). \square

Lemma 6.5. *The carrier I of a multideal (I_1, \dots, I_n) of a nBA \mathbf{A} is a subalgebra of \mathbf{A} .*

Proof. The constants $\mathbf{e}_1, \dots, \mathbf{e}_n$ belong to I by (m1). If $a \in I_r$ and $b \in I_k$, then $q(a, c_1, \dots, c_{r-1}, b, c_{r+1}, \dots, c_n) \in I_k$, for all $c_1, \dots, c_n \in A$, by (m2). Hence I is a subalgebra of \mathbf{A} . \square

Any component I_i of a multideal (I_1, \dots, I_n) determines the multideal completely, as shown in the following lemma.

Lemma 6.6. *If (I_1, \dots, I_n) is a multideal of a nBA \mathbf{A} , then $I_k = I_r^{(rk)}$ for all r, k .*

Proof. Let $a \in I_r$. Then $a^{(rk)} = t_r(a, t_k(a, a, \mathbf{e}_r), \mathbf{e}_k) \in I_k$ by Lemma 4.8(5) and Proposition 6.3(I2). Then we have

$$I_r^{(rk)} \subseteq I_k; \quad I_k^{(rk)} \subseteq I_r.$$

The conclusion follows because $(a^{(rk)})^{(rk)} = a$, by Lemma 4.8(6) and (B4). \square

Multideals are closed under arbitrary nonempty componentwise intersection. The minimum multideal is the sequence $(\{\mathbf{e}_k\})_{k \in \hat{n}}$. Given a nBA \mathbf{A} , and $A_1, \dots, A_n \subseteq A$, let us consider the set \mathcal{A} of multideals containing (A_1, \dots, A_n) . The *ideal closure* of (A_1, \dots, A_n) is the componentwise intersection of the elements of \mathcal{A} , if $\mathcal{A} \neq \emptyset$. Otherwise, the ideal closure of (A_1, \dots, A_n) is the constant n -tuple $I^\top = (A, \dots, A)$, that we consider as a degenerate multideal, by a small abuse of terminology.

As a matter of fact, I^\top is the only degenerate multideal.

Lemma 6.7. *Let \mathbf{A} be a nBA and $I = (I_1, \dots, I_n)$ be a tuple of subsets of A satisfying the closure properties of Definition 6.1. The following are equivalent:*

- (i) *there exist $a \in A$ and $r \neq k$ such that $a \in I_r \cap I_k$.*
- (ii) *there exist $r \neq k$ such that $\mathbf{e}_k \in I_r$.*
- (iii) *$I = I^\top$.*

Proof. (i) \Rightarrow (ii): since $a \in I_k$, by Lemma 6.6 we have that $a^{(rk)} \in I_r$. By Definition 6.1(m2), we conclude that $q(a, \mathbf{e}_k/\bar{r}, a^{(rk)}/r) \stackrel{(B2)}{=} q(a, \mathbf{e}_k, \dots, \mathbf{e}_k) = \mathbf{e}_k \in I_r$.
(ii) \Rightarrow (iii): given $b \in A$, we have $b = q(\mathbf{e}_k, b/\bar{r}, \mathbf{e}_r/r) \in I_r$ by Definition 6.1(m2). Hence $I_r = A$ and the result follows from Lemma 6.6 since $A^{(rk)} = A$ for all $1 \leq k \leq n$.
(iii) \Rightarrow (i): trivial. \square

7. THE RELATIONSHIP BETWEEN MULTIDEALS AND CONGRUENCES

For any congruence θ on a nBA, the equivalence classes \mathbf{e}_i/θ form a multideal (see Lemma 6.1), exactly as in the Boolean case $0/\theta$ is an ideal and $1/\theta$ the corresponding filter. Conversely, in the Boolean case, any ideal I (resp. filter F) defines the congruence $x\theta_I y \Leftrightarrow x \oplus y \in I$ (resp. $x\theta_F y \Leftrightarrow x \leftrightarrow y \in F$). Rephrasing this latter correspondence in the n -ary case is a bit more complicated.

7.1. The Boolean algebra of coordinates. Let \mathbf{A} be a n BA, $a \in A$ and $i \in \hat{n}$. We consider the factor congruence $\theta_a^i = \theta(a, \mathbf{e}_i) = \{(x, y) : t_i(a, x, y) = x\}$ generated by a . By Lemma 6.1 the tuple

$$(\mathbf{e}_1/\theta_a^i, \dots, \mathbf{e}_n/\theta_a^i)$$

is a multideal of \mathbf{A} .

We recall that $\preceq_{\mathcal{R}}^i$ and \leq^i denote the right preorder and the partial order of the SBA $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathbf{e}_i)$, respectively (see Section 2.4).

Lemma 7.1. $\mathbf{e}_i/\theta_a^i = \{b \in A : b \preceq_{\mathcal{R}}^i a\}$.

