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Nonextensive Bose–Einstein condensation model

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The imperfect Boson gas supplemented with a gentle repulsive interaction is completely solved. In particular, it is proved that it has nonextensive Bose–Einstein condensation, i.e., there is condensation without macroscopic occupation of the ground ($k=0$) state level. © *1999 American Institute of Physics.* [S0022-2488(99)03902-X]

I. INTRODUCTION

The search for microscopic models of interacting bosons showing Bose–Einstein condensation is an ever challenging problem. It is known that the phenomenon only appears for space dimensions $d\geqslant 3$.[1] A general two-body interacting Bose system in a finite centered cubic box $\Lambda\subset\mathbb{R}^d$, with volume $V=L^d$, is given by a Hamiltonian,

$$H_\Lambda=T_\Lambda+U_\Lambda, \tag{1}$$

where

$$T_\Lambda=\sum_{k\in\Lambda^*}\epsilon_k a_k^* a_k, \quad \epsilon_k=\frac{|k|^2}{2m},$$

$$U_\Lambda=\frac{1}{2V}\sum_{q,k,k'\in\Lambda^*}v(q)a_{k+q}^* a_{k'-q}^* a_{k'} a_k, \quad a(x)=\frac{1}{\sqrt{V}}\sum_{k\in\Lambda^*}a_k e^{ik\cdot x}.$$

The $a^\sharp(x)$ are the Boson operators satisfying the commutation rules

$$[a(x),a^*(y)]=\delta(x-y), \quad [a(x),a(y)]=0,$$

and

$$\Lambda^*=\left\{k:k=\frac{2\pi}{L}n, n\in\mathbb{Z}^d\right\}.$$

We limit ourself to periodic boundary conditions.

Rigorous results on the existence of Bose–Einstein condensation are known for very special potentials v in (1), in particular, of course, for $v=0$, the free Bose gas, and for v in the δ-function limit[2] or in the van der Waals limit.[3] Another class of models that are treatable is this for which the Hamiltonian is a function of the number operators $N_k=a_k^* a_k$ only. These models are called the diagonal models.[4] The Hamiltonian is a function of a set of mutually commuting operators with a spectrum consisting of the integers. The operators can be considered as random variables taking values in the integers. The equilibrium states are looked for among the measures minimizing the

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free energy. This method, developed in a series of papers (Ref. 4, and references therein), opened the possibility to derive rigorous results for so far unsolved interacting Bose gas models. The method is a powerful application of the large deviation principle for quantum systems.

In this paper we derive some rigorous results for another diagonal model, inspired by Ref. 5, where the pressure is computed. We are not using the large deviation technique of Ref. 4, but the full quantum mechanical technology, in particular, correlation inequalities, in order to prove the existence of Bose–Einstein condensation. In Sec. II, we first rederive the result of Ref. 5, and give a concise, rigorous, and direct proof of the pressure formula. Some arguments of Ref. 6 are translated into our situation. Our main contribution is in Sec. III, where we prove the occurrence of Bose–Einstein condensation, and where we study in detail the type of condensation.

There exist different types of condensation. The best known is macroscopic occupation of the ground state, but there is also so-called generalized condensation, when the number of particles distributed over a set of arbitrary small energies above the lowest energy level becomes macroscopic, proportional to the volume. This notion has been put into a rigorous and workable form in Ref. 7.

As far as our results are concerned, this notion of generalized condensation is crucial. We prove that in our model generalized condensation occurs without macroscopic occupation of the ground state. As far as we know, this is the first model of an interacting Bose gas for which this type of condensation is found. The only existing result is for the free Bose gas, considering a special thermodynamic limit, not of the type of increasing, absorbing cubes.[8,9]

The result of Sec. III also allows us, using the technique of Ref. 10, to give an explicit form of the equilibrium states in the thermodynamic limit. One verifies that they are of the same type as the equilibrium state of the imperfect Bose gas.

II. THE MODEL

In Ref. 5 Schröder considers a Bose gas contained in a d-dimensional ($d\geqslant 3$) cubic box with Dirichlet boundary conditions on two opposite faces and periodic boundary conditions on the remaining surface. This can be interpreted as the model of a Bose gas enclosed between two hard walls at a macroscopic distance. An interaction term is introduced that behaves locally like the mean field interaction. This gives rise to the following Hamiltonian:

$$H_\Lambda=\sum_{k\in\Lambda^*}\epsilon_k N_{k,\Lambda}+\frac{\lambda}{V}\left(N_\Lambda^2+\frac{1}{2}\sum_{j\in\mathbb{N}}\tilde{N}_{j,\Lambda}^2\right), \tag{2}$$

where

$$\Lambda=\left\{x\in\mathbb{R}^d:-\frac{L}{2}\leqslant x_i\leqslant\frac{L}{2},i=1,\ldots,d-1;0\leqslant x_d\leqslant L\right\};V=L^d,$$

