Modelling replenishment and transhipment decisions in periodic review multi-location inventory systems

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Abstract: Effective models of key operational decisions in multilocation inventory systems are important for a successful retail sector. This paper argues that much of the existing research in this area is not applicable to a highly competitive retail environment, particularly if periodic review replenishment policies are used. The paper develops a model of a periodic review multilocation inventory system that is suitable for this environment and investigates the characteristics of optimal replenishment and transshipment decisions. This motivates the development of three simple heuristic transshipment policies that are practical for systems with many locations. The results of a numerical study involving systems with five locations suggest that the performance of these heuristic policies is often close to optimal and can be considerably better than the performance of commonly used policies.

Keywords: Inventory, Dynamic Programming, Markov Decision Processes, Transshipment, Retail

1 Introduction

It is common for a retail company to use several outlets or *locations* to distribute its products throughout a region. The efficient management of multilocation inventory systems is therefore important to retail companies as they strive to satisfy the conflicting objectives of maximizing customer service and minimizing cost. The key issues in the management of such systems are how to organize the ordering process and how to organize the distribution of inventory between locations. Changes to the distribution of inventory involve transshipments between the locations in the system. Models of inventory systems that allow such transshipments can generally be classified as one of the following three types.

1. **Periodic review systems that allow transshipments at a single point during a period before the demand for the period is fully known.** One of the first such models is due to Gross (1963) who characterizes an optimal policy for a two location system in which replenishment and transshipment decisions are taken together at the beginning of each period. Das (1975) analyses a variant of this model in which the transshipment decision is taken at a fixed point during each period. Jonsson and Silver (1987) examine a model in which the objective is to minimize backorders rather than cost. The transshipment decision is taken a fixed time before the replenishment decision and the model allows for non-zero transshipment lead time and an arbitrary number of locations. Bertrand and Bookbinder (1998) consider this model with the objective of

minimizing cost for the case of zero transshipment lead time. It is common for retailers to initiate a transshipment after a stockout has arisen, so these models do not fully satisfy the requirements of the retail industry.

2. **Periodic review systems that allow transshipments after the demand for the period is known, but before it has to be satisfied.** The model of Krishnan and Rao (1965) minimizes cost in a multilocation system with zero replenishment and transshipment lead times. Robinson (1990) characterizes the form of close-to-optimal policies for similar systems. Tagaras and Cohen (1992) examine a model with two locations and non-zero replenishment lead time. Rudi et al. (2001) investigate the conflict between maximizing location and system profits in a two location model. These models require a non-negligible lead time on the service of customer requests to allow the total demand to become apparent before transshipments are arranged. In highly competitive retail situations, such a delay would often lead to lost sales and so these models would not be appropriate.

3. **Continuous review systems that allow transshipments in response to stockouts and use a one-for-one replenishment policy.** These models extend the multiechelon inventory model for repairable items with Poisson demand developed by Sherbrooke (1968) by allowing transshipments. Lee (1987) develops a method of determining the minimum cost inventory position for a system that allows transshipments between identical locations and finds approximations to measures of system performance including the expected number of backorders and transshipments. Axsäter (1990) and Sherbrooke (1992) propose similar approximations for systems that allow transshipments between non-identical locations. Kukreja et al. (2001) develop a heuristic to determine replenishment and transshipment policies for a system with non-identical locations under the objective of minimizing cost. These models are only appropriate for slow moving, expensive and/or repairable items, but transshipments are more widely used.

There are a few recent papers that do not fall into any of the categories above. Evers (2001) and Minner et al. (2003) develop heuristics to determine whether or not to make a transshipment in a continuous review multilocation inventory system facing a stockout. Axsäter (2003a) develops an approximate method of determining the replenishment policy for a continuous review multilocation inventory system in which a location facing a stockout sources items from locations with lower shortage costs whenever possible. Axsäter (2003b) extends this model by relaxing the assumption that the decision rule for transshipments is given and develops a heuristic to determine whether or not to make a transshipment in response to a stockout. This heuristic is based on the assumption that no further transshipments will be possible. It is not uncommon for retail companies to use periodic review replenishment policies, so these models are not always appropriate. Archibald et al. (1997) analyse a periodic review inventory system with Poisson demand and unlimited transshipments during a period in response to stockouts, but their model only allows two locations.

The above review shows that there is a gap between existing mathematical models and the requirements of a retail company operating in highly competitive environment and using periodic review replenishment policies. This paper aims to address this issue by developing a model of a periodic review inventory system with an arbitrary number of locations in which transshipments may occur at any time in response to stockouts.

Due to the difficulty of determining optimal transshipment policies for complex multilocation inventory systems, simple heuristic policies are often used in practice. The most common of these are: "no pooling" in which case transshipments are never used; and "complete pooling" in which case, when a stockout arises, a transshipment is used to satisfy demand whenever possible. The main weakness of these heuristics is that they do not consider the likelihood of items being used before the next replenishment. For example if the item to be transshipped is almost certain to be required at some other location before the next replenishment, transshipment effectively moves the location of the stockout as well as the item. On the other hand, if the item is almost certainly not required, transshipment is likely to improve customer satisfaction and reduce system holding cost. Effective heuristic transshipment policies should depend on inventory levels and the time until the next replenishment, but it is not obvious what form this dependency might take in general. This paper develops three heuristics with this property and demonstrates via a numerical investigation that these can be highly effective for a range of problems with different characteristics.

The model considered in this paper assumes a periodic review replenishment policy. Such policies may be used due to practical constraints such as the frequency with which a supplier delivers to an area. It is also assumed that locations are not open 24 hours per day and that all replenishment orders and deliveries are completed during periods in which the locations are closed (e.g. overnight). Hence the lead time for replenishment orders is effectively zero. Ideally all demand arising at a location would be satisfied from stock held at that location. However due to uncertainty in demand, this will not always be possible. When a stockout occurs at a location, it is assumed that the company can either place an emergency order with the supplier or transship an item from another location in the area. Customers are assumed always to accept the alternative offered. Such an assumption is reasonable only if the lead times for emergency orders and transshipments are negligible. Figure 1 shows the flow of inventory in the system resulting from the decisions that can be taken. The unit cost of transshipment is less than that of an emergency order, so transshipment is better in a myopic sense. However this benefit must be weighed against the increased risk of further stockouts in the system. The objective is to minimize the long-run average cost of managing the system. (An alternative interpretation of this model is that failure to supply demand from stock held in the system results in a lost sale. In this case the cost of an "emergency order" would be interpreted as the cost of a lost sale.)

