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Copula-Based Formulas to Estimate Unexpected Credit Losses (The Future of Basel Accords?)

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COPULA-BASED FORMULAS TO ESTIMATE UNEXPECTED CREDIT LOSSES (THE FUTURE OF BASEL ACCORDS?)

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ABSTRACT

The model used to estimate the capital required to cover unexpected credit losses in financial institutions (Basel II) has some drawbacks that reduce its ability to capture potential joint extreme losses in downturns.

This paper suggests an alternative approach based on Copula Theory to overcome such flaws. Similarly to Basel II, the suggested model assumes that defaults are driven by a latent variable which varies as a response to an unobserved factor. On the other hand, the use of copulas allows the identification of asymmetric dependence between defaults which has been registered in the literature.

As an example, a specific copula family (Clayton) is adopted to represent the association between the latent variables and a formula to estimate potential unexpected losses at a certain level of confidence is derived.

Simulations reveal that, in most of the cases, the alternative model outperforms Basel II for portfolios with right-tail-dependent probabilities of default (supposedly, a good representation for real loan portfolios).

1. INTRODUCTION

The current rule to calculate capital necessary to cover unexpected credit losses is based on structural models which define that joint defaults are driven by a latent variable which, in turn, is driven by an unobserved (economic) factor.

The economic factor, the latent variable, and the specific (idiosyncratic) risk for each obligor are assumed to follow the standard normal distribution but there is vast evidence in the literature that those variables are not normally distributed.

The dependence across pairs of latent variables and between each latent variable and the economic factor is measured by the correlation coefficient which is accurate only for normal data and does not detect tail dependence. So, the current model used to calculate the regulatory capital is deficient because it may not identify conjunct extreme occurrences.

To overcome this problem, this paper proposes the application of copulas to link distributions of latent variables and evaluate unexpected credit losses in financial institutions.

Copulas are functions used to express several types of dependence (with or without tail association) between variables regardless of their distributions. Hence, the suggested approach relaxes the assumption of normality and is able to identify tail dependence.

The latent variables are considered to be survival functions of the probabilities of default (PDs), i.e., high PDs indicate low values of latent variables and vice versa.

While traditional credit risk models use percent values, the copula approach is based on percentiles (ranks) of the variables. Considering that portfolios/segments are taken for homogenous, the levels (percentiles) of the latent variables that imply default are equal for all loans. Then, for each pair of debtors, the copula will associate two equal variables (percentiles of latent variables) in extreme conditions and will return the likelihood of both percentiles being simultaneously below a specific level (percentile of the latent variable's historical average in this case). This is equivalent to the probability of potential losses being above the rank of the average (expected) PD.

The method implementation is relatively simple and, alike models derived from Merton's approach (Merton, 1974), is based on the interpretation that default happens when the

latent variable falls below a cutoff value. The suggested method focuses on joint defaults which occur when the latent variables of loans become smaller than their limit percentile at the same time. Losses are unexpected (above the average) when such underlying variables drop even more and reach percentiles smaller than their average's percentile among the values that indicate default. Thus, for a particular level of confidence, "high" unexpected losses will be estimated by a copula that gives the joint probability of the historical latent variable's average being below an extreme percentile.

In principle, a general approach is presented to derive formulas based on any copula found to be representative of loan portfolios. If large datasets on PDs are available, precise models may be built according to the steps proposed in this study.

An example is given for the case where PDs are assumed to be right-tail associated and, consequently, the latent variables present left-tail dependence. For convenience, the relationship between the latent variables is represented by the Clayton Copula.

Simulations reveal that, in most of the cases, when compared to Basel II, the alternative model yields better estimations of the effective losses in portfolios with tail-dependent probabilities of default (which is expected to be a property of most credit portfolios in the financial market – see some references in section 5.1).

In around 73% of the scenarios, the copula-based approach outperformed Basel II for at least one of the three credit classes analysed (revolving consumer, mortgage, and "other retail"). On average, the new method was better for all three categories in 52% of the cases. The results were sensitive to the confidence specified and the shape of the loss distribution. Normally-distributed losses generated the worst estimations for the suggested model at the confidence level used while the other three distributions tested (exponential, beta, and gamma) resulted in an outperformance ratio of 75%.

The remainder of the paper is organized as follows. Basel Accords are addressed in the next section. Then Copula Theory is discussed. Section 4 summarizes a general approach to derive formulas based on assumed or empirically found dependence between probabilities of default. In section 5, PDs are presumed to be right-tail dependent (i.e. high losses are more associated) and a formula based on Clayton Copula is derived to estimate unexpected losses. Next, the results from the formula

presented in the prior section are compared to capital calculated by the Basel formula. Section 7 concludes.

2. BASEL ACCORDS

The Basel Accord from 1988 stipulated that the capital charge on assets was 8% of the risk weighted assets. But due to many drawbacks in this Accord (see De Servigny and Renault, 2004), new rules were issued in June 2004.

The Basel II Accord is based on three "pillars": minimum capital requirements, Supervisory Review, and market discipline. Banks are allowed to use Internal Ratings Based approaches (IRB) to calculate the capital required and to do so, institutions should group their assets into homogenous "buckets" (segments, classes) with respect to credit quality.

However Basel II also has some limitations. It assumes normally distributed loans' performances and uses the correlation coefficient that does not capture oscillations in dependence when the level of variables changes. Thus, this may lead to excessive capital required in good economic scenarios or scarce requirements in downturns.

Basically, for each segment, the capital required to cover unexpected losses in credit portfolios is calculated as the unexpected losses adjusted by the portfolio maturity.

In mathematical terms:

$$[LGD*K_V - LGD*PD]*Maturity = [LGD*(K_V - PD)]*Maturity \qquad [1]$$

where:

LGD is the "loss given default", i.e. the percentage of exposure the lender will lose if borrowers default;

PD stands for probability of default;

K_V is the expected default rate at the 99.9% percentile of the PD distribution ("Vasicek Formula") - see formula ahead;

$Maturity$ corresponds to the maturity of corporate loans and is added to the calculation in order to give higher weight to long-term credits which are known to be riskier. See formula ahead;

K_V is calculated by the formula:

$$K_V = N\left(\frac{N^{-1}(PD) + \sqrt{\rho}N^{-1}(0.999)}{\sqrt{1-\rho}} \right) \qquad [2]$$

being that:

N represents the standard normal cumulative distribution;
N^{-1} is the inverse of the standard normal cumulative distribution;
PD, as before, is the probability of default of the loan portfolio (average);
$N^{-1}(PD)$ is used to derive the default threshold (i.e. the cutoff level of obligors' assets below which default occurs);
$N^{-1}(0.999)$, which is equal to $-N^{-1}(0.001)$, is the level of the economy chosen to represent an extreme scenario in which unexpected losses may occur. Therefore, the systematic factor is assumed to be normally distributed. Intuitively, this is the confidence level (99.9%) for the default rate; and
Rho (ρ) is the correlation between returns of obligors' assets. $\sqrt{\rho}$ is the linear correlation between the unobserved systematic factor and those asset returns. In Basel II, the correlation between asset returns is calculated as a function of PD and (in the case of corporate debt) the size of debtors (measured in terms of annual sales). For the sake of brevity, the formula and parameters used to estimate ρ will not be presented here. See BCBS (2005, 2006) for more details.

Readers interested in the derivation of models that assume that correlation between defaults is driven by an unobserved factor, should consult, for instance, Schönbucher (2000), Perli and Nayda (2004), and Crook and Bellotti (2010). For details on Vasicek distribution (K_V), see Vasicek (1991, 2002).