$$\Lambda^*=\frac{2\pi}{L}\mathbb{Z}^{d-1}\times\frac{\pi}{L}\mathbb{N},\qquad N_{k,\Lambda}=a^*(f_{k,\Lambda})a(f_{k,\Lambda}),$$

$$f_{k,\Lambda}=\left(\frac{2}{V}\right)^{1/2}\exp[i(k_1x_1+\cdots+k_{d-1}x_{d-1})]\sin(k_dx_d),$$

$$\lambda\in\mathbb{R}^+,\qquad \tilde{N}_{j,\Lambda}=\sum_{\{k\in\Lambda^*:k_d=(\pi/L)j\}}N_{k,\Lambda},$$

$$N_\Lambda=\sum_{k\in\Lambda^*}N_{k,\Lambda}.$$

Schröder shows that the grand-canonical pressure of this so-called local mean field model coincides with the grand-canonical pressure of the usual mean field model, or imperfect Bose gas, with Hamiltonian

$$H_\Lambda^{\mathrm{MF}} = \sum_{k\in\Lambda^*} \epsilon_k N_{k,\Lambda} + \frac{\lambda}{V} N_\Lambda^2, \tag{3}$$

which is a soluble model.

From this result, Schröder concludes that his model exhibits a phase transition with the same critical behavior as the imperfect Bose gas, although macroscopic occupation of the ground state may not occur, and opens the question of whether generalized condensation, as defined in Ref. 7, does take place.

We study a model of an interacting Bose gas that is inspired by Schröder's model, but that contains a nontrivial part of the self-interaction terms appearing in the general two-body repulsive interaction (1). More precisely, we consider a system of identical bosons in a centered cubic box $\Lambda\in\mathbb{R}^d$, $d\geqslant 3$, with volume $V=L^d$, with periodic boundary conditions for the wave functions, and described by the Hamiltonian

$$H_\Lambda = \sum_{k\in\Lambda^*} \epsilon_k N_{k,\Lambda} + \frac{\lambda}{V}\left(N_\Lambda^2 + \frac{1}{2}\sum_{k\in\Lambda^*} N_{k,\Lambda}^2 \right), \tag{4}$$

where now

$$\Lambda^* = \frac{2\pi}{L}\mathbb{Z}^d, \quad N_{k,\Lambda} = a_{k,\Lambda}^* a_{k,\Lambda},$$

$$a_{k,\Lambda}^* = \frac{1}{\sqrt{V}}\int_\Lambda dx\, e^{ik\cdot x} a^*(x),$$

$$\lambda\in\mathbb{R}^+, \quad N_\Lambda = \sum_{k\in\Lambda^*} N_{k,\Lambda}.$$

Our model can also be compared to the Huang–Yang–Luttinger model, rigorously studied in Ref. 11. Compared to our model, here the interaction terms $N_{k,\Lambda}^2$ appear with a minus sign and are therefore attractive perturbations of the imperfect Bose gas. The attractive character enhances (see Ref. 11) the condensation in the zero mode. The repulsive character of these terms in our model should make condensation in the zero mode more difficult. Heuristically one might expect that our model is a candidate for nonextensive Bose–Einstein condensation.

First we give a new proof, inspired by a proof in Ref. 6, of the main result of Schröder, i.e., the equality of the grand-canonical pressure of this model and the grand-canonical pressure of the imperfect Bose gas. From this we can immediately prove that there is no macroscopic occupation of any single-particle state.

For every μ in $\mathbb{R}$, denote

$$H_\Lambda(\mu) = \sum_{k\in\Lambda^*} \epsilon_k N_{k,\Lambda} + \frac{\lambda}{V}\left(N_\Lambda^2 + \frac{1}{2}\sum_{k\in\Lambda^*} N_{k,\Lambda}^2 \right) - \mu N_\Lambda, \tag{5}$$

and

$$H_\Lambda^{\mathrm{MF}}(\mu) = \sum_{k\in\Lambda^*} \epsilon_k N_{k,\Lambda} + \frac{\lambda}{V} N_\Lambda^2 - \mu N_\Lambda. \tag{6}$$

For $\alpha \leqslant 0$, let

$$\mathcal{C}^{\alpha} = \{ t \in \mathcal{C}^b(\mathbb{R}^d) : \inf_{k \in \mathbb{R}^d} (\epsilon_k - t_k - \alpha) > 0 \},$$

with $\mathcal{C}^b(\mathbb{R}^d)$ the space of continuous bounded functions on $\mathbb{R}^d$. For $t \in \mathcal{C}^{\alpha}$, let

$$H_\Lambda^{t+\alpha} = \sum_{k \in \Lambda^*} (\epsilon_k - t_k - \alpha) N_{k,\Lambda}.$$

First, we prove the following.