Proof. By definition of θ_a^i , we have $b\theta_a^i c$ iff $t_i(a, b, c) = b$. Then $b \in \mathbf{e}_i/\theta_a^i$ iff $b = t_i(a, b, \mathbf{e}_i) = a \wedge_i b$ iff $b \preceq_{\mathcal{R}}^i a$ (by definition of $\preceq_{\mathcal{R}}^i$). \square

The following proposition is a consequence of [7, Proposition 4.15].

Proposition 7.2. (i) *The set \mathbf{e}_i/θ_a^i is a subalgebra of the right Church i -reduct (A, t_i, \mathbf{e}_i) .*
(ii) *The algebra $(\mathbf{e}_i/\theta_a^i, t_i, \mathbf{e}_i, a)$ is a 2CA.*
(iii) *The set $\downarrow_i a = \{b : b \leq^i a\}$ is the Boolean algebra of 2-central elements of $(\mathbf{e}_i/\theta_a^i, t_i, \mathbf{e}_i, a)$.*

Proof. (i) Let $b, c, d \in \mathbf{e}_i/\theta_a^i$. Then $b\theta_a^i \mathbf{e}_i$, $c\theta_a^i \mathbf{e}_i$ and $d\theta_a^i \mathbf{e}_i$. By applying the properties of the congruences, we derive $t_i(b, c, d)\theta_a^i t_i(\mathbf{e}_i, \mathbf{e}_i, \mathbf{e}_i) = \mathbf{e}_i$.

(ii) By $t_i(a, b, c) = b$ and $t_i(\mathbf{e}_i, b, c) = c$, for every $b, c \in \mathbf{e}_i/\theta_a^i$.

(iii) By Lemma 4.3(i) b is a factor element for every $b \in \mathbf{e}_i/\theta_a^i$. Then b is 2-central iff $t_i(b, a, \mathbf{e}_i) = b$ iff $b \wedge_i a = b$ iff $b \leq^i a$, because $a \wedge_i b = t_i(a, b, \mathbf{e}_i) = b$ for all $b \in \mathbf{e}_i/\theta_a^i$. \square

Notice that a is maximal (w.r.t. \leq^i) because, if $a \leq^i b \in \mathbf{e}_i/\theta_a^i$, then $a = a \wedge_i b = t_i(a, b, \mathbf{e}_i) = b$.

We now specialise the above construction to the case $a = \mathbf{e}_j$ for a given $j \neq i$.

Definition 7.1. Let \mathbf{A} be a n BA and $i \neq j$. The *Boolean centre of \mathbf{A}* , denoted by B_{ij} , is the Boolean algebra of 2-central elements of the 2CA $(A, t_i, \mathbf{e}_i, \mathbf{e}_j)$.

By Proposition 7.2 the carrier set of B_{ij} is the set $\downarrow_i \mathbf{e}_j = \{b \in A : b \leq^i \mathbf{e}_j\}$ and we call *Boolean* any element of B_{ij} .

Remark 2. The Boolean algebra B_{ij} was defined in [5] in a different but equivalent way (see [5, Section 6.1, Lemma 7(iii)]).

Lemma 7.3. *Let \mathbf{A} be a n BA, B_{ij} be the Boolean centre of \mathbf{A} , $S_i(\mathbf{A}) = (A, \wedge_i, \vee_i, \setminus_i, \mathbf{e}_i)$ be the skew i -reduct of \mathbf{A} , and $i \neq j$. Then, for all $b, c \in B_{ij}$, we have $b \vee_i c = t_i(b, \mathbf{e}_j, c)$.*

Proof. $b \vee_i c = t_i(b, b, c) =_{(b \leq^i \mathbf{e}_j)} t_i(b, t_i(b, \mathbf{e}_j, \mathbf{e}_i), c) =_{(B2)} t_i(b, \mathbf{e}_j, c)$. \square

By Lemma 7.3 and by [21] the Boolean operations on B_{ij} are $\wedge_i, \vee_i, \neg_{ij}$, where \wedge_i, \vee_i are the corresponding operation of $S_i(\mathbf{A})$ restricted to B_{ij} , and $\neg_{ij}(b) = t_i(b, \mathbf{e}_i, \mathbf{e}_j)$ for every $b \in B_{ij}$.

In [5] a representation theorem is proved, showing that any given n BA \mathbf{A} can be embedded into the n BA of the n -central elements of the Boolean vector space $B_{ij} \times \dots \times B_{ij} = B_{ij}^n$ (see Example 2). The proof of this result makes an essential use of the notion of *coordinates* of elements of \mathbf{A} , that are n -tuples of elements of B_{ij}^n , codifying the elements of \mathbf{A} as linear combinations (see Lemma 7.9(5)). In this paper, the notion of coordinate is again a central one, being used to define the congruence associated to a multideal. In order to highlight their relationship with the skew reducts of \mathbf{A} , here we define the coordinates in terms of the t_k operations.

