Exceptional del Pezzo hypersurfaces

Citation for published version:
Cheltsov, I, Park, J & Shramov, C 2010, 'Exceptional del Pezzo hypersurfaces' Journal of Geometric Analysis, vol. 20, no. 4, pp. 787-816. DOI: 10.1007/s12220-010-9135-2

Digital Object Identifier (DOI):
10.1007/s12220-010-9135-2

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
Journal of Geometric Analysis

EXCEPTIONAL DEL PEZZO HYPERSURFACES

IVAN CHELTSOV, JIHUN PARK, CONSTANTIN SHRAMOV

Abstract. We compute global log canonical thresholds of a large class of quasismooth well-formed del Pezzo weighted hypersurfaces in $\mathbb{P}(a_1, a_2, a_3, a_4)$. As a corollary we obtain the existence of orbifold Kähler–Einstein metrics on many of them, and classify exceptional and weakly exceptional quasismooth well-formed del Pezzo weighted hypersurfaces in $\mathbb{P}(a_1, a_2, a_3, a_4)$.

Contents

Part 1. Introduction

All varieties are always assumed to be complex, algebraic, projective and normal unless otherwise stated.

1.1. Background

The multiplicity of a nonzero polynomial $f \in \mathbb{C}[z_1, \ldots, z_n]$ at a point $P \in \mathbb{C}^n$ is the nonnegative integer m such that $f \in \mathfrak{m}_P^m \setminus \mathfrak{m}_P^{m+1}$, where $\mathfrak{m}_P$ is the maximal ideal of polynomials vanishing at the point P in $\mathbb{C}[z_1, \ldots, z_n]$. It can be also defined by derivatives: the multiplicity of f at the point P is the number

$$\mathrm{mult}_P(f) = \min\left\{m \;\middle|\; \frac{\partial^m f}{\partial^{m_1} z_1 \partial^{m_2} z_2 \ldots \partial^{m_n} z_n}(P) \neq 0\right\}.$$

On the other hand, we have a similar invariant that is defined by integrations. This invariant, which is called the complex singularity exponent of f at the point P, is given by

$$c_P(f) = \sup\left\{c \;\middle|\; |f|^{-c} \text{ is locally } L^2 \text{ near the point } P \in \mathbb{C}^n\right\}.$$

In algebraic geometry this invariant is usually called a log canonical threshold. Let X be a variety with at most log canonical singularities, let $Z \subseteq X$ be a closed subvariety, and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on the variety X. Then the number

$$\mathrm{lct}_Z\big(X, D\big) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \text{the log pair } \big(X, \lambda D\big) \text{ is log canonical along } Z\right\}$$

is called a log canonical threshold of the divisor D along Z. It follows from [26] that for a polynomial f in n variables over $\mathbb{C}$ and a point $P \in \mathbb{C}^n$

$$\mathrm{lct}_P\Big(\mathbb{C}^n, D\Big) = c_P\big(f\big),$$

where the divisor D is defined by the equation $f = 0$ on $\mathbb{C}^n$. We can define the log canonical threshold of D on X by

$$\begin{aligned}\mathrm{lct}_X\big(X, D\big) &= \inf\left\{\mathrm{lct}_P\big(X, D\big) \;\middle|\; P \in X\right\}\\ &= \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \text{the log pair } \big(X, \lambda D\big) \text{ is log canonical}\right\}.\end{aligned}$$

For simplicity, the log canonical threshold $\mathrm{lct}_X(X, D)$ will be denoted by $\mathrm{lct}(X, D)$.

Example 1.1.1. Let D be a cubic curve on the projective plane $\mathbb{P}^2$. Then

$$\mathrm{lct}\big(\mathbb{P}^2, D\big) = \begin{cases} 1 & \text{if } D \text{ is a smooth curve,}\\ 1 & \text{if } D \text{ is a curve with ordinary double points,}\\ \dfrac{5}{6} & \text{if } D \text{ is a curve with one cuspidal point,}\\ \dfrac{3}{4} & \text{if } D \text{ consists of a conic and a line that are tangent,}\\ \dfrac{2}{3} & \text{if } D \text{ consists of three lines intersecting at one point,}\\ \dfrac{1}{2} & \text{if } \mathrm{Supp}\big(D\big) \text{ consists of two lines,}\\ \dfrac{1}{3} & \text{if } \mathrm{Supp}\big(D\big) \text{ consists of one line.} \end{cases}$$

Now we suppose that X is a Fano variety with at most log terminal singularities.

Definition 1.1.2. The global log canonical threshold of the Fano variety X is the number

$$\mathrm{lct}\big(X\big) = \inf\Big\{\mathrm{lct}\big(X, D\big)\ \Big|\ D \text{ is an effective } \mathbb{Q}\text{-divisor on } X \text{ with } D\ \sim_{\mathbb{Q}} -K_X\Big\}.$$

The number $\mathrm{lct}(X)$ is an algebraic counterpart of the α-invariant introduced in [44] and [48] (see [14, Appendix A]). Because X is rationally connected (see [50]), we have

$$\mathrm{lct}\big(X\big) = \sup\left\{\lambda\in\mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } \Big(X, \lambda D\Big) \text{ is log canonical for every}\\ \text{effective } \mathbb{Q}\text{-divisor numerically equivalent to } -K_X\end{array}\right\}.$$

It immediately follows from Definition 1.1.2 that

$$\mathrm{lct}\big(X\big) = \sup\left\{\varepsilon\in\mathbb{Q}\ \middle|\ \begin{array}{l}\text{the log pair } \Big(X, \dfrac{\varepsilon}{n}D\Big) \text{ is log canonical for every}\\ \text{divisor } D\in\big|-nK_X\big| \text{ and every positive integer } n\end{array}\right\}.$$

Example 1.1.3 ([14])**.** Suppose that $\mathbb{P}(a_0, a_1, \ldots, a_n)$ is a well-formed weighted projective space with $a_0 \leqslant a_1 \leqslant \ldots \leqslant a_n$ (see [22]). Then

$$\mathrm{lct}\Big(\mathbb{P}\big(a_0, a_1, \ldots, a_n\big)\Big) = \frac{a_0}{\sum_{i=0}^{n} a_i}.$$

Example 1.1.4. Let X be a smooth hypersurface in $\mathbb{P}^n$ of degree $m \leqslant n$. The paper [6] shows that

$$\mathrm{lct}\big(X\big) = \frac{1}{n+1-m}$$

if $m < n$. For the case $m = n \geqslant 2$ it also shows that

$$1 - \frac{1}{n} \leqslant \mathrm{lct}(X) \leqslant 1$$

and the left equality holds if X contains a cone of dimension $n-2$. Meanwhile, the papers [13] and [38] show that

$$1 \geqslant \mathrm{lct}\big(X\big) \geqslant \begin{cases} 1 \text{ if } n \geqslant 6, \\ \dfrac{22}{25} \text{ if } n = 5, \\ \dfrac{16}{21} \text{ if } n = 4, \\ \dfrac{3}{4} \text{ if } n = 3, \end{cases}$$

if X is general.

Example 1.1.5. Let X be a smooth hypersurface in the weighted projective space $\mathbb{P}(1^{n+1}, d)$ of degree $2d \geqslant 4$. Then

$$\mathrm{lct}\big(X\big) = \frac{1}{n+1-d}$$

in the case when $d < n$ (see [8, Proposition 20]). Suppose that $d = n$. Then the inequalities

$$\frac{2n-1}{2n} \leqslant \mathrm{lct}\big(X\big) \leqslant 1$$

hold (see [13]). But $\mathrm{lct}(X) = 1$ if X is general and $n \geqslant 3$. Furthermore for the case $n = 3$ the papers [13] and [38] prove that

$$\mathrm{lct}\big(X\big) \in \left\{\frac{5}{6}, \frac{43}{50}, \frac{13}{15}, \frac{33}{38}, \frac{7}{8}, \frac{33}{38}, \frac{8}{9}, \frac{9}{10}, \frac{11}{12}, \frac{13}{14}, \frac{15}{16}, \frac{17}{18}, \frac{19}{20}, \frac{21}{22}, \frac{29}{30}, 1\right\}$$

and all these values are attained. For instance, if the hypersurface X is given by

$$w^2 = x^6 + y^6 + z^6 + t^6 + x^2y^2zt \subset \mathbb{P}\big(1,1,1,1,3\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x,y,z,t,w\big]\Big),$$

where $\mathrm{wt}(x) = \mathrm{wt}(y) = \mathrm{wt}(z) = \mathrm{wt}(t) = 1$ and $\mathrm{wt}(w) = 3$, then $\mathrm{lct}(X) = 1$ (see [13]).

Example 1.1.6 ([21])**.** Let X be a rational homogeneous space such that the Picard group of X is generated by an ample Cartier divisor D and $-K_X \sim rD$ for some positive integer r. Then $\mathrm{lct}(X) = \frac{1}{r}$.

Example 1.1.7. Let X be a quasismooth well-formed (see [22]) hypersurface in $\mathbb{P}(1, a_1, a_2, a_3, a_4)$ of degree $\sum_{i=1}^{4} a_i$ with at most terminal singularities, where $a_1 \leqslant \ldots \leqslant a_4$. Then there are exactly 95 possibilities for the quadruple (a_1, a_2, a_3, a_4) (see [22], [24]). For a general hypersurface X, it follows from [7], [9], [10] and [13] that

$$1 \geqslant \mathrm{lct}\big(X\big) \geqslant \begin{cases} \dfrac{16}{21} & \text{if } a_1 = a_2 = a_3 = a_4 = 1, \\ \dfrac{7}{9} & \text{if } (a_1, a_2, a_3, a_4) = (1,1,1,2), \\ \dfrac{4}{5} & \text{if } (a_1, a_2, a_3, a_4) = (1,1,2,2), \\ \dfrac{6}{7} & \text{if } (a_1, a_2, a_3, a_4) = (1,1,2,3), \\ 1 & \text{otherwise.} \end{cases}$$

The global log canonical threshold of the hypersurface

$$w^2 = t^3 + z^9 + y^{18} + x^{18} \subset \mathbb{P}(1,1,2,6,9) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x,y,z,t,w\big]\Big)$$

is equal to $\frac{17}{18}$, where $\mathrm{wt}(x) = \mathrm{wt}(y) = 1$, $\mathrm{wt}(z) = 2$, $\mathrm{wt}(t) = 6$, $\mathrm{wt}(w) = 9$ (see [7]).

Example 1.1.8 ([12])**.** Let X be a singular cubic surface in $\mathbb{P}^3$ with at most canonical singularities. The possible singularities of X are listed in [5]. The global log canonical threshold of X is as follows:

$$\mathrm{lct}\big(X\big) = \begin{cases} \dfrac{2}{3} & \text{if } \mathrm{Sing}\big(X\big) = \big\{\mathbb{A}_1\big\}, \\ \dfrac{1}{3} & \text{if } \mathrm{Sing}\big(X\big) \supseteq \big\{\mathbb{A}_4\big\},\ \mathrm{Sing}\big(X\big) = \big\{\mathbb{D}_4\big\} \text{ or } \mathrm{Sing}\big(X\big) \supseteq \big\{\mathbb{A}_2, \mathbb{A}_2\big\}, \\ \dfrac{1}{4} & \text{if } \mathrm{Sing}\big(X\big) \supseteq \big\{\mathbb{A}_5\big\} \text{ or } \mathrm{Sing}\big(X\big) = \big\{\mathbb{D}_5\big\}, \\ \dfrac{1}{6} & \text{if } \mathrm{Sing}\big(X\big) = \big\{\mathbb{E}_6\big\}, \\ \dfrac{1}{2} & \text{otherwise.} \end{cases}$$

So far we have not seen any single variety whose global log canonical threshold is irrational. In general, it is unknown whether global log canonical thresholds are rational numbers or not(cf. Question 1 in [46]). Even for del Pezzo surfaces with log terminal singularities the rationality of their global log canonical thresholds is unknown. However, we expect more than this as follows:

Conjecture 1.1.9. There is an effective $\mathbb{Q}$-divisor D on the variety X such that it is $\mathbb{Q}$-linearly equivalent to $-K_X$ and

$$\mathrm{lct}\big(X\big) = \mathrm{lct}\big(X, D\big).$$

The following definition is due to [42] (cf. [23], [29], [32], [37]).

Definition 1.1.10. The Fano variety X is exceptional (resp. weakly exceptional, strongly exceptional) if for every effective $\mathbb{Q}$-divisor D on the variety X such that $D \sim_{\mathbb{Q}} -K_X$ and the pair (X, D) is log terminal (resp. $\mathrm{lct}(X) \geqslant 1$, $\mathrm{lct}(X) > 1$).

It is easy to see the implications

$$\text{strongly exceptional} \implies \text{exceptional} \implies \text{weakly exceptional}.$$

However, if Conjecture 1.1.9 holds for X, then we see that X is exceptional if and only if X is strongly exceptional. Exceptional del Pezzo surfaces, which are called del Pezzo surfaces without tigers in [27], lie in finitely many families (see [42], [37]). We expect that strongly exceptional Fano varieties with quotient singularities enjoy very interesting geometrical properties (cf. [41, Theorem 3.3], [35, Theorem 1]).

The main motivation for this article is that the global log canonical threshold turns out to play important roles both in birational geometry and in complex geometry. We have two significant applications of the global log canonical threshold of a Fano variety X. The first one is for the case when $\mathrm{lct}(X) \geqslant 1$. This inequality has serious applications to rationality problems for Fano varieties in birational geometry. The other is for the case when $\mathrm{lct}(X) > \frac{\dim(X)}{1+\dim(X)}$. This has important applications to Kähler-Einstein metrics on Fano varieties in complex geometry.

For a simple application of the first inequality, we can mention the following.

Theorem 1.1.11 ([7] and [38])**.** Let X_i be birationally super-rigid Fano variety with $\mathrm{lct}(X_i) \geqslant 1$ for each $i = 1, \ldots, r$. Then the variety $X_1 \times \ldots \times X_r$ is non-rational and

$$\mathrm{Bir}\Big(X_1 \times \ldots \times X_r\Big) = \mathrm{Aut}\Big(X_1 \times \ldots \times X_r\Big).$$

For every dominant map $\rho\colon X_1 \times \ldots \times X_r \dashrightarrow Y$ whose general fiber is rationally connected, there is a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$ and a commutative diagram

$$\begin{array}{ccc} X_1 \times \ldots \times X_r & \overset{\sigma}{\dashrightarrow} & X_1 \times \ldots \times X_r \\ \downarrow{\scriptstyle \pi} & & \searrow{\scriptstyle \rho} \\ X_{i_1} \times \ldots \times X_{i_k} & \underset{\xi}{\dashrightarrow} & Y, \end{array}$$

where ξ and σ are birational maps, and π is the natural projection.

This theorem may be more generalized so that we could obtain the following

Example 1.1.12 ([7])**.** Let X_i be a threefold satisfying hypotheses of Example 1.1.7 with $\mathrm{lct}(X_i) = 1$ for each $i = 1, \ldots r$. Suppose, in addition, that each X_i is general in its deformation family. Then the variety $X_1 \times \ldots \times X_r$ is non-rational and

$$\mathrm{Bir}\Big(X_1 \times \ldots \times X_r\Big) = \Big\langle \prod_{i=1}^{r} \mathrm{Bir}(X_i),\ \mathrm{Aut}\Big(X_1 \times \ldots \times X_r\Big)\Big\rangle.$$

For every dominant map $\rho\colon X_1 \times \ldots \times X_r \dashrightarrow Y$ whose general fiber is rationally connected, there is a subset $\{i_1, \ldots, i_k\} \subseteq \{1, \ldots, r\}$ and a commutative diagram

$$\begin{array}{ccc} X_1 \times \ldots \times X_r & \overset{\sigma}{\dashrightarrow} & X_1 \times \ldots \times X_r \\ \downarrow{\scriptstyle \pi} & & \searrow{\scriptstyle \rho} \\ X_{i_1} \times \ldots \times X_{i_k} & \underset{\xi}{\dashrightarrow} & Y, \end{array}$$

where ξ and σ are birational maps, and π is the natural projection.

The following result that gives strong connection between global log canonical thresholds and Kähler-Einstein metrics was proved in [16], [34],[44] (see [14, Appendix A]).

Theorem 1.1.13. Suppose that X is a Fano variety with at most quotient singularities. Then it admits an orbifold Kähler–Einstein metric if

$$\mathrm{lct}\big(X\big) > \frac{\dim\big(X\big)}{\dim\big(X\big) + 1}.$$

Examples 1.1.4, 1.1.5 and 1.1.7 are good examples to which we may apply Theorem 1.1.13.

There are many known obstructions for the existence of orbifold Kähler–Einstein metrics on Fano varieties with quotient singularities (see [17], [19], [31], [33], [40], [47]).

Example 1.1.14 ([20])**.** Let X be a quasismooth hypersurface in $\mathbb{P}(a_0,\ldots,a_n)$ of degree $d < \sum_{i=0}^{n} a_i$, where $a_0 \leqslant \ldots \leqslant a_n$. Suppose that X is well-formed and has a Kähler–Einstein metric. Then

$$d\left(\sum_{i=0}^{n} a_i - d\right)^n \leqslant n^n \prod_{i=0}^{n} a_i,$$

and $\sum_{i=0}^{n} a_i \leqslant d + na_0$ (see [2], [43]).

The problem of existence of Kähler–Einstein metrics on smooth del Pezzo surfaces is completely solved by [45] as follows:

Theorem 1.1.15. If X is a smooth del Pezzo surface, then the following conditions are equivalent:

- the automorphism group $\mathrm{Aut}(X)$ is reductive;
- the surface X admits a Kähler–Einstein metric;
- the surface X is not a blow up of $\mathbb{P}^2$ at one or two points.

Acknowledgments. The first author is grateful to the Max Plank Institute for Mathematics at Bonn for the hospitality and excellent working conditions. The first author was supported by the grants NSF DMS-0701465 and EPSRC EP/E048412/1, the third author was supported by the grants RFFI No. 08-01-00395-a, N.Sh.-1987.1628.1 and EPSRC EP/E048412/1. The second author has been supported by the Korea Research Foundation Grant funded by the Korean Government (KRF-2008-313-C00024).

The authors thank I. Kim, B. Sea, and J. Won for their pointing out numerous mistakes in the first version of this paper.

1.2. Results

Let X_d be a quasismooth and well-formed hypersurface in $\mathbb{P}(a_0,a_1,a_2,a_3)$ of degree d, where $a_0 \leqslant a_1 \leqslant a_2 \leqslant a_4$. Then the hypersurface X_d is given by a quasihomogeneous polynomial equation $f(x,y,z,t)=0$ of degree d. The quasihomogeneous equation

$$f\big(x,y,z,t\big) = 0 \subset \mathbb{C}^4 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x,y,z,t\big]\Big),$$

defines an isolated quasihomogeneous singularity (V,O) with the Milnor number $\prod_{i=0}^{n}(\frac{d}{a_i}-1)$, where O is the origin of $\mathbb{C}^4$. It follows from the adjunction formula that

$$K_{X_d} \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(a_0,a_1,a_2,a_3)}\Big(d - \sum_{i=0}^{3} a_i\Big),$$

and it follows from [18], [26, Proposition 8.14], [39] that the following conditions are equivalent:

- the inequality $d \leqslant \sum_{i=0}^{3} a_i - 1$ holds;
- the surface X_d is a del Pezzo surface;
- the singularity (V,O) is rational;
- the singularity (V,O) is canonical.

Blowing up $\mathbb{C}^4$ at the origin O with weights (a_0, a_1, a_2, a_3), we get a purely log terminal blow up of the singularity (V, O) (see [28], [36]). The paper [36] shows that the following conditions are equivalent:

- the surface X_d is exceptional (weakly exceptional, respectively);
- the singularity (V, O) is exceptional[1] (weakly exceptional, respectively).

From now on we suppose that $d \leqslant \sum_{i=0}^{3} a_i - 1$. Then X_d is a del Pezzo surface. Put $I = \sum_{i=0}^{3} a_i - d$. The list of possible values of (a_0, a_1, a_2, a_3, d) with $2I < 3a_0$ can be found in [4] and [15]. For the case $I = 1$, we can obtain the complete list of del Pezzo surfaces $X_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ from [25] as follows:

- smooth del Pezzo surfaces
  $X_3 \subset \mathbb{P}(1,1,1,1)$, $X_4 \subset \mathbb{P}(1,1,1,2)$, $X_6 \subset \mathbb{P}(1,1,2,3)$,
- singular del Pezzo surfaces
  $X_{8n+4} \subset \mathbb{P}(2, 2n+1, 2n+1, 4n+1)$, where n is a positive integer,
  $X_{10} \subset \mathbb{P}(1,2,3,5)$, $X_{15} \subset \mathbb{P}(1,3,5,7)$, $X_{16} \subset \mathbb{P}(1,3,5,8)$, $X_{18} \subset \mathbb{P}(2,3,5,9)$,
  $X_{15} \subset \mathbb{P}(3,3,5,5)$, $X_{25} \subset \mathbb{P}(3,5,7,11)$, $X_{28} \subset \mathbb{P}(3,5,7,14)$,
  $X_{36} \subset \mathbb{P}(3,5,11,18)$, $X_{56} \subset \mathbb{P}(5,14,17,21)$, $X_{81} \subset \mathbb{P}(5,19,27,31)$,
  $X_{100} \subset \mathbb{P}(5,19,27,50)$, $X_{81} \subset \mathbb{P}(7,11,27,37)$, $X_{88} \subset \mathbb{P}(7,11,27,44)$,
  $X_{60} \subset \mathbb{P}(9,15,17,20)$, $X_{69} \subset \mathbb{P}(9,15,23,23)$, $X_{127} \subset \mathbb{P}(11,29,39,49)$,
  $X_{256} \subset \mathbb{P}(11,49,69,128)$, $X_{127} \subset \mathbb{P}(13,23,35,57)$, $X_{256} \subset \mathbb{P}(13,35,81,128)$.

The global log canonical thresholds of such del Pezzo surfaces have been considered either implicitly or explicitly in [1], [3], [11], [16], [25]. For example, the papers [1], [3], [16] and [25] gives us lower bounds for global log canonical thresholds of singular del Pezzo surfaces with $I = 1$. Meanwhile, the paper [11] deals with the exact values of the global log canonical thresholds of smooth del Pezzo surfaces with $I = 1$.

Theorem 1.2.1. Suppose that $I = 1$ and X_d is smooth. Then

$$\mathrm{lct}\big(X_d\big) = \begin{cases} 1 & \text{if } \big(a_0, a_1, a_2, a_3\big) = \big(1,1,2,3\big) \text{ and } |-K_{X_6}| \text{ contains no cuspidal curves,} \\ \frac{5}{6} & \text{if } \big(a_0, a_1, a_2, a_3\big) = \big(1,1,2,3\big) \text{ and } |-K_{X_6}| \text{ contains a cuspidal curve,} \\ \frac{5}{6} & \text{if } \big(a_0, a_1, a_2, a_3\big) = \big(1,1,1,2\big) \text{ and } |-K_{X_4}| \text{ contains no tacnodal curves,} \\ \frac{3}{4} & \text{if } \big(a_0, a_1, a_2, a_3\big) = \big(1,1,1,2\big) \text{ and } |-K_{X_4}| \text{ contains a tacnodal curve,} \\ \frac{3}{4} & \text{if } X_3 \text{ is a cubic in } \mathbb{P}^3 \text{ with no Eckardt points,} \\ \frac{2}{3} & \text{if } X_3 \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point.} \end{cases}$$

However, for singular del Pezzo surfaces, the exact values of global log canonical thresholds have not been considered seriously.

A singular del Pezzo hypersurface $X_d \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ must satisfy exclusively one of the following properties:

[1]For notions of exceptional and weakly exceptional singularities see [36, Definition 4.1], [42], [23].

(1) $2I \geqslant 3a_0$;
(2) $2I < 3a_0$ and

$$\big(a_0, a_1, a_2, a_3, d\big) = \big(I - k, I + k, a, a + k, 2a + k + I\big)$$

for some non-negative integer $k < I$ and some positive integer $a \geqslant I + k$.
(3) $2I < 3a_0$ but

$$\big(a_0, a_1, a_2, a_3, d\big) \neq \big(I - k, I + k, a, a + k, 2a + k + I\big)$$

for any non-negative integer $k < I$ and any positive integer $a \geqslant I + k$.

For the first two cases one can check that $\mathrm{lct}(X_d) \leqslant \frac{2}{3}$ (for instance, see [4] and [15]). All the quintuples (a_0, a_1, a_2, a_3, d) such that the hypersurface X_d is singular and satisfies the last condition are listed in Section 4. They are taken from [4] and [15]. Note that we rearranged a little the quintuples taken from [4] by putting some cases that were contained in the infinite series of [4] into the sublist of sporadic cases; on the other hand, we removed two sporadic cases, because they are contained in the additional infinite series found in [15]. The completeness of this list is proved in [15] by using [49].

We already know the global log canonical thresholds of smooth del Pezzo surfaces. For del Pezzo surfaces satisfying one of the first two conditions, their global log canonical thresholds are relatively too small to enjoy the condition of Theorem 1.1.13. However, the global log canonical thresholds of del Pezzo surfaces satisfying the last condition have not been investigated sufficiently. In the present paper we compute all of them and then we obtain the following result.

Theorem 1.2.2. Let X_d be a quasismooth well-formed singular del Pezzo surface in the weighted projective space $\mathrm{Proj}(\mathbb{C}[x, y, z, t])$ with weights $\mathrm{wt}(x) = a_0 \leqslant \mathrm{wt}(y) = a_1 \leqslant \mathrm{wt}(z) = a_2 \leqslant \mathrm{wt}(t) = a_3$ such that $2I < 3a_0$ but $(a_0, a_1, a_2, a_3, d) \neq (I - k, I + k, a, a + k, 2a + k + I)$ for any non-negative integer $k < I$ and any positive integer $a \geqslant I + k$, where $I = \sum_{i=0}^{3} a_i - d$. Then if $a_0 \neq a_1$, then

$$\mathrm{lct}(X_d) = \min\left\{\mathrm{lct}\Big(X_d, \frac{I}{a_0}C_x\Big),\ \mathrm{lct}\Big(X_d, \frac{I}{a_1}C_y\Big),\ \mathrm{lct}\Big(X_d, \frac{I}{a_2}C_z\Big)\right\},$$

where C_x (resp. C_y, C_z) is the divisor on X_d defined by $x = 0$ (resp. $y = 0$, $z = 0$). If $a_0 = a_1$, then

$$\mathrm{lct}(X_d) = \mathrm{lct}\Big(X_d, \frac{I}{a_0}C\Big),$$

where C is a reducible divisor in $|\mathcal{O}_{X_d}(a_0)|$.

In particular, we obtain the value of $\mathrm{lct}(X_d)$ for every del Pezzo surface X_d listed in Section 4. As a result, we obtain the following corollaries.

Corollary 1.2.3. The following assertions are equivalent:

- the surface X_d is exceptional;
- $\mathrm{lct}(X_d) > 1$;

- the quintuple (a_0, a_1, a_2, a_3, d) lies in the set

$$\left\{\begin{aligned}&(2,3,5,9,18),(3,3,5,5,15),(3,5,7,11,25),(3,5,7,14,28),\\&(3,5,11,18,36),(5,14,17,21,56),(5,19,27,31,81),(5,19,27,50,100),\\&(7,11,27,37,81),(7,11,27,44,88),(9,15,17,20,60),(9,15,23,23,69),\\&(11,29,39,49,127),(11,49,69,128,256),(13,23,35,57,127),\\&(13,35,81,128,256),(3,4,5,10,20),(3,4,10,15,30),(5,13,19,22,57),\\&(5,13,19,35,70),(6,9,10,13,36),(7,8,19,25,57),(7,8,19,32,64),\\&(9,12,13,16,48),(9,12,19,19,57),(9,19,24,31,81),(10,19,35,43,105),\\&(11,21,28,47,105),(11,25,32,41,107),(11,25,34,43,111),(11,43,61,113,226),\\&(13,18,45,61,135),(13,20,29,47,107),(13,20,31,49,111),(13,31,71,113,226),\\&(14,17,29,41,99),(5,7,11,13,33),(5,7,11,20,40),(11,21,29,37,95),\\&(11,37,53,98,196),(13,17,27,41,95),(13,27,61,98,196),(15,19,43,74,148),\\&(9,11,12,17,45),(10,13,25,31,75),(11,17,20,27,71),(11,17,24,31,79),\\&(11,31,45,83,166),(13,14,19,29,71),(13,14,23,33,79),(13,23,51,83,166),\\&(11,13,19,25,63),(11,25,37,68,136),(13,19,41,68,136),(11,19,29,53,106),\\&(13,15,31,53,106),(11,13,21,38,76)\end{aligned}\right\}.$$

Corollary 1.2.4. The following assertions are equivalent:

- the surface X_d is weakly exceptional and not exceptional;
- $\operatorname{lct}(X_d) = 1$;
- one of the following holds
  - the quintuple (a_0, a_1, a_2, a_3, d) lies in the set

$$\left\{\begin{aligned}&(2,2n+1,2n+1,4n+1,8n+4),\\&(3,3n,3n+1,3n+1,9n+3),(3,3n+1,3n+2,3n+2,9n+6),\\&(3,3n+1,3n+2,6n+1,12n+5),(3,3n+1,6n+1,9n,18n+3),\\&(3,3n+1,6n+1,9n+3,18n+6),(4,2n+1,4n+2,6n+1,12n+6),\\&(4,2n+3,2n+3,4n+4,8n+12),(6,6n+3,6n+5,6n+5,18n+15),\\&(6,6n+5,12n+8,18n+9,36n+24),(6,6n+5,12n+8,18n+15,36n+30),\\&(8,4n+5,4n+7,4n+9,12n+23),(9,3n+8,3n+11,6n+13,12n+35),\\&(1,3,5,8,16),(2,3,4,7,14),(5,6,8,9,24),(5,6,8,15,30)\end{aligned}\right\},$$

    where n is a positive integer,
  - $(a_0, a_1, a_2, a_3, d) = (1, 1, 2, 3, 6)$ and the pencil $|-K_X|$ does not have cuspidal curves,
  - $(a_0, a_1, a_2, a_3, d) = (1, 2, 3, 5, 10)$ and $C_x = \{x = 0\}$ has an ordinary double point,
  - $(a_0, a_1, a_2, a_3, d) = (1, 3, 5, 7, 15)$ and the defining equation of X contains yzt,
  - $(a_0, a_1, a_2, a_3, d) = (2, 3, 4, 5, 12)$ and the defining equation of X contains yzt.

Corollary 1.2.5. In the notation and assumptions of Theorem 1.2.2, the surface X_d has an orbifold Kähler–Einstein metric with the following possible exceptions: $X_{45} \subset \mathbb{P}(7,10,15,19)$, $X_{81} \subset \mathbb{P}(7,18,27,37)$, $X_{64} \subset \mathbb{P}(7,15,19,32)$, $X_{82} \subset \mathbb{P}(7,19,25,41)$, $X_{117} \subset \mathbb{P}(7,26,39,55)$, $X_{15} \subset \mathbb{P}(1,3,5,7)$ whose defining equation does not contain yzt, and $X_{12} \subset \mathbb{P}(2,3,4,5)$ whose defining equation does not contain yzt.

Corollary 1.2.3 illustrates the fact that exceptional del Pezzo surfaces lie in finitely many families (see [42], [37]). On the other hand, Corollary 1.2.3 shows that weakly-exceptional del Pezzo surfaces do not enjoy this property. Note also that Corollary 1.2.3 follows from [29].

1.3. Preliminaries

For the basic definitions and properties concerning singularities of pairs we refer the reader to [26]. To prove Theorem 1.2.2 we need to compute the log canonical thresholds of individual effective divisors. The following two lemmas are rather basic properties of log canonical thresholds but will be useful to compute them. For the proofs the reader is referred to [26] and [30].

Lemma 1.3.1. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$ and $D = (f = 0)$. Suppose that the polynomial f vanishes at the origin O in $\mathbb{C}^n$. Set $d = \mathrm{mult}_O(f)$ and let f_d denote the degree d homogeneous part of f. Let $T_0D = (f_d = 0) \subset \mathbb{C}^n$ be the tangent cone of D and $\mathbb{P}(T_0D) = (f_d = 0) \subset \mathbb{P}^{n-1}$ be the projectivised tangent cone of D. Then

(1) $\frac{1}{d} \leqslant \mathrm{lct}_O(\mathbb{C}^n, D) \leqslant \frac{n}{d}$.
(2) The log pair $(\mathbb{P}^{n-1}, \frac{n}{d}\mathbb{P}(T_0D))$ is log canonical if and only if $\mathrm{lct}_O(\mathbb{C}^n, D) = \frac{n}{d}$.
(3) If $\mathbb{P}(T_0D)$ is smooth (or even log canonical) then $\mathrm{lct}_O(\mathbb{C}^n, D) = \min\{1, \frac{n}{d}\}$.

Lemma 1.3.2. Let f be a polynomial in $\mathbb{C}[z_1, z_2]$. Suppose that the polynomial defines an irreducible curve C passing through the origin O in $\mathbb{C}^2$. We then have

$$\mathrm{lct}_O(\mathbb{C}^2, C) = \min\left(1, \frac{1}{m} + \frac{1}{n}\right),$$

where (m, n) is the first pair of Puiseux exponents of f. We also have

$$\mathrm{lct}_O\left(\mathbb{C}^2, (z_1^{n_1} z_2^{n_2}(z_1^{m_1} + z_2^{m_2}) = 0)\right) = \min\left(\frac{1}{n_1}, \frac{1}{n_2}, \frac{m_1 + m_2}{m_1m_2 + m_1n_2 + m_2n_1}\right),$$

where n_1, n_2, m_1, m_2 are non-negative integers.

Throughout the proof of Theorem 1.2.2, Inversion of Adjunction that enables us to compute log canonical thresholds on lower dimensional varieties will be frequently utilized. Let X be a normal (but not necessarily projective) variety. Let S be a smooth Cartier divisor on X and B be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X such that $K_X + S + B$ is $\mathbb{Q}$-Cartier and $S \not\subseteq \mathrm{Supp}(B)$.

Theorem 1.3.3. The log pair $(X, S + B)$ is log canonical along S if and only if the log pair $(S, B|_S)$ is log canonical.

In the case when X is a surface, Theorem 1.3.3 can be stated in terms of local intersection numbers.

Lemma 1.3.4. Suppose that X is a surface. Let P be a smooth point of X such that it is also a smooth point of S. Then the log pair $(X, S+B)$ is log canonical at the point P if and only if the local intersection number of B and S at the point P is at most 1. In particular, if the log pair $(X, mS+B)$ is not log canonical at the point P for $m \leqslant 1$, then $B \cdot S > 1$.

Lemma 1.3.5. Let D be an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. For a smooth point P of X, the log pair (X, D) is log canonical at the point P if $\operatorname{mult}_P(D) \leqslant 1$.

Throughout the proof of Theorem 1.2.2, we interrelate Lemma 1.3.5 with Lemma 1.3.4 to get some contradictions. To do so, we need the following lemma that plays the role of a bridge between them.

Lemma 1.3.6. Let D_1 and D_2 be effective $\mathbb{Q}$-divisors on Y with $D_1 \sim_{\mathbb{Q}} D_2$. Suppose that the pair (X, D_1) is not log canonical at a point $P \in Y$ but the pair (X, D_2) is log canonical at the point P. Then there is an effective $\mathbb{Q}$-divisor D on Y such that

- $D \sim_{\mathbb{Q}} D_1$;
- at least one irreducible component of D_2 is not contained in the support of D;
- the pair (X, D) is not log canonical at the point P.

Proof. Write $D_2 = \sum_{i=1}^{r} b_i C_i$ where b_i's are positive rational numbers and C_i's are distinct irreducible and reduced divisors. Also, we write $D_1 = \Delta + \sum_{i=1}^{r} e_i C_i$ where e_i's are non-negative rational numbers and Δ is an effective $\mathbb{Q}$-divisor whose support contains none of C_i's. Suppose that $e_i > 0$ for each i. If not, then we put $D = D_1$. Let

$$\alpha = \min\left\{ \frac{e_i}{b_i} \;\middle|\; i = 1, 2, \dots, r \right\}.$$

Then the positive rational number α is less than 1 since $D_1 \sim_{\mathbb{Q}} D_2$. Put

$$\begin{aligned} D &= \frac{1}{1-\alpha} D_1 - \frac{\alpha}{1-\alpha} D_2 \\ &= \frac{1}{1-\alpha}\Delta + \sum_{i=1}^{r} \left(\frac{e_i - \alpha b_i}{1-\alpha} \right) C_i. \end{aligned}$$

It is easy to see that the divisor D satisfies the first two conditions. If the pair (X, D) is log canonical at the point P, then the pair $(X, D_1) = (X, (1-\alpha)D + \alpha D_2)$ must be log canonical at the point P. Therefore, the divisor D also satisfies the last condition. □

In the present paper, we deal with surfaces with at most quotient singularities. However, the statements mentioned so far require smoothness of the ambient space for us to utilize them to the fullest. Fortunately, the following proposition enables us to apply the statements with ease since we have a natural finite morphism of a germ of the origin in $\mathbb{C}^2$ to a germ of a quotient singularity that is ramified only at a point.

Proposition 1.3.7 ([26])**.** Let $f\colon Y \to X$ be a finite morphism between normal varieties and assume that f is unramified outside a set of codimension two. Let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Then a log pair (X, D) is log canonical (resp. Kawamata log terminal) if and only if the log pair (Y, f^*D) is log canonical (resp. Kawamata log terminal).

The following two lemmas will be useful for this paper. The first lemma is just a reformulation of Lemma 1.3.4 mixed with Proposition 1.3.7 that we can apply to our cases immediately.

Suppose that X is a quasismooth well-formed hypersurface in $\mathbb{P} = \mathbb{P}(a_0, a_1, a_2, a_3)$ of degree d.

Lemma 1.3.8. Let C be a reduced and irreducible curve on X and D be an effective $\mathbb{Q}$-divisor on X. Suppose that for a given positive rational number λ we have $\lambda\mathrm{mult}_C(D) \leqslant 1$. If $\lambda(C \cdot D - (\mathrm{mult}_C(D))C^2) \leqslant 1$, then the pair $(X, \lambda D)$ is log canonical at each smooth point P of C not in $\mathrm{Sing}(X)$. Furthermore, if the point P of C is a singular point of X of type $\frac{1}{r}(a, b)$ and $r\lambda(C \cdot D - (\mathrm{mult}_C(D))C^2) \leqslant 1$, then the pair $(X, \lambda D)$ is log canonical at P.

Proof. We may write $D = mC + \Omega$, where Ω is an effective divisor whose support does not contain the curve C. Suppose that the pair $(X, \lambda D)$ is not log canonical at a smooth point P of C not in $\mathrm{Sing}(X)$. Since $\lambda m \leqslant 1$, the pair $(X, C + \lambda\Omega)$ is not log canonical at the point P. Then by Lemma 1.3.4 we obtain an absurd inequality

$$1 < \lambda\Omega \cdot C = \lambda C \cdot (D - mC) \leqslant 1.$$

Also, if the point P is a singular point of X, then we obtain from Lemma 1.3.4 and Proposition 1.3.7

$$\frac{1}{r} < \lambda\Omega \cdot C = \lambda C \cdot (D - mC) \leqslant \frac{1}{r}.$$

This proves the second statement. □

Let D be an effective $\mathbb{Q}$-divisor on X such that

$$D \sim_{\mathbb{Q}} \mathcal{O}_{\mathbb{P}(a_0,\, a_1,\, a_2,\, a_3)}(I)\Big|_X.$$

The next lemma will be applied to show that the log pair (X, D) is log canonical at some smooth points on X.

Lemma 1.3.9. Let k be a positive integer. Suppose that $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k))$ contains

- at least two different monomials of the form $x^\alpha y^\beta$,
- at least two different monomials of the form $x^\gamma z^\delta$.

For a smooth point P of X in the outside of C_x,

$$\mathrm{mult}_P\big(D\big) \leqslant \frac{Ikd}{a_0 a_1 a_2 a_3}$$

if either $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(k))$ contains at least two different monomials of the form $x^\mu t^\nu$ or the point P is not contained in a curve contracted by the projection $\psi : X \dashrightarrow \mathbb{P}(a_0, a_1, a_2)$. Here, α, β, γ, δ, μ and ν are non-negative integers.

Proof. The first case follows from [1, Lemma 3.3]. Arguing as in the proof of [1, Corollary 3.4], we can also obtain the second case. □

Let us conclude this section by mentioning two results that are never used in this paper, but nevertheless can be used to give shorter proofs of Corollaries 1.2.3 and 1.2.5. Suppose that X is given by a quasihomogeneous equation

$$f\big(x, y, z, t\big) = 0 \subset \mathbb{P}\big(a_0, a_1, a_2, a_3\big) \cong \mathrm{Proj}\Big(\mathbb{C}\big[x, y, z, t\big]\Big),$$

where $\mathrm{wt}(x) = a_0 \leqslant \mathrm{wt}(y) = a_1 \leqslant \mathrm{wt}(z) = a_2 \leqslant \mathrm{wt}(t) = a_3$.