Lemma 1:

$$\frac{1}{\beta V} \ln \operatorname{tr} e^{-\beta H_\Lambda(\mu)} \geqslant \frac{1}{\beta V} \ln \operatorname{tr} e^{-\beta H_\Lambda^{t+\alpha}} - \frac{1}{V} \omega_\Lambda^{t+\alpha}(H_\Lambda(\mu) - H_\Lambda^{t+\alpha}), \tag{7}$$

with

$$\omega_\Lambda^{t+\alpha}(A) = \frac{\operatorname{tr} e^{-\beta H_\Lambda^{t+\alpha}} A}{\operatorname{tr} e^{-\beta H_\Lambda^{t+\alpha}}}.$$

Proof: The function $x \in [0,1] \mapsto \ln \operatorname{tr} e^{C + xD}$, for C and D self-adjoint is convex. Hence, define the convex function f on $[0,1]$ by

$$f(x) = \ln \operatorname{tr} e^{-\beta (x H_\Lambda(\mu) + (1-x) H_\Lambda^{t+\alpha})}.$$

For all *a,b* in $[0,1]$, $f(a) - f(b) - (a-b) f'(b) \geqslant 0$, in particular,

$$f(1) \geqslant f(0) + f'(0),$$

which immediately yields the stated inequality. □

We can now prove a first result.

Theorem 1: *The grand-canonical pressure at chemical potential* μ,

$$\tilde{p}(\mu) = \lim_{V \to \infty} \tilde{p}_\Lambda(\mu) = \lim_{V \to \infty} \frac{1}{\beta V} \ln \operatorname{tr} e^{-\beta H_\Lambda(\mu)},$$

exists for every μ *in* $\mathbb{R}$ *and is given by*

$$\tilde{p}(\mu) = p^{\mathrm{MF}}(\mu) = \inf_{\alpha \leqslant 0} \left(p(\alpha) + \frac{(\mu - \alpha)^2}{4\lambda} \right),$$

with $p^{MF}(\mu)$ *the grand-canonical pressure of the imperfect Bose gas at chemical potential* μ *and* $p(\alpha)$ *the free-gas grand-canonical pressure at chemical potential* α.

[The expression for $p^{\mathrm{MF}}(\mu)$ is computed in Ref. 3.]

Proof: Since for every $\mu \in \mathbb{R}$, $H_\Lambda(\mu) \geqslant H_\Lambda^{\mathrm{MF}}(\mu)$, we have

$$\tilde{p}_\Lambda(\mu) \leqslant p_\Lambda^{\mathrm{MF}}(\mu),$$

and hence

$$\limsup_{V \to \infty} \tilde{p}_\Lambda(\mu) \leqslant \lim_{V \to \infty} p_\Lambda^{\mathrm{MF}}(\mu) = p^{\mathrm{MF}}(\mu).$$

To prove the lower bound, we make use of Lemma 1. For $\alpha \leqslant 0$ and $t \in \mathcal{C}^{\alpha}$, let

$$\rho(k;t,\alpha)=\frac{1}{e^{\beta(\epsilon_k-t_k-\alpha)}-1}.$$

Then

$$\omega_\Lambda^{t+\alpha}(N_{k,\Lambda})=\rho(k;t,\alpha),$$

$$\omega_\Lambda^{t+\alpha}(N_{k,\Lambda}N_{k',\Lambda})=\rho(k;t,\alpha)\rho(k';t,\alpha),\quad \text{if } k\neq k',$$

$$\omega_\Lambda^{t+\alpha}(N_{k,\Lambda}^2)=\rho(k;t,\alpha)(2\rho(k;t,\alpha)+1).$$

We calculate the rhs of (7). The first term gives

$$\frac{1}{\beta V}\ln \operatorname{tr} e^{-\beta H_\Lambda^{t+\alpha}}=-\frac{1}{\beta V}\sum_{k\in\Lambda^*}\ln(1-e^{-\beta(\epsilon_k-t_k-\alpha)}).$$

To calculate $(1/V)\omega_\Lambda^{t+\alpha}(H_\Lambda(\mu))$, we write

$$H_\Lambda(\mu)=\sum_{k\in\Lambda^*}(\epsilon_k-\mu)N_{k,\Lambda}+\frac{\lambda}{V}\sum_{k\in\Lambda^*}\sum_{k'\neq k\in\Lambda^*}N_{k,\Lambda}N_{k',\Lambda}+\frac{3\lambda}{2V}\sum_{k\in\Lambda^*}N_{k,\Lambda}^2,$$

hence

$$\frac{1}{V}\omega_\Lambda^{t+\alpha}(H_\Lambda(\mu))=\frac{1}{V}\sum_{k\in\Lambda^*}(\epsilon_k-\mu)\rho(k;t,\alpha)+\frac{\lambda}{V^2}\sum_{k\in\Lambda^*}\sum_{k'\neq k\in\Lambda^*}\rho(k;t,\alpha)\rho(k';t,\alpha)+\frac{c_V}{V},$$

where

$$c_V=\frac{3\lambda}{2V}\sum_{k\in\Lambda^*}\rho(k;t,\alpha)(2\rho(k;t,\alpha)+1).$$