In general, the term K_V follows the main presumptions of structural models (see, e.g., Gordy, 2003). Each latent variable (Y_i) is a linear function of an unobserved single factor (systematic risk, E) and specific characteristics of the respective obligor (idiosyncratic risk, ε_i). The single factor is assumed to be standard normally distributed and equally impacts all obligors (same correlation $\sqrt{\rho}$) and the latent variables are considered equicorrelated (same ρ for all pairs) and also follow the standard normal distribution:

$$Y_i = E\sqrt{\rho} + \varepsilon_i \sqrt{1-\rho} \qquad [\,3\,]$$

The maturity is applied only to corporate debt and is given by:

$$maturity = \frac{1+(M-2.5)*b(PD)}{1-1.5*b(PD)} \qquad [\,4\,]$$

M is the average maturity of the credit portfolio; and
$$b(PD) = (0.11852 - 0.5478 * \log(PD))^2$$

Thus, in expression [1], $LGD*K_V$ gives the total potential loss and $LGD*PD$ represents the expected losses. The difference between them is therefore the unexpected losses.

The proposed formulas in this study are limited to replace the term $(K_V - PD)$, which expresses the unexpected default rate, and do not consider possible shortcomings in the computation of LGD and the maturity adjustment.

3. A BRIEF VIEW ON COPULAS

Broadly speaking, copulas are functions that link marginal (individual) distributions of variables to their joint distributions.

In usual notation:

$$H(x, y) = C(F(x), G(y)) \qquad [5]$$

The univariate functions $F(x)$ and $G(y)$ transform the variables x and y into their correspondent percentiles (ranks, commonly represented by "u" and "v"), i.e. they become uniformly distributed on the interval (0,1). Such transformations are explained by the "Probability Integral Transformation" (PIT) which states that a random variable "X", with continuous cumulative distribution function F_X, applied to its own function generates a uniform variable between 0 and 1. That is[1], $F_X(x) \sim U(0,1)$.

Copulas give the probability that the percentiles of x and y are simultaneously below the specified percentiles u and v.

Notwithstanding it is also possible to use copulas in order to calculate the probability that percentiles will be jointly *above* a specific point. These are the so called Survival Copulas and have the form:

$$\overline{H}(x, y) = \hat{C}(\overline{F}(x), \overline{G}(y)) \qquad [6]$$

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[1] The proof of PIT is given, for example, in Casella and Berger (2002, pp. 54-55).

where $\overline{H}(x,y)$ is the joint probability $\Pr(X > x, Y > y)$ and $\overline{F}(x)$ and $\overline{G}(y)$ are survival (or reliability) functions $\Pr(X > x) = 1 - F(x)$ and $\Pr(Y > y) = 1 - G(y)$, respectively.

Whilst the linear correlation coefficient ρ is accurate only for spherically or elliptically distributed data[2] (see Embrechts *et al.*, 2002)[3], copulas are suitable for any type of distribution since they are based on ranks.

The shape of the dependence (e.g. lower/upper tail association, symmetry/asymmetry) is defined by the family (formula) of the copula. The strength of the dependence is measured by the copula parameter θ. Many families of copulas are described in Joe (1997, chapter 5) and Nelsen (2006, chapter 4).

The parameter θ is closely related to rank correlations Kendall's tau (τ) and Spearman's rho (ρ_S) and can be inferred from[4]:

$$\tau = 4\iint_{[0,1]^2} C(u,v)dC(u,v) - 1 \qquad [7]$$

and

$$\rho_S = 12\iint_{[0,1]^2} uv\,dC(u,v) - 3 \qquad [8]$$

4. USING COPULAS TO ESTIMATE UNEXPECTED CREDIT LOSSES: A GENERAL APPROACH

4.1 Characterization of default in the copula approach

Traditional approaches employed in the financial industry, such as CreditMetrics® and KMV®, incorporate the basic idea of structural models and assume that default happens when a latent variable (for example, the log-return of debtors' assets) falls below a cutoff point. The probability of default (PD) is given by the area on the left side of the cutoff under the curve of the latent variable's (Normal) distribution. In other words, it is

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[2] For a technical concept of spherical and elliptical distributions, see item 3.3 of Embrechts et al. (2002). Intuitively, in the bivariate case, we can identify such distributions through their contour diagrams (graphs of level curves) which have spherical and elliptical shapes respectively. The Normal distribution is an example of this class.
[3] This study appeared first in 1999 as a working paper.
[4] The proofs are given in Nelsen (2006, chapter 5).

the likelihood of the latent variable (Y) being smaller than that particular value (the threshold y_c) as shown in Figure 1.
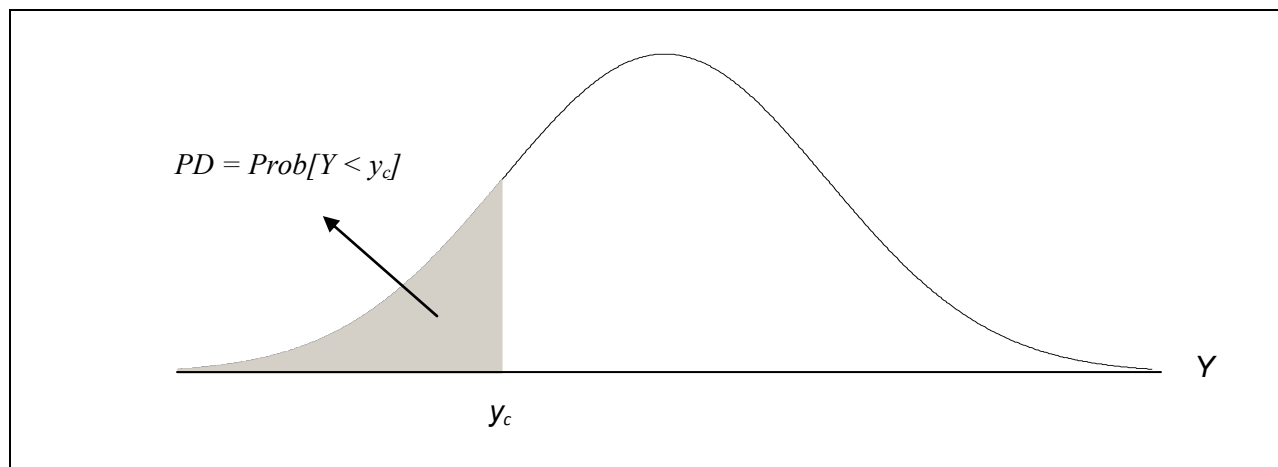

FIGURE 5.1 – Diagram representing default in structural models
as the probability of a latent variable being below a cutoff value (area below y_c).

In this alternative method, PD is viewed in a portfolio perspective and is defined as the area below a cutoff in the *joint* distribution of latent variables relative to the loans that compose the portfolio. Here, such latent variables are supposed to have a symmetrically inverse relationship with the probability of default. This means that when PD increases (decreases), the latent variable decreases (increases) at the same "degree". The identical magnitude ("degree") of the variables' opposite movements will be expressed by their percentiles in their respective distributions. Hence high (low) levels of PDs are associated with low (high) levels of the latent variables and when PD moves "*p*" percentiles in its distribution, Y moves *p* percentiles in its respective distribution in the opposite direction.

This symmetric inverse behaviour may be captured by representing each latent variable (Y) as a survival function[5] of PD, which implies that the *percentile* of the latent variable is equal to one minus the *percentile* of PD:

$$F(y) = \overline{F}(pd) = 1 - F(pd) \qquad [\,9\,]$$

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[5] The use of the subscribed "*t*" to indicate the time dependence in survival functions was relaxed.

Note that $F(y)$ and $F(pd)$ are cumulative distributions of Y and PD, respectively, and give which percentiles are represented by the points y and pd. For instance, $F(y) = 0.20$ indicates that y is the 20th percentile of F.

According to [9], Y may be interpreted as the probability of non-default and expresses the "quality" of debtors. This idea resembles the survival function used by Li (2000) to define the likelihood that a security will reach a specific age. The higher this probability, the higher the asset quality.

Since we are using percentiles of the variables Y and PD, $F(y)$ and $F(pd)$ respectively, Copula Theory may be applied and the resultant calculations are suitable for any kind of loss distribution.

Like in the Basel approach, the capital needed to cover unexpected losses will be separately determined for each segment considered homogeneous in terms of credit quality. This means that PD is presumed identical for all loans in each segment and $F(pd)$ values are also equal. Therefore, the average Y is the same for every debtor within the segment and so is $F(y)$. This is true regardless of the number of debtors.