Section 2 presents a full description of the problem and formulates the problem as a Markov decision process. Due to the curse of dimensionality, it is not practical to solve the Markov decision process for instances of the problem with more than 5 or 6 locations. Instead sections 3 and 4 analyse the Markov decision process formulation to establish characteristics of optimal replenishment and transshipment policies for certain cases, and use these to develop three heuristic transshipment policies for the general problem. Section 5 presents a numerical comparison of the performance of the proposed heuristic policies. This shows that the proposed heuristics are often close to optimal in terms of long-run average cost and are generally considerably better than the no pooling and complete pooling heuristics used in practice.

2 A model of a multilocation single product inventory system

Consider a multilocation inventory system consisting of m locations that uses a periodic review replenishment policy in which all locations are replenished simultaneously from an external supplier with infinite supply. Replenishment lead time is zero and the cost of replenishment is c per item, independent of the location. The time between successive review epochs is fixed and divided into T intervals of length δ, where δ is chosen so that the probability of more than one customer demand in the system in an interval of length δ is negligible. Generally as the average system demand increases, more intervals will be required to approximate the demand process. However the complexity of the dynamic programming solution methods proposed in this paper increase linearly with the number of intervals, so this is not a major issue. Let p_k, $1 \leq k \leq m$, denote the probability that a customer demand arises at location k in an interval of length δ. Define $p_0 = 1 - \sum_{k=1}^{m} p_k$ to be the probability that there is no customer demand in an interval of length δ. Hence the probability distribution of the system demand between successive review epochs is binomial $(T, 1-p_0)$, and the probability distribution of demand at location k between successive review epochs is binomial (T, p_k). Location k has a limited storage capacity for M_k items. A holding cost of h_k is incurred for each item in stock at location k immediately before a review epoch.

Whenever possible, customer demand is met from local stock because this action involves no additional cost. When a customer demand cannot be met from local stock, an item is sourced either from the supplier via an emergency order or another location via transshipment. An emergency order costs E and is assumed to cost more than a regular order (i.e. $E > c$), as otherwise it would be optimal to use emergency orders to satisfy all demand. Let $T_{i,j}$ denote the cost of transshipping an item from location i to location j. Lead time for emergency orders and transshipments is zero.

For a given replenishment decision, the problem of minimizing the expected cost until the next review epoch is modelled as a Markov decision process. The state of the system is defined to be a vector describing the stock level in each of the locations at the beginning of an interval. When the state of the system is described by the vector $\boldsymbol{i}$, the stock level in location k is i_k. Due to the limited storage capacity at the locations, the number of possible states is finite. Define $Y(\boldsymbol{i})$ and $Z(\boldsymbol{i})$ to be the set of locations with local stock and no local stock respectively when the system is in state $\boldsymbol{i}$ (i.e. $Y(\boldsymbol{i}) = \{k : i_k > 0, 1 \leq k \leq m\}$ and $Z(\boldsymbol{i}) = \{k : i_k = 0, 1 \leq k \leq m\}$).

Define $\omega_t(\boldsymbol{i})$ to be the minimum expected cost until the next review epoch when the time until the next review epoch is δt and the state of the system is $\boldsymbol{i}$. Following the assumptions of the model above, $\omega_0(\boldsymbol{i})$ must account for the cost of holding the stock remaining at the next review epoch. Define $x_t(k, \boldsymbol{i})$ to be the minimum expected cost until the next review epoch when the time until the next review epoch is δt, the state of the system is $\boldsymbol{i}$ where $i_k = 0$ and a customer demand will arise at location k in the next interval. There are at most m ways of satisfying such demand, by emergency order or by transshipment from one of at most $m-1$ locations in $Y(\boldsymbol{i})$. The problem is therefore a finite horizon, discrete time Markov decision process with finite state and action spaces (see e.g. Puterman, 1994), and the value functions must satisfy the following optimality equations.

$$\omega_t(\boldsymbol{i}) = p_0\omega_{t-1}(\boldsymbol{i}) + \sum_{k\in Y(\boldsymbol{i})} p_k\omega_{t-1}(\boldsymbol{i}-\boldsymbol{e_k}) + \sum_{k\in Z(\boldsymbol{i})} p_k x_t(k,\boldsymbol{i}) \quad \text{for } \boldsymbol{0} \le \boldsymbol{i} \le \boldsymbol{M},\ 1 \le t \le T \tag{1}$$

$$x_t(k,\boldsymbol{i}) = \min\left\{E + \omega_{t-1}(\boldsymbol{i}),\ \min_{j\in Y(\boldsymbol{i})}\left(T_{j,k} + \omega_{t-1}(\boldsymbol{i}-\boldsymbol{e_j})\right)\right\} \quad \text{for } 1 \le k \le m,\ \boldsymbol{0} \le \boldsymbol{i} \le \boldsymbol{M} \text{ with } i_k = 0,\ 1 \le t \le T \tag{2}$$

$$\omega_0(\boldsymbol{i}) = \sum_{k=1}^{m} h_k i_k \text{ for } \boldsymbol{0} \le \boldsymbol{i} \le \boldsymbol{M} \tag{3}$$

(Notation: $\boldsymbol{M}$ is the m-vector with M_k in position k and $\boldsymbol{e_k}$ is the m-vector with 1 in position k and 0 elsewhere.)

Let the T-vector $\boldsymbol{d}$ represent the pattern of demand between successive review epochs in the following way. When the time until the next review epoch is δt, let $d_t > 0$ indicate that a demand occurs at location d_t in the next interval and $d_t = 0$ indicate that no demand occurs in the next interval. Let D be the finite set of all such demand patterns and define $P(\boldsymbol{d}) = p_{d_1}p_{d_2}\ldots p_{d_T}$ to be the probability of demand pattern $\boldsymbol{d}$ occurring. Define $u_k(t, \boldsymbol{i}, \boldsymbol{d})$ to be the stock remaining in location k at the next review epoch under the optimal transshipment policy (determined by equations (1)–(3)) when the time until the next review epoch is δt, the current stock levels are $\boldsymbol{i}$ and the pattern of demand between review epochs is $\boldsymbol{d}$.