Definition 7.2. The *coordinates* of $a \in A$ are the elements $a_k = t_k(a, \mathbf{e}_i, \mathbf{e}_j)$, for $1 \leq k \leq n$.

Notice that $a_k \in B_{ij}$ for every $1 \leq k \leq n$, because $a_k \leq^i e_j$:

$$a_k \wedge_i e_j = t_i(t_k(a, e_i, e_j), e_j, e_i) =_{(B3)} q(a, e_i/\bar{k}, e_j/k) = t_k(a, e_i, e_j) = a_k.$$

Lemma 7.4. *Let $a, b, b_1, \dots, b_n \in A$. We have:*

- (i) $a_k \wedge_i a_r = e_i$ for all $k \neq r$.
- (ii) $a_1 \vee_i a_2 \vee_i \dots \vee_i a_n = e_j$.
- (iii) $q(a, b_1, \dots, b_n)_k = q(a, (b_1)_k, \dots, (b_n)_k) = (a_1 \wedge_i (b_1)_k) \vee_i \dots \vee_i (a_n \wedge_i (b_n)_k)$.
- (iv) $(a \wedge_i b)_k = a \wedge_i b_k$, for every $k \neq i$.
- (v) $a_k \wedge_i a = a_k \wedge_i e_k$, for every $k \neq i$.
- (vi) $a_i \wedge_i a = e_i$.
- (vii) If $a \in B_{ij}$, then

$$a_k = \begin{cases} \neg_{ij}(a) & \text{if } k = i \\ a & \text{if } k = j \\ e_i & \text{otherwise} \end{cases}$$

Proof. (i)-(vi) It is sufficient to check in the generator \mathbf{n} of the variety $n\mathbf{BA}$, where $B_{ij} = \{e_i, e_j\}$, $(e_r)_r = e_j$ and $(e_r)_k = e_i$ if $r \neq k$.

(vii) ($k = i$): By definition of \neg_{ij} : $a_i = t_i(a, e_i, e_j) = \neg_{ij}(a)$.

($k \neq i, j$): $a_k = t_k(a, e_i, e_j) = t_k(a \wedge_i e_j, e_i, e_j) = t_k(t_i(a, e_j, e_i), e_i, e_j) = t_i(a, e_i, e_i) = e_i$.

($k = j$): $a_j = t_j(a, e_i, e_j) = t_j(a \wedge_i e_j, e_i, e_j) = t_j(t_i(a, e_j, e_i), e_i, e_j) = t_i(a, e_j, e_i) = a$. \square

Proposition 7.5. *The following conditions are equivalent for an element $a \in A$:*

- (a) a is Boolean;
- (b) $a \wedge_i e_j = a$;
- (c) $a = b_k$, for some $b \in A$ and index $1 \leq k \leq n$;
- (d) $a = a_j$;
- (e) $a_k = e_i$, for every $k \neq i, j$;
- (f) $a = (a_i)_i$.

Proof. (a) \Leftrightarrow (b): We have that $a \leq^i e_j$ iff $a \wedge_i e_j = a$ and $e_j \wedge_i a = a$. The conclusion is obtained because the latter equality is trivially true.

(c) \Rightarrow (b): $b_k \wedge_i e_j = t_i(t_k(b, e_i, e_j), e_j, e_i) =_{(B3)} q(b, e_i/\bar{k}, e_j/k) = t_k(b, e_i, e_j) = b_k$.

(b) \Rightarrow (d): If $a \wedge_i e_j = a$, then $a_j = t_j(a, e_i, e_j) = t_j(a \wedge_i e_j, e_i, e_j) = t_j(t_i(a, e_j, e_i), e_i, e_j) =_{(B3)} t_i(a, e_j, e_i) = a \wedge_i e_j = a$.

(d) \Rightarrow (c): Trivial.

(a) \Rightarrow (e): By Lemma 7.4(vii).

(e) \Rightarrow (d): By Lemma 7.4(ii) the join of all coordinates of $a \in A$ in B_{ij} is the top element e_j . By hypothesis (e) we derive $a_i \vee_i a_j = e_j$. Then, by applying the strong distributive property of \wedge_i w.r.t. \vee_i in the SBA $S_i(\mathbf{A})$, we obtain: $a = e_j \wedge_i a =_{L.7.4(ii)} (a_i \vee_i a_j) \wedge_i a = (a_i \wedge_i a) \vee_i (a_j \wedge_i a) =_{L.7.4(vi)} e_i \vee_i (a_j \wedge_i a) = a_j \wedge_i a =_{L.7.4(v)} a_j \wedge_i e_j =_{(a_j \leq^i e_j)} a_j$.

