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ACCELERATED FINITE DIFFERENCE SCHEMES FOR SECOND ORDER DEGENERATE ELLIPTIC AND PARABOLIC PROBLEMS IN THE WHOLE SPACE

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Abstract. We give sufficient conditions under which the convergence of finite difference approximations in the space variable of possibly degenerate second order parabolic and elliptic equations can be accelerated to any given order of convergence by Richardson's method.

1. Introduction

This is the third article of a series studying a class of finite difference equations, related to finite difference approximations in *the space variable* of second order parabolic and elliptic PDEs in $\mathbb{R}^d$. These PDEs are given on the whole $\mathbb{R}^d$ in the space variable, and may degenerate and become first order PDEs. Denote by u_h the solutions of the finite difference equations corresponding to a given grid with mesh-size h. By shifting the grid so that x becomes a grid point we define u_h for all $x \in \mathbb{R}^d$ rather than only at the points of the original grid. In [5] and [6], the first and second articles of the series, we focus on the smoothness in x of u_h, rather than their convergence for $h \to 0$. The main results in [5] and [6] give estimates, independent of h, for the first order derivatives Du_h and for derivatives $D^k u_h$ in x of any order k, respectively.

In the present paper one of our main concerns is the smoothness of the approximations u_h in (x, h). In particular, we are interested in the convergence of u_h, and their *derivatives* in x, in the supremum norm, as $h \to 0$. We give conditions ensuring that for any given integer $k \geq 0$ the approximations u_h admit power series expansions up to order $k+1$ in h near 0 like

$$u_h = \sum_{j=0}^{k} h^j u^{(j)} + h^{k+1} r_h, \tag{1.1}$$

and such that the coefficients are bounded functions of $(t, x) \in [0, T] \times \mathbb{R}^d$ for fixed $T > 0$ in the case of parabolic equations, and, with the exception of r_h, are independent of h. This is Theorem 2.3, our first result on Taylor's

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formula for u_h in h. We obtain it by proving first Theorems 2.1 and 2.2 below on the solvability of the PDE that is being approximated, and of a system of degenerate parabolic PDEs, respectively, for the coefficients $u^{(j)}$, $j = 0, \dots, k$. Of course, $u^{(0)}$ is the true solution of the corresponding PDE. The remainder term r_h satisfies a finite difference equation, with the same difference operator appearing in the equation for u_h, and we estimate r_h by making use of the maximum principle enjoyed by this operator. This is a standard approach to get power series expansions for finite difference approximations in general, and it works well in many situations, when suitable results regarding the equations for the coefficients $u^{(j)}$ are available. In our situation it requires some facts either from the theory of diffusion processes or from the theory of degenerate parabolic equations. However, we do not use any facts from these theories. We prove Theorem 2.1, and hence Theorem 2.2, relying on results on finite difference schemes, obtained in [6] by elementary techniques. It is worth saying that since long ago finite difference equations were already used to prove the solvability of partial differential equations (see, for instance, [8] and [9]). Our contribution lies in considering degenerate equations.

After establishing the expansions of u_h in h not only we obtain the possibility to prove the convergence of u_h to the true solution in the sup norm as $h \to 0$ but also the possibility to accelerate it to any order under appropriate assumptions. We prove the latter by taking linear combinations of finite difference approximations corresponding to different mesh-sizes. This method is especially effective when many of the coefficients in the expansion of u_h are zero. These results are given by Theorem 2.21 and Corollary 2.8. Their counterparts in the elliptic case are presented by Corollary 3.7.

The idea of accelerating the convergence of finite difference approximations in the above way is well-known in numerical analysis. It is due to L.F. Richardson, who showed that it works in some cases and demonstrated its usefulness in [15] and [16]. This method is often called *Richardson's method* or *extrapolation to the limit*, and is applied to various types of approximations. The reader is referred to the survey papers [2] and [4] for a review on the history of the method and on the scope of its applicability and to the textbooks (for instance, [10] and [11]) concerning finite difference methods and their accelerations.

We are interested in approximating in the sup norm not only the true solution but also its derivatives. Note that even if the coefficients $u^{(j)}$ are bounded smooth functions of (t, x), the derivatives $D^k u_h$ of u_h in x may not admit similar expansions, since the derivatives of r_h may not be bounded in h near 0. Note also that the bounds on the sups of $u^{(j)}$ and r_h generally depend on T, and may grow exponentially in T. This becomes a big obstacle on the way of extending our results to the elliptic case.

Our next result on power series expansions, Theorem 2.7, improves the previous theorem in two directions. It gives sufficient conditions such that for any given integer $k \geq 0$

(a) $D^k u_h$ admits an expansion similar to (1.1),

(b) the bounds on the coefficients are independent of T.

Having (a) we can approximate the k-th derivatives of the true solution by $D^k u_h$ with rate of order h and accelerate the rate under appropriate assumptions. We can also approximate the k-th derivatives of the true solution with finite difference operators in place of D^k applied to u_h, which is more convenient in applications because it does not require computing the derivatives of u_h.

We ensure (a) and (b) by relying heavily on derivative estimates, independent of T, obtained in [5] and [6] for solutions of finite difference equations. Property (b) of the expansions allows us to extend Theorem 2.7 to the elliptic case. This extension is Theorem 3.5.

As a consequence of the derivative estimates proved in [6] we obtain also, see Theorem 2.9 below, estimates, independent of h and T, for the derivatives of u_h in x *and* h. Clearly, Theorem 2.9 immediately implies Taylor's formula for u_h in h, up to appropriate order, with bounded coefficients. It is interesting to notice that the converse implication does not hold: If for $k \geq 1$ the function u_h admits a power series expansion up to order $k+1$ in h near 0 with bounded coefficients, it does not imply, in general, that the derivative of u_h in h up to order $k+1$ are bounded functions. That is why Theorem 2.7 does not imply Theorem 2.9, and the latter implies the former only if condition (i) in Theorem 2.7 is satisfied. Additional information on the behaviour of the derivatives of u_h in x and h when h is near 0 is given by Theorem 2.11. The corresponding result in the elliptic case is Theorem 3.4.

In this article we are working with equations in the whole space having in mind considering equations in bounded smooth domains in a subsequent article. Still it may be worth noting that the results of this article are applicable to the one dimensional ODE

$$(1-x^2)^2 u''(x) - c(x)u(x) = f(x), \quad x \in (-1,1).$$

The point is that one need not prescribe any boundary value of u at the points ± 1 and if one considers this equation on all of $\mathbb{R}$, the values of its coefficients and f outside $(-1,1)$ do not affect the values of $u(x)$ for $|x| < 1$.

2. Formulation of the main results for parabolic equations

We fix some numbers $h_0, T \in (0,\infty)$ and for each number $h \in (0, h_0]$ we consider the integral equation

$$u(t,x) = g(x) + \int_0^t \big(L_h u(s,x) + f(s,x)\big)\,ds, \quad (t,x) \in H_T \tag{2.1}$$

for u, where $g(x)$ and $f(s,x)$ are given real-valued Borel functions of $x \in \mathbb{R}^d$ and $(s,x) \in H_T = [0,T] \times \mathbb{R}^d$, respectively, and L_h is a linear operator defined by

$$L_h \varphi(t,x) = L_h^0 \varphi(t,x) - c(t,x)\varphi(x), \tag{2.2}$$

$$L_h^0\varphi(t,x)=\frac{1}{h}\sum_{\lambda\in\Lambda_1}q_\lambda(t,x)\delta_{h,\lambda}\varphi(x)+\sum_{\lambda\in\Lambda_1}p_\lambda(t,x)\delta_{h,\lambda}\varphi(x), \tag{2.3}$$

for functions φ on $\mathbb{R}^d$. Here Λ_1 is a finite subset of $\mathbb{R}^d$ such that $0\notin\Lambda_1$,

$$\delta_{h,\lambda}\varphi(x)=\frac{1}{h}(\varphi(x+h\lambda)-\varphi(x)),\quad \lambda\in\Lambda_1,$$

$q_\lambda(t,x)\geq 0$, $p_\lambda(t,x)$, and $c(t,x)$ are given real-valued Borel functions of $(t,x)\in H_\infty=[0,\infty)\times\mathbb{R}^d$ for each $\lambda\in\Lambda_1$. Set $|\Lambda_1|^2=\sum_{\lambda\in\Lambda_1}|\lambda|^2$.

As usual, we denote

$$D^\alpha=D_1^{\alpha_1}...D_d^{\alpha_d},\quad D_i=\frac{\partial}{\partial x_i},\quad |\alpha|=\sum_i\alpha_i,\quad D_{ij}=D_iD_j$$

for multi-indices $\alpha=(\alpha_1,\dots\alpha_d)$, $\alpha_i\in\{0,1,\dots\}$. For smooth φ and integers $k\geq 0$ we introduce $D^k\varphi$ as the collection of partial derivatives of φ of order k, and define

$$|D^k\varphi|^2=\sum_{|\alpha|=k}|D^\alpha\varphi|^2,\quad [\varphi]_k=\sup_{x\in\mathbb{R}^d}|D^k\varphi(x)|,\quad |\varphi|_k=\sum_{i\leq k}[\varphi]_i.$$

For functions ψ_h depending on $h\in(0,h_0]$ the notation $D_h^k\psi_h$ means the k-th derivative of ψ in h. For Borel measurable bounded functions $\psi=\psi(t,x)$ on H_T we write $\psi\in\mathfrak{B}^m=\mathfrak{B}_T^m$ if, for each $t\in[0,T]$, $\psi(t,x)$ is continuous in $\mathbb{R}^d$ and for all multi-indices α with $|\alpha|\leq m$ the generalized functions $D^\alpha\psi(t,x)$ are bounded on H_T. In this case we use the notation

$$\|\psi\|_m^2=\sup_{H_T}\sum_{|\alpha|\leq m}|D^\alpha\psi(t,x)|^2.$$

This notation will be also used for functions ψ independent of t.

Let $m\geq 0$ be a fixed integer. We make the following assumptions.

Assumption 2.1. For any $\lambda\in\Lambda_1$, we have $p_\lambda,q_\lambda,c,f,g\in\mathfrak{B}^m$ and, for $k=0,...,m$ and some constants M_k we have

$$\sup_{H_T}\Big(\sum_{\lambda\in\Lambda_1}(|D^kq_\lambda|^2+|D^kp_\lambda|^2)+|D^kc|^2\Big)\leq M_k^2. \tag{2.4}$$

Remark 2.1. By Theorem 2.3 of [5] under Assumption 2.1 for each $h\in(0,h_0]$, there exists a unique bounded solution u_h of (2.1), this solution is continuous in H_T, and all its derivatives in x up to order m are bounded. Actually, in Theorem 2.3 of [5] it is required that the derivatives of the data up to order m be continuous in H_T, but its proof can be easily adjusted to include our case (see Remark 2.6 below).