Also,

$$\frac{1}{V}\omega_\Lambda^{t+\alpha}(H_\Lambda^{t+\alpha})=\frac{1}{V}\sum_{k\in\Lambda^*}(\epsilon_k-t_k-\alpha)\rho(k;t,\alpha).$$

Substituting all this in (7), we get

$$\tilde{p}_\Lambda(\mu)\geqslant-\frac{1}{\beta V}\sum_{k\in\Lambda^*}\ln(1-e^{-\beta(\epsilon_k-t_k-\alpha)})+\frac{1}{V}\sum_{k\in\Lambda^*}(\mu-t_k-\alpha)\rho(k;t,\alpha)$$
$$-\frac{\lambda}{V^2}\sum_{k\in\Lambda^*}\sum_{k'\neq k\in\Lambda^*}\rho(k;t,\alpha)\rho(k';t,\alpha)-\frac{c_V}{V}.$$

Since $\rho(k;t,\alpha)$ and c_V, for V large enough, are bounded

$$\liminf_{V\to\infty}\tilde{p}_\Lambda(\mu)\geqslant-\beta^{-1}\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\ln(1-e^{-\beta(\epsilon_k-t_k-\alpha)})$$
$$+\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}(\mu-t_k-\alpha)\rho(k;t,\alpha)-\lambda\left(\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\rho(k;t,\alpha)\right)^2. \tag{8}$$

For $\alpha\leqslant 0$ the free-gas pressure is given by

$$p(\alpha)=-\beta^{-1}\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\ln(1-e^{-\beta(\epsilon_k-\alpha)})$$

and

$$p'(\alpha)=\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\frac{1}{e^{\beta(\epsilon_k-\alpha)}-1}.$$

Also, let $\rho_c=p'(0)$ as usual.

First, consider the case $\mu<2\lambda\rho_c$. Taking $\alpha<0$ and $t=0$ in (8) we get

$$\liminf_{V\to\infty}\tilde{p}_\Lambda(\mu)\geqslant p(\alpha)+(\mu-\alpha)p'(\alpha)-\lambda(p'(\alpha))^2. \tag{9}$$

For $\mu<2\lambda\rho_c$, since $p'(\alpha)$ is increasing and $p'(0)=\rho_c$, the equation

$$p'(\alpha)=\frac{\mu-\alpha}{2\lambda}$$

has a unique solution $\alpha^*<0$. Taking $\alpha=\alpha^*$ in (9), we get

$$\liminf_{V\to\infty}\tilde{p}_\Lambda(\mu)\geqslant p(\alpha^*)+\frac{(\mu-\alpha^*)^2}{4\lambda}=\inf_{\alpha\leqslant 0}\left\{p(\alpha)+\frac{(\mu-\alpha)^2}{4\lambda}\right\}=p^{\mathrm{MF}}(\mu),$$

which proves the theorem for $\mu<2\lambda\rho_c$.

Consider now the case $\mu\geqslant 2\lambda\rho_c$. Take $\alpha=0$ and an appropriate t in (8):

$$\liminf_{V\to\infty}\tilde{p}_\Lambda(\mu)\geqslant-\beta^{-1}\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\ln(1-e^{-\beta(\epsilon_k-t_k)})$$
$$+\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}(\mu-t_k)\rho(k;t)-\lambda\left(\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\rho(k;t)\right)^2, \tag{10}$$

with

$$\rho(k;t)=\frac{1}{e^{\beta(\epsilon_k-t_k)}-1}.$$

For all $\delta>0$, take $t_\delta\in\mathcal{C}^0$ such that

$$t_\delta(k)=0,\quad |k|>\delta.$$

Then

$$\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\rho(k;t_\delta)=\int_{|k|\leqslant\delta}\frac{dk}{(2\pi)^d}\rho(k;t_\delta)+\int_{|k|>\delta}\frac{dk}{(2\pi)^d}\frac{1}{e^{\beta\epsilon_k}-1}.$$

Letting $\delta\to 0$, the second term on the rhs converges to ρ_c. Take t_δ such that the first term on the rhs converges to $\mu/2\lambda-\rho_c$ as $\delta\to 0$. Such a sequence of t_δ's can be constructed rigorously by using the Approximation theorem proved in Ref. 12. It certainly means that $t_\delta\to 0$ as $\delta\to 0$. Hence we get

$$\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\rho(k;t_\delta)\to\frac{\mu}{2\lambda},$$

as $\delta \to 0$, and thus

$$\liminf_{V\to\infty} \tilde{p}_\Lambda(\mu) \geq p(0) + \frac{\mu^2}{4\lambda} \geq \inf_{\alpha\leq 0}\left\{ p(\alpha) + \frac{(\mu-\alpha)^2}{4\lambda}\right\} = p^{\mathrm{MF}}(\mu),$$

so that the theorem is proved for $\mu \geq 2\lambda\rho_c$ as well. □

From Theorem 1 we can immediately derive that there is no macroscopic occupation of any single-particle state, in particular, the following.