(f) \Leftrightarrow (b): $(a_i)_i = t_i(t_i(a, e_i, e_j), e_i, e_j) =_{(B3)} t_i(a, e_j, e_i) = a \wedge_i e_j$. Then $(a_i)_i = a$ iff $a \wedge_i e_j = a$. \square

By Lemma 7.4(iv) and Lemma 7.5(d) $a \wedge_i b$ is a Boolean element, for every $a \in A$ and $b \in B_{ij}$.

7.2. The congruence defined by a multideal. Let \mathbf{A} be a $n\mathbf{BA}$ and B_{ij} be the Boolean centre of \mathbf{A} .

Lemma 7.6. *Let I be a multideal on \mathbf{A} . Then $I_* = B_{ij} \cap I_i$ is a Boolean ideal and $I^* = B_{ij} \cap I_j$ is the Boolean filter complement of I_* .*

Proof. Recall that, in B_{ij} , \mathbf{e}_i is the bottom element, \mathbf{e}_j is the top element and $b \in B_{ij}$ iff $b \wedge_i \mathbf{e}_j = b$. We prove that I_* is a Boolean ideal. First $\mathbf{e}_i \in I_*$. If $b, c \in I_*$ and $d \in B_{ij}$, then we prove that $b \vee_i c$ and $b \wedge_i d$ belong to I_* . By Proposition 6.3(I2) $b \vee_i c$ and $b \wedge_i d$ belong to I_i . Moreover, since $b, c, d \in B_{ij}$ and \wedge_i, \vee_i are respectively the meet and the join of B_{ij} , then $b \vee_i c$ and $b \wedge_i d$ also belong to B_{ij} . We now show that I^* is the Boolean filter complement of I_* .

($b \in I_* \Rightarrow \neg_{ij} b \in I^*$): As $b \in I_i \cap B_{ij}$, then by Proposition 6.3(I2) $\neg_{ij} b = t_i(b, \mathbf{e}_i, \mathbf{e}_j) \in I_j \cap B_{ij}$.
($\neg_{ij} b \in I^* \Rightarrow b \in I_*$): As $t_i(b, \mathbf{e}_i, \mathbf{e}_j) \in I_j$, then $b = \neg_{ij} \neg_{ij} b = t_i(t_i(b, \mathbf{e}_i, \mathbf{e}_j), \mathbf{e}_i, \mathbf{e}_j) \in I_i$. \square

The following lemma characterises multideals in terms of coordinates.

Lemma 7.7. *Let (I_1, \dots, I_n) be a multideal on a nBA \mathbf{A} and let $b \in A$. Then we have:*

- (a) $b \in I_r$ if and only if the coordinate b_r of b belongs to I_j .
- (b) If $b \in I_i$, then the coordinate b_k of b belongs to I_i , for every $k \neq i$.

Proof. (a) We start with $r = i$.

(\Rightarrow) Let $b \in I_i$. By Proposition 6.3(I2) we have $b_i = t_i(b, \mathbf{e}_i, \mathbf{e}_j) \in I_j$, because $b \in I_i$ and $\mathbf{e}_j \in I_j$.
(\Leftarrow) By hypothesis $b_i \in I_j$. Then by Lemma 6.6 $b_i^{(ij)} \in I_i$. Now $b_i^{(ij)} = q(b_i, \mathbf{e}_{(ij)1}, \dots, \mathbf{e}_{(ij)n}) = q(t_i(b, \mathbf{e}_i, \mathbf{e}_j), \mathbf{e}_{(ij)1}, \dots, \mathbf{e}_{(ij)n}) =_{(B3)} t_i(b, \mathbf{e}_j, \mathbf{e}_i) = b \wedge_i \mathbf{e}_j \in I_i$. We conclude $b \in I_i$ by applying Proposition 6.3(I2) to $t_i(b \wedge_i \mathbf{e}_j, b, \mathbf{e}_i)$, because $b \wedge_i \mathbf{e}_j \in I_i$ and $b =_{L.4.2(3)} b \wedge_i b =_{(b=\mathbf{e}_j \wedge_i b)} b \wedge_i \mathbf{e}_j \wedge_i b = t_i(b \wedge_i \mathbf{e}_j, b, \mathbf{e}_i)$.

We analyse $r \neq i$. Let σ be equal to the transposition (ir) . By definition of b^σ we derive $(b^\sigma)_i = t_i(q(b, \mathbf{e}_{\sigma 1}, \dots, \mathbf{e}_{\sigma n}), \mathbf{e}_i, \mathbf{e}_j) =_{(B3)} q(b, \mathbf{e}_j/r, \mathbf{e}_i/\bar{r}) = t_r(b, \mathbf{e}_i, \mathbf{e}_j) = b_r$. Then, $b_r \in I_j \Leftrightarrow b_r = (b^\sigma)_i \in I_j \Leftrightarrow b^\sigma \in I_i \Leftrightarrow_{L.6.6} b = (b^\sigma)^\sigma \in I_r$.

