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Polynomial C^1 shape functions on the triangle

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ABSTRACT: We derive generic formulae for all possible C^1 continuous polynomial interpolations for triangular elements, by considering individual shape functions, without the need to prescribe the type of the degrees of freedom in advance. We then consider the possible ways in which these shape functions can be combined to form finite elements with given properties. The simplest case of fifth-order polynomial functions is presented in detail, showing how two existing elements can be obtained, as well as two new elements, one of which shows good numerical behaviour in numerical tests.

KEYWORDS: finite elements; C^1 continuity; shape functions; triangular element; gradient elasticity

1 INTRODUCTION

A displacement-only discretisation of continuum mechanics problems involving fourth-order partial differential equations introduces the need for finite elements with C^1 continuous interpolation of the displacements. Elements of this type were developed initially mostly for use in modelling bending of thin plates, while more recently they have been succesfully employed for modelling strain-gradient dependent materials. Among others, Argyris *et al.* [1] presented a C^1 triangular element with polynomial interpolation and 21 degrees of freedom (named TUBA 6), from which a simpler element with 18 degrees of freedom (named TUBA 3) has also been derived. The shape functions for these elements were not given explicitly at first, but they are available in more recent publications (see for example [2, 3]).

In a recent paper [4] we presented an algorithm for directly deriving the shape functions of elements like TUBA 3, thus creating a new family of "TRF" elements. In that algorithm, the interpolation is obtained by determining a priori the location and type of the degrees of freedom. This restriction is relaxed here, by presenting a method that allows us to derive formulae for all possible C^1 continuous polynomial shape functions for triangular elements without first prescribing the type of the degrees of freedom.

The generic formulae thus derived for the shape functions are expressed in terms of a number of unknown functions of the element geometry. These unknown functions can be determined by considering the way individual shape functions are combined in specific elements. We discuss here the general problem of determining appropriate combinations of shape functions to obtain elements with specific properties and then consider in detail the simplest case in which fifth-order polynomial shape functions are used. As expected, the generic approach described in this paper allows us to obtain in this case all known elements (such as TUBA 3 and TUBA 6), while we also obtain two additional new elements.

Since C^1 continuity is a necessary, but not sufficient, condition for correct numerical behaviour of the resulting elements, we test the newly obtained elements using two benchmark problems of gradient elasticity and compare their behaviour to that of existing elements. We thus see that while one of the new elements fails to perform adequately, the second one exhibits a good numerical behaviour, comparable to that of the existing elements.

2 ELEMENT GEOMETRY

We consider finite elements which are triangles with straight sides, like the one shown in Figure 1. The only additional constraint we impose is that the elements must have "rotational symmetry", so that cycling the vertex and side numbers does not affect the interpolation.

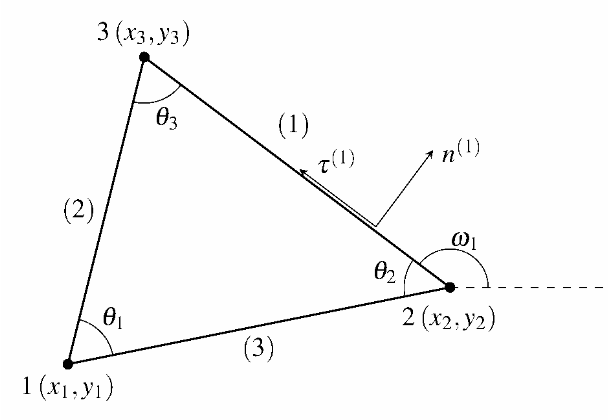

Figure 1: Geometry of a triangular element

The geometry of the side from vertex b to vertex a, where a and b can take a value of 1, 2 or 3, is given by the quantities

$$x_{ab} = x_a - x_b, \qquad y_{ab} = y_a - y_b \tag{1}$$

so that $x_{aa} = y_{aa} = 0$, $x_{ab} = -x_{ba}$ and $y_{ab} = -y_{ba}$. In general only four quantities are needed to define the element geometry, for example the quantities x_{21}, y_{21}, x_{31} and y_{31} obtained by singling out vertex 1 suffice.