Lemma 1.3.10. Suppose that $I=\sum_{i=0}^{3}a_i-d>0$. Then

$$\mathrm{lct}(X)\geqslant\begin{cases}\dfrac{a_0a_1}{dI},\\ \dfrac{a_0a_2}{dI}\ \text{if}\ f(0,0,z,t)\neq 0,\\ \dfrac{a_0a_3}{dI}\ \text{if}\ f(0,0,0,t)\neq 0.\end{cases}$$

Proof. See [4, Corollary 5.3] (cf. [25, Proposition 11]). □

Lemma 1.3.11. Suppose that $I=\sum_{i=0}^{3}a_i-d>0$, the curve $C_x=\{x=0\}$ is irreducible and reduced. Then

$$\mathrm{lct}(X)\geqslant\begin{cases}\min\left(\dfrac{a_1a_2}{dI},\ \mathrm{lct}\left(X,\dfrac{I}{a_0}C_x\right)\right),\\ \min\left(\dfrac{a_1a_3}{dI},\ \mathrm{lct}\left(X,\dfrac{I}{a_0}C_x\right)\right)\ \text{if}\ f(0,0,0,t)\neq 0.\end{cases}$$

Proof. Arguing as in the proof of [25, Proposition 11] and using Lemma 1.3.6, we obtain the required assertion. □

1.4. Notation

We reserve the following notation that will be used throughout the paper:

- $\mathbb{P}(a_0,a_1,a_2,a_3)$ denotes the well-formed weighted projective space $\mathrm{Proj}(\mathbb{C}[x,y,z,t])$ with weights $\mathrm{wt}(x)=a_0$, $\mathrm{wt}(y)=a_1$, $\mathrm{wt}(z)=a_2$, $\mathrm{wt}(t)=a_3$, where we always assume the inequalities $a_0\leqslant a_1\leqslant a_2\leqslant a_3$. We may use simply $\mathbb{P}$ instead of $\mathbb{P}(a_0,a_1,a_2,a_3)$ when this does not lead to confusion.
- X denotes a quasismooth and well-formed hypersurface in $\mathbb{P}(a_0,a_1,a_2,a_3)$ (see Definitions 6.3 and 6.9 in [22], respectively).
- O_x is the point in $\mathbb{P}(a_0,a_1,a_2,a_3)$ defined by $y=z=t=0$. The points O_y, O_z and O_t are defined in the similar way.
- C_x is the curve on X cut out by the equation $x=0$. The curves C_y, C_z and C_t are defined in the similar way.
- L_{xy} is the curve in $\mathbb{P}(a_0,a_1,a_2,a_3)$ defined by $x=y=0$. The curves L_{xz}, L_{xt}, L_{yz}, L_{yt} and L_{zt} are defined in the similar way.
- Let D be a divisor on X and $P\in X$. Choose an orbifold chart $\pi\colon\tilde{U}\to U$ for some neighborhood $P\in U\subset X$. We put $\mathrm{mult}_P(D)=\mathrm{mult}_Q(\pi^*D)$, where Q is a point on $\tilde{U}$ with $\pi(Q)=P$, and refer to this quantity as the multiplicity of D at P.

1.5. The scheme of the proof

We have 83 families[2] of del Pezzo hypersurfaces in The Big Table. In the present section we explain the methods to compute the global log canonical thresholds of the del Pezzo hypersurfaces in The Big Table.

[2]By family we mean either one-parameter series (which actually gives rise to an infinite number of deformation families) or a sporadic case. We hope that this would not lead to a confusion.

Let $X \subset \mathbb{P}(a_0, a_1, a_2, a_3)$ be a del Pezzo surface of degree d in one of the 83 families (actually, one infinite series has been treated in [15], so we will omit the computations in this case). Set $I = a_0 + a_1 + a_2 + a_3 - d$. There are two exceptional cases where $a_0 = a_1$. The method for these two cases is a bit different from the other cases. Both cases will be individually dealt with (Lemmas 2.2.4 and 3.1.5).

If $a_0 \neq a_1$, then we will take steps as follows:

Step 1. Using Lemmas 1.3.1 and 1.3.2 with Proposition 1.3.7, we compute the log canonical thresholds $\mathrm{lct}(X, \frac{I}{a_0}C_x)$, $\mathrm{lct}(X, \frac{I}{a_0}C_y)$, $\mathrm{lct}(X, \frac{I}{a_0}C_z)$ and $\mathrm{lct}(X, \frac{I}{a_0}C_t)$. Set

$$\lambda = \min\left\{\mathrm{lct}(X, \frac{I}{a_0}C_x), \mathrm{lct}(X, \frac{I}{a_0}C_y), \mathrm{lct}(X, \frac{I}{a_0}C_z), \mathrm{lct}(X, \frac{I}{a_0}C_t)\right\}.$$

Then the global log canonical threshold $\mathrm{lct}(X)$ is at most λ.

Step 2. We claim that the global log canonical threshold $\mathrm{lct}(X)$ is equal to λ. To prove this assertion, we suppose $\mathrm{lct}(X) < \lambda$. Then there is an effective $\mathbb{Q}$-divisor D equivalent to the anticanonical divisor $-K_X$ of X such that the log pair $(X, \lambda D)$ is not log canonical at some point $P \in X$. In particular, we obtain

$$\mathrm{mult}_P(\lambda D) > \begin{cases} 1 & \text{if the point } P \text{ is a smooth point of } X, \\ \dfrac{1}{r} & \text{if the point } P \text{ is a singular point of } X \text{ of type } \dfrac{1}{r}(a,b). \end{cases}$$

from Lemma 1.3.5 and Proposition 1.3.7.

Step 3. We show that the point P cannot be a singular point of X using the following methods.

Method 3.1. (Multiplicity) We may assume that a suitable irreducible component C of C_x, C_y, C_z, and C_t is not contained in the support of the divisor D. We derive a possible contradiction from the inequality

$$C \cdot D \geqslant \mathrm{mult}_P(C) \cdot \frac{\mathrm{mult}_P(D)}{r} > \frac{\mathrm{mult}_P(C)}{r\lambda},$$

where r is the index of the quotient singular point P. The last inequality follows from the assumption that $(X, \lambda D)$ is not log canonical at P. This method can be applied to exclude a smooth point.

Method 3.2. (Inversion of Adjunction) We consider a suitable irreducible curve C smooth at P. We then write $D = \mu C + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain C. We check $\lambda\mu \leqslant 1$. If so, then the log pair $(X, C + \lambda\Omega)$ is not log canonical at the point P either. By Lemma 1.3.8 we have

$$\lambda(D - \mu C) \cdot C = \lambda C \cdot \Omega > \frac{1}{r}.$$

We try to derive a contradiction from this inequality. The curve C is taken usually from an irreducible component of C_x, C_y, C_z, or C_t. This method can be applied to exclude a smooth point.

Method 3.3. (Weighted Blow Up) Sometimes we cannot exclude a singular point P only with the previous two methods. In such a case, we take a suitable weighted blow up $\pi : Y \to X$ at the point P. We can write

$$K_Y + D^Y \sim_{\mathbb{Q}} \pi^*(K_X + \lambda D),$$

where D^Y is the log pull-back of λD by π. Using method 3.1 we obtain that D^Y is effective. Then we apply the previous two methods to the pair (Y, D^Y), or repeat this method until we get a contradictory inequality.

Step 4. We show that the point P cannot be a smooth point of X. To do so, we first apply Lemma 1.3.9. However, this method does not work always. If the method fails, then we try to find a suitable pencil $\mathcal{L}$ on X. The pencil has a member F which passes through the point P. We show that the pair $(X, \lambda F)$ is log canonical at the point P. Then, we may assume that the support of D does not contain at least one irreducible component of F. If the divisor D itself is irreducible, then we use Method 3.1 to exclude the point P. If F is reducible, then we use Method 3.2.

Part 2. Infinite series

2.1. Infinite series with $I = 1$

Lemma 2.1.1. Let X be a quasismooth hypersurface of degree $8n+4$ in $\mathbb{P}(2, 2n+1, 2n+1, 4n+1)$ for a natural number n. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X is singular at the point O_t, which is of type $\frac{1}{4n+1}(1,1)$. It has also four singular points O_1, O_2, O_3, O_4, which are cut out on X by L_{xt}. Each O_i is a singular point of type $\frac{1}{2n+1}(1,n)$ on the surface X.

The curve C_x is reducible. We see

$$C_x = L_1 + L_2 + L_3 + L_4,$$

where L_i is a smooth rational curves such that

$$-K_X \cdot L_i = \frac{1}{(2n+1)(4n+1)},$$

and $L_1 \cap L_2 \cap L_3 \cap L_4 = \{O_t\}$. Then

$$L_i \cdot L_j = \frac{1}{4n+1}$$

for $i \neq j$. Also, we have

$$L_i^2 = C_x \cdot L_i - \frac{3}{4n+1} = \frac{2}{(2n+1)(4n+1)} - \frac{3}{4n+1} = -\frac{6n+1}{(2n+1)(4n+1)}.$$

It is easy to see $\mathrm{lct}(X, \frac{1}{2}C_x) = 1$. Therefore, $\mathrm{lct}(X) \leqslant 1$. Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Since

$$\frac{(4n+2)(8n+4)}{2(2n+1)^2(4n+1)} = \frac{4}{4n+1} < 1$$

and $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(4n+2))$ contains x^{2n+1}, y^2 and z^2, Lemma 1.3.9 implies that $P \in C_x$.

It follows from Lemma 1.3.6 that we may assume that $L_i \not\subset \mathrm{Supp}(D)$ for some i. Also, $P \in L_j$ for some j. Put $D = mL_j + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_j \not\subset \mathrm{Supp}(\Omega)$. Since

$$\frac{1}{(2n+1)(4n+1)} = D \cdot L_i = \big(mL_j + \Omega\big) \cdot L_i \geqslant mL_i \cdot L_j = \frac{m}{4n+1},$$

we have $0 \leqslant m \leqslant \frac{1}{2n+1}$. Since

$$(2n+1)\Omega \cdot L_j = (2n+1)(D - mL_j) \cdot L_j = (2n+1)\frac{1 + m(6n+1)}{(2n+1)(4n+1)} \leqslant \frac{2}{(2n+1)} < 1$$

Lemma 1.3.8 implies the point P must be O_t. Note that the inequality

$$\mathrm{mult}_{O_t}\big(D\big) \leqslant (4n+1)D \cdot L_i = \frac{1}{2n+1} \leqslant 1,$$

shows that the point P cannot be the point O_t. This is a contradiction. □

2.2. Infinite series with $I = 2$

Lemma 2.2.1. Let X be a quasismooth hypersurface of degree $8n+12$ in $\mathbb{P}(4, 2n+3, 2n+3, 4n+4)$ for a natural number n. Then $\mathrm{lct}(X) = 1$.

Proof. The only singularities of X are a singular point O_t of index $4n+4$, two singular points P_1, P_2 of index 4 on L_{yz}, and four singular points Q_1, Q_2, Q_3, Q_4 of index $2n+3$ on L_{xt}.

The curve C_x is reduced and splits into four irreducible components $L_1, \ldots, L_4$. Each L_i passes through Q_i. They intersect each other at O_t. One can easily see that $\mathrm{lct}(X, \frac{1}{2}C_x) = 1$, and hence $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

By Lemma 1.3.6 we may assume that $L_i \not\subset \mathrm{Supp}\,(D)$ for some i. Since

$$(4n+4)L_i \cdot D = \frac{4n+4}{(2n+2)(2n+3)} < 1$$

for all $n \geqslant 1$, the point P cannot belong to the curve L_i.

For $j \neq i$, put $D = \mu L_j + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_j \not\subset \mathrm{Supp}\,(\Omega)$. Since

$$\frac{\mu}{4n+4} = \mu L_i \cdot L_j \leqslant D \cdot L_i = \frac{1}{2(n+1)(2n+3)},$$

we have

$$\mu \leqslant \frac{2}{2n+3}.$$

Note that

$$L_j^2 = C_x \cdot L_j - 3L_i \cdot L_j = \frac{2}{2(n+1)(2n+3)} - \frac{3}{4(n+1)} = -\frac{6n+5}{4(n+1)(2n+3)}.$$

By Lemma 1.3.4 the inequality

$$(2n+3)\Omega\cdot L_j=(2n+3)(D-\mu L_j)\cdot L_j=\frac{2+(6n+5)\mu}{4(n+1)}\leqslant\frac{4}{2n+3}<1$$

for all $n\geqslant 1$ shows that P cannot be contained in L_j. Consequently, the point P is located in the outside of C_x.

By a suitable coordinate change we may assume that $P_1=O_x$. Then, the curve C_t is reduced and splits into four irreducible components $L_1',\ldots,L_4'$. Each L_i' passes through the point Q_i. They intersect each other at O_x. We can easily see that the log pair $(X,\frac{2}{4n+4}C_t)$ is log canonical. By Lemma 1.3.6 we may assume that $L_i'\not\subset\mathrm{Supp}\,(D)$. Since

$$\mathrm{mult}_{O_x}(D)\leqslant 4L_i'\cdot D=\frac{2}{2n+3}<1$$

for all $n\geqslant 1$, the point P cannot be O_x. The point P_2 can be excluded in a similar way.

Therefore, P is a smooth point of $X\setminus C_x$. Applying Lemma 1.3.9, we see that

$$1<\mathrm{mult}_P(D)\leqslant\frac{2(8n+12)^2}{4(2n+3)^2(4n+4)}\leqslant 1$$

for $n\geqslant 1$ since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(8n+12))$ contains x^{2n+3}, y^4 and z^4. The obtained contradiction completes the proof. □

Lemma 2.2.2. Let X be a quasismooth hypersurface of degree $18n+6$ in $\mathbb{P}(3,3n+1,6n+1,9n+3)$ for a natural number $n\geqslant 1$. Then $\mathrm{lct}(X)=1$.

Proof. The only singularities of X are a singular point O_z of index $6n+1$, two singular points P_1, P_2 of index 3 on L_{yz}, and two singular points Q_1, Q_2 of index $3n+1$ on L_{xz}.

The curve C_x is reduced and splits into two components L_1 and L_2 that intersect at O_z. It is easy to see that $\mathrm{lct}(X,\frac{2}{3}C_x)=1$. Therefore, $\mathrm{lct}(X)\leqslant 1$. Note that

$$L_1\cdot L_2=\frac{3}{6n+1}\text{ and }L_1^2=L_2^2=-\frac{9n-3}{(3n+1)(6n+1)}.$$

Suppose that $\mathrm{lct}(X)<1$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair (X,D) is not log canonical at some point $P\in X$.

We may assume that L_2 is not contained the support of D. The inequality

$$D\cdot L_2=\frac{2}{(3n+1)(6n+1)}\leqslant\frac{1}{6n+1}$$

shows that the point P cannot belong to the curve L_2. Put $D=\mu L_1+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_1. Since

$$\frac{3\mu}{6n+1}=\mu L_1\cdot L_2\leqslant D\cdot L_2=\frac{2}{(3n+1)(6n+1)},$$

we have

$$0\leqslant\mu\leqslant\frac{2}{3(3n+1)}.$$

Lemma 1.3.8 and the inequality

$$\Omega \cdot L_1 = (D - \mu L_1) \cdot L_1 = \frac{2 + \mu(9n-3)}{(3n+1)(6n+1)} < \frac{4}{(3n+1)(6n+1)}$$

show that the point P is located in the outside of L_1. Therefore, $P \notin C_x$.

The curve C_y is irreducible. It is easy to see that the log pair $(X, \frac{2}{3n+1}C_y)$ is log canonical. Therefore, we may assume that the support of D does not contain the curve C_y. Note that $P_1, P_2 \in C_y$. The inequality

$$3D \cdot C_y = \frac{4}{6n+1} \leqslant 1$$

shows that neither P_1 not P_2 can be the point P.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.3.9, we get an absurd inequality

$$1 < \mathrm{mult}_P(D) \leqslant \frac{2(18n+6)(18n+3)}{3(3n+1)(6n+1)(9n+3)} \leqslant 1$$

since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(18n+3))$ contains x^{6n+1}, $x^{3n}y^3$ and z^3. The obtained contradiction completes the proof. □

Lemma 2.2.3. Let X be a quasismooth hypersurface of degree $18n+3$ in $\mathbb{P}(3, 3n+1, 6n+1, 9n)$ for a natural number $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. The singularities of X are a singular point O_y of index $3n+1$, a singular point O_t of index $9n$, and two singular points Q_1, Q_2 of index 3 on L_{yz}.

The curve C_x is reduced and irreducible and has the only singularity at O_t. It is easy to see that $\mathrm{lct}(X, \frac{2}{3}C_x) = 1$, and hence $\mathrm{lct}(X) \leqslant 1$. The curve C_y is quasismooth. Therefore, the log pair $(X, \frac{2}{3n+1}C_y)$ is log canonical.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. By Lemma 1.3.6 we may assume that neither C_x nor C_y is contained in $\mathrm{Supp}\,(D)$.

The inequalities

$$C_x \cdot D < (3n+1)C_x \cdot D = \frac{2}{3n} < 1,$$

$$\mathrm{mult}_{O_t}(D) = \frac{\mathrm{mult}_{O_t}(C_x)\mathrm{mult}_{O_t}(D)}{3} \leqslant \frac{9nC_x \cdot D}{3} = \frac{2}{3n+1} < 1$$

show that the point P must be located in the outside of C_x.

Also, the inequality

$$3C_y \cdot D = \frac{2}{3n} < 1$$

implies that neither Q_1 not Q_2 can be the point P. Hence P is a smooth point of $X \setminus C_x$. We see that $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(9n+3))$ contains x^{3n+1}, y^3 and xt. Also, the projection of X from the point O_z has only finite fibers. Therefore, Lemma 1.3.9 implies a contradictory inequality

$$1 < \mathrm{mult}_P(D) \leqslant \frac{2(18n+3)(9n+3)}{3(3n+1)(6n+1) \cdot 9n} = \frac{2}{3n} < 1.$$

The obtained contradiction completes the proof. □

Lemma 2.2.4. Let X be a quasismooth hypersurface of degree 12 in $\mathbb{P}(3,3,4,4)$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$\prod_{i=1}^{4}(\alpha_i x + \beta_i y) = \prod_{j=1}^{3}(\gamma_j z + \delta_j t),$$

where $[\alpha_i : \beta_i]$ define four distinct points and $[\gamma_j : \delta_j]$ define three distinct points in $\mathbb{P}^1$.

Let P_i be the point in X given by $z = t = \alpha_i x + \beta_i y = 0$. These are singular point of X of type $\frac{1}{3}(1,1)$. Let Q_j be the point in X that is given by $x = y = \gamma_j z + \delta_j t = 0$. Then each of them is a singular point of X of type $\frac{1}{4}(1,1)$.

Let L_{ij} be the curve in X defined by $\alpha_i x + \beta_i y = \gamma_j z + \delta_j t = 0$, where $i = 1, \ldots, 4$ and $j = 1, \ldots, 3$.

The divisor C_i cut out by the equation $\alpha_i x + \beta_i y = 0$ consists of three smooth curves L_{i1}, L_{i2}, L_{i3}. These divisors C_i, $i = 1,2,3,4$, are the only reducible members in the linear system $|\mathcal{O}_X(3)|$. Meanwhile, the divisor B_j cut out by $\gamma_j z + \delta_j t = 0$ consists of four smooth curves L_{1j}, L_{2j}, L_{3j}, L_{4j}. Note that $L_{i1} \cap L_{i2} \cap L_{i3} = \{P_i\}$ and $L_{1j} \cap L_{2j} \cap L_{3j} \cap L_{4j} = \{Q_j\}$. We have $L_{ij} \cdot L_{ik} = \frac{1}{3}$ and $L_{ji} \cdot L_{ki} = \frac{1}{4}$ if $k \neq j$. But $L_{ij}^2 = -\frac{5}{12}$.

Since $\mathrm{lct}\left(X, \frac{2}{3}C_i\right) = \mathrm{lct}\left(X, \frac{2}{4}B_j\right) = 1$, we have $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair (X, D) is not log canonical at some point P. For every $i = 1, \ldots, 4$, we may assume that the support of the divisor D does not contain at least one curve among L_{i1}, L_{i2}, L_{i3}. Suppose $L_{ik} \not\subset \mathrm{Supp}\,(D)$. Then the inequality

$$\mathrm{mult}_{P_i}(D) \leqslant 3D \cdot L_{ik} = \frac{1}{2}$$

implies that none of the points P_i can be the point P. For every $j = 1,2,3$, we may also assume that the support of the divisor D does not contain at least one curve among $L_{1j}, L_{2j}, L_{3j}, L_{4j}$. Suppose $L_{lj} \not\subset \mathrm{Supp}\,(D)$. Then the inequality

$$\mathrm{mult}_{Q_j}(D) \leqslant 4D \cdot L_{lj} = \frac{2}{3}$$

implies that none of the points Q_j can be the point P. Therefore, the point must be a smooth point of X.

Write $D = \mu L_{ij} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain L_{ij}. If $\mu > 0$, then we have $\mu L_{ij} \cdot L_{ik} \leqslant D \cdot L_{ik}$, and hence $\mu \leqslant \frac{1}{2}$. Since

$$\Omega \cdot L_{ij} = \frac{2 + 5\mu}{12} < 1,$$

Lemma 1.3.4 implies the point P cannot be on the curve L_{ij}. Consequently,

$$P \notin \bigcup_{i=1}^{4}\bigcup_{j=1}^{3} L_{ij}.$$

There is a unique curve $C \subset X$ cut out by $\lambda x + \mu y = 0$, where $[\lambda : \mu] \in \mathbb{P}^1$, passing through the point P. Then the curve C is irreducible and quasismooth. Thus, we may assume that C is not contained in the support of D. Then

$$1 < \mathrm{mult}_P(D) \leqslant D \cdot C = \frac{1}{2}.$$

This is a contradiction. □

Lemma 2.2.5. Let X be a quasismooth hypersurface of degree $9n+3$ in $\mathbb{P}(3, 3n, 3n+1, 3n+1)$ for $n \geqslant 2$. Then $\mathrm{lct}(X) = 1$.

Proof. We may assume that the surface X is defined by the equation

$$xy(y - ax^n)(y - bx^n) + zt(z - ct) = 0,$$

where a, b, c are non-zero constants and $b \neq c$. The point O_y is a singular point of of index $3n$ on X. The three points O_x, $P_a = [1 : a : 0 : 0]$, $P_b = [1 : b : 0 : 0]$ are singular points of index 3 on X. Also, X has three singular points O_z, O_t, $P_c = [0 : 0 : c : 1]$ of index $3n+1$ on L_{xy}.

The curve C_x consists of three irreducible components L_{xz}, L_{xt} and $L_c = \{x = z - ct = 0\}$. These three components intersect each other at O_y. It is easy to check $\mathrm{lct}(X, \frac{2}{3}C_x) = 1$. Thus, $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

By Lemma 1.3.6 we may assume that at least one of the components of C_x is not contained in Supp (D). Then, the inequality

$$3nL_{xz} \cdot D = 3nL_{xt} \cdot D = 3nL_c \cdot D = \frac{2}{3n+1} < 1$$

implies that the point P cannot be the point O_y.

Put $D = \mu L_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{xz}. We claim that

$$\mu \leqslant \frac{2}{3n+1}.$$

Indeed, if the inequality fails, one of the curves L_{xt} and L_c is not contained in Supp (D). Then either

$$\frac{\mu}{3n} = \mu L_{xz} \cdot L_{xt} \leqslant D \cdot L_{xt} = \frac{2}{3n(3n+1)}, \text{ or } \quad \frac{\mu}{3n} = \mu L_{xz} \cdot L_c \leqslant D \cdot L_c = \frac{2}{3n(3n+1)}$$

holds. This is a contradiction. Note that

$$L_{xz}^2 = -\frac{6n-1}{3n(3n+1)}.$$

The inequality

$$\Omega \cdot L_{xz} = \frac{2 + (6n-1)\mu}{3n(3n+1)} < \frac{1}{3n+1}$$

holds for all $n \geqslant 2$. Therefore, Lemma 1.3.8 implies the point P cannot belong to L_{xz}. By the same way, we can show that $P \notin L_{xt} \cup L_c$.

Let C be the curve on X cut out by the equation $z-\alpha t=0$, where α is non-zero constant different from c. Then the curve C is quasismooth and hence $\mathrm{lct}(X, \frac{2}{3n+1}C)\geqslant 1$. Therefore, we may assume that the support of D does not contain the curve C. Then

$$\mathrm{mult}_{O_x}(D),\ \mathrm{mult}_{P_a}(D),\ \mathrm{mult}_{P_b}(D)\leqslant 3D\cdot C=\frac{2}{n}\leqslant 1$$

for $n\geqslant 2$. Therefore, P cannot be a singular point of X. Hence P is a smooth point of $X\setminus C_x$. Applying Lemma 1.3.9, we get an absurd inequality

$$1<\mathrm{mult}_P(D)\leqslant \frac{2(9n+3)^2}{3\cdot 3n(3n+1)(3n+1)}\leqslant 1$$

for $n\geqslant 2$ since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(9n+3))$ contains x^{3n+1}, xy^3 and z^3. The obtained contradiction completes the proof. ☐

Lemma 2.2.6. Let X be a quasismooth hypersurface of degree $9n+6$ in $\mathbb{P}(3,3n+1,3n+2,3n+2)$ for $n\geqslant 1$. Then $\mathrm{lct}(X)=1$.

Proof. The only singularities of X are a singular point O_y of index $3n+1$, and three singular points P_i, $i=1,2,3$, of index $3n+2$ on L_{xy}.

The divisor C_x consists of three distinct irreducible and reduced curves L_1, L_2, L_3, where each L_i contains the singular point P_i. Then $L_1\cap L_2\cap L_3=\{O_y\}$. It is obvious that $\mathrm{lct}(X,\frac{2}{3}C_x)=1$, and hence $\mathrm{lct}(X)\leqslant 1$.

Suppose that $\mathrm{lct}(X)<1$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair (X,D) is not log canonical at some point $P\in X$. By Lemma 1.3.6 we may assume that L_1 is not contained in $\mathrm{Supp}\,(D)$.

Since

$$L_1\cdot D<(3n+1)L_1\cdot D=\frac{2}{3n+2}<1$$

for all $n\geqslant 1$, we see that $P\notin L_1$. In particular, we see that $P\neq O_y$.

Put $D=\mu L_2+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_2\not\subset\mathrm{Supp}\,(\Omega)$. Then the inequality

$$\frac{\mu}{3n+1}=\mu L_1\cdot L_2\leqslant D\cdot L_1=\frac{2}{(3n+1)(3n+2)},$$

implies that $\mu\leqslant\frac{2}{3n+2}$. The intersection number

$$L_1^2=-\frac{6n+1}{(3n+1)(3n+2)}$$

shows

$$(3n+2)\Omega\cdot L_2=(3n+2)(D-\mu L_2)\cdot L_2=\frac{2+\mu(6n+1)}{(3n+1)}\leqslant\frac{6}{(3n+2)}$$

for all $n\geqslant 1$. Therefore, Lemma 1.3.8 excludes all the smooth point on L_2 in the case where $n\geqslant 1$ and the singular point P_2 in the case where $n\geqslant 2$. For the case $n=1$, let C_2 be the unique curve in the pencil $|\mathcal{O}_X(5)|$ that passes through the point P_2. Then the divisor C_2 consists of two distinct irreducible and reduced curve L_2 and R_2. The curve R_2 is singular at the point

P_2. Moreover, the log pair $(X, \frac{2}{5}C_2)$ is log canonical at the point P_2. By Lemma 1.3.6, we may assume that $R_2 \not\subset \operatorname{Supp}(D)$. Then the inequality

$$2\mathrm{mult}_{P_2}(D) \leqslant \mathrm{mult}_{P_2}(D)\mathrm{mult}_{P_2}(R_2) \leqslant 5D \cdot R_2 = 2$$

excludes the point P_2 in the case where $n = 1$. By the same method, we can show $P \notin L_3$.

Hence the point P must be a smooth point in $X \setminus C_x$. For the case $n \geqslant 2$, we can use Lemma 1.3.9 to get a contradiction

$$1 < \mathrm{mult}_P(D) \leqslant \frac{2(9n+6)^2}{3(3n+1)(3n+2)(3n+2)} = \frac{6}{3n+1} < 1,$$

since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(9n+6))$ contains x^{3n+2}, y^3x and z^3. For the case $n = 1$, let R_P be the unique curve in the pencil $|\mathcal{O}_X(5)|$ that passes through the point P. The log pair $(X, \frac{2}{5}R_P)$ is log canonical at the point P. By Lemma 1.3.6, we may assume that $\operatorname{Supp}(D)$ does not contain at least one irreducible component of R_P. Note that either R_P is irreducible or $P_k \in R_P$ for some $k = 1, 2, 3$. If R_P is irreducible, then we can obtain a contradiction

$$1 < \mathrm{mult}_P(D) \leqslant D \cdot R_P = \frac{1}{2}.$$

Thus, $P_k \in R_P$. Then R_P consists of two distinct irreducible curves L_k and Z. Since we already showed that P is located in the outside of L_k, the point P must belong to the curve Z. We have

$$L_k^2 = -\frac{7}{20}, \quad L_k \cdot Z = \frac{3}{5}, \quad Z^2 = \frac{2}{5}.$$

Put $D = mZ + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z \not\subset \operatorname{Supp}(\Delta)$. If $m > 0$, then

$$\frac{3m}{5} = mZ \cdot L_k \leqslant D \cdot L_k = \frac{1}{10},$$

and hence $\mu \leqslant \frac{1}{6}$. Then Lemma 1.3.8 gives us a contradiction

$$1 < \Delta \cdot Z = \frac{2-2m}{5} < 1.$$

□

Lemma 2.2.7. Let X be a quasismooth hypersurface of degree $12n+6$ in $\mathbb{P}(4, 2n+1, 4n+2, 6n+1)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. We may assume that the surface X is defined by the equation

$$xt^2 + x^{2n+1}z + ax^{2n+1}y^2 - (z - a_1y^2)(z - a_2y^2)(z - a_3y^2) = 0,$$

where a_1, a_2, a_3 are distinct constants and a is a constant.

The only singularities of X are a singular point O_x of index 4, a singular point O_t of index $6n+1$, a singular point $Q = [1:0:1:0]$ of index 2, and three singular points $P_1 = [0:1:a_1:0]$, $P_2 = [0:1:a_2:0]$, $P_3 = [0:1:a_3:0]$ of index $2n+1$.

The divisor C_x consists of three distinct irreducible curves $L_i = \{x = z - a_iy^2 = 0\}$, $i = 1, 2, 3$. Note that each L_i passes through the point P_i and $L_1 \cap L_2 \cap L_3 = \{O_t\}$. We can easily check $\mathrm{lct}(X, \frac{1}{2}C_x) = 1$, and hence $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

By Lemma 1.3.6 we may assume that L_1 is not contained in $\mathrm{Supp}\,(D)$. Since

$$(6n+1)L_1 \cdot D = \frac{2}{2n+1} < 1$$

the point P is located in the outside of L_1.

Put $D = \mu L_2 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_2 \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\frac{2\mu}{6n+1} = \mu L_1 \cdot L_2 \leqslant D \cdot L_2 = \frac{2}{(2n+1)(6n+1)},$$

and hence $\mu \leqslant \frac{1}{2n+1}$. Since

$$L_2^2 = -\frac{8n}{(2n+1)(6n+1)}$$

we have

$$(2n+1)\Omega \cdot L_2 = (2n+1)(D - \mu L_2) \cdot L_2 = \frac{2+8n\mu}{6n+1} \leqslant \frac{2}{2n+1} < 1$$

for all $n \geqslant 1$. Then Lemma 1.3.8 excludes all the points on L_2. Furthermore, the same method works for L_3.

The curve C_y is quasismooth. Thus the log pair $(X, \frac{2}{2n+1}C_y)$ is log canonical. By Lemma 1.3.6 we may assume that C_y is not contained in $\mathrm{Supp}\,(D)$. Then the inequality

$$4C_y \cdot D = \frac{6}{6n+1} < 1$$

implies that the point P is neither O_x nor Q. Hence P is a smooth point of $X \setminus C_x$. However, Lemma 1.3.9 gives us

$$\mathrm{mult}_P(D) \leqslant \frac{144n(2n+1)}{8(2n+1)^2(6n+1)} < 1$$

since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(12n))$ contains x^{3n}, y^4x^{n-1} and z^2x^{n-1}. This is a contradiction. □

2.3. Infinite series with $I = 4$

Lemma 2.3.1. Let X be a quasismooth hypersurface of degree $18n+15$ in $\mathbb{P}(6, 6n+3, 6n+5, 6n+5)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. We may assume that the surface X is defined by the equation

$$(z - a_1t)(z - a_2t)(z - a_3t) + xy(y^2 - x^{2n+1}) = 0,$$

where a_1, a_2, a_3 are distinct constants. The only singularities of X are a singular point O_x of index 6, a singular point O_y of index $6n+3$, a singular point $Q = [1:1:0:0]$ of index 3, and three singular points $P_i = [0:0:a_i:1]$, $i = 1, 2, 3$, of index $6n+5$.

The divisor C_x consists of three distinct irreducible curves $L_i = \{x = z - a_it = 0\}$, $i = 1, 2, 3$. Note that each L_i passes through the point P_i and $L_1 \cap L_2 \cap L_3 = \{O_y\}$. We can easily check $\mathrm{lct}(X, \frac{2}{3}C_x) = 1$, and hence $\mathrm{lct}(X) \leqslant 1$.

The divisor C_y consists of three distinct irreducible curves $L'_i = \{y = z - a_it = 0\}$, $i = 1, 2, 3$. Each L'_i passes through the point P_i and $L'_1 \cap L'_2 \cap L'_3 = \{O_x\}$. The log pair $(X, \frac{4}{6n+3}C_y)$ is log canonical.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

For a general member C in $|\mathcal{O}_X(6n+5)|$, we have

$$\mathrm{mult}_Q(D) \leqslant 3D \cdot C = \frac{6}{6n+3} < 1.$$

Therefore the point P cannot be the point Q.

By Lemma 1.3.6 we may assume that L_1 and L_1' are not contained in $\mathrm{Supp}\,(D)$. The two inequalities $(6n+5)D \cdot L_1 = \frac{4}{6n+3} < 1$ and $6D \cdot L_1' = \frac{4}{6n+5} < 1$ show that the point P is located in the outside of $L_1 \cup L_1'$.

Write $D = \mu L_2 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_2 \not\subset \mathrm{Supp}\,(\Omega)$. Then

$$\frac{\mu}{6n+3} = \mu L_1 \cdot L_2 \leqslant D \cdot L_1 = \frac{4}{(6n+3)(6n+5)},$$

and hence $\mu \leqslant \frac{4}{6n+5}$. Note that

$$L_2^2 = -\frac{12n+4}{(6n+3)(6n+5)}.$$

Therefore, we have

$$(6n+5)\Omega \cdot L_2 = (6n+5)(D - \mu L_2) \cdot L_2 = \frac{4 + (12n+4)\mu}{6n+3} \leqslant \frac{12}{6n+5}.$$

Therefore, Lemma 1.3.8 excludes all the smooth point on L_2 in the case where $n \geqslant 1$ and the singular point P_2 in the case where $n \geqslant 2$. For the case $n = 1$, let C_2 be the unique curve in the pencil $|\mathcal{O}_X(11)|$ that passes through the point P_2. Then the divisor C_2 consists of three distinct irreducible and reduced curve L_2, L_2' and R_2. The log pair $(X, \frac{4}{11}C_2)$ is log canonical at the point P_2. If $\mu = 0$, then the inequality above immediately excludes the point P_2 for the case $n = 1$. Therefore we may assume that either $L_2' \not\subset \mathrm{Supp}\,(D)$ or $R_2 \not\subset \mathrm{Supp}\,(D)$. In the former case, the intersection number

$$D \cdot L_2' = \frac{2}{33}$$

shows that the point P cannot be P_2. In the latter case, the intersection number

$$D \cdot R_2 = \frac{1}{11}$$

excludes the point P_2. By the same method, we can show $P \notin L_3$.

Hence the point P must be a smooth point in $X \setminus C_x$. For the case $n \geqslant 2$, we can use Lemma 1.3.9 to get a contradiction

$$1 < \mathrm{mult}_P(D) \leqslant \frac{4(18n+15) \cdot 6(6n+5)}{6(6n+3)(6n+5)(6n+5)} = \frac{4}{2n+1} < 1,$$

since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6(6n+5))$ contains x^{6n+5}, y^6x^2 and z^6. For the case $n = 1$, let R_P be the unique curve in the pencil $|\mathcal{O}_X(11)|$ that passes through the point P. The log pair $(X, \frac{4}{11}R_P)$ is log canonical at the point P. By Lemma 1.3.6, we may assume that $\mathrm{Supp}\,(D)$ does not contain at least one irreducible component of R_P. Note that either R_P is irreducible or $P_k \in R_P$ for some $k = 1, 2, 3$. However, if R_P is irreducible, then we can obtain a contradiction

$$1 < \mathrm{mult}_P(D) \leqslant D \cdot R_P = \frac{2}{9}.$$

Thus, $P_k \in R_P$. Then R_P consists of three distinct irreducible curves L_k, L'_k and Z. We have

$$D \cdot L'_k = \frac{2}{33}, \quad D \cdot Z = \frac{4}{33}, \quad L'^2_k = -\frac{13}{66}, \quad Z^2 = -\frac{4}{33}.$$

Put $D = m_1 Z + m_2 L'_k + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither Z nor L'_k. Since the pair (X, D) is log canonical at the point P_k, we have $m_1, m_2 \leqslant 1$. Since we already showed that P is located in the outside of L_k, the point P must belong to either L'_k or Z. However, Lemma 1.3.8 shows that the pair (X, D) is log canonical at the point P since

$$(D - m_1 Z) \cdot Z = \frac{4 + 4m_1}{33} < 1, \quad (D - m_2 L'_k) \cdot L'_k = \frac{4 + 13m_2}{66} < 1.$$

This is a contradiction. □

Lemma 2.3.2. Let X be a quasismooth hypersurface of degree $36n + 24$ in $\mathbb{P}(6, 6n + 5, 12n + 8, 18n + 9)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. We may assume that the surface X is defined by the equation

$$z^3 + y^3 t + xt^2 - x^{6n+4} + ax^{2n+1} y^2 z = 0,$$

where a is a constant. The only singularities of X are a singular point O_y of index $6n + 5$, a singular point O_t of index $18n + 9$, a singular point $Q = [1 : 0 : 0 : 1]$ of index 3, and a singular point $Q' = [1 : 0 : 1 : 0]$ of index 2.

The curve C_x is reduced and irreducible with $\mathrm{mult}_{O_t}(C_x) = 3$. Clearly, $\mathrm{lct}(X, \frac{2}{3} C_x) = 1$, and hence $\mathrm{lct}(X) \leqslant 1$. The curve C_y is quasismooth, and hence the log pair $(X, \frac{4}{6n+5} C_y)$ is log canonical.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(36n + 30))$ contains x^{6n+5}, y^6 and $z^3 x$, Lemma 1.3.9 implies

$$\mathrm{mult}_P(D) \leqslant \frac{4(36n + 24)(36n + 30)}{6(6n + 5)(12n + 8)(18n + 9)} < 1.$$

Therefore, the point P cannot be a smooth point in the outside of C_x.

By Lemma 1.3.6 we may assume that neither C_x nor C_y is contained in $\mathrm{Supp}\,(D)$. Then the inequality

$$3D \cdot C_y = \frac{2}{6n + 3} \leqslant 1$$

implies that the point P is neither Q nor Q'. One the other hand, the inequality

$$(6n + 5) D \cdot C_x = \frac{4}{6n + 3} < 1$$

shows that the point P can be neither a smooth point on C_x nor the point O_y. Therefore, it must be O_t. However, this is a contradiction since

$$\mathrm{mult}_{O_t}(D) = \frac{\mathrm{mult}_{O_t}(D)\mathrm{mult}_{O_t}(C_x)}{3} \leqslant \frac{18n+9}{3} D \cdot C_x = \frac{4}{6n+5} < 1.$$

The obtained contradiction completes the proof. □

Lemma 2.3.3. Let X be a quasismooth hypersurface of degree $36n+30$ in $\mathbb{P}(6, 6n+5, 12n+8, 18n+15)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. We may assume that the surface X is defined by the equation

$$(t - a_1 y^3)(t - a_2 y^3) + xz^3 - x^{6n+5} + ax^{2n+1} y^2 z = 0,$$

where $a_1 \neq a_2$ and a are constants. The only singularities of X are a singular point O_z of index $12n+8$, a singular point $Q = [1:0:1:0]$ of index 2, a singular point $Q' = [1:0:0:1]$ of index 3, and two singular points $P_1 = [0:1:0:a_1]$, $P_2 = [0:1:0:a_2]$ of index $6n+5$.

The curve C_x consists of two distinct irreducible curves $L_i = \{x = t - a_i y^3 = 0\}$, $i = 1, 2$. Each L_i passes through the point P_i. These two curves meet each other at the point O_z. It is easy to see $\mathrm{lct}(X, \frac{2}{3} C_x) = 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

By Lemma 1.3.6 we may assume that L_1 is not contained in $\mathrm{Supp}\,(D)$. Then the inequality

$$(12n+8) D \cdot L_1 = \frac{4}{6n+5} < 1$$

shows that the point P must be located in the outside of L_1.

Write $D = \mu L_2 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_2 \not\subset \mathrm{Supp}\,(\Omega)$. Then, the inequality

$$\frac{3\mu}{12n+8} = \mu L_2 \cdot L_1 \leqslant D \cdot L_1 = \frac{1}{(3n+2)(6n+5)},$$

implies

$$\mu \leqslant \frac{4}{3(6n+5)}.$$

Note that

$$L_2^2 = -\frac{18n+9}{(12n+8)(6n+5)}.$$

Since

$$(6n+5)\Omega \cdot L_2 = \frac{4 + (18n+9)\mu}{12n+8} < \frac{4}{6n+5},$$

Lemma 1.3.8 excludes all the points of $L_2 \setminus \{O_z\}$. Consequently, the point P is in the outside of C_x.