(b) By Proposition 6.3(I2), $k \neq i$ and $b \in I_i$ we get $b_k = t_k(b, \mathbf{e}_i, \mathbf{e}_j) \in I_i$. \square

We consider the homomorphism $f_I : B_{ij} \rightarrow B_{ij}/I_*$ and we define on A the following equivalence relation:

$$b\theta_I c \Leftrightarrow \forall k. f_I(b_k) = f_I(c_k),$$

where b_k, c_k are the k -coordinates of b and c , respectively (see Definition 7.2).

Proposition 7.8. θ_I is a congruence on \mathbf{A} .

Proof. Let $a, b, c_1, d_1, \dots, c_n, d_n$ be elements of A such that $a\theta_I b$ and $c_k\theta_I d_k$, for every k . Then $q(a, c_1, \dots, c_n)\theta_I q(b, d_1, \dots, d_n)$ iff $\forall k. f_I(q(a, c_1, \dots, c_n)_k) = f_I(q(b, d_1, \dots, d_n)_k)$. The conclusion follows, because f_I is a Boolean homomorphism and $q(a, c_1, \dots, c_n)_k =_{L.7.4(iii)} q(a, (c_1)_k, \dots, (c_n)_k) =_{L.7.4(iii)} (a_1 \wedge_i (c_1)_k) \vee_i \dots \vee_i (a_n \wedge_i (c_n)_k)$. \square

We define a new term operation to be used in Theorem 7.10:

$$x +_i y = q(x, t_i(y, \mathbf{e}_i, \mathbf{e}_1), \dots, t_i(y, \mathbf{e}_i, \mathbf{e}_{i-1}), y, t_i(y, \mathbf{e}_i, \mathbf{e}_{i+1}), \dots, t_i(y, \mathbf{e}_i, \mathbf{e}_n)); \quad (y \text{ at position } i).$$

Lemma 7.9. *Let $a, b \in A$ and a_1, a_2, \dots, a_n be the coordinates of a . Then*

- (1) $a +_i \mathbf{e}_i = \mathbf{e}_i +_i a = a$;
- (2) $a +_i b = b +_i a$;
- (3) $a +_i \mathbf{e}_k = \mathbf{e}_k +_i a = a_i \wedge_i \mathbf{e}_k$ ($k \neq i$).
- (4) $a +_i a = \mathbf{e}_i$;
- (5) *The value of the expression $E \equiv (a_1 \wedge_i \mathbf{e}_1) +_i ((a_2 \wedge_i \mathbf{e}_2) +_i (\dots +_i (a_n \wedge_i \mathbf{e}_n))) \dots$ is independent of the order of its parentheses. Without loss of generality, we write $(a_1 \wedge_i \mathbf{e}_1) +_i (a_2 \wedge_i \mathbf{e}_2) +_i \dots +_i (a_n \wedge_i \mathbf{e}_n)$ for the expression E . Then we have:*

$$(a_1 \wedge_i \mathbf{e}_1) +_i (a_2 \wedge_i \mathbf{e}_2) +_i \dots +_i (a_n \wedge_i \mathbf{e}_n) = a.$$

- (6) *If a and b have the same coordinates, then $a = b$.*

Proof. (1) $a +_i \mathbf{e}_i = q(a, t_i(\mathbf{e}_i, \mathbf{e}_i, \mathbf{e}_1), \dots, t_i(\mathbf{e}_i, \mathbf{e}_i, \mathbf{e}_{i-1}), \mathbf{e}_i, t_i(\mathbf{e}_i, \mathbf{e}_i, \mathbf{e}_{i+1}), \dots, t_i(\mathbf{e}_i, \mathbf{e}_i, \mathbf{e}_n)) = q(a, \mathbf{e}_1, \dots, \mathbf{e}_n) \stackrel{(B4)}{=} a.$

(2)

$$\begin{aligned} a +_i b &= q(a, t_i(b, \mathbf{e}_i, \mathbf{e}_1), \dots, t_i(b, \mathbf{e}_i, \mathbf{e}_{i-1}), b, t_i(b, \mathbf{e}_i, \mathbf{e}_{i+1}), \dots, t_i(b, \mathbf{e}_i, \mathbf{e}_n)) \\ &\stackrel{(B4)}{=} q(t_i(b, a, a), t_i(b, \mathbf{e}_i, \mathbf{e}_1), \dots, t_i(b, \mathbf{e}_i, \mathbf{e}_{i-1}), q(b, \mathbf{e}_1, \dots, \mathbf{e}_n), t_i(b, \mathbf{e}_i, \mathbf{e}_{i+1}), \dots, t_i(b, \mathbf{e}_i, \mathbf{e}_n)) \\ &\stackrel{(B3)}{=} q(b, t_i(a, \mathbf{e}_i, \mathbf{e}_1), \dots, t_i(a, \mathbf{e}_i, \mathbf{e}_{i-1}), q(a, \mathbf{e}_1, \dots, \mathbf{e}_n), t_i(a, \mathbf{e}_i, \mathbf{e}_{i+1}), \dots, t_i(a, \mathbf{e}_i, \mathbf{e}_n)) \\ &= b +_i a. \end{aligned}$$