The problem of minimizing the long run average cost per period is modelled as a Markov decision process. The state of the system is the stock level in each of the locations at a review epoch. The decision is the number of items to order for each location. Due to the limited storage capacity at the locations, the number of states and decisions are finite. The problem is an infinite horizon, average cost Markov decision process with finite state and action spaces (see e.g. Puterman, 1994). Define g to be the minimum average cost per period and $v(\boldsymbol{i})$ to be the bias term associated with starting the system in state $\boldsymbol{i}$. The optimality equation of the model of the system under the above assumptions is as follows.

$$g + v(\boldsymbol{i}) = \min_{\boldsymbol{i}\le\boldsymbol{j}\le\boldsymbol{M}}\left(\sum_{k=1}^{m}(j_k - i_k)c + \omega_T(\boldsymbol{j}) + \sum_{\boldsymbol{d}\in D} P(\boldsymbol{d})v(\boldsymbol{u}(T,\boldsymbol{j},\boldsymbol{d}))\right) \text{ for } \boldsymbol{0} \le \boldsymbol{i} \le \boldsymbol{M} \tag{4}$$

(Notation: $\boldsymbol{u}(T, \boldsymbol{i}, \boldsymbol{d})$ is the m-vector with $u_k(T, \boldsymbol{i}, \boldsymbol{d})$ in position k.)

3 An optimal replenishment policy

This section establishes the form of an optimal replenishment policy for the model developed in Section 2.

Proposition 1

There exist non-negative integers $S_1, \ldots, S_m$ such that, provided the initial state of the process is less than or equal to $\boldsymbol{S}$, an optimal replenishment policy is to order $S_k - i_k$ items for location k whenever the process is in state $\boldsymbol{i}$, satisfying $0 \leq \boldsymbol{i} \leq \boldsymbol{S}$, at a review epoch.

Proof

Define $S_1, \ldots, S_m$ to be the values of $j_1, \ldots, j_m$ that minimize the right hand side of (4) for $\boldsymbol{i} = \boldsymbol{0}$. Note that the right hand side of (4) can be written as

$$\begin{aligned} -\sum_{k=1}^{m} i_k c + \min_{\boldsymbol{i} \leq \boldsymbol{j} \leq \boldsymbol{M}} \left(\sum_{k=1}^{m} j_k c + \omega_T(\boldsymbol{j}) + \sum_{\boldsymbol{d} \in D} P(\boldsymbol{d}) v(\boldsymbol{u}(T, \boldsymbol{j}, \boldsymbol{d})) \right) \\ = \quad -\sum_{k=1}^{m} i_k c + \sum_{k=1}^{m} S_k c + \omega_T(\boldsymbol{S}) + \sum_{\boldsymbol{d} \in D} P(\boldsymbol{d}) v(\boldsymbol{u}(T, \boldsymbol{j}, \boldsymbol{d})) \end{aligned}$$

whenever $\boldsymbol{i} \leq \boldsymbol{S}$ from the definition of $\boldsymbol{S}$. Hence the policy that orders up to S_k items at location k is optimal for any state $\boldsymbol{i}$ satisfying $\boldsymbol{0} \leq \boldsymbol{i} \leq \boldsymbol{S}$. Between review epochs the stock level at a location cannot increase, so $\{\boldsymbol{i} : \boldsymbol{0} \leq \boldsymbol{i} \leq \boldsymbol{S}\}$ forms a closed set of recurrent states in the Markov chain corresponding to this optimal policy. Hence, provided the initial state of the process lies in this set, the long run average cost per period is minimized by ordering up to S_k items at location k at each review epoch. $\circ$

4 A heuristic transshipment policy

This section proves results about the form of an optimal transshipment policy for the special case of a two location system. These results provide useful insight into the important problem of organizing transshipments in a cost effective manner. A heuristic is developed that uses an instance of the two location problem to determine if it is advisable to transship an item between a pair of locations in the general multilocation problem.

Proposition 2

In the two location problem ($m = 2$), if it is optimal to use transshipment to satisfy a demand at one location when there are i items in stock at the other location and t intervals until the next review epoch, then it is also optimal to use transshipment to satisfy such demand when the other location:

(i) has more than i items in stock and there are t intervals until the next review epoch;

(ii) has i items in stock and there are fewer than t intervals until the next review epoch.

Proof

See the appendix for details. ◦

The importance of Proposition 2 is that it establishes the existence of an optimal transshipment policy for two location systems with the following form. There exist threshold times $\tau_j(1), \tau_j(2), \ldots, \tau_j(M_j)$ such that location j should agree to satisfy a transshipment request when it has i items in stock if and only if the time until the next replenishment is less than $\tau_j(i)$. A policy of this form is straightforward to implement with or without computer support because the criterion for transshipment is easy to understand and verify. This form of policy is particularly attractive for systems with many locations for two reasons. Firstly the location facing the stockout can always contact the other locations to request a transshipment in the same order, for example in order of increasing transshipment cost. Secondly the criterion for transshipment depends only on factors local to the location receiving the transshipment request. Unfortunately results similar to Proposition 2 do not hold for general multilocation inventory systems as the following simple example shows.

Example

Consider a three location inventory system ($m = 3$) with $p_k = 0.25$ for $0 \leq k \leq 3$, $E = 30$, $T_{1,3} = 12$, $T_{2,3} = 16$ and $T_{2,1} = 28$. Assume $h_k = c$ for $1 \leq k \leq 3$, so that $\omega_0(\boldsymbol{i}) = 0$ for all $\boldsymbol{i}$.

Let $i_3 = 0$ and $i_2 > 0$. If $i_1 > 0$ then $\omega_1(\boldsymbol{i}) = 3$ and the optimal decision is to meet demand at location 3 by transshipment from location 1. If $i_1 = 0$ then $\omega_1(\boldsymbol{i}) = 11$ and the optimal decision is to meet demand at locations 1 and 3 by transshipment from location 2.