To obtain simpler expressions we use the areal coordinates L_1, L_2 and L_3, related to the Cartesian coordinates x, y through the equations

$$\begin{aligned} x &= L_1 x_1 + L_2 x_2 + L_3 x_3 \\ y &= L_1 y_1 + L_2 y_2 + L_3 y_3 \\ 1 &= L_1 + L_2 + L_3 \end{aligned} \tag{2}$$

Using the quantities x_{21}, y_{21}, x_{31} and y_{31}, the relation between Cartesian and areal derivatives is written as

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{x_{21}y_{31} - y_{21}x_{31}}\left(y_{31}\left(\frac{\partial}{\partial L_2} - \frac{\partial}{\partial L_1}\right) - y_{21}\left(\frac{\partial}{\partial L_3} - \frac{\partial}{\partial L_1}\right)\right) \\ \frac{\partial}{\partial y} &= \frac{1}{x_{21}y_{31} - y_{21}x_{31}}\left(-x_{31}\left(\frac{\partial}{\partial L_2} - \frac{\partial}{\partial L_1}\right) + x_{21}\left(\frac{\partial}{\partial L_3} - \frac{\partial}{\partial L_1}\right)\right) \end{aligned} \tag{3}$$

A different set of quantities that can be used to define the element geometry is the length $l_1 = \sqrt{x_{32}^2 + y_{32}^2}$ and the three angles ω_1, θ_2 and θ_3 shown in Figure 1. Setting

$$\kappa_2 = \cot\theta_2 = \frac{x_{32}x_{12} + y_{32}y_{12}}{x_{32}y_{12} - y_{32}x_{12}} \tag{4}$$

and similarly for κ_3 and κ_1 by cycling the indices, we obtain the following simple expressions for the tangential and normal derivatives on side 1

$$\frac{\partial}{\partial \mathbf{t}^{(1)}} = \frac{1}{l_1}\left(\frac{\partial}{\partial L_3} - \frac{\partial}{\partial L_2}\right)\Big|_{L_1=0}$$

$$\frac{\partial}{\partial \mathbf{n}^{(1)}} = \frac{1}{l_1}\left(\kappa_3\left(\frac{\partial}{\partial L_2} - \frac{\partial}{\partial L_1}\right) + \kappa_2\left(\frac{\partial}{\partial L_3} - \frac{\partial}{\partial L_1}\right)\right)\Big|_{L_1=0} \tag{5}$$

3 SHAPE FUNCTIONS

The interpolation w within an element is given by the linear combination

$$w = \sum_{\alpha=1}^{n_d} \phi^{(\alpha)} d^{(\alpha)} \tag{6}$$

where n_d is the number of degrees of freedom, $\phi^{(\alpha)}$ are the shape functions and $d^{(\alpha)}$ are the values of the degrees of freedom.

To obtain a C^1 interpolation, we must ensure C^1 continuity of w both within an element and at the boundary between elements with common sides or common vertices. Since the degrees of freedom can assume any value, C^1 continuity is required for all individual shape functions. Thus we can examine each shape function separately.

We consider here the case of polynomial shape functions of arbitrary order n_p. The generic form of the shape function ϕ can then be written as

$$\phi = \sum_{i=0}^{n_p} \sum_{k=0}^{n_p - i} t_{i,k} L_1^{n_p - i - k} L_2^i L_3^k \tag{7}$$

where the coefficients $t_{i,k}$ depend only on the geometry of the element. Note that Bézier polynomials can also be used [5, 3], instead of the form in equation (7). In the present case, however, they would result in more complex expressions without providing any actual benefit.

The use of polynomial shape functions ensures that C^1 continuity is obtained within the element, thus it must only be enforced at the boundary between adjacent elements. To determine the coefficients $t_{i,k}$ which yield C^1 continuity for a given shape function ϕ associated with a given node, we distinguish between the sides which include the node ("near" sides) and those that do not include it ("far" sides). The value of ϕ and its normal derivative must be zero on the far sides and they must match the respective values of the shape function on neighbouring elements on near sides (see Figure 2 and the description in subsections 3.2 and 3.3). It is important to note that, since the normal on each side is defined as pointing outside the triangle, the normal derivatives for two adjacent elements are matched on the common side when they have equal absolute value and opposite sign.