(3) $a +_i \mathbf{e}_k = q(a, t_i(\mathbf{e}_k, \mathbf{e}_i, \mathbf{e}_1), \dots, t_i(\mathbf{e}_k, \mathbf{e}_i, \mathbf{e}_{i-1}), \mathbf{e}_k, t_i(\mathbf{e}_k, \mathbf{e}_i, \mathbf{e}_{i+1}), \dots, t_i(\mathbf{e}_k, \mathbf{e}_i, \mathbf{e}_n)) \stackrel{(k \neq i)}{=} q(a, \mathbf{e}_1, \dots, \mathbf{e}_i, \mathbf{e}_k, \mathbf{e}_i, \dots, \mathbf{e}_i) = t_i(a, \mathbf{e}_i, \mathbf{e}_k) = \mathbf{e}_k +_i a.$ Moreover, $a_i \wedge_i \mathbf{e}_k = t_i(t_i(a, \mathbf{e}_i, \mathbf{e}_j), \mathbf{e}_k, \mathbf{e}_i) \stackrel{(B3)}{=} t_i(a, \mathbf{e}_i, \mathbf{e}_k).$

(4) $a +_i a = q(a, t_i(a, \mathbf{e}_i, \mathbf{e}_1), \dots, t_i(b, \mathbf{e}_i, \mathbf{e}_{i-1}), a, t_i(a, \mathbf{e}_i, \mathbf{e}_{i+1}), \dots, t_i(a, \mathbf{e}_i, \mathbf{e}_n)) \stackrel{(B2)}{=} q(a, \mathbf{e}_1, \dots, \mathbf{e}_i, a, \mathbf{e}_i, \dots, \mathbf{e}_i) \stackrel{(B4)}{=} q(a, \mathbf{e}_1, \dots, \mathbf{e}_i, q(a, \mathbf{e}_1, \dots, \mathbf{e}_n), \mathbf{e}_i, \dots, \mathbf{e}_i) = q(a, \mathbf{e}_1, \dots, \mathbf{e}_i) = \mathbf{e}_i.$

(5) It is easy to check the identity in the generator \mathbf{n} of the variety $n\mathbf{BA}$.

(6) is a consequence of (5). \square

Theorem 7.10. *Let ϕ be a congruence, $I(\phi) = (\mathbf{e}_1/\phi, \dots, \mathbf{e}_n/\phi)$ be the multideal of \mathbf{A} determined by ϕ , $H = (H_1, \dots, H_n)$ be a multideal and θ_H be the congruence on \mathbf{A} determined by H . Then*

$$\theta_{I(\phi)} = \phi \quad \text{and} \quad I(\theta_H) = H.$$

Proof. We first prove $I(\theta_H)_k = H_k$, for all k . Recall that $\mathbf{e}_j = (\mathbf{e}_i)_i$ is the i -coordinate of \mathbf{e}_i and $\mathbf{e}_i = (\mathbf{e}_i)_k$ is the k -coordinate of \mathbf{e}_i for every $k \neq i$.

(1) First we provide the proof for $k = i$. Let $a \in I(\theta_H)_i$. If $a\theta_H \mathbf{e}_i$ then $f_H(a_i) = f_H((\mathbf{e}_i)_i) = f_H(\mathbf{e}_j)$, that implies $a_i \in H^*$. By Lemma 7.7(a) we get the conclusion $a \in H_i$.

For the converse, let $a \in H_i$. By Lemma 7.7(a) we have $a_i \in H^*$ and by Lemma 7.7(b) $a_k \in H_*$ for all $k \neq i$. This implies $f_H(a_i) = f_H(\mathbf{e}_j) = f_H((\mathbf{e}_i)_i)$ and $f_H(a_k) = f_H(\mathbf{e}_i) = f_H((\mathbf{e}_i)_k)$ for all $k \neq i$, that implies $a\theta_H \mathbf{e}_i$. Since $I(\theta_H)_i = \mathbf{e}_i/\theta_H$, we conclude.

(2) Let now $k \neq i$. By Lemma 6.6 we have $H_k = H_i^{(ik)}$. Let $a \in H_k$. Then $a = b^{(ik)}$ for some $b \in H_i$. As, by (1), $b\theta_H \mathbf{e}_i$, then we have $a = b^{(ik)}\theta_H(\mathbf{e}_i)^{(ik)} = \mathbf{e}_k$. Since $I(\theta_H)_k = \mathbf{e}_k/\theta_H$, we conclude. Now, assuming $a\theta_H \mathbf{e}_k$, we have: $b = (a)^{(ik)}\theta_H(\mathbf{e}_k)^{(ik)} = \mathbf{e}_i$. Then $b \in H_i$ and $a = b^{(ik)} \in H_k$.