If $i_3 = 0$, $i_2 > 1$ and $i_1 = 1$ then $\omega_2(\boldsymbol{i}) = 0.25\,(3 + 11 + 3 + \min(30 + 3, 12 + 11, 16 + 3)) = 9$ and the optimal decision is to meet demand at location 3 by transshipment from location 2. ◦

In this example it is only optimal for location 2 to transship to location 3 when there is 1 interval until replenishment if location 1 has no stock. This shows that, in an optimal transshipment policy, locations must consider the inventory levels of other locations when deciding whether or not to agree to a transshipment request. Further when the state of the system is $\boldsymbol{i} = (1\ 2\ 0)'$, the optimal location from which to transship to location 3 depends on the number of intervals until replenishment. This shows that even if it were practical to consider every state of the system, it would not be possible to characterize an optimal transshipment policy using a single threshold time for each state. Hence the optimal transshipment policy has no obvious structure that would simplify the formulation of the problem. The number of states that must be considered in the calculation of an optimal transshipment policy directly from equations (1)–(3) increases exponentially with the number of locations in the system. Hence this approach is not practical in general due to the excessive time required to compute an optimal policy.

The discussion above motivates the study of a heuristic for multilocation problems which models every pair of locations as a two location system and finds an optimal transshipment policy of the form described above for each pair independently. One advantage of this approach is that the time required to calculate the policy is quadratic in the number of locations, while, as noted above, the time required to calculate an optimal policy is exponential in the number of locations. Define $\tau_{j,k}(i)$ to be a threshold time with the property that, in the two location system consisting of locations j and k, it is optimal to transship from location j to location k when location j has i items in stock whenever the time until the next replenishment is less than $\tau_{j,k}(i)$. Under the heuristic a location facing a stockout requests a transshipment from other locations in order of increasing transshipment cost. When location j has i items in stock, it will agree to a transshipment request from location k if and only if the time until the next transshipment is less than $\tau_{j,k}(i)$. In many cases this approach will also limit the delay to the customer, as the cost of transshipment between locations is often proportional to the time required for transshipment. This heuristic is referred to as the τ-heuristic. The τ-heuristic requires that each location knows its own inventory level, the location making the transshipment request and the time until replenishment. In most systems this information is readily available, so the τ-heuristic would be easy and cheap to implement.

Under the τ-heuristic it is likely that, for a given time and stock level, a location will agree to transship to some locations and refuse to transship to others. This might be confusing to staff and create the impression that some locations are less important. Consequently the τ-heuristic could be considered impractical by some companies. Consider two modifications of the τ-heuristic which remove the dependency of the transshipment decision on the location making the request. The first modification says that if one is willing to transship to one location then one should be willing to transship to all locations. Define $\alpha_j(i) = \max_k\{\tau_{j,k}(i)\}$. Location j will agree to any transshipment request when it has i items in stock provided the time until the next transshipment is less than $\alpha_j(i)$. The second modification says that until one is willing to transship to all locations, one should not transship to any. Define $\omega_j(i) = \min_k\{\tau_{j,k}(i)\}$. Location j will agree to any transshipment request when it has i items in stock provided the time until the next transshipment is less than $\omega_j(i)$. These modifications are referred to as the α-heuristic and the ω-heuristic respectively.

5 Numerical comparison

This section compares the performance of the proposed heuristics with that of an optimal transshipment policy and the commonly used heuristics of complete pooling and no pooling. The comparison is based on a five-location inventory system with no location having storage capacity for more than 8 units. Limiting the number of locations and the storage capacity in this way means it is possible to compute an optimal transshipment policy using equations (1)–(3) for comparison. By varying the storage capacity, demand rates, holding costs and transshipment costs a total of 1512 problems were created. This allows the effect of the problem parameters on the performance of the heuristics to be investigated.

The time between successive review epochs is one week and this is divided into $T = 5000$ intervals for the purposes of modelling demand. Replenishment orders cost 1 per item and emergency orders cost 10 per item. Four sets of storage capacities are considered with the capacity at a location always in the range 4 to 8. Seven sets of demand rates are considered with the average weekly demand at a location (i.e. $p_k T$ at location k) in the range 1 to 5. Three values for the holding cost per item per week in the range 0.002 to 0.008 are considered. This corresponds to an annual holding cost of between 10% and 50% of the cost of inventory. Two distinct transshipment cost structures are considered. Firstly, the transshipment cost is modelled as a fixed cost plus a multiple of the distance between the locations (with $|i - j|$ being used as a proxy for the distance between locations i and j). Secondly, the transshipment costs are assumed to be symmetric (i.e. $T_{i,j} = T_{j,i}$), but otherwise random. Transshipment costs are always in the range 5 to 9. The data for the test problems is summarized in Table 1.

Tables 2 to 6 compare the performance of the five heuristic transshipment policies considered in this paper on the test problems. The performance measure used in the comparison is the "percentage suboptimality" for a problem which is defined as:

$$100\left(\frac{\text{expected cost under heuristic}}{\text{optimal expected cost}} - 1\right).$$

Each row of a table compares the performance of the heuristics on a different subset of the test problems. The following notation is used:

n — number of problems in the subset;

w c — worst case performance, i.e. maximum percentage suboptimality;

ave — average percentage suboptimality;

s d — standard deviation of percentage suboptimality.

Table 2 shows that overall the three proposed heuristics perform far better than complete pooling and no pooling in terms of worst case, average and standard deviation of percentage suboptimality. In fact the expected cost with no pooling is never lower than the expected cost under any of the proposed heuristics, while the expected cost with complete pooling is lower than the expected costs under the τ-, α- and ω- heuristics for only 3%, 2% and 23% of the cases respectively. Figure 2 depicts one case of the proposed heuristic transshipment policies. It is interesting to note that, under the proposed heuristics, transshipments are not used until 60% of the period between replenishments has elapsed and the last item of inventory is never used in a transshipment until at least 95% of this period has elapsed. So the proposed heuristics all start off like no pooling, but become more willing to agree to transshipment as the time to replenishment decreases until, by the end of the period, they all behave like complete pooling. This is typical of the behaviour of the proposed heuristics, but the point at which transshipments are first used and the speed of transition to complete pooling vary greatly from problem to problem.