The estimation of unexpected losses depends on an average point considering only the occurrences below the latent variable's cutoff. Figure 2 – Panel A is a level curve of the joint cumulative distribution of the latent variables and represents the distinction between expected and unexpected losses in this context. When the percentile of each latent variable Y, $F(y)$, in a homogeneous portfolio falls below the percentile of the cutoff $F(y_c)$, obligors default at the same time. The losses are considered expected (area EL in Figure 2 – Panel A) while each $F(y)$ keeps falling from $F(y_c)$ until $F(y_A)$, which represents the percentile of the average of historical values of the latent variable Y *in scenarios of default*. On the other hand, when the latent variable becomes smaller than that average (area UL in Figure 2 – Panel A), the losses are unexpected (meaning that the obligors' conditions – whose proxy is the latent variable – got worse than usual). Given that the latent variables and the losses for each loan have inversely symmetric percentiles, $F(y) = 1 - F(pd)$, the joint function $H(y_A, y_A)$ is equivalent to the area above the percentiles of the average probability of default in a complete PD distribution

($F(pd_A)$ in Figure 2 – panel B), i.e. a distribution that includes non-default status. Thus, both areas UL in Panels A and B of Figure 2 are equal to each other and indicate unexpected losses.
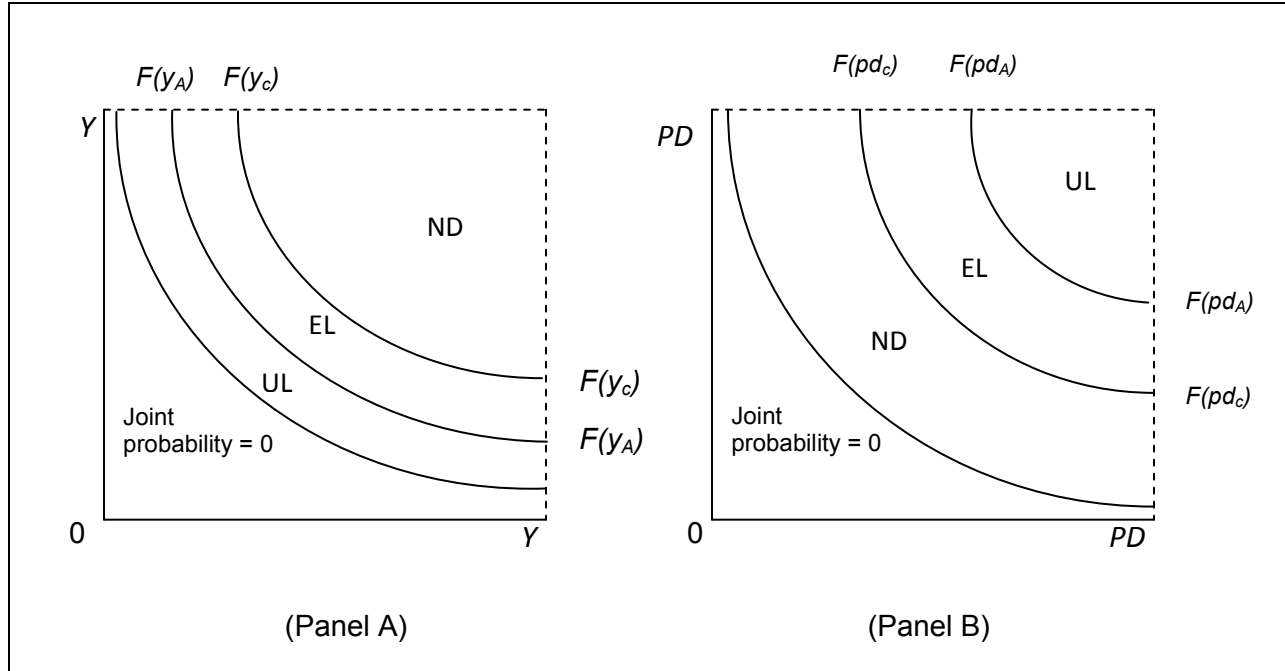
FIGURE 2 – Contour plots of cumulative distributions representing expected losses (EL), unexpected losses (UL), and non-default (ND) in a copula context (for homogenous portfolios). In Panel A, default at the portfolio level happens if the percentiles of the latent variable, *F(y)*, of both loans fall below a specific point, *F(y_c)*; when each *F(y)* is smaller than *F(y_c)* but greater than *F(y_A)*, the losses are expected; and when those percentiles drop below the point that indicates average default, *F(y_A)*, the losses are unexpected. Panel B shows ND, EL, and UL under the perspective of the probability of default which are equal to the equivalent areas in Panel A: *F(pd_c)* =1-*F(y_c)* and *F(pd_A)* =1- *F(y_A)* are, respectively, the percentiles of *PD* above which default and unexpected losses happen. Since the focus is on percentiles, both panels are valid regardless of the PD distribution's family.

What we should estimate is the likelihood of the joint probability of default for two obligors being above its average. Recalling the concept of Survival Copulas in [6] and that each debtor has the same PD, we have:

$$\overline{H}(pd_A, pd_A) = \hat{C}(\overline{F}(pd_A), \overline{F}(pd_A)) = \hat{C}(1 - F(pd_A), 1 - F(pd_A)) \qquad [\ 10\ ]$$

In the prior formula, pd_A stands for the average of the historical probability of default and the notation $\bar{H}$ refers to a joint survival function. The expression gives the probability of both PDs being above the historical average pd_A at the same time. The Survival Copula $\hat{C}$ links the two univariate survival functions of $\bar{F}(pd_A) = 1 - F(pd_A)$ to the bivariate function.

Now, applying the definition introduced in [9] to the average PD, we have $F(y_A) = 1 - F(pd_A)$ and [10] becomes:

$$\bar{H}(pd_A, pd_A) = \hat{C}(F(y_A), F(y_A)) = H(y_A, y_A) \qquad [\,11\,]$$

where pd_A and y_A are the historical average of PD and of the latent variable, respectively; for homogenous pools of borrowers i and j, such that $PD_i = PD_j = PD$ and $Y_i = Y_j = Y$, $\hat{C}$ is a copula that returns $\Pr(PD > pd_A, PD > pd_A)$ = $\Pr(Y < y_A, Y < y_A)$ the probability of both PDs (latent variables) being above (below) their observed average up to the moment or, in other words, the probability of unexpected losses.

4.2 Finding the percentiles of the latent variable

To apply this copula model we need the *whole* distribution of the latent variable so that we can calculate the percentile of Y associated with the point of historical average loss, $F(y_A)$. Given a group of obligors in default, the percentile of the cutoff $F(y_c)$ would be obviously 1 and any area calculated under this circumstance, would return the likelihood of PD being below or above a point and not the PD itself.

Based on Figure 2 – Panel A, we see that to find the unexpected losses (UL) we would need to know the percentile $F(y_A)$ in the *complete* distribution of Y (i.e. including non-default status, ND). However, in principle, we do not have enough information to find that value. Otherwise, we could use $F(y_c)$ to calculate the total losses (EL + UL) and subtract EL, which is known (the average PD of the portfolio). But, again, we cannot find $F(y_c)$ using solely the information available so far.

One way to start solving this problem is considering a relationship between $F(y_A)$ and $F(y_c)$. Figure 3 – Panel A illustrates the density $f^{default}$ of a latent variable Y that includes only default cases (i.e. all observations have values below the cutoff that indicates default). $F^{default}$ is the correspondent cumulative distribution and $F^{default}(y_A)$ is the percentile of the average latent variable in that distribution. The latter can be calculated from datasets of PDs taken over several periods by finding the percentile of the average PD and applying [9]: $F^{default}(y_A) = 1 - F^{default}(pd_A)$. The cutoff y_c is the largest value in that density function, so $F^{default}(y_c) = 1$.

The distribution of the latent variable becomes complete if we add the non-default cases (when the latent variable is higher than the cutoff value) like in Figure 3 – Panel B. The complete distribution F is not observable and may have any shape.