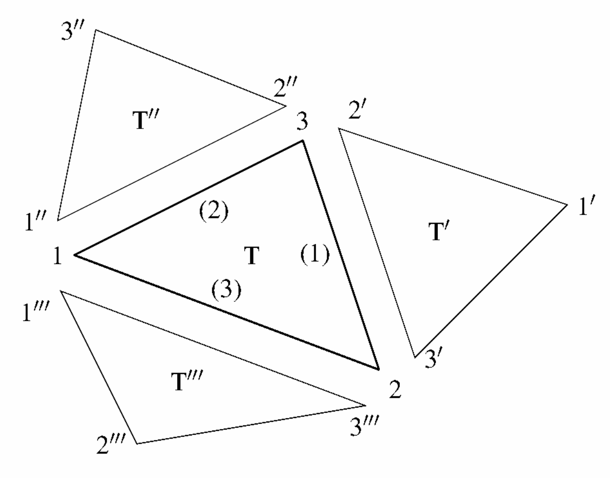

Figure 2: A triangle element with its neighbouring elements

The value of ϕ on the three sides is easily calculated as

$$\phi|_{L_1=0} = \sum_i t_{i,n_p-i} L_2^i (1-L_2)^{n_p-i}$$

$$\phi|_{L_2=0} = \sum_i t_{0,i} L_1^{n_p-i} (1-L_1)^i \qquad (8)$$

$$\phi|_{L_3=0} = \sum_i t_{i,0} L_1^{n_p-i} (1-L_1)^i$$

while the values of the normal derivative of ϕ on the three sides are

$$l_1 \frac{\partial \phi}{\partial \mathbf{n}^{(1)}} = \sum_j L_2^j (1-L_2)^{n_p-1-j} (\kappa_2 (n_p - j) t_{j,n_p-j} + \kappa_3 (j+1) t_{j+1,n_p-1-j} - (\kappa_2 + \kappa_3) t_{j,n_p-1-j})$$

$$l_2 \frac{\partial \phi}{\partial \mathbf{n}^{(2)}} = \sum_j L_1^{n_p-1-j} (1-L_1)^j (\kappa_3 (n_p - j) t_{0,j} + \kappa_1 (j+1) t_{0,j+1} - (\kappa_3 + \kappa_1) t_{1,j}) \qquad (9)$$

$$l_3 \frac{\partial \phi}{\partial \mathbf{n}^{(3)}} = \sum_j L_1^{n_p-1-j} (1-L_1)^j (\kappa_2 (n_p - j) t_{j,0} + \kappa_1 (j+1) t_{j+1,0} - (\kappa_2 + \kappa_1) t_{j,1})$$

In equations (8) and (9), as well as in the rest of this paper, the free indices i and j always take the values $i = 0, \ldots, n_p$ and $j = 0, \ldots, n_p - 1$.

Equation (7) shows that a polynomial shape function of order n_p has $(n_p + 2)(n_p + 1)/2$ coefficients $t_{i,k}$. As will be seen later, no C^1 polynomial functions exist for $n_p < 5$. For $n_p \geq 5$, it can be verified from equations (8) and (9) that only $6n_p - 9$ of the $t_{i,k}$ coefficients have an influence on the value of ϕ or its normal derivative at the boundary, and have thus a role to play in ensuring C^1 continuity. These are called here the "boundary coefficients". The remaining $(n_p - 4)(n_p - 5)/2$ of the $t_{i,k}$ coefficients do not affect the existence of C^1 continuity and are called here the "internal coefficients".

A given shape function (and respective degree of freedom) of an element is associated with a node that can be located on a vertex, on a side (excluding the vertices) or in the interior of the element. We can then distinguish between *vertex* shape functions, *side* shape functions and *internal* shape functions.

An important assumption, which is often only implied when discussing C^1 continuity, is that the elements are part of a regular mesh, where two elements having two points in common must share a whole side. An interesting implication is that all side nodes on a given side can be replaced with a single node, where all degrees of freedom of the original nodes are transferred to the new node. The same holds for all internal nodes, thus we can consider only vertex, midside and centroid nodes.

3.1 Internal shape functions

For internal nodes, all three sides are far sides, therefore the shape function and its normal derivative must be zero on all sides. From equations (8) and (9) this means that all $6n_p - 9$ boundary coefficients must be zero, i.e.

$$t_{i,n_p-i} = t_{0,i} = t_{i,0} = 0, \qquad t_{j,n_p-1-j} = t_{1,j} = t_{j,1} = 0 \tag{10}$$

For $n_p \leq 5$ all coefficients $t_{i,k}$ are zero, therefore there exist no internal C^1 shape functions, while for $n_p > 5$ the resulting shape functions are "C^1 bubble functions".

3.2 Side shape functions

We consider a node on side 1 of the triangle T in Figure 2, and the adjacent triangle $\mathrm{T'}$ whose side 1 coincides with side 1 of T. Due to the rotational symmetry of the element, this is enough to cover all possible combinations of sides of T and matching sides of T'.