The figures in Table 2 suggest that the best heuristic is the α-heuristic which performs slightly better than the τ-heuristic. Both the α- and τ- heuristics appear to be considerably better than the ω-heuristic. This is reinforced by the fact that the α-heuristic is the best performing heuristic in 67% of cases while the τ- and ω- heuristics are only best in 32% and 27% of cases respectively. (Note these percentages do not sum to 100 because of ties.) When evaluating a possible transshipment, the τ-heuristic only considers the inventory level at the source location. Hence if other locations have plenty of inventory on hand, the heuristic is likely to be rather conservative. It is perhaps not surprising therefore that the α-heuristic, which effectively relaxes the criterion for transshipments, performs better than the τ-heuristic. The ω-heuristic on the other hand tightens the criterion for transshipments and so makes the situation worse. The arrows in Figure 2 illustrate this effect. The left arrow shows that the boundary between transshipment and no transshipment for the ω-heuristic generally lies to the left of that for the τ-heuristic, while the right arrow shows that for the α-heuristic it generally lies to the right. Hence compared to the τ-heuristic, the ω-heuristic is less likely to use transshipment and the α-heuristic is more likely to use transshipment.

Table 3 shows the effect of varying the storage capacities of the locations. As the depot capacities increase from $M_k = 4$ for all k to $M_k = 6$ for all k, the performances of all the heuristics improve. This is due to higher order quantities which reduce the opportunities for transshipments. For all depot capacities considered, the α-heuristic is the best heuristic and no pooling is the worst.

Table 4 shows the effect of varying the average weekly demand at the locations. With low demand ($p_k T = 1$ for all k) all five heuristics are very close to optimal. This is to be expected as there are very few opportunities for transshipments in this case. As demand increases, the performances of all the heuristics deteriorate. With high demand ($p_k T = 5$ for all k) complete pooling is never within 3.7% of optimal making it the worst heuristic. This is because it frequently transships items from locations that subsequently suffer stockouts. This demonstrates clearly that when demand is high relative to the depot capacity, it can pay to be selective about transshipments. All the heuristics apart from no pooling perform better when each location faces a different demand rate than they do when the demand rate is constant and greater than or equal to 4 units per week. When the demand rates vary, no pooling is never within 2.2% of optimal and is sometimes more than 30% worse than optimal. No pooling is also the worst heuristic by far when the demand parameters are constant and less than 5 units per week. These observations demonstrate that transshipments can be an important factor in the management of inventory costs. For all choices of demand parameters the α-heuristic is the best performing heuristic.

Tables 5 and 6 show the effect of varying the holding costs and the transshipment costs respectively. The holding cost has no noticeable effect on the performance of any of the heuristics in the test problems. With the structured transshipment costs, all the heuristics apart from complete pooling perform better as the mean and variance of the transshipment costs increase. Complete pooling is the worst heuristic in several cases. (No pooling is the worst heuristic in all other cases of transshipment cost.) This is further evidence that being selective about transshipments can help to keep inventory costs down. The α-heuristic is the best heuristic for all cases of transshipment cost.

6 Conclusions

A model of a multilocation inventory system, found in highly competitive retail environments, has been analysed to show that an "order-up-to" replenishment policy is optimal, but that the form of an optimal transshipment policy is not easy to characterize for more than two locations. It is not practical to use the model to compute an optimal transshipment policy for systems with many locations due to the high-dimensional state and action spaces involved. Therefore three heuristic transshipment policies, based on the simple form of optimal transshipment policies for two-location systems, have been developed. These heuristic policies are practical to compute for large systems, because the complexity of the solution algorithm is only quadratic in the number of locations. The heuristic policies are also easy to implement, only requiring information that is local to the location receiving a request for a transshipment.

Comparing the structure of the proposed heuristics, the main differences lie in their readiness to agree to transshipment requests and their ease of implementation. When deciding whether or not to transship, the τ-heuristic concentrates on the locations at the source and destination of the potential transshipment, and so neglects the total inventory held in the system and the possibility of other locations facing stockouts. This simplification may result in the heuristic being too willing or too reluctant to agree to a transshipment request. Hence the α-heuristic and the ω-heuristic, which are, respectively, more and less willing to agree to transshipment requests, were developed. Under the τ-heuristic, the decision to transship is based on the inventory level at the location receiving the request, the time until the next replenishment and the identity of the location making the request. However under the other heuristics, this decision is based only on the inventory level at the location receiving the request and the time until the next replenishment. Consequently the τ-heuristic is slightly less convenient to operate in practice.

The computational results show that the proposed heuristic transshipment policies work well across a range of test problems with different characteristics. Importantly the results show that the proposed heuristics can be much better than the simple policies of complete pooling and no pooling, which are often used in practice. Based on the computational results, the best heuristic appears to be the α-heuristic, which is the least conservative of the three proposed heuristics and one of the easiest to apply.

References

T. W. Archibald, S. A. E. Sassen, and L. C. Thomas (1997). An optimal policy for a two depot inventory problem with stock transfer. *Mngt Sci* **43**:173–183.

S. Axsäter (2003a). Evaluation of unidirectional transshipments and substitutions in inventory systems. *Eur J Opl Res* **149**:438–447.

S. Axsäter (2003b). A new decision rule for lateral transshipments in inventory systems. *Mngt Sci* **49**:1168–1179.

S. Axsäter (1990). Modelling emergency lateral transshipments in inventory systems. *Mngt Sci* **36**:1329–1338.

L. P. Bertrand and J. H. Bookbinder (1998). Stock redistribution in two-echelon logistics systems. *J Opl Res Soc* **49**:966–975.

C. Das (1975). Supply and redistribution rules for two-location inventory systems — one period analysis. *Mngt Sci* **21**:765–776.

P. T. Evers (2001). Heuristics for assessing emergency transshipments. *Eur J Opl Res* **129**: 311–316.

D. Gross (1963). Centralized inventory control in multilocation systems. In: H. E. Scarf, D. M. Gilford, and M. W. Shelly (eds). *Multistage inventory models and techniques.* Stanford University Press, Stanford pp 47–84.

H. Jonsson and E. A. Silver (1987). Analysis of a two echelon inventory control system with complete redistribution. *Mngt Sci* **33**:215–227.

K. S. Krishnan and V. R. K. Rao (1965). Inventory control in n warehouses. *Journal of Industrial Engineering* **16**:212–215.

A. Kukreja, C. P. Schmidt, and D. M. Miller (2001). Stocking decisions for low usage items in a multilocation inventory system. *Mngt Sci* **47**:1371–1383.