Theorem 2: *For every $\epsilon > 0$ and for V large enough, we have, for every $k \in \Lambda^*$:*

$$\frac{1}{V}\omega_\Lambda(N_{k,\Lambda}) < \epsilon,$$

where ω_Λ is the finite-volume Gibbs state of $H_\Lambda(\mu)$.

Proof: We have

$$e^{\beta V p_\Lambda^{\mathrm{MF}}(\mu)} = \mathrm{tr}\, e^{-\beta H_\Lambda^{\mathrm{MF}}(\mu)} = \mathrm{tr}(e^{-\beta H_\Lambda(\mu)} e^{(\beta\lambda/2V)\Sigma_{k\in\Lambda^*} N_{k,\Lambda}^2}) = \omega_\Lambda(e^{(\beta\lambda/2V)\Sigma_{k\in\Lambda^*} N_{k,\Lambda}^2}) e^{\beta V \tilde{p}_\Lambda(\mu)}.$$

Hence,

$$p_\Lambda^{\mathrm{MF}} = \tilde{p}_\Lambda(\mu) + \frac{1}{\beta V}\ln \omega_\Lambda(e^{(\beta\lambda/2V)\Sigma_{k\in\Lambda^*} N_{k,\Lambda}^2}).$$

By Theorem 1 we get

$$\lim_{V\to\infty}\frac{1}{\beta V}\ln \omega_\Lambda(e^{(\beta\lambda/2V)\Sigma_{k\in\Lambda^*} N_{k,\Lambda}^2}) = 0.$$

From the Jensen inequality, i.e., for F a convex function and ω a normal state,

$$\omega(F(X)) \geq F(\omega(X)),$$

we get

$$\omega_\Lambda(e^{(\beta\lambda/2V)\Sigma_{k\in\Lambda^*} N_{k,\Lambda}^2}) \geq e^{(\beta\lambda/2V)\Sigma_{k\in\Lambda^*}\omega_\Lambda(N_{k,\Lambda}^2)},$$

or

$$0 \leq \frac{\lambda}{2V^2}\sum_{k\in\Lambda^*}\omega_\Lambda(N_{k,\Lambda}^2) \leq \frac{1}{\beta V}\ln \omega_\Lambda(e^{(\beta\lambda/2V)\Sigma_{k\in\Lambda^*} N_{k,\Lambda}^2}).$$

Hence

$$\lim_{V\to\infty}\frac{1}{V^2}\sum_{k\in\Lambda^*}\omega_\Lambda(N_{k,\Lambda}^2) = 0.$$

Since for each $k \in \Lambda^*$,

$$0 \leq \left(\frac{1}{V}\omega_\Lambda(N_{k,\Lambda})\right)^2 \leq \frac{1}{V^2}\omega_\Lambda(N_{k,\Lambda}^2) \leq \frac{1}{V^2}\sum_{k'\in\Lambda^*}\omega_\Lambda(N_{k',\Lambda}^2),$$

we get the Theorem. □

III. BOSE–EINSTEIN CONDENSATION

In Ref. 7 it is stressed that Bose condensation does not necessarily manifest itself through a macroscopic occupation of a single-particle state (the ground state usually), but that there are, in fact, two good candidates for the concept of macroscopic occupation of the zero-kinetic energy state. Macroscopic occupation of the ground state is said to occur when the number of particles in the ground state becomes proportional to the volume; generalized condensation is said to occur when the number of particles whose energy levels lie in an arbitrary small band above zero becomes proportional to the volume. Obviously, the first implies the second. However, the second can occur without the first; this is called nonextensive condensation. The concept of generalized condensation was first introduced in Ref. 13. More precisely, we have the following.