On the far sides 2 and 3 of triangle T the value of ϕ and its normal derivative must be zero, therefore

$$t_{0,i} = t_{i,0} = 0, \qquad t_{1,j} = t_{j,1} = 0 \tag{11}$$

The value of ϕ on side 1 must only depend on the geometry of side 1, so

$$t_{i,n_p-i} = \overline{F}_i(x_{32}, y_{32}) \tag{12}$$

The value of the normal derivative of ϕ on side 1 must also depend only on the geometry of the side, so we can write the first of equations (9) as

$$l_1 \frac{\partial \phi}{\partial \mathbf{n}^{(1)}} = \sum_j L_2^j (1 - L_2)^{n_p - 1 - j} \overline{G}_j(x_{32}, y_{32}) \tag{13}$$

where we have set

$$\overline{G}_j(x_{32}, y_{32}) = \kappa_2 (n_p - j) \overline{F}_j + \kappa_3 (j+1) \overline{F}_{j+1} - (\kappa_2 + \kappa_3) t_{j, n_p - 1 - j} \tag{14}$$

so that

$$t_{j,n_p-1-j} = \frac{1}{\kappa_2 + \kappa_3} \left(-\overline{G}_j(x_{32}, y_{32}) + \kappa_2 (n_p - j) \overline{F}_j(x_{32}, y_{32}) + \kappa_3 (j+1) \overline{F}_{j+1}(x_{32}, y_{32}) \right) \tag{15}$$

Matching the value of ϕ and its normal derivative on side 1 of T and T' yields

$$\overline{F}_i(x_{32}, y_{32}) = \overline{F}_{n_p - i}(-x_{32}, -y_{32})$$

$$\overline{G}_j(x_{32}, y_{32}) = -\overline{G}_{n_p - 1 - j}(-x_{32}, -y_{32}) \tag{16}$$

Using the above equations we also obtain $\overline{G}_0 = \overline{G}_1 = \overline{G}_{n_p-2} = \overline{G}_{n_p-1} = 0$ and $t_{2,n_p-2} = t_{n_p-2,2} = \overline{F}_2 = \overline{F}_{n_p-2} = 0$. Therefore for $n_p \le 4$ there exist no side C^1 shape functions.

For $n_p \geq 5$ there are $4n_p$ border coefficients equal to zero, while the remaining ($2n_p - 9$) border coefficients are given by equations (12) and (15). Due to the "symmetries" (16), only $n_p - 4$ indeterminate functions $\bar{F}_3, \ldots, \bar{F}_{\lfloor n_p/2 \rfloor}$ and $\bar{G}_2, \ldots, \bar{G}_{\lfloor (n_p-1)/2 \rfloor}$ (which depend only on the geometry of the side) are needed.

3.3 Vertex shape functions

Exploiting once more the rotational symmetry of the element, we can consider only the case of a shape function ϕ for vertex 1 of the triangle T in Figure 2, with sides 2 and 3 of T coinciding respectively with side 3 of triangle T'' and side 2 of triangle T'''.

The value of ϕ and its normal derivative on side 1 is zero, therefore

$$t_{i,n_p-i} = 0, \quad t_{j,n_p-1-j} = 0 \tag{17}$$

The value of ϕ on the near sides 2 and 3 must depend only on the geometry of the side and must be equal to the value computed on the adjacent triangle, therefore

$$t_{i,0} = \hat{F}_i(x_{21}, y_{21}), \quad t_{0,i} = \hat{F}_i(x_{31}, y_{31}) \tag{18}$$

A similar procedure for the normal derivative of ϕ yields, after some computations,

$$t_{1,j} = \frac{1}{\kappa_1 + \kappa_3}(-\hat{G}_j(x_{31}, y_{31}) + \kappa_3(n_p - j)\hat{F}_j(x_{31}, y_{31}) + \kappa_1(j+1)\hat{F}_{j+1}(x_{31}, y_{31}))$$

$$t_{j,1} = \frac{1}{\kappa_1 + \kappa_2}(\quad \hat{G}_j(x_{21}, y_{21}) + \kappa_2(n_p - j)\hat{F}_j(x_{21}, y_{21}) + \kappa_1(j+1)\hat{F}_{j+1}(x_{21}, y_{21})) \tag{19}$$

where $\hat{F}_j$ and $\hat{G}_j$ are indeterminate functions.