Naturally, we view (2.1) as a finite difference schemes for the problem

$$\frac{\partial}{\partial t}u(t,x)=\mathcal{L}u(t,x)+f(t,x),\quad t\in(0,T],\,x\in\mathbb{R}^d, \tag{2.5}$$

$$u(0,x)=g(x),\quad x\in\mathbb{R}^d, \tag{2.6}$$

where

$$\mathcal{L} := \tfrac{1}{2} \sum_{\lambda\in\Lambda_1} \sum_{i,j=1}^{d} q_\lambda \lambda_i \lambda_j D_i D_j + \sum_{\lambda\in\Lambda_1} \sum_{i=1}^{d} p_\lambda \lambda_i D_i - c. \tag{2.7}$$

By a solution of (2.5)-(2.6) we mean a bounded continuous function $u(t,x)$ on H_T, such that it belongs to $\mathfrak{B}^2$ and satisfies

$$u(t,x) = g(x) + \int_0^t [\mathcal{L}u(s,x) + f(s,x)]\,ds \tag{2.8}$$

in H_T in the sense of generalized functions, that is, for any $t \in [0,T]$ and $\phi \in C_0^\infty(\mathbb{R}^d)$

$$\int_{\mathbb{R}^d} \phi(x)u(t,x)\,dx = \int_{\mathbb{R}^d} \phi(x)g(x)\,dx + \int_0^t \int_{\mathbb{R}^d} \phi(-cu+f)(s,x)\,dxds$$
$$+ \int_0^t \int_{\mathbb{R}^d} \phi \sum_{\lambda\in\Lambda_1} \big(\tfrac{1}{2} \sum_{i,j=1}^{d} q_\lambda \lambda_i \lambda_j D_i D_j u + \sum_{i=1}^{d} p_\lambda \lambda_i D_i u\big)(s,x)\,dxds. \tag{2.9}$$

Observe that if $u \in \mathfrak{B}^2$, then (2.9) implies that (2.8) holds almost everywhere with respect to x and if $u \in \mathfrak{B}^3$ then the second derivatives of u in x are continuous in x and (2.8) holds everywhere.

The reader can find in [7] a discussion showing that in all practically interesting cases of parabolic equations like (2.8) the operator $\mathcal{L}$ can be represented as in (2.7), so that considering operators L_h^0 in form (2.3) is rather realistic.

The following theorem on existence and uniqueness of solutions is a classical result (see, for instance, [12], [13], [14]) which we are going to obtain by using finite-difference approximations.

Theorem 2.1. *Let Assumption 2.1 hold with $m \geq 2$. Then equation* (2.8) *has a unique solution $u^{(0)} \in \mathfrak{B}^2 = \mathfrak{B}^2_T$. Moreover, $u^{(0)} \in \mathfrak{B}^m_T$ and*

$$\|u^{(0)}\|_m \leq N(\|f\|_m + \|g\|_m), \tag{2.10}$$

where N is a constant, depending only on d, m, $|\Lambda_1|$, $M_0,\dots,$ M_m, and T.

Observe that this result is rather sharp in what concerns the smoothness of solutions, which is seen if all the coefficients of $\mathcal{L}$ are identically zero and f is independent of t in which case the solution is $tf(x) + g(x)$.

The existence part in Theorem 2.1 is proved in Section 6 and uniqueness in Section 4.

In Section 6 a repeated application of this theorem allows us to prove a result on the solvability of (2.13) below. First introduce

$$\mathcal{L}^{(i)} := \tfrac{1}{(i+1)(i+2)} \sum_{\lambda\in\Lambda_1} q_\lambda \partial_\lambda^{i+2} + \tfrac{1}{i+1} \sum_{\lambda\in\Lambda_1} p_\lambda \partial_\lambda^{i+1}, \tag{2.11}$$

where

$$\partial_\lambda \varphi := \sum_i \lambda_i D_i \varphi \tag{2.12}$$

is the derivative of φ in the direction of λ. Consider the system of equations

$$u^{(j)}(t,x)=\int_0^t\big(\mathcal{L}u^{(j)}(s,x)+\sum_{i=1}^{j}C_j^i\mathcal{L}^{(i)}u^{(j-i)}(s,x)\big)\,ds, \tag{2.13}$$

$(t,x)\in H_T$, $j=1,\dots,k$.

Remark 2.2. Quite often in the article we use the following symmetry condition:

(S) $\Lambda_1=-\Lambda_1$ and $q_\lambda=q_{-\lambda}$ for all $\lambda\in\Lambda_1$.

Notice that, if condition (S) holds, then

$$h^{-1}\sum_{\lambda\in\Lambda_1}q_\lambda(t,x)\delta_{h,\lambda}\varphi(x)=(1/2)\sum_{\lambda\in\Lambda_1}q_\lambda(t,x)\Delta_{h,\lambda}\varphi(x),$$

where

$$\Delta_{h,\lambda}\varphi(x)=h^{-2}(\varphi(x+h\lambda)-2\varphi(x)+\varphi(x-h\lambda)).$$

Theorem 2.2. *Let $k\geq 1$ be an integer. (i) If Assumption 2.1 is satisfied with $m\geq 3k+2$, then* (2.13) *has a unique solution $\{u^{(j)}\}_{j=1}^k$, such that*

$$u^{(j)}\in\mathfrak{B}^{m-3j},\quad \|u^{(j)}\|_{m-3j}\leq N(\|f\|_m+\|g\|_m) \tag{2.14}$$

for $j=1,\dots,k$.

(ii) If the symmetry condition (S) holds and Assumption 2.1 is satisfied with $m\geq 2k+2$, then (2.13) *has a unique solution $\{u^{(j)}\}_{j=1}^k$, such that*

$$u^{(j)}\in\mathfrak{B}^{m-2j},\quad \|u^{(j)}\|_{m-2j}\leq N(\|f\|_m+\|g\|_m) \tag{2.15}$$

for $j=1\dots,k$. In addition, if

$$p_{-\lambda}=-p_\lambda,\quad \textit{for } \lambda\in\Lambda_1, \tag{2.16}$$

then

$$u^{(j)}=0, \tag{2.17}$$

for odd numbers $j\leq k$.

In all cases the constants N depends only on d, m, $|\Lambda_1|$, $M_0,\dots,M_m$, and T.

The next series of results is related to the possibility of expansion

$$u_h(t,x)=u^{(0)}(t,x)+\sum_{1\leq j\leq k}\frac{h^j}{j!}u^{(j)}(t,x)+h^{k+1}r_h(t,x), \tag{2.18}$$

for all $(t,x)\in H_T$ and $h\in(0,h_0]$, where u_h is the unique bounded solution of (2.1) (see Remark 2.1) and r_h is a function on H_T defined for each $h\in(0,h_0]$ such that

$$|r_h(t,x)|\leq N(\|f\|_m+\|g\|_m) \tag{2.19}$$

for all $(t,x)\in H_T$, $h\in(0,h_0]$.

Introduce

$$\chi_{h,\lambda}=q_\lambda+hp_\lambda.$$

Assumption 2.2. For all $(t,x) \in H_T$, $h \in (0,h_0]$, and $\lambda \in \Lambda_1$,

$$\chi_{h,\lambda}(t,x) \geq 0. \tag{2.20}$$

Assumption 2.3. We have

$$\sum_{\lambda \in \Lambda_1} \lambda q_\lambda(t,x) = 0 \quad \text{for all } (t,x) \in H_T.$$

Notice that condition (S) is stronger than Assumption 2.3.

Theorem 2.3. *Let Assumption 2.1 with $m \geq 3$ and Assumption 2.2 hold. Let $k \geq 0$ be an integer. Then expansion* (2.18) *holds with r_h satisfying* (2.19)*, provided one of the following conditions is met:*

(i) $m \geq 3k+3$ and Assumption 2.3 holds;

(ii) $m \geq 2k+3$ and condition (S) holds;

(iii) k is odd, $m \geq 2k+2$, and conditions (S) and (2.16) *are satisfied.*

In each of the cases (i)-(iii) the constant N depends only on d, m, $|\Lambda_1|$, $M_0, \dots, M_m$, and T. In case (iii) we have $u^{(j)} = 0$ for all odd j in expansion (2.18).

We prove this theorem in Section 7. The following corollary is one of the results of [3] proved there by using the theory of diffusion processes. We obtain it immediately from case (iii) with $k = 1$. Of course, the result is well known for uniformly nondegenerate equations but we do not assume any nondegeneracy of $\mathcal{L}$, which becomes just a zero operator at those points where $q_\lambda = p_\lambda = c = 0$.

Corollary 2.4. *Let conditions (S) and* (2.16) *be satisfied. Let Assumption 2.1 with $m = 4$ and Assumption 2.2 hold. Then we have $|u_h - u_0| \leq Nh^2$.*

Actually, in [3] a full discretization in time and space is considered for parabolic equations, so that, formally, Corollary 2.4 does not yield the corresponding result of [3]. On the other hand, a similar corollary can be derived from Theorem 3.5 below which treats elliptic equations and it does imply the corresponding result of [3]. It also generalizes it because in [3] one of the assumptions, unavoidable for the methods used there, is that $q_\lambda = r_\lambda^2$ with functions r_λ that have four bounded derivatives in x, which may easily be not the case under the assumptions of Theorem 3.5.

To formulate our main result about acceleration for parabolic equations we fix an integer $k \geq 0$ and set

$$\bar{u}_h = \sum_{j=0}^{k} b_j u_{2^{-j}h},, \tag{2.21}$$

where, naturally, $u_{2^{-j}h}$ are the solutions to (2.1), with $2^{-j}h$ in place of h,

$$(b_0, b_1, ..., b_k) := (1, 0, 0, ..., 0)V^{-1} \tag{2.22}$$

and V^{-1} is the inverse of the Vandermonde matrix with entries

$$V^{ij} := 2^{-(i-1)(j-1)}, \quad i,j = 1, ..., k+1.$$

The following result is a simple corollary of Theorem 2.3.

Theorem 2.5. *In each situation when Theorem 2.3 is applicable we have that the estimate*

$$|\bar{u}_h(t,x) - u^{(0)}(t,x)| \le N(\|f\|_m + \|g\|_m)h^{k+1} \tag{2.23}$$

holds for all $(t,x) \in H_T$, $h \in (0, h_0]$, *where* N *is a constant depending only on* d, m, $|\Lambda_1|$, $M_0, \dots, M_m$, *and* T.

Proof. By Theorem 2.3

$$u_{2^{-j}h} = u^{(0)} + \sum_{i=1}^{k} \frac{h^i}{i! 2^{ji}} u^{(i)} + \bar{r}_{2^{-j}h} h^{k+1}, \quad j = 0, 1, ..., k,$$

with $\bar{r}_{2^{-j}h} := 2^{-j(k+1)} r_{2^{-j}h}$, which gives

$$\bar{u}_h = \sum_{j=0}^{k} b_j u_{2^{-j}h} = (\sum_{j=0}^{k} b_j) u^{(0)} + \sum_{j=0}^{k} \sum_{i=1}^{k} b_j \frac{h^i}{i! 2^{ij}} u^{(i)} + \sum_{j=0}^{k} b_j \bar{r}_{2^{-j}h} h^{k+1}$$

$$= u^{(0)} + \sum_{i=1}^{k} \frac{h^i}{i!} u^{(i)} \sum_{j=0}^{k} \frac{b_j}{2^{ij}} + \sum_{j=0}^{k} b_j \bar{r}_{2^{-j}h} = u^{(0)} + \sum_{j=0}^{k} b_j \bar{r}_{2^{-j}h} h^{k+1},$$

since

$$\sum_{j=0}^{k} b_j = 1, \quad \sum_{j=0}^{k} b_j 2^{-ij} = 0, \quad i = 1, 2, ...k$$

by the definition of $(b_0, ..., b_k)$. Hence,

$$\sup_{H_T} |\bar{u}_h - u^{(0)}| = \sup_{H_T} |\sum_{j=0}^{k} b_j \bar{r}_{2^{-j}h}| h^{k+1} \le N(\|f\|_m + \|g\|_m) h^{k+1},$$

and the theorem is proved. □

Sometimes it suffices to combine fewer terms $u_{2^{-j}h}$ to get accuracy of order $k+1$. To consider such a case for odd integers $k \ge 1$ define

$$\tilde{u}_h = \sum_{j=0}^{\tilde{k}} \tilde{b}_j u_{2^{-j}h}\,, \tag{2.24}$$

where

$$(\tilde{b}_0, \tilde{b}_1, ..., \tilde{b}_{\tilde{k}}) := (1, 0, 0, ..., 0)\tilde{V}^{-1}, \quad \tilde{k} = \tfrac{k-1}{2}, \tag{2.25}$$

and $\tilde{V}^{-1}$ is the inverse of the Vandermonde matrix with entries

$$\tilde{V}^{ij} := 4^{-(i-1)(j-1)}, \quad i, j = 1, ..., \tilde{k}+1.$$

Theorem 2.6. *Suppose that the assumptions of Theorem 2.3 are satisfied and condition (iii) is met. Then for* $\tilde{u}_h$ *we have*

$$\sup_{H_T} |u^{(0)} - \tilde{u}_h| \le N(\|f\|_m + \|g\|_m) h^{k+1}$$

for all $h \in (0, h_0]$, *where* N *depends only on* d, m, $|\Lambda_1|$, $M_0, \dots, M_m$, *and* T.