As an example, consider that the latent variables represent borrowers' asset returns. For the sake of simplicity, the debt of all obligors will be assumed equal but this presumption can be easily relaxed if we work with the percentage of asset returns over the (different) liability of each obligor. If debts are equal to 100 monetary units, $y_c = 100$ and default happens when $Y < 100$. Assume also that among all obligors that failed their payments, the average asset return (y_A) was 75 monetary units.

We cannot observe Y or its distribution but we know that its percentile is equal to one minus the percentile of the associated PD in the PD distribution. So, if expected losses (average PD = pd_A) are, for instance, 5% and this value is the 60th percentile in its distribution restricted to default cases, according to [9], the percentile of y_A in the latent variable's distribution, $F^{default}(y_A)$, will be $1 - F^{default}(pd_A) = 1 - 0.60 = 0.40$. This reasoning also works for the complete unobservable distribution F.

Regardless of the size of the non-default area (ND) in Figure 3 – Panel B, y_A and y_c are the same in both distributions (F and $F^{default}$) and their percentiles indicate the proportion of data occurrences below those specific points. $F^{default}(y_A)$, for instance,

gives the number of Y observations in distribution $F^{default}$ below y_A, $n_A^{default}$, divided by

the total observations, $n^{default}$. Thus, $\dfrac{F^{default}(y_A)}{F^{default}(y_c)} = \dfrac{n_A^{default}/n^{default}}{n_c^{default}/n^{default}} = \dfrac{n_A^{default}}{n_c^{default}}$.
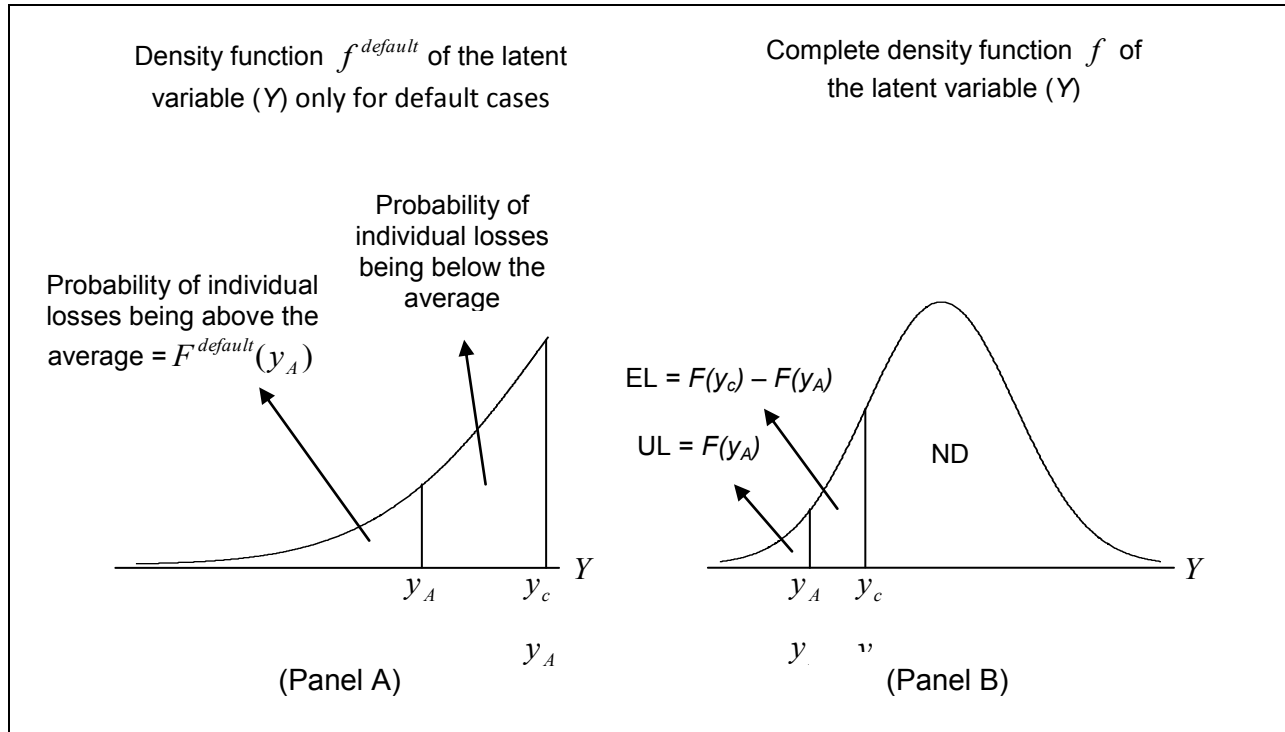
FIGURE 3 – Density function of the latent variable (Y). The shapes are merely illustrative. Panel A displays only the cases where losses happened whilst Panel B includes levels of Y that did not result in default (above the cutoff y_c). The percentile of each $y_A = F^{default}(y_A)$ is equal to one minus the percentile of the average PD, which can be inferred from datasets. $F^{default}(y_c) = 1$. UL, EL, and ND represent unexpected losses, expected losses, and non-default, in that order. $F(y_A)$ and $F(y_c)$ are, respectively, the percentiles of the latent variable related to the historical average losses and to the cutoff value below which defaults happen.

Regarding F, let n_A, n_c, and n denote, respectively, the number of observations below

y_A, below y_c, and in the complete distribution. Following the reasoning in the prior

paragraph: $\dfrac{F(y_A)}{F(y_c)} = \dfrac{n_A/n}{n_c/n} = \dfrac{n_A}{n_c}$.

Since no data is included below y_c when the non-default area (ND) is added to $F^{default}$

in order to generate the entire distribution F, $n_A^{default} = n_A$ and $n_c^{default} = n_c$. Therefore,

$\dfrac{F^{default}(y_A)}{F^{default}(y_c)} = \dfrac{F(y_A)}{F(y_c)}$. As stated before, $F^{default}(y_c) = 1$, thus $F(y_c)$ is always equal to

$F(y_A)/F^{default}(y_A)$.

From Figure 2 – Panel A that represents homogenous segments/portfolios (same PD for all loans), it is easy to see that the joint area below y_c minus the joint area below y_A is equal to the expected probability of default (EL). In copula terms,

$$\hat{C}(F(y_c), F(y_c)) - \hat{C}(F(y_A), F(y_A)) =$$

$$\hat{C}(F(y_A)/F^{default}(y_A), F(y_A)/F^{default}(y_A)) - \hat{C}(F(y_A), F(y_A)) = PD$$

[12]

where $F(y_A)$, the percentile of the historical average latent variable, is the only unknown variable, $F(y_c) = F(y_A)/F^{default}(y_A)$, $F^{default}(y_A)$ is the percentile of the historical average of Y in the distribution restricted to $Y < y_c$, and PD expresses the expected (average) probability of default (EL). The notation $\hat{C}$ (from [11] and based on Nelsen, 2006) was kept to indicate that we are dealing with a survival copula from a PD standpoint. The existence of a closed-form solution to calculate $F(y_A)$ will depend on the copula chosen or empirically found to represent the association between the latent variables of the loans.

After $F(y_A)$ is estimated, the joint distribution $H(y_A, y_A)$ may be calculated as the copula $\hat{C}(F(y_A), F(y_A))$ and will express the mean unexpected losses in a particular period (the sum of percent losses above the average in a period divided by the number of unit times considered – months, for instance).

However, in bank regulation, the major concern is the maximum potential loss. In this copula-based method, the risk of severe unexpected losses comes from possible oscillations in the percentile of the past average (= expected) latent variable, i.e. changes in $F(y_A)$ that may reach extreme values in spite of y_A being constant. The augment of that percentile is interpreted as a response to the deterioration of the economic status.

This situation can be depicted with the support of Figure 3 – Panel B. In downturns, latent variables smaller than y_A tend to appear more frequently. In these circumstances, the percentage of non-default (ND) drops and, as the expected losses (EL) stay unaltered, the unexpected losses (UL) rise. Therefore the ratio UL/EL goes up and so does $F(y_A)$. It is worth noting that y_A remains steady and each new $Y < y_A$ makes y_A "move" to the right side at the density representation and get closer to y_c. The risk is "how far" y_A can go, i.e. how close to $F(y_c)$ $F(y_A)$ can get.