Meanwhile, the curve C_y is quasismooth, and hence the log pair $(X, \frac{4}{6n+5} C_y)$ is log canonical. Lemma 1.3.6 enables us to assume that C_y is not contained in $\mathrm{Supp}\,(D)$. Then the inequality

$$3 C_y \cdot D = \frac{1}{3n+2} \leqslant 1,$$

excludes the singular points Q and Q'.

Hence P is a smooth point of $X \setminus C_x$. Applying Lemma 1.3.9, we see that

$$1 < \mathrm{mult}_P(D) \leqslant \frac{4(36n+30)(3(12n+8)+6)}{6(6n+5)(12n+8)(18n+15)} < 1,$$

because $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(3(12n+8)+6))$ contains x^{6n+5}, y^6 and z^3x. The obtained contradiction completes the proof. □

2.4. Infinite series with $I = 6$

Lemma 2.4.1. Let X be a quasismooth hypersurface of degree $12n+23$ in $\mathbb{P}(8, 4n+5, 4n+7, 4n+9)$ for $n \geqslant 3$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be given by the equation

$$z^2t + yt^2 + xy^3 + x^{n+2}z = 0.$$

The surface X is singular only at O_x, O_y, O_z and O_t.

The curve C_x (resp. C_y, C_z, C_t) consists of the irreducible curve L_{xt} (resp. L_{yz}, L_{yz}, L_{xt}) and a residual curve $R_x = \{x = z^2+yt = 0\}$ (resp. $R_y = \{y = x^{n+2}+zt = 0\}$, $R_z = \{z = t^2+xy^2 = 0\}$, $R_t = \{t = y^3 + x^{n+1}z = 0\}$) . These two curves intersect each other at O_y (resp. O_t, O_x, O_z).

We can easily see that

$$\mathrm{lct}(X, \frac{3}{4}C_x) = 1, \quad \mathrm{lct}(X, \frac{6}{4n+5}C_y) = \frac{(n+3)(4n+5)}{12(n+2)},$$

$$\mathrm{lct}(X, \frac{6}{4n+7}C_z) = \frac{4n+7}{9}, \quad \mathrm{lct}(X, \frac{6}{4n+9}C_t) = \frac{(4n+9)(n+4)}{6(3n+6)}.$$

Therefore, $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

We have the following intersection numbers:

$$L_{xt} \cdot D = \frac{6}{(4n+5)(4n+7)}, \quad L_{yz} \cdot D = \frac{6}{8(4n+9)}, \quad R_x \cdot D = \frac{12}{(4n+5)(4n+9)},$$

$$R_y \cdot D = \frac{6(n+2)}{(4n+7)(4n+9)}, \quad R_z \cdot D = \frac{12}{8(4n+5)}, \quad R_t \cdot D = \frac{18}{8(4n+7)},$$

$$L_{xt} \cdot R_x = \frac{2}{4n+5}, \quad L_{xt} \cdot R_t = \frac{3}{4n+7}, \quad L_{yz} \cdot R_y = \frac{n+2}{4n+9}, \quad L_{yz} \cdot R_z = \frac{1}{4},$$

$$L_{xt}^2 = -\frac{8n+6}{(4n+5)(4n+7)}, \quad L_{yz}^2 = -\frac{4n+11}{8(4n+9)}, \quad R_x^2 = -\frac{8n+2}{(4n+5)(4n+9)},$$

$$R_y^2 = -\frac{2n+4}{(4n+7)(4n+9)}, \quad R_z^2 = \frac{1}{2(4n+5)}, \quad R_t^2 = \frac{12n+3}{8(4n+7)}.$$

By Lemma 1.3.6 we may assume that either $L_{xt} \not\subset \mathrm{Supp}\,(D)$ or $R_x \not\subset \mathrm{Supp}\,(D)$. Then at least one of the inequalities

$$\mathrm{mult}_{O_y}(D) \leqslant (4n+5)L_{xt} \cdot D = \frac{6}{4n+7}, \quad \mathrm{mult}_{O_y}(D) \leqslant (4n+5)R_x \cdot D = \frac{12}{4n+9}$$

holds. Therefore, the point P cannot be the point O_y. We also may assume that either $L_{yz} \not\subset \mathrm{Supp}\,(D)$ or $R_z \not\subset \mathrm{Supp}\,(D)$. Then at least one of the inequalities

$$\mathrm{mult}_{O_x}(D) \leqslant 8L_{yz}\cdot D = \frac{6}{4n+9}, \quad \mathrm{mult}_{O_x}(D) \leqslant \frac{8}{2}R_z\cdot D = \frac{6}{4n+5}$$

holds. Note that the curve R_z is singular at the point O_x. Therefore, the point P cannot be the point O_x. We also may assume that either $L_{xt} \not\subset \mathrm{Supp}\,(D)$ or $R_t \not\subset \mathrm{Supp}\,(D)$. Then at least one of the inequalities

$$\mathrm{mult}_{O_z}(D) \leqslant (4n+7)L_{xt}\cdot D = \frac{6}{4n+5}, \quad \mathrm{mult}_{O_z}(D) \leqslant \frac{4n+7}{3}R_t\cdot D = \frac{3}{4}$$

holds. Note that the curve R_t has multiplicity 3 at the point O_z if $n \geqslant 2$. Therefore, the point P cannot be the point O_z.

Write $D = m_1L_{xt} + m_2L_{yz} + m_3R_x + m_4R_y + m_5R_z + m_6R_t + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of L_{xt}, L_{yz}, R_x, R_y, R_z, R_t.

If $m_1 > 0$, then $m_3 = 0$. Therefore, the inequality

$$\frac{2m_1}{4n+5} = m_1L_{xt}\cdot R_x \leqslant D\cdot R_x = \frac{12}{(4n+5)(4n+9)}$$

shows $0 \leqslant m_1 \leqslant \frac{6}{4n+9}$. By Lemma 1.3.8 the inequality

$$(D - m_1L_{xt})\cdot L_{xt} = \frac{6 + m_1(8n+6)}{(4n+5)(4n+7)} \leqslant \frac{18}{(4n+7)(4n+9)} < 1$$

implies that the point P cannot be a smooth point on L_{xt}.

If $m_2 > 0$, then $R_z \not\subset \mathrm{Supp}\,(D)$. Therefore, the inequality

$$\frac{m_2}{4} = m_2L_{yz}\cdot R_z \leqslant D\cdot R_z = \frac{3}{2(4n+5)}$$

shows $0 \leqslant m_2 \leqslant \frac{6}{4n+5}$. By Lemma 1.3.8 the inequality

$$(D - m_2L_{yz})\cdot L_{yz} = \frac{6 + m_2(4n+11)}{8(4n+9)} \leqslant \frac{6(n+2)}{(4n+5)(4n+9)} < 1$$

implies that the point P cannot be a smooth point on L_{yz}.

If $m_3 > 0$, then $m_1 = 0$, and hence

$$\frac{2m_3}{4n+5} = m_3L_{xt}\cdot R_x \leqslant D\cdot L_{xt} = \frac{6}{(4n+5)(4n+7)}.$$

Therefore, $0 \leqslant m_3 \leqslant \frac{3}{4n+7}$. The inequality

$$(D - m_3R_x)\cdot R_x = \frac{12 + m_3(8n+2)}{(4n+5)(4n+9)} \leqslant \frac{18}{(4n+7)(4n+9)} < 1$$

implies that the point P cannot be a smooth point on R_x. Moreover, this inequality shows that the point P cannot be the point O_t since $n \geqslant 3$.

If $m_4 > 0$, then we may assume that $m_2 = 0$. We then obtain

$$\frac{m_4(n+2)}{4n+9} = m_4R_y\cdot L_{yz} \leqslant D\cdot L_{yz} = \frac{3}{4(4n+9)}.$$

Therefore, $0 \leqslant m_4 \leqslant \frac{3}{4(n+2)}$. The inequality

$$(D - m_4 R_y) \cdot R_y = \frac{6(n+2) + 2m_4(n+2)}{(4n+7)(4n+9)} \leqslant \frac{3}{2(4n+7)} < 1$$

implies that the point P cannot be a smooth point on R_y.

Since the pair (X, D) is log canonical at the point O_x and the curve R_z contains the point O_x, we have $m_5 \leqslant 1$. By Lemma 1.3.8, the inequality

$$(D - m_5 R_z) \cdot R_z \leqslant D \cdot R_z = \frac{3}{2(4n+5)} < 1$$

shows that the point P cannot be a smooth point on R_z.

The pair (X, D) is log canonical at the point O_x and the curve R_t contains the point O_x. Thus $m_6 \leqslant 1$. By Lemma 1.3.8, the inequality

$$(D - m_6 R_t) \cdot R_t \leqslant D \cdot R_t = \frac{9}{4(4n+7)} < 1$$

implies that the point P cannot be a smooth point of R_t.

Consider the pencil $\mathcal{L}$ defined by the equations $\lambda xy^2 + \mu t^2 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. Note that the curve L_{xt} is the only base component of the pencil $\mathcal{L}$. There is a unique divisor C_α in $\mathcal{L}$ passing through the point P. This divisor must be defined an equation $xy^2 + \alpha t^2 = 0$, where α is a non-zero constant, since the point P is located in the outside of $C_x \cup C_y \cup C_z \cup C_t$. Note that the curve C_y does not contain any component of C_α. Therefore, to see all the irreducible components of C_α, it is enough to see the affine curve

$$\left\{\begin{aligned} & x + \alpha t^2 = 0 \\ & z^2 t + t^2 + x + x^{n+2} z = 0 \end{aligned}\right\} \subset \mathbb{C}^3 \cong \operatorname{Spec}\Big(\mathbb{C}\big[x, z, t\big]\Big).$$

This is isomorphic to the plane affine curve defined by the equation

$$t\{z^2 + (1-\alpha)t^2 + (-\alpha)^{n+2} t^{2n+1} z\} = 0 \subset \mathbb{C}^2 \cong \operatorname{Spec}\Big(\mathbb{C}\big[z, t\big]\Big).$$

Thus, if $\alpha \neq 1$, then the divisor C_α consists of two reduced and irreducible curves L_{xt} and Z_α. If $\alpha = 1$, then it consists of three reduced and irreducible curves L_{xt}, R_z, R. Moreover, Z_α and R contain the point P and they are smooth at the point P.

Suppose that $\alpha \neq 1$. It is easy to check

$$D \cdot Z_\alpha = \frac{3(12n+19)}{2(4n+5)(4n+7)}.$$

We also see that

$$Z_\alpha^2 = C_\alpha \cdot Z_\alpha - L_{xt} \cdot Z_\alpha \geqslant C_\alpha \cdot Z_\alpha - (L_{xt} + R_x) \cdot Z_\alpha = \frac{4n+5}{3} D \cdot Z_\alpha > 0$$

since Z_α is different from the curve R_x. Put $D = \epsilon Z_\alpha + \Xi$, where Ξ is an effective $\mathbb{Q}$-divisor such that $Z_\alpha \not\subset \operatorname{Supp}(\Xi)$. Since the pair (X, D) is log canonical at the point O_y and the curve Z_α passes through the point O_y, we have $\epsilon \leqslant 1$. But

$$(D - \epsilon Z_\alpha) \cdot Z_\alpha \leqslant D \cdot Z_\alpha = \frac{3(12n+19)}{2(4n+5)(4n+7)} < 1$$

and hence Lemma 1.3.8 implies that the point P cannot belong to the curve Z_α.

Suppose that $\alpha = 1$. Then we have

$$D \cdot R = \frac{6(2n+3)}{(4n+5)(4n+7)}.$$

Since R is different from L_{yz} and R_x,

$$R^2 = C_\alpha \cdot R - L_{xt} \cdot R - R_z \cdot R \geqslant C_\alpha \cdot R - (L_{xt} + R_x) \cdot R - (L_{yz} + R_z) \cdot R \geqslant \frac{4n+3}{6} D \cdot R > 0$$

Put $D = \epsilon_1 R + \Xi'$, where Ξ' is an effective $\mathbb{Q}$-divisor such that $R \not\subset \mathrm{Supp}(\Xi')$. Since the curve R passes through the point O_y at which the pair (X, D) is log canonical, we have $\epsilon_1 \leqslant 1$. Since

$$(D - \epsilon_1 R) \cdot R \leqslant D \cdot R = \frac{6(2n+3)}{(4n+5)(4n+7)} < 1.$$

Lemma 1.3.8 implies that the point P cannot belong to R. □

Lemma 2.4.2. Let X be a quasismooth hypersurface of degree 47 in $\mathbb{P}(8, 13, 15, 17)$. Then $\mathrm{lct}(X) = 1$.

Proof. If we exclude the point O_t, then the proof of Lemma 2.4.1 works for this case. Thus we suppose that $P = O_t$. Then $L_{yz} \not\subset \mathrm{Supp}(D)$; otherwise we would have a contradictory inequality

$$\frac{3}{4 \cdot 17} = D \cdot L_{yz} \geqslant \mathrm{mult}_P(D) > \frac{1}{17}.$$

By Lemma 1.3.6, we may assume that $R_y \not\subset \mathrm{Supp}(D)$. Put

$$D = mL_{yz} + cR_x + \Omega,$$

where $m > 0$ and $c \geqslant 0$, and Ω is an effective $\mathbb{Q}$-divisor whose support contains neither L_{yz} nor R_x. Then

$$\frac{24}{15 \cdot 17} = D \cdot R_y = \big(mL_{yz} + cR_x + \Omega\big) \cdot R_y \geqslant \frac{4m}{17} + \frac{\mathrm{mult}_{O_t}(D) - m}{17} > \frac{3m+1}{17},$$

and hence

$$m < \frac{1}{5}.$$

Then it follows from Lemma 1.3.8 that

$$\frac{6+19m}{8 \cdot 17} = (D - mL_{yz}) \cdot L_{yz} > \frac{1}{17},$$

and hence

$$\frac{2}{19} < m.$$

On the other hand, if $c > 0$, then

$$\frac{6}{13 \cdot 15} = D \cdot L_{xt} \geqslant cR_x \cdot L_{xt} = \frac{2c}{13}.$$

Therefore, $0 \leqslant c \leqslant \frac{1}{5}$.

Let $\pi\colon \bar{X}\to X$ be the weighted blow up at the point O_t with weights $(6,7)$. Let E be the exceptional curve of π. Also we let $\bar{\Omega}$, $\bar{L}_{yz}$ and $\bar{R}_x$ be the proper transforms of Ω, L_{yz} and R_x, respectively. Then

$$K_{\bar{X}}\sim_{\mathbb{Q}}\pi^*(K_X)-\frac{4}{17}E,\ \bar{L}_{yz}\sim_{\mathbb{Q}}\pi^*(L_{yz})-\frac{7}{17}E,\ \bar{R}_x\sim_{\mathbb{Q}}\pi^*(R_x)-\frac{6}{17}E,\ \bar{\Omega}\sim_{\mathbb{Q}}\pi^*(\Omega)-\frac{a}{17}E,$$

where a is a non-negative rational number.

The curve E contains two singular points Q_7 and Q_6 of $\bar{X}$. The point Q_7 is a singular point of type $\frac{1}{7}(1,3)$ and the point Q_6 is a singular point of type $\frac{1}{6}(1,1)$. Then the point Q_7 is contained in $\bar{R}_x$ but not in $\bar{L}_{yz}$, on the other hand, Q_6 is contained in $\bar{L}_{yz}$ but not in $\bar{R}_x$. We also see that $\bar{L}_{yz}\cap\bar{R}_x=\varnothing$. The log pull back of the log pair (X,D) is the log pair

$$\left(\bar{X},\ \bar{\Omega}+m\bar{L}_{yz}+c\bar{R}_x+\frac{4+a+7m+6c}{17}E\right).$$

This pair must have non-log canonical singularity at some point $Q\in E$. Then

$$0\leqslant\bar{R}_x\cdot\bar{\Omega}=R_x\cdot\Omega+\frac{6a}{17^2}E^2=\frac{12-13m+18c}{13\cdot17}-\frac{a}{7\cdot17},$$
$$0\leqslant\bar{L}_{yz}\cdot\bar{\Omega}=L_{yz}\cdot\Omega+\frac{7a}{17^2}E^2=\frac{6+19m-8c}{8\cdot17}-\frac{a}{6\cdot17},$$

and hence $0\leqslant 84-13a+126c-91m$ and $0\leqslant 18-4a-24c+57m$. In particular, we see that $a\leqslant\frac{259}{40}$. Then $4+a+7m+6c<17$ since $m\frac{1}{5}$ and $c\leqslant\frac{1}{5}$.

Suppose that the point Q is neither Q_6 nor Q_7. Then the point Q must be located in the outside of $\bar{L}_{yz}$ and $\bar{R}_x$. By Lemma 1.3.8, we have

$$\frac{a}{42}=-\frac{a}{17}E^2=\bar{\Omega}\cdot E>1,$$

and hence $a>42$. This is a contradiction since $a<\frac{259}{40}$. Therefore, either $Q=Q_6$ or $Q=Q_7$.

Suppose that $Q=Q_7$. Then $Q\notin\bar{L}_{yz}$. Hence, it follows from Lemma 1.3.8 that

$$\frac{1}{7}\leqslant\left(\bar{\Omega}+m\bar{L}_{yx}+\frac{4+a+7m+6c}{17}E\right)\cdot\bar{R}_x=\frac{136+204c}{7\cdot13\cdot17},$$

and hence $c>\frac{5}{12}$. But $c\leqslant\frac{1}{5}$. This is a contradiction.

Finally, we suppose that $Q=Q_6$. Then $Q\notin\bar{R}_x$. It follows from Lemma 1.3.8 that

$$\frac{1}{6}\leqslant\left(\bar{\Omega}+c\bar{R}_x+\frac{4+a+7m+6c}{17}E\right)\cdot\bar{L}_{yz}=\frac{34+85m}{3\cdot8\cdot17},$$

and hence $m>\frac{2}{5}$. This contradiction completes the proof. □

Lemma 2.4.3. Let X be a quasismooth hypersurface of degree 35 in $\mathbb{P}(8,9,11,13)$. Then $\mathrm{lct}(X)=1$.

Proof. If we exclude the points O_z and O_t, then the proof of Lemma 2.4.1 works also for this case.

Suppose that $P=O_z$. Then $L_{xt}\subset\mathrm{Supp}(D)$, since otherwise we would have an absurd inequality

$$\frac{6}{9\cdot11}=D\cdot L_{xt}>\frac{1}{11}.$$

We may assume that $M_t \not\subset \mathrm{Supp}(D)$ by Lemma 1.3.6. Put

$$D = mL_{xt} + cM_y + \Omega,$$

where $m > 0$ and $c \geqslant 0$, and Ω is an effective $\mathbb{Q}$-divisor whose support contains neither L_{xt} nor R_y. Then

$$\frac{18}{8\cdot 11} = D\cdot R_t = \big(mL_{xt} + cR_y + \Omega\big)\cdot R_t \geqslant \frac{3m}{11} + \frac{2(\mathrm{mult}_{O_z}(D) - m)}{11} > \frac{m+2}{11},$$

and hence $m < \frac{1}{4}$. Note that $\mathrm{mult}_{O_z}(R_t) = 2$. It follows from Lemma 1.3.8 that

$$\frac{6+14m}{9\cdot 11} = \big(D - mL_{xt}\big)\cdot L_{xt} > \frac{1}{11}.$$

Therefore, $\frac{3}{14} < m < \frac{1}{4}$. On the other hand, if $c > 0$, then

$$\frac{6}{8\cdot 13} = D\cdot L_{yz} \geqslant cR_y\cdot L_{yz} = \frac{3c}{13},$$

and hence $c \leqslant \frac{1}{4}$.

Let $\pi\colon \bar{X} \to X$ be the weighted blow up at the point O_z with weights $(3,2)$. Let E be the exceptional curve of π and let $\bar{\Omega}$, $\bar{L}_{xt}$ and $\bar{R}_y$ be the proper transforms of Ω, L_{xt} and R_y, respectively. Then

$$K_{\bar{X}} \sim_{\mathbb{Q}} \pi^*(K_X) - \frac{6}{11}E,\ \bar{L}_{xt} \sim_{\mathbb{Q}} \pi^*(L_{xt}) - \frac{3}{11}E,\ \bar{R}_y \sim_{\mathbb{Q}} \pi^*(R_y) - \frac{2}{11}E,\ \bar{\Omega} \sim_{\mathbb{Q}} \pi^*(\Omega) - \frac{a}{11}E.$$

where a is a non-negative rational number.

The curve E contains two singular points Q_2 and Q_3 of $\bar{X}$. The point Q_2 is a singular point of type $\frac{1}{2}(1,1)$. It is contained in $\bar{L}_{xt}$ but not in $\bar{R}_y$. On the other hand, the point Q_3 is a singular point of type $\frac{1}{3}(2,1)$. It is contained in $\bar{R}_y$ but not in $\bar{L}_{xt}$. But $\bar{L}_{xt} \cap \bar{R}_y = \varnothing$.

The log pull back of the log pair (X, D) is the log pair

$$\left(\bar{X},\ \bar{\Omega} + m\bar{L}_{xt} + c\bar{R}_y + \frac{6+a+3m+2c}{11}E\right),$$

which must have non-log canonical singularity at some point $Q \in E$. We have

$$0 \leqslant \bar{\Omega}\cdot\bar{R}_y = \frac{18+6c}{11\cdot 13} - \frac{m}{11} - \frac{a}{33},$$
$$0 \leqslant \bar{\Omega}\cdot\bar{L}_{xt} = \frac{6+14m}{9\cdot 11} - \frac{c}{11} - \frac{a}{22}.$$

Then, $a \leqslant \frac{12+28m}{9} < \frac{19}{9}$ since $m < \frac{1}{4}$. Also, we obtain $6 + a + 3m + 2c < 11$ since $c \leqslant \frac{1}{4}$.

Suppose that the point Q is neither Q_2 nor Q_3. Then $Q \notin \bar{L}_{xt} \cup \bar{R}_y$. By Lemma 1.3.4, we have

$$\frac{a}{2\cdot 3} = -\frac{a}{11}E^2 = \bar{\Omega}\cdot E > 1,$$

and hence $a > 6$. This contradicts to the inequality $a < \frac{19}{9}$. Therefore, we see that either $Q = Q_2$ or $Q = Q_3$.

Suppose that $Q=Q_2$. Then $Q\not\in\bar{R}_y$. Lemma 1.3.8 shows that

$$\frac{1}{2}<\Big(\bar{\Omega}+c\bar{R}_y+\frac{6+a+3m+2c}{11}E\Big)\cdot\bar{L}_{xt}=\frac{66+55m}{2\cdot 9\cdot 11}$$

and hence $m>\frac{3}{5}$. But $m<\frac{1}{4}$. This is a contradiction.

Thus, the point Q must be Q_3. Then $Q\not\in\bar{L}_{xt}$. It follows from Lemma 1.3.8 that

$$\frac{1}{3}<\Big(\bar{\Omega}+m\bar{L}_{xt}+\frac{6+a+3m+2c}{11}E\Big)\cdot\bar{R}_y=\frac{132+44c}{13\cdot 33}.$$

Therefore, $c>\frac{1}{4}$. But we have seen $c\leqslant\frac{1}{4}$. The obtained contradiction shows that $P\neq O_z$. The point P must be the point O_t. Then $L_{yz}\not\subset\mathrm{Supp}(D)$ since otherwise we would have

$$\frac{6}{8\cdot 13}=D\cdot L_{yz}>\frac{1}{13}.$$

By Lemma 1.3.6, we may assume that $R_y\not\subset\mathrm{Supp}(D)$. Put

$$D=mL_{yz}+cR_x+\Omega,$$

where $m>0$ and $c\geqslant 0$, and Ω is an effective $\mathbb{Q}$-divisor whose support contains neither L_{yz} nor R_x. Then

$$\frac{18}{11\cdot 13}=D\cdot R_y=\big(mL_{yz}+cR_x+\Omega\big)\cdot R_y\geqslant\frac{3m}{13}+\frac{\mathrm{mult}_{O_t}(D)-m}{13}>\frac{2m+1}{13},$$

and hence $m<\frac{7}{22}$. On the other hand, Lemma 1.3.8 implies

$$\frac{6+15m}{8\cdot 13}=\big(D-mL_{yz}\big)\cdot L_{yz}>\frac{1}{13},$$

and hence $\frac{2}{15}<m<\frac{7}{22}$. If $c>0$, then

$$\frac{6}{9\cdot 11}=D\cdot L_{xt}=cR_x\cdot L_{xt}\geqslant\frac{2c}{9}.$$

Therefore, $c\leqslant\frac{3}{11}$.

Let $\pi\colon\bar{X}\to X$ be the weighted blow up at the point O_t with weights $(5,2)$. Let E be the exceptional curve of π. Let $\bar{\Omega}$, $\bar{L}_{yz}$ and $\bar{R}_x$ be the proper transforms of Ω, L_{yz} and R_x, respectively. Then

$$K_{\bar{X}}\sim_{\mathbb{Q}}\pi^*(K_X)-\frac{6}{13}E,\ \bar{L}_{yz}\sim_{\mathbb{Q}}\pi^*(L_{yz})-\frac{2}{13}E,\ \bar{R}_x\sim_{\mathbb{Q}}\pi^*(R_x)-\frac{5}{13}E,\ \bar{\Omega}\sim_{\mathbb{Q}}\pi^*(\Omega)-\frac{a}{13}E,$$

where a is a non-negative rational number.

The curve E contains two singular points Q_5 and Q_2 of $\bar{X}$. The point Q_5 is a singular point of type $\frac{1}{5}(1,1)$. It belongs to $\bar{L}_{yz}$ but not to $\bar{R}_x$. The point Q_2 is a singular point of type $\frac{1}{2}(1,1)$. It belongs to $\bar{R}_x$ but not to $\bar{L}_{yz}$. Note that $\bar{L}_{yz}\cap\bar{R}_x=\varnothing$.

The log pull back of the log pair (X,D) is the log pair

$$\Big(\bar{X},\ \bar{\Omega}+m\bar{L}_{yz}+c\bar{R}_x+\frac{6+a+2m+5c}{13}E\Big).$$

It must have non-log canonical singularity at some point $Q \in E$. We have

$$0 \leqslant \bar{\Omega} \cdot \bar{R}_x = \frac{12+10c}{9 \cdot 13} - \frac{m}{13} - \frac{a}{26},$$
$$0 \leqslant \bar{\Omega} \cdot \bar{L}_{yz} = \frac{6+15m}{8 \cdot 13} - \frac{c}{13} - \frac{a}{65}.$$

Therefore, $30 + 75m \geqslant 40c + 8a$ and $24 + 20c \geqslant 18m + 9a$. In particular, we see that $a \leqslant \frac{240}{77}$. Then $6 + a + 2m + 5c < 13$ since $c \leqslant \frac{3}{11}$ and $m \leqslant \frac{7}{22}$.

Suppose that $Q \neq Q_2$ and $Q \neq Q_5$. Then $Q \notin \bar{L}_{yz} \cup \bar{R}_x$. By Lemma 1.3.8, we have

$$\frac{a}{10} = -\frac{a}{13}E^2 = \bar{\Omega} \cdot E > 1,$$

and hence $a > 10$. This is a contradiction since $a < \frac{240}{77}$. Therefore, the point Q is either the point Q_2 or the point Q_5.

Suppose that $Q = Q_2$. Then $Q \notin \bar{L}_{yz}$. It follows from Lemma 1.3.8 that

$$\frac{1}{2} < \left(\bar{\Omega} + m\bar{L}_{yz} + \frac{6+a+2m+5c}{13}E\right) \cdot \bar{R}_x = \frac{78+65c}{9 \cdot 26},$$

and hence $c > \frac{3}{5}$. However, $c \leqslant \frac{3}{11}$. Thus, the point Q must be Q_5. Then $Q \notin \bar{R}_x$. Again, Lemma 1.3.8 shows that

$$\frac{1}{5} < \left(\bar{\Omega} + c\bar{R}_x + \frac{6+a+2m+5c}{13}E\right) \cdot \bar{L}_{yz} = \frac{78+91m}{5 \cdot 8 \cdot 13},$$
$$\frac{1}{5} < \left(\bar{\Omega} + m\bar{L}_{yz}\right) \cdot E = \frac{a}{10} + \frac{m}{5}.$$

Therefore, $m > \frac{2}{7}$ and $a + 2m > 2$. In particular, $\frac{2}{7} < m < \frac{7}{22}$.

Let $\psi\colon \tilde{X} \to \bar{X}$ be the weighted blow up at the point Q_5 with weights $(1,1)$. Let G be the exceptional curve of ψ and let $\tilde{\Omega}$, $\tilde{L}_{yz}$, $\tilde{R}_x$ and $\tilde{E}$ be the proper transforms of Ω, L_{yz}, R_x and E, respectively. Then

$$K_{\tilde{X}} \sim_{\mathbb{Q}} \psi^*(K_{\bar{X}}) - \frac{3}{5}G,\ \tilde{L}_{yz} \sim_{\mathbb{Q}} \psi^*(\bar{L}_{yz}) - \frac{1}{5}G,\ \tilde{E} \sim_{\mathbb{Q}} \psi^*(E) - \frac{1}{5}G,\ \tilde{\Omega} \sim_{\mathbb{Q}} \psi^*(\bar{\Omega}) - \frac{b}{5}G,$$

where b is a non-negative rational number.

The surface is smooth along G. The log pull back of (X, D) is the log pair

$$\left(\tilde{X},\ \tilde{\Omega} + m\tilde{L}_{yz} + c\tilde{R}_x + \frac{6+a+2m+5c}{13}\tilde{E} + \theta G\right),$$

where

$$\theta = \frac{15m + 45 + a + 13b + 5c}{65}.$$

Then the log pair is not log canonical at some point $O \in G$. We have

$$0 \leqslant \tilde{E} \cdot \tilde{\Omega} = \frac{a}{10} - \frac{b}{5},$$
$$0 \leqslant \tilde{L}_{yz} \cdot \tilde{\Omega} = \frac{6+15m}{8 \cdot 13} - \frac{c}{13} - \frac{a}{65} - \frac{b}{5},$$

and hence $30+75m\geqslant 8(a+13b+5c)$ and $a\geqslant 2b$. In particular, we obtain

$$\theta=\frac{15m+45+a+13b+5c}{65}\leqslant\frac{195m+390}{8\cdot 65}\leqslant\frac{195\cdot 7+390\cdot 22}{8\cdot 22\cdot 65}<1$$

since $m\leqslant\frac{7}{22}$.

Suppose that $O\not\in\tilde{E}\cup\tilde{L}_{yz}$. Then it follows from Lemma 1.3.8 that

$$b=-\frac{b}{5}G^2=\tilde{\Omega}\cdot G>1.$$

However, this gives an absurd inequality $104<104b\leqslant 30+75m-8a-40c\leqslant 30+75m<104$ since $m\leqslant\frac{7}{22}$. Therefore, $O\in\tilde{E}\cup\tilde{L}_{yz}$. Note that $\tilde{E}\cap\tilde{L}_{yz}=\varnothing$.

Suppose that $O\in\tilde{L}_{yz}$. Then it follows from Lemma 1.3.8 that

$$1<\big(\tilde{\Omega}+c\tilde{R}_x+\frac{6+a+2m+5c}{13}\tilde{E}+\theta G\big)\cdot\tilde{L}_{yz}=\big(\tilde{\Omega}+\theta G\big)\cdot\tilde{L}_{yz}=\frac{3m+6}{8},$$

and hence $m>\frac{2}{3}$. But $m\leqslant\frac{7}{22}$. Thus, we see that $O\in\tilde{E}$. Lemma 1.3.8 implies that

$$1<\Bigg(\tilde{\Omega}+\frac{6+a+2m+5c}{13}\tilde{E}\Bigg)\cdot G=b+\frac{6+a+2m+5c}{13},$$
$$1<\big(\tilde{\Omega}+\theta G\big)\cdot\tilde{E}=\frac{a}{10}-\frac{b}{5}+\theta.$$

Therefore, we obtain $13b+a+2m+5c>7$ and $3a+2c+6m>8$.

Let $\phi\colon\hat{X}\to\tilde{X}$ be the blow up at the point O. Let F be the exceptional curve of ϕ. Let $\hat{\Omega}$, $\hat{L}_{yz}$, $\hat{R}_x$, $\hat{E}$ and $\hat{G}$ be the proper transforms of Ω, L_{yz}, R_x, E and G, respectively. Then

$$K_{\hat{X}}\sim_{\mathbb{Q}}\phi^*(K_{\tilde{X}})+F,\ \hat{G}\sim_{\mathbb{Q}}\phi^*(G)-F,\ \hat{E}\sim_{\mathbb{Q}}\phi^*(\tilde{E})-F,\ \hat{\Omega}\sim_{\mathbb{Q}}\phi^*(\tilde{\Omega})-dF,$$

where d is a non-negative rational number. The log pull back of (X,D) is the log pair

$$\Bigg(\hat{X},\ \hat{\Omega}+m\hat{L}_{yz}+c\hat{R}_x+\frac{6+a+2m+5c}{13}\hat{E}+\theta\hat{G}+\nu F\Bigg),$$

where

$$\nu=\frac{65d+25m+6a+13b+30c+10}{65}.$$

It is not log canonical at some point $A\in F$. We have

$$0\leqslant\hat{E}\cdot\hat{\Omega}=\frac{a}{10}-\frac{b}{5}-d,$$
$$0\leqslant\hat{G}\cdot\hat{\Omega}=b-d,$$

and hence $b \geqslant d$ and $a \geqslant 2b + 10d$. In particular,

$$\begin{aligned}\nu &= \frac{65d + 25m + 6a + 13b + 30c + 10}{65} = \\ &= \frac{13(5d + b) + 25m + 6a + 30c + 10}{65} \leqslant \\ &\leqslant \frac{5a + 10m + 12c + 4}{26} \leqslant \\ &\leqslant \frac{6 + 8c}{9} < 1\end{aligned}$$

since we have $24 + 20c \geqslant 18m + 9a$ and $c \leqslant \frac{3}{11}$.

Suppose that $A \notin \hat{E} \cup \hat{G}$. Then Lemma 1.3.8 shows that $d = \hat{\Omega} \cdot F > 1$. This is impossible since

$$10d \leqslant a - 2b \leqslant a \leqslant \frac{240}{77}.$$

Thus, we see that $A \in \hat{E} \cup \hat{G}$. Note that $\hat{E} \cap \hat{G} = \varnothing$.

Suppose that $A \in \hat{E}$. Then it follows from Lemma 1.3.8 that

$$\frac{a}{10} - \frac{b}{5} - d + \nu = \big(\hat{\Omega} + \nu F\big) \cdot \hat{E} > 1,$$

which implies that $5a + 10m + 12c > 22$. However, this inequality with $24 + 20c \geqslant 18m + 9a$ gives

$$\frac{9}{5}(22 - 12c) < \frac{9}{5}(5a + 10m) \leqslant 24 + 20c,$$

and hence $\frac{3}{8} < c$. But $c \leqslant \frac{3}{11}$. Thus, the point A cannot belong to $\hat{E}$. Then $A \in \hat{G}$. By Lemma 1.3.8, we see that

$$b - d + \nu = \big(\hat{\Omega} + \nu F\big) \cdot \hat{G} > 1,$$

and hence $6a + 25m + 30c + 78b > 55$. But

$$55 < 25m + 6a + 78b + 3c = 25m + \frac{3}{4}(8a + 104b + 40c) \leqslant 25m + \frac{3}{4}(30 + 75m) < 55$$

since $8a + 104b + 40c \leqslant 30 + 75m$ and $m \leqslant \frac{7}{22}$. The obtained contradiction completes the proof. ☐

Lemma 2.4.4. Let X be a quasismooth hypersurface of degree $12n + 35$ in $\mathbb{P}(9, 3n + 8, 3n + 11, 6n + 13)$ for $n \geqslant 1$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be defined by the equation

$$z^2t + y^3z + xt^2 + x^{n+3}y = 0.$$

It is singular only at the points O_x, O_y, O_z and O_t.

The curve C_x (resp. C_y, C_z, C_t) consists of two irreducible and reduced curves L_{xz} (resp. L_{yt}, L_{xz}, L_{yt}) and $R_x = \{x = zt + y^3 = 0\}$ (resp. $R_y = \{y = z^2 + xt = 0\}$, $R_z = \{z = t^2 + x^{n+2}y = 0\}$, $R_t = \{t = y^2z + x^{n+3} = 0\}$). These two curves intersect at the point O_t (resp. O_x, O_y, O_z).

It is easy to see that $\mathrm{lct}(X, \frac{2}{3}C_x) = 1$ is less than each of the numbers

$$\mathrm{lct}(X, \frac{6}{3n+8}C_y), \quad \mathrm{lct}(X, \frac{6}{3n+11}C_z), \quad \mathrm{lct}(X, \frac{6}{6n+13}C_t).$$

We have the following intersection numbers.

$$-L_{xz}\cdot K_X = \frac{6}{(3n+8)(6n+13)}, \quad -L_{yt}\cdot K_X = \frac{2}{3(3n+11)}, \quad -R_x\cdot K_X = \frac{18}{(3n+11)(6n+13)},$$

$$-R_y\cdot K_X = \frac{4}{3(6n+13)}, \quad -R_z\cdot K_X = \frac{4}{3(3n+8)}, \quad -R_t\cdot K_X = \frac{6(n+3)}{(3n+8)(3n+11)},$$

$$L_{xz}\cdot R_x = \frac{3}{6n+13}, \quad L_{yt}\cdot R_y = \frac{2}{9}, \quad L_{xz}\cdot R_z = \frac{2}{3n+8}, \quad L_{yt}\cdot R_t = \frac{n+3}{3n+11},$$

$$L_{xz}^2 = -\frac{9n+15}{(3n+8)(6n+13)}, \quad L_{yt}^2 = -\frac{3n+14}{9(3n+11)}, \quad R_x^2 = -\frac{9n+6}{(3n+11)(6n+13)},$$

$$R_y^2 = -\frac{6n+10}{9(6n+13)}, \quad R_z^2 = \frac{6n+4}{9(3n+8)}, \quad R_t^2 = \frac{(n+3)(3n+5)}{(3n+8)(3n+11)}.$$

Now we suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$.

By Lemma 1.3.6 we may assume that $\mathrm{Supp}\,(D)$ does not contain either the curve L_{yt} or the curve R_y. Since these two curves intersect at the point O_x, the inequalities

$$L_{yt}\cdot D = \frac{2}{3(3n+11)} < \frac{1}{9},$$

$$R_y\cdot D = \frac{4}{3(6n+13)} < \frac{1}{9}$$

show that the point P cannot be the point O_x.

By Lemma 1.3.6 we may assume that $\mathrm{Supp}\,(D)$ does not contain either the curve L_{xz} or the curve R_z. Therefore, one of the following inequalities must hold:

$$\mathrm{mult}_{O_y}(D) \leqslant (3n+8)L_{xz}\cdot D = \frac{6}{6n+13} < 1,$$

$$\mathrm{mult}_{O_y}(D) \leqslant \frac{3n+8}{2}R_z\cdot D = \frac{2}{3}.$$

Therefore, the point P cannot be the point O_y.

Suppose that $P = O_z$. If $L_{yt} \not\subset \mathrm{Supp}\,(D)$, then we get an absurd inequality

$$\frac{6}{9(3n+11)} = L_{yt}\cdot D > \frac{1}{3n+11}.$$

Therefore $\mathrm{Supp}\,(D)$ must contain the curve L_{yt}. By Lemma 1.3.6 we may assume that $M_t \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu L_{yt} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{yt}. Then

$$\frac{6(n+3)}{(3n+8)(3n+11)} = D\cdot R_t \geqslant \mu L_{yt}\cdot R_t + \frac{(\mathrm{mult}_P(D)-\mu)\mathrm{mult}_P(R_t)}{3n+11} > \frac{\mu(n+3)}{3n+11} + \frac{2(1-\mu)}{3n+11},$$

and hence

$$\mu < \frac{2}{(3n+8)(n+1)}.$$

On the other hand, Theorem 1.3.3 shows

$$\frac{1}{3n+11} < \Omega \cdot L_{yt} = D \cdot L_{yt} - \mu L_{yt}^2 = \frac{6+\mu(3n+14)}{9(3n+11)}.$$

It implies $\frac{3}{3n+14} < \mu$. Consequently, the point P cannot be the point O_z.

Suppose that $P = O_t$. Since $L_{xz} \cdot D < \frac{1}{6n+13}$, the curve L_{xz} must be contained in $\mathrm{Supp}\,(D)$. Then, we may assume that $R_x \not\subset \mathrm{Supp}\,(D)$. Put $D = \mu L_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{xz}. Then

$$\frac{18}{(3n+11)(6n+13)} = D \cdot R_x \geqslant \mu L_{xz} \cdot R_x + \frac{\mathrm{mult}_P(D) - \mu}{6n+13} > \frac{1+2\mu}{6n+13},$$

and hence

$$\mu < \frac{7-3n}{6n+22}.$$

However, Theorem 1.3.3 implies

$$\frac{1}{6n+13} < \Omega \cdot L_{xz} = D \cdot L_{xz} - \mu L_{xz}^2 = \frac{6+(9n+15)\mu}{(3n+8)(6n+13)},$$

and hence $\frac{3n+2}{9n+15} < \mu$. This is a contradiction. Therefore, the point P cannot be the point O_t.