Proof. We obtain this result from Theorem 2.3 by a straightforward modification of the proof of the previous result, taking into account that for odd j the terms with h^j vanish in expansion (2.18) when condition (iii) holds in Theorem 2.3. □

Example 2.1. Assume that in the situation of Theorem 2.6 we have $m = 8$. Then

$$\tilde{u}_h := \tfrac{4}{3}u_{h/2} - \tfrac{1}{3}u_h$$

satisfies

$$\sup_{H_T} |u^{(0)} - \tilde{u}_h| \le Nh^4$$

for all $h \in (0, h_0]$.

The above results show that if the data in equation (2.8) are sufficiently smooth, then the order of accuracy in approximating the solution $u^{(0)}$ can be as high as we wish if we use suitable mixtures of finite difference approximations calculated along nested grids with different mesh-sizes. Assume now that we need to approximate not only $u^{(0)}$ but its derivative $D^\alpha u^{(0)}$ for some multi-index α as well. What accuracy can we achieve? The answer is closely related to the question whether the expansion

$$D^\alpha u_h(t,x) = D^\alpha u^{(0)}(t,x) + \sum_{1\le j\le k} \frac{h^j}{j!} D^\alpha u^{(j)}(t,x) + h^{k+1} D^\alpha r_h(t,x) \quad (2.26)$$

holds for all $(t, x) \in H_T$ and $h \in (0, h_0]$, such that

$$|D^\alpha r_h(t,x)| \le N(\|f\|_m + \|g\|_m) \quad (2.27)$$

for all $(t, x) \in H_T$, $h \in (0, h_0]$.

The result concerning this expansion and the following series of results appeared after the authors tried to extend the above theorems from the parabolic to the elliptic case. The main and rather hard obstacle is that the constants in our estimates depend on T and, actually, may grow exponentially in T. By the way, this obstacle is caused by possible degeneration of our equations and exists even if we consider equations in bounded smooth domain.

To be able to give some conditions under which this does not happen, we introduce new notation and investigate smoothness properties of u_h with respect to x. As a simple byproduct of this investigation we also obtain smoothness of u_h with respect to h, which, by the way, cannot be derived from (2.18).

Take a function τ_λ defined on Λ_1 taking values in $[0, \infty)$ and for $\lambda \in \Lambda_1$ introduce the operators

$$T_{h,\lambda}\varphi(x) = \varphi(x + h\lambda), \quad \bar{\delta}_{h,\lambda} = \tau_\lambda h^{-1}(T_{h,\lambda} - 1).$$

Set

$$\|\Lambda_1\|^2 = \sum_{\lambda\in\Lambda_1} |\tau_\lambda \lambda|^2.$$

For uniformity of notation we also introduce Λ_2 as the set of fixed distinct vectors $\ell^1, ..., \ell^d$ none of which is in Λ_1 and define

$$\bar{\delta}_{h,\ell^i} = \tau_0 D_i, \quad T_{h,\ell^i} = 1, \quad \Lambda = \Lambda_1 \cup \Lambda_2,$$

where $\tau_0 > 0$ is a fixed parameter. For $\lambda = (\lambda^1, \lambda^2) \in \Lambda^2$ introduce the operators

$$T_{h,\lambda} = T_{h,\lambda^1} T_{h,\lambda^2}, \quad \bar{\delta}_{h,\lambda} = \bar{\delta}_{h,\lambda^1} \bar{\delta}_{h,\lambda^2}.$$

For $k = 1, 2$, $\mu \in \Lambda^k$ we set

$$Q_{h,\mu}\varphi = h^{-1} \sum_{\lambda \in \Lambda_1} (\bar{\delta}_{h,\mu} q_\lambda) \delta_\lambda \varphi, \quad L^0_{h,\mu}\varphi = Q_{h,\mu}\varphi + \sum_{\lambda \in \Lambda_1} (\bar{\delta}_{h,\mu} p_\lambda) \delta_\lambda \varphi,$$

$$A_h(\varphi) = 2 \sum_{\lambda \in \Lambda} (\bar{\delta}_{h,\lambda}\varphi) L^0_{h,\lambda} T_{h,\lambda}\varphi, \quad \mathcal{Q}_h(\varphi) = \sum_{\lambda \in \Lambda_1} \chi_{h,\lambda} (\delta_{h,\lambda}\varphi)^2.$$

Below $B(\mathbb{R}^d)$ is the set of bounded Borel functions on $\mathbb{R}^d$ and $\mathfrak{K}$ is the set of bounded operators $\mathcal{K}_h = \mathcal{K}_h(t)$ mapping $B(\mathbb{R}^d)$ into itself preserving the cone of nonnegative functions and satisfying $\mathcal{K}_h 1 \leq 1$.

Finally, fix some constants $\delta \in (0, 1]$ and $K \in [1, \infty)$.

Assumption 2.4. There exists a constant $c_0 > 0$ such that $c \geq c_0$.

Remark 2.3. The above assumption is almost irrelevant if we only consider (2.1) on a finite time interval. Indeed, if c is just bounded, say $|c| \leq C =$ const, by introducing a new function $v(t, x) = u(t, x)e^{-2Ct}$ we will have an equation for v similar to (2.1) with $L^0_h v - (c + 2C)v$ and fe^{-2Ct} in place of $L_h u$ and f, respectively. Now for the new c we have $c + 2C \geq C$.

Assumption 2.5. We have $m \geq 1$ and for any $h \in (0, h_0]$, there exists an operator $\mathcal{K}_h = \mathcal{K}_{h,m} \in \mathfrak{K}$, such that

$$mA_h(\varphi) \leq (1 - \delta) \sum_{\lambda \in \Lambda} \mathcal{Q}_h(\bar{\delta}_{h,\lambda}\varphi) + K\mathcal{Q}_h(\varphi) + 2(1 - \delta)c\mathcal{K}_h\big(\sum_{\lambda \in \Lambda} |\bar{\delta}_{h,\lambda}\varphi|^2\big) \tag{2.28}$$

on H_T for all smooth functions φ.

Assumption 2.6. We have $m \geq 2$ and, for any $h \in (0, h_0]$ and $n = 1, ..., m$, there exists an operator $\mathcal{K}_h = \mathcal{K}_{h,n} \in \mathfrak{K}$, such that

$$n \sum_{\nu \in \Lambda} A_h(\bar{\delta}_{h,\nu}\varphi) + n(n - 1) \sum_{\lambda \in \Lambda^2} (\bar{\delta}_{h,\lambda}\varphi) Q_{h,\lambda} T_{h,\lambda}\varphi \leq (1 - \delta) \sum_{\lambda \in \Lambda^2} \mathcal{Q}_h(\bar{\delta}_{h,\lambda}\varphi)$$

$$+ K \sum_{\lambda \in \Lambda} \mathcal{Q}_h(\bar{\delta}_{h\lambda}\varphi) + 2(1 - \delta)c\mathcal{K}_h\big(\sum_{\lambda \in \Lambda^2} |\bar{\delta}_{h,\lambda}\varphi|^2\big) + K\mathcal{K}\big(\sum_{\lambda \in \Lambda} |\bar{\delta}_{h,\lambda}\varphi|^2\big) \tag{2.29}$$

on H_T for all smooth functions φ.

Obviously Assumptions 2.5 and 2.6 are satisfied if q_λ and p_λ are independent of x. In the general case, as it is discussed in [5], the above assumptions impose not only analytical conditions, but they are related also to some structural conditions, which can somewhat easier be analized under the symmetry condition (S).

Assumption 2.7. For all $t \in [0,T]$

$$\sum_{\lambda\in\Lambda_1} \lambda q_\lambda(t,x) \quad \text{is independent of} \quad x. \tag{2.30}$$

In the main case of applications we will require the last sum to be identically zero as in Assumption 2.3.

Remark 2.4. Assumptions 2.5 and 2.6 are discussed at length and in many details in [5] and [6], and sufficient conditions, without involving test functions φ are given for these assumptions to be satisfied. In particular, it is shown in [6] that if condition (S) holds, $m \geq 2$, $\tau_\lambda = 1$, Assumptions 2.1 and 2.2 are satisfied, and $q_\lambda \geq \kappa$ for a constant $\kappa > 0$, then both Assumptions 2.5 and 2.6 are satisfied for any $c_0 > 0$ and $\delta \in (0,1)$, if h_0 is sufficiently small and τ_0, K, and $\mathcal{K}$ are chosen appropriately. Moreover, the condition $\kappa > 0$ can be dropped, provided, additionally, that c_0 is large enough (this time we need not assume that h is small). Remember, that by Remark 2.3 the condition that c_0 be large is, actually, harmless as long as we are concerned with equations on a finite time interval. Mixed situations, when c is large at those points where some of q_λ can vanish are also considered in [6].

In [5] we have seen that Assumption 2.5 imposes certain nontrivial *structural* conditions on q_λ which cannot be guaranteed by the size of c_0 if q_λ is only once continuously differentiable. In contrast, even without condition (S), given that Assumptions 2.1, 2.5, 2.7 are satisfied and $m \geq 2$, as is shown in [6], Assumption 2.6 is also satisfied if c_0 is large enough.

Theorem 2.7. *Let Assumption 2.1 through 2.6 hold with $m \geq 3$. Let $k \geq 0$ and $l \in [0,m]$ be integers. Then for every multi-index α such that $|\alpha| \leq l$ the function $D^\alpha u_h$ is a continuous function on H_T and expansion* (2.26) *holds with $D^\alpha r_h$ satisfying* (2.27)*, provided one of the following conditions is met:*

(i) $m \geq 3k + 3 + l$;

(ii) $m \geq 2k + 3 + l$ and condition (S) holds;

(iii) k is odd, $m \geq 2k + 2 + l$, and conditions (S) and (2.16) *are satisfied.*

In each of the cases (i)-(iii) the constant N depends only on d, m, δ, K, τ_0, c_0, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0,\dots,M_m$. In case (iii) we have $u^{(j)} = 0$ for all odd j in the expansion.

We prove this theorem in Section 7. Remember the definition of $\bar{u}_h$ and $\tilde{u}_h$ in (2.21) and (2.24). The following is an obvious consequence of Theorem 2.7.