Let ϕ be a congruence.

(a) Let $a\phi b$. Then $\forall h. a_h\phi b_h$. Since ϕ restricted to B_{ij} is also a Boolean congruence, then we obtain $(a_h \oplus_{ij} b_h)\phi \mathbf{e}_i$, where \oplus_{ij} denotes the symmetric difference in the Boolean centre B_{ij} . We now prove that $a\theta_{I(\phi)} b$. We have $a\theta_{I(\phi)} b$ iff $\forall h. f_{I(\phi)}(a_h) = f_{I(\phi)}(b_h)$ iff $\forall h. a_h \oplus_{ij} b_h \in I(\phi)_* = B_{ij} \cap \mathbf{e}_i/\phi$ iff $\forall h. a_h \oplus_{ij} b_h \in \mathbf{e}_i/\phi$ iff $\forall h. (a_h \oplus_{ij} b_h)\phi \mathbf{e}_i$. This last relation is proved above and we conclude $a\theta_{I(\phi)} b$.

(b) Let $a\theta_{I(\phi)} b$. Then $\forall h. a_h \oplus_{ij} b_h \in \mathbf{e}_i/\phi$ that implies $\forall h. a_h\phi b_h$, because ϕ restricted to B_{ij} is a Boolean congruence. Since by Lemma 7.9(5) there is a n -ary term u such that $a = u(a_1, \dots, a_n)$ and $b = u(b_1, \dots, b_n)$, then we conclude $a\phi b$ by using $\forall h. a_h\phi b_h$. \square

7.3. Ultramultideals. In the Boolean case, there is a bijective correspondence between maximal ideals and homomorphisms onto $\mathbf{2}$. In this section we show that every multideal can be extended to an ultramultideal, and that there exists a bijective correspondence between ultramultideals and homomorphisms onto \mathbf{n} . We also show that prime multideals coincide with ultramultideals.

Let (I_1, \dots, I_n) be a multideal of a $n\mathbf{BA}$ \mathbf{A} and U be a Boolean ultrafilter of B_{ij} that extends $I^* = B_{ij} \cap I_j$, and so the maximal ideal $\bar{U} = B_{ij} \setminus U$ extends $I_* = B_{ij} \cap I_i$.

Lemma 7.11. *For all $a \in A$, there exists a unique k such that $a_k \in U$.*

Proof. By Lemma 7.4(ii) the meet of two distinct coordinates is the bottom element e_i . Then at most one coordinate may belong to U . On the other hand, if all coordinates belong to \bar{U} , then the top element e_j belong to \bar{U} . \square

Let $(G_k)_{k \in \hat{n}}$ be the sequence such that $G_k = \{a \in A : a_k \in U\}$, which, by Lemma 7.11, is well defined.

Lemma 7.12. $(G_k)_{k \in \hat{n}}$ is a ultramultideal which extends $(I_k)_{k \in \hat{n}}$.

Proof. (m1) $e_k \in G_k$ because $(e_k)_k = e_j \in U$.

(m2): let $a \in G_r$, $b \in G_k$, and $c^1, \dots, c^n \in A$. By Lemma 7.4(ii),

$$q(a, c^1, \dots, c^{r-1}, b, c^{r+1}, \dots, c^n)_k = [\bigvee_{s \neq r} (a_s \wedge_i (c^s)_k)] \vee_i (a_r \wedge_i b_k).$$

Since $a_r, b_k \in U$, then $a_r \wedge_i b_k \in U$, and so $a_r \wedge_i b_k \sqsubseteq [\bigvee_{s \neq r} (a_s \wedge_i (c^s)_k)] \vee_i (a_r \wedge_i b_k) \in U$, where \sqsubseteq is the Boolean order of the Boolean algebra B_{ij} . Hence, $q(a, c^1, \dots, c^{r-1}, b, c^{r+1}, \dots, c^n) \in G_k$.

(m3) can be proved similarly.

We now prove that $(G_k)_{k \in \hat{n}}$ extends $(I_k)_{k \in \hat{n}}$. It is sufficient to show that, for every $x \in I_k$, we have that $x_k \in U$. We get the conclusion by Lemma 7.7(a). \square

Theorem 7.13. (i) Every multideal can be extended to an ultramultideal.

(ii) There is a bijective correspondence between ultramultideals and homomorphisms onto \mathbf{n} .

Proof. (i) follows from Lemma 7.12. Regarding (ii), we remark that the algebra \mathbf{n} is the unique simple n BA. \square

We conclude this section by characterising prime multideals.