Write $D = aL_{xz} + bR_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither L_{xz} nor R_x. Since the log pair (X, D) is log canonical at the point O_t, we have $0 \leqslant a, b \leqslant 1$. Then by Theorem 1.3.3 the following two inequalities

$$(bR_x + \Delta) \cdot L_{xz} = (D - aL_{xz}) \cdot L_{xz} = \frac{6+a(9n+15)}{(3n+8)(6n+13)} < 1,$$

$$(aL_{xz} + \Delta) \cdot R_x = (D - bR_x) \cdot R_x = \frac{18+b(9n+6)}{(3n+11)(6n+13)} < 1$$

show that the point P cannot belong to the curve C_x. By the same way, we can show $P \notin C_y \cup C_z \cup C_t$.

Consider the pencil $\mathcal{L}$ defined by the equations $\lambda xt + \mu z^2 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. Note that the curve L_{xz} is the only base component of the pencil $\mathcal{L}$. There is a unique divisor C_α in $\mathcal{L}$ passing through the point P. This divisor must be defined an equation $xt + \alpha z^2 = 0$, where α is a non-zero constant, since the point P is located in the outside of $C_x \cup C_z \cup C_t$. Note that the curve C_t does not contain any component of C_α. Therefore, to see all the irreducible components of C_α, it is enough to see the affine curve

$$\left\{\begin{array}{l} x + \alpha z^2 = 0 \\ z^2 + y^3 z + x + x^{n+3} y = 0 \end{array}\right\} \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z\big]\Big).$$

This is isomorphic to the plane affine curve defined by the equation

$$z\{(1-\alpha)z + y^3 + (-\alpha)^{n+3} y z^{2n+5}\} = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big).$$

Thus, if $\alpha \neq 1$, then the divisor C_α consists of two reduced and irreducible curves L_{xz} and Z_α. If $\alpha = 1$, then it consists of three reduced and irreducible curves L_{xz}, R_y, R. Moreover, Z_α and R are smooth at the point P.

Suppose that $\alpha \neq 1$. Then we have

$$D \cdot Z_\alpha = \frac{2(24n+61)}{3(3n+8)(6n+13)}.$$

Since Z_α is different from R_x,

$$Z_\alpha^2 = C_\alpha \cdot Z_\alpha - L_{xz} \cdot Z_\alpha \geqslant C_\alpha \cdot Z_\alpha - (L_{xz} + R_x) \cdot Z_\alpha = \frac{6n+13}{6} D \cdot Z_\alpha > 0.$$

Put $D = \epsilon Z_\alpha + \Xi$, where Ξ is an effective $\mathbb{Q}$-divisor such that $Z_\alpha \not\subset \mathrm{Supp}(\Xi)$. Since the pair (X, D) is log canonical at the point O_t and the curve Z_α passes through the point O_t, we have $\epsilon \leqslant 1$. But

$$(D - \epsilon Z_\alpha) \cdot Z_\alpha \leqslant D \cdot Z_\alpha = \frac{2(24n+61)}{3(3n+8)(6n+13)} < 1$$

and hence Lemma 1.3.8 implies that the point P cannot belong to the curve Z_α.

Suppose that $\alpha = 1$. We have

$$D \cdot R = \frac{6(2n+5)}{(3n+8)(6n+13)}.$$

Since R is different from R_x and L_{yt},

$$R^2 = C_\alpha \cdot R - L_{xz} \cdot R - R_y \cdot R \geqslant C_\alpha \cdot R - (L_{xz} + R_x) \cdot R - (L_{yt} + R_y) \cdot R = \frac{3n+5}{6} D \cdot D > 0.$$

Put $D = \epsilon_1 R + \Xi'$, where Ξ' is an effective $\mathbb{Q}$-divisor such that $R \not\subset \mathrm{Supp}(\Xi')$. Since the curve R passes through the point O_t at which the pair (X, D) is log canonical, $\epsilon_1 \leqslant 1$. Since

$$(D - \epsilon_1 R) \cdot R \leqslant D \cdot R = \frac{6(2n+5)}{(3n+8)(6n+13)} < 1,$$

Lemma 1.3.8 implies that the point P cannot belong to R. □

Part 3. Sporadic cases

3.1. Sporadic cases with $I = 1$

Lemma 3.1.1. Let X be a quasismooth hypersurface of degree 10 in $\mathbb{P}(1, 2, 3, 5)$. Then

$$\mathrm{lct}\big(X\big) = \begin{cases} 1 \text{ if } C_x \text{ has an ordinary double point,} \\ \dfrac{7}{10} \text{ if } C_x \text{ has a non-ordinary double point.} \end{cases}$$

Proof. The surface X is singular only at the point O_z. The curve C_x is reduced and irreducible. Moreover, we have

$$\mathrm{lct}\big(X, C_x\big) = \begin{cases} 1 \text{ if the curve } C_x \text{ has an ordinary double point at the point } O_z, \\ \dfrac{7}{10} \text{ if the curve } C_x \text{ has a non-ordinary double point at the point } O_z. \end{cases}$$

Suppose that $\mathrm{lct}\big(X\big) < \mathrm{lct}\big(X, C_x\big)$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. By Lemma 1.3.6 we may assume that the support of D does not contain the curve C_x. Also Lemma 1.3.9 shows that $P \in C_x$. However, we obtain absurd inequalities

$$\frac{1}{3} = D \cdot C_x \geqslant \begin{cases} \mathrm{mult}_P\big(D\big) > 1 & \text{if } P \neq O_z, \\ \dfrac{\mathrm{mult}_P\big(D\big)}{3} > \dfrac{1}{3} & \text{if } P = O_z. \end{cases}$$

Therefore, $\mathrm{lct}\big(X\big) = \mathrm{lct}\big(X, C_x\big)$. □

Lemma 3.1.2. Let X be the quasismooth hypersurface defined by a quasihomogeneous polynomial $f(x, y, z, t)$ of degree 15 in $\mathbb{P}(1, 3, 5, 7)$. Then

$$\mathrm{lct}\big(X\big) = \begin{cases} 1 & \text{if } f(x, y, z, t) \text{ contains } yzt, \\ \dfrac{8}{15} & \text{if } f(x, y, z, t) \text{ does not contain } yzt. \end{cases}$$

Proof. The surface X is singular only at the point O_t. The curve C_x is reduced and irreducible. It is easy to check

$$\mathrm{lct}\big(X, C_x\big) = \begin{cases} 1 & \text{if } f(x, y, z, t) \text{ contains } yzt, \\ \dfrac{8}{15} & \text{if } f(x, y, z, t) \text{ does not contain } yzt. \end{cases}$$

The proof is exactly the same as the proof of Lemma 3.1.1. The contradictory inequalities

$$\frac{1}{7} = D \cdot C_x \geqslant \begin{cases} \mathrm{mult}_P\big(D\big) > 1 & \text{if } P \neq O_t, \\ \dfrac{\mathrm{mult}_P\big(D\big)}{7} > \dfrac{1}{7} & \text{if } P = O_t. \end{cases}$$

complete the proof. □

Lemma 3.1.3. Let X be a quasismooth hypersurface of degree 16 in $\mathbb{P}(1, 3, 5, 8)$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X is singular only at the points O_y and O_z. The former is a singular point of type $\frac{1}{3}(1, 1)$ and the latter is of type $\frac{1}{5}(1, 1)$.

The curve C_x consists of two distinct irreducible curves L_1 and L_2. It is easy to see that $\mathrm{lct}(X, C_x) = 1$.

Suppose that $\mathrm{lct}\big(X\big) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. By Lemma 1.3.6 we may assume that the support of D does not contain the curve L_1 without loss of generality. Moreover, Lemma 1.3.9 implies $P \in C_x$.

We have

$$D \cdot L_1 = D \cdot L_2 = \frac{1}{15},$$

and $L_1 \cap L_2 = \{O_y, O_z\}$. We also have

$$L_1^2 = L_2^2 = -\frac{7}{15}, \quad L_1 \cdot L_2 = \frac{8}{15}.$$

Since $5D\cdot L_1=\frac{1}{3}$, the point P cannot belong to L_1. Therefore, the point P is a smooth point on L_2. Put

$$D=mL_2+\Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $L_2\not\subset \mathrm{Supp}(\Omega)$. Since the log pair (X,D) is log canonical at O_y, we must have $m\leqslant 1$. Then it follows from Lemma 1.3.8 that

$$1<\Omega\cdot L_2=\big(D-mL_2\big)\cdot L_2=\frac{1+7m}{15}.$$

This gives us $m>2$. This is a contradiction. Consequently, $\mathrm{lct}(X)=1$. □

Lemma 3.1.4. Let X be a quasismooth hypersurface of degree 18 in $\mathbb{P}(2,3,5,9)$. Then

$$\mathrm{lct}\big(X\big)=\begin{cases}2 \text{ if } C_y \text{ has a tacnodal point,}\\ \dfrac{11}{6} \text{ if } C_y \text{ has no tacnodal points.}\end{cases}$$

Proof. The surface X is singular at the point O_z. This is a singular point of type $\frac{1}{5}(1,2)$. The surface X also has two singular points O_1 and O_2 that are cut out by the equations $x=z=0$. These are of type $\frac{1}{3}(1,1)$ on the surface X.

The curves C_x and C_y are reduced and irreducible. The curve C_y is always singular at the point O_z. We can see $\mathrm{lct}(X,C_x)=1$ and

$$\mathrm{lct}\big(X,C_y\big)=\begin{cases}\dfrac{3}{4} \text{ if } C_y \text{ has a tacnodal singularity at the point } O_z,\\ \dfrac{11}{18} \text{ if } C_y \text{ has a non-tacnodal singularity at the point } O_z.\end{cases}$$

Therefore, if C_y has a tacnodal singularity at the point O_z, then

$$2=\mathrm{lct}\left(X,\frac{1}{2}C_x\right)<\mathrm{lct}\left(X,\frac{1}{3}C_y\right)=\frac{9}{4}.$$

If C_y has a non-tacnodal singularity at the point O_z, then

$$2=\mathrm{lct}\left(X,\frac{1}{2}C_x\right)>\mathrm{lct}\left(X,\frac{1}{3}C_y\right)=\frac{11}{6}.$$

Let $\epsilon=\min\left\{\mathrm{lct}\left(X,\frac{1}{2}C_x\right),\mathrm{lct}\left(X,\frac{1}{3}C_y\right)\right\}$. Then $\mathrm{lct}(X)\leqslant\epsilon$.

Suppose that $\mathrm{lct}(X)<\epsilon$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\epsilon D)$ is not log canonical at some point $P\in X$. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither the curve C_x nor the curve C_y.

The inequalities

$$\mathrm{mult}_{O_z}(\epsilon D)\leqslant\frac{\epsilon}{2}\mathrm{mult}_{O_z}(D)\mathrm{mult}_{O_z}(C_y)\leqslant 5D\cdot C_y=1$$

imply that the point P cannot be the point O_z. If the point P is a smooth point on C_y, then we have obtain a contradictory inequalities

$$\frac{1}{5}=D\cdot C_y\geqslant\mathrm{mult}_P\left(D\right)>\frac{1}{\epsilon}\geqslant\frac{1}{2}.$$

Therefore, the point P is located in the outside of the curve C_y.

Suppose that $P \in C_x$. Then we obtain the following contradictory inequalities

$$\frac{2}{15} = D \cdot C_x \geqslant \begin{cases} \mathrm{mult}_P\big(D\big) > \dfrac{1}{2} & \text{if } P \in X \setminus \mathrm{Sing}(X), \\ \dfrac{\mathrm{mult}_P\big(D\big)}{3} > \dfrac{1}{6} & \text{if } P = O_1 \text{ or } P = O_2. \end{cases}$$

Therefore, $P \notin C_x \cup C_y$. Then P is a smooth point. There is a unique curve C in the pencil $|-5K_X|$ passing through the point P. The curve C is a hypersurface in $\mathbb{P}(1,2,3)$ of degree 6 such that the natural projection

$$C \longrightarrow \mathbb{P}(1,2) \cong \mathbb{P}^1$$

is a double cover. Thus, we have $\mathrm{mult}_P(C) \leqslant 2$. In particular, the log pair $(X, \frac{\epsilon}{5}C)$ is log canonical. Thus, it follows from Lemma 1.3.6 that we may assume that the support of the divisor D does not contain one of the irreducible components of the curve C. Then

$$\frac{1}{3} = D \cdot C \geqslant \mathrm{mult}_P\big(D\big) > \frac{1}{2}$$

in the case when C is irreducible (but possibly non-reduced). Therefore, the curve C must be reducible and reduced. Then

$$C = C_1 + C_2,$$

where C_1 and C_2 are irreducible and reduced smooth rational curves such that

$$C_1^2 = C_2^2 = -\frac{2}{3}, \quad C_1 \cdot C_2 = \frac{3}{2}.$$

Without loss of generality we may assume that $P \in C_1$. Put

$$D = mC_1 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $C_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then $C_2 \not\subset \mathrm{Supp}(\Omega)$ and

$$\frac{1}{6} = D \cdot C_2 = \big(mC_1 + \Omega\big) \cdot C_2 \geqslant mC_1 \cdot C_2 = \frac{3m}{2},$$

and hence $m \leqslant \frac{1}{9}$. Thus, it follows from Lemma 1.3.8 that

$$\frac{1+4m}{6} = \big(D - mC_1\big) \cdot C_1 = \Omega \cdot C_1 > \frac{1}{\epsilon} \geqslant \frac{1}{2}.$$

Therefore, $m > \frac{1}{2}$. But $m \leqslant \frac{1}{9}$. Consequently, $\mathrm{lct}(X) = \epsilon$.

□

Lemma 3.1.5. Let X be a quasismooth hypersurface of degree 15 in $\mathbb{P}(3,3,5,5)$. Then $\mathrm{lct}(X) = 2$.

Proof. The surface X has five singular points $O_1, \ldots, O_5$ of type $\frac{1}{3}(1,1)$. They are cut out by the equations $z = t = 0$. The surface also has three singular points Q_1, Q_2, Q_3 of type $\frac{1}{5}(1,1)$. These three points are cut out by the equations $x = y = 0$.

Let C_i be the curve in the pencil $|-3K_X|$ passing through the point O_i, where $i = 1, \ldots, 5$. The curve C_i consists of three reduced and irreducible smooth rational curves

$$C_i = L_1^i + L_2^i + L_3^i.$$

The curve L^i_j contains the point Q_j. Furthermore, $L^i_1 \cap L^i_2 \cap L^i_3 = \{O_i\}$. We see that

$$-K_X \cdot L^i_j = \frac{1}{15}, \quad \left(L^i_j\right)^2 = -\frac{7}{15}, \quad L^i_j \cdot L^i_k = \frac{1}{3}$$

where $j \neq k$.

Note that $\mathrm{lct}(X, C_i) = \frac{2}{3}$. Thus $\mathrm{lct}(X) \leqslant 2$.

Suppose that $\mathrm{lct}(X) < 2$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, 2D)$ is not log canonical at some point $P \in X$. Then, $\mathrm{mult}_P(D) > \frac{1}{2}$.

Suppose that $P \notin C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$. Then P is a smooth point of X. There is a unique curve $C \in |-3K_X|$ passing through point P. Then C is different from the curves $C_1, \ldots, C_5$ and hence C is irreducible. Furthermore, the log pair (X, C) is log canonical. Thus, it follows from Lemma 1.3.6 that we may assume that $C \not\subset \mathrm{Supp}(D)$. Then we obtain an absurd inequality

$$\frac{1}{5} = D \cdot C \geqslant \mathrm{mult}_P\big(D\big) > \frac{1}{2},$$

since the log pair $(X, 2D)$ is not log canonical at the point P. Therefore, $P \in C_1 \cup C_2 \cup C_3 \cup C_4 \cup C_5$. However, we may assume that $P \in C_1$ without loss of generality. Furthermore, by Lemma 1.3.6, we may assume that $L^1_i \not\subset \mathrm{Supp}(D)$ for some $i = 1, 2, 3$.

Since

$$\frac{1}{5} = 3D \cdot L^1_i \geqslant \mathrm{mult}_{O_1}\big(D\big),$$

the point P cannot be the point O_1.

Without loss of generality, we may assume that $P \in L^1_1$.

Let Z be the curve in the pencil $|-5K_X|$ passing through the point Q_1. Then

$$Z = Z_1 + Z_2 + Z_3 + Z_4 + Z_5,$$

where Z_i is a reduced and irreducible smooth rational curve. The curve Z_i contains the point O_i. Moreover, $Z_1 \cap Z_2 \cap Z_3 \cap Z_4 \cap Z_5 = \{Q_1\}$. It is easy to check $\mathrm{lct}(X, Z) = \frac{2}{5}$. By Lemma 1.3.6, we may assume that $Z_k \not\subset \mathrm{Supp}(D)$ for some $k = 1, \ldots, 5$. Then

$$\frac{1}{3} = 5D \cdot Z_k \geqslant \mathrm{mult}_{Q_1}\big(D\big),$$

and hence the point P cannot be the point Q_1.

Thus, the point P is a smooth point on L^1_1. Put

$$D = mL^1_1 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $L^1_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{1}{15} = D \cdot L^1_i = \big(mL^1_1 + \Omega\big) \cdot L^1_i \geqslant mL^1_1 \cdot L^1_i = \frac{m}{3},$$

and hence $m \leqslant \frac{1}{5}$. Then it follows from Lemma 1.3.8 that

$$\frac{1 + 7m}{15} = \big(D - mL^1_1\big) \cdot L^1_1 = \Omega \cdot L^1_1 > \frac{1}{2}.$$

This implies that $m > \frac{13}{14}$. But $m \leqslant \frac{1}{5}$. The obtained contradiction completes the proof. □

Lemma 3.1.6. Let X be a quasismooth hypersurface of degree 25 in $\mathbb{P}(3, 5, 7, 11)$. Then $\mathrm{lct}(X) = \frac{21}{10}$.

Proof. The curve C_x is irreducible and reduced. It is easy to see that $\mathrm{lct}(X, \frac{1}{3}C_x) = \frac{21}{10}$. Therefore, $\mathrm{lct}(X) \leqslant \frac{21}{10}$.

Suppose that $\mathrm{lct}(X) < \frac{21}{10}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{21}{10}D)$ is not log canonical at some point $P \in X$. We may assume that the support of D does not contain the curve C_x by Lemma 1.3.6.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(21))$ contains x^7, x^2y^3, z^3, by Lemma 1.3.9 we have

$$\mathrm{mult}_P(D) \leqslant \frac{21\cdot 25}{3\cdot 5\cdot 7\cdot 11} < \frac{10}{21}$$

if P is a smooth point in the outside of the curve C_x. Thus, either $P = O_x$ or $P \in C_x$.

If $P \in C_x$, then we obtain a contradictory inequalities

$$\frac{5}{77} = D\cdot C_x \geqslant \begin{cases} \mathrm{mult}_P(D)\mathrm{mult}_P(C_x) = \mathrm{mult}_P(D) > \dfrac{10}{21} & \text{if } P \in X\setminus \mathrm{Sing}(X),\\ \dfrac{\mathrm{mult}_P(D)\mathrm{mult}_P(C_x)}{7} = \dfrac{\mathrm{mult}_P(D)}{7} > \dfrac{10}{147} & \text{if } P = O_z,\\ \dfrac{\mathrm{mult}_P(D)\mathrm{mult}_P(C_x)}{11} = \dfrac{2\mathrm{mult}_P(D)}{11} > \dfrac{20}{231} & \text{if } P = O_t. \end{cases}$$

Therefore, we see that $P = O_x$.

Since the curve C_y is irreducible and the log pair $(X, \frac{1}{5}C_y)$ is log canonical at the point O_x, we may assume that the support of D does not contain the curve C_y. Then

$$\frac{10}{63} < \frac{\mathrm{mult}_{O_x}(D)}{3} \leqslant D\cdot C_y = \frac{25}{231} < \frac{10}{63}.$$

This is a contradiction. □

Lemma 3.1.7. Let X be a quasismooth hypersurface of degree 28 in $\mathbb{P}(3,5,7,14)$. Then $\mathrm{lct}(X) = \frac{9}{4}$.

Proof. The surface X is singular at the point O_x and the point O_y. The former is a singular point of type $\frac{1}{3}(1,1)$ and the latter is of type $\frac{1}{5}(1,2)$. Let O_1 and O_2 be the two points cut out on X by the equations $x = y = 0$. The points O_1 and O_2 are singular points of type $\frac{1}{7}(3,5)$ on the surface X.

The curve C_x consists of two reduced and irreducible smooth rational curves L_1 and L_2. These two curves intersect each other only at the point O_y. Each curve L_i contains the singular point O_i. We have

$$-K_X\cdot L_i = \frac{1}{35},\quad L_1\cdot L_2 = \frac{2}{5},\quad L_1^2 = L_2^2 = -\frac{11}{35}.$$

Since $\mathrm{lct}(X, C_x) = \frac{3}{4}$, $\mathrm{lct}(X) \leqslant \frac{9}{4}$.

Suppose that $\mathrm{lct}(X) < \frac{9}{4}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{9}{4}D)$ is not log canonical at some point $P \in X$.

If P is a smooth point in the outside of C_x, then

$$\mathrm{mult}_P\big(D\big) \leqslant \frac{588}{1470} < \frac{4}{9}$$

by Lemma 1.3.9 since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(21))$ contains x^7, z^3, x^2y^3. Therefore, either P belongs to the curve C_x or $P = O_x$.

By Lemma 1.3.6, we may assume that $L_i \not\subset \mathrm{Supp}(D)$ for some $i = 1, 2$. Similarly, we may assume that $C_y \not\subset \mathrm{Supp}(D)$ since (X, C_y) is log canonical and the curve C_y is irreducible.

The inequalities

$$\mathrm{mult}_{O_x}(D) \leqslant 3D \cdot C_y = \frac{2}{7} < \frac{4}{9}$$

show that the point P cannot be the point O_x. Therefore, the point P belongs to the curve C_x.

The inequalities

$$\mathrm{mult}_{O_y}(D) \leqslant 5D \cdot L_i = \frac{1}{7} < \frac{4}{9}$$

show that the point P cannot be the point O_y.

Without loss of generality, we may assume that $P \in L_1$. Put $D = mL_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{1}{35} = D \cdot L_2 = \big(mL_1 + \Omega\big) \cdot L_2 \geqslant mL_1 \cdot L_2 = \frac{2m}{5},$$

and hence $m \leqslant \frac{1}{14}$. Then Lemma 1.3.8 implies an absurd inequality

$$\frac{5}{98} \geqslant \frac{1+11m}{35} = \big(D - mL_1\big) \cdot L_1 = \Omega \cdot L_1 > \begin{cases} \dfrac{4}{9} \text{ if } P \neq O_1, \\ \dfrac{4}{63} \text{ if } P = O_1. \end{cases}$$

The obtained contradiction completes the proof. □

Lemma 3.1.8. Let X be a quasismooth hypersurface of degree 36 in $\mathbb{P}(3, 5, 11, 18)$. Then $\mathrm{lct}(X) = \frac{21}{10}$.

Proof. The surface X is singular at the points O_y and O_z. It is also singular at two points P_1 and P_2 on the curve L_{yz}. These two points P_1 and P_2 are contained in C_y.

The curve C_x is irreducible and reduced. It is easy to see that $\mathrm{lct}(X, \frac{1}{3}C_x) = \frac{21}{10}$. Also, the curve C_y is always irreducible and the pair $(X, \frac{21}{5\cdot 10}C_y)$ is log canonical. We see that $\mathrm{lct}(X) \leqslant \frac{21}{10}$.

Suppose that $\mathrm{lct}(X) < \frac{21}{10}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{21}{10}D)$ is not log canonical at some point $P \in X$. By Lemma 1.3.6, we may assume that the support of D contains neither the curve C_x nor C_y.

Then the following inequalities

$$\mathrm{mult}_{O_y}(D) \leqslant 5D \cdot C_x = \frac{5 \cdot 3 \cdot 36}{3 \cdot 5 \cdot 11 \cdot 18} < \frac{10}{21},$$
$$\mathrm{mult}_{O_z}(D) \leqslant 11D \cdot C_x = \frac{11 \cdot 3 \cdot 36}{3 \cdot 5 \cdot 11 \cdot 18} < \frac{10}{21},$$
$$\mathrm{mult}_{P_i}(D) \leqslant 3D \cdot C_y = \frac{3 \cdot 5 \cdot 36}{3 \cdot 5 \cdot 11 \cdot 18} < \frac{10}{21},$$

show that the point P is a smooth point P on X. Furthermore, the first two inequalities also show that the point P cannot belong to the curve C_x. Therefore, the point P is a smooth point in the outside of the curve C_x.

However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(39))$ contains x^{13}, x^3y^6, x^2z^3, by Lemma 1.3.9 we have

$$\frac{10}{21} < \mathrm{mult}_P(D) \leqslant \frac{36 \cdot 39}{3 \cdot 5 \cdot 11 \cdot 18} < \frac{10}{21}.$$

The obtained contradiction completes the proof. □

Lemma 3.1.9. Let X be a quasismooth hypersurface of degree 56 in $\mathbb{P}(5,14,17,21)$. Then $\mathrm{lct}(X)=\frac{25}{8}$.

Proof. The surface X is singular at the points O_x, O_z and O_t. The first point is a singular point of type $\frac{1}{5}(2,1)$, the second of type $\frac{1}{17}(7,2)$, the last of type $\frac{1}{21}(5,17)$. There is one more singular point O of type $\frac{1}{7}(5,3)$ on L_{xz} that is different from the singular point O_t.

The curve C_x (resp. C_y) consists of two reduced and irreducible curves L_{xy} and R_x (resp. R_y). The curve L_{xy} intersects the curve R_x at the point O_z. The curve R_x is singular at the point O_z. On the other hand, it intersects the curve R_y at the point O_t. The curve R_y is singular at O_t. We have

$$L_{xy}^2=-\frac{37}{357},\quad L_{xy}\cdot R_x=\frac{2}{17},\quad R_x^2=-\frac{9}{119},\quad L_{xy}\cdot R_y=\frac{1}{7},\quad R_y^2=\frac{9}{35}.$$

It is easy to check $\mathrm{lct}(X,C_x)=\frac{5}{8}$ and $\mathrm{lct}(X,C_y)=\frac{3}{7}$, and hence $\mathrm{lct}(X)\leqslant\frac{25}{8}$.

Suppose that $\mathrm{lct}(X)<\frac{25}{8}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\frac{25}{8}D)$ is not log canonical at some point $P\in X$. By Lemma 1.3.6 we may assume that either the support of the divisor D does not contain the curve L_{xy} or it contains neither R_x nor R_y.

Suppose that $P\notin C_x\cup C_y$. Then P is a smooth point and

$$\mathrm{mult}_P\big(D\big)\leqslant\frac{4}{21}<\frac{8}{25}$$

by Lemma 1.3.9 since the natural projection $X\dashrightarrow\mathbb{P}(5,14,17)$ is a finite morphism outside of the curve C_y and $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(85))$ contains monomials x^{17},z^5,x^3y^5. This is a contradiction. Thus, the point P must belong to $C_x\cup C_y$.

The curve C_z is irreducible and the log pair $(X,\frac{25}{8\cdot 17}C_z)$ is log canonical. By Lemma 1.3.6 we may assume that $C_z\not\subset\mathrm{Supp}(D)$. Then

$$\frac{8}{25}>\frac{4}{21}=5D\cdot C_z\geqslant\mathrm{mult}_{O_x}\big(D\big),$$

and hence the point P cannot be O_x.

Suppose that $P\in L_{xy}$. Put $D=mL_{xy}+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xy}\not\subset\mathrm{Supp}(\Omega)$. If $m\neq 0$, then

$$\frac{1}{119}=D\cdot R_x=\big(mL_{xy}+\Omega\big)\cdot R_x\geqslant mL_{xy}\cdot R_x=\frac{2m}{17},$$

and hence $m\leqslant\frac{1}{14}$. Then it follows from Lemma 1.3.8 that

$$\frac{1+37m}{357}=\big(D-mL_{xy}\big)\cdot L_{xy}=\Omega\cdot L_{xy}>\begin{cases}\dfrac{8}{525}\text{ if }P=O_t,\\ \dfrac{8}{425}\text{ if }P=O_z,\\ \dfrac{8}{25}\text{ if }P\neq O_z\text{ and }P\neq O_t.\end{cases}$$

This implies $m>\frac{3}{25}$. But $m\leqslant\frac{1}{14}$. The obtained contradiction implies that $P\notin L_{xy}$.

Suppose that $P \in R_x$. Put $D = aR_x + \Upsilon$, where Υ is an effective $\mathbb{Q}$-divisor such that $R_x \not\subset \mathrm{Supp}(\Upsilon)$. If $a \neq 0$, then

$$\frac{1}{357} = D \cdot L_{xy} = \big(aR_x + \Upsilon\big) \cdot L_{xy} \geqslant aL_{xy} \cdot R_x = \frac{2a}{17},$$

and hence $a \leqslant \frac{1}{42}$. Then it follows from Lemma 1.3.8 that

$$\frac{1+9a}{119} = \big(D - aR_x\big) \cdot R_x = \Upsilon \cdot R_x > \begin{cases} \dfrac{8}{175} \text{ if } P = O, \\ \dfrac{8}{25} \text{ if } P \neq O. \end{cases}$$

This is impossible because $a \leqslant \frac{1}{42}$. Thus, we see that $P \notin C_x$.

We see that $P \in R_y$ and $P \in X \setminus \mathrm{Sing}(X)$. Put $D = bR_y + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $R_y \not\subset \mathrm{Supp}(\Delta)$. If $b \neq 0$, then

$$\frac{1}{357} = D \cdot L_{xy} = \big(bR_y + \Delta\big) \cdot L_{xy} \geqslant bL_{xy} \cdot R_y = \frac{b}{7},$$

and hence $b \leqslant \frac{1}{51}$. Then it follows from Lemma 1.3.8 that

$$\frac{1+9b}{35} = \big(D - bR_y\big) \cdot R_y = \Delta \cdot R_y > \frac{8}{25}.$$

This is impossible because $b \leqslant \frac{1}{51}$. The obtained contradiction completes the proof. □

Lemma 3.1.10. Let X be a quasismooth hypersurface of degree 81 in $\mathbb{P}(5, 19, 27, 31)$. Then $\mathrm{lct}(X) = \frac{25}{6}$.

Proof. The curve C_x is irreducible and reduced. Moreover, the curve C_x is smooth outside of the singular locus of the surface X. It is easy to see that $\mathrm{lct}(X, \frac{1}{5}C_x) = \frac{25}{6}$. Hence, we have $\mathrm{lct}(X) \leqslant \frac{25}{6}$. The curve C_y is irreducible and reduced. The log pair $(X, \frac{1}{19}C_y)$ is log canonical.

Suppose that $\mathrm{lct}(X) < \frac{25}{6}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{25}{6}D)$ is not log canonical at some point $P \in X$. We may assume that the support of D contains neither C_x nor C_y by Lemma 1.3.6.

The inequality

$$31D \cdot C_x = \frac{3}{19} < \frac{6}{25}$$

shows that the point P cannot be on the curve C_x. On the other hand, the inequality

$$5D \cdot C_y = \frac{3}{31} < \frac{6}{25}$$

shows that the point P cannot be on the curve C_y. In particular, the point P cannot be the point O_x.

Therefore, the point P must be a smooth point in the outside of C_x. However, Lemma 1.3.9 implies

$$\mathrm{mult}_P(D) \leqslant \frac{190 \cdot 81}{5 \cdot 19 \cdot 27 \cdot 31} < \frac{6}{25}$$

since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(190))$ contains $x^{38}, x^{11}z, y^{10}$. This is a contradiction. □

Lemma 3.1.11. Let X be a quasismooth hypersurface of degree 100 in $\mathbb{P}(5,19,27,50)$. Then $\mathrm{lct}(X) = \frac{25}{6}$.

Proof. The surface X is singular at the points O_y and O_z. Also, it is singular at two points P_1 and P_2 on L_{yz}. The point O_y is a singular point of type $\frac{1}{19}(2,3)$ on X. The point O_z is of type $\frac{1}{27}(5,23)$. The last two points are of type $\frac{1}{5}(2,1)$.

The curve C_x is irreducible and reduced. It is easy to see that $\mathrm{lct}(X, \frac{1}{5}C_x) = \frac{25}{6}$. Therefore, $\mathrm{lct}(X) \leqslant \frac{25}{6}$. The curve C_z is irreducible and reduced. The log pair $(X, \frac{25}{6\cdot 27}C_z)$ is log canonical.

Suppose that $\mathrm{lct}(X) < \frac{25}{6}$. Then it follows from Lemma 1.3.6 that there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that $C_x, C_z \not\subset \mathrm{Supp}(D)$ and the pair $(X, \frac{25}{6}D)$ is not log canonical at some point $P \in X$.

The inequality

$$27D \cdot C_x = \frac{2}{19} < \frac{6}{25}$$

shows that the point P cannot be on the curve C_x. On the other hand, the inequality

$$5D \cdot C_z = \frac{2}{19} < \frac{6}{25}$$

shows that the point P cannot be on the curve C_z. In particular, the point P can be neither the point P_1 nor the point P_2.

Consequently, the point P must be a smooth point in the outside of C_x. However, $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(270))$ contains $x^{54}, x^{16}y^{10}, z^{10}$. Then, Lemma 1.3.9 implies a contradictory inequality

$$\frac{6}{25} < \mathrm{mult}_P(D) \leqslant \frac{270 \cdot 100}{5 \cdot 19 \cdot 27 \cdot 50} < \frac{6}{25}.$$

□

Lemma 3.1.12. Let X be a quasismooth hypersurface of degree 81 in $\mathbb{P}(7,11,27,37)$. Then $\mathrm{lct}(X) = \frac{49}{12}$.

Proof. The surface X is singular only at the points O_x, O_y and O_t.

The curve C_x is irreducible and reduced. It is easy to see that $\mathrm{lct}(X, \frac{1}{7}C_x) = \frac{49}{12}$, and hence $\mathrm{lct}(X) \leqslant \frac{49}{12}$. The curve C_y is irreducible and reduced. Moreover, the log pair $(X, \frac{49}{11\cdot 12}C_y)$ is log canonical.

Suppose that $\mathrm{lct}(X) < \frac{49}{12}$. By Lemma 1.3.6, there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the support of D contains neither the curve C_x nor the curve C_y, and the log pair $(X, \frac{49}{12}D)$ is not log canonical at some point $P \in X$.

The three inequalities

$$11D \cdot C_x = \frac{3}{37} < \frac{12}{49},$$

$$7D \cdot C_y = \frac{3}{37} < \frac{12}{49},$$

$$\mathrm{mult}_{O_t}(D) = \frac{\mathrm{mult}_{O_t}(D)\mathrm{mult}_{O_t}(C_x)}{3} \leqslant \frac{37}{3} D \cdot C_x = \frac{1}{11} < \frac{12}{49}$$

show that the point P is a smooth point in the outside of C_x.

However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(189))$ contains $x^{27}, x^{16}y^7, z^7$, Lemma 1.3.9 implies an absurd inequalities

$$\frac{12}{49} < \mathrm{mult}_P(D) \leqslant \frac{189 \cdot 81}{7 \cdot 11 \cdot 27 \cdot 37} < \frac{12}{49}.$$

Therefore, $\mathrm{lct}(X) = \frac{49}{12}$. □

Lemma 3.1.13. Let X be a quasismooth hypersurface of degree 88 in $\mathbb{P}(7, 11, 27, 44)$. Then $\mathrm{lct}(X) = \frac{35}{8}$.

Proof. The surface X is singular at the points O_x and O_z. The former is a singular point of type $\frac{1}{7}(3, 1)$ and the latter is of type $\frac{1}{27}(11, 17)$. The surface is also singular at the points O_1 and O_2 on L_{xz}. They are of type $\frac{1}{11}(7, 5)$.

The curve C_x consists of two smooth rational curves L_1 and L_2. Each curve L_i contains the singular point O_i. The curves L_1 and L_2 intersects each other only at the point O_z. We have

$$L_1^2 = L_2^2 = -\frac{37}{297}, \quad L_1 \cdot L_2 = \frac{4}{27}.$$

It is easy to check $\mathrm{lct}(X, \frac{1}{7}C_x) = \frac{35}{8}$. Meanwhile, the curve C_y is irreducible and reduced. Also, the log pair $(X, \frac{35}{88}C_y)$ is log canonical.

Suppose that $\mathrm{lct}(X) < \frac{35}{8}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{35}{8}D)$ is not log canonical at some point $P \in X$. By Lemma 1.3.6 we may assume that the support of D contains neither C_y nor L_2 without loss of generality.

The inequality

$$27D \cdot L_2 = \frac{1}{11} < \frac{8}{35}$$

shows that the point P is located in the outside of L_2. The inequality

$$7D \cdot C_y = \frac{2}{27} < \frac{8}{35}$$

implies that the point P cannot be O_x. Write

$$D = mL_1 + \Omega,$$

where Ω is an effective $\mathbb{Q}$-divisor such that $L_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{1}{297} = D \cdot L_2 = \big(mL_1 + \Omega\big) \cdot L_2 \geqslant mL_1 \cdot L_2 = \frac{4m}{27},$$

and hence $m \leqslant \frac{1}{44}$. Then

$$(D - mL_1) \cdot L_1 = \frac{1 + 37m}{297} \leqslant \frac{3}{484} < \frac{8}{35 \cdot 11},$$

and hence Lemma 1.3.8 implies that the point P cannot be on the curve L_1. Therefore, the point P is a smooth point in the outside of C_x. However, Lemma 1.3.9 shows

$$\mathrm{mult}_P\big(D\big) \leqslant \frac{2}{11} < \frac{8}{35}$$

since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(189))$ contains monomials $x^{27}, z^7, x^{16}y^7$. This is a contradiction. □

Lemma 3.1.14. Let X be a quasismooth hypersurface of degree 60 in $\mathbb{P}(9,15,17,20)$. Then $\mathrm{lct}(X)=\frac{21}{4}$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$xz^3+x^5y-y^4+t^3=0.$$

Note that the surface X is singular at O_x and O_z. It is also singular at the point $P_1=[1:1:0:0]$ and the point $P_2=[0:1:0:1]$.

The curves C_x, C_y, and C_z are irreducible and reduced. We have

$$\mathrm{lct}(X,\frac{1}{9}C_x)=\frac{21}{4},\quad \mathrm{lct}(X,\frac{1}{15}C_y)=10,\quad \mathrm{lct}(X,\frac{1}{17}C_z)=17.$$

The curve C_x is singular at the point O_z with multiplicity 3.

Suppose that $\mathrm{lct}(X)<\frac{21}{4}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{21}{4}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of D contains none of the curves C_x, C_y, C_z.

The three inequalities

$$\frac{17}{3}D\cdot C_x=\frac{17\cdot 9\cdot 60}{3\cdot 9\cdot 15\cdot 17\cdot 20}<\frac{4}{21},$$
$$9D\cdot C_y=\frac{9\cdot 15\cdot 60}{9\cdot 15\cdot 17\cdot 20}<\frac{4}{21},$$
$$3D\cdot C_z=\frac{3\cdot 17\cdot 60}{9\cdot 15\cdot 17\cdot 20}<\frac{4}{21}$$

imply that the point P is located in the outside of $C_x\cup C_y\cup C_z$.

Let $\mathcal{L}$ be the pencil on X that is cut out by the equations

$$\lambda z^3+\mu x^4y=0,$$

where $[\lambda:\mu]\in\mathbb{P}^1$. Then the base locus of the pencil $\mathcal{L}$ consists of the points P_2 and O_x. Let C be the unique curve in $\mathcal{L}$ that passes through the point P. Then C is cut out on X by an equation

$$x^4y=\alpha z^3,$$

where α is a non-zero constant, since the point P is located in the outside of $C_x\cup C_y\cup C_z$. The curve C is smooth outside of the points P_2 and O_x by the Bertini theorem because C is isomorphic to a general curve in the pencil $\mathcal{L}$ unless $\alpha=-1$. In the case when $\alpha=-1$, the curve C is smooth outside the points P_2 and O_x as well.

We claim that the curve C is irreducible. If so, then we may assume that the support of D does not contain the curve C and hence we obtain a contradiction

$$\frac{4}{21}<\mathrm{mult}_Q(D)\leqslant D\cdot C=\frac{51\cdot 60}{9\cdot 15\cdot 17\cdot 20}<\frac{4}{21}.$$

For the irreducibility of the curve C, we may consider the curve C as a surface in $\mathbb{C}^4$ defined by the equations $t^3+y^4+(1+\alpha)xz^3=0$ and $x^4y=\alpha z^3$. This surface is isomorphic to the surface in $\mathbb{C}^4$ defined by the equations $t^3+y^4+\beta xz^3=0$ and $x^4y=z^3$, where $\beta=1$ or 0. Then, we consider the surface in $\mathbb{P}^4$ defined by the equations $t^3w+y^4+\beta xz^3=0$ and $x^4y=z^3w^2$. We take the affine piece defined by $t\neq 1$. This affine piece is isomorphic to the surface defined by the equation $x^4y+z^3(y^4+\beta xz^3)^2=0$ in $\mathbb{C}^3$. If $\beta=1$, the surface is irreducible. If $\beta=0$, then it has an extra component defined by $y = 0$. However, this component originates from the hyperplane $w = 0$ in $\mathbb{P}^4$. Therefore, the surface in $\mathbb{C}^4$ defined by the equations $t^3 + y^4 = 0$ and $x^4 y = z^3$ is also irreducible. □

Lemma 3.1.15. Let X be a quasismooth hypersurface of degree 69 in $\mathbb{P}(9, 15, 23, 23)$. Then $\mathrm{lct}(X) = 6$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$zt(z - t) + xy(y^3 - x^5) = 0.$$

The surface X is singular at three distinct points O_x, O_y, $P_1 = [1 : 1 : 0 : 0]$. Also, it is singular at three distinct points O_z, O_t, $Q_1 = [0 : 0 : 1 : 1]$.