Corollary 2.8. *Suppose that the assumptions of Theorem 2.7 are satisfied. Then*

$$\sup_{H_T} |D^\alpha \bar{u}_h - D^\alpha u^{(0)}| \leq N h^{k+1}(\|f\|_m + \|g\|_m),$$

and if condition (iii) is met then

$$\sup_{H_T} |D^\alpha \tilde{u}_h - D^\alpha u^{(0)}| \leq N h^{k+1}(\|f\|_m + \|g\|_m),$$

where N depends only on on d, m, δ, K, τ_0, c_0, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0,\dots,M_m$.

Remark 2.5. Observe that for $k=0$ Theorem 2.7 implies that

$$\sup_{H_T}|D^\alpha u_h - D^\alpha u^{(0)}| \le Nh \tag{2.31}$$

if $m \ge 3+|\alpha|$ and Assumption 2.1 through 2.6 hold. In addition one can replace $D^\alpha u_h$ in (2.31) with δ^α_h, where

$$\delta^\alpha_h = \delta^{\alpha_1}_{h,e_1}\cdot \ldots \cdot \delta^{\alpha_d}_{h,e_d}$$

and e_i is the ith basis vector in $\mathbb{R}^d$. This follows easily from the mean value theorem and Theorem 2.9 below. The reader understands that similar assertion is true in case of Corollary 2.8 with the only difference that one needs larger m and better finite-difference approximations of D^α.

Next we investigate the smoothness of u_h in x and h. Recall that for functions φ depending on h we use the notation $D^r_h\varphi$ for the r-th derivative of φ in h. As usual, $D^0_h\varphi := \varphi$.

Remark 2.6. Suppose that Assumption 2.1 is satisfied. Take an $h_1 \in (0,h_0)$, consider equation (2.1) as an equation about a function $u_h(t,x)$ as function of $(h,t,x)\in[h_1,h_0]\times H_T$ and look for solutions in the space $\mathfrak{B}^m(h_1)=\mathfrak{B}^m_T(h_1)$ which is defined as the space of functions on $[h_1,h_0]\times H_T$ with finite norm

$$\sum_{|\alpha|+3r\le m}\ \sup_{[h_1,h_0]\times H_T}|D^\alpha D^r_h u_h(t,x)|. \tag{2.32}$$

It is obvious that the integrand in (2.1) can be considered as the result of application of an operator, which is bounded in $\mathfrak{B}^m(h_1)$, to $u_h(s,x)$. Therefore, a standard abstract theorem on solvability of ODEs in Banach spaces shows that there exists a solution of (2.1) in $\mathfrak{B}^m(h_1)$. Since just bounded solutions are uniquely defined by (2.1), we conclude that our u_h belongs to $\mathfrak{B}^m(h_1)$ for any $h_1\in(0,h_0)$. Obviously, if the derivatives of the data are continuous in x, the same will hold for u_h.

The above argument, actually, works if we replace $|\alpha|+3r\le m$ with $|\alpha|+r\le m$ in (2.32). We talk about (2.32) in the above form because we will show that under our future assumptions the quantity (2.32) is bounded independently of h_1.

Theorem 2.9. *Let $k\ge 0$ and $m\ge 2$ be integers and suppose that Assumptions 2.1 through 2.6 are satisfied. Then, for each integer $r\ge 0$ such that*

$$3k+r\le m,$$

the generalized derivatives $D^r D^k_h u_h$ exist on $(0,h_0]\times H_T$, are bounded and we have

$$|D^r D^k_h u_h| \le N(\|f\|_m+\|g\|_m), \tag{2.33}$$

where N is a constant depending only on m,δ,c_0,τ_0,K, M_0, ..., M_m, $|\Lambda_1|$, and $\|\Lambda_1\|$. In particular, $u_h\in\mathfrak{B}^m$ and

$$\|u_h\|_m \le N(\|f\|_m+\|g\|_m).$$

We prove this theorem in Section 5, and in Section 6 we show that the following fact, used when we come to the elliptic case, is a simple corollary of it.

Theorem 2.10. *Suppose that Assumptions 2.1 through 2.6 hold with* $m \geq 2$*. Then the constant* N *in* (2.10) *depends only on* $m, \delta, c_0, \tau_0, K$*,* M_0*, ...,* M_m*,* $|\Lambda_1|$*, and* $\|\Lambda_1\|$ *(thus, is independent of* T*). The same is true for the constants* N *in Theorems 2.2, 2.3, 2.5, and 2.6.*

Additional information on the behavior of $D^r D_h^k u_h$ for small h is provided by the following result which we prove in Section 5.

Theorem 2.11. *Let* $k \geq 1$ *be an* odd *number and suppose that Assumptions 2.1 through 2.6 hold with* $m \geq 3k + 1$*. Assume that the symmetry condition (S) and* (2.16) *are satisfied.*

Then, for any integer $r \geq 0$ *such that*

$$3k + r \leq m - 1$$

we have

$$\sup_{H_T} |D^r D_h^k u_h| \leq N(\|f\|_m + \|g\|_m)h \tag{2.34}$$

for all $h \in (0, h_0]$*, where* N *depends only on* m*,* δ*,* c_0*,* τ_0*,* K*,* $|\Lambda_1|$*,* $\|\Lambda_1\|$*,* M_0*,...,* M_m*.*

3. Main results for elliptic equations

Here we assume that p_λ, q_λ, c, and f are independent of t and turn now our attention to the equations

$$L_h v_h(x) + f(x) = 0 \quad x \in \mathbb{R}^d, \tag{3.1}$$

$$\mathcal{L} v(x) + f(x) = 0 \quad x \in \mathbb{R}^d. \tag{3.2}$$

Naturally by a solution of (3.2) we mean a function v on $\mathbb{R}^d$ such that it belongs to $\mathfrak{B}^2$ and (3.2) holds almost everywhere. Clearly, if a solution v belongs to $\mathfrak{B}^3$ and q_λ, p_λ, c, and f are continuous functions on $\mathbb{R}^d$, then (3.2) holds everywhere.

First we prove the existence and uniqueness of the solutions of equations (3.1) and (3.2).

Theorem 3.1. *Suppose that Assumption 2.1 is satisfied with an* $m \geq 0$ *and let Assumptions 2.2 and 2.4 hold. Then equation* (3.1) *has a unique bounded solution* v_h*. Moreover,* v_h *belongs to* $\mathfrak{B}^m$*.*

Proof. Observe that (3.1) is equivalent to

$$v_h(x) = h^2 \xi(x) f(x) + \xi(x) \sum_{\lambda \in \Lambda_1} \chi_\lambda v_h(x + \lambda h),$$

where

$$\xi^{-1} = h^2 c + \sum_{\lambda \in \Lambda_1} \chi_\lambda.$$

It is seen that the existence and uniqueness of bounded solution of (3.1) follows by contraction principle. Using smooth successive iterations yields that $v_h \in \mathfrak{B}^m$. □

Theorem 3.2. *Let Assumptions 2.1 through 2.6 hold with an $m \geq 2$. Then equation* (3.2) *has a unique solution v in the space $\mathfrak{B}^2$. Moreover, $v \in \mathfrak{B}^m$ and there is a constant N depending only on m, δ, c_0, τ_0, K, M_0, ..., M_m, $|\Lambda_1|$, and $\|\Lambda_1\|$ such that*

$$\|v\|_m \leq N\|f\|_m. \tag{3.3}$$

Proof. First we prove uniqueness. Let $v \in \mathfrak{B}^2$ satisfy (3.2) with $f = 0$. Take a constant $\nu > 0$, so small that $c-\nu \geq c_0/2$ and conditions (2.28) and (2.29) hold with $c-\nu$ and $\delta/2$ in place of c and δ, respectively. Then for each $T > 0$ the function $u(t,x) := e^{\nu t}v(x))$, $(t,x) \in H_T$, is a solution of class $\mathfrak{B}^2_T$ of the equation

$$\frac{\partial}{\partial t}u = (\mathcal{L} + \nu)u \quad \text{on } H_T \tag{3.4}$$

with initial condition $u(0,x) = v(x)$. Hence by virtue of Theorem 2.10 for every $T > 0$

$$e^{\nu T}|v(x)| = |u(T,x)| \leq N\|v\|_2,$$

where N is independent of (T,x). Multiplying both sides of the above inequality by $e^{-\nu T}$ and letting $T \to \infty$ we get $v = 0$, which proves uniqueness.

To show the existence of a solution in $\mathfrak{B}^m$, let u be a function defined on H_∞ such that for each $T > 0$ its restriction onto H_T is the unique solution in $\mathfrak{B}^m_T$ of (3.4) with initial condition $u(0,x) = f(x)$ (see Theorem 2.1). By Theorem 2.10

$$\sup_{H_\infty} \sum_{r \leq m} |D^r u| \leq N\|f\|_m$$

with a constant N depending only on m, δ, c_0, τ_0, K, M_0, ..., M_m, $|\Lambda_1|$, and $\|\Lambda_1\|$. Hence

$$v(x) := \int_0^\infty e^{-\nu t}u(t,x)\,dt, \quad x \in \mathbb{R}^d$$

is a well-defined function on $\mathbb{R}^d$, $v \in \mathfrak{B}^m$, and

$$\mathcal{L}v(x) = \int_0^\infty e^{-\nu t}\mathcal{L}u(t,x)\,dt$$

$$= \int_0^\infty e^{-\nu t}(\frac{\partial}{\partial t}u(t,x) - \nu u(t,x))\,dt = -f(x),$$

where the last equality is obtained by integration by parts. Consequently, v is a solution of (3.4) and it satisfies estimate (3.3). □

Theorem 3.3. *Let $k \geq 0$ and suppose that Assumptions 2.1 through 2.6 are satisfied with an $m \geq 3k$. Then, for any $h \in (0, h_0]$ and for each integer $r \geq 0$, such that*

$$3k + r \leq m,$$

for the unique bounded solution v_h *of* (3.1) *we have*

$$\sup_{(0,h_0]\times\mathbb{R}^d} |D^r D_h^k v_h| \le N\|f\|_m, \tag{3.5}$$

where N *is a constant depending only on* $m, \delta, c_0, \tau_0, K$, $|\Lambda_1|$, $\|\Lambda_1\|$, M_0, ..., M_m. *In particular,*

$$\|v_h\|_m \le N\|f\|_m.$$

Proof. To prove (3.5), take a constant $\nu > 0$ as in the proof of Theorem 3.2, define $u(t,x) := v_h(x)e^{\nu t}$, and observe that u is the unique bounded solution of

$$\frac{\partial}{\partial t}u = L_h^0 u - (c-\nu)u + e^{\nu t}f, \quad u(0,x) = v_h(x).$$

By Theorem 2.9 for any $T > 0$

$$e^{\nu T}|D^r D_h^k v_h(x)| = |D^r D_h^k u(T,x)| \le N e^{\nu T}\|f\|_m + N\|v_h\|_m,$$

where N is a constant, depending only on $m, \delta, c_0, \tau_0, K$, $|\Lambda_1|$, $\|\Lambda_1\|$, M_0, ..., M_m. By multiplying the extreme terms by $e^{-\nu T}$ and letting $T \to \infty$, we get the result. □

From estimate (2.34) we obtain the corresponding estimate for the derivatives of v_h.

Theorem 3.4. *Let the conditions of Theorem 2.11 hold. Then for any integer* $r \ge 0$ *such that*

$$3k + r \le m - 1,$$

for the solution v_h *of* (3.1) *we have*

$$\sup_{\mathbb{R}^d} |D^r D_h^k v_h| \le N\|f\|_m h$$

for all $h \in (0, h_0]$, *where* N *depends only on* $m, \delta, c_0, \tau_0, K$, $|\Lambda_1|$, $\|\Lambda_1\|$ *and* M_0,..., M_m.