(i) Macroscopic occupation of the ground state if the limit

$$\lim_{V\to\infty}\frac{1}{V}\,\omega_\Lambda(N_{0,\Lambda})$$

exists and is strictly positive; (ii) generalized condensation if the limit

$$\lim_{\delta\to 0}\lim_{V\to\infty}\frac{1}{V}\sum_{\{k\in\Lambda^*:\epsilon_k<\delta\}}\omega_\Lambda(N_{k,\Lambda})$$

exists and is strictly positive; (iii) nonextensive condensation if the limit

$$\lim_{V\to\infty}\frac{1}{V}\,\omega_\Lambda(N_{0,\Lambda})=0,$$

but nevertheless the limit

$$\lim_{\delta\to 0}\lim_{V\to\infty}\frac{1}{V}\sum_{\{k\in\Lambda^*:\epsilon_k<\delta\}}\omega_\Lambda(N_{k,\Lambda})$$

exists and is strictly positive.

Examples of these different occurrences of Bose condensation in the free Bose gas, depending on how the bulk limit is taken, can be found in Refs. 7–9.

As is proved in Theorem 2, there is no macroscopic occupation of the ground state in our system. However, as we will show, there is generalized condensation. In other words, we have a model for an interacting Bose gas displaying nonextensive condensation.

Our approach is based on Ref. 10, where the imperfect Bose gas is treated. The system is given by the local Hamiltonian H_Λ, with periodic boundary conditions

$$H_\Lambda=\sum_{k\in\Lambda^*}\epsilon_k N_{k,\Lambda}+\frac{\lambda}{V}\left(N_\Lambda^2+\frac{1}{2}\sum_{k\in\Lambda^*}N_{k,\Lambda}^2\right)-\mu_\Lambda N_\Lambda, \tag{11}$$

as specified before, and μ_Λ is determined by the constant density $\rho>0$ equation:

$$\frac{1}{V}\,\omega_\Lambda(N_\Lambda)=\rho.$$

We study the equilibrium state of this system in the grand-canonical ensemble. The key technique is the equivalence of the equilibrium condition or Gibbs state ω_Λ with the correlation inequalities[14,15]

$$\beta\omega_\Lambda(X^*[H_\Lambda,X]) \geqslant \omega_\Lambda(X^*X)\ln\frac{\omega_\Lambda(X^*X)}{\omega_\Lambda(XX^*)}, \tag{12}$$

for all local observables X belonging to the domain of $[H_\Lambda,\cdot]$. In particular, we take for X polynomials in the creation and annihilation operators. We prove the occurrence of nonextensive condensation in this model, and follow closely the method used in Ref. 10.

Lemma 2: $\forall k,\ j\in\Lambda^*$:

(i)

$$\beta\omega_\Lambda\left(-\epsilon_k N_{k,\Lambda}+\left(\mu_\Lambda-\frac{2\lambda}{V}N_\Lambda\right)N_{k,\Lambda}-\frac{\lambda}{V}N_{k,\Lambda}^2+\frac{3\lambda}{2V}N_{k,\Lambda}\right)\geqslant\omega_\Lambda(N_{k,\Lambda})\ln\frac{\omega_\Lambda(N_{k,\Lambda})}{\omega_\Lambda(N_{k,\Lambda})+1}; \tag{13}$$

(ii)

$$\omega_\Lambda\left(\left(\mu_\Lambda-\frac{2\lambda}{V}N_\Lambda\right)N_{k,\Lambda}\right)\leqslant\omega_\Lambda\left(\epsilon_j N_{k,\Lambda}+\frac{4\lambda}{V}N_{j,\Lambda}N_{k,\Lambda}+\frac{3\lambda}{2V}N_{k,\Lambda}\right). \tag{14}$$

Proof: For *(i)*, the result follows by taking $X=a_k$ in the correlation inequality (12). One gets *(ii)* by taking

$$X=a_j N_{k,\Lambda}^{1/2},$$

in the inequality

$$\omega_\Lambda([X^*,[H_\Lambda,X]])\geqslant 0,$$

which follows immediately from (12) by adding the correlation inequality for X and the complex conjugate of the correlation inequality for X^*. □

Lemma 3: For every $\delta>0$, *for every* V *and for every* $k\in\Lambda^*$, $|k|\geqslant\delta$,

$$\omega_\Lambda(N_{k,\Lambda})\leqslant\frac{1}{e^{c_k(\Lambda)}-1}+\frac{4\lambda}{V}\omega_\Lambda(N_{j,\Lambda}N_{k,\Lambda})\frac{1}{1-e^{-c_\delta(\Lambda)}},$$

with

$$c_k(\Lambda)=\beta\left(\epsilon_k-\frac{\delta^2}{8m}-\frac{3\lambda}{V},\right),$$

$c_\delta(\Lambda)=c_k(\Lambda)|_{|k|=\delta}$ *and* $j\in\Lambda^*$, $|j|\leqslant\delta/2$.