The curve C_x consists of three distinct curves L_{xz}, L_{xt} and $R_x = \{x = z - t = 0\}$ that intersect altogether at the point O_y. Similarly, the curve C_y consists of three curves L_{yz}, L_{yt} and $R_y = \{y = z - t = 0\}$ that intersect altogether at the point O_x. The curve C_z consists of L_{xz}, L_{yz}, and $R_z = \{z = y^3 - x^5 = 0\}$. The curve R_z is singular at the point O_t with multiplicity 3. The curve C_t consists of L_{xt}, L_{yt} and $R_t = \{t = y^3 - x^5 = 0\}$. The curve R_t is singular at the point O_z with multiplicity 3.

Note that $\mathrm{lct}(X, \frac{1}{9}C_x) = 6$. The log pairs $(X, \frac{6}{15}C_y)$, $(X, \frac{6}{23}C_z)$ and $(X, \frac{6}{23}C_t)$ are log canonical.

Suppose that $\mathrm{lct}(X) < 6$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, 6D)$ is not log canonical at some point $P \in X$. Lemma 1.3.6 implies that we may assume that the support of D contains neither R_x nor R_y by a linear coordinate change. Furthermore, we may assume that the support of D does not contain at least one component of C_z. Also, it may be assumed not to contain at least one component of C_t.

The inequalities

$$15D \cdot R_x = \frac{15 \cdot 23 \cdot 9}{9 \cdot 15 \cdot 23 \cdot 23} = \frac{1}{23} < \frac{1}{6},$$
$$23D \cdot R_y = \frac{23 \cdot 23 \cdot 15}{9 \cdot 15 \cdot 23 \cdot 23} = \frac{1}{9} < \frac{1}{6}$$

show that the point P is located in the outside of $R_x \cup R_y$.

Then the inequalities

$$23D \cdot L_{xz} = \frac{1}{15} < \frac{1}{6}, \quad 23D \cdot L_{yz} = \frac{1}{9} < \frac{1}{6}, \quad \frac{23}{3}D \cdot R_z = \frac{1}{9} < \frac{1}{6}$$

show that $\mathrm{mult}_{O_t}(D) < \frac{1}{6}$, and hence the point P cannot be the point O_t. By the same way, we can show that $P \neq O_z$.

Write $D = mR_z + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $R_z \not\subset \mathrm{Supp}(\Omega)$. Then $m \leqslant \frac{1}{6}$ since $(X, 6D)$ is log canonical at O_t. We have

$$R_z \cdot (L_{xz} + L_{yz}) = \frac{8}{23}, \quad R_z \cdot D = \frac{1}{69},$$

and hence $R_z^2 = -\frac{1}{69}$. Then

$$\Omega \cdot R_z = D \cdot R_z - mR_z^2 = \frac{1 + m}{3 \cdot 23} \leqslant \frac{7}{6 \cdot 3 \cdot 23} < \frac{1}{3 \cdot 6}.$$

Lemma 1.3.8 implies that the point P cannot belong to R_z. In particular, the point P cannot be the point P_1.

Write $D = aL_{xz} + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{xz}. Then $a \leqslant \frac{1}{6}$. Then

$$\Omega \cdot L_{xz} = 6(D \cdot L_{xz} - aL_{xz}^2) = \frac{6 \cdot (1 + 37a)}{345} \leqslant \frac{6 + 37}{345} = \frac{43}{345} < 1,$$

because $L_{xz}^2 = -\frac{37}{345}$. Thus, we see that $P \notin L_{xz}$. Similarly, we can show that $P \notin L_{yz}$. Thus, we see that $P \notin C_z$. In the same way, we can see that P is not contained in the curves C_t and $\{z - t = 0\}$.

Therefore, the point P is a smooth point in the outside of $C_z \cup C_t \cup \{z - t = 0\}$. Let E be the unique curve on X such that E is given by the equation $z = \lambda t$ and $P \in E$, where λ is a non-zero constant different from 1. Then E is quasismooth and hence irreducible. Therefore, we may assume that the support of D does not contain the curve E. Then

$$\mathrm{mult}_P(D) \leqslant D \cdot E = \frac{23 \cdot 69}{9 \cdot 15 \cdot 23 \cdot 23} < \frac{1}{6}.$$

This is a contradiction. □

Lemma 3.1.16. Let X be a quasismooth hypersurface of degree 127 in $\mathbb{P}(11, 29, 39, 49)$. Then $\mathrm{lct}(X) = \frac{33}{4}$.

Proof. We may assume that the hypersurface X is defined by the equation

$$z^2t + yt^2 + xy^4 + x^8z = 0.$$

The singularities of X consist of a singular point of type $\frac{1}{11}(7, 5)$ at O_x, a singular point of type $\frac{1}{29}(1, 2)$ at O_y, a singular point of type $\frac{1}{39}(11, 29)$ at O_z, and a singular point of type $\frac{1}{49}(11, 39)$ at O_t.

The curve C_x (resp. C_y, C_z, C_t) consists of two irreducible curves L_{xt} (resp. L_{yz}, L_{yz}, L_{xt}) and $R_x = \{x = z^2 + yt = 0\}$ (resp. $R_y = \{y = x^8 + zt = 0\}$, $R_z = \{z = t^2 + xy^3 = 0\}$, $R_t = \{t = y^4 + x^7z = 0\}$). We can see that

$$L_{xt} \cap R_x = \{O_y\},\ L_{yz} \cap R_y = \{O_t\},\quad L_{yz} \cap R_z = \{O_x\},\ L_{xt} \cap R_t = \{O_z\}.$$

It is easy to check $\mathrm{lct}(X, \frac{1}{11}C_x) = \frac{33}{4}$. The log pairs $(X, \frac{33}{4 \cdot 29}C_y)$, $(X, \frac{33}{4 \cdot 39}C_z)$ and $(X, \frac{33}{4 \cdot 49}C_t)$ are log canonical.

Suppose that $\mathrm{lct}(X) < \frac{33}{4}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{33}{4}D)$ is not log canonical at some point $P \in X$.

By Lemma 1.3.6, we may assume that the support of D does not contain L_{xt} or R_x. Then one of the following two inequalities must hold:

$$\frac{4}{33} > \frac{1}{39} = 29L_{xt} \cdot D \geqslant \mathrm{mult}_{O_y}(D),$$
$$\frac{4}{33} > \frac{2}{49} = 29R_x \cdot D \geqslant \mathrm{mult}_{O_y}(D).$$

Therefore, the point P cannot be the point O_y. For the same reason, one of two inequalities

$$\frac{4}{33} > \frac{1}{49} = 11L_{yz} \cdot D \geqslant \mathrm{mult}_{O_x}(D),$$
$$\frac{4}{33} > \frac{2}{29} = 11R_z \cdot D \geqslant \mathrm{mult}_{O_x}(D)$$

must hold, and hence the point P cannot be the point O_x. Since R_t is singular at the point O_z with multiplicity 4, we can apply the same method to C_t, i.e., one of the following inequalities must be satisfied:

$$\frac{4}{33} > \frac{1}{29} = 39L_{xt}\cdot D \geqslant \mathrm{mult}_{O_z}(D),$$

$$\frac{4}{33} > \frac{1}{11} = \frac{39}{4}R_t\cdot D \geqslant \frac{1}{4}\mathrm{mult}_{O_z}(D)\mathrm{mult}_{O_z}(R_t) = \mathrm{mult}_{O_z}(D).$$

Thus, the point P cannot be O_z.

Write $D = \mu R_x + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $R_x \not\subset \mathrm{Supp}\,(\Omega)$. If $\mu > 0$, then L_{xt} is not contained in the support of D. Thus,

$$\frac{2}{29}\mu = \mu R_x\cdot L_{xt} \leqslant D\cdot L_{xt} = \frac{1}{29\cdot 39},$$

and hence $\mu \leqslant \frac{1}{78}$. We have

$$49\Omega\cdot R_x = 49(D\cdot R_x - \mu R_x^2) = \frac{2+76\mu}{29} < \frac{4}{33}.$$

Then Lemma 1.3.8 shows that the point P cannot belong to R_x. In particular, the point P cannot be O_t.

Put $D = \epsilon L_{xt} + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $L_{xt} \not\subset \mathrm{Supp}\,(\Delta)$. Since $(X, \frac{33}{4}D)$ is log canonical at the point O_y, $\epsilon \leqslant \frac{4}{33}$ and hence

$$\Delta\cdot L_{xt} = D\cdot L_{xt} - \epsilon L_{xt}^2 = \frac{1+67\epsilon}{29\cdot 39} < \frac{4}{33}.$$

Then Lemma 1.3.8 implies that the point P cannot belong to L_{xt}.

Consequently, the point P must be a smooth point in the outside of C_x. Then an absurd inequality

$$\frac{4}{33} < \mathrm{mult}_P(D) \leqslant \frac{539\cdot 127}{11\cdot 29\cdot 39\cdot 49} < \frac{4}{33}$$

follows from Lemma 1.3.9 since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(539))$ contains $x^{20}y^{11}$, x^{49}, $x^{10}z^{11}$ and t^{11}. The obtained contradiction completes the proof. □

Lemma 3.1.17. Let X be a quasismooth hypersurface of degree 256 in $\mathbb{P}(11, 49, 69, 128)$. Then $\mathrm{lct}(X) = \frac{55}{6}$.

Proof. The curve C_x is irreducible and reduced. Moreover, it is easy to see $\mathrm{lct}(X, \frac{1}{11}C_x) = \frac{55}{6}$. The curve C_y is also irreducible and reduced and the log pair $(X, \frac{1}{49}C_y)$ is log canonical.

Suppose that $\mathrm{lct}(X) < \frac{55}{6}$. By Lemma 1.3.6, there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that $C_x, C_y \not\subset \mathrm{Supp}(D)$ and the log pair $(X, \frac{55}{6}D)$ is not log canonical at some point $P \in X$.

The inequalities

$$69D\cdot C_x = \frac{69\cdot 11\cdot 256}{11\cdot 49\cdot 69\cdot 128} < \frac{6}{55},$$

$$11D\cdot C_y = \frac{11\cdot 49\cdot 256}{11\cdot 49\cdot 69\cdot 128} < \frac{6}{55},$$

imply that the point P is a smooth point in the outside of C_x. However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(759))$ contains $x^{69}, x^{20}y^{11}, z^{11}$, we obtain

$$\mathrm{mult}_P(D) \leqslant \frac{759\cdot 256}{11\cdot 49\cdot 69\cdot 128} < \frac{6}{55}$$

from Lemma 1.3.9. This is a contradiction. □

Lemma 3.1.18. Let X be a quasismooth hypersurface of degree 127 in $\mathbb{P}(13, 23, 35, 57)$. Then $\mathrm{lct}(X) = \frac{65}{8}$.

Proof. We may assume that the hypersurface X is given by the equation

$$z^2t + y^4z + xt^2 + x^8y = 0.$$

The only singularities of X are a singular point of type $\frac{1}{13}(9, 5)$ at O_x, a singular point of type $\frac{1}{23}(13, 11)$ at O_y, a singular point of type $\frac{1}{35}(13, 23)$ at O_z, and a singular point of type $\frac{1}{57}(23, 35)$ at O_t.

The curve C_x (resp. C_y, C_z, C_t) consists of two irreducible curves L_{xz} (resp. L_{yt}, L_{xz}, L_{yt}) and $R_x = \{x = y^4 + zt = 0\}$ (resp. $R_y = \{y = z^2 + xt = 0\}$, $R_z = \{z = t^2 + x^7y = 0\}$, $R_t = \{t = y^3z + x^8 = 0\}$). We can see that

$$L_{xt} \cap R_x = \{O_t\},\ L_{yz} \cap R_y = \{O_x\},\quad L_{yz} \cap R_z = \{O_y\},\ L_{xt} \cap R_t = \{O_z\}.$$

It is easy to check $\mathrm{lct}(X, \frac{1}{13}C_x) = \frac{65}{8}$. The log pairs $(X, \frac{65}{8\cdot 23}C_y)$, $(X, \frac{65}{8\cdot 35}C_z)$ and $(X, \frac{65}{8\cdot 57}C_t)$ are log canonical.

Suppose that $\mathrm{lct}(X) < \frac{65}{8}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{65}{8}D)$ is not log canonical at some point $P \in X$.

By Lemma 1.3.6, we may assume that the support of D does not contain L_{xz} or R_x. Then one of the following two inequalities must hold:

$$\frac{8}{65} > \frac{1}{23} = 57L_{xz}\cdot D \geqslant \mathrm{mult}_{O_t}(D),$$
$$\frac{8}{65} > \frac{4}{35} = 57R_x\cdot D \geqslant \mathrm{mult}_{O_t}(D).$$

Therefore, the point P cannot be the point O_t. For the same reason, one of two inequalities

$$\frac{8}{65} > \frac{1}{35} = 13L_{yt}\cdot D \geqslant \mathrm{mult}_{O_x}(D),$$
$$\frac{8}{65} > \frac{2}{57} = 13R_y\cdot D \geqslant \mathrm{mult}_{O_x}(D)$$

must hold, and hence the point P cannot be the point O_x.

To apply the same method to C_z and C_t, we note that R_z is singular at O_y with multiplicity 2 and R_t is singular at O_z with multiplicity 3. Then we can see that one inequality from each of the pairs

$$\frac{8}{65} > \frac{1}{13} = 35L_{yt}\cdot D \geqslant \mathrm{mult}_{O_z}(D),$$
$$\frac{8}{65} > \frac{8}{23\cdot 3} = \frac{35}{3}R_t\cdot D \geqslant \frac{1}{3}\mathrm{mult}_{O_z}(D)\mathrm{mult}_{O_z}(R_t) = \mathrm{mult}_{O_z}(D);$$

$$\frac{8}{65} > \frac{1}{57} = 23L_{xz} \cdot D \geqslant \mathrm{mult}_{O_y}(D),$$
$$\frac{8}{65} > \frac{1}{13} = \frac{23}{2}R_z \cdot D \geqslant \frac{1}{2}\mathrm{mult}_{O_y}(D)\mathrm{mult}_{O_y}(R_z) = \mathrm{mult}_{O_y}(D)$$

must be satisfied. Therefore, the point P can be neither O_y nor O_z.

To apply Lemma 1.3.8 to L_{xz} and R_x, we compute

$$L_{xz}^2 = -\frac{79}{23 \cdot 57}, \quad R_x^2 = -\frac{88}{35 \cdot 57}.$$

Put $D = aL_{xz} + bR_x + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz}, R_x \not\subset \mathrm{Supp}\,(\Omega)$. Then $a, b \leqslant \frac{8}{65}$ since the log pair $(X, \frac{65}{8}D)$ is log canonical at the point O_t. Therefore,

$$D \cdot L_{xz} - aL_{xz}^2 = \frac{1+79a}{23 \cdot 57} < \frac{8}{65},$$
$$D \cdot R_x - bR_x^2 = \frac{4+88b}{35 \cdot 57} < \frac{8}{65}.$$

Then, Lemma 1.3.8 implies that the point P is a smooth point in the outside of C_x.

Applying Lemma 1.3.9, we see that

$$\frac{8}{65} < \mathrm{mult}_P(D) \leqslant \frac{741 \cdot 127}{13 \cdot 23 \cdot 35 \cdot 57} < \frac{8}{65},$$

since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(455))$ contains $x^{35}, x^{12}y^{13}, z^{13}$ and the point P is in the outside of L_{xz}. The obtained contradiction completes the proof. □

Lemma 3.1.19. Let X be a quasismooth hypersurface of degree 256 in $\mathbb{P}(13, 35, 81, 128)$. Then $\mathrm{lct}(X) = \frac{91}{10}$.

Proof. We may assume that the surface X is given by the equation

$$t^2 + y^5z + xz^3 + x^{17}y = 0.$$

It has a singular point of type $\frac{1}{13}(3, 11)$ at O_x, a singular point of type $\frac{1}{35}(13, 23)$ at O_y, and a singular point of type $\frac{1}{81}(35, 47)$ at O_z.

The curve C_x is reduced and irreducible. The curve is singular at the point O_z. It is easy to check that $\mathrm{lct}(X, C_x) = \frac{7}{10}$. Therefore, $\mathrm{lct}(X) \leqslant \frac{91}{10}$. The curve C_y is also reduced and irreducible. The curve C_y is singular only at O_x. Moreover, the log pair $(X, \frac{91}{10 \cdot 35}C_y)$ is log canonical.

Suppose that $\mathrm{lct}(X) < \frac{91}{10}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{91}{10}D)$ is not log canonical at some point $P \in X$. By Lemma 1.3.6 we may assume neither C_x nor C_y is contained in $\mathrm{Supp}\,(D)$.

The following two inequalities show that the point P is located in the outside of $C_x \cup C_y$:

$$\frac{81}{2}C_x \cdot D = \frac{1}{35} < \frac{10}{91},$$
$$\frac{13}{2}C_x \cdot D = \frac{1}{81} < \frac{10}{91}.$$

However, applying Lemma 1.3.9, we can obtain

$$\mathrm{mult}_P(D) \leqslant \frac{1053 \cdot 256}{13 \cdot 35 \cdot 81 \cdot 128} < \frac{10}{91},$$

since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1053))$ contains x^{81}, $x^{11}y^{26}$ and z^{13}. This is a contradiction. □

3.2. Sporadic cases with $I = 2$

Lemma 3.2.1. Let X be the quasismooth hypersurface defined by a quasihomogeneous polynomial $f(x,y,z,t)$ of degree 12 in $\mathbb{P}(2,3,4,5)$. Then

$$\mathrm{lct}\big(X\big) = \begin{cases} 1 & \text{if } f(x,y,z,t) \text{ contains the term } yzt, \\ \dfrac{7}{12} & \text{if } f(x,y,z,t) \text{ does not contain the term } yzt. \end{cases}$$

Proof. We may assume

$$f(x,y,z,t) = z(z-x^2)(z-\epsilon x^2) + y^4 + xt^2 + ayzt + bxy^2z + cx^2yt + dx^3y^2,$$

where ϵ ($\neq 0, 1$), a, b, c, d are constants. Note that X is singular at the point O_t and three points $Q_1 = [1:0:0:0]$, $Q_2 = [1:0:1:0]$, $Q_3 = [1:0:\epsilon:0]$. The curve C_x always is irreducible and reduced. We can easily check that

$$\mathrm{lct}\big(X, C_x\big) = \begin{cases} 1 & \text{if } a \neq 0, \\ \dfrac{7}{12} & \text{if } a = 0. \end{cases}$$

Suppose that $\mathrm{lct}(X) < \lambda := \mathrm{lct}(X, C_x)$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \lambda D)$ is not log canonical at some point $P \in X$. We may assume that the curve C_x is not contained in the support of D.

First, we consider the case where $a = 0$. Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6))$ contains x^3, y^2, and xz, Lemma 1.3.9 implies that for a smooth point $O \in X \setminus C_x$

$$\mathrm{mult}_O(D) < \frac{2 \cdot 12 \cdot 6}{2 \cdot 3 \cdot 4 \cdot 5} < \frac{12}{7}.$$

Therefore, the point P cannot be a smooth point in $X \setminus C_x$. Since the curve C_x is not contained in the support of D and it is singular at O_t with multiplicity 3, the inequality

$$\frac{5}{3} D \cdot C_x = \frac{5 \cdot 2 \cdot 2 \cdot 12}{3 \cdot 2 \cdot 3 \cdot 4 \cdot 5} < \frac{12}{7}$$

implies that the point P is located in the outside of C_x. Thus, the point P must be one of the point Q_1, Q_2, Q_3. The curve C_y is quasismooth. Therefore, we may assume that the support of D does not contain the curve C_y. Then the inequality

$$\mathrm{mult}_{Q_i}(D) \leqslant 2D \cdot C_y = \frac{2 \cdot 2 \cdot 3 \cdot 12}{2 \cdot 3 \cdot 4 \cdot 5} < \frac{12}{7}$$

gives us a contradiction.

From now we consider the case where $a \neq 0$. Note that the curve C_x is not contained in the support of D and it is singular at O_t with multiplicity 2. Since

$$\frac{5}{2} D \cdot C_x = \frac{5 \cdot 2 \cdot 2 \cdot 12}{2 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = 1$$

the point P is located in the outside of C_x.

The curve C_z is irreducible and the log pair $(X, \frac{1}{2}C_z)$ is log canonical. Therefore, we may assume that the support of D does not contain the curve C_z. The curve C_z is singular at the point Q_1. The inequality

$$\mathrm{mult}_{Q_1}(D) \leqslant D \cdot C_z = \frac{2 \cdot 4 \cdot 12}{2 \cdot 3 \cdot 4 \cdot 5} < 1$$

implies that P cannot be the point Q_1. We consider the curves C_{z-x^2} defined by $z = x^2$ and $C_{z-\epsilon x^2}$ defined by $z = \epsilon x^2$. Then by coordinate changes we can see that they have the same properties as that of C_z. Moreover, we can see that the point P can be neither Q_2 nor Q_3. Therefore, the point P must be located in the outside of $C_x \cup C_z \cup C_{z-x^2} \cup C_{z-\epsilon x^2}$.

Let $\mathcal{L}$ be the pencil on X defined by $\lambda x^2 + \mu z = 0$, where $[\lambda : \mu] \in \mathbb{P}^1$. Let C the curve in $\mathcal{L}$ that passes through the point P. Then it is cut out by $z = \alpha x^2$, where $\alpha \neq 0, 1, \epsilon$. The curve C is isomorphic to the curve in $\mathbb{P}(2, 3, 5)$ defined by

$$x^6 + y^4 + xt^2 + \beta x^2 yt + \gamma x^3 y^2 = 0,$$

where β and γ are constants. We can easily see that the curve C is irreducible. Moreover, we can check $\mathrm{mult}_P(C) \leqslant 2$ and hence the log pair $(X, \frac{1}{2}C)$ is log canonical. Therefore, we may assume that the support of D does not contain the curve C. Then, the inequality

$$\mathrm{mult}_P(D) \leqslant D \cdot C = \frac{2 \cdot 4 \cdot 12}{2 \cdot 3 \cdot 4 \cdot 5} < 1$$

gives us a contradiction. □

Lemma 3.2.2. Let X be a quasismooth hypersurface of degree 14 in $\mathbb{P}(2, 3, 4, 7)$. Then $\mathrm{lct}(X) = 1$.

Proof. We may assume that X is defined by the quasihomogeneous equation

$$t^2 - y^2 z^2 + x(z - \beta_1 x^2)(z - \beta_2 x^2)(z - \beta_3 x^2) + \epsilon x y^2 (y^2 - \gamma x^3)$$

where $\epsilon \neq 0$, β_1, β_2, β_3, γ are constants. Note that X is singular at the points O_y, O_z and three points $Q_1 = [1 : 0 : \beta_1 : 0]$, $Q_2 = [1 : 0 : \beta_2 : 0]$, $Q_3 = [1 : 0 : \beta_3 : 0]$. The constants β_1, β_2 and β_3 are distinct since X is quasismooth. The curve C_x consists of two irreducible reduced curves C_- and C_+. However, the curves C_y and C_z are irreducible. We can easily see that $\mathrm{lct}(X, C_x) = 1$, $\mathrm{lct}(X, \frac{2}{3}C_y) = \frac{3}{2}$ and $\mathrm{lct}(X, \frac{1}{2}C_z) > 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair (X, D) is not log canonical at some point $P \in X$. Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(6))$ contains x^3, y^2 and xz, Lemma 1.3.9 implies that the point P is either a singular point of X or a point of C_x. Furthermore, C_y is irreducible and hence we may assume that the support of D does not contain the curve C_y. Hence the equality

$$2C_y \cdot D = \frac{2 \cdot 3 \cdot 2 \cdot 14}{2 \cdot 3 \cdot 4 \cdot 5} = 1$$

implies that $P \neq Q_i$ for each $i = 1, 2, 3$. In particular, the point P must belong to C_x.

We have the following intersection numbers:

$$C_x \cdot C_- = C_x \cdot C_+ = \frac{1}{6}, \quad C_- \cdot C_+ = \frac{7}{12}, \quad C_-^2 = C_+^2 = -\frac{5}{12}.$$

We may assume that the support of D cannot contain either C_- or C_+. If D does not contain the curve C_+, then we obtain

$$\mathrm{mult}_{O_y}(D) \leqslant 4D \cdot C_+ = \frac{2}{3} < 1,$$
$$\mathrm{mult}_{O_z}(D) \leqslant 4D \cdot C_+ = \frac{2}{3} < 1.$$

On the other hand, if D does not contain the curve C_-, then we obtain

$$\mathrm{mult}_{O_y}(D) \leqslant 4D \cdot C_- = \frac{2}{3} < 1,$$
$$\mathrm{mult}_{O_z}(D) \leqslant 4D \cdot C_- = \frac{2}{3} < 1.$$

Therefore, the point P must be in $C_x \setminus \mathrm{Sing}(X)$.

We write $D = mC_+ + \Omega$, where the support of Ω does not contain the curve C_+. Then $m \geqslant \frac{2}{7}$ since $D \cdot C_- \geqslant mC_+ \cdot C_-$. Then we see $C_+ \cdot D - mC_+^2 < 1$. By the same method, we also obtain $C_- \cdot D - mC_-^2 < 1$. Then Lemma 1.3.8 completes the proof. □

Lemma 3.2.3. Let X be a quasismooth hypersurface of degree 20 in $\mathbb{P}(3,4,5,10)$. Then $\mathrm{lct}(X) = \frac{3}{2}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 = y^5 + z^4 + x^5 z + \epsilon_1 xy^3 z + \epsilon_2 x^2 yz^2 + \epsilon_3 x^4 y^2,$$

where $\epsilon_i \in \mathbb{C}$. Note that the surface X is singular only at the point O_x, $O = [0:1:0:1]$, $P_1 = [0:0:1:1]$ and $P_2 = [0:0:1:-1]$.

The curves C_x, C_y and C_z are irreducible. Moreover, we have

$$\frac{3}{2} = \mathrm{lct}(X, \frac{2}{3}C_x) < \mathrm{lct}(X, \frac{2}{4}C_y) = 2,$$

and hence $\mathrm{lct}(X) \leqslant \frac{3}{2}$. We also see that $\mathrm{lct}(X, \frac{2}{5}C_z) > \frac{3}{2}$.

Suppose that $\mathrm{lct}(X) < \frac{3}{2}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{3}{2}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D does not contain the curves C_x, C_y and C_z.

Suppose that $P \notin C_x \cup C_y \cup C_z$. Then we consider the pencil $\mathcal{L}$ on X cut out by the equations $\lambda y^2 + \mu xz = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. There is a unique member Z in the pencil $\mathcal{L}$ with $P \in Z$. The curve Z is cut out by an equation of the form $\alpha y^2 + xz$, where α is a non-zero constant. There is a natural double cover $\omega\colon Z \to C$, where C is the curve in $\mathbb{P}(3,4,5)$ given by the equation $\alpha y^2 + xz$. The curve C is quasismooth and $\omega(P)$ is a smooth point of $\mathbb{P}(3,4,5)$. Thus, we see that $\mathrm{mult}_P(Z) \leqslant 2$, the curve Z consists of at most 2 components, each component of Z is a smooth rational curve. In particular, $(X, \frac{3}{8}Z)$ is log canonical. Therefore, we may assume that $\mathrm{Supp}\,(D)$ does not contain at least one irreducible component of Z. Thus, if Z is irreducible, then we obtain an absurd inequality

$$\frac{8}{15} = D \cdot Z \geqslant \mathrm{mult}_P(D) \geqslant \frac{2}{3}.$$

So, we see that $Z = Z_1 + Z_2$, where Z_1 and Z_2 are smooth irreducible rational curves. Then

$$Z_1^2 = Z_2^2 = -\frac{4}{15}, \quad Z_1 \cdot Z_2 = \frac{4}{3}.$$

Without loss of generality we may assume that $P \in Z_1$. Put $D = mZ_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $Z_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{4}{15} = D \cdot Z_2 \geqslant mZ_1 \cdot Z_2 = \frac{4m}{3},$$

and hence $m \leqslant \frac{1}{5}$. On the other hand, Lemma 1.3.8 shows that

$$\frac{4+4m}{15} = \big(D - mZ_1\big) \cdot Z_1 = \Omega \cdot Z_1 > \frac{2}{3},$$

and hence $m > \frac{3}{2}$. This is a contradiction. Therefore, $P \in C_x \cup C_y \cup C_z$.

The inequalities

$$D \cdot C_x = \frac{1}{5} < \frac{2}{3}, \quad D \cdot C_y = \frac{4}{15} < \frac{2}{3}, \quad D \cdot C_z = \frac{1}{3} < \frac{2}{3}$$

imply that the point P must be a singular point of X.

The curve C_z is singular at the point O_x. Thus, we have

$$\frac{1}{2} = \frac{3}{2} D \cdot C_z \geqslant \frac{\mathrm{mult}_{O_x}(D)\mathrm{mult}_{O_x}(C_z)}{2} = \mathrm{mult}_{O_x}(D).$$

Therefore, the point P cannot be O_x.

Also, we have

$$\frac{2}{5} = 2D \cdot C_x \geqslant \mathrm{mult}_O(D).$$

This inequality shows that the point P cannot be the point O. Consequently, the point P must be either P_1 or P_2.

Without loss of generality we may assume that $P = P_1$. Note that $C_x \cap C_y = \{P_1, P_2\}$.

Let $\pi\colon \bar{X} \to X$ be the weighted blow up at the point P_1 with weights $(3,4)$. Let E be the exceptional curve of π and let $\bar{D}$, $\bar{C}_x$ and $\bar{C}_y$ be the proper transforms of D, C_x and C_y, respectively. Then

$$K_{\bar{X}} \sim_{\mathbb{Q}} \pi^*(K_X) + \frac{2}{5}E, \ \bar{C}_x \sim_{\mathbb{Q}} \pi^*(C_x) - \frac{3}{5}E, \ \bar{C}_y \sim_{\mathbb{Q}} \pi^*(C_y) - \frac{4}{5}E, \ \bar{D} \sim_{\mathbb{Q}} \pi^*(D) - \frac{a}{5}E,$$

where a is a non-negative rational number. The curve E contains one singular point Q_3 of type $\frac{1}{3}(1,1)$ and one singular point of Q_4 of type $\frac{1}{4}(1,1)$ on the surface $\bar{X}$. The point Q_3 is contained in $\bar{C}_y$ but not in $\bar{C}_x$. On the other hand, the point Q_4 is contained in $\bar{C}_x$ but not in $\bar{C}_y$. The intersection $\bar{C}_x \cap \bar{C}_y$ consists of a single point that dominates the point P_2.

The log pull back of the log pair $(X, \frac{3}{2}D)$ is the log pair

$$\left(\bar{X}, \frac{3}{2}\bar{D} + \frac{3a-4}{10}E\right).$$

This is not log canonical at some point $Q \in E$. We see that

$$0 \leqslant \bar{C}_x \cdot \bar{D} = C_x \cdot D + \frac{3a}{25}E^2 = \frac{1}{5} - \frac{a}{20},$$

and hence $a \leqslant 4$. In particular,

$$\frac{3a-4}{10} < 1.$$

This implies that the log pull back of the log pair $(X, \frac{3}{2}D)$ is log canonical in a punctured neighborhood of the point Q.

If $a \leqslant \frac{4}{3}$, then the log pair $(\bar{X}, \frac{3}{2}\bar{D})$ is not log canonical at Q as well. We then obtain

$$\frac{a}{12} = \bar{D} \cdot E > \begin{cases} \frac{2}{3} \text{ if } Q \neq Q_3 \text{ and } Q \neq Q_4, \\ \frac{2}{3} \cdot \frac{1}{3} \text{ if } Q = Q_3, \\ \frac{2}{3} \cdot \frac{1}{4} \text{ if } Q = Q_4. \end{cases}$$

In particular, we have $a > 2$. This contradicts the assumption $a \leqslant \frac{4}{3}$. Therefore, $a > \frac{4}{3}$ and the log pull back of the log pair $(X, \frac{3}{2}D)$ is effective. Then

$$\mathrm{mult}_Q(\bar{D}) > \frac{2}{3}\left(1 - \frac{3a-4}{10}\right) = \frac{14-3a}{15}.$$

Since $\bar{D} \cdot E = \frac{a}{12} \leqslant \frac{2}{3}$, Lemma 1.3.8 implies that the point Q cannot be a smooth point. Therefore, the point Q is either Q_3 or Q_4. However, two inequalities

$$\frac{4}{5} - \frac{a}{5} = 4\bar{D} \cdot \bar{C}_x \geqslant \mathrm{mult}_{Q_4}(\bar{D}) > \frac{14-3a}{15},$$
$$\frac{4}{5} - \frac{a}{5} = 3\bar{D} \cdot \bar{C}_y \geqslant \mathrm{mult}_{Q_3}(\bar{D}) > \frac{14-3a}{15}$$

give us a contradiction. □

Lemma 3.2.4. Let X be a quasismooth hypersurface of degree 30 in $\mathbb{P}(3, 4, 10, 15)$. Then $\mathrm{lct}(X) = \frac{3}{2}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 = z^3 - y^5 z - x^{10} + \epsilon_1 x^2 y z^2 + \epsilon_2 x^2 y^6 + \epsilon_3 x^4 y^2 z + \epsilon_4 x^6 y^3,$$

where $\epsilon_i \in \mathbb{C}$. The surface X is singular at the points O_y, $O_2 = [0 : 1 : 1 : 0]$, $O_5 = [0 : 0 : 1 : 1]$, $P_1 = [1 : 0 : 0 : 1]$ and $P_2 = [1 : 0 : 0 : -1]$.

The curves C_x and C_y are irreducible. Moreover, we have

$$\frac{3}{2} = \mathrm{lct}\left(X, \frac{2}{3}C_x\right) < \mathrm{lct}\left(X, \frac{2}{4}C_y\right) = 2.$$

Suppose that $\mathrm{lct}(X) < \frac{3}{2}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{3}{2}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither C_x nor C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(20))$ contains the monomials y^5, y^2x^4, z^2, it follows from Lemma 1.3.9 that the point P is either a singular point of X or a smooth point in C_y. However, the point P cannot belong to C_y since $\frac{2}{3} = 5D \cdot C_y$. Therefore, the point P must be either the point O_y or

O_2. On the other hand, we have $4D\cdot C_x = \frac{2}{5}$. This means that the pair $(X, \frac{3}{2}D)$ is log canonical at the points O_y and O_2. Consequently, $\mathrm{lct}(X) = \frac{3}{2}$. □

Lemma 3.2.5. Let X be a quasismooth hypersurface of degree 57 in $\mathbb{P}(5,13,19,22)$. Then $\mathrm{lct}(X) = \frac{25}{12}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + yt^2 + xy^4 + x^7t + \epsilon x^5yz = 0,$$

where $\epsilon \in \mathbb{C}$. The surface X is singular only at the points O_x, O_y and O_t.

The curves C_x and C_y are irreducible. Moreover, we have

$$\frac{25}{12} = \mathrm{lct}\left(X, \frac{2}{5}C_x\right) < \mathrm{lct}\left(X, \frac{2}{13}C_y\right) = \frac{65}{21}.$$

Suppose that $\mathrm{lct}(X) < \frac{25}{12}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{25}{12}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither C_x nor C_y.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(110))$ contains the monomials x^9y^5, x^{22} and t^5, it follows from Lemma 1.3.9 that the point P is either a singular point of X or a smooth point on C_x. However, this is impossible since $22D\cdot C_x = \frac{6}{13} < \frac{12}{25}$ and $5D\cdot C_y = \frac{3}{11} < \frac{12}{25}$. □

Lemma 3.2.6. Let X be a quasismooth hypersurface of degree 70 in $\mathbb{P}(5,13,19,35)$. Then $\mathrm{lct}(X) = \frac{25}{12}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + yz^3 + xy^5 - x^{14} + \epsilon x^5y^2z = 0,$$

where $\epsilon \in \mathbb{C}$. The surface X is singular at the points O_y and O_z. It is also singular at two points $P_1 = [1:0:0:1]$ and $P_2 = [1:0:0:-1]$.

The curves C_x is irreducible. On the other hand, the curve C_y consists of two smooth curves $C_1 = \{y = x^7 - t = 0\}$ and $C_2 = \{y = x^7 + t = 0\}$. Moreover, we have

$$\frac{25}{12} = \mathrm{lct}\left(X, \frac{2}{5}C_x\right) < \mathrm{lct}\left(X, \frac{2}{13}C_y\right) = \frac{26}{7}.$$

Suppose that $\mathrm{lct}(X) < \frac{25}{12}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{25}{12}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D dose not contains C_x. Also, we may assume that the support of D does not contain either C_1 or C_2.

Since $19D\cdot C_x = \frac{4}{13} < \frac{12}{25}$, the point P cannot belong to C_x.

We put $m_1C_1 + m_2C_2 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains neither C_1 nor C_2. Since the pair $(X, \frac{25}{12}D)$ is log canonical at the point O_z, we see that $m_i \leqslant \frac{12}{25}$. Since

$$5(D - m_iC_i)\cdot C_i = \frac{2 - m_i}{19} < \frac{12}{25}$$

for each i, Lemma 1.3.8 implies that the point P can be neither P_1 nor P_2. Therefore, the point P is a smooth point of X in the outside of C_x. However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(95))$ contains the monomials x^6y^5, x^{19} and z^5, it follows from Lemma 1.3.9 that the point P is either a singular point of X or a smooth point on C_x. This is a contradiction. □

Lemma 3.2.7. Let X be a quasismooth hypersurface of degree 36 in $\mathbb{P}(6,9,10,13)$. Then $\mathrm{lct}(X)=\frac{25}{12}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$zt^2+y^4+xz^3+x^6+\epsilon x^3y^2=0,$$

where ϵ is a constant different from ± 2. The surface X is singular at the points O_z and O_t. It is also singular at two points P_1 and P_2 on L_{zt}. The surface X is also singular at one point Q on L_{yt}.

The curves C_x and C_y are irreducible and reduced. However, the curve C_z consists of two irreducible and reduced curves C_1 and C_2. The curve C_1 contains the point P_1 but not P_2. On the other hand, C_2 contains the point P_2 but not P_1. We also see

$$C_1^2=C_2^2=-\frac{8}{39},\quad C_1\cdot C_2=\frac{6}{13}.$$

It is easy to check

$$\frac{25}{12}=\mathrm{lct}\left(X,\frac{2}{10}C_z\right)<\frac{9}{4}=\mathrm{lct}\left(X,\frac{2}{6}C_x\right)<\frac{9}{2}=\mathrm{lct}\left(X,\frac{2}{9}C_y\right).$$

Suppose that $\mathrm{lct}(X)<\frac{25}{12}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{25}{12}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither C_x nor C_y. In addition, we may assume that it cannot contain either C_1 or C_2.

Since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(30))$ contains the monomials x^2y^2, x^5 and z^3, it follows from Lemma 1.3.9 that $P\in\mathrm{Sing}(X)\cup C_x\cup C_z$. However, $2D\cdot C_y=\frac{12}{65}<\frac{12}{25}$ and hence the point P cannot be the point Q. Note that the curve C_x passes through the point O_z with multiplicity 2. Then the inequality $5D\cdot C_x=\frac{4}{13}<\frac{12}{25}$ shows that the point P cannot be a point on $C_x\setminus\{O_t\}$.

Put $D=mC_1+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $C_1\not\subset\mathrm{Supp}(\Omega)$. If $m\neq 0$, then

$$\frac{2}{39}=D\cdot C_2=\big(mC_1+\Omega\big)\cdot C_2\geqslant mC_1\cdot C_2=\frac{6m}{13},$$

and hence $m\leqslant\frac{1}{9}$. Then

$$3\big(D-mC_1\big)\cdot C_1=\frac{2+8m}{13}\leqslant\frac{12}{25}.$$

Therefore, it follows from Lemma 1.3.8 that the point P cannot be a point on $C_1\setminus\{O_t\}$. By the same method, we can show that the point P cannot be a point on $C_2\setminus\{O_t\}$. Therefore, the point P must be the point O_t.

Let $\pi\colon\bar{X}\to X$ be the weighted blow up at the point O_t with weights $(2,3)$. Let E be the exceptional curve of π and let $\bar{D}$, $\bar{C}_x$ and $\bar{C}_y$ be the proper transforms of D, C_x and C_y, respectively. Then

$$K_{\bar{X}}\sim_{\mathbb{Q}}\pi^*(K_X)-\frac{8}{13}E,\ \bar{C}_x\sim_{\mathbb{Q}}\pi^*(C_x)-\frac{2}{13}E,\ \bar{C}_y\sim_{\mathbb{Q}}\pi^*(C_y)-\frac{3}{13}E,\ \bar{D}\sim_{\mathbb{Q}}\pi^*(D)-\frac{a}{13}E,$$

where a is a non-negative rational number. The curve E contains one singular point Q_3 of type $\frac{1}{3}(1,1)$ and one singular point of Q_2 of type $\frac{1}{2}(1,1)$ on the surface $\bar{X}$. The point Q_2 is contained in $\bar{C}_y$ but not in $\bar{C}_x$. On the other hand, the point Q_3 is contained in $\bar{C}_x$ but not in $\bar{C}_y$.