H. L. Lee (1987). A multi-echelon inventory model for repairable items with emergency lateral transshipments. *Mngt Sci* **33**:1302–1316.

S. Minner, E. A. Silver, and D. J. Robb (2003). An improved heuristic for deciding on emergency transshipments. *Eur J Opl Res* **149**:384–400.

M. L. Puterman (1994). *Markov decision processes: Discrete stochastic dynamic programming.* John Wiley: New York.

L. W. Robinson (1990). Optimal and approximate policies in multiperiod multilocation inventory models with transshipment. *Opns Res* **38**:278–295.

N. Rudi, S. Kapur, and D. F. Pyke (2001). A two-location inventory model with transshipment and local decision making. *Mngt Sci* **47**:1668–1680.

C. C. Sherbrooke (1968). METRIC: A multi-echelon technique for recoverable item control. *Opns Res* **16**:122–141.

C. C. Sherbrooke (1992). Multiechelon inventory systems with lateral supply. *Naval Res Logist* **39**:29–40.

G. Tagaras and M. A. Cohen (1992). Pooling in two location inventory systems with non-negligible lead times. *Mngt Sci* **38**:1067–1083.

Appendix

Proof of Proposition 2

Suppose location k has no local stock, so that $i_k = 0$, where $k = 1$ or $k = 2$. Let j be the other location, so $j = 3 - k$. Define:

$$\begin{aligned}
W_t(\lambda) &= \omega_t((\lambda+1)\boldsymbol{e_j}) - \omega_t(\lambda\boldsymbol{e_j}) \text{ for } 0 \le \lambda < M_j, 0 \le t \le T \qquad (5)\\
X_t(\lambda) &= x_t(k,(\lambda+1)\boldsymbol{e_j}) - x_t(k,\lambda\boldsymbol{e_j}) \text{ for } 0 \le \lambda < M_j, 0 < t \le T\\
X_t^E(\lambda) &= E + \omega_{t-1}((\lambda+1)\boldsymbol{e_j}) - E - \omega_{t-1}(\lambda\boldsymbol{e_j}) = W_{t-1}(\lambda) \text{ for } 0 \le \lambda < M_j, 0 < t \le T\\
X_t^T(\lambda) &= T_{j,k} + \omega_{t-1}(\lambda\boldsymbol{e_j}) - T_{j,k} - \omega_t((\lambda-1)\boldsymbol{e_j}) = W_{t-1}(\lambda-1) \text{ for } 0 < \lambda < M_j, 0 < t \le T
\end{aligned}$$

For any t and λ satisfying $0 < t \le T$ and $0 < \lambda \le M_j$, an emergency order is preferable to a transshipment from location j to satisfy a demand at location k when the state of the system is $\lambda\boldsymbol{e_j}$ and the time until the next review epoch is δt if and only if

$$E + \omega_{t-1}(\lambda\boldsymbol{e_j}) < T_{j,k} + w_{t-1}((\lambda-1)\boldsymbol{e_j}) \Leftrightarrow W_{t-1}(\lambda-1) < T_{j,k} - E \qquad (6)$$

Hence to prove (i) it is sufficient to prove that $W_t(\lambda)$ is non-decreasing in λ for $0 \le \lambda < M_j$ and $0 \le t < T$, while to prove (ii) it is sufficient to prove that $W_t(\lambda)$ is non-increasing in t for $0 \le t < T$ and $0 \le \lambda < M_j$.

First prove that for $0 \le t \le T$, $W_t(\lambda)$ is non-decreasing in λ for $0 \le \lambda < M_j$ and $W_t(0) \ge -E$. The proof is by induction on t. From (3) and (5), $W_0(\lambda) = h_j - c$ which is constant, and hence non-decreasing in λ, and $W_0(0) \ge -E$ since $h_j \ge 0$ and $E > c$. Hence the result holds for $t = 0$.

Assume that for some t satisfying $0 < t \le T$, $W_{t-1}(\lambda)$ is non-decreasing in λ for $0 \le \lambda < M_j$ and $W_{t-1}(0) \ge -E$. From (6), either $x_t(k,\lambda\boldsymbol{e_j}) = E + \omega_{t-1}(\lambda\boldsymbol{e_j})$ for $0 \le \lambda \le M_j$, in which case $X_t(\lambda) = X_t^E(\lambda) = W_{t-1}(\lambda)$ is non-decreasing in λ by the inductive hypothesis, or there exists λ^* such that $x_t(k,\lambda\boldsymbol{e_j}) = E + \omega_{t-1}(\lambda\boldsymbol{e_j})$ for $0 \le \lambda \le \lambda^*$ and $x_t(k,\lambda\boldsymbol{e_j}) = T_{j,k} + \omega_{t-1}((\lambda-1)\boldsymbol{e_j})$ for $\lambda^* < \lambda \le M_j$. In this latter case:

$$X_t(\lambda) = \begin{cases} X_t^E(\lambda) = W_{t-1}(\lambda) \text{ for } 0 \le \lambda < \lambda^* \\ T_{j,k} + \omega_{t-1}(\lambda\boldsymbol{e_j}) - E - \omega_{t-1}(\lambda\boldsymbol{e_j}) = T_{j,k} - E \text{ for } \lambda = \lambda^* \\ X_t^T(\lambda) = W_{t-1}(\lambda-1) \text{ for } \lambda^* < \lambda < M_j \end{cases} \qquad (7)$$

From (6) and the definition of λ^*, $W_{t-1}(\lambda^*-1) < T_{j,k} - E \leq W_{t-1}(\lambda^*)$. This fact and the inductive hypothesis show that $X_t(\lambda)$ is non-decreasing in λ for $0 \leq \lambda < M_j$.