Proof. This theorem can be deduced from Theorem 2.11 in the same way as Theorem 3.3 is obtained from Theorem 2.9. □

Now we want to establish an expansion for v_h, i.e., to show for an integer $k \ge 0$ the existence of some functions $v^{(0)}$,...,$v^{(k)}$ on $\mathbb{R}^d$, and a function R_h on $\mathbb{R}^d$ for each $h \in (0, h_0]$ such that for all $x \in \mathbb{R}^d$ and $h \in (0, h_0]$

$$v_h(x) = v^{(0)}(x) + \sum_{1\le j\le k} \frac{h^j}{j!} v^{(j)}(x) + h^{k+1}R_h(x), \tag{3.6}$$

$$\sup_{h\in(0,h_0]} \sup_{\mathbb{R}^d} |R_h| \le N\|f\|_m \tag{3.7}$$

with a constant N.

Theorem 3.5. *Suppose that Assumptions 2.1 through 2.6 are satisfied with an $m \geq 3$. Let $k \geq 0$ be an integer. Then expansion* (3.6) *holds with $v^{(0)}$ being the unique $\mathfrak{B}^m$ solution of* (3.2) *and R_h satisfying* (3.7) *provided one of the following conditions is met:*

(i) $m \geq 3k+3$;

(ii) $m \geq 2k+3$ and condition (S) holds;

(iii) k is odd, $m \geq 2k+2$, and conditions (S) and (2.16) *are satisfied.*

In each of the cases (i)-(iii) the constant N in (3.7) *depends only on d, m, δ, c_0, τ_0, K, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0, \dots, M_m$. Moreover, when (iii) holds we have $v^{(j)} = 0$ for all odd j.*

Proof. Take a small constant $\nu > 0$, as in the proof of Theorem 3.2, let u be a function defined on H_∞ such that for each $T > 0$ its restriction onto H_T is the unique solution in $\mathfrak{B}^m_T$ of

$$\frac{\partial}{\partial t} u_h = (L_h + \nu) u_h \quad (t,x) \in H_\infty$$

$$u_h(0,x) = f(x) \quad x \in \mathbb{R}^d,$$

(see Remark 2.1). As in the proof of Theorem 3.2 we get that

$$v_h(x) = \int_0^\infty e^{-\nu t} u_h(t,x)\, dt.$$

By Theorem 2.3 in each of the cases (i)-(iii) we have

$$u_h(t,x) = u^{(0)}(t,x) + \sum_{1 \leq j \leq k} \frac{h^j}{j!} u^{(j)}(t,x) + h^{k+1} r_h(t,x), \tag{3.8}$$

for all $(t,x) \in H_\infty$, $h \in (0, h_0]$, and by Theorem 2.10 we have

$$\sup_{h \in (0,h_0]} \sup_{H_\infty} \{|u_h| + \sum_{j=0}^{k} |u^{(j)}| + |r_h|\} \leq N \|f\|_m \tag{3.9}$$

with a constant N depending only on d, m, δ, c_0, τ_0, K, M_0,...,M_m, $|\Lambda_1|$ and $\|\Lambda_1\|$. Multiplying both sides of equation (3.8) by $e^{-\nu t}$ and then integrating them over $[0,\infty)$ with respect to dt, we get expansion (3.6) with

$$R_h(x) := \int_0^\infty e^{-\nu t} r_h(t,x)\, dt,$$

$$v^{(j)}(x) := \int_0^\infty e^{-\nu t} u^{(j)}(t,x)\, dt, \quad \text{for } j = 0, \dots, k.$$

Clearly, (3.9) implies that (3.7) holds with N depending only on d, m, δ, c_0, τ_0, K, M_0,...,M_m, $|\Lambda_1|$, and $\|\Lambda_1\|$. As we know the function $u^{(0)}$ in (3.8) is the $\mathfrak{B}^m$ solution of

$$\frac{\partial}{\partial t} u = (\mathcal{L} + \nu) u \quad (t,x) \in H_\infty,$$

$$u(0,x) = f(x) \quad x \in \mathbb{R}^d,$$

which as we have seen in the proof of Theorem 3.2 guarantees that $v^{(0)}$ is the unique $\mathfrak{B}^m$ solution of equation (3.2). □

Remark 3.1. We can show similarly that $v^{(i)}$, $i = 1, ..., k$, is the unique solution of the system

$$\mathcal{L}v^{(j)}(s,x) + \sum_{i=1}^{j} C_j^i \mathcal{L}^{(i)} v^{(j-i)} = 0$$

in an appropriate class of functions (cf. Theorem 2.2).

The following result can be obtained easily from Theorem 2.7 by inspecting the proof of the previous theorem.

Theorem 3.6. *Let p_λ, q_λ, c, and f satisfy the conditions of Theorem 3.5, with $m-l$ in place of m in each of the conditions (i)–(iii) for an integer $l \in [0, m]$. Then $D^\alpha v_h$ is a bounded continuous function on $\mathbb{R}^d$ for every multi-index α, $|\alpha| \le l$, and the expansion* (3.6) *is valid with $D^\alpha v_h$, $\{D^\alpha v^{(j)}\}_{j=0}^k$ and $D^\alpha R_h$ in place of v_h, $\{v^{(j)}\}_{j=0}^k$ and R_h, respectively. Furthermore,* (3.7) *holds with $D^\alpha R_h$ in place of R_h and a constant N depending only on d, m, δ, c_0, τ_0, K, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0, \dots, M_m$. In case (iii) we have $v^{(j)} = 0$ for all odd j in the expansion.*

Set

$$\bar{v}_h = \sum_{j=0}^{k} b_j v_{2^{-j}h}\,, \quad \tilde{v}_h = \sum_{j=0}^{\tilde{k}} \tilde{b}_j v_{2^{-j}h}\,,$$

where $(b_0, b_1, \dots, b_k)$ and $\tilde{k}$, $(\tilde{b}_0, \tilde{b}_1, \dots, \tilde{b}_{\tilde{k}})$ are defined in (2.22) and in (2.25). Then we have the following corollary.

Corollary 3.7. *Suppose that the assumptions of Theorem 3.6 are satisfied. Then for every multi-index α with $|\alpha| \le l$,*

$$\sup_{\mathbb{R}^d} |D^\alpha \bar{v}_h - D^\alpha v^{(0)}| \le N\|f\|_m h^{k+1},$$

and if condition (iii) is met then

$$\sup_{\mathbb{R}^d} |D^\alpha \tilde{v}_h - D^\alpha v^{(0)}| \le N\|f\|_m h^{k+1},$$

where N depends only on on d, m, δ, K, τ_0, c_0, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0, \dots, M_m$.

4. Proof of uniqueness in Theorem 2.1 and a stipulation

We will see later that the proof of Theorem 2.3 only uses the existence of sufficiently smooth solutions of (2.8) and (2.13). Therefore, if $m \ge 3$, uniqueness of $u^{(0)}$ follows from expansion (2.18). If $m = 2$, one can use simple ideas based on integrating by parts. We briefly outline these ideas referring for details to [12], [13], [14].

First, one may assume that $g = f = 0$ and let $u^{(0)}$ be the corresponding solution. Then, by introducing a new function $v = u^{(0)}(\cosh|x|)^{-1}$ one reduces the issue to uniqueness of v, which satisfies an equation similar to (2.5) with $g = f = 0$ and different coefficients which we denote by $\hat{q}_\lambda$, $\hat{p}_\lambda$, and $\hat{c} = c$, and, moreover, $v, Dv, D^2v \in L_2(H_T)$. After that one multiplies the equation for v by v and integrates over H_T. One uses integration by parts, and the fact that due to the assumption $q_\lambda \geq 0$ we have $|D\hat{q}_\lambda|^2 \leq 4\hat{q}_\lambda \sup |D^2\hat{q}_\lambda|$. One also uses Young's inequality implying that

$$|v(\partial_\lambda \hat{q}_\lambda)\partial_\lambda v| \leq N|v\hat{q}_\lambda^{1/2}\partial_\lambda v| \leq \hat{q}_\lambda(\partial_\lambda v)^2 + Nv^2,$$

and the fact that $2\hat{v}\hat{p}_\lambda\partial_\lambda \hat{v} = \hat{p}_\lambda\partial_\lambda(\hat{v})^2$. Then one quickly arrives at a relation like

$$\int_{H_T}(N-c)|v|^2\,dxdt \geq \int_{\mathbb{R}^d}|v(T,x)|^2\,dx \geq 0,$$

where N is a constant independent of c. If c is large enough, the above inequality is only possible if $v = 0$, which proves uniqueness if c is large enough. In the general case it only remains to observe that the usual change of the unknown function taking $v(t,x)e^{\lambda t}$ in place of v for an appropriate λ will lead to as large c as we like.

Remark 4.1. Notice that apart from uniqueness in Theorems 2.1 and 2.2 all our other assertions and assumptions are stable under applying mollifications of the data with respect to x. For instance, take a nonnegative $\zeta \in C_0^\infty(\mathbb{R}^d)$ with unit integral, for $\varepsilon > 0$ define $\zeta_\varepsilon(x) = \varepsilon^{-d}\zeta(x/\varepsilon)$ and for locally summable $\psi(x)$ use the notation

$$\psi^{(\varepsilon)} = \psi * \zeta_\varepsilon.$$

Then $q_\lambda^{(\varepsilon)}, p_\lambda^{(\varepsilon)}, c^{(\varepsilon)}, f^{(\varepsilon)}$, and $g^{(\varepsilon)}$ will satisfy the same assumptions with the same constants as the original ones and will be infinitely differentiable in x.

It is not hard to see that if our assertions are true for the mollified data, then they are also true for the original ones. For instance, let v^ε be the solution of (2.5) with the new data. The uniform in ε estimates of the derivatives in x and the equation itself, guaranteeing that the first derivatives in time are bounded, show that v^ε are uniformly continuous in $[0,T]\times\{|x| \leq R\}$ for any R. Then there is a sequence $\varepsilon_n \downarrow 0$ such that v^{ε_n} converges uniformly in $[0,T]\times\{|x| \leq R\}$ for any R to a bounded continuous function v.

This along with uniform boundedness of $|D^\alpha v^\varepsilon|$, $|\alpha| \leq m$, lead to the fact that the generalized derivatives $|D^\alpha v|$, $|\alpha| \leq m$, are bounded and admit the same estimates as those of v^ε. Also since $D^\alpha v^{\varepsilon_n} \to D^\alpha v$ in the sense of distributions and all of them are uniformly bounded, we conclude that this convergence is true in the weak sense in any $L_2([0,T]\times\{|x| \leq R\})$. Now it is easy to pass to the limit in equation (2.9) written for modified coefficients and v^ε in place of u concluding that since the derivatives converge weakly and $q_\lambda^{(\varepsilon)} \to q_\lambda$,..., $f^{(\varepsilon)} \to f$ uniformly on H_T, v satisfies (2.9).

Similar argument takes care of Theorem 2.2 (in which uniqueness will be derived from uniqueness in Theorem 2.1).

Our claim about stability of other results is almost obvious and *from this moment on we will assume that the data are as smooth in x as we like.*

5. Proof of Theorems 2.9 and 2.11

In [5] (see there Theorems 2.3 and 2.1 and Corollary 3.2 if $m = 0$) and [6] we obtained the following result on the smoothness in x of the solution u_h to equation (2.1).