Definition 7.3. We say that a multideal (I_1, \dots, I_n) is *prime* if, for every i , $a \wedge_i b \in I_i$ implies $a \in I_i$ or $b \in I_i$.

Proposition 7.14. A multideal is prime iff it is an ultramultideal.

Proof. (\Rightarrow) Let (I_1, \dots, I_n) be a prime ideal. If $a \in B_{ij}$, then $a \wedge_i \neg_{ij}(a) = e_i \in I_i$. Then either a or $\neg_{ij}(a) \in I_i$. This implies that $I_* = B_{ij} \cap I_i$ is a maximal Boolean ideal and the complement $I^* = B_{ij} \cap I_j$ is a Boolean ultrafilter.

Let now $b \in A$ such that $b \notin I = \bigcup_{k=1}^n I_k$. By Lemma 7.7(a) we have that $b \in I_r$ iff $b_r \in I^*$. Then $b_r \notin I^*$ for all r . Since I^* is a Boolean ultrafilter, then $b_r \in I_*$ for all r . Hence $e_j = \bigvee_{r=1}^n b_r \in I_*$, contradicting the fact that the top element does not belong to a maximal ideal. In conclusion, $b \in I = \bigcup_{k=1}^n I_k$ for an arbitrary b , so that $I = A$.

(\Leftarrow) Let I be an ultramultideal. Let $a \wedge_i b \in I_i$ with $a \in I_r$ and $b \in I_k$ (with $r \neq i$ and $k \neq i$). Then by property (m2) of multideals we get $a \wedge_i b = t_i(a, b, e_i) = q(a, b, \dots, b, e_i, b, \dots, b) \in I_k$. Contradiction. \square

CONCLUSION

Boolean-like algebras have been introduced in [21, 5] as a generalisation of Boolean algebras to any finite number of truth values. Boolean-like algebras provide a new characterisation of primal varieties exhibiting a perfect symmetry of the values of the generator of the variety. In this paper we have investigated the relationships between skew Boolean algebras and Boolean-like algebras. We have shown that any n -dimensional Boolean-like algebra is a skew cluster of n isomorphic right-handed skew Boolean algebras, and that the variety of skew star algebras is term equivalent to the variety of Boolean-like algebras. Moreover, we have got a representation theorem for right-handed skew Boolean algebras, and developed a general theory of multideals for Boolean-like algebras. Several further works are worth mentioning:

- (1) How the duality theory of SBAs and BAs are related to a possible duality theory of n BAs (a Stone-like topology on ultramultideals).
- (2) Find a more satisfactory axiomatisation of skew star algebras.
- (3) Each SBA living inside a n BA has a bottom element 0 and several maximal elements. The construction could be made symmetric, by defining “skew-like” algebras having several minimal and several maximal elements.
- (4) For each n BA \mathbf{A} , the algebras $S_1(\mathbf{A}), \dots, S_n(\mathbf{A})$, constituting the skew cluster of \mathbf{A} , are isomorphic. This result is also of technical interest for the following open problem in the theory of skew Boolean algebras:

Problem 1. Given a SBA \mathbf{A} with a maximal class $M \supseteq \{m_1, m_2\}$, let \mathbf{A}_{m_1} and \mathbf{A}_{m_2} be the algebras obtained from \mathbf{A} on distinguishing the elements m_1 and m_2 respectively. Are the algebras \mathbf{A}_{m_1} and \mathbf{A}_{m_2} isomorphic?

This problem is part of the folklore and it does not appear in any published work to date. It is implicit in Leech [15], where both skew Boolean algebras (as they are now understood) and skew Boolean algebras possessing a maximal class are introduced.

The difficulty in obtaining a solution to Problem 1 evidently lies in constructing the required isomorphism. For skew Boolean \cap -algebras, a related problem has been considered and resolved in the positive by Bignall [2]. The proof exploits sheaf (Boolean product) representations to obtain the desired isomorphism; as skew Boolean algebras admit only a weak Boolean product representation, the proof does not seem readily adaptable.

Problem 1 is of purely technical interest in the theory of skew Boolean algebras. However, it assumes greater prominence in logics arising from (structurally enriched) skew Boolean algebras. Very roughly speaking, let S be an algebraisable logic arising from a quasivariety K of 1-regular (necessarily structurally enriched) skew Boolean algebras. Given n residually distinct constant terms of K , $1 < n < \omega$ (working with the finite case for simplicity), S admits $n - 1$ negation connectives via implication into m , for each m a constant term distinct from 1. A positive solution to Problem 1 would imply that these $n - 1$ negations are not essentially different, and hence that it is enough to fix a single such negation univocally when studying S ; whereas a negative solution to Problem 1 would imply that these $n - 1$ negations are all distinct, and hence that they must all be accounted for in any study of S .

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