Equations (17), (18) and (19) are partly coupled. Solving them we obtain

$$\hat{F}_{n_p} = \hat{F}_{n_p-1} = \hat{F}_{n_p-2} = \hat{G}_{n_p-1} = \hat{G}_{n_p-2} = 0 \tag{20}$$

and

$$\begin{aligned}
&\hat{F}_0(x,y) = C_0 \\
&\hat{F}_1(x,y) = n_p C_0 + C_1 x + C_2 y \\
&\hat{F}_2(x,y) = \tfrac{1}{2} n_p (n_p - 1) C_0 + (n_p - 1) C_1 x + (n_p - 1) C_2 y + \tfrac{1}{2} C_3 x^2 + \tfrac{1}{2} C_4 y^2 + C_5 xy \\
&\hat{G}_0(x,y) = C_2 x - C_1 y \\
&\hat{G}_1(x,y) = (n_p - 1)(C_2 x - C_1 y) + (C_4 - C_3) xy + C_5 (x^2 - y^2)
\end{aligned} \tag{21}$$

where $C_0, \ldots, C_5$ are arbitrary constants. From these values we obtain the values of the coefficients

$$t_{n_p-2,0} = t_{0,n_p-2} = 0 \tag{22}$$

and

$$\begin{aligned}
&t_{0,0} = C_0 \\
&t_{0,1} = n_p C_0 + C_1 x_{31} + C_2 y_{31} \\
&t_{1,0} = n_p C_0 + C_1 x_{21} + C_2 y_{21} \\
&t_{0,2} = \tfrac{1}{2} n_p (n_p - 1) C_0 + (n_p - 1) C_1 x_{31} + (n_p - 1) C_2 y_{31} + \tfrac{1}{2} C_3 x_{31}^2 + \tfrac{1}{2} C_4 y_{31}^2 + C_5 x_{31} y_{31} \\
&t_{2,0} = \tfrac{1}{2} n_p (n_p - 1) C_0 + (n_p - 1) C_1 x_{21} + (n_p - 1) C_2 y_{21} + \tfrac{1}{2} C_3 x_{21}^2 + \tfrac{1}{2} C_4 y_{21}^2 + C_5 x_{21} y_{21} \\
&t_{1,1} = n_p (n_p - 1) C_0 + (n_p - 1) C_1 (x_{31} + x_{21}) + (n_p - 1) C_2 (y_{31} + y_{21}) + \\
&\qquad C_3 x_{31} x_{21} + C_4 y_{21} y_{31} + C_5 (x_{21} y_{31} + x_{31} y_{21})
\end{aligned} \tag{23}$$

It should be noted that the constants $C_0, \ldots, C_5$, which are dimensionless, must be combined so that there is dimensional consistency in the terms added in each one of the equations (23).

For $n_p < 5$ all coefficients will be zero, therefore no C^1 shape functions exist. For $n_p \geq 5$ out of the $6n_p - 9$ border coefficients $2n_p + 3$ must be zero, 6 coefficients are given by equations (23) and the remaining $2(2n_p - 9)$ coefficients are computed using the $2n_p - 9$ functions $\hat{F}_3, \ldots, \hat{F}_{n_p-3}$ and $\hat{G}_2, \ldots, \hat{G}_{n_p-3}$.

4 COMBINING SHAPE FUNCTIONS

4.1 Possible combinations of shape functions

In Section 3 we presented general formulas which can be used to derive all possible C^1 polynomial shape functions over the triangle. We see that such shape functions only exist for $n_p \geq 5$ (see also [6]). Note that C^1 continuity is ensured only if the element has rotational symmetry, therefore for each shape function we must also use the two shape functions obtained by cycling the indices of the areal coordinates in the polynomial (7). This means that shape functions can only be used in groups of three, with the exception of internal shape functions such as $(L_1 L_2 L_3)^\beta$, with $\beta \geq 2$.

Any combination of C^1 shape functions that respects rotational symmetry will therefore result into a C^1 element. However, not all such elements will offer a good, or even acceptable, numerical behaviour. It is therefore necessary to consider how C^1 shape functions can be combined to produce finite elements with appropriate properties.

We consider here the case of C^1 elements where all shape functions are polynomials of the same order n_p (by setting some coefficients to zero, this covers also the case where some shape functions have lower order). As already mentioned, a polynomial of order n_p has $m_p = (n_p + 2)(n_p + 1)/2$ coefficients. Since we require that the shape functions should be linearly independent, we can combine at most m_p shape functions. Indeed, any combination of exactly m_p linearly independent shape functions will yield an element that can exactly interpolate any polynomial of order n_p.

It may however be advantageous to use fewer than m_p shape functions. In this case the element can provide exact interpolation only for polynomials up to an order n_e with $n_e < n_p$, where n_e is called the "order of exact interpolation". Once a given combination of shape functions is selected, it is relatively straightforward to determine the order of exact interpolation of the resulting element. However, there is not currently a simple way to determine *a priori* which shape functions should be selected in order to obtain a given order of exact interpolation. A trial-and-error method is therefore used, as seen in Section 5, where the order and number of the shape functions is first selected, and then the highest possible order of exact interpolation is sought by assigning specific values to the undetermined polynomial coefficients of each shape function.