Theorem 5.1. *Suppose that Assumptions 2.1 and 2.4 are satisfied. Suppose that (i) if $m = 1$, then Assumptions 2.2 and 2.5 are satisfied, and (ii) if $m \geq 2$, then Assumptions 2.2, 2.5, 2.6, and 2.7 are satisfied. Then for $h \in (0, h_0]$ we have that $D^k u_h$, $k = 0, ..., m$, are continuous in x and*

$$\sup_{H_T} \sum_{k=0}^{m} |D^k u_h| \leq N(F_m + G_m), \tag{5.1}$$

where

$$F_n = \sum_{k \leq n} \sup_{H_T} |D^k f_h|, \quad G_n = \sum_{k \leq n} \sup_{\mathbb{R}^d} |D^k g_h|,$$

and N depends only on τ_0, m, δ, c_0, K, $|\Lambda_1|$, $\|\Lambda_1\|$, $M_0, ..., M_m$ (N depends on fewer parameters if $m \leq 1$).

To proceed further we need a few formulas.

Lemma 5.2. *Let φ be a function on H_T and $n \geq 0$ be an integer.*

(i) Assume that the derivatives of φ in $x \in \mathbb{R}^d$ up to order $n + 1$ are continuous functions in x. Then for each $h > 0$

$$D_h^n \sum_{\lambda \in \Lambda_1} p_\lambda \delta_{h,\lambda} \varphi = \sum_{\lambda \in \Lambda_1} p_\lambda \int_0^1 \theta^n \partial_\lambda^{n+1} \varphi(t, x + h\theta\lambda)\, d\theta \tag{5.2}$$

on H_T, where $\partial_\lambda \varphi$ is introduced in (2.12).

(ii) Assume that the derivatives of φ in x up to order $n+2$ are continuous functions in x, and that Assumption 2.3 holds. Then

$$D_h^n \sum_{\lambda \in \Lambda_1} h^{-1} q_\lambda \delta_{h,\lambda} \varphi = \sum_{\lambda \in \Lambda_1} q_\lambda \int_0^1 (1 - \theta)\theta^n \partial_\lambda^{n+2} \varphi(t, x + h\theta\lambda)\, d\theta, \tag{5.3}$$

on H_T.

Proof. By Taylor's formula applied to $\varphi(t, x+h\theta\lambda)$ as a function of $\theta \in [0, 1]$

$$\delta_{h,\lambda} \varphi(t, x) = \int_0^1 \partial_\lambda \varphi(t, x + h\theta\lambda)\, d\theta$$

and

$$\delta_{h,\lambda} \varphi(t, x) = \partial_\lambda \varphi(t, x) + h \int_0^1 (1 - \theta) \partial_\lambda^2 \varphi(t, x + h\theta\lambda)\, d\theta.$$

Multiplying the first equality by p_λ and summing up in λ over Λ_1 we obtain (5.2) for $n = 0$. Multiplying the second equality by q_λ, summing up in λ over Λ_1 we obtain (5.3) for $n = 0$ since

$$\sum_{\lambda\in\Lambda_1} q_\lambda \partial_\lambda \varphi = 0$$

due to Assumption 2.3.

After that it only remains to differentiate n times in h both parts of the particular case of formulas (5.2) and (5.3). The lemma is proved. □

Introduce

$$u_h^{(j)} = D_h^j u_h$$

and observe that by Remark 2.6 under Assumption 2.1 the functions $\partial_\lambda^n u^j$ are well defined if $n + j \le m$. By combining this with Lemma 5.2 and the Leibnitz formula we obtain the following.

Corollary 5.3. *Let Assumptions 2.1 and 2.3 be satisfied. Let $k \ge 1$ be an integer such that $k + 2 \le m$. Then*

$$u_h^{(k)}(t,x) = \int_0^t \big(L_h u_h^{(k)}(s,x) + R_h^k(s,x)\big)\,ds \tag{5.4}$$

on $(0,h_0] \times H_T$, where

$$R_h^k(t,x) = \sum_{i=1}^k C_k^i \sum_{\lambda\in\Lambda_1} \int_0^1 \theta^i \big[p_\lambda(t,x)(\partial_\lambda^{i+1} u_h^{(k-i)})(t,x+h\theta\lambda)$$

$$+(1-\theta)q_\lambda(t,x)(\partial_\lambda^{i+2} u_h^{(k-i)})(t,x+h\theta\lambda)\big]\,d\theta.$$

Now we are ready to prove Theorems 2.9 and 2.11.

Proof of Theorem 2.9. If $m = 2$ or $k = 0$, our assertion follow directly from Theorem 5.1. Therefore, in the rest of the proof we assume that $m \ge 3$ and $k \ge 1$.

We will be using (5.4). Observe that if $1 \le i \le k$, then

$$(i+2) + r + (k-i) = k + 2 + r \le 3k + r \le m.$$

Thus by Remark 2.6 we know that $D^{i+2+r} u_h^{(k-i)}$ are bounded and continuous on H_T. It follows that $R_h^k \in \mathfrak{B}^r$. By Theorem 5.1 with r in place of m we obtain

$$I_{kr} := \sup_{H_T} \sum_{j\le r} |D^j u_h^{(k)}| \le N \sup_{H_T} \sum_{j\le r} |D^j R_h^k|.$$

It is not hard to see that

$$|D^j R_h^k| \le N \sup_{H_T} \sum_{i=1}^k \sum_{n=1}^{i+2+j} |D^n u_h^{(k-i)}| \le N \sum_{i=1}^k I_{k-i,i+2+j}.$$

Hence,

$$I_{kr} \le N \sum_{i=1}^{k} I_{k-i,i+2+r}.$$

Here on the right the first index of I_{kr} is reduced by at least 1 and the sum of indices increased by 2. Therefore, after k iterations we will come to the inequality

$$I_{kr} \le N I_{0,k+2k+r}.$$

It only remains to observe that $I_{0,3k+r} \le I_{0,m}$ and the latter quantity is estimated in Theorem 5.1. The theorem is proved.

Proof of Theorem 2.11. First of all observe that the symmetry assumption and (2.16) imply that for any smooth function $\varphi(x)$, odd $i \ge 0$, and any multi-index α, such that $|\alpha| \le m$, we have

$$\sum_{\lambda\in\Lambda_1} (D^\alpha p_\lambda)\partial_\lambda^{i+1}\varphi = \sum_{\lambda\in\Lambda_1} (D^\alpha q_\lambda)\partial_\lambda^{i+2}\varphi = 0. \tag{5.5}$$

If $k = 1$ and an integer $n \le r$, then owing to (5.5)

$$\big|D^n \sum_{\lambda\in\Lambda_1} q_\lambda(t,x)(\partial_\lambda^3 u_h)(t,x+h\theta\lambda)\big|$$

$$= \big|D^n \sum_{\lambda\in\Lambda_1} q_\lambda(t,x)\big[(\partial_\lambda^3 u_h)(t,x+h\theta\lambda) - \partial_\lambda^3 u_h(t,x)\big]\big|$$

$$\le Nh \sup_{H_T} \sum_{i\le r} |D^{i+4}u_h| \le Nh\|u\|_m \le N(\|f\|_m + \|g\|_m)h =: NJh,$$

where the last two estimates follow from the fact that $r+4 = r+3k+1 \le m$ and from Theorem 2.9, respectively. Similarly,

$$\big|D^n \sum_{\lambda\in\Lambda_1} p_\lambda(t,x)(\partial_\lambda^2 u_h)(t,x+h\theta\lambda)\big|$$

$$= \big|D^n \sum_{\lambda\in\Lambda_1} p_\lambda(t,x)\big[(\partial_\lambda^2 u_h)(t,x+h\theta\lambda) - \partial_\lambda^2 u_h(t,x)\big]\big| \le NJh.$$

Hence,

$$\sup_{H_T} \sum_{n\le r} |D^n R_h^1| \le N(\|f\|_m + \|g\|_m)h \le NJh$$

and applying Theorem 5.1 to (5.4) yields (2.34).

Now we proceed by induction on k. Assume that for an odd number j estimate (2.34) holds whenever $3k + r \le m - 1$ *and* odd $k \le j$. This hypothesis is justified by the above for $j = 1$ and to prove the theorem it suffices to show that the hypothesis also holds with $j+2$ in place of j. Take an odd k and an integer r such that

$$k \le j+2, \quad 3k + r \le m - 1$$

and again use (5.4). As above, to obtain (2.34) it suffices to prove that

$$\sup_{H_T}\sum_{n\le r}|D^n R^k_h|\le NJh. \tag{5.6}$$

Take an integer $n\le r$. Observe that if $1\le i\le k$ and i is even, then $k-i$ is odd and $k-i\le j+2-i\le j$ and

$$3(k-i)+i+2+n=3k+n-2i+2\le m-1-2i+2\le m-1$$

so that by the induction hypothesis

$$\sup_{H_T}\big|D^n\sum_{\lambda\in\Lambda_1}q_\lambda(t,x)(\partial_\lambda^{i+2}u_h^{(k-i)})(t,x+h\theta\lambda)\big|\le NJh. \tag{5.7}$$

If $1\le i\le k$ and i is odd, then $i+2$ is odd too and as in the beginning of the proof

$$\begin{aligned}&\big|D^n\sum_{\lambda\in\Lambda_1}q_\lambda(t,x)(\partial_\lambda^{i+2}u_h^{(k-i)})(t,x+h\theta\lambda)\big|\\ &=\big|D^n\sum_{\lambda\in\Lambda_1}q_\lambda(t,x)\big[(\partial_\lambda^{i+2}u_h^{(k-i)})(t,x+h\theta\lambda)-\partial_\lambda^{i+2}u_h^{(k-i)}(t,x)\big]\big|\\ &\le Nh\sup_{H_T}\sum_{i\le k,l\le r}|D^{l+i+3}u_h^{(k-i)}|,\end{aligned}$$

where the last sup is majorated by NJ owing to Theorem 2.9 since

$$3(k-i)+r+i+3\le m-1-2i+3\le m.$$

In both situations we have (5.7). Similarly, if $1\le i\le k$ and i is odd, then $i+1$ is even and

$$\begin{aligned}&\big|D^n\sum_{\lambda\in\Lambda_1}p_\lambda(t,x)(\partial_\lambda^{i+1}u_h^{(k-i)})(t,x+h\theta\lambda)\big|\\ &=\big|D^n\sum_{\lambda\in\Lambda_1}p_\lambda(t,x)\big[(\partial_\lambda^{i+1}u_h^{(k-i)})(t,x+h\theta\lambda)-\partial_\lambda^{i+1}u_h^{(k-i)}(t,x)\big]\big|\\ &\le Nh\sup_{H_T}\sum_{i\le k,l\le r}|D^{l+i+2}u_h^{(k-i)}|,\end{aligned}$$

where the last sup is majorated by NJ again owing to Theorem 2.9 since

$$3(k-i)+r+i+2\le m-1-2i+2\le m.$$

Finally, if $1\le i\le k$ and i is even, then $k-i$ is odd, $k-i\le j+2-i\le j$, and

$$3(k-i)+r+i+1\le m-1-2i+1<m-1,$$

so that by the induction hypothesis

$$\big|D^n\sum_{\lambda\in\Lambda_1}p_\lambda(t,x)(\partial_\lambda^{i+1}u_h^{(k-i)})(t,x+h\theta\lambda)\big|\le NJh,$$

which is now shown to hold in both subcases. By combining this with (5.7) we come to (5.6) and the theorem is proved.