The log pull back of the log pair $(X, \frac{25}{12}D)$ is the log pair

$$\left(\bar{X}, \frac{25}{12}\bar{D} + \frac{25a+96}{12\cdot 13}E\right).$$

This is not log canonical at some point $Q \in E$. We see that

$$0 \leqslant \bar{C}_y \cdot \bar{D} = C_y \cdot D + \frac{3a}{169}E^2 = \frac{6}{5\cdot 13} - \frac{a}{2\cdot 13},$$

and hence $a \leqslant \frac{12}{5}$. In particular,

$$\frac{25a+96}{12\cdot 13} \leqslant 1.$$

This implies that the log pull back of the log pair $(X, \frac{25}{12}D)$ is log canonical in a punctured neighborhood of the point Q. Then

$$\mathrm{mult}_Q(\bar{D}) > \frac{12}{25}\left(1 - \frac{25a+96}{12\cdot 13}\right) = \frac{12}{5\cdot 13} - \frac{a}{13}.$$

Since $\bar{D}\cdot E = \frac{a}{6} \leqslant \frac{12}{25}$, Lemma 1.3.8 implies that the point Q cannot be a smooth point. Therefore, the point Q is either Q_2 or Q_3. However, two inequalities

$$\frac{12}{5\cdot 13} - \frac{a}{13} = 3\bar{D}\cdot\bar{C}_x \geqslant \mathrm{mult}_{Q_3}(\bar{D}) > \frac{12}{5\cdot 13} - \frac{a}{13},$$
$$\frac{12}{5\cdot 13} - \frac{a}{13} = 2\bar{D}\cdot\bar{C}_y \geqslant \mathrm{mult}_{Q_2}(\bar{D}) > \frac{12}{5\cdot 13} - \frac{a}{13}$$

give us a contradiction. □

Lemma 3.2.8. Let X be a quasismooth hypersurface of degree 57 in $\mathbb{P}(7,8,19,25)$. Then $\mathrm{lct}(X) = \frac{49}{24}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + y^4t + xt^2 + x^7y + \epsilon x^2y^3z = 0,$$

where $\epsilon \in \mathbb{C}$. The surface X is singular at the points O_x, O_y and O_t. The curves C_x, C_y and C_z are irreducible. We have

$$\frac{49}{24} = \mathrm{lct}\left(X, \frac{2}{7}C_x\right) < \mathrm{lct}\left(X, \frac{2}{8}C_y\right) = \frac{10}{3} < \mathrm{lct}\left(X, \frac{2}{19}C_z\right) = \frac{19}{2}.$$

Thus, $\mathrm{lct}(X) \leqslant \frac{49}{24}$.

Suppose that $\mathrm{lct}(X) < \frac{49}{24}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{49}{24}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains none of the curves C_x, C_y and C_z. The curve C_x is singular at the point O_t. Since $\frac{25}{2}D\cdot C_x = \frac{3}{8} < \frac{24}{49}$, $7D\cdot C_y = \frac{6}{25} < \frac{24}{49}$ and $D\cdot C_z = \frac{57}{700} < \frac{24}{49}$, the point P cannot belong to the set $C_x \cup C_y \cup C_z$.

Consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda y^2 z + \mu x^5 = 0$, $[\mu, \lambda] \in \mathbb{P}^1$. Then there is a unique curve Z in the pencil $\mathcal{L}$ passing through the point P. Then the curve Z is defined by an equation of the form $y^2 z - \alpha x^5 = 0$, where α is a non-zero constant.

We see that $C_y \not\subset \mathrm{Supp}(Z)$. But the open subset $Z \setminus C_y$ of the curve Z is a $\mathbb{Z}_8$-quotient of the affine curve

$$z - \alpha x^5 = z^3 + t + xt^2 + x^7 + \epsilon x^2 z = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, z, t\big]\Big)$$

that is isomorphic to the plane affine curve defined by the equation

$$\alpha^3 x^{15} + t + xt^2 + x^7 + \epsilon\alpha x^7 = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, t\big]\Big).$$

This curve is irreducible and hence the curve Z is also irreducible. Thus $\mathrm{mult}_P(Z) \leqslant 14$. We may assume that $\mathrm{Supp}(D)$ does not contain the curve Z by Lemma 1.3.6. Then we obtain an absurd inequality

$$\frac{3}{20} = D \cdot Z \geqslant \mathrm{mult}_P\big(D\big) > \frac{24}{49}.$$

□

Lemma 3.2.9. Let X be a quasismooth hypersurface of degree 64 in $\mathbb{P}(7, 8, 19, 32)$. Then $\mathrm{lct}(X) = \frac{35}{16}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 - y^8 + xz^3 + x^8 y + \epsilon x^3 y^3 z,$$

where $\epsilon \in \mathbb{C}$. Note that X is singular at the points O_x and O_z. The surface X also has two singular points $P_1 = [0:1:0:1]$ and $P_2 = [0:1:0:-1]$ of type $\frac{1}{8}(7,3)$.

The curve C_x is reducible. We have $C_x = C_1 + C_2$, where C_1 and C_2 are irreducible and reduced curves. The curve C_1 contains the point P_1 but not the point P_2. On the other hand, the curve C_2 contains the point P_2 but not the point P_1. However, these two curves meet each other only at the point O_z. We also have

$$C_1^2 = C_2^2 = -\frac{25}{8 \cdot 19}, \quad C_1 \cdot C_2 = \frac{4}{19}.$$

The curve C_y is irreducible. It is easy to check

$$\mathrm{lct}\left(X, \frac{2}{7}C_x\right) = \frac{35}{16} < \mathrm{lct}\left(X, \frac{2}{8}C_y\right) = \frac{10}{3}.$$

Therefore, $\mathrm{lct}(X) \leqslant \frac{35}{16}$.

Suppose that $\mathrm{lct}(X) < \frac{35}{16}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{35}{16}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of D does not contain the curve C_y. Moreover, we may assume that the support of D does not contain either the curve C_1 or the curve C_2.

Since $C_i \not\subset \mathrm{Supp}(D)$ for either $i = 1$ or 2, we have

$$\mathrm{mult}_{O_z}(D) \leqslant 19 D \cdot C_i = \frac{1}{4} < \frac{16}{35},$$

and hence $P \neq O_z$. Meanwhile, the inequality $7D \cdot C_y = \frac{4}{19} < \frac{16}{25}$ implies that the point P cannot belong to C_y.

Suppose that $P \in C_1$. Then we write $D = mC_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $C_1 \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{1}{4 \cdot 19} = D \cdot C_2 = \big(mC_1 + \Omega\big) \cdot C_2 \geqslant mC_1 \cdot C_2 = \frac{4m}{19},$$

and hence $m \leqslant \frac{1}{16}$. Then it follows from Lemma 1.3.8 that

$$\frac{2 + 25m}{8 \cdot 19} = \big(D - mC_1\big) \cdot C_1 = \Omega \cdot C_1 > \begin{cases} \frac{16}{35} \text{ if } P \neq P_1, \\ \frac{16}{35} \cdot \frac{1}{8} \text{ if } P = P_1. \end{cases}$$

This is impossible since $m \leqslant \frac{1}{16}$. Thus, $P \notin C_1$. Similarly, we can show that $P \notin C_2$.

Consequently, the point P is located in the outside of $C_x \cup C_y$. In particular, it is a smooth point of X. But $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(64))$ contains monomials y^8, x^8y, y^4t and t^2. This is impossible by Lemma 1.3.9. The obtained contradiction completes the proof. $\square$

Lemma 3.2.10. Let X be a quasismooth hypersurface of degree 48 in $\mathbb{P}(9, 12, 13, 16)$. Then $\mathrm{lct}(X) = \frac{63}{24}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^3 - y^4 + xz^3 + x^4y = 0.$$

The surface X is singular at the points O_x, O_z, $Q_4 = [0 : 1 : 0 : 1]$ and $Q_3 = [1 : 1 : 0 : 0]$.

The curves C_x, C_y, C_z and C_t are irreducible and reduced. We have

$$\frac{63}{24} = \mathrm{lct}\left(X, \frac{2}{9}C_x\right) < \mathrm{lct}\left(X, \frac{2}{12}C_y\right) = 4 < \mathrm{lct}\left(X, \frac{2}{13}C_z\right) = \frac{13}{2} < \mathrm{lct}\left(X, \frac{2}{16}C_t\right) = \frac{16}{2}.$$

Therefore, $\mathrm{lct}(X) \leqslant \frac{63}{24}$.

Suppose that $\mathrm{lct}(X) < \frac{63}{24}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{63}{24}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains none of the curves C_x, C_y, C_z and C_t.

Note that the curve C_x is singular at O_z with multiplicity 3 and the curve C_y is singular at O_x with multiplicity 3. Then the inequalities

$$\frac{13}{3}D \cdot C_x = \frac{1}{6} < \frac{24}{63}, \quad \frac{9}{3}D \cdot C_y = \frac{2}{13} < \frac{24}{63}, \quad 3D \cdot C_z = \frac{1}{6} < \frac{24}{63}, \quad D \cdot C_t = \frac{8}{9 \cdot 13} < \frac{24}{63}$$

show that the point P must be located in the outside of $C_x \cup C_y \cup C_z \cup C_t$.

Consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda xt + \mu yz = 0$, $[\mu, \lambda] \in \mathbb{P}^1$. Then there is a unique curve Z in the pencil $\mathcal{L}$ passing through the point P. Then the curve Z is defined by an equation of the form $xt - \alpha yz = 0$, where α is a non-zero constant. We see that $C_x \not\subset \mathrm{Supp}(Z)$. But the open subset $Z \setminus C_x$ of the curve Z is a $\mathbb{Z}_9$-quotient of the affine curve

$$t - \alpha yz = t^3 + y^4 + z^3 + y = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big),$$

which is isomorphic to the plane affine curve given by the equation

$$\alpha^3y^3z^3+y^4+z^3+y=0\subset\mathbb{C}^2\cong\mathrm{Spec}\Big(\mathbb{C}\big[y,z\big]\Big).$$

Then, it is easy to see that the curve Z is irreducible and $\mathrm{mult}_P(Z)\leqslant 4$. Thus, we may assume that $\mathrm{Supp}(D)$ does not contain the curve Z by Lemma 1.3.6. However,

$$\frac{25}{18\cdot 13}=D\cdot Z\geqslant \mathrm{mult}_P\big(D\big)>\frac{24}{63}.$$

Consequently, $\mathrm{lct}(X)=\frac{63}{24}$. □

Lemma 3.2.11. Let X be a quasismooth hypersurface of degree 57 in $\mathbb{P}(9,12,19,19)$. Then $\mathrm{lct}(X)=3$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$zt(z-t)-xy^4+x^5y=0.$$

The surface X is singular at three distinct points O_x, O_y, $Q_3=[1:1:0:0]$. Also, it is singular at three distinct points O_z, O_t, $Q_{19}=[0:0:1:1]$.

The curve C_x consists of three distinct curves L_{xz}, L_{xt} and $R_x=\{x=z-t=0\}$ that intersect altogether at the point O_y. Similarly, the curve C_y consists of three curves L_{yz}, L_{yt} and $R_y=\{y=z-t=0\}$ that intersect altogether at the point O_x. The curve C_z consists of three distinct curves L_{xz}, L_{yz} and $R_z=\{z=x^4-y^3=0\}$ that intersect altogether at the point O_t. The curve C_t consists of three distinct curves L_{xt}, L_{yt} and $R_t=\{t=x^4-y^3=0\}$ that intersect altogether at the point O_z. Let C_{z-t} be the curve cut out on X by the equation $z=t$. Then C_{z-t} consists of three distinct curves R_x, R_y and $R_{z-t}=\{z-t=x^4-y^3=0\}$ that intersect altogether at the point Q_{19}.

We have the following intersection numbers:

$$L_{xz}^2=L_{xt}^2=R_x^2=-\frac{29}{19\cdot 12},\quad L_{yz}^2=L_{yt}^2=R_y^2=-\frac{26}{19\cdot 9},\quad R_z^2=R_t^2=R_{z-t}^2=-\frac{2}{19\cdot 3}$$

$$-K_X\cdot L_{xz}=-K_X\cdot L_{xt}=-K_X\cdot R_x=\frac{1}{19\cdot 6},\quad -K_X\cdot L_{yz}=-K_X\cdot L_{yt}=-K_X\cdot R_y=\frac{2}{19\cdot 9},$$

$$-K_X\cdot R_z=-K_X\cdot R_t=-K_X\cdot R_{z-t}=\frac{2}{19\cdot 3}.$$

Since $\mathrm{lct}(X,\frac{2}{9}C_x)=3$, we have $\mathrm{lct}(X)\leqslant 3$. Suppose that $\mathrm{lct}(X)<3$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,3D)$ is not log canonical at some point $P\in X$.

The pairs $(X,\frac{6}{9}C_x)$ and $(X,\frac{6}{12}C_y)$ are log canonical. By Lemma 1.3.6, we may assume that the support of D does not contain at least one component of C_x. Then one of the inequalities

$$\mathrm{mult}_{O_y}(D)\leqslant 12D\cdot L_{xz}=\frac{6}{57}<\frac{1}{3},$$

$$\mathrm{mult}_{O_y}(D)\leqslant 12D\cdot L_{xt}=\frac{6}{57}<\frac{1}{3},$$

$$\mathrm{mult}_{O_y}(D)\leqslant 12D\cdot R_x=\frac{6}{57}<\frac{1}{3}$$

must hold, and hence the point P cannot be the point O_y. Also, we may assume that the support of D does not contain at least one component of C_y. By the same reason, the point P cannot be the point O_x.

We have

$$\mathrm{lct}\left(X, \frac{2}{19}C_z\right) = \mathrm{lct}\left(X, \frac{2}{19}C_t\right) = \mathrm{lct}\left(X, \frac{2}{19}C_1\right) = \frac{7}{2}.$$

By Lemma 1.3.6, we may assume that the support of D does not contain at least one component of each curve C_z, C_t and C_{z-t}. Since the curve R_z is singular at the point O_t with multiplicity 3, Then one of the inequalities

$$\mathrm{mult}_{O_t}(D) \leqslant 19D \cdot L_{xz} = \frac{1}{6} < \frac{1}{3},$$
$$\mathrm{mult}_{O_t}(D) \leqslant 19D \cdot L_{yz} = \frac{2}{9} < \frac{1}{3},$$
$$\mathrm{mult}_{O_t}(D) \leqslant \frac{19}{3}D \cdot R_z = \frac{2}{9} < \frac{1}{3}$$

must hold, and hence the point P cannot be the point O_t. By applying the same method to C_t and C_{z-t}, we see that the point P can neither O_z not Q_{19}.

The three curves R_z, R_t, and R_{z-t} intersects only at the point Q_3. The log pair

$$\left(X,\ \frac{3}{18}\Big(R_z + R_t + R_{z-t}\Big)\right)$$

is log canonical at Q_3, and $R_z + R_t + R_{z-t} \sim -18K_X$. By Lemma 1.3.6, we may assume that the support of D does not contain at least one curve among R_z, R_t and R_{z-t}. Without loss of generality, we may assume that the support of D does not contain the curve R_z. Then

$$\mathrm{mult}_{Q_3}(D) \leqslant 3D \cdot R_z = \frac{2}{19} < \frac{1}{3},$$

and hence the point P cannot be Q_3.

Write $D = m_1L_{xz} + m_2L_{yz} + m_3R_z + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains none of the curves L_{xz}, L_{yz}, R_z. Since the pair $(X, 3D)$ is log canonical at the point O_t, we have $m_i \leqslant \frac{1}{3}$ for each $i = 1, 2, 3$. By Lemma 1.3.8, the inequalities

$$(D - m_1L_{xz}) \cdot L_{xz} = \frac{2 + 29m_1}{12 \cdot 19} < \frac{1}{3},$$
$$(D - m_2L_{yz}) \cdot L_{yz} = \frac{2 + 26m_2}{9 \cdot 19} < \frac{1}{3},$$
$$(D - m_3R_z) \cdot R_z = \frac{2 + 2m_3}{3 \cdot 19} < \frac{1}{3}$$

show that the point P cannot belong to C_z. By the same way, we can show that the point P is not contained in $C_t \cup C_{z-t}$. Therefore, the point P is a smooth point of X in the outside of the set $C_z \cup C_t \cup C_{z-t}$. Then there is a unique quasismooth irreducible curve $E \subset X$ passing through the point P and defined by the equation $z = \lambda t$, where λ is a non-zero constant different from 1. By Lemma 1.3.6, we may assume that the support of D does not contain the curve E. Then

$$\frac{1}{3} < \mathrm{mult}_P(D) \leqslant D \cdot E = \frac{1}{18}.$$

This is a contradiction. □

Lemma 3.2.12. Let X be a quasismooth hypersurface of degree 81 in $\mathbb{P}(9,19,24,31)$. Then $\mathrm{lct}(X)=3$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$yt^2+y^3z+xz^3-x^9=0.$$

It is singular at the point O_y, O_z and O_t. The surface X is also singular at the point $Q_3=[1:0:1:0]$.

The curve C_x (resp. C_y) consists of two irreducible curves L_{xy} and $R_x=\{x=t^2+y^2z=0\}$ (resp. $R_y=\{y=z^3-x^8=0\}$). The curve L_{xy} intersects R_x (resp. R_y) only at the point O_z (resp. O_t). We have the following intersection numbers:

$$-K_X\cdot L_{xy}=\frac{1}{12\cdot 31},\quad -K_X\cdot R_x=\frac{1}{6\cdot 19},\quad -K_X\cdot R_y=\frac{2}{3\cdot 31},\quad L_{xy}\cdot R_x=\frac{1}{12},$$
$$L_{xy}\cdot R_y=\frac{3}{31},\quad L_{xy}^2=-\frac{53}{24\cdot 31},\quad R_x^2=-\frac{5}{6\cdot 19},\quad R_y^2=\frac{10}{3\cdot 31}.$$

Meanwhile, the curve C_z is irreducible. We see that $\mathrm{lct}(X)\leqslant 3$ since

$$3=\mathrm{lct}\left(X,\frac{2}{9}C_x\right)<\mathrm{lct}\left(X,\frac{2}{19}C_y\right)=\frac{209}{54}<\mathrm{lct}\left(X,\frac{2}{24}C_z\right)=\frac{22}{3}.$$

Suppose that $\mathrm{lct}(X)<3$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,3D)$ is not log canonical at some point P. We may assume that the support of D does not contain at least one component of each of C_x and C_y by Lemma 1.3.6. One of the inequalities

$$\mathrm{mult}_{O_z}D\leqslant 24D\cdot L_{xy}=\frac{2}{31}<\frac{1}{3},\quad \mathrm{mult}_{O_z}D\leqslant 24D\cdot R_x=\frac{4}{19}<\frac{1}{3}$$

must hold, and hence the point P cannot be the point O_z. Since the curve R_y is singular at the point O_t with multiplicity 3, one of the inequalities

$$\mathrm{mult}_{O_t}D\leqslant 31D\cdot L_{xy}=\frac{1}{12}<\frac{1}{3},\quad \mathrm{mult}_{O_t}D\leqslant \frac{31}{3}D\cdot R_y=\frac{2}{9}<\frac{1}{3}$$

must hold, and hence the point P cannot be the point O_t.

By Lemma 1.3.6, we may also assume that the curve C_z is not contained in the support of D. The curve C_z is singular at the point O_y. Then the inequality

$$\frac{19}{2}D\cdot C_z=\frac{9}{31}<\frac{1}{3}$$

shows that the point P cannot be the point O_y.

Write $D=m_0L_{xy}+m_1R_x+m_2R_y+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of L_{xy}, R_x, R_y. If $m_0\neq 0$, then we obtain

$$\frac{1}{6\cdot 19}=D\cdot R_x\geqslant m_0L_{xy}\cdot R_x=\frac{m_0}{12},$$

and hence $m_0\leqslant\frac{2}{19}$. Similarly, we see that $m_1\leqslant\frac{1}{31}$ and $m_2\leqslant\frac{1}{36}$. Since we have

$$(D-m_0L_{xy})\cdot L_{xy}=\frac{2+53m_0}{24\cdot 31}<\frac{1}{3},$$

$$(D - m_1 R_x) \cdot R_x = \frac{1 + 5m_1}{6 \cdot 19} < \frac{1}{3},$$
$$3(D - m_2 R_y) \cdot R_y = \frac{2 - 10m_2}{31} < \frac{1}{3},$$

it follows from Lemma 1.3.8 that the point P is located in the outside of C_x and C_y. Therefore, the point P is a smooth point in the outside of C_x and C_y. However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(171))$ contains the monomials y^9, x^{19}, x^3z^6 and $x^{11}z^3$, it follows from Lemma 1.3.9 that the point P must be either a singular point of X or a point in $C_x \cup C_y$. This is a contradiction. □

Lemma 3.2.13. Let X be a quasismooth hypersurface of degree 105 in $\mathbb{P}(10, 19, 35, 43)$. Then $\mathrm{lct}(X) = \frac{57}{14}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + yt^2 + xy^5 - x^7 z = 0.$$

The surface X is singular at the points O_x, O_y, O_t and $Q_5 = [1 : 0 : 1 : 0]$.

The curve C_x is irreducible. However, the curve C_y consists of two irreducible curves L_{yz} and $R_y = \{y = z^2 - x^7 = 0\}$. The curve L_{yz} intersects R_y at the point O_t. We have

$$L_{yz}^2 = -\frac{51}{10 \cdot 43},\ R_y^2 = -\frac{16}{5 \cdot 43},\ L_{yz} \cdot R_y = \frac{7}{43}.$$

We also have $\mathrm{lct}(X) \leqslant \frac{57}{14}$ since

$$\frac{57}{14} = \mathrm{lct}\left(X, \frac{2}{19}C_y\right) < \mathrm{lct}\left(X, \frac{2}{10}C_x\right) = \frac{25}{6}.$$

Suppose that $\mathrm{lct}(X) < \frac{57}{14}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{57}{14}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D does not contain the curve C_x. Similarly, we may assume that the support of the divisor D does not contain either L_{yz} or R_y.

Since the support of the divisor D does not contain either L_{yz} or R_y and the curve R_y is singular at the point O_t, one of the inequalities

$$\mathrm{mult}_{O_t}(D) \leqslant 43 D \cdot L_{yz} = \frac{1}{5} < \frac{14}{57},\ \ \mathrm{mult}_{O_t}(D) \leqslant \frac{43}{2} D \cdot R_y = \frac{1}{5} < \frac{14}{57}$$

must hold, and hence the point P cannot be O_t.

We write $D = m_0 L_{yz} + m_1 R_y + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains neither L_{yz} nor R_y. If $m_0 \neq 0$, then $m_1 = 0$ and hence

$$\frac{2}{5 \cdot 43} = D \cdot R_y \geqslant m_0 L_{yz} \cdot R_y = \frac{7m_0}{43}.$$

Therefore, $m_0 \leqslant \frac{2}{35}$. Similarly, we have $m_1 \leqslant \frac{1}{35}$. Since

$$10(D - m_0 L_{yz}) \cdot L_{yz} = \frac{2 + 51m_0}{43} < \frac{14}{57},$$
$$5(D - m_1 R_y) \cdot R_y = \frac{2 + 16m_1}{43} < \frac{14}{57},$$

it follows from Lemma 1.3.8 that the point P is located in the outside of C_y.

Since the divisor D does not contain the curve C_x, $\mathrm{mult}_{O_y}(D) \leqslant 19D\cdot C_x = \frac{6}{43} < \frac{14}{57}$, and hence the point P cannot belong to the curve C_x. Therefore, the point P is a smooth point in the outside of $C_x \cup C_y$. However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(190))$ contains x^{19}, y^{10}, x^5z^4 and $x^{12}z^2$, it follows from Lemma 1.3.9 that the point P must be either a singular point of X or a point in $C_x \cup C_y$. This is a contradiction. □

Lemma 3.2.14. Let X be a quasismooth hypersurface of degree 105 in $\mathbb{P}(11, 21, 28, 47)$. Then $\mathrm{lct}(X) = \frac{77}{30}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$yz^3 - y^5 + xt^2 + x^7z = 0.$$

The surface X is singular at the point O_x, O_z, O_t and $Q_7 = [0 : 1 : 1 : 0]$.

The curve C_x (resp. C_y) consists of two irreducible curves L_{xy} and $R_x = \{x = z^3 - y^4 = 0\}$ (resp. $R_y = \{y = t^2 + x^6z = 0\}$. The curve L_{xy} intersects R_x (resp. R_y) only at the point O_t (resp. O_z). We have the following intersection numbers:

$$-K_X\cdot L_{xy} = \frac{1}{14\cdot 47},\quad -K_X\cdot R_x = \frac{2}{7\cdot 47},\quad -K_X\cdot R_y = \frac{1}{7\cdot 11},\quad L_{xy}\cdot R_x = \frac{3}{47},$$
$$L_{xy}\cdot R_y = \frac{1}{14},\quad L_{xy}^2 = -\frac{73}{28\cdot 47},\quad R_x^2 = -\frac{10}{7\cdot 47},\quad R_y^2 = \frac{5}{7\cdot 11}.$$

We see that $\mathrm{lct}(X) \leqslant \frac{77}{30}$ since

$$\frac{77}{30} = \mathrm{lct}\left(X, \frac{2}{11}C_x\right) < \mathrm{lct}\left(X, \frac{2}{21}C_y\right) = 6.$$

Suppose that $\mathrm{lct}(X) < \frac{77}{30}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{77}{30}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of D does not contain at least one component of each of C_x and C_y. Note that the curve R_x is singular at the point O_t with multiplicity 3 and the curve R_y is singular at the point O_z. Then one of two inequalities

$$\mathrm{mult}_{O_t}(D) \leqslant 47D\cdot L_{xy} = \frac{1}{14} < \frac{30}{77},\quad \mathrm{mult}_{O_t}(D) \leqslant \frac{47}{3}D\cdot R_x = \frac{2}{21} < \frac{30}{77}$$

must hold, and hence the point P cannot be O_t. Applying the same method to C_y, we show that the point P cannot be the point O_z.

Write $D = m_0L_{xy} + m_1R_x + m_2R_y + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of L_{xy}, R_x, R_y. If $m_0 \neq 0$, then we obtain

$$\frac{2}{7\cdot 47} = D\cdot R_x \geqslant m_0L_{xy}\cdot R_x = \frac{3m_0}{47},$$

and hence $m_0 \leqslant \frac{2}{21}$. Similarly, we see that $m_1 \leqslant \frac{1}{42}$ and $m_2 \leqslant \frac{1}{47}$. Since we have

$$(D - m_0L_{xy})\cdot L_{xy} = \frac{2+73m_0}{28\cdot 47} < \frac{30}{77},$$
$$7(D - m_1R_x)\cdot R_x = \frac{2+10m_1}{47} < \frac{30}{77},$$
$$11(D - m_2R_y)\cdot R_y = \frac{1-5m_2}{7} < \frac{30}{77},$$

it follows from Lemma 1.3.8 that the point P is located in the outside of C_x and C_y. Therefore, the point P is a smooth point in the outside of C_x. However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(517))$ contains x^5y^{22}, $x^{26}y^{11}$, x^{47}, $x^{19}z^{11}$, x^{47}, t^{11}, it follows from Lemma 1.3.9 that the point P must be either a singular point of X or a point in C_x. This is a contradiction. □

Lemma 3.2.15. Let X be a quasismooth hypersurface of degree 107 in $\mathbb{P}(11, 25, 32, 41)$. Then $\mathrm{lct}(X) = \frac{11}{3}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$yt^2 + y^3z + xz^3 + x^6t = 0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xy} (resp. L_{xy}, L_{zt}, L_{zt}) and $R_x = \{x = t^2 + y^2z = 0\}$ (resp. $R_y = \{y = z^3 + x^5t = 0\}$, $R_z = \{z = x^6 + yt = 0\}$, $R_t = \{t = y^3 + xz^2 = 0\}$). Also, we see that

$$L_{xy} \cap R_x = \{O_z\},\ L_{xy} \cap R_y = \{O_t\},\ L_{zt} \cap R_z = \{O_y\},\ L_{zt} \cap R_t = \{O_x\}.$$

We have the following intersection numbers:

$$-K_X \cdot L_{xy} = \frac{1}{16 \cdot 41},\quad -K_X \cdot L_{zt} = \frac{2}{11 \cdot 25},\quad -K_X \cdot R_x = \frac{1}{8 \cdot 25},\quad -K_X \cdot R_y = \frac{6}{11 \cdot 41},$$

$$-K_X \cdot R_z = \frac{12}{25 \cdot 41},\quad -K_X \cdot R_t = \frac{3}{11 \cdot 16},\quad L_{xy} \cdot R_x = \frac{1}{16},\ L_{xy} \cdot R_y = \frac{3}{41},\quad L_{zt} \cdot R_z = \frac{6}{25}$$

$$L_{xy}^2 = -\frac{71}{32 \cdot 41},\quad L_{zt}^2 = -\frac{34}{11 \cdot 25},\quad R_x^2 = -\frac{7}{8 \cdot 25},\quad R_y^2 = \frac{42}{11 \cdot 41}$$

We see $\mathrm{lct}(X) \leqslant \frac{11}{3}$ since

$$\frac{11}{3} = \mathrm{lct}\left(X, \frac{2}{11}C_x\right) < \frac{50}{9} = \mathrm{lct}\left(X, \frac{2}{25}C_y\right) < \frac{28}{3} = \mathrm{lct}\left(X, \frac{2}{32}C_z\right) < \frac{205}{18} = \mathrm{lct}\left(X, \frac{2}{41}C_t\right).$$

Suppose that $\mathrm{lct}(X) < \frac{11}{3}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{11}{3}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that either $\mathrm{Supp}(D)$ does not contain at least one irreducible component of each of C_x, C_y, C_z and C_t. Since the curve R_y is singular at the point O_t with multiplicity 3, one of the inequalities

$$\mathrm{mult}_{O_t}(D) \leqslant 41D \cdot L_{xy} = \frac{1}{16} < \frac{3}{11},\quad \mathrm{mult}_{O_t}(D) \leqslant \frac{41}{3}D \cdot R_y = \frac{2}{11} < \frac{3}{11}$$

must hold, and hence the point P cannot be the point O_t. Applying the same method to each of C_x and C_t, we can show that the point P can be neither O_z nor O_x.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(352))$ contains the monomials x^7y^{11}, x^{32} and z^{11}, it follows from Lemma 1.3.9 that the point P is either the point O_t or a smooth point on C_x.

Write $D = m_0L_{xy} + m_1R_x + m_2L_{zt} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of L_{xy}, R_x, R_z. If $m_0 \neq 0$, then $m_1 = 0$ and hence we obtain

$$\frac{1}{8 \cdot 25} = D \cdot R_x \geqslant m_0L_{xy} \cdot R_x = \frac{m_0}{16}.$$

Therefore, $m_0 \leqslant \frac{2}{25}$. Similarly, we get $m_1 \leqslant \frac{1}{41}$. Since we have

$$(D - m_0 L_{xy}) \cdot L_{xy} = \frac{2 + 71m_0}{32 \cdot 41} < \frac{3}{11},$$

$$(D - m_1 R_x) \cdot R_x = \frac{1 + 7m_1}{8 \cdot 25} < \frac{3}{11}$$

it follows from Lemma 1.3.8 that the point P must be the point O_y.

Suppose that $m_2 = 0$. Then the inequality

$$\mathrm{mult}_{O_y}(D) \leqslant 25 D \cdot L_{zt} = \frac{2}{11} < \frac{3}{11}$$

gives us a contradiction. Therefore, $m_2 \neq 0$ and hence the curve R_z is not contained in the support of D. Then

$$\frac{12}{25 \cdot 41} = D \cdot R_z \geqslant m_2 L_{zt} \cdot R_z + \frac{\mathrm{mult}_{O_y}(D) - m_2}{25} > \frac{5m_2}{25} + \frac{3}{11 \cdot 25},$$

and hence $m_2 < \frac{9}{5 \cdot 11 \cdot 41}$. Since

$$25(D - m_2 L_{zt}) \cdot L_{zt} = \frac{2 + 34m_2}{11} < \frac{3}{11}$$

the pair $(X, \frac{11}{3}D)$ is log canonical at the point O_y by Lemma 1.3.8. This is a contradiction. □

Lemma 3.2.16. Let X be a quasismooth hypersurface of degree 111 in $\mathbb{P}(11, 25, 34, 43)$. Then $\mathrm{lct}(X) = \frac{33}{8}$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2 y + t z^2 + x y^4 + x^7 z = 0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xt} (resp. L_{yz}, L_{yz}, L_{xt}) and $R_x = \{x = yt + z^2 = 0\}$ (resp. $R_y = \{y = zt + x^7 = 0\}$, $R_z = \{z = xy^3 + t^2 = 0\}$, $R_t = \{t = y^4 + x^6 z = 0\}$). Also, we see that

$$L_{xt} \cap R_x = \{O_y\},\ L_{yz} \cap R_y = \{O_t\},\ L_{yz} \cap R_z = \{O_x\},\ L_{xt} \cap R_t = \{O_z\}.$$

The intersection numbers among the divisors D, L_{xt}, L_{yz}, R_x, R_y, R_z, R_t are as follows:

$$-K_X \cdot L_{xt} = \frac{1}{17 \cdot 25}, \quad -K_X \cdot R_x = \frac{4}{25 \cdot 43}, \quad -K_X \cdot R_y = \frac{7}{17 \cdot 43},$$

$$-K_X \cdot L_{yz} = \frac{2}{11 \cdot 43}, \quad -K_X \cdot R_z = \frac{4}{11 \cdot 25}, \quad -K_X \cdot R_t = \frac{4}{11 \cdot 17},$$

$$L_{xt} \cdot R_x = \frac{2}{25}, \quad L_{yz} \cdot R_y = \frac{7}{43}, \quad L_{yz} \cdot R_z = \frac{2}{11}, \quad L_{xt} \cdot R_t = \frac{2}{17},$$

$$L_{xt}^2 = -\frac{57}{34 \cdot 25}, \quad R_x^2 = -\frac{64}{25 \cdot 43}, \quad R_y^2 = -\frac{63}{34 \cdot 43},$$

$$L_{yz}^2 = -\frac{52}{11 \cdot 43}, \quad R_z^2 = \frac{18}{11 \cdot 25}, \quad R_t^2 = \frac{64}{11 \cdot 17}.$$

We can easily see that $\mathrm{lct}(X, \frac{2}{11}C_x) = \frac{33}{8}$ is less than each of the numbers $\mathrm{lct}(X, \frac{2}{25}C_y)$, $\mathrm{lct}(X, \frac{2}{34}C_z)$ and $\mathrm{lct}(X, \frac{2}{43}C_t)$. Therefore, $\mathrm{lct}(X) \leqslant \frac{33}{8}$.

Suppose that $\mathrm{lct}(X) < \frac{33}{8}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{33}{8}D)$ is not log canonical at some point $P \in X$.

By Lemma 1.3.6 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. The inequalities

$$25D \cdot L_{xt} = \frac{1}{17} < \frac{8}{33}, \quad 25D \cdot R_x = \frac{4}{43} < \frac{8}{33}$$

imply that $P \neq O_y$. The inequalities

$$11D \cdot L_{yz} = \frac{2}{43} < \frac{8}{33}, \quad 11D \cdot R_z = \frac{4}{25} < \frac{8}{33}$$

imply that $P \neq O_x$. Since the curve R_t is singular at the point O_z, the inequalities

$$34D \cdot L_{xt} = \frac{34}{17 \cdot 25} < \frac{8}{33}, \quad \frac{34}{4}D \cdot R_t = \frac{2}{11} < \frac{8}{33}$$

imply that $P \neq O_z$.

We write $D = a_1L_{xt} + a_2L_{yz} + a_3R_x + a_4R_y + a_5R_z + a_6R_t + \Omega$, where Ω is an effective divisor whose support contains none of the curves L_{xt}, L_{yz}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{33}{8}D)$ is log canonical at the points O_x, O_y, O_z, the numbers a_i are at most $\frac{8}{33}$. Then by Lemma 1.3.8 the following inequalities enable us to conclude that either the point P is in the outside of $C_x \cup C_y \cup C_z \cup C_t$ or $P = O_t$:

$$\frac{33}{8}D \cdot L_{xt} - L_{xt}^2 = \frac{261}{8 \cdot 17 \cdot 25} < 1, \quad \frac{33}{8}D \cdot L_{yz} - L_{xt}^2 = \frac{241}{4 \cdot 11 \cdot 43} < 1,$$

$$\frac{33}{8}D \cdot R_x - R_x^2 = \frac{161}{2 \cdot 25 \cdot 43} < 1, \quad \frac{33}{8}D \cdot R_y - R_y^2 = \frac{483}{4 \cdot 34 \cdot 43} < 1,$$

$$\frac{33}{8}D \cdot R_z - R_z^2 \leqslant \frac{33}{8}D \cdot R_z = \frac{3}{2 \cdot 25} < 1, \quad \frac{33}{8}D \cdot R_t - R_t^2 \leqslant \frac{33}{8}D \cdot R_t = \frac{3}{34} < 1.$$

Suppose that $P \neq O_t$. Then we consider the pencil $\mathcal{L}$ defined by $\lambda yt + \mu z^2 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. The base locus of the pencil consists of the curve L_{yz} and the point O_y. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P \notin C_x \cup C_y \cup C_z \cup C_t$, the divisor E is defined by the equation $z^2 = \alpha yt$, where $\alpha \neq 0$.

Suppose that $\alpha \neq -1$. Then the curve E is isomorphic to the curve defined by the equations $yt = z^2$ and $t^2y + xy^4 + x^7z = 0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t \neq 0$ is the curve given by $z(z^3 + xz^7 + x^7) = 0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{yz} and C. We have

$$D \cdot C = D \cdot E - D \cdot L_{yz} = \frac{394}{11 \cdot 25 \cdot 43}.$$

Also, we see

$$C^2 = E \cdot C - C \cdot L_{yz} \geqslant E \cdot C - (L_{yz} + R_y) \cdot C = \frac{43}{2}D \cdot C > 0.$$

By Lemma 1.3.8 the inequality $D \cdot C < \frac{8}{33}$ gives us a contradiction.

Suppose that $\alpha = -1$. Then divisor E consists of three irreducible and reduced curves L_{yz}, R_x, and M. Note that the curve M is different from the curves R_y and L_{xt}. Also, it is smooth at the point P. We have

$$D \cdot M = D \cdot E - D \cdot L_{yz} - D \cdot R_x = \frac{14}{11 \cdot 43},$$

$$M^2 = E \cdot M - L_{yz} \cdot M - R_x \cdot M \geqslant E \cdot M - C_y \cdot M - C_x \cdot M > 0.$$

By Lemma 1.3.8 the inequality $D \cdot M < \frac{8}{33}$ gives us a contradiction. Therefore, $P = O_t$.

Put $D = bR_x + \Delta$, where Δ is an effective divisor whose support does not contain R_x. By Lemma 1.3.6, we may assume that $R_x \not\subseteq \mathrm{Supp}(\Delta)$ if $b > 0$. Thus, if $b > 0$, then

$$\frac{2}{25 \cdot 34} = D \cdot L_{xt} \geqslant bR_x \cdot L_{xt} = \frac{2b}{25},$$

and hence $b \leqslant \frac{1}{34}$. On the other hand, it follows from Lemma 1.3.8 that

$$\frac{4 + 64b}{25 \cdot 43} = \Delta \cdot R_x > \frac{8}{33 \cdot 43}.$$

Therefore, $b > \frac{17}{528}$. Since $\frac{17}{528} > \frac{1}{34}$, this is a contradiction. □

Lemma 3.2.17. Let X be a quasismooth hypersurface of degree 226 in $\mathbb{P}(11, 43, 61, 113)$. Then $\mathrm{lct}(X) = \frac{55}{12}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + yz^3 + xy^5 + x^{15}z = 0.$$

The surface X is singular at the points O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{55}{12} = \mathrm{lct}\left(X, \frac{2}{11}C_x\right) < \mathrm{lct}\left(X, \frac{2}{43}C_y\right) = \frac{17 \cdot 43}{60}.$$

Therefore, $\mathrm{lct}(X) \leqslant \frac{55}{12}$.

Suppose that $\mathrm{lct}(X) < \frac{55}{12}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{55}{12}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither C_x nor C_y. Then the inequalities

$$61D \cdot C_x = \frac{4}{43} < \frac{12}{55}, \quad 11D \cdot C_y = \frac{4}{61} < \frac{12}{55}$$

show that the point P must be a smooth point of X in the outside of C_x. However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(671))$ contains the monomials $x^{18}y^{11}$, x^{61} and z^{11}, it follows from Lemma 1.3.9 that the point P is either a singular point of X or a point on C_x. This is a contradiction. □

Lemma 3.2.18. Let X be a quasismooth hypersurface of degree 135 in $\mathbb{P}(13, 18, 45, 61)$. Then $\mathrm{lct}(X) = \frac{91}{30}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 - y^5z + xt^2 + x^9y = 0.$$

The surface X is singular at the points O_x, O_y, O_t, $Q_9 = [0 : 1 : 1 : 0]$.