From (1) and (5):

$$W_t(\lambda) = \begin{cases} p_0 W_{t-1}(\lambda) + p_j W_{t-1}(\lambda-1) + p_k X_t(\lambda) \text{ for } 0 < \lambda < M_j \\ p_0 W_{t-1}(0) - p_j E + p_k X_t(0) \text{ for } \lambda = 0 \end{cases} \tag{8}$$

Hence, using the inductive hypothesis, $W_t(0) \leq p_0 W_{t-1}(0) + p_j W_{t-1}(0) + p_k X_t(0)$. Since $W_{t-1}(\lambda)$ and $X_t(\lambda)$ are non-decreasing in λ and $p_\ell \geq 0$ for $0 \leq \ell \leq 2$, it follows that $W_t(\lambda)$ is non-decreasing in λ for $0 \leq \lambda < M_j$. Note that either $X_t(0) = W_{t-1}(0) \geq -E$ by the inductive hypothesis or $X_t(0) = T_{j,k} - E \geq -E$ since $T_{j,k}$ is a transshipment cost and so non-negative. From (8) it follows that $W_t(0) \geq -E$. Therefore it is proved by induction that for $0 \leq t \leq T$, $W_t(\lambda)$ is non-decreasing in t for $0 \leq \lambda < M_j$ and $W_t(0) \geq -E$. Hence (i) is proved.

Finally use these properties of $W_t(\lambda)$ to prove that $W_t(\lambda)$ is non-increasing in t for $0 \leq t \leq T$ and $0 \leq \lambda < M_j$. The argument above shows that for $0 < t \leq T$, either $X_t(\lambda) = W_{t-1}(\lambda)$ for $0 \leq \lambda < M_j$ or

$$X_t(\lambda) = \begin{cases} W_{t-1}(\lambda) \text{ for } 0 \leq \lambda < \lambda^* \\ T_{j,k} - E \leq W_{t-1}(\lambda^*) \text{ for } \lambda = \lambda^* \\ W_{t-1}(\lambda-1) \leq W_{t-1}(\lambda) \text{ for } \lambda^* < \lambda < M_j \end{cases}$$

Hence $X_t(\lambda) \leq W_{t-1}(\lambda)$. From (8) it follows that $W_t(\lambda) \leq W_{t-1}(\lambda)$ and (ii) is proved. ∘

Figure 1: Structure of the inventory system showing the inventory flows caused by replenishment orders, emergency orders and transshipments.

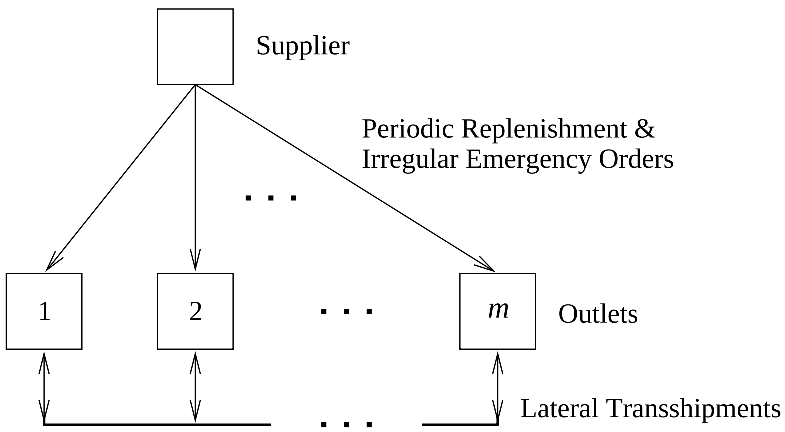

Figure 2: An example of the transshipment decision from location 1 to location 2 as a function of inventory level and time to replenishment for the proposed heuristics (when $M_k = 5$, $p_k T = 6-k$, $h_k = 0.002$ and $T_{i,j} \sim U[5, 9]$). A heuristic only uses transshipment in situations corresponding to points above and to the left of its line.

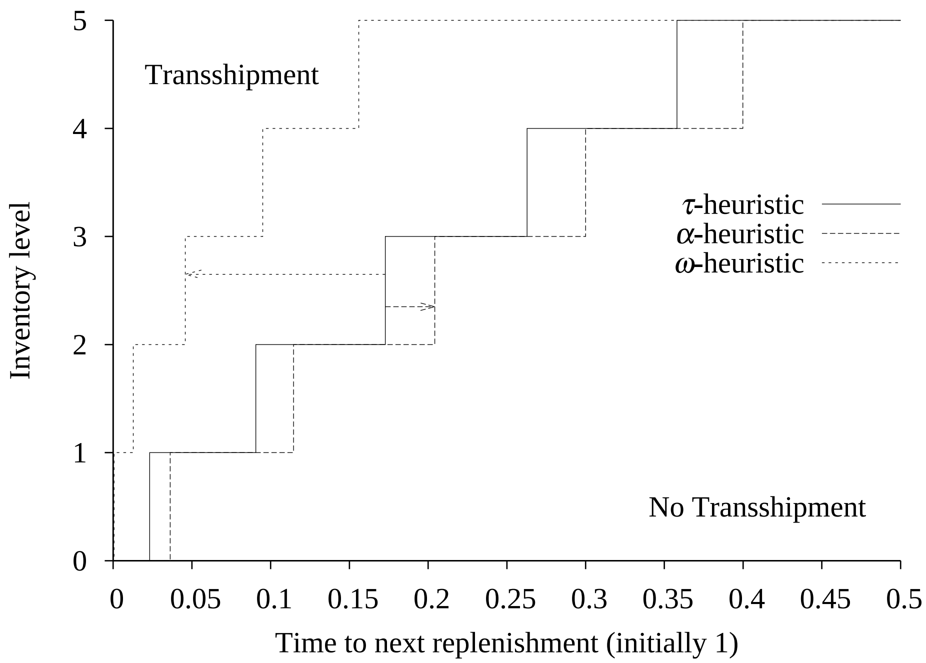

Table 1: Characteristics of the 1512 test problems. In each case, $T = 5000$, $c = 1$ and $E = 10$.

Storage capacities:	$M_k = 4$, $M_k = 5$, $M_k = 6$ and $M_k = 3 + k$ where $1 \leq k \leq 5$.
Demand parameters:	$p_k T = 1$, $p_k T = 2$, $p_k T = 3$, $p_k T = 4$, $p_k T = 5$, $p_k T = k$, and $p_k T = 6 - k$ where $1 \leq k \leq 5$.
Holding costs:	$h_k = 0.002$, $h_k = 0.005$ and $h_k = 0.008$ where $1 \leq k \leq 5$.
Transshipment costs:	9 cases with $T_{i,j} = a + b\|i - j\|$ for $a = 5$, 6 & 7 and $b = 0$, 0.25 & 0.5 where $1 \leq i, j \leq 5$ and $i \neq j$. 9 cases with $T_{i,j} = T_{j,i}$ drawn from a Uniform $[5, 9]$ distribution where $1 \leq i < j \leq 5$.