Proof: Substitution of (14) in (13), changing the sign, and using the trivial bound $\omega_\Lambda(N_{k,\Lambda}^2)\geqslant 0$ we get

$$\beta\left(\epsilon_k-\epsilon_j-\frac{3\lambda}{V}\right)\omega_\Lambda(N_{k,\Lambda})-\frac{4\lambda}{V}\omega_\Lambda(N_{j,\Lambda}N_{k,\Lambda})\leqslant\omega_\Lambda(N_{k,\Lambda})\ln\frac{\omega_\Lambda(N_{k,\Lambda})+1}{\omega_\Lambda(N_{k,\Lambda})}. \tag{15}$$

Take $\delta>0$ arbitrary, $|k|\geqslant\delta$, and $|j|\leqslant\delta/2$.

Using $\epsilon_j\leqslant\delta^2/8m$, (15) becomes

$$c_k(\Lambda)\omega_\Lambda(N_{k,\Lambda})-\frac{4\lambda}{V}\omega_\Lambda(N_{j,\Lambda}N_{k,\Lambda})\leqslant\omega_\Lambda(N_{k,\Lambda})\ln\frac{\omega_\Lambda(N_{k,\Lambda})+1}{\omega_\Lambda(N_{k,\Lambda})}.$$

The lemma now follows from convexity arguments on the rhs: we want to solve for t the inequality

$$ct-b\leq t\ln\frac{t+1}{t},$$

with c and b positive constants and $t\in\mathbb{R}^+$. It follows that $t\leq t_2$, with t_2, satisfying

$$ct_2-b=t_2\ln\frac{t_2+1}{t_2}.$$

One can write this as $t\leq t_1+(t_2-t_1)$, with

$$t_1=\frac{1}{e^c-1}.$$

Let $f(t)=t\ln(t+1)/t$, f is concave, hence

$$f(t_2)-f(t_1)-(t_2-t_1)f'(t_1)\leq 0,$$

and

$$t_2-t_1\leq b\,\frac{1}{t-e^{-c}}.$$

Substitute this into the inequality $t\leq t_1+(t_2-t_1)$, one gets

$$t\leq\frac{1}{e^c-1}+b\,\frac{1}{1-e^{-c}}.$$

Finally, use $|k|\geq\delta$ in the second term on the rhs to prove the lemma. □

Lemma 4: For every $\epsilon>0$, *V large enough and* $j\in\Lambda^*$:

$$\frac{1}{V^2}\,\omega_\Lambda(N_\Lambda N_{j,\Lambda})<\epsilon.$$

Proof: (13) gives

$$\beta\omega_\Lambda\left(-\epsilon_j N_{j,\Lambda}+\left(\mu_\Lambda-\frac{2\lambda}{V}N_\Lambda\right)N_{j,\Lambda}+\frac{3\lambda}{2V}N_{j,\Lambda}\right)\geq\omega_\Lambda(N_{j,\Lambda})\ln\frac{\omega_\Lambda(N_{j,\Lambda})}{\omega_\Lambda(N_{j,\Lambda})+1}\geq -1.$$

This can be rewritten in the form

$$\frac{2\lambda}{V^2}\,\omega_\Lambda(N_\Lambda N_{j,\Lambda})\leq\frac{1}{\beta V}+\left(\mu_\Lambda+\frac{3\lambda}{2V}-\epsilon_j\right)\frac{1}{V}\,\omega_\Lambda(N_{j,\Lambda}). \tag{16}$$

Taking $X=a_j$ in the inequality

$$\omega_\Lambda([X^*,[H_\Lambda,X]])\geq 0,$$

gives

$$\mu_\Lambda\leq 2\lambda\rho+\epsilon_j+\frac{4\lambda}{V}\,\omega_\Lambda(N_{j,\Lambda})+\frac{3\lambda}{2V}.$$

Putting this into (16) gives

$$\frac{2\lambda}{V^2}\omega_\Lambda(N_\Lambda N_{j,\Lambda}) \leqslant \frac{1}{\beta V} + \left(2\lambda\rho + \frac{3\lambda}{V}\right)\frac{1}{V}\omega_\Lambda(N_{j,\Lambda}) + \frac{4\lambda}{V^2}\omega_\Lambda(N_{j,\Lambda})^2.$$

Using Theorem 2 proves the lemma. □

We now prove the existence of generalized condensation in the thermodynamic limit $V \to\infty$, taken with constant particle density ρ.