The curve C_x consists of two irreducible and reduced curves L_{xz} and $R_x = \{x = z^2 - y^5 = 0\}$. The curve L_{xz} intersects R_x at the point O_t. It is easy to check

$$L_{xz}^2 = -\frac{77}{18 \cdot 61},\ R_x^2 = -\frac{32}{9 \cdot 61},\ L_{xz} \cdot R_x = \frac{5}{61}.$$

Meanwhile, the curve C_y is irreducible. We have

$$\frac{91}{30} = \mathrm{lct}\left(X, \frac{2}{13}C_x\right) < \mathrm{lct}\left(X, \frac{2}{18}C_y\right) = \frac{15}{2}.$$

Therefore, $\mathrm{lct}(X) \leqslant \frac{91}{30}$.

Suppose that $\mathrm{lct}(X) < \frac{91}{30}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{91}{30}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D does not contain the curve C_y. Similarly, we may assume that either $L_{xz} \not\subset \mathrm{Supp}(D)$ or $R_x \not\subset \mathrm{Supp}(D)$.

Since the support of D cannot contain either L_{xz} or R_x one of the inequalities

$$\mathrm{mult}_{O_t}(D) \leqslant 61D \cdot L_{xz} = \frac{1}{9} < \frac{30}{91},\ \ \mathrm{mult}_{O_t}(D) \leqslant 61D \cdot R_x = \frac{2}{9} < \frac{30}{91}$$

must hold, and hence the point P cannot be O_t. Also, the inequality

$$13D \cdot C_y = \frac{6}{61} < \frac{30}{91}$$

implies that the point P cannot be O_x.

We write $D = m_0 L_{xz} + m_1 R_x + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains neither L_{xz} nor R_x. If $m_0 \neq 0$, then we obtain

$$\frac{2}{9 \cdot 61} = D \cdot R_x \geqslant m_0 L_{xz} \cdot R_x = \frac{5m_0}{61}$$

and hence $m_0 \leqslant \frac{2}{45}$. By the same way, we get $m_1 \leqslant \frac{1}{45}$. Since

$$18(D - m_0 L_{xz}) \cdot L_{xz} = \frac{2 + 77m_0}{61} < \frac{30}{91},\ \ 9(D - m_1 R_x) \cdot R_x = \frac{2 + 32m_0}{61} < \frac{30}{91}$$

it follows from Lemma 1.3.8 that the point P is a smooth point in the outside of C_x. However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(585))$ contains x^{45}, $x^{27}y^{13}$, z^{13}, this is impossible by Lemma 1.3.9. $\square$

Lemma 3.2.19. Let X be a quasismooth hypersurface of degree 107 in $\mathbb{P}(13, 20, 29, 47)$. Then $\mathrm{lct}(X) = \frac{65}{18}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$yz^3 + y^3t + xt^2 + x^6z = 0.$$

The surface X is singular at the points O_x, O_y, O_z and O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xy} (resp. L_{xy}, L_{zt}, L_{zt}) and $R_x = \{x = z^3 + y^2t = 0\}$ (resp. $R_y = \{y = t^2 + x^5z = 0\}$, $R_z = \{z = y^3 + xt = 0\}$, $R_t = \{t = x^6 + yz^2 = 0\}$). The curve L_{xy} intersects R_x (resp. R_y) only at the point O_t (resp. O_z). Also, the curve L_{zt} intersects R_z (resp. R_t) only at the point O_x (resp. O_y). It is easy to check

$$-K_X\cdot L_{xy}=\frac{2}{29\cdot 47},\quad -K_X\cdot L_{zt}=\frac{1}{13\cdot 10},\quad -K_X\cdot R_x=\frac{3}{10\cdot 47},$$
$$-K_X\cdot R_y=\frac{4}{13\cdot 29},\quad -K_X\cdot R_z=\frac{6}{13\cdot 47},\quad -K_X\cdot R_t=\frac{3}{5\cdot 29},$$
$$L_{xy}^2=-\frac{74}{29\cdot 47},\quad R_x^2=-\frac{21}{20\cdot 47},\quad L_{xy}\cdot R_x=\frac{3}{47}.$$

We see $\mathrm{lct}(X)\leqslant\frac{65}{18}$ since

$$\frac{65}{18}=\mathrm{lct}\left(X,\frac{2}{13}C_x\right)<\frac{70}{12}=\mathrm{lct}\left(X,\frac{2}{20}C_y\right)<\frac{29}{3}=\mathrm{lct}\left(X,\frac{2}{29}C_z\right)<\frac{94}{9}=\mathrm{lct}\left(X,\frac{2}{47}C_t\right).$$

Suppose that $\mathrm{lct}(X)<\frac{65}{18}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{65}{18}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of D does not contain at least one irreducible component of each of the curves C_x, C_y, C_z and C_t. The curve R_x (resp. R_y, R_t) is singular at the point O_t (resp. O_z, O_y). Then in each of the following pairs of inequalities, at least one of two must hold:

$$\mathrm{mult}_{O_t}(D)\leqslant 47D\cdot L_{xy}=\frac{2}{29}<\frac{18}{65},\quad \mathrm{mult}_{O_t}(D)\leqslant\frac{47}{2}D\cdot R_x=\frac{3}{20}<\frac{18}{65};$$
$$\mathrm{mult}_{O_z}(D)\leqslant 29D\cdot L_{xy}=\frac{2}{47}<\frac{18}{65},\quad \mathrm{mult}_{O_z}(D)\leqslant\frac{29}{2}D\cdot R_y=\frac{2}{13}<\frac{18}{65};$$
$$\mathrm{mult}_{O_x}(D)\leqslant 13D\cdot L_{zt}=\frac{1}{10}<\frac{18}{65},\quad \mathrm{mult}_{O_x}(D)\leqslant 13D\cdot R_z=\frac{6}{47}<\frac{18}{65};$$
$$\mathrm{mult}_{O_y}(D)\leqslant 20D\cdot L_{zt}=\frac{2}{13}<\frac{18}{65},\quad \mathrm{mult}_{O_y}(D)\leqslant\frac{20}{2}D\cdot R_t=\frac{6}{29}<\frac{18}{65}.$$

Therefore, the point P must be a smooth point of X.

We write $D=m_0L_{xy}+m_1R_x+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of L_{xy}, R_x. If $m_0\neq 0$, then $m_1=0$ and hence we obtain

$$\frac{3}{10\cdot 47}=D\cdot R_x\geqslant m_0L_{xy}\cdot R_x=\frac{3m_0}{47}.$$

Therefore, $m_0\leqslant\frac{1}{10}$. Similarly, we get $m_1\leqslant\frac{2}{87}$. Since

$$(D-m_0L_{xy})\cdot L_{xy}=\frac{2+74m_0}{29\cdot 47}<\frac{18}{65},$$
$$(D-m_1R_x)\cdot R_x=\frac{6+21m_1}{20\cdot 47}<\frac{18}{65}$$

it follows from Lemma 1.3.8 that the point P is a smooth point in the outside of C_x. However, since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(377))$ contains the monomials x^9y^{13}, x^{29} and z^{13}, this is impossible by Lemma 1.3.9. □

Lemma 3.2.20. Let X be a quasismooth hypersurface of degree 111 in $\mathbb{P}(13,20,31,49)$. Then $\mathrm{lct}(X)=\frac{65}{16}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^2t + y^4z + xt^2 + x^7y = 0.$$

It is singular at the point O_x, O_y, O_z and O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xz} (resp. L_{yt}, L_{xz}, L_{yt}) and $R_x = \{x = y^4 + zt = 0\}$ (resp. $R_y = \{y = z^2 + xt = 0\}$, $R_z = \{z = t^2 + x^6y = 0\}$, $R_t = \{t = x^7 + y^3z = 0\}$). The curve L_{xz} intersects R_x (resp. R_z) only at the point O_t (resp. O_y). Also, the curve L_{yt} intersects R_y (resp. R_t) only at the point O_x (resp. O_z). It is easy to check

$$-K_X \cdot L_{xz} = \frac{1}{10 \cdot 49},\ -K_X \cdot L_{yt} = \frac{2}{13 \cdot 31},\ -K_X \cdot R_x = \frac{8}{31 \cdot 49},$$
$$-K_X \cdot R_y = \frac{4}{13 \cdot 49},\ -K_X \cdot R_z = \frac{1}{5 \cdot 13},\ -K_X \cdot R_t = \frac{7}{10 \cdot 31},$$
$$L_{xz}^2 = -\frac{67}{20 \cdot 49},\ R_x^2 = -\frac{72}{31 \cdot 49},\ L_{xz} \cdot R_x = \frac{4}{49}.$$

We have $\mathrm{lct}(X) \leqslant \frac{65}{16}$ since

$$\frac{65}{16} = \mathrm{lct}\left(X, \frac{2}{13}C_x\right) < \frac{30}{4} = \mathrm{lct}\left(X, \frac{2}{20}C_y\right) < \frac{245}{28} = \mathrm{lct}\left(X, \frac{2}{49}C_t\right) < \frac{62}{7} = \mathrm{lct}\left(X, \frac{2}{31}C_z\right).$$

Suppose that $\mathrm{lct}(X) < \frac{65}{16}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{65}{16}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of D does not contain at least one irreducible component of each of the curves C_x, C_y, C_z and C_t. The curve R_z is singular at the point O_y. The curve R_t is singular at O_z with multiplicity 3. Then in each of the following pairs of inequalities, at least one of two must hold:

$$\mathrm{mult}_{O_x}(D) \leqslant 13D \cdot L_{yt} = \frac{2}{31} < \frac{16}{65},\ \ \mathrm{mult}_{O_x}(D) \leqslant 13D \cdot R_y = \frac{4}{49} < \frac{16}{65};$$
$$\mathrm{mult}_{O_y}(D) \leqslant 20D \cdot L_{xz} = \frac{2}{49} < \frac{16}{65},\ \ \mathrm{mult}_{O_y}(D) \leqslant \frac{20}{2}D \cdot R_z = \frac{2}{13} < \frac{16}{65};$$
$$\mathrm{mult}_{O_z}(D) \leqslant 31D \cdot L_{yt} = \frac{2}{13} < \frac{16}{65},\ \ \mathrm{mult}_{O_z}(D) \leqslant \frac{31}{3}D \cdot R_t = \frac{7}{30} < \frac{16}{65}.$$

Therefore, the point P can be none of O_x, O_y, O_z.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(403))$ contains the monomials $x^{11}y^{13}$, x^{31} and z^{13}, it follows from Lemma 1.3.9 that the point P is either the point O_t or a smooth point of X in C_x.

Write $D = m_0L_{xz} + m_1R_x + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of L_{xz}, R_x. If $m_0 \neq 0$, then $m_1 = 0$ and hence we obtain

$$\frac{8}{31 \cdot 49} = D \cdot R_x \geqslant m_0L_{xz} \cdot R_x = \frac{4m_0}{49}.$$

Therefore, $m_0 \leqslant \frac{2}{31}$. Similarly, we get $m_1 \leqslant \frac{1}{40}$. Since we have

$$(D - m_0L_{xz}) \cdot L_{xz} = \frac{2 + 67m_0}{20 \cdot 49} < \frac{16}{65},$$
$$(D - m_1R_x) \cdot R_x = \frac{8 + 72m_1}{31 \cdot 49} < \frac{16}{65}$$

it follows from Lemma 1.3.8 that the point P must be the point O_t.

Suppose that $m_0 = 0$. Then the inequality

$$\mathrm{mult}_{O_t}(D) \leqslant 49D \cdot L_{xz} = \frac{1}{10} < \frac{16}{65}$$

gives us a contradiction. Therefore, $m_0 \neq 0$ and hence the curve R_x is not contained in the support of D. Then

$$\frac{8}{31 \cdot 49} = D \cdot R_x \geqslant m_0 L_{xz} \cdot R_x + \frac{\mathrm{mult}_{O_t}(D) - m_0}{49} > \frac{3m_0}{49} + \frac{16}{65 \cdot 49},$$

and hence $m_0 < \frac{8}{31 \cdot 65}$. Since

$$49(D - m_0 L_{xz}) \cdot L_{xz} = \frac{2 + 67m_0}{20} < \frac{16}{65}$$

the pair $(X, \frac{65}{16}D)$ is log canonical at the point O_t by Lemma 1.3.8. This is a contradiction. □

Lemma 3.2.21. Let X be a quasismooth hypersurface of degree 226 in $\mathbb{P}(13, 31, 71, 113)$. Then $\mathrm{lct}(X) = \frac{91}{20}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + y^5 z + xz^3 + x^{15} y = 0.$$

It is singular at the points O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{91}{20} = \mathrm{lct}\left(X, \frac{2}{13}C_x\right) < \mathrm{lct}\left(X, \frac{2}{31}C_y\right) = \frac{155}{12}.$$

Therefore, $\mathrm{lct}(X) \leqslant \frac{91}{20}$.

Suppose that $\mathrm{lct}(X) < \frac{91}{20}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{91}{20}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither C_x nor C_y. Then the inequalities

$$71D \cdot C_x = \frac{4}{31} < \frac{20}{91}, \quad 13D \cdot C_y = \frac{4}{71} < \frac{20}{91}$$

show that the point P is a smooth point in the outside of C_x. However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(923))$ contains x^{71}, $y^{26}x^9$, $y^{13}x^{40}$ and z^{13}, it follows from Lemma 1.3.9 that the point P is either a singular point of X or a point on C_x. This is a contradiction. □

Lemma 3.2.22. Let X be a quasismooth hypersurface of degree 99 in $\mathbb{P}(14, 17, 29, 41)$. Then $\mathrm{lct}(X) = \frac{51}{10}$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2 y + tz^2 + xy^5 + x^5 z = 0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xt} (resp. L_{yz}, L_{yz}, L_{xt}) and $R_x = \{x = yt + z^2 = 0\}$ (resp. $R_y = \{y = zt + x^5 = 0\}$, $R_z = \{z = xy^4 + t^2 = 0\}$, $R_t = \{t = y^5 + x^4 z = 0\}$). Also, we see that

$$L_{xt} \cap R_x = \{O_y\},\ L_{yz} \cap R_y = \{O_t\},\ L_{yz} \cap R_z = \{O_x\},\ L_{xt} \cap R_t = \{O_z\}.$$

We can easily check that $\mathrm{lct}(X, \frac{2}{17}C_y) = \frac{51}{10}$ is less than each of the numbers $\mathrm{lct}(X, \frac{2}{14}C_y)$, $\mathrm{lct}(X, \frac{2}{29}C_z)$ and $\mathrm{lct}(X, \frac{2}{41}C_t)$. Therefore, $\mathrm{lct}(X) \leqslant \frac{51}{10}$. Suppose $\mathrm{lct}(X) < \frac{51}{10}$. Then, there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{51}{10}D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xt}, L_{yz}, R_x, R_y, R_z, R_t are as follows:

$$D \cdot L_{xt} = \frac{2}{17 \cdot 29}, \quad D \cdot R_x = \frac{4}{17 \cdot 41}, \quad D \cdot R_y = \frac{10}{29 \cdot 41},$$
$$D \cdot L_{yz} = \frac{1}{7 \cdot 41}, \quad D \cdot R_z = \frac{2}{7 \cdot 17}, \quad D \cdot R_t = \frac{5}{7 \cdot 29},$$
$$L_{xt} \cdot R_x = \frac{2}{17}, \quad L_{yz} \cdot R_y = \frac{5}{41}, \quad L_{yz} \cdot R_z = \frac{1}{7}, \quad L_{xt} \cdot R_t = \frac{5}{29},$$
$$L_{xt}^2 = -\frac{44}{17 \cdot 29}, \quad R_x^2 = -\frac{54}{17 \cdot 41}, \quad R_y^2 = -\frac{60}{29 \cdot 41},$$
$$L_{yz}^2 = -\frac{53}{14 \cdot 41}, \quad R_z^2 = \frac{12}{7 \cdot 17}, \quad R_t^2 = \frac{135}{14 \cdot 29}.$$

By Lemma 1.3.6 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. The inequalities

$$17D \cdot L_{xt} = \frac{2}{29} < \frac{10}{51}, \quad 17D \cdot R_x = \frac{4}{41} < \frac{10}{51}$$

imply that $P \neq O_y$. The inequalities

$$14D \cdot L_{yz} = \frac{2}{41} < \frac{10}{51}, \quad 7D \cdot R_z = \frac{2}{17} < \frac{10}{51}$$

imply that $P \neq O_x$. The curve R_z is singular at the point O_x. The inequalities

$$29D \cdot L_{xt} = \frac{2}{17} < \frac{10}{51}, \quad \frac{29}{4}D \cdot R_t = \frac{5}{28} < \frac{10}{51}$$

imply that $P \neq O_z$. The curve R_t is singular at the point O_z.

We write $D = m_0L_{xt} + m_1L_{yz} + m_2R_x + m_3R_y + m_4R_z + m_5R_t + \Omega$, where Ω is an effective divisor whose support contains none of the curves L_{xt}, L_{yz}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{51}{10}D)$ is log canonical at the points O_x, O_y, O_z, the numbers m_i are at most $\frac{10}{51}$. Then by Lemma 1.3.8 the following inequalities enable us to conclude that either the point P is in the outside of $C_x \cup C_y \cup C_z \cup C_t$ or $P = O_t$:

$$(D - m_0L_{xt}) \cdot L_{xz} = \frac{2 + 44m_0}{17 \cdot 29} \leqslant \frac{10}{51}, \quad (D - m_1L_{yz}) \cdot L_{yt} = \frac{2 + 53m_1}{14 \cdot 41} \leqslant \frac{10}{51},$$
$$(D - m_2R_x) \cdot R_x = \frac{4 + 54m_2}{17 \cdot 41} \leqslant \frac{10}{51}, \quad (D - m_3R_y) \cdot R_y = \frac{10 + 60m_3}{29 \cdot 41} \leqslant \frac{10}{51},$$
$$(D - m_4R_z) \cdot R_z = \frac{2 - 12m_4}{7 \cdot 17} \leqslant \frac{10}{51}, \quad (D - m_5R_t) \cdot R_t = \frac{10 - 135m_5}{14 \cdot 29} \leqslant \frac{10}{51}.$$

Suppose that $P \neq O_t$. Then we consider the pencil $\mathcal{L}$ on X defined by $\lambda yt + \mu z^2 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. The base locus of the pencil $\mathcal{L}$ consists of the curve L_{yz} and the point O_y. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P \notin C_x \cup C_y \cup C_z \cup C_t$, the divisor E is defined by the equation $z^2 = \alpha yt$, where $\alpha \neq 0$.

Suppose that $\alpha \neq -1$. Then the curve E is isomorphic to the curve defined by the equations $yt = z^2$ and $t^2y + xy^5 + x^5z = 0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t \neq 0$ is the curve given by $z(z + xz^9 + x^5) = 0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{yz} and C. We have the intersection number

$$D \cdot C = D \cdot E - D \cdot L_{yz} = \frac{181}{7 \cdot 17 \cdot 41}.$$

Also, we see

$$C^2 = E \cdot C - C \cdot L_{yz} \geqslant E \cdot C - C_y \cdot C > 0$$

since C is different from R_y. By Lemma 1.3.8 the inequality $D \cdot C < \frac{10}{51}$ gives us a contradiction.

Suppose that $\alpha = -1$. Then divisor E consists of three irreducible and reduced curves L_{yz}, R_x, and M. Note that the curve M is different from the curves R_y and L_{xt}. Also, it is smooth at the point P. We have

$$D \cdot M = D \cdot E - D \cdot L_{yz} - D \cdot R_x = \frac{153}{7 \cdot 17 \cdot 41},$$

$$M^2 = E \cdot M - L_{yz} \cdot M - R_x \cdot M \geqslant E \cdot M - C_y \cdot M - C_x \cdot M > 0.$$

By Lemma 1.3.8 the inequality $D \cdot M < \frac{10}{51}$ gives us a contradiction. Therefore, the log pair $(X, \frac{51}{10}D)$ is not log canonical at the point O_t.

Put $D = aL_{yz} + bR_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither L_{yz} nor R_x. Then $a > 0$ since otherwise we would have a contradictory inequality

$$\frac{1}{7} = 41 D \cdot L_{yz} \geqslant \mathrm{mult}_{O_t}(D) > \frac{10}{51}.$$

Therefore, we may assume that $R_y \not\subset \mathrm{Supp}(\Delta)$ by Lemma 1.3.6. Similarly, we may assume that $L_{xt} \not\subset \mathrm{Supp}(\Delta)$ if $b > 0$.

If $b > 0$, then

$$\frac{2}{17 \cdot 29} = D \cdot L_{xt} \geqslant bR_x \cdot L_{xt} = \frac{2b}{17},$$

and hence $b \leqslant \frac{1}{29}$. Similarly, we have

$$\frac{10}{29 \cdot 41} = D \cdot R_y \geqslant \frac{5a}{41} + \frac{b}{41} + \frac{\mathrm{mult}_{O_t}(D) - a - b}{41} > \frac{4a}{41} + \frac{4}{21 \cdot 41}.$$

Therefore, $a < \frac{47}{2 \cdot 21 \cdot 29}$.

Let $\pi \colon \bar{X} \to X$ be the weighted blow up at the point O_t with weights $(9, 4)$ and let F be the exceptional curve of the morphism π. Then F contains two singular points Q_9 and Q_4 of $\bar{X}$ such that Q_9 is a singular point of type $\frac{1}{9}(1, 1)$, and Q_4 is a singular point of type $\frac{1}{4}(3, 1)$. Then

$$K_{\bar{X}} \sim_{\mathbb{Q}} \pi^*(K_X) - \frac{28}{41}F, \quad \bar{L}_{yz} \sim_{\mathbb{Q}} \pi^*(L_{yz}) - \frac{4}{41}F, \quad \bar{R}_x \sim_{\mathbb{Q}} \pi^*(R_x) - \frac{9}{41}F, \quad \bar{\Delta} \sim_{\mathbb{Q}} \pi^*(\Delta) - \frac{c}{41}F,$$

where $\bar{L}_{yz}$, $\bar{R}_x$ and $\bar{\Delta}$ are the proper transforms of L_{yz}, R_x and Δ by π, respectively, and c is a non-negative rational number. Note that $F \cap \bar{R}_x = \{Q_4\}$ and $F \cap \bar{L}_{yz} = \{Q_9\}$.

The log pull-back of the log pair $(X, \frac{51}{10}D)$ by π is the log pair

$$\left(\bar{X},\ \frac{51a}{10}\bar{L}_{yz}+\frac{51b}{10}\bar{R}_x+\frac{51}{10}\bar{\Delta}+\theta_1 F\right),$$

where

$$\theta_1=\frac{280+51(4a+9b+c)}{10\cdot 41}.$$

This is not log canonical at some point $Q\in F$.

We have

$$0\leqslant \bar{\Delta}\cdot\bar{R}_x=\frac{4+54b}{17\cdot 41}-\frac{a}{41}-\frac{c}{4\cdot 41}.$$

This inequality shows $4a+c\leqslant\frac{4}{17}(4+54b)$. Since $b\leqslant\frac{1}{29}$, we obtain

$$\theta_1=\frac{280+51(4a+9b+c)}{10\cdot 41}\leqslant\frac{4760+51(16+369b)}{10\cdot 17\cdot 41}<1.$$

Suppose that $Q\notin \bar{R}_x\cup\bar{L}_{yz}$. Then the log pair $(F, \frac{51}{10}\bar{\Delta}|_F)$ is not log canonical at the point Q, and hence

$$\frac{17c}{120}=\frac{51}{10}\bar{\Delta}\cdot F>1$$

by Lemma 1.3.8. Thus, we see that $c>\frac{120}{17}$. However, since $b\leqslant\frac{1}{29}$, we obtain

$$c\leqslant 4a+c\leqslant\frac{4}{17}(4+54b)<\frac{120}{17}.$$

Therefore, the point Q must be either Q_4 or Q_9.

Suppose that $Q=Q_4$. The pair $(\bar{R}_x, (\frac{51}{10}\bar{\Delta}+\theta_1 F)|_{\bar{R}_x})$ is not log canonical at Q. It then follows from Lemma 1.3.8 that

$$1<4\left(\frac{51}{10}\bar{\Delta}+\theta_1 F\right)\cdot\bar{R}_x=\frac{4\cdot 51}{10}\left(\frac{4+54b}{17\cdot 41}-\frac{a}{41}-\frac{c}{4\cdot 41}\right)+\theta_1.$$

However,

$$\frac{4\cdot 51}{10}\left(\frac{4+54b}{17\cdot 41}-\frac{a}{41}-\frac{c}{4\cdot 41}\right)+\theta_1=\frac{4760+51(16+369b)}{10\cdot 17\cdot 41}<1.$$

This is a contradiction. Consequently, the point Q must be Q_9.

Let $\psi\colon\tilde{X}\to\bar{X}$ be the blow up at the point Q_9 and let E be the exceptional curve of the morphism ψ. The surface $\tilde{X}$ is smooth along the exceptional divisor E. Then

$$K_{\tilde{X}}\sim_{\mathbb{Q}}\psi^*(K_{\bar{X}})-\frac{7}{9}E,\quad \tilde{L}_{yz}\sim_{\mathbb{Q}}\psi^*(\bar{L}_{yz})-\frac{1}{9}E,\quad \tilde{F}\sim_{\mathbb{Q}}\psi^*(F)-\frac{1}{9}E,\quad \tilde{\Delta}\sim_{\mathbb{Q}}\psi^*(\bar{\Delta})-\frac{d}{9}E,$$

where $\tilde{L}_{yz}$, $\tilde{F}$ and $\tilde{\Delta}$ are the proper transforms of $\bar{L}_{yz}$, F and $\bar{\Delta}$ by ψ, respectively, and d is a non-negative rational number.

The log pull-back of the log pair $(X, \frac{51}{10}D)$ by $\pi\circ\psi$ is the log pair

$$\left(\tilde{X},\ \frac{51a}{10}\tilde{L}_{yz}+\frac{51b}{10}\tilde{R}_x+\frac{51}{10}\tilde{\Delta}+\theta_1\tilde{F}+\theta_2 E\right),$$

where $\tilde{R}_x$ is the proper transform of $\bar{R}_x$ by ψ and

$$\theta_2 = \frac{70 + 51(a+d) + 10\theta_1}{90} = \frac{3150 + 51(45a + 9b + c + 41d)}{90 \cdot 41}.$$

This is not log canonical at some point $O \in E$.

We have

$$0 \leqslant \tilde{\Delta} \cdot \tilde{L}_{yx} = \bar{\Delta} \cdot \bar{L}_{yz} - \frac{d}{9} = \frac{2 + 53a}{14 \cdot 41} - \frac{b}{41} - \frac{c}{9 \cdot 41} - \frac{d}{9},$$

and hence $9b + c + 41d \leqslant \frac{9}{14}(2 + 53a)$. Therefore, this inequality together with $a < \frac{47}{2 \cdot 21 \cdot 29}$ gives us

$$\begin{aligned}\theta_2 &= \frac{3150 + 51(45a + 9b + c + 41d)}{90 \cdot 41} = \\ &= \frac{3150 + 2295a}{90 \cdot 41} + \frac{51(9b + c + 41d)}{90 \cdot 41} \leqslant \\ &\leqslant \frac{5002 + 6273a}{10 \cdot 14 \cdot 41} < 1.\end{aligned}$$

Suppose that the point O is in the outside of $\tilde{L}_{yz}$ and $\tilde{F}$. Then the log pair $(E, \frac{51}{10}\tilde{\Delta}|_E)$ is not log canonical at the point O and hence

$$1 < \frac{51}{10}\tilde{\Delta} \cdot E = \frac{51d}{10}.$$

However,

$$41d \leqslant 9b + c + 41d \leqslant \frac{9}{14}(2 + 53a) < \frac{10 \cdot 41}{51}$$

since $a < \frac{47}{2 \cdot 21 \cdot 29}$. This is a contradiction.

Suppose that the point O belongs to $\tilde{L}_{yz}$ Then the log pair $(E, (\frac{51a}{10}\tilde{L}_{yz} + \frac{51}{10}\tilde{\Delta})|_E)$ is not log canonical at the point O and hence

$$1 < (\frac{51a}{10}\tilde{L}_{yz} + \frac{51}{10}\tilde{\Delta}) \cdot E = \frac{51}{10}(a + d).$$

However,

$$\frac{51}{10}(a + d) \leqslant \frac{51}{10}\left(a + \frac{9}{14 \cdot 41}(2 + 53a)\right) < 1$$

since $a < \frac{47}{2 \cdot 21 \cdot 29}$. This is a contradiction. Therefore, the point O is the intersection point of $\tilde{F}$ and E.

Let $\xi \colon \hat{X} \to \tilde{X}$ be the blow up at the point O and let H be the exceptional divisor of ξ. We also let $\hat{L}_{yz}$, $\hat{R}_x$, $\hat{\Delta}$, $\hat{E}$, and $\hat{F}$ be the proper transforms of $\tilde{L}_{yz}$, $\tilde{R}_x$, $\tilde{\Delta}$, E and $\tilde{F}$ by ξ, respectively. We have

$$K_{\hat{X}} \sim_{\mathbb{Q}} \xi^*(K_{\tilde{X}}) + H,\ \hat{E} \sim_{\mathbb{Q}} \xi^*(E) - H,\ \hat{F} \sim_{\mathbb{Q}} \xi^*(\tilde{F}) - H,\ \hat{\Delta} \sim_{\mathbb{Q}} \xi^*(\tilde{\Delta}) - eH,$$

where e is a non-negative rational number. The log pull-back of the log pair $(X, \frac{51}{10}D)$ via $\pi \circ \phi \circ \xi$ is

$$\left(\hat{X}, \frac{51a}{10}\hat{L}_{yz} + \frac{51b}{10}\hat{R}_x + \frac{51}{10}\hat{\Delta} + \theta_1\hat{F} + \theta_2\hat{E} + \theta_3 H\right),$$

where

$$\theta_3 = \theta_1 + \theta_2 + \frac{51e}{10} - 1 = \frac{1980 + 51(81a + 90b + 10c + 41d + 369e)}{90 \cdot 41}.$$

This log pair is not log canonical at some point $A \in H$. We have

$$\frac{c}{9 \cdot 4} - \frac{d}{9} - e = \hat{\Delta} \cdot \hat{F} \geqslant 0.$$

Therefore, $4d + 36e \leqslant c$. Then

$$\begin{aligned}\theta_3 &= \frac{1980 + 51(81a + 90b + 10c)}{90 \cdot 41} + \frac{51(d + 9e)}{90} \leqslant \\ &\leqslant \frac{7920 + 51(324a + 360b + 81c)}{4 \cdot 90 \cdot 41} = \\ &= \frac{22 + 51b}{41} + \frac{51 \cdot 81(4a + c)}{4 \cdot 90 \cdot 41} \leqslant \\ &\leqslant \frac{22 + 51b}{41} + \frac{9 \cdot 51(2 + 27b)}{5 \cdot 17 \cdot 41} < 1\end{aligned}$$

since $b \leqslant \frac{1}{29}$ and $4a + c \leqslant \frac{4}{17}(4 + 54b)$.

Suppose that $A \notin \hat{F} \cup \hat{E}$. Then the log pair $\left(\hat{X}, \frac{51}{10}\hat{\Delta} + \theta_3 H\right)$ is not log canonical at the point A. Applying Lemma 1.3.4, we get

$$1 < \frac{51}{10}\hat{\Delta} \cdot H = \frac{51e}{10}.$$

However,

$$e \leqslant \frac{1}{36}(4d + 36e) \leqslant \frac{c}{36} \leqslant \frac{1}{36}(4a + c) \leqslant \frac{4 + 54b}{17 \cdot 9} < \frac{10}{51}.$$

Therefore, the point A must be either in $\hat{F}$ or in $\hat{E}$.

Suppose that $A \in \hat{F}$. Then the log pair $\left(\hat{X}, \frac{51}{10}\hat{\Delta} + \theta_1 \hat{F} + \theta_3 H\right)$ is not log canonical at the point A. Applying Lemma 1.3.4, we get

$$1 < \left(\frac{51}{10}\hat{\Delta} + \theta_3 H\right) \cdot \hat{F} = \frac{51}{10}\left(\frac{c}{9 \cdot 4} - \frac{d}{9} - e\right) + \theta_3 = \frac{7920 + 51(324a + 360b + 81c)}{4 \cdot 90 \cdot 41}.$$

However,

$$\frac{7920 + 51(324a + 360b + 81c)}{4 \cdot 90 \cdot 41} < 1$$

as seen in the previous. Therefore, the point A is the intersection point of H and $\hat{E}$. Then the log pair $\left(\hat{X}, \frac{51}{10}\hat{\Delta} + \theta_2 \hat{E} + \theta_3 H\right)$ is not log canonical at the point A. From Lemma 1.3.4, we obtain

$$1 < \left(\frac{51}{10}\hat{\Delta} + \theta_3 H\right) \cdot \hat{E} = \frac{51}{10}(d - e) + \theta_3 = \frac{1980 + 51(81a + 90b + 10c + 410d)}{90 \cdot 41}.$$

However,

$$\frac{1980+51(81a+90b+10c+410d)}{90\cdot 41}=\frac{220+459a}{10\cdot 41}+\frac{51(9b+c+41d)}{9\cdot 41}\leqslant$$
$$\leqslant\frac{220+459a}{10\cdot 41}+\frac{51(2+53a)}{14\cdot 41}<1$$

since $9b+c+41d\leqslant\frac{9}{14}(2+53a)$ and $a<\frac{47}{2\cdot 21\cdot 29}$. The obtained contradiction completes the proof. □

3.3. Sporadic cases with $I=3$

Lemma 3.3.1. Let X be a quasismooth hypersurface of degree 33 in $\mathbb{P}(5,7,11,13)$. Then $\mathrm{lct}(X)=\frac{49}{36}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3+yt^2+xy^4+x^4t+\epsilon x^3yz=0,$$

where $\epsilon\in\mathbb{C}$. Note that the surface X is singular at O_x, O_y and O_t.

The curves C_x, C_y are irreducible. Moreover, we have

$$\frac{25}{18}=\mathrm{lct}(X,\frac{3}{5}C_x)>\mathrm{lct}(X,\frac{3}{7}C_y)=\frac{49}{36}.$$

Therefore, $\mathrm{lct}(X)\leqslant\frac{49}{36}$.

Suppose that $\mathrm{lct}(X)<\frac{49}{36}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{49}{36}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of D contains neither C_x nor C_y. Since the curve C_y is singular at the point O_t, the three inequalities

$$5D\cdot C_y=\frac{9}{13}<\frac{36}{49},\quad 7D\cdot C_x=\frac{63}{91}<\frac{36}{49}$$

show that the point P is located in the outside of the set $C_x\cup C_y$.

Let $\mathcal{L}$ be the pencil on X that is cut out by the equations

$$\lambda x^7+\mu y^5=0,$$

where $[\lambda:\mu]\in\mathbb{P}^1$. Then the base locus of the pencil $\mathcal{L}$ consists of the point O_t. Let C be the unique curve in $\mathcal{L}$ that passes through the point P. Since the point P is in the outside of the set $C_x\cup C_y$, the curve C is defined by an equation of the form $y^5-\alpha x^7=0$, where α is a non-zero constant. Suppose that C is irreducible and reduced. Then $\mathrm{mult}_P(C)\leqslant 3$ since the curve C is a triple cover of the curve

$$y^5-\alpha x^7=0\subset\mathbb{P}\big(5,7,13\big)\cong\mathrm{Proj}\Big(\mathbb{C}\big[x,y,t\big]\Big).$$

In particular, $\mathrm{lct}(X,\frac{3}{35}C)>\frac{49}{36}$. Thus, we may assume that the support of D does not contain the curve C and hence we obtain

$$\frac{36}{49}<\mathrm{mult}_P(D)\leqslant D\cdot C=\frac{9}{13}<\frac{36}{49}.$$

This is a contradiction. Thus, to conclude the proof it suffices to prove that the curve C is irreducible and reduced.

Let $S\subset\mathbb{C}^4$ be the affine variety defined by the equations

$$y^5-\alpha x^7=z^3+yt^2+xy^4+x^4t+\epsilon x^3yz=0\subset\mathbb{C}^4\cong\mathrm{Spec}\Big(\mathbb{C}\big[x,y,z,t\big]\Big).$$

To conclude the proof, it is enough to prove that the variety S is irreducible.

Consider the projectivised surface $\bar{S}$ of S defined by the homogeneous equations

$$y^5w^2-\alpha x^7=z^3w^2+yt^2w^2+xy^4+x^4t+\epsilon x^3yz=0\subset\mathbb{P}^4\cong\mathrm{Proj}\Big(\mathbb{C}\big[x,y,z,t,w\big]\Big).$$

Then we consider the affine piece S' of $\bar{S}$ defined by $y\neq 0$. The affine surface S' is defined by the equations

$$w^2-\alpha x^7=z^3w^2+t^2w^2+x+x^4t+\epsilon x^3z=0\subset\mathbb{C}^4\cong\mathrm{Spec}\Big(\mathbb{C}\big[x,z,t,w\big]\Big).$$

This is isomorphic to the affine hypersurface defined by

$$x(\alpha x^6z^3+\alpha x^6t^2+1+x^3t+\epsilon x^2z)=0\subset\mathbb{C}^3\cong\mathrm{Spec}\Big(\mathbb{C}\big[x,z,t\big]\Big).$$

This affine hypersurface has two irreducible components. However, the component defined by $x=0$ originates from the hyperplane section of $\bar{S}$ by $w=0$. Therefore, the original affine surface S must be irreducible and reduce. □

Lemma 3.3.2. Let X be a quasismooth hypersurface of degree 40 in $\mathbb{P}(5,7,11,20)$. Then $\mathrm{lct}(X)=\frac{25}{18}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t(t-x^4)+yz^3+xy^5+\epsilon x^3y^2z,$$

where $\epsilon\in\mathbb{C}$. Note that X is singular at the points O_x, O_y, O_z and $Q_5=[1:0:0:1]$.

The curve C_x is irreducible. We have

$$\mathrm{lct}(X,\frac{3}{5}C_x)=\frac{25}{18}.$$

Therefore, $\mathrm{lct}(X)\leqslant\frac{25}{18}$. Meanwhile, the curve C_y is reducible. It consists of two irreducible components L_{yt} and $R_y=\{y=t-x^4=0\}$. The curve L_{yt} intersects R_y only at the point O_z. It is easy to see

$$L_{yt}^2=R_y^2=-\frac{13}{55},\ L_{yt}\cdot R_y=\frac{4}{11}.$$

Suppose that $\mathrm{lct}(X)<\frac{25}{18}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{25}{18}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of D does not contain the curve C_x. Moreover, we may assume that the support of D does not contain either L_{yt} or R_y since

$$\mathrm{lct}(X,\frac{3}{7}C_y)=\frac{35}{24}>\frac{25}{18}.$$

Then one of the inequalities

$$\mathrm{mult}_{O_z}(D)\leqslant 11D\cdot L_{yt}=\frac{3}{5}<\frac{18}{25},\ \ \mathrm{mult}_{O_z}(D)\leqslant 11D\cdot R_y=\frac{3}{5}<\frac{18}{25}$$

must hold, and hence the point P cannot be the point O_z. Also, since $7D\cdot C_x=\frac{6}{11}<\frac{18}{25}$, the point P cannot belong to the curve C_x.

We write $D=aL_{yt}+bR_y+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains neither L_{yt} nor R_y. If $a\neq 0$, then we have

$$\frac{3}{55}=D\cdot R_y\geqslant aL_{yt}\cdot R_y=\frac{4a}{11}.$$

Therefore, $a\leqslant\frac{3}{20}$. By the same way, we also obtain $b\leqslant\frac{3}{20}$.

Since we have

$$5(D-aL_{yt})\cdot L_{yt}=\frac{3+13a}{11}<\frac{18}{25},\quad 5(D-bR_y)\cdot R_y=\frac{3+13a}{11}<\frac{18}{25}$$

Lemma 1.3.8 implies that the point P is in the outside of C_y. Consequently, the point P is located in the outside of $C_x\cup C_y$. However, since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(40))$ contains monomials x^8, xy^5, x^4t and the natural projection $X\dashrightarrow\mathbb{P}(5,7,20)$ is a finite morphism outside of the curve C_y, Lemma 1.3.9 shows that the point P must belong to the set $C_x\cup C_y$. This is a contradiction. □

Lemma 3.3.3. Let X be a quasismooth hypersurface of degree 95 in $\mathbb{P}(11,21,29,37)$. Then $\mathrm{lct}(X)=\frac{11}{4}$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2y+tz^2+xy^4+x^6z=0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xt} (resp. L_{yz}, L_{yz}, L_{xt}) and $R_x=\{x=yt+z^2=0\}$ (resp. $R_y=\{y=zt+x^6=0\}$, $R_z=\{z=xy^3+t^2=0\}$, $R_t=\{t=y^4+x^5z=0\}$). Also, we see that

$$L_{xt}\cap R_x=\{O_y\},\ L_{yz}\cap R_y=\{O_t\},\ L_{yz}\cap R_z=\{O_x\},\ L_{xt}\cap R_t=\{O_z\}.$$

It is easy to check that $\mathrm{lct}(X,\frac{3}{11}C_x)=\frac{11}{4}$ is less than each of the numbers $\mathrm{lct}(X,\frac{3}{21}C_y)$, $\mathrm{lct}(X,\frac{3}{29}C_z)$ and $\mathrm{lct}(X,\frac{3}{37}C_t)$. Therefore, $\mathrm{lct}(X)\leqslant\frac{11}{4}$.

Suppose that $\mathrm{lct}(X)<\frac{11}{4}$. Then, there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\frac{11}{4}D)$ is not log canonical at some point $P\in X$.