6. Proof of Theorems 2.1, 2.2, and 2.10

Proof of Theorem 2.1. First we replace q_λ with symmetric ones using the fact that the symmetrization does not affect formula (2.7). To this end introduce

$$\Lambda_1^s = \Lambda_1 \cap (-\Lambda_1), \quad \hat{\Lambda}_1 = \Lambda_1 \cup (-\Lambda_1).$$

On Λ_1^s we set $\hat{q}_{\pm\lambda} = (1/2)(q_\lambda + q_{-\lambda})$. If $\lambda \in \pm(\Lambda_1 \setminus \Lambda_1^s)$ we set $\hat{q}_\lambda = (1/2)q_{\pm\lambda}$. Then $\hat{\Lambda}_1$ and $\hat{q}_\lambda$ satisfy the symmetry condition (S) and can be used to represent the first term on the right in (2.7) in place of the original ones. Next, we redefine and extend p_λ introducing $\hat{p}_\lambda$ on $\hat{\Lambda}_1$, so that $\hat{p}_\lambda = M_0 + p_\lambda$ on Λ_1^s, for $\lambda \in \Lambda_1 \setminus \Lambda_1^s$ we set $\hat{p}_{\pm\lambda} = M_0 \pm (1/2)p_\lambda$, and for $-\lambda \in \Lambda_1 \setminus \Lambda_1^s$ we set $\hat{p}_{\pm\lambda} = M_0 \mp (1/2)p_{-\lambda}$. (Remember that for the constant M_0 from Assumption 2.1 we have $|p_\lambda| \leq M_0$.) Then $\hat{\Lambda}_1$ and $\hat{p}_\lambda$ can be used to represent the second term on the right in (2.7) in place of the original ones. One of the advantages of the new $\hat{p}_\lambda$ is that $\hat{p}_\lambda \geq 0$, which implies that the new χ_λ satisfies Assumption 2.2.

Define $\tau_\lambda > 0$ arbitrarily. As in Remark 6.4 of [5] and Remark 4.3 of [6] one shows that Assumptions 2.5 and 2.6 are also satisfied for any $\delta \in (0,1)$, say $\delta = 1/2$, if c is sufficiently large (independently of h) and $\tau_0 > 0, K$, and $\mathcal{K}$ are chosen appropriately and depending only on $d, |\Lambda_1|, \|\Lambda_1\|, M_0, M_1, M_2$. We first concentrate on the case that c is indeed sufficiently large. In that case by Theorem 2.9, for $h \in (0, h_0]$, there exists a unique solution $u_h(t,x)$ of class $\mathfrak{B}_T^m$ satisfying equation (2.1) with $\hat{L}_h$ in place of L_h, where $\hat{L}_h$ is constructed from $\hat{\Lambda}_1$, $\hat{q}_\lambda$, and $\hat{p}_\lambda$. Furthermore,

$$\|u_h\|_m \leq N(\|f\|_m + \|g\|_m), \tag{6.1}$$

where N is a constant depending only on $m, \inf c$, $|\Lambda_1|$, M_0,..., M_m, and $\|\Lambda_1\|$. Upon observing that owing to Remark 2.2

$$|\hat{L}_h u_h| \leq N(\sup_{H_T} |D^2 u_h| + \sup_{H_T} |Du_h| + \sup_{H_T} |u_h|)$$

with N independent of h, we conclude from the equation for u_h that their first derivatives in t are bounded uniformly in h. Therefore, there exists a sequence $h(n) \downarrow 0$ such that $u_{h(n)}$ converges uniformly on $[0,T] \times \{x : |x| \leq R\}$ for any R to a continuous function v. Then (6.1) implies that $v \in \mathfrak{B}^m$ and

$$\|v\|_m \leq N(\|f\|_m + \|g\|_m) \tag{6.2}$$

with the same N as in (6.1). If we take $\tau_\lambda \equiv 1$, then Remark 6.4 of [5] and Remark 4.3 of [6] imply that both N's can be chosen to depend only on d, m, $\inf c$, $|\Lambda_1|$, and $M_0, ..., M_m$.

Next, the modified equation (2.1) yields that for any $\phi \in C_0^\infty(\mathbb{R}^d)$ and $t \in [0,T]$

$$\int_{\mathbb{R}^d} u_h(t,x)\phi(x)\,dx = \int_{\mathbb{R}^d} g(x)\phi(x)\,dx$$

$$+\int_0^t \int_{\mathbb{R}^d} \sum_{\lambda\in\hat\Lambda_1} u_h(s,x)[(1/2)\Delta_{h,\lambda}(\hat q_\lambda\phi)+\delta_{h,-\lambda}(\hat p_\lambda\phi)](s,x)\,dxds$$

$$+\int_0^t \int_{\mathbb{R}^d}(-cu_h+f)\phi(s,x)\,dx\,ds.$$

We pass to the limit in this equation and find that v satisfies an integral equation, integrating by parts in which proves that v is a solution of (2.8).

Finally, we notice that the case that c is not large is reduced to the above one by usual change of the unknown function taking $v(t,x)e^{\lambda t}$ in place of v for an appropriate λ, which leads to subtracting λv from the right-hand side of (2.5). For the new equation we then find a solution admitting estimate (6.2) with N independent of T but coming back to the solution of the original equation will bring an exponential factor depending on T.

This and uniqueness proved in Section 4 finish proving the theorem. □

Remark 6.1. In the above proof we considered arbitrary $\tau_\lambda>0$ for the following reason. If Assumptions 2.1 through 2.6 hold with $m\geq 2$, then by Theorem 2.9 estimate (6.1) and hence (6.2) hold with N depending only on m,δ,c_0,τ_0,K, M_0, ..., M_m, $|\Lambda_1|$, and $\|\Lambda_1\|$. This proves the assertion of Theorem 2.10 regarding the constant N in Theorem 2.1.

Proof of Theorem 2.2. Notice that for each $j=1,\dots,k$ equation (2.13) does not involve the unknown functions $u^{(l)}$ with indices $l>j$. Therefore we can solve (2.13) and prove the statements (i) and (ii) recursively on j.

First we prove that there is at most one solution $(u^{(1)},\dots,u^{(k)})$ in the space $\mathfrak{B}^2\times\dots\times\mathfrak{B}^2$. Denote

$$S_j=\sum_{i=1}^{j} C_j^i\mathcal{L}^{(i)}u^{(j-i)}.$$

We may assume that $u^{(0)}=0$. Then clearly $S_1=0$ and by Theorem 2.1 we have $u^{(1)}=0$. If for a $j\in\{2,\dots k\}$ we have $u^{(1)}=u^{(2)}=\dots=u^{(j-1)}=0$, then clearly $S_j=0$ which by Theorem 2.1 yields $u^{(j)}=0$. Hence the statements on uniqueness follow because for every $j=1,2,\dots,k$ we obviously have $\mathfrak{B}^{m-3j}\subset\mathfrak{B}^2$ when $m\geq 3k+2$ and $\mathfrak{B}^{m-2j}\subset\mathfrak{B}^2$ when $m\geq 2k+2$.

While dealing with the existence of a solution first take $j=1$. Observe that by Theorem 2.1 we have $u^{(0)}\in\mathfrak{B}^m$ with $m\geq 5$ in case (i) and with $m\geq 4$ in case (ii). Thus in case (i) we have $S_1\in\mathfrak{B}^{m-3}\subset\mathfrak{B}^2$ and by Theorem 2.1 it follows that there exists $u^{(1)}\in\mathfrak{B}^{m-3}$ satisfying (2.13) and admitting the estimate

$$\|u^{(1)}\|_{m-3}\leq N\|u^{(0)}\|_m.$$

Taking the estimate of the last term again from Theorem 2.1 we obtain (2.14) for $j=1$. In case (ii) we have actually better smoothness of S_1, because the first sum in (2.11) is zero for $i=1$ and, for that matter, for all odd i. It follows that $S_1 \in \mathfrak{B}^{m-2}$ and this leads to (2.15) for $j=1$ as above. By adding that under the conditions (S) and (2.16) we have $\mathcal{L}^{(1)}=0$, $S_1=0$, and $u^{(1)}=0$, we obtain (2.17) for $j=1$.

Passing to higher j we assume that $k \geq 2$. Suppose that, for a $j \in \{2,...,k\}$ we have found $u^{(1)}$,...,$u^{(j-1)}$ with the asserted properties. Then in the case (i) we have

$$\mathcal{L}^{(i)}u^{(j-i)} \in \mathfrak{B}^{m-3j} \subset \mathfrak{B}^2$$

for $i=1,\dots,j$, since

$$m-3(j-i)-(i+2)=m-3j+2i-2 \geq m-3j \geq 2.$$

Hence $S_j \in \mathfrak{B}^{m-3j}$ and therefore by Theorem 2.1 there exists $u^{(j)} \in \mathfrak{B}^{m-3j}$ satisfying (2.13) and admitting the estimate

$$\|u^{(j)}\|_{m-3j} \leq N\sum_{i=1}^{j}\|u^{(j-i)}\|_{m-3j+3}$$

$$\leq N\sum_{i=1}^{j}\|u^{(j-i)}\|_{m-3(j-i)} \leq N(\|f\|_m+\|g\|_m),$$

where the last inequality follows by the induction hypothesis.

In case (ii) we take into account that due to condition (S) we have

$$\sum_{\lambda\in\Lambda_1} q_\lambda \partial_\lambda^{i+2}\varphi = 0, \tag{6.3}$$

and due to condition (2.16) we have

$$\sum_{\lambda\in\Lambda_1} p_\lambda \partial_\lambda^{i+1}\varphi = 0 \tag{6.4}$$

for odd numbers i and sufficiently smooth functions φ. It follows that in case (ii) for $i=1,...,j$ we have

$$\mathcal{L}^{(i)}u^{(j-i)} \in \mathfrak{B}^{m-2j} \subset \mathfrak{B}^2,$$

since $\mathcal{L}^{(1)}u^{(j-1)} \in \mathfrak{B}^{m-2(j-1)-2}$ and for $i \geq 2$

$$m-2(j-i)-(i+2)=m-2j+i-2 \geq m-2j \geq 2.$$

Hence $S_j \in \mathfrak{B}^{m-2j}$ and therefore by Theorem 2.1 there exists $u^{(j)} \in \mathfrak{B}^{m-2j}$ satisfying (2.13) and admitting the estimate

$$\|u^{(j)}\|_{m-2j} \leq N\|u^{(j-1)}\|_{m-2j+2} + N\sum_{i=2}^{j}\|u^{(j-i)}\|_{m-2j+3}$$

$$\leq N\sum_{i=1}^{j}\|u^{(j-i)}\|_{m-2(j-i)},$$

and by using the induction hypothesis we come to (2.15).

Furthermore, in case (ii) if (2.16) is satisfied, our induction hypothesis says that $u^{(l)}=0$ for all odd $l \leq j-1$. If j is even, then, obviously, $u^{(l)}=0$

for all odd $l \leq j$ as well. If j is odd then to carry the induction forward it only remains to prove that $u^{(j)} = 0$. However, for odd i we have

$$\mathcal{L}^{(i)} u^{(j-i)} = 0$$

due to (6.3)-(6.4). This equality also holds if $i \geq 2$ and i is even, since then $j - i$ is odd and $u^{(j-i)} = 0$ by assumption. Thus, $S_j = 0$ and $u^{(j)} = 0$. □

Remark 6.2. The above proof is based on Theorem 2.1 and leads to estimates (2.14) and (2.15) with N depending only on the same parameters as in Theorem 2.1. Therefore, according to Remark 6.1 if Assumptions 2.1 through 2.6 are satisfied and the restrictions on m and k from Theorem 2.2 are met, then the constants N in estimates (2.14) and (2.15) depend only on $m, \delta, c_0, \tau_0, K$, M_0, ..., M_m, $|\Lambda_1|$, and $\|\Lambda_1\|$. This proves the part of assertions of Theorem 2.10 concerning Theorem 2.2. The proof of its remaining assertions can be obtained in the same way and is left to the reader.