In order to estimate this potential increment of $F(y_A)$, we should find an extreme percentile of the average latent variable in the distribution resultant from the inclusion of smaller latent variables, $F^{EXT}(y_A)$, as illustrated in Figure 4 that follows the intuition of Figure 3. The confidence required will express the ratio UL/EL and may be understood as a measure of the economy's degradation. In this fashion, when confidence equals 100%, all losses are unexpected. Conversely, when it approaches zero (upturns), small unexpected losses are supposed to happen. So, like in Basel II and factor models, the latent variables of loans are driven by the (unobserved) economic status. Here, the latter is captured by the oscillation of the former which, in turn, are inferred from available data on probabilities of default.

Using the example mentioned earlier, in which the latent variable is interpreted as obligors' asset returns, y_A is still 75 (the historical average) but due to the severe economical conditions, asset returns lower than that value are included in the distribution and the percentile of y_A increases, i.e. $F^{EXT}(y_A) > F(y_A)$.

The copula calculated for the extreme percentiles of y_A will give the maximum unexpected loss with the confidence demanded which defines the location of the average latent variable in the new distribution F^{EXT}. Following the same reasoning in [12], we can find the extreme percentile for each loan, $F^{EXT}(y_A)$, doing:

$$\hat{C}(F^{EXT}(y_A)/confidence, F^{EXT}(y_A)/confidence) - \hat{C}(F^{EXT}(y_A), F^{EXT}(y_A)) = PD \qquad [13]$$

PD is the average probability of default (EL), $F^{EXT}(y_c) = F^{EXT}(y_A)/confidence$ and $confidence \in (0,1]$ establishes the percentile of the average latent variable for each obligor in an adverse economic scenario.

The final formula is intended to replace the term $(K_V - PD)$ in [1]. Thus, the capital to cover unexpected losses will be:

$$[LGD * \hat{C}(F^{EXT}(y_A), F^{EXT}(y_A))] * Maturity \qquad [14]$$
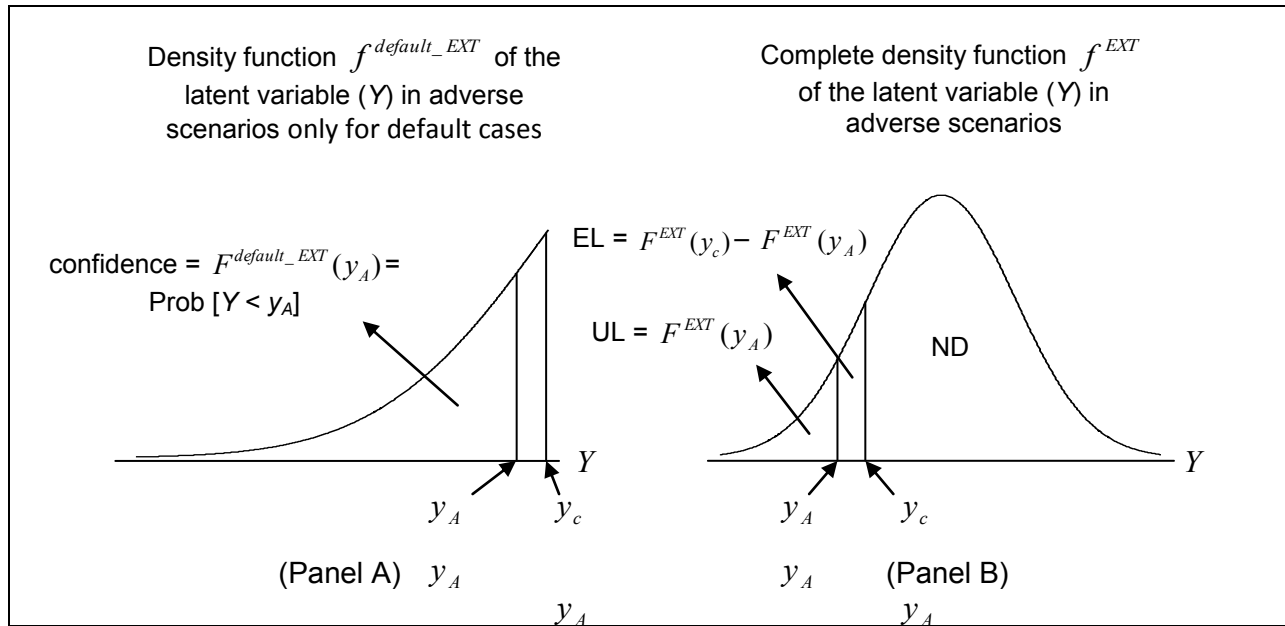
FIGURE 4 – Density function of the latent variable (Y) depicting a situation where the average of the historical latent variable (y_A) reaches an extreme percentile in an unfavourable scenario. The shapes are merely illustrative. In Panel A, the area below the extreme percentile of each loan, $F^{default\_EXT}(y_A)$, is the model confidence, i.e., the probability that $Y < y_A$ in $f^{default\_EXT}$. The percentile of the cutoff value of the latent variable below which default occurs is equal to one, $F^{default\_EXT}(y_c) = 1$. In Panel B, values of Y above y_c (non-default status) are included in the density function. UL, EL, and ND stand for unexpected losses, expected losses, and non-default, respectively. UL is given by $F^{EXT}(y_A)$ which is associated with $F^{EXT}(y_c)$ as

$$F^{EXT}(y_c) = F^{EXT}(y_A)/F^{default\_EXT}(y_A) = F^{EXT}(y_A)/confidence.$$

4.3 Defining the copula to be used

If large datasets on probabilities of default are available, the dependence across pairs of latent variables may be found through the estimation of the best copula for "$1 - PD$".

Therefore it is not necessary estimating the best copula that expresses the dependence between PDs. What matters is the copula that will represent the dependence across the latent variables (which may be interpreted as returns of debtors' assets or "time until default", for instance). To estimate such dependence it suffices to have a series of PDs from a "homogeneous" credit segment/portfolio.

Durrleman et al. (2000) and Cherubini *et al*. (2004, chapter 5) present some methods that can be used to empirically find the parameter, for each copula family, with the best fit to a dataset. A practical way to find the copula's parameter is estimating it from the kendall's tau of $1-PD$ (by using [7][6]) which is the same kendall's tau for PDs (which are observable).

Berg (2009) and Genest *et al*. (2009) describe some goodness-of-fit tests that allow us to decide which copula (considering the estimated parameters) gives the best expression of the dependence related to the variables analysed.

The use of empirically-found copulas gives more realistic results because the probability of unexpected losses and the dependence between the variables come from "real" data (PDs).

Following, an example shows the application of the model if we assume that high PDs are more linked.

5. MODEL APPLICATION: AN EXAMPLE FOR RIGHT-TAIL-DEPENDENT LOSSES

5.1 Assumptions

Mandelbrot (1963) and Fama (1965) showed that asset returns in general are not normally distributed and therefore are more subject to extreme events than returns estimated by models based on assumptions of normality. Since then many empirical studies confirmed this behaviour for several classes of investments, including loan portfolios (Rosenberg and Schuermann, 2006).

Moreover, it has also been found that returns are more correlated in the left tail (i.e. when investments result in losses or lower returns). See Ang and Bekaert (2002),

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[6] Alternatively, [8] can be used to estimate the copula parameter as a function of Spearman's rho (ρ_S). In the simulations run for this study, the results usually matched up to the second decimal place.

Patton (2006) and Ning (2006) who cites many other studies that reach this same conclusion.

According to Di Clemente and Romano (2004) and Das and Geng (2006), amongst others, returns of credit assets also present asymmetric (tail) dependence.