Obviously, we want the order of exact interpolation to be as high as possible while the number of shape functions, and thus degrees of freedom, should be as low as possible. It is not however possible to determine *a priori* the optimum balance between these two contrasting requirements. For this reason, either a theoretical study or a series of numerical tests are required to provide indications as to the relative performance of the resulting different C^1 elements.

4.2 Degrees of freedom

Up to now, the nature of the degrees of freedom used in the element under consideration has not been discussed. Indeed, one useful property of the finite element method is that it is *not* necessary to know this information in order to apply the method. Nonetheless, we are interested in determining the nature of the degrees of freedom corresponding to each node and shape function. In the general case, this can only be achieved when considering the combined shape functions within a given element type and not each one of them individually.

An important special case is obtained when considering the value of a vertex shape function and its derivatives when calculated on the vertex. Consider for example a vertex shape function ϕ for vertex 1. The value of ϕ and its first and second derivatives on vertex 1 are calculated as

$$\phi = C_0, \quad \frac{\partial \phi}{\partial x} = C_1, \quad \frac{\partial \phi}{\partial y} = C_2, \quad \frac{\partial^2 \phi}{\partial x^2} = C_3, \quad \frac{\partial^2 \phi}{\partial y^2} = C_4, \quad \frac{\partial^2 \phi}{\partial x \partial y} = C_5 \tag{24}$$

Additionally, the respective values (always calculated at vertex 1) for vertex shape functions of vertices 2 and 3, as well as for all side and internal shape functions, are all zero. We can therefore set one of the constants $C_0, C_1, \ldots, C_5$ equal to one and the others equal to zero to obtain six shape functions whose corresponding degrees of freedom are the value at the vertex of the function being interpolated and its first and second derivatives.

The nature of these degrees of freedom is not affected by the choice of the remaining shape functions.

The six shape functions mentioned above will depend on some indeterminate functions $\hat{F}$ and $\hat{G}$, whose value however does not appear in equation (24) and thus it does not affect the nature of the respective degree of freedom.

5 SPECIFIC ELEMENTS

Since no C^1 polynomial shape functions exist for $n_p < 5$, the first C^1 elements are obtained for $n_p = 5$. Using the formulae developed in Section 3, we see that in this case there exist no internal shape functions, while each side and vertex shape functions depends on a single indeterminate function ($\bar{G}_2$ and $\hat{G}_2$ respectively).

Side shape functions have only a single non-zero coefficient. For side 1 this coefficient is $t_{2,2}$ so that the corresponding shape function is

$$\bar{\phi}_1 = -\frac{1}{\kappa_2 + \kappa_3} \bar{G}_2(x_{32}, y_{32}) L_1 L_2^2 L_3^2 \tag{25}$$

where $\bar{G}_2(x_{32}, y_{32}) = -\bar{G}_2(-x_{32}, -y_{32})$. By using different functions $\bar{G}_2$ we obtain different shape functions for side 1, which are however linearly dependent. Therefore for each side we actually obtain only one shape function, for a total of three side shape functions for the element. These three shape functions are linearly independent, provided that $\bar{G}_2(x, y) \neq 0$.

Vertex shape functions have 8 non-zero coefficients. Each vertex shape function depends on an indeterminate function $\hat{G}_2$ and six coefficients $C_0, \ldots, C_5$. By setting one of the coefficients $C_0,\ldots,C_5$ equal to one and all other coefficients equal to zero we obtain six shape functions for each vertex, with the corresponding degrees of freedom being the value at the vertex of the function being interpolated and its first and second derivatives, as mentioned in Section 4.2. These six shape functions can be written as

$$\begin{aligned}
\hat{\phi}_1^{(1)} &= L_1^3(6L_1^2-15L_1+10)+30(v_2+v_3)+g^{(1)}\\
\hat{\phi}_1^{(2)} &= L_1^3(4-3L_1)(L_3x_{31}+L_2x_{21})+12(x_{31}v_3+x_{21}v_2)+g^{(2)}\\
\hat{\phi}_1^{(3)} &= L_1^3(4-3L_1)(L_3y_{31}+L_2y_{21})+12(y_{31}v_3+y_{21}v_2)+g^{(3)}\\
\hat{\phi}_1^{(4)} &= (1/2)L_1^3(L_3x_{31}+L_2x_{21})^2+(3/2)(x_{31}^2v_3+x_{21}^2v_2)+g^{(4)}\\
\hat{\phi}_1^{(5)} &= (1/2)L_1^3(L_3y_{31}+L_2y_{21})^2+(3/2)(y_{31}^2v_3+y_{21}^2v_2)+g^{(5)}\\
\hat{\phi}_1^{(6)} &= L_1^3(L_3x_{31}+L_2x_{21})(L_3y_{31}+L_2y_{21})+3(x_{31}y_{31}v_3+x_{21}y_{21}v_2)+g^{(6)}
\end{aligned} \tag{26}$$