7. Proof of Theorem 2.3 and 2.7

We need some lemmas. The first one is a simple lemma from undergraduate calculus on Taylor's expansion.

Lemma 7.1. *Let F be a real-valued function on $(0,1]$ such that for an integer $m \geq 0$ the derivative $F^{(m+1)}(h)$ of order $m+1$ exists for all $h \in (0,1]$, and $F^{(m+1)}$ is a bounded function on $(0,1]$. Then*

$$F^{(k)}(0) := \lim_{s \downarrow 0} F^{(k)}(s)$$

exist for $0 \leq k \leq m$, and

$$F(h) = \sum_{k=0}^{m} \frac{h^k}{k!} F^{(k)}(0) + R_m(h)$$

holds for $h \in [0,1]$ with

$$R_m(h) = \int_0^h \frac{(h-s)^m}{m!} F^{(m+1)}(s)\, ds,$$

so that

$$|R_m(h)| \leq \sup_{s \in (0,1]} |F^{(m+1)}(s)| \frac{h^{m+1}}{(m+1)!} \quad \text{for all } h \in [0,1].$$

To formulate our next lemma we recall the operators L_h, $\mathcal{L}$ and $\mathcal{L}^{(i)}$, defined in (2.2), (2.7), and (2.11), respectively, and for each $h \in (0, h_0]$ and integer $j \geq 0$ introduce the operator

$$\mathcal{O}_h^{(j)} = L_h - \mathcal{L} - \sum_{1 \leq i \leq j} \frac{h^i}{i!} \mathcal{L}^{(i)}.$$

Lemma 7.2. *Let Assumption 2.3 hold. Assume that for some integer $l \geq 0$ the functions p_λ, q_λ belong to $\mathfrak{B}^l$ for all $\lambda \in \Lambda_1$. Then for any integer $j \geq 0$*

$$\|\mathcal{O}_h^{(j)}\varphi\|_l \leq N\|\varphi\|_{l+j+3}h^{j+1} \tag{7.1}$$

for all $h \in (0, h_0]$ and $\varphi \in \mathfrak{B}^{l+j+3}$, where N is a constant depending only on $|\Lambda_1|, M_0, ..., M_l$.

Proof. We may assume that the derivatives in x of φ up to order $l + j + 3$ are bounded continuous functions on H_T. By Lemma 5.2 the derivatives of the function $L_h\varphi$ in h up to the $(l + j + 1)$st order are bounded functions on $(0, h_0] \times H_T$ and

$$(\mathcal{L}\phi)(t, x) = \lim_{h\to 0}(L_h\varphi)(t, x),$$

$$(\mathcal{L}^{(i)}\phi)(t, x) = \lim_{h\to 0}(D_h^i L_h\phi)(t, x).$$

Thus applying Lemma 7.1 to $F(h) := L_h\varphi(t, x)$ for fixed (t, x) and using Lemma 5.2, we have

$$\mathcal{O}_h^{(j)}\varphi = \int_0^h \frac{(h-\vartheta)^j}{j!} L_\vartheta^{(j+1)}\varphi \, d\vartheta$$

$$= \sum_{\lambda\in\Lambda_1} q_\lambda \int_0^h \frac{(h-\vartheta)^j}{j!} \int_0^1 (1-\theta)\theta^{j+1}\partial_\lambda^{j+3}\varphi(t, x + \vartheta\theta\lambda)\, d\theta\, d\vartheta$$

$$+ \sum_{\lambda\in\Lambda_1} p_\lambda \int_0^h \frac{(h-\vartheta)^j}{j!} \int_0^1 \theta^{j+1}\partial_\lambda^{j+2}\varphi(t, x + \vartheta\theta\lambda)\, d\theta\, d\vartheta.$$

Now estimate (7.1) follows easily. □

The next lemma is a version of the maximum principle for $\partial/\partial t - L_h$. It is a special case of Corollary 3.2 in [5].

Lemma 7.3. *Let Assumption 2.1 with $m = 0$ be satisfied and let $\chi_{h,\lambda} \geq 0$ for all $\lambda \in \Lambda_1$. Let v be a bounded function on H_T, such that the partial derivative $\partial v(t, x)/\partial t$ exists in H_T. Let F be a nonnegative integrable function on $[0, T]$, and let C be a nonnegative bounded function on H_T such that*

$$\nu := \sup_{H_T}(C - c) < 0.$$

Assume that for all $(t, x) \in H_T$ we have

$$\frac{\partial}{\partial t}v \leq L_h v + C\bar{v}_+ + F, \tag{7.2}$$

where $\bar{v}(t) = \sup\{v(t, x) : x \in \mathbb{R}^d\}$. Then in $[0, T]$ we have

$$\bar{v}(t) \leq \bar{v}_+(0) + |\nu|^{-1} \sup_{[0,t]} F, \tag{7.3}$$

where $a_+ := (|a| + a)/2$ for real numbers a.

Proof of Theorem 2.3. By taking $u_h e^{-(M_0+1)t}$ in place of u_h, we may assume that $c \geq 1$. Consider first the case $k = 0$. Since $m \geq 3$, by Theorem 2.1 equation (2.7) has a solution $u^{(0)}$, which belongs to $\mathfrak{B}^m$ and estimate (2.10) holds. Clearly, $w := u_h - u^{(0)}$ is the unique bounded solution of the equation

$$w(t,x) = \int_0^t \big(L_h w(s,x) + F(s,x)\big)\, ds, \quad (t,x) \in H_T, \tag{7.4}$$

where $F := \mathcal{O}_h^{(0)} u^{(0)} = L_h u^{(0)} - \mathcal{L} u^{(0)}$. By Lemma 7.2 and estimate (2.10)

$$\|\mathcal{O}_h^{(0)} u^{(0)}\|_0 \leq N \sum_{\lambda \in \Lambda_1} (\|p_\lambda\|_0 + \|q_\lambda\|_0) \|u^{(0)}\|_3 h \leq N(\|f\|_3 + \|g\|_3) h$$

with constants N depending only on d, $|\Lambda_1|$ M_0, M_1, M_3, and T. After that an application of Lemma 7.3 to equation (7.4) proves the statement of Theorem 2.3 for $k = 0$.

Let $k \geq 1$. Then by Theorem 2.2 the system of equations (2.13) has a bounded solution $\{u^{(i)}\}_{i=1}^k$. Observe that for

$$w := u_h - \sum_{j=0}^k u^{(j)} \frac{h^j}{j!} \tag{7.5}$$

we have equation (7.4) with

$$F := L_h u^{(0)} - \mathcal{L} u^{(0)} + \sum_{j=1}^k L_h u^{(j)} \frac{h^j}{j!} - \sum_{j=1}^k \mathcal{L} u^{(j)} \frac{h^j}{j!} - G,$$

and

$$G := \sum_{j=1}^k \sum_{i=1}^j \frac{1}{i!(j-i)!} \mathcal{L}^{(i)} u^{(j-i)} h^j = \sum_{i=1}^k \sum_{j=i}^k \frac{1}{i!(j-i)!} \mathcal{L}^{(i)} u^{(j-i)} h^j$$

$$= \sum_{i=1}^k \sum_{l=0}^{k-i} \frac{1}{i!l!} \mathcal{L}^{(i)} u^{(l)} h^{l+i} = \sum_{l=0}^{k-1} \frac{h^l}{l!} \sum_{i=1}^{k-l} \frac{h^i}{i!} \mathcal{L}^{(i)} u^{(l)}$$

$$= \sum_{j=0}^k \frac{h^j}{j!} \sum_{1 \leq i \leq k-j} \frac{h^i}{i!} \mathcal{L}^{(i)} u^{(j)}.$$

Hence by simple arithmetics

$$F = \sum_{j=0}^k \frac{h^j}{j!} \mathcal{O}_h^{(k-j)} u^{(j)}. \tag{7.6}$$

Notice that

$$k - j + 3 \leq m - 3j \ \text{ for } j = 0, 1, \ldots, k \ \text{ in case (i)},$$

$$k - j + 3 \leq m - 2j \ \text{ for } j = 0, 1, \ldots, k \ \text{ in case (ii)},$$

$$k - j + 3 \leq m - 2j \ \text{ for } j = 0, 1, \ldots, k-1 \ \text{ in case (iii)}.$$

Therefore by Theorem 2.2 under each of (i), (ii), and (iii)

$$\|u^{(j)}\|_{k-j+3} \le N(\|f\|_m + \|g\|_m)$$

for $j = 0, 1 \dots, k$ ($u^{(k)} = 0$ in the case (iii)). Thus by Lemma 7.2

$$\|\mathcal{O}_h^{(k-j)} u^{(j)}\|_0 \le N h^{k-j+1} \|u^{(j)}\|_{k-j+3} \le N h^{k+1-j}(\|f\|_m + \|g\|_m).$$

Consequently,

$$\|F\|_0 \le N(\|f\|_m + \|g\|_m) h^{k+1} \quad \text{for } h \in (0, h_0],$$

where N depends only on d, m, $|\Lambda_1|$, $M_0, \dots, M_m$, and T. Hence we get (2.18) by Lemma 7.3, and the proof is complete. □

Proof of Theorem 2.7. Coming back to the above proof of Theorem 2.3 we see that function (7.5) satisfies (7.4) with F given by (7.6). We notice that

$$k - j + 3 + l \le m - 3j \ \text{ for } j = 0, 1, \dots, k \ \text{ in case (i)},$$

$$k - j + 3 + l \le m - 2j \ \text{ for } j = 0, 1, \dots, k \ \text{ in case (ii)},$$

$$k - j + 3 + l \le m - 2j \ \text{ for } j = 0, 1, \dots, k-1 \ \text{ in case (iii)}.$$

Therefore by Theorem 2.1, when $k = 0$, and by Theorem 2.2, when $k \ge 1$, under each of (i), (ii), and (iii)

$$\|u^{(j)}\|_{k-j+3+l} \le N(\|f\|_m + \|g\|_m)$$

for $j = 0, 1 \dots, k$ ($u^{(k)} = 0$ in case (iii)). By Theorem 2.10 the constant N depends only on m, δ, c_0, τ_0, K, M_0, ..., M_m, $|\Lambda_1|$, and $\|\Lambda_1\|$. By Lemma 7.2

$$\|\mathcal{O}_h^{(k-j)} u^{(j)}\|_l \le N h^{k-j+1} \|u^{(j)}\|_{k-j+l+3},$$

where N is a constant depending only on $|\Lambda_1|$, $M_0, \dots M_l$. Hence

$$\|F\|_l \le N(\|f\|_m + \|g\|_m) h^{k+1} \quad \text{for } h \in (0, h_0].$$

Consequently, applying Theorem 2.9 to equation (7.4), for any multi-index α, $|\alpha| \le l$, for

$$r_h^{(\alpha)} := h^{-(k+1)} \big(D^\alpha u_h - \sum_{j=0}^{k} D^\alpha u^{(j)} \frac{h^j}{j!} \big)$$

we have

$$\|r_h^{(\alpha)}\|_0 = h^{-(k+1)} \|D^\alpha w\|_0 \le N(\|f\|_m + \|g\|_m),$$

with a constant N depending only on m, d, δ, c_0, τ_0, K, M_0, ..., M_m, $|\Lambda_1|$ and $\|\Lambda_1\|$, which proves the theorem. □

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