Based on this, it is assumed in this section that PDs (probabilities of default, credit losses) have upper tail dependence (which means that high PDs are more correlated than the other levels or, in other words, large losses of different obligors tend to be more associated whereas small losses are not very linked). This relationship can be represented by copulas such as Gumbel, Joe, Galambos, and Hüsler-Reiss. The Gumbel was chosen because, among those copulas cited, it has been more studied and its properties are better known.

The scatter plot of a Gumbel-dependent random variable "X" ($0 \leq X \leq 1$) looks like Figure 5. Consequently, the plot of the symmetrically inverse variable "$1 - X$" will be like Figure 6.
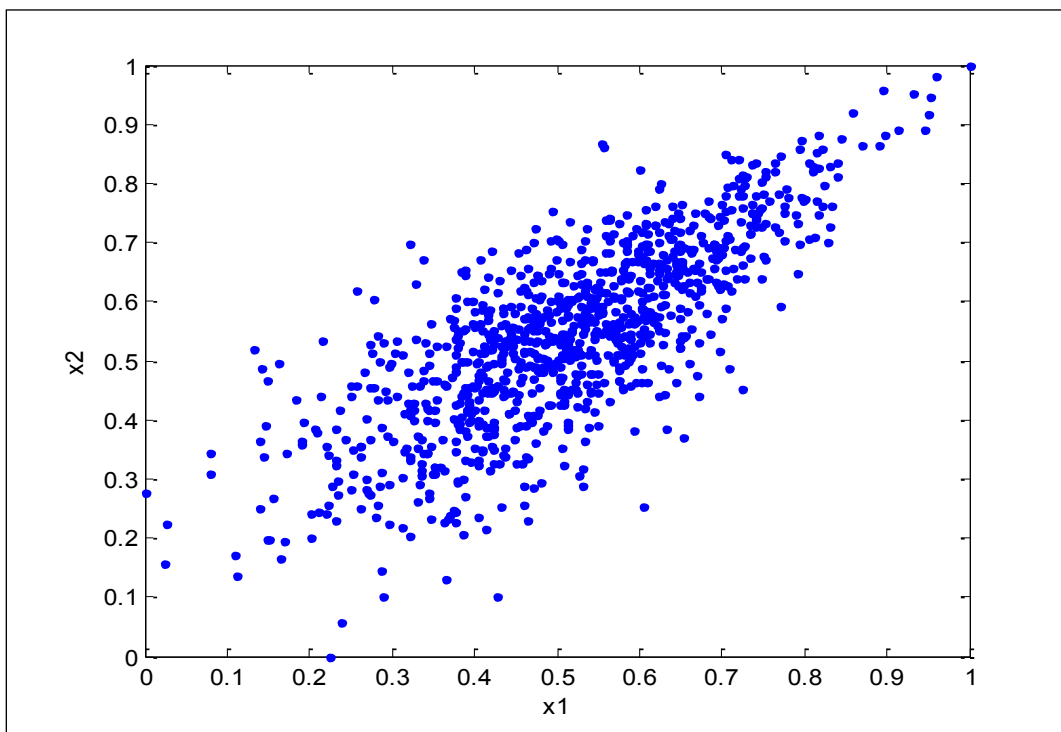
FIGURE 5 – Two random variables with Gumbel dependence (upper tail dependence).

Those figures are suitable for representing the dependence between PD of loans and their latent variable Y respectively, such that $F(y) = 1 - F(pd)$ as defined in [9].

The Clayton Copula is a good representation for the second type of dependence (between latent variables) that indicates lower tail association. This relationship could be expressed by other copulas that express lower tail dependence (Raftery, for instance) but the Clayton Copula was chosen because it has been more studied and its formula is more tractable than the other alternatives.
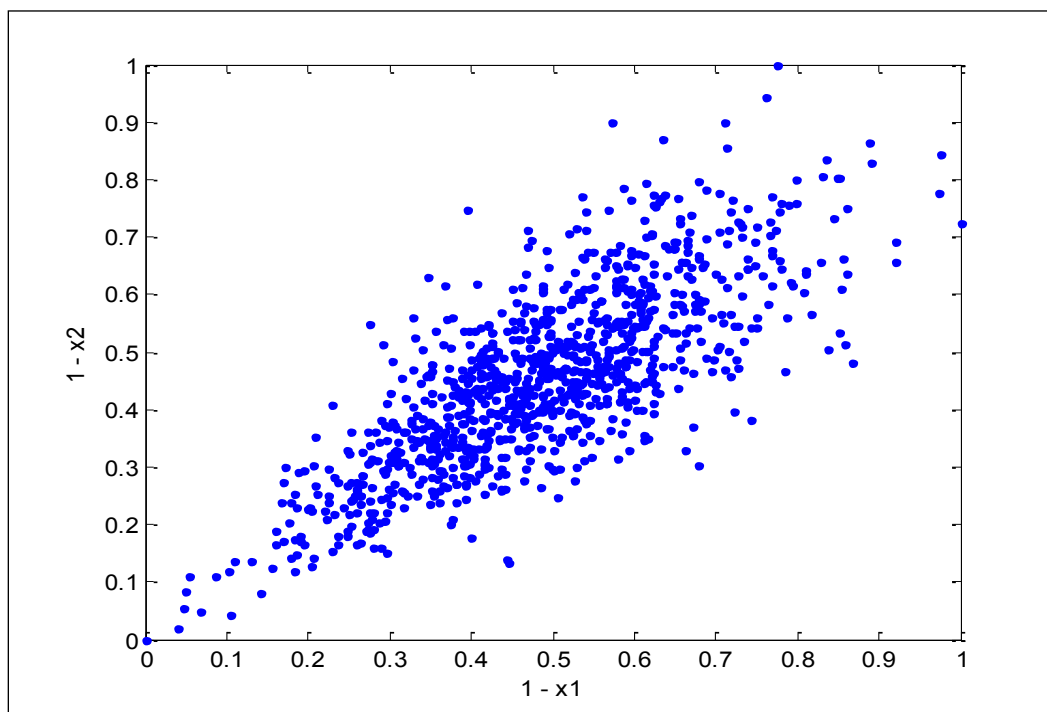
FIGURE 6 – Two random variables with lower tail dependence.

5.2 The formula

We are interested in calculating the joint probability of the latent variable's historical average being below the percentile of an extreme point that indicates joint unexpected losses in adverse scenarios. To do so, we should estimate the copula $\hat{C}(F^{EXT}(y_A), F^{EXT}(y_A))$ where $F^{EXT}(y_A)$ is the percentile of the historical average latent variable of individual loans at an extreme location and refers to the confidence demanded.

Recall that both variables y_{EXT} used to calculate the probability are equal to each other because the segment/portfolio is assumed to be homogenous, so the percentile of the average Y in the extreme distribution $(=F^{EXT}(y_A))$ is the same for all loans. Consequently, the extreme percentile of $PD (= F^{EXT}(pd_A))$ is also the same for all loans. $\hat{C}$ is assumed to be a Clayton Copula to detect the supposed lower tail dependence of the latent variables: they are more related in downturns when their levels are lower.

For this particular case, the Clayton Copula with parameter θ is:

$$\hat{C}(F^{EXT}(y_A),F^{EXT}(y_A)) = [F^{EXT}(y_A)^{-\theta} + F^{EXT}(y_A)^{-\theta} - 1]^{-1/\theta} = [2*F^{EXT}(y_A)^{-\theta} - 1]^{-1/\theta} \quad [\,15\,]$$

This formula gives the probability of the latent variable being jointly smaller than its historical average when the latter reaches an unusually high percentile, $F^{EXT}(y_A)$, in the respective distribution. This corresponds to the likelihood of losses being simultaneously above an extreme point and the expression above substitutes $(K_V - PD)$ in [1]. Therefore the capital to cover unexpected losses is:

$$[LGD*(2*F^{EXT}(y_A)^{-\theta} - 1]^{-1/\theta})]*Maturity \quad\quad\quad [\,16\,]$$

where LGD and $Maturity$ are defined as in [1], $F^{EXT}(y_A)$ is the extreme percentile of the average latent variable calculated in [13] according to the confidence required, and θ is the parameter of the Clayton Copula, estimated from the rank correlation (Kendall's tau or Spearman's rho) of PD (following [7] and [8], respectively).