Table 2: Overall performance of the heuristic transshipment policies on the test problems.

	n	Complete Pooling			No Pooling			τ-heuristic			α-heuristic			ω-heuristic		
		w c	ave	s d	w c	ave	s d	w c	ave	s d	w c	ave	s d	w c	ave	s d
All problems	1512	19.2	3.6	4.3	31.2	8.0	6.3	4.4	0.8	0.9	4.0	0.7	0.8	7.9	1.2	1.5

Table 3: Effect of storage capacity on the performance of the heuristic transshipment policies.

Problems with $M_k =$	n	Complete Pooling			No Pooling			τ-heuristic			α-heuristic			ω-heuristic		
		w c	ave	s d	w c	ave	s d	w c	ave	s d	w c	ave	s d	w c	ave	s d
4	328	19.2	6.5	5.4	31.2	8.9	7.0	4.4	1.2	1.1	3.7	1.2	0.9	7.9	2.0	1.7
5	328	14.6	3.5	3.8	25.5	8.4	6.2	3.5	0.8	0.7	3.5	0.7	0.7	6.1	1.3	1.2
6	328	7.8	1.5	2.0	17.5	6.4	4.8	3.3	0.4	0.7	3.3	0.4	0.7	5.1	0.8	1.1
$3 + k$	328	12.7	2.8	3.4	31.0	8.4	6.9	4.0	0.6	0.8	4.0	0.6	0.8	6.9	0.9	1.3

Table 4: Effect of demand parameters on the performance of the heuristic transshipment policies.

Problems with $p_kT =$	n	Complete Pooling w c	Complete Pooling ave	Complete Pooling s d	No Pooling w c	No Pooling ave	No Pooling s d	τ-heuristic w c	τ-heuristic ave	τ-heuristic s d	α-heuristic w c	α-heuristic ave	α-heuristic s d	ω-heuristic w c	ω-heuristic ave	ω-heuristic s d
1	216	0.0	0.0	0.0	1.7	0.5	0.5	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0
2	216	1.5	0.3	0.4	12.2	3.6	2.8	0.6	0.1	0.2	0.6	0.1	0.2	2.3	0.3	0.5
3	216	6.4	1.9	1.7	19.8	7.9	3.5	3.6	0.7	0.8	3.6	0.7	0.8	7.0	1.3	1.4
4	216	14.7	5.5	3.6	20.6	9.3	3.9	3.5	1.3	0.6	3.5	1.3	0.6	5.6	2.1	1.1
5	216	19.2	10.0	4.1	22.5	7.8	4.7	4.0	1.4	0.9	4.0	1.4	0.8	6.9	2.2	1.4
k	216	10.1	2.5	2.7	31.2	11.6	6.4	3.6	0.8	1.0	3.3	0.7	0.8	7.5	1.4	1.7
$6-k$	216	14.0	4.8	3.6	31.2	15.4	6.2	4.4	0.9	1.0	3.7	0.8	0.8	7.9	1.5	1.6

Table 5: Effect of holding costs on the performance of the heuristic transshipment policies.

Problems with $h_k =$	n	Complete Pooling w c	ave	s d	No Pooling w c	ave	s d	τ-heuristic w c	ave	s d	α-heuristic w c	ave	s d	ω-heuristic w c	ave	s d
0.002	504	19.2	3.5	4.2	31.2	8.0	6.4	4.4	0.8	0.9	4.0	0.7	0.8	7.4	1.2	1.4
0.005	504	19.2	3.6	4.3	31.2	8.0	6.3	4.4	0.8	0.9	4.0	0.7	0.8	7.9	1.3	1.5
0.008	504	19.2	3.6	4.3	31.2	8.0	6.4	4.4	0.7	0.9	4.0	0.7	0.8	7.2	1.2	1.4

Table 6: Effect of transshipment costs on the performance of the heuristic transshipment policies.

Problems with $T_{i,j} =$	n	Complete Pooling w c	Complete Pooling ave	Complete Pooling s d	No Pooling w c	No Pooling ave	No Pooling s d	τ-heuristic w c	τ-heuristic ave	τ-heuristic s d	α-heuristic w c	α-heuristic ave	α-heuristic s d	ω-heuristic w c	ω-heuristic ave	ω-heuristic s d
5	84	11.3	3.4	3.7	31.2	13.7	9.7	4.4	1.4	1.5	4.0	1.3	1.4	4.5	1.4	1.5
6	84	13.8	4.0	4.4	21.5	9.5	6.6	3.2	1.0	1.0	2.8	0.9	0.9	3.3	1.0	1.0
7	84	16.9	4.7	5.2	13.4	5.8	4.0	2.1	0.6	0.6	1.8	0.6	0.5	2.2	0.6	0.6
$5 + \|i-j\|/4$	84	11.8	3.5	3.5	25.6	11.5	7.8	3.1	1.1	1.2	2.9	1.0	1.0	3.3	1.1	1.2
$6 + \|i-j\|/4$	84	14.7	4.3	4.3	16.8	7.6	5.0	2.1	0.7	0.7	1.8	0.6	0.6	2.3	0.7	0.8
$7 + \|i-j\|/4$	84	17.9	5.0	5.1	9.2	4.1	2.6	1.1	0.4	0.3	0.8	0.3	0.3	1.2	0.4	0.4
$5 + \|i-j\|/2$	84	12.9	3.5	3.6	21.1	9.7	6.3	2.4	0.9	0.9	2.3	0.8	0.8	2.9	1.0	1.1
$6 + \|i-j\|/2$	84	15.9	4.2	4.4	12.9	6.0	3.7	1.5	0.5	0.5	1.3	0.5	0.4	1.8	0.7	0.7
$7 + \|i-j\|/2$	84	19.2	5.1	5.4	5.8	2.8	1.6	0.6	0.3	0.2	0.6	0.2	0.2	1.4	0.6	0.5
$U[5,9]$	756	19.2	3.0	3.9	27.2	8.2	6.0	3.3	0.8	0.8	3.6	0.7	0.8	7.9	1.7	1.7

Captions for Figures and Tables

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