where

$$v_2 = L_1^2L_2^2L_3\frac{\kappa_2}{\kappa_1+\kappa_2}, \qquad v_3 = L_1^2L_2L_3^2\frac{\kappa_3}{\kappa_1+\kappa_3} \tag{27}$$

and

$$g^{(c)} = L_1^2L_2^2L_3\frac{\hat{G}_2^{(c)}(x_{21},y_{21})}{\kappa_1+\kappa_2} - L_1^2L_2L_3^2\frac{\hat{G}_2^{(c)}(x_{31},y_{31})}{\kappa_1+\kappa_3}, \qquad c=1,\ldots,6 \tag{28}$$

An additional shape function can be obtained for each vertex by setting all coefficients $C_0,\ldots,C_5$ equal to zero. The resulting shape function for vertex 1 is then

$$\hat{\phi}_1^{(7)} = -\frac{1}{\kappa_1+\kappa_3}\hat{G}_2^{(7)}(x_{31},y_{31})L_1^2L_2L_3^2 + \frac{1}{\kappa_1+\kappa_2}\hat{G}_2^{(7)}(x_{21},y_{21})L_1^2L_2^2L_3 \tag{29}$$

The vertex shape functions $\hat{\phi}^{(7)}$ and the side shape functions $\bar{\phi}$ are not linearly independent (for example, $\hat{\phi}_1^{(7)}$ is a linear combination of $\overline{\phi}_2$ and $\overline{\phi}_3$). Therefore it is not possible to use both types of shape functions together. Moreover, the function $\hat{G}_2^{(7)}$ must satisfy the inequality $\hat{G}_2^{(7)}(x,y) \neq \hat{G}_2^{(7)}(-x,-y)$, otherwise the three shape functions $\hat{\phi}_1^{(7)}$, $\hat{\phi}_2^{(7)}$ and $\hat{\phi}_3^{(7)}$ are not linearly independent (their sum is zero).

Given the above results, we can now proceed to combine the individual shape functions of polynomial order $n_p = 5$ to obtain specific elements. Any element with 21 linearly independent shape functions will exactly interpolate any fifth-order polynomial. There are only two ways to obtain such an element: either by using all seven vertex shape functions $\hat{\phi}^{(1)},\ldots,\hat{\phi}^{(7)}$ or by using the six vertex shape functions $\hat{\phi}^{(1)},\ldots,\hat{\phi}^{(6)}$ together with the side shape functions $\overline{\phi}$. In the second case we obtain elements like the TUBA 6 element [1], while in the first case we obtain a new element with fifth-order complete interpolation using exclusively vertex nodes, which we call TRV21C1. Note that these elements depend on seven G_2 functions, but these functions only change the nature of the seventh degree of freedom on each vertex, while the six first degrees of freedom are not affected.

Elements with fewer shape functions can be obtained by requiring a lower order of complete interpolation n_e. While a fourth-order polynomial has 15 independent coefficients, it is not possible to obtain an element with 15 fifth-order shape functions and $n_e = 4$, as in this case the fifth-order shape functions would degenerate to fourth-order ones which, as demonstrated, do not exist. Since the shape functions can only appear in groups of three, an element with $n_p = 5$ and $n_e = 4$ would therefore have 18 shape functions. By trial and error we see that such an element can only be obtained using the vertex functions $\hat{\phi}^{(1)},\ldots,\hat{\phi}^{(6)}$ with specific forms of the functions $\hat{G}_2^{(1)},\ldots,\hat{G}_2^{(6)}$ so that the resulting element is actually the TRF254C1 element [4], a special case of which is the TUBA 3 element [1].

By trial and error we can also determine that there exist no elements with $n_e = 3$ or $n_e = 2$. There exists however an element with $n_e = 1$. This element has nine vertex shape functions, with its degrees of freedom being the value of the interpolated function and its first derivatives at each node, so it is actually a TRF151C1 element [4].

Indicative numerical results for the elements described in this section are given in Figures 3 and 4, using the theory of gradient elasticity [7] and the two benchmark tests described in [4]. It is seen at once that the numerical performance of the TRF151C1 element makes it unsuitable for practical use. The TRV21C1 element, on the other hand, performs very well in both tests (though it is outperformed by the TUBA 3 element) demonstrating how the method proposed in this paper can lead to the formulation of new C^1 elements with good numerical behaviour.