5.3 Additional comments on this alternative model

A prior use of copulas in order to suggest some improvements to Basel II was registered in Benvegnù et al. (2006). The main purpose was to capture diversification effects, since the Basel II determines the simple addition of all capital requirements for segments without taking correlations into account. Their analysis was focused on

corporate loans and concluded that the copula approach reduces the capital required by 10 to 30%.

However, *"to be in line with the model used in the Basel II credit framework and the major industry models"* (p. 497), the authors assumed that the loans have Vasicek distributions, the underlying factors that drive credit losses were joint normally distributed, and the dependence between them was also normal (Gaussian Copula). Such assumptions restricted the identification of joint extreme occurrences.

Here, the relation between the latent variables is assumed to be satisfactorily represented by the Clayton Copula in order to find their lower tail dependence (i.e. lower levels of latent variables, which lead to defaults, are more correlated in lower economic levels).

In short words, although copulas enable us to capture the diversification effects among different segments (which tend to reduce the capital necessary to cover unexpected losses, as in Benvegnù et al., 2006) some of their families identify higher level of dependence at the extremes (which may increase the capital needed). Thus, due to the assumption of tail dependence, the proposed formula in this paper is more conservative and it is aligned with regulators' point of view (and practitioners who want to guarantee adequate capital to cover losses in severe scenarios).

It could be said that if "real" data do not present intense tail dependence, the capital calculated by [16] will be excessive. But even if there are chances of overestimation, regulators and/or institutions that adopt this approach may reduce the confidence of the extreme average latent variable used as an input in the formula.

Also, the current Basel Accord assumes an unrealistic distribution for the variables involved and measures the dependence between them by using the linear correlation coefficient which does not capture tail dependence.

Furthermore, the copula-based approach has other advantages: it may be used for negatively correlated losses (provided that the rank correlation is positive) while Basel II's model does not admit negative correlation; and it does not assume any specific type of distribution for credit losses, the latent variable, and the unobserved economic factor.

6. SIMULATIONS AND RESULTS FOR REQUIRED CAPITAL

Simulations were used to test the efficiency of the alternative model. The capital required according to Basel method was computed for three types of consumer loans (to which the maturity adjustment is not applied)[7]: revolving credit, mortgage, and "other retail". For simplicity, LGD was assumed equal to 100% (i.e. the Recovery Rate is 0%).

The simulations were controlled for three variables, PD (15 rates between 1% and 15%, inclusive), PD dependence expressed by the Gumbel Copula's parameter[8] (11 values from 1.05 to 2), and the shape of PDs' distributions (Normal/Gaussian, Exponential, Beta, and Gamma) which totalized 660 scenarios. Apart from the case of normal PDs, the other three distributions were simulated in such way that their parameters resulted in the mean (PD) chosen and in distributions skewed to the right indicating asymmetric high losses (following Kalyvas et al., 2006 who stated that credit losses present distributions skewed to the right).

Note that the selection of Gumbel Copula implies the existence of upper-tail dependence for losses. The higher the parameter, the higher that dependence. The confidence[9] of the proposed model was set at 0.90. Each scenario contained 1,000 observations (equivalent to 1,000 periods) and was run 1,000 times to minimize possible randomness effects on results[10].

To calculate the "true" joint unexpected losses, two "correlated" variables (probabilities of default $= PDs$) were simulated with the same features (mean, distribution's family, and its parameters) since the segment or portfolio in terms of calculation of capital is assumed to be homogenous. Such pair of variables represents all pairs of dependent loans (all pairs have the same dependence) in the simulation criteria. Then it was checked the maximum loss when the variables were simultaneously above the mean (average PD).

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[7] These simulations can be run for corporate debt as well but some scenarios for the maturity adjustment should be also defined.

[8] The smallest value allowed for the Gumbel parameter is 1 (which represents independence).

[9] Other confidence levels were tested (not displayed here) but yielded lower ratios of outperformance over Basel II, mainly due to overestimations of the copula-based method.

[10] The results presented in Table 2 are the averages of each variable simulated. Furthermore, the codes for data generation include some commands to guarantee that the loss dependence's parameters are close enough to the stated values (divergence no greater than 0.01).

The performance of the models was measured according to the magnitude of the ratio between the "true" maximum unexpected losses and the capital estimated without taking into account if the capital was excessive or insufficient to cover the losses. Thus, for instance, if the real maximum unexpected losses were 20%, one particular method resulted in 25% and other method estimated 16%, the latter was considered better because the magnitude of its divergence (= 1 - (0.16/0.20) = 0.20 deficient) was less than the difference generated by the former model (= (0.25/0.20) - 1 = 0.25 in excess).

Considering all 660 scenarios, Basel estimations for the three categories of consumer loans were concurrently better than the alternative model's results in 26.52% of the cases. On the other hand, the copula approach was more efficient than traditional calculations for the three (at least one of the) consumer credit classes in 33.79% (73.48%) of the cases. However these ratios rise to 45.05% (92.32%) if the normally distributed losses are excluded. Therefore the performance of the copula-based method was directly related to the shape of the marginal loss distributions.

Table 1 – Panel A presents the proportions of the alternative method's outperformance tabulated according to loss distributions. The forecasts pertaining to exponential (normal) losses presented the best (worst) results. However, it was noticed in other simulations (not displayed here) that the results for normal PDs could be improved if lower levels of confidence were employed.

As for the classes of loans, revolving credit and "other retail" had superior performance: they were better than Basel II in around 68% and 66% of the scenarios, respectively (these figures go up to 90% and 87% if normal losses are not taken into account). The formula for mortgage was more accurate because the correlation for this group is, in general, higher and this avoided excessive underestimation in some circumstances.

So, if the assumptions followed to generate the scenarios are valid for "real" portfolios, the alternative approach is liable to outperform Basel II especially for revolving credit and "other retail" whose losses are not normally distributed.

A special warning about Basel results is the high percentage of underestimated maximum potential losses: 85% with respect to revolving credit and "other retail" and 61% in mortgage portfolios. Typically, this drawback happened for non-normal losses.

As an additional analysis to get results closer to what financial institutions might experience in practice, the comparison was limited to levels of dependence of PDs that are likely more representative of empirical credit portfolios. The proxy for the dependence of "real" consumer loans is based on the results of the empirical studies that supported Basel II Accord and found that retail credit's (linear) correlation varies from[11] 0.03 to 0.16. The outperformance proportion of the alternative model with regard to portfolios correlated in that restricted range is displayed in Table 1 – Panel B.

On average, the alternative approach yielded worse results for portfolios less correlated (Panel B compared to Panel A). This result is explained by the fact that the main benefit of using the Clayton Copula method is the identification of left tail dependence and the consequent higher number of joint occurrences in the extreme left side of the latent variable's distribution (or, equivalently, in the right tail of the loss distribution). Since loans presenting lower correlation (as those in Panel B of Table 1) also have lower tail dependence, the poorer performance of the suggested model in these cases was expected.

TABLE 1 – Proportion of outperformance of the alternative method over Basel II estimations for consumer loans

Loss distribution	Revolving credit	Mortgage	"Other retail"	Three classes	At least one class
Panel A: All scenarios simulated					
Normal	0.00%	16.97%	3.64%	0.00%	16.97%
Exponential	100.00%	84.85%	90.91%	78.79%	100.00%
Beta	96.36%	38.79%	95.15%	38.79%	98.79%
Gamma	73.94%	17.58%	74.55%	17.58%	78.18%
Average	67.58%	39.55%	66.06%	33.79%	73.48%
Panel B: Scenarios with correlation lower than or equal to 0.16					
Normal	0.00%	43.33%	10.00%	0.00%	43.33%
Exponential	100.00%	93.33%	100.00%	93.33%	100.00%
Beta	86.67%	33.33%	73.33%	33.33%	86.67%
Gamma	66.67%	33.33%	53.33%	26.67%	66.67%
Average	50.67%	49.33%	49.33%	30.67%	68.00%

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[11] The correlations adopted by Basel II model are: 0.04 for revolving credit, 0.15 for mortgages and from 0.03 to 0.16 (as a decreasing function of PD) for "other retail credit".