Theorem 3: *One has (i)*

$$\lim_{\delta\to 0}\lim_{V\to\infty}\frac{1}{V}\sum_{\{k\in\Lambda^*:|k|<\delta\}}\omega_\Lambda(N_{k,\Lambda}) \geqslant \rho - \int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\frac{1}{e^{\beta\epsilon_k}-1};$$

(ii) for every $\rho>0$, there is a β_c such that for all $\beta>\beta_c$:

$$0 < \lim_{\delta\to 0}\liminf_{V\to\infty}\frac{1}{V}\sum_{\{k\in\Lambda^*:|k|<\delta\}}\omega_\Lambda(N_{k,\Lambda}) \leqslant \lim_{\delta\to 0}\limsup_{V\to\infty}\frac{1}{V}\sum_{\{k\in\Lambda^*:|k|<\delta\}}\omega_\Lambda(N_{k,\Lambda}) \leqslant \rho.$$

Proof: We have clearly

$$\frac{1}{V}\sum_{\{k\in\Lambda^*:|k|<\delta\}}\omega_\Lambda(N_{k,\Lambda}) = \rho - \frac{1}{V}\sum_{\{k\in\Lambda^*:|k|\geqslant\delta\}}\omega_\Lambda(N_{k,\Lambda}).$$

Applying Lemma 3 gives

$$\frac{1}{V}\sum_{\{k\in\Lambda^*:|k|<\delta\}}\omega_\Lambda(N_{k,\Lambda}) \geqslant \rho - \frac{1}{V}\sum_{\{k\in\Lambda^*:|k|\geqslant\delta\}}\frac{1}{e^{c_k(\Lambda)}-1} - \frac{4\lambda}{V^2}\sum_{\{k\in\Lambda^*:|k|\geqslant\delta\}}\omega_\Lambda(N_{j,\Lambda}N_{k,\Lambda})\frac{1}{1-e^{-c_\delta(\Lambda)}}. \tag{17}$$

Take $\epsilon>0$ arbitrary, and V large enough such that Lemma 4 is satisfied. This implies that

$$\frac{1}{V^2}\sum_{\{k\in\Lambda^*:|k|\geqslant\delta\}}\omega_\Lambda(N_{j,\Lambda}N_{k,\Lambda}) \leqslant \frac{1}{V^2}\omega_\Lambda(N_\Lambda N_{j,\Lambda}) < \epsilon.$$

Hence taking V large enough, the second term on the rhs of (17) can be made arbitrarily close to

$$\int_{|k|\geqslant\delta}\frac{dk}{(2\pi)^d}\frac{1}{e^{\beta(\epsilon_k-\delta^2/8m)}-1},$$

whereas the third term is made arbitrarily small.

Hence in the limit $V\to\infty$, one gets

$$\lim_{V\to\infty}\frac{1}{V}\sum_{\{k\in\Lambda^*:|k|<\delta\}}\omega_\Lambda(N_{k,\Lambda}) \geqslant \rho - \int_{|k|\geqslant\delta}\frac{dk}{(2\pi)^d}\frac{1}{e^{\beta(\epsilon_k-\delta^2/8m)}-1}.$$

Now take the limit $\delta\to 0$ to get *(i)*.

The function

$$\beta\mapsto f(\beta) = \int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\frac{1}{e^{\beta\epsilon_k}-1}$$

is clearly decreasing in $\beta>0$ and, furthermore, $f(\beta)\to\infty$ for $\beta\to 0$, and $f(\beta)\to 0$ for $\beta\to\infty$. Hence, for every $\rho>0$ there exists $\beta_c>0$, defined by

$$\rho=\int_{\mathbb{R}^d}\frac{dk}{(2\pi)^d}\,\frac{1}{e^{\beta_c\epsilon_k}-1}.$$

Together with *(i)* this proves *(ii)*. □

Theorem 1 proves that the model (11) has the same pressure as the imperfect Bose gas. Theorems 2 and 3 prove that the model (11) shows a Bose–Einstein condensation exactly as the imperfect Bose gas, be it that the nature of the condensation is different. One aspect of this is that the ground state $(k=0)$ condensation of the imperfect Bose gas is unstable against any arbitrary small repulsive perturbation of the type $(\gamma/V)\Sigma_{k\in\Lambda*}N_{k,\Lambda}^2$, for any $\gamma>0$. The condensation becomes nonextensive. However, on the level of the thermodynamics the models are similar.

The natural question to ask is, whether the equilibrium states of the two models coincide. For the imperfect Bose gas, this problem is solved, e.g., in Ref. 10. We are not going into the details, but the technique of Ref. 10 can also be used in order to solve rigorously the equilibrium—or KMS—equations of our model. The result is that all equilibrium states are of the same type as the ones of the imperfect Bose gas. In particular, the equilibrium states are also integrals over a set of quasifree or generalized free states.

On the other hand, it is interesting to remark the following. Given this result, one might ask whether the variational principle of statistical mechanics, formulated in the thermodynamic limit, but restricted to the set of quasifree states, does also give the results of this paper, namely, the existence of condensation and the equilibrium states. Performing this program, one remarks that the particular type of condensation is not recovered by this method. Hence, for the time being, the only way to keep track of it is to follow closely the details of the thermodynamic limit, as is done above. In this work we illustrate clearly that care must be taken of this limit and that statistical mechanics remains the theory of really handling the thermodynamic limit.

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