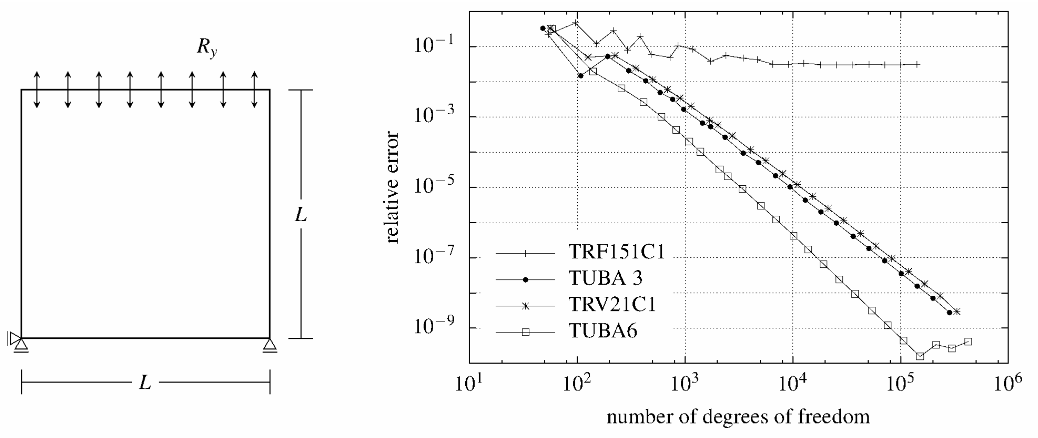

Figure 3: Square gradient-elastic domain with applied double traction: benchmark description and results

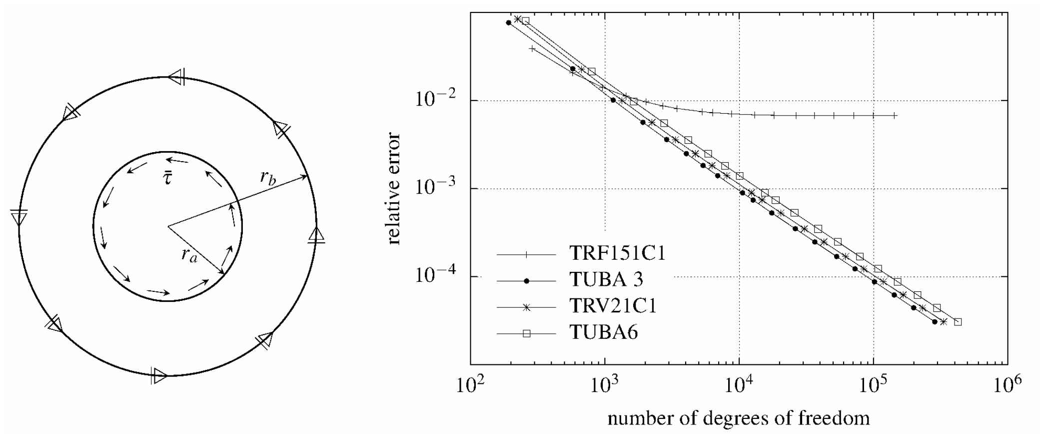

Figure 4: Gradient elastic hollow cylinder with applied tangential traction: benchmark description and results

6 CONCLUSIONS

By considering individual shape functions and the way these match between adjacent elements, we have derived generic formulae for all possible C^1 polynomial shape functions of a given order. These are then used as the building blocks to construct C^1 finite elements with given properties. In the simplest case, involving fifth-order polynomials, existing C^1 elements are successfully obtained, as well as some new elements.

Additionally, since all polynomial C^1 shape functions are described by the formulae presented here, the proposed method also allows us to exclude the existence of other elements, once all possible combinations have been examined. For fifth-order elements, for example, it is seen that we cannot obtain an element which does not use second derivatives as degrees of freedom and yet has good numerical behaviour.

The formulae presented in this paper can be used for any polynomial order of the shape functions. For increasing polynomial orders, we see that the number of indeterminate functions present in the shape functions increases as well, thus widening the range of possible elements but also increasing the effort needed to consider all possible combinations of shape functions.

7 Acknowledgements

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Figure Captions

Figure 1: Geometry of a triangular element

Figure 2: A triangle element with its neighbouring elements

Figure 3: Square gradient-elastic domain with applied double traction: benchmark

description and results

Figure 4: Gradient elastic hollow cylinder with applied tangential traction: benchmark description and results