Table 2 shows some examples[12] of capital estimated using the copula technique and Basel II for consumer portfolios with correlation compatible with empirical tests run by the Basel Committee on Banking Supervision (BCBS) – see the fourth column. The best approximations for the maximum unexpected losses observed in the simulated portfolios are highlighted in boldface.

If regulators and/or practitioners wish to set particular dependence values for each type of loans instead of calculating them directly from every single portfolio, the copula model may still be used successfully through the definition of a copula parameter for each credit category (which may be inferred from rank correlations between losses by utilizing [7] or [8]).

7. FINAL COMMENTS

Due to the assumptions of normally distributed variables and the use of a linear measure of dependence (correlation coefficient), Basel II is not able to identify extreme events accurately. Therefore, the capital demanded to cover unexpected losses may be misestimated.

The main contribution of this paper is considering potential tail dependence between related variables to calculate the probability of credit losses in adverse situations. By capturing joint extreme events more precisely without assuming any particular type of loss distribution, the alternative model improves the accuracy of estimations related to simultaneous large losses which usually happen in downturns.

The formulas proposed can be easily implemented and are intended to replace the term in Basel II referent to the subtraction of the extreme default rate (K_V) by PD (see [1] and [2]). Nevertheless, some basic assumptions of Basel II approach are kept, namely: the homogeneity of segments/portfolios and the fact that defaults are driven by latent variables which are impacted by an unobserved (economic) factor. Also, possible pitfalls related to the calculation of the loss given default (LGD) and the maturity adjustment are not investigated.

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[12] Among the 15 PDs simulated, seven were selected: 0.01, 0.03, 0.05, 0.07, 0.10, 0.12, and 0.15.

TABLE 2 – Comparison between capital calculated by Basel Model and the alternative formula for some of the simulated credit portfolios (with linear correlation between 0.03 and 0.16, inclusive)*

PD	PD dependence (Gumbel θ)	Latent variable dependence (Clayton θ)	Linear correlation	"True" maximum unexpected losses	Alternative required capital	Capital required Basel II (revolving)	Capital required Basel II (mortgage)	Capital required Basel II ("other retail")
				Panel A : Normal distribution				
0.01	1.05	0.0997	0.0770	**0.0132**	0.0494	**0.0306**	0.1003	0.0814
0.01	1.10	0.1999	0.1472	**0.0141**	0.0555	**0.0306**	0.1003	0.0814
0.03	1.05	0.1010	0.0787	**0.0396**	0.1411	**0.0687**	0.1991	0.1116
0.03	1.10	0.2005	0.1475	**0.0425**	0.1529	**0.0687**	0.1991	0.1116
0.05	1.05	0.1003	0.0783	**0.0670**	0.2293	**0.0973**	0.2635	0.1181
0.05	1.10	0.2003	0.1475	**0.0701**	0.2440	**0.0973**	0.2635	0.1181
0.07	1.05	0.1029	0.0798	**0.0927**	0.3162	**0.1207**	0.3111	0.1231
0.07	1.10	0.2011	0.1473	**0.0993**	0.3317	**0.1207**	0.3111	0.1231
0.10	1.05	0.1037	0.0805	**0.1320**	0.4436	0.1491	0.3634	**0.1343**
0.10	1.10	0.2024	0.1488	**0.1406**	0.4586	0.1491	0.3634	**0.1343**
0.12	1.05	0.1038	0.0803	**0.1589**	0.5273	0.1649	0.3895	**0.1434**
0.12	1.10	0.2013	0.1479	**0.1693**	0.5407	**0.1649**	0.3895	0.1434
0.15	1.05	0.1023	0.0795	**0.1996**	0.6512	**0.1847**	0.4191	0.1575
0.15	1.10	0.1998	0.1469	**0.2115**	0.6614	**0.1847**	0.4191	0.1575
				Panel B: Exponential distribution				
0.01	1.05	0.1033	0.1012	**0.0469**	**0.0496**	0.0306	0.1003	0.0814
0.03	1.05	0.1017	0.1001	**0.1446**	**0.1412**	0.0687	0.1991	0.1116
0.05	1.05	0.1006	0.0971	**0.2366**	**0.2293**	0.0973	0.2635	0.1181
0.07	1.05	0.1012	0.1014	**0.3346**	**0.3158**	0.1206	0.3111	0.1231
0.10	1.05	0.1039	0.1010	**0.4808**	**0.4438**	0.1492	0.3634	0.1343
0.12	1.05	0.0984	0.0973	**0.5706**	**0.5264**	0.1649	0.3895	0.1434
0.15	1.05	0.1046	0.0991	**0.7012**	**0.6513**	0.1847	0.4191	0.1575

(continued on next page)
(*) The maximum unexpected losses observed and the best estimation for each scenario are highlighted.

TABLE 2 (continued) – Comparison between capital calculated by Basel Model and the alternative formula for some of the simulated credit portfolios (with linear correlation between 0.03 and 0.16, inclusive)*

PD	PD dependence (Gumbel θ)	Latent variable dependence (Clayton θ)	Linear correlation	"True" unexpected losses	Alternative required capital	Capital required Basel II (revolving)	Capital required Basel II (mortgage)	Capital required Basel II ("other retail")
Panel C: Beta distribution								
0.01	1.05	0.1024	0.1040	**0.2136**	0.0496	0.0306	**0.1003**	0.0814
0.03	1.05	0.1043	0.1029	**0.2813**	0.1416	0.0688	**0.1991**	0.1116
0.05	1.05	0.1040	0.1003	**0.3149**	0.2298	0.0973	**0.2635**	0.1181
0.07	1.05	0.1029	0.0972	**0.3344**	0.3163	0.1207	0.3111	0.1231
0.10	1.05	0.1010	0.0947	**0.3574**	0.4432	0.1491	**0.3634**	0.1343
0.12	1.05	0.1040	0.0940	**0.3717**	0.5272	0.1649	**0.3895**	0.1434
0.15	1.05	0.1024	0.0917	**0.3817**	0.6511	0.1847	**0.4191**	0.1575
Panel D: Gamma distribution								
0.01	1.05	0.1015	0.1056	**0.1494**	0.0495	0.0306	**0.1003**	0.0814
0.03	1.05	0.1026	0.1034	**0.2027**	0.1413	0.0688	**0.1991**	0.1116
0.05	1.05	0.1025	0.0983	**0.2336**	0.2296	0.0973	0.2635	0.1181
0.07	1.05	0.1050	0.0993	**0.2654**	0.3164	0.1206	0.3111	0.1231
0.10	1.05	0.1012	0.0957	**0.2980**	0.4433	0.1492	**0.3634**	0.1343
0.12	1.05	0.1041	0.0955	**0.3126**	0.5274	0.1649	**0.3895**	0.1434
0.15	1.05	0.0985	0.0907	**0.3383**	0.6508	0.1847	**0.4191**	0.1575

(*) The maximum unexpected losses observed and the best estimation for each scenario are highlighted.

Simulations of right-tail-dependent losses that controlled for several levels of PDs, their dependencies and marginal distributions confirmed the superiority of the suggested method when losses are not normally distributed. Hence, given that the literature has presented some evidence that credit losses do not follow the normal distribution and have tail dependence, the copula-based model is likely to outperform the current method in many (or most) of the loan portfolios held by financial institutions.

Even if the dependence structure adopted in the exemplary model (Clayton Copula) is considered too rigorous, it still can be used without major concerns if the confidence is reduced.

Naturally, the higher performance of the alternative model shown for some scenarios in section 6 is valid only if losses have upper-tail dependence. The next step to consolidate the application of this approach is the empirical search for the copula family and respective parameter(s) that best represent the relationship between latent variables (which may result in different families and parameters for distinct classes of credit, such as corporate, mortgage, revolving, and so on).

Another promissory extension of this study is the use of Copula Theory to evaluate another component in the Basel formula: the loss given default (LGD).

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