The intersection numbers among the divisors D, L_{xt}, L_{yz}, R_x, R_y, R_z, R_t are as follows:

$$D\cdot L_{xt}=\frac{1}{7\cdot 29},\quad D\cdot R_x=\frac{2}{7\cdot 37},\quad D\cdot R_y=\frac{18}{29\cdot 37},$$

$$D\cdot L_{yz}=\frac{3}{11\cdot 37},\quad D\cdot R_z=\frac{2}{7\cdot 11},\quad D\cdot R_t=\frac{12}{11\cdot 29},$$

$$L_{xt}\cdot R_x=\frac{2}{21},\quad L_{yz}\cdot R_y=\frac{6}{37},\quad L_{yz}\cdot R_z=\frac{2}{11},\quad L_{xt}\cdot R_t=\frac{4}{29},$$

$$L_{xt}^2=-\frac{47}{21\cdot 29},\quad R_x^2=-\frac{52}{21\cdot 37},\quad R_y^2=-\frac{48}{29\cdot 37},$$

$$L_{yz}^2=-\frac{45}{11\cdot 37},\quad R_z^2=\frac{16}{11\cdot 21},\quad R_t^2=\frac{104}{11\cdot 29}.$$

By Lemma 1.3.6 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. The inequalities

$$21D\cdot L_{xt}=\frac{3}{29}<\frac{4}{11},\quad 21D\cdot R_x=\frac{6}{37}<\frac{4}{11}$$

imply that $P\neq O_y$. The inequalities

$$11D\cdot L_{yz}=\frac{3}{37}<\frac{4}{11},\quad 11D\cdot R_z=\frac{2}{7}<\frac{4}{11}$$

imply that $P\neq O_x$. The inequalities

$$29D\cdot L_{xt}=\frac{1}{7}<\frac{4}{11},\quad \frac{29}{4}D\cdot R_t=\frac{3}{11}<\frac{4}{11}$$

imply that $P\neq O_z$. The curve R_t is singular at the point O_z with multiplicity 4.

We write $D=m_0L_{xt}+m_1L_{yz}+m_2R_x+m_3R_y+m_4R_z+m_5R_t+\Omega$, where Ω is an effective divisor whose support contains none of the curves L_{xt}, L_{yz}, R_x, R_y, R_z, R_t. Since the pair $(X,\frac{11}{4}D)$ is log canonical at the points O_x, O_y, O_z, the numbers m_i are at most $\frac{4}{11}$. Then by Lemma 1.3.8 the following inequalities enable us to conclude that either the point P is in the outside of $C_x\cup C_y\cup C_z\cup C_t$ or $P=O_t$:

$$(D-m_0L_{xt})\cdot L_{xt}=\frac{3+47m_0}{21\cdot 29}\leqslant\frac{4}{11},\quad (D-m_1L_{yz})\cdot L_{yz}=\frac{3+45m_1}{11\cdot 37}\leqslant\frac{4}{11},$$

$$(D-m_2R_x)\cdot R_x=\frac{6+52m_2}{21\cdot 37}\leqslant\frac{4}{11},\quad (D-m_3R_y)\cdot R_y=\frac{18+48m_3}{29\cdot 37}\leqslant\frac{4}{11},$$

$$(D-m_4R_z)\cdot R_z=\frac{6-16m_4}{11\cdot 21}\leqslant\frac{4}{11},\quad (D-m_5R_t)\cdot R_t=\frac{12-104m_5}{11\cdot 29}\leqslant\frac{4}{11}.$$

Suppose that $P\neq O_t$. Then we consider the pencil $\mathcal{L}$ defined by $\lambda yt+\mu z^2=0$, $[\lambda:\mu]\in\mathbb{P}^1$. The base locus of the pencil $\mathcal{L}$ consists of the curve L_{yz} and the point O_y. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P\notin C_x\cup C_y\cup C_z\cup C_t$, the divisor E is defined by the equation $z^2=\alpha yt$, where $\alpha\neq 0$.

Suppose that $\alpha\neq -1$. Then the curve E is isomorphic to the curve defined by the equations $yt=z^2$ and $t^2y+xy^4+x^6z=0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t\neq 0$ is the curve given by $z(z+xz^7+x^6)=0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{yz} and C. We have the intersection number

$$D\cdot C=D\cdot E-D\cdot L_{yz}=\frac{169}{7\cdot 11\cdot 37}.$$

Also, we see

$$C^2=E\cdot C-C\cdot L_{yz}\geqslant E\cdot C-C_y\cdot C>0$$

since C is different from R_y. The multiplicity of D along the curve C is at most $\frac{4}{11}$ since the intersection number $C\cdot C_t$ is positive and the pair $(X,\frac{11}{4}D)$ is log canonical along the curve C_t. Then by Lemma 1.3.8 the inequality $D\cdot C<\frac{4}{11}$ gives us a contradiction.

Suppose that $\alpha=-1$. Then divisor E consists of three irreducible and reduced curves L_{yz}, R_x, and M. Note that the curve M is different from the curves R_y and L_{xt}. Also, it is smooth at the point P. We have

$$D\cdot M=D\cdot E-D\cdot L_{yz}-D\cdot R_x=\frac{147}{7\cdot 11\cdot 37},$$

$$M^2=E\cdot M-L_{yz}\cdot M-R_x\cdot M\geqslant E\cdot M-C_y\cdot M-C_x\cdot M>0.$$

The multiplicity of D along the curve M is at most $\frac{4}{11}$ since the intersection number $M\cdot C_t$ is positive and the pair $(X,\frac{11}{4}D)$ is log canonical along the curve C_t. By Lemma 1.3.8 the inequality $D\cdot M<\frac{4}{11}$ gives us a contradiction. Therefore, $P=O_t$.

Put $D=aL_{yz}+bR_x+\Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither L_{yz} nor R_x. Then $a>0$ since otherwise we would obtain an absurd inequality

$$\frac{3}{11}=37D\cdot L_{yz}\geqslant \mathrm{mult}_{O_t}(D)>\frac{4}{11}.$$

Therefore, we may assume that $R_y\not\subset\mathrm{Supp}(\Delta)$ by Lemma 1.3.6.

If $b>0$, the curve L_{xt} is not contained in the support of D, and hence

$$\frac{3}{21\cdot 29}=D\cdot L_{xt}\geqslant bR_x\cdot L_{xt}=\frac{2b}{21}.$$

Therefore, $b\leqslant\frac{3}{58}$. Similarly, we have

$$\frac{18}{29\cdot 37}=D\cdot R_y\geqslant\frac{6a}{37}+\frac{b}{37}+\frac{\mathrm{mult}_{O_t}(D)-a-b}{37}>\frac{5a}{37}+\frac{4}{11\cdot 37},$$

and hence $a<\frac{82}{5\cdot 11\cdot 29}$.

Let $\pi\colon\bar{X}\to X$ be the weighted blow up of the point O_t with weights $(13,4)$ and let F be the exceptional curve of the morphism π. Then F contains two singular points Q_{13} and Q_4 of $\bar{X}$ such that Q_{13} is a singular point of type $\frac{1}{13}(1,2)$ and Q_4 is a singular point of type $\frac{1}{4}(3,1)$. Then

$$K_{\bar{X}}\sim_{\mathbb{Q}}\pi^*(K_X)-\frac{20}{37}F,\ \bar{L}_{yz}\sim_{\mathbb{Q}}\pi^*(L_{yz})-\frac{4}{37}F,\ \bar{R}_x\sim_{\mathbb{Q}}\pi^*(R_x)-\frac{13}{37}F,\ \bar{\Delta}\sim_{\mathbb{Q}}\pi^*(\Delta)-\frac{c}{37}F,$$

where $\bar{L}_{yz}$, $\bar{R}_x$ and $\bar{\Delta}$ are the proper transforms of L_{yz}, R_x and Δ by π, respectively, and c is a non-negative rational number.

The log pull-back of the log pair $(X,\frac{11}{4}D)$ by π is the log pair

$$\left(\bar{X},\ \frac{11a}{4}\bar{L}_{yz}+\frac{11b}{4}\bar{R}_x+\frac{11}{4}\bar{\Delta}+\theta_1F\right),$$

where

$$\theta_1=\frac{11(4a+13b+c)+80}{4\cdot 37}.$$

This pair is not log canonical at some point $Q\in F$. We have

$$0\leqslant\bar{\Delta}\cdot\bar{R}_x=\frac{6+52b}{21\cdot 37}-\frac{a}{37}-\frac{c}{4\cdot 37}.$$

This inequality shows $4a+c\leqslant\frac{4}{21}(6+52b)$. Then

$$\theta_1=\frac{11(4a+c)}{4\cdot 37}+\frac{143b}{4\cdot 37}+\frac{20}{37}\leqslant\frac{11}{21\cdot 37}(6+52b)+\frac{143b}{4\cdot 37}+\frac{20}{37}=\frac{1944+5291b}{4\cdot 21\cdot 37}<1$$

since $b\leqslant\frac{3}{58}$. Note that $F\cap\bar{R}_x=\{Q_4\}$ and $F\cap\bar{L}_{yz}=\{Q_{13}\}$.

Suppose that the point Q is neither Q_4 nor Q_{13}. Then the pair $\big(\bar{X},\frac{11}{4}\bar{\Delta}+F\big)$ is not log canonical at the point Q. Then

$$\frac{11c}{16\cdot 13}=\frac{11}{4}\bar{\Delta}\cdot F>1$$

by Lemma 1.3.4. However, $c\leqslant 4a+c\leqslant\frac{4}{21}(6+52b)$. This is a contradiction since $b\leqslant\frac{3}{58}$. Therefore, the point Q is either Q_4 or Q_{13}.

Suppose that the point Q is the point Q_4. Then the log pair $\big(\bar{X},\frac{11b}{4}\bar{R}_x+\frac{11}{4}\bar{\Delta}+\theta_1F\big)$ is not log canonical at the point Q. It then follows from Lemma 1.3.4 that

$$1<4\left(\frac{11}{4}\bar{\Delta}+\theta_1F\right)\cdot\bar{R}_x=11\left(\frac{6+52b}{21\cdot 37}-\frac{a}{37}-\frac{c}{4\cdot 37}\right)+\theta_1.$$

However,

$$11\left(\frac{6+52b}{21\cdot 37}-\frac{a}{37}-\frac{c}{4\cdot 37}\right)+\theta_1=\frac{1944+5291b}{4\cdot 21\cdot 37}<1.$$

Therefore, the point Q must be the point Q_{13}.

Let $\phi\colon\tilde{X}\to\bar{X}$ be the weighted blow up at the point Q_{13} with weights $(1,2)$. Let G be the exceptional divisor of the morphism ϕ. Then G contains one singular point Q_2 of the surface $\tilde{X}$ that is a singular point of type $\frac{1}{2}(1,1)$. Let $\tilde{L}_{yz}$, $\tilde{R}_x$, $\tilde{\Delta}$ and $\tilde{F}$ be the proper transforms of L_{yz}, R_x, Δ and F by ϕ, respectively. We have

$$K_{\tilde{X}}\sim_{\mathbb{Q}}\phi^*(K_{\bar{X}})-\frac{10}{13}G,\ \tilde{L}_{yz}\sim_{\mathbb{Q}}\phi^*(\bar{L}_{yz})-\frac{2}{13}G,\ \tilde{F}\sim_{\mathbb{Q}}\phi^*(F)-\frac{1}{13}G,\ \tilde{\Delta}\sim_{\mathbb{Q}}\phi^*(\bar{\Delta})-\frac{d}{13}G,$$

where d is a non-negative rational number. The log pull-back of the log pair $(X,\frac{11}{4}D)$ via $\pi\circ\phi$ is

$$\left(\tilde{X},\frac{11a}{4}\tilde{L}_{yz}+\frac{11b}{4}\tilde{R}_x+\frac{11}{4}\tilde{\Delta}+\theta_1\tilde{F}+\theta_2G\right),$$

where

$$\theta_2=\frac{11}{4\cdot 13}(2a+d)+\frac{\theta_1}{13}+\frac{10}{13}=\frac{1560+11(78a+13b+c+37d)}{4\cdot 13\cdot 37}.$$

This log pair is not log canonical at some point $O\in G$. We have

$$0\leqslant\tilde{\Delta}\cdot\tilde{L}_{yz}=\frac{3+45a}{11\cdot 37}-\frac{b}{37}-\frac{c}{13\cdot 37}-\frac{d}{13}.$$

We then obtain $13b+c+37d\leqslant\frac{13}{11}(3+45a)$. Since $a<\frac{82}{5\cdot 11\cdot 29}$, we see

$$\theta_2=\frac{1560+11(78a+13b+c+37d)}{4\cdot 13\cdot 37}\leqslant\frac{1560+858a}{4\cdot 13\cdot 37}+\frac{3+45a}{4\cdot 37}<1.$$

Note that $\tilde{F}\cap G=Q_2$ and $Q_2\not\in\tilde{L}_{yz}$. Suppose that $O\not\in\tilde{F}\cup\tilde{L}_{yz}$. The log pair $\left(\tilde{X},\frac{11}{4}\tilde{\Delta}+G\right)$ is not log canonical at the point O. Applying Lemma 1.3.4, we get

$$1<\frac{11}{4}\tilde{\Delta}\cdot G=\frac{11d}{4\cdot 2},$$

and hence $d>\frac{8}{11}$. However, $d\leqslant\frac{1}{37}(13b+c+37d)\leqslant\frac{13}{11\cdot 37}(3+45a)$. This is a contradiction since $a<\frac{82}{5\cdot 11\cdot 29}$. Therefore, the point O is either the point Q_2 or the intersection point of G and $\tilde{L}_{yz}$. In the latter case, the pair $\left(\tilde{X},\frac{11a}{4}\tilde{L}_{yz}+\frac{11}{4}\tilde{\Delta}+\theta_2 G\right)$ is not log canonical at the point O. Then, applying Lemma 1.3.4, we get

$$1<\left(\frac{11}{4}\tilde{\Delta}+\theta_2 G\right)\cdot\tilde{L}_{yz}=\frac{11}{4}\left(\frac{3+45a}{11\cdot 37}-\frac{b}{37}-\frac{c}{13\cdot 37}-\frac{d}{13}\right)+\theta_2.$$

However,

$$\frac{11}{4}\left(\frac{3+45a}{11\cdot 37}-\frac{b}{37}-\frac{c}{13\cdot 37}-\frac{d}{13}\right)+\theta_2=\frac{11}{4}\left(\frac{3+45a}{11\cdot 37}\right)+\frac{1560+858a}{4\cdot 13\cdot 37}<1$$

since $a<\frac{82}{5\cdot 11\cdot 29}$. Therefore, the point O must be the point Q_2.

Let $\xi\colon\hat{X}\to\tilde{X}$ be the blow up at the point Q_2 and let H be the exceptional divisor of ξ. We also let $\hat{L}_{yz}$, $\hat{R}_x$, $\hat{\Delta}$, $\hat{G}$, and $\hat{F}$ be the proper transforms of $\tilde{L}_{yz}$, $\tilde{R}_x$, $\tilde{\Delta}$, G and $\tilde{F}$ by ξ, respectively. Then $\hat{X}$ is smooth along the exceptional divisor H. We have

$$K_{\hat{X}}\sim_{\mathbb{Q}}\xi^*(K_{\tilde{X}}),\ \hat{G}\sim_{\mathbb{Q}}\xi^*(G)-\frac{1}{2}H,\ \hat{F}\sim_{\mathbb{Q}}\xi^*(\tilde{F})-\frac{1}{2}H,\ \hat{\Delta}\sim_{\mathbb{Q}}\xi^*(\tilde{\Delta})-\frac{e}{2}H,$$

where e is a non-negative rational number. The log pull-back of the log pair $(X,\frac{11}{4}D)$ via $\pi\circ\phi\circ\xi$ is

$$\left(\hat{X},\frac{11a}{4}\hat{L}_{yz}+\frac{11b}{4}\hat{R}_x+\frac{11}{4}\hat{\Delta}+\theta_1\hat{F}+\theta_2\hat{G}+\theta_3 H\right),$$

where

$$\theta_3=\frac{\theta_1+\theta_2}{2}+\frac{11e}{8}=\frac{2600+11(130a+182b+14c+37d+481e)}{8\cdot 13\cdot 37}.$$

This log pair is not log canonical at some point $A\in H$. We have

$$\frac{c}{13\cdot 4}-\frac{d}{13\cdot 2}-\frac{e}{2}=\hat{\Delta}\cdot\hat{F}\geqslant 0.$$

Therefore, $2d+26e\leqslant c$. Then

$$\begin{aligned}\theta_3&=\frac{2600+11(130a+182b+14c)}{8\cdot 13\cdot 37}+\frac{11(d+13e)}{8\cdot 13}\leqslant\\&\leqslant\frac{5200+11(260a+364b+65c)}{16\cdot 13\cdot 37}=\\&=\frac{5200+4004b}{16\cdot 13\cdot 37}+\frac{11\cdot 65(4a+c)}{16\cdot 13\cdot 37}\leqslant\\&\leqslant\frac{100+77b}{4\cdot 37}+\frac{5\cdot 11(6+52b)}{4\cdot 21\cdot 37}=\frac{2430+4477b}{4\cdot 21\cdot 37}<1\end{aligned}$$

since $b\leqslant\frac{3}{58}$ and $4a+c\leqslant\frac{4}{21}(6+52b)$.

Suppose that $A \notin \hat{F} \cup \hat{G}$. Then the log pair $\left(\hat{X}, \frac{11}{4}\hat{\Delta} + \theta_3 H\right)$ is not log canonical at the point A. Applying Lemma 1.3.4, we get

$$1 < \frac{11}{4}\hat{\Delta} \cdot H = \frac{11e}{4}.$$

However,

$$e \leqslant \frac{1}{26}(2d + 26e) \leqslant \frac{c}{26} \leqslant \frac{1}{26}(4a + c) \leqslant \frac{4(6 + 52b)}{21 \cdot 26} \leqslant \frac{4}{11}.$$

Therefore, the point A must be either in $\hat{F}$ or in $\hat{G}$.

Suppose that $A \in \hat{F}$. Then the log pair $\left(\hat{X}, \frac{11}{4}\hat{\Delta} + \theta_1 \hat{F} + \theta_3 H\right)$ is not log canonical at the point A. Applying Lemma 1.3.4, we get

$$1 < \left(\frac{11}{4}\hat{\Delta} + \theta_3 H\right) \cdot \hat{F} = \frac{11}{4}\left(\frac{c}{4 \cdot 13} - \frac{d}{2 \cdot 13} - \frac{e}{2}\right) + \theta_3 = \frac{5200 + 11(260a + 364b + 65c)}{16 \cdot 13 \cdot 37}.$$

However,

$$\frac{5200 + 11(260a + 364b + 65c)}{16 \cdot 13 \cdot 37} = \frac{400 + 11 \cdot 28b}{16 \cdot 37} + \frac{11 \cdot 5(4a + c)}{16 \cdot 37} \leqslant \frac{2430 + 4477b}{4 \cdot 21 \cdot 37} < 1.$$

Therefore, the point A is the intersection point of H and $\hat{G}$. Then the log pair $\left(\hat{X}, \frac{11}{4}\hat{\Delta} + \theta_2 \hat{G} + \theta_3 H\right)$ is not log canonical at the point A. From Lemma 1.3.4, we obtain

$$1 < \left(\frac{11}{4}\hat{\Delta} + \theta_3 H\right) \cdot \hat{G} = \frac{11}{4}\left(\frac{d}{2} - \frac{e}{2}\right) + \theta_3 = \frac{2600 + 11(130a + 182b + 14c)}{8 \cdot 13 \cdot 37} + \frac{77d}{4 \cdot 13}.$$

However,

$$\frac{2600 + 11(130a + 182b + 14c)}{8 \cdot 13 \cdot 37} + \frac{77d}{4 \cdot 13} = \frac{100 + 55a}{4 \cdot 37} + \frac{77(13b + c + 37d)}{4 \cdot 13 \cdot 37} \leqslant \frac{121 + 370a}{4 \cdot 37} < 1$$

since $a < \frac{82}{5 \cdot 11 \cdot 29}$ and $13b + c + 37d \leqslant \frac{13}{11}(3 + 45a)$. The obtained contradiction completes the proof. □

Lemma 3.3.4. Let X be a quasismooth hypersurface of degree 196 in $\mathbb{P}(11, 37, 53, 98)$. Then $\mathrm{lct}(X) = \frac{55}{18}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + yz^3 + xy^5 + x^{13}z = 0.$$

It is singular at the points O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{55}{18} = \mathrm{lct}\left(X, \frac{3}{11}C_x\right) < \mathrm{lct}\left(X, \frac{3}{37}C_y\right) = \frac{37 \cdot 5}{26},$$

and hence $\mathrm{lct}(X) \leqslant \frac{55}{18}$.

Suppose that $\mathrm{lct}(X) < \frac{55}{18}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{55}{18}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither C_x nor C_y. Then the inequalities

$$53D \cdot C_x = \frac{6}{37} < \frac{18}{55}, \quad 11D \cdot C_y = \frac{6}{53} < \frac{18}{55}$$

show that the point P is a smooth point in the outside of C_x. However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(583))$ contains the monomials x^{53}, $y^{11}x^{16}$ and z^{11}, it follows from Lemma 1.3.9 that the point P is either a singular point of X or a point on C_x. This is a contradiction. □

Lemma 3.3.5. Let X be a quasismooth hypersurface of degree 95 in $\mathbb{P}(13, 17, 27, 41)$. Then $\mathrm{lct}(X) = \frac{65}{24}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^2t + y^4z + xt^2 + x^6y = 0.$$

The surface X is singular at the point O_x, O_y, O_z and O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xz} (resp. L_{yt}, L_{xz}, L_{yt}) and $R_x = \{x = y^4 + zt = 0\}$ (resp. $R_y = \{y = z^2 + xt = 0\}$, $R_z = \{z = t^2 + x^5y = 0\}$, $R_t = \{t = x^6 + y^3z = 0\}$). The curve L_{xz} intersects R_x (resp. R_z) only at the point O_t (resp. O_y). Also, the curve L_{yt} intersects R_y (resp. R_t) only at the point O_x (resp. O_z).

It is easy to check

$$-K_X \cdot L_{xz} = \frac{3}{17 \cdot 41},\ -K_X \cdot L_{yt} = \frac{1}{9 \cdot 13},\ -K_X \cdot R_x = \frac{4}{9 \cdot 41},$$

$$-K_X \cdot R_y = \frac{6}{13 \cdot 41},\ -K_X \cdot R_z = \frac{6}{13 \cdot 17},\ -K_X \cdot R_t = \frac{2}{3 \cdot 17},$$

$$L_{xz}^2 = -\frac{55}{17 \cdot 41},\ L_{yt}^2 = -\frac{37}{13 \cdot 27},\ R_x^2 = -\frac{56}{27 \cdot 41},\ R_y^2 = -\frac{48}{13 \cdot 41},\ R_z^2 = \frac{28}{13 \cdot 17}$$

$$R_t^2 = \frac{16}{3 \cdot 17},\ L_{xz} \cdot R_x = \frac{4}{41},\ L_{yt} \cdot R_y = \frac{2}{13},\ L_{xz} \cdot R_z = \frac{2}{17},\ L_{yt} \cdot R_t = \frac{2}{9}.$$

We have $\mathrm{lct}(X) \leqslant \frac{65}{24}$ since

$$\frac{65}{24} = \mathrm{lct}\left(X, \frac{3}{13}C_x\right) < \frac{51}{12} = \mathrm{lct}\left(X, \frac{3}{17}C_y\right) < \frac{41}{8} = \mathrm{lct}\left(X, \frac{3}{41}C_t\right) < \frac{21}{4} = \mathrm{lct}\left(X, \frac{3}{27}C_z\right).$$

Suppose that $\mathrm{lct}(X) < \frac{65}{24}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{65}{24}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of D does not contain at least one irreducible component of each of the curves C_x, C_y, C_z and C_t. The curve R_z is singular at the point O_y. The curve C_t is singular at O_z with multiplicity 3. Then in each of the following pairs of inequalities, at least one of two must hold:

$$\mathrm{mult}_{O_x}(D) \leqslant 13D \cdot L_{yt} = \frac{1}{9} < \frac{24}{65},\quad \mathrm{mult}_{O_x}(D) \leqslant 13D \cdot R_y = \frac{6}{41} < \frac{24}{65};$$

$$\mathrm{mult}_{O_y}(D) \leqslant 17D \cdot L_{xz} = \frac{3}{41} < \frac{24}{65},\quad \mathrm{mult}_{O_y}(D) \leqslant \frac{17}{2}D \cdot R_z = \frac{3}{13} < \frac{24}{65};$$

$$\mathrm{mult}_{O_z}(D) \leqslant 27D \cdot L_{yt} = \frac{3}{13} < \frac{24}{65},\quad \mathrm{mult}_{O_z}(D) \leqslant \frac{27}{3}D \cdot R_t = \frac{6}{17} < \frac{24}{65};$$

Therefore, the point P can be none of O_x, O_y, O_t.

Put $D = m_0L_{xz} + m_1L_{yt} + m_2R_x + m_3R_y + m_4R_z + m_5R_t + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of L_{xz}, L_{yt}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{65}{24}D)$ is log canonical at the points O_x, O_y, O_z, we have $m_i \leqslant \frac{24}{65}$ for each i. Since

$$(D - m_0L_{xz}) \cdot L_{xz} = \frac{3 + 55m_0}{17 \cdot 41} \leqslant \frac{24}{65}, \quad (D - m_1L_{yt}) \cdot L_{yt} = \frac{3 + 37m_1}{13 \cdot 27} \leqslant \frac{24}{65},$$
$$(D - m_2R_x) \cdot R_x = \frac{12 + 56m_2}{27 \cdot 41} \leqslant \frac{24}{65}, \quad (D - m_3R_y) \cdot R_y = \frac{6 + 48m_3}{13 \cdot 41} \leqslant \frac{24}{65},$$
$$(D - m_4R_z) \cdot R_z = \frac{6 - 28m_4}{13 \cdot 17} \leqslant \frac{24}{65}, \quad (D - m_5R_t) \cdot R_t = \frac{2 - 16m_5}{3 \cdot 17} \leqslant \frac{24}{65}$$

Lemma 1.3.8 implies that the point P cannot be a smooth point of X on $C_x \cup C_y \cup C_z \cup C_t$. Therefore, the point P is either a point in the outside of $C_x \cup C_y \cup C_z \cup C_t$ or the point O_t.

Suppose that the point P is not the point O_t. We consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda xt + \mu z^2 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. Then there is a unique curve Z_α in the pencil $\mathcal{L}$ passing through the point P. Since the point P is located in the outside of $C_x \cup C_z \cup C_t$, the curve Z_α is defined by an equation of the form

$$xt + \alpha z^2 = 0,$$

where α is a non-zero constant. Note that any component of C_t is not contained in Z_α. The open subset $Z_\alpha \setminus C_t$ is a $\mathbb{Z}_{41}$-quotient of the affine curve

$$x + \alpha z^2 = z^2 + y^4z + x + x^6y = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[x, y, z\big]\Big)$$

that is isomorphic to the plane affine curve defined by the equation

$$z(y^4 + (1 - \alpha)z + \alpha^6z^{11}y) = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big).$$

Therefore, if $\alpha \neq 1$, then the curve Z_α consists of two irreducible components L_{xz} and C_α. On the other hand, if $\alpha = 1$, then the curve Z_α consists of three irreducible components L_{xz}, R_y, and C_1. Since $P \notin C_x \cup C_y \cup C_z \cup C_t$, the point P must be contained in C_α (including $\alpha = 1$). Also, the curve C_α is smooth at the point P. By Lemma 1.3.6, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve Z_α.

Write $D = mC_\alpha + \Gamma$, where Γ is an effective $\mathbb{Q}$-divisor whose support contains C_α. Suppose that $m \neq 0$. If $\alpha \neq 1$, then we obtain

$$\frac{3}{17 \cdot 41} = D \cdot L_{xz} \geqslant mC_\alpha \cdot L_{xz} = \frac{109m}{17 \cdot 41}$$

and hence $m \leqslant \frac{3}{109}$. If $\alpha = 1$, then one of the inequalities

$$\frac{3}{17 \cdot 41} = D \cdot L_{xz} \geqslant mC_1 \cdot L_{xz} = \frac{92m}{17 \cdot 41}, \quad \frac{6}{13 \cdot 41} = D \cdot R_y \geqslant mC_1 \cdot R_y = \frac{11m}{41}$$

must hold, and hence $m \leqslant \frac{6}{11 \cdot 13}$. We also see that

$$D \cdot C_\alpha = \begin{cases} D \cdot (Z_\alpha - L_{xz}) = \dfrac{531}{13 \cdot 17 \cdot 41} & \text{if } \alpha \neq 1, \\ D \cdot (Z_\alpha - L_{xz} - R_y) = \dfrac{33}{17 \cdot 41} & \text{if } \alpha = 1. \end{cases}$$

Also, if $\alpha \neq 1$, then

$$C_\alpha^2 = Z_\alpha \cdot C_\alpha - L_{xz} \cdot C_\alpha \geqslant Z_\alpha \cdot C_\alpha - (L_{xz} + R_x) \cdot C_\alpha = \frac{41}{3} D \cdot C_\alpha.$$

If $\alpha = 1$,

$$C_1^2 = Z_\alpha \cdot C_1 - (L_{xz} + R_y) \cdot C_1 \geqslant Z_\alpha \cdot C_1 - (L_{xz} + R_x + L_{yt} + R_y) \cdot C_1 = 8D \cdot C_1.$$

In both cases, we have $C_\alpha^2 > 0$. Since

$$(D - mC_\alpha) \cdot C_\alpha \leqslant D \cdot C_\alpha < \frac{24}{65}$$

Lemma 1.3.8 gives us a contradiction. Therefore, the point P must be the point O_t.

If L_{xz} is not contained in the support of D, then the inequality

$$\mathrm{mult}_{O_t}(D) \leqslant 41 D \cdot L_{xz} = \frac{3}{17} < \frac{24}{65}$$

is a contradiction. Therefore, the irreducible component L_{xz} must be contained in the support of D, and hence the curve R_x is not contained in the support of D. Put $D = aL_{xz} + bR_y + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither L_{xz} nor R_y. Then

$$\frac{4}{9 \cdot 41} = D \cdot R_x \geqslant aL_{xz} \cdot R_x + \frac{\mathrm{mult}_{O_t}(D) - a}{41} > \frac{3a}{41} + \frac{24}{41 \cdot 65}$$

and hence $a \leqslant \frac{44}{585}$. If $b \neq 0$, then L_{yt} is not contained in the support of D. Therefore,

$$\frac{1}{9 \cdot 13} = D \cdot L_{yt} \geqslant bR_y \cdot L_{yt} = \frac{2b}{13},$$

and hence $b \leqslant \frac{1}{18}$.

Let $\pi\colon \bar{X} \to X$ be the weighted blow up at the point O_t with weights $(1, 4)$ and let F be the exceptional curve of the morphism π. Then F contains one singular point Q_4 of $\bar{X}$ such that Q_4 is a singular point of type $\frac{1}{4}(3, 1)$. Then

$$K_{\bar{X}} \sim_{\mathbb{Q}} \pi^*(K_X) - \frac{36}{41}F, \quad \bar{L}_{xz} \sim_{\mathbb{Q}} \pi^*(L_{xz}) - \frac{4}{41}F, \quad \bar{R}_y \sim_{\mathbb{Q}} \pi^*(R_y) - \frac{1}{41}F, \quad \bar{\Delta} \sim_{\mathbb{Q}} \pi^*(\Delta) - \frac{c}{41}F,$$

where $\bar{L}_{xz}$, $\bar{R}_y$ and $\bar{\Delta}$ are the proper transforms of L_{xz}, R_y and Δ by π, respectively, and c is a non-negative rational number. Note that $F \cap \bar{R}_y = \{Q_4\}$.

The log pull-back of the log pair $(X, \frac{65}{24}D)$ by π is the log pair

$$\left(\bar{X},\ \frac{65a}{24}\bar{L}_{xz} + \frac{65b}{24}\bar{R}_y + \frac{65}{24}\bar{\Delta} + \theta_1 F\right),$$

where

$$\theta_1 = \frac{864 + 65(4a + b + c)}{24 \cdot 41}.$$

This is not log canonical at some point $Q \in F$. We have

$$0 \leqslant \bar{\Delta} \cdot \bar{L}_{xz} = \frac{3 + 55a}{17 \cdot 41} - \frac{b}{41} - \frac{c}{41}.$$

This inequality shows $b + c \leqslant \frac{1}{17}(3 + 55a)$. Since $a \leqslant \frac{44}{585}$, we obtain

$$\theta_1 = \frac{864 + 260a}{24 \cdot 41} + \frac{65(b + c)}{24 \cdot 41} \leqslant \frac{864 + 260a}{24 \cdot 41} + \frac{65(3 + 55a)}{17 \cdot 24 \cdot 41} = \frac{121 + 65a}{8 \cdot 17} < 1.$$

Suppose that the point Q is neither Q_4 nor the intersection point of F and $\bar{L}_{xz}$. Then, the point Q is not in $\bar{L}_{xz}\cup\bar{R}_y$. Therefore, the pair $\big(\bar{X},\frac{65}{24}\bar{\Delta}+F\big)$ is not log canonical at the point Q, and hence

$$1<\frac{65}{24}\bar{\Delta}\cdot F=\frac{65c}{4\cdot 24}.$$

But $c\leqslant b+c\leqslant\frac{1}{17}(3+55a)<\frac{4\cdot 24}{65}$ since $a\leqslant\frac{44}{585}$. Therefore, the point Q is either Q_4 or the intersection point of F and $\bar{L}_{xz}$.

Suppose that the point Q is the intersection point of F and $\bar{L}_{xz}$. Then the point Q is in $\bar{L}_{xz}$ but not in $\bar{R}_y$. Therefore, the pair $\big(\bar{X},\bar{L}_{xz}+\frac{65}{24}\bar{\Delta}+\theta_1F\big)$ is not log canonical at the point Q. Then

$$1<\left(\frac{65}{24}\bar{\Delta}+\theta_1F\right)\cdot\bar{L}_{xz}=\frac{65}{24}\left(\frac{3+55a}{17\cdot 41}-\frac{b+c}{41}\right)+\theta_1=\frac{121+65a}{8\cdot 17}.$$

However, this is impossible since $a\leqslant\frac{44}{585}$. Therefore, the point Q must be the point Q_4.

Let $\psi\colon\tilde{X}\to\bar{X}$ be the weighted blow up at the point Q_4 with weights $(3,1)$ and let E be the exceptional curve of the morphism ψ. The exceptional curve E contains one singular point O_3 of $\tilde{X}$. This singular point is of type $\frac{1}{3}(1,2)$. Then

$$K_{\tilde{X}}\sim_{\mathbb{Q}}\psi^*(K_{\bar{X}}),\quad\tilde{R}_y\sim_{\mathbb{Q}}\psi^*(\bar{R}_y)-\frac{3}{4}E,\quad\tilde{F}\sim_{\mathbb{Q}}\psi^*(F)-\frac{1}{4}E,\quad\tilde{\Delta}\sim_{\mathbb{Q}}\psi^*(\bar{\Delta})-\frac{d}{4}E,$$

where $\tilde{R}_y$, $\tilde{F}$ and $\tilde{\Delta}$ are the proper transforms of $\bar{R}_y$, F and $\bar{\Delta}$ by ψ, respectively, and d is a non-negative rational number.

The log pull-back of the log pair $(X,\frac{65}{24}D)$ by $\pi\circ\psi$ is the log pair

$$\left(\tilde{X},\ \frac{65a}{24}\tilde{L}_{xz}+\frac{65b}{24}\tilde{R}_y+\frac{65}{24}\tilde{\Delta}+\theta_1\tilde{F}+\theta_2E\right),$$

where

$$\theta_2=\frac{65(3b+d)}{4\cdot 24}+\frac{1}{4}\theta_1.$$

This is not log canonical at some point $O\in E$.

We have

$$0\leqslant\tilde{\Delta}\cdot\tilde{R}_y=\bar{\Delta}\cdot\bar{R}_y-\frac{d}{4}=\frac{6+48b}{13\cdot 41}-\frac{4a+c}{4\cdot 41}-\frac{d}{4},$$

and hence $4a+c+41d\leqslant\frac{4}{13}(6+48b)$. Therefore, this inequality together with $b\leqslant\frac{1}{18}$ gives us

$$\begin{aligned}\theta_2&=\frac{65(3b+d)}{4\cdot 24}+\frac{864+65(4a+b+c)}{4\cdot 24\cdot 41}=\\&=\frac{864+8060b+65(4a+c+41d)}{4\cdot 24\cdot 41}\leqslant\\&\leqslant\frac{6+55b}{24}<1.\end{aligned}$$

Suppose that the point O is in the outside of $\tilde{R}_y$ and $\tilde{F}$. Then the log pair $(E,\frac{65}{24}\tilde{\Delta}|_E)$ is not log canonical at the point O, and hence

$$1<\frac{65}{24}\tilde{\Delta}\cdot E=\frac{65d}{72}.$$

However,

$$41d \leqslant 4a + c + 41d \leqslant \frac{4}{13}(6 + 48b) < \frac{41 \cdot 72}{65}$$

since $b \leqslant \frac{1}{18}$. This is a contradiction.

Suppose that the point O belongs to $\tilde{R}_y$ Then the log pair $\left(E, \left(\frac{65b}{24}\tilde{R}_y + \frac{65}{24}\tilde{\Delta}\right)\Big|_E\right)$ is not log canonical at the point O, and hence

$$1 < \left(\frac{65b}{24}\tilde{R}_y + \frac{65}{24}\tilde{\Delta}\right) \cdot E = \frac{65}{24}\left(b + \frac{d}{3}\right).$$

However,

$$\frac{65}{24}\left(b + \frac{d}{3}\right) \leqslant \frac{65}{24}\left(b + \frac{4}{3 \cdot 13 \cdot 41}(6 + 48b)\right) < 1$$

since $b \leqslant \frac{1}{18}$. This is a contradiction. Therefore, the point O is the point O_3 which is the intersection point of E and $\tilde{F}$.

Let $\xi\colon \hat{X} \to \tilde{X}$ be the weighted blow up at the point O with weights $(1, 2)$ and let H be the exceptional divisor of ξ. The exceptional divisor H contains a singular point of $\hat{X}$. This singular point is of type $\frac{1}{2}(1, 1)$. We have

$$K_{\hat{X}} \sim_{\mathbb{Q}} \xi^*(K_{\tilde{X}}),\ \hat{E} \sim_{\mathbb{Q}} \xi^*(E) - \frac{1}{3}H,\ \hat{F} \sim_{\mathbb{Q}} \xi^*(\tilde{F}) - \frac{2}{3}H,\ \hat{\Delta} \sim_{\mathbb{Q}} \xi^*(\tilde{\Delta}) - \frac{e}{3}H,$$

where $\hat{E}$, $\hat{F}$, $\hat{\Delta}$, be the proper transforms of E, $\tilde{F}$, $\tilde{\Delta}$ by ξ, respectively, and e is a non-negative rational number. The log pull-back of the log pair $(X, \frac{65}{24}D)$ via $\pi \circ \psi \circ \xi$ is

$$\left(\hat{X}, \frac{65a}{24}\hat{L}_{xz} + \frac{65b}{24}\hat{R}_y + \frac{65}{24}\hat{\Delta} + \theta_1\hat{F} + \theta_2\hat{E} + \theta_3 H\right),$$

where $\hat{L}_{xz}$ and $\hat{R}_y$ are the proper transforms of $\tilde{L}_{xz}$ and $\tilde{R}_y$ by ξ, respectively, and

$$\theta_3 = \frac{1}{3}(2\theta_1 + \theta_2) + \frac{65e}{3 \cdot 24}.$$

This log pair is not log canonical at some point $A \in H$. We have

$$0 \leqslant \hat{\Delta} \cdot \hat{F} = \bar{\Delta} \cdot F - \frac{d}{12} - \frac{e}{3} = \frac{c}{4} - \frac{d}{12} - \frac{e}{3},$$

and hence $d + 4e \leqslant 3c$. Then

$$\begin{aligned}
\theta_3 = \frac{1}{3}(2\theta_1 + \theta_2) + \frac{65e}{3 \cdot 24} = & = \frac{3}{4}\theta_1 + \frac{65(3b + d)}{3 \cdot 4 \cdot 24} + \frac{65e}{3 \cdot 24} \leqslant \\
& = \frac{2592 + 65(12a + 44b + 3c)}{3 \cdot 32 \cdot 41} + \frac{65(d + 4e)}{3 \cdot 4 \cdot 24} \leqslant \\
& \leqslant \frac{2592 + 65(12a + 44b + 3c)}{3 \cdot 32 \cdot 41} + \frac{65c}{4 \cdot 24} = \\
& = \frac{2592 + 65(12a + 44b + 44c)}{3 \cdot 32 \cdot 41} \leqslant \\
& \leqslant \frac{216 + 65a}{8 \cdot 41} + \frac{65 \cdot 11(3 + 55a)}{3 \cdot 8 \cdot 17 \cdot 41} = \frac{321 + 1040a}{3 \cdot 8 \cdot 17} < 1
\end{aligned}$$

since $b+c\leqslant\frac{1}{17}(3+55a)$ and $a\leqslant\frac{44}{585}$.

Suppose that $A\not\in\hat{F}\cup\hat{E}$. Then the log pair $\left(\hat{X},\frac{65}{24}\hat{\Delta}+\theta_3H\right)$ is not log canonical at the point A. Applying Lemma 1.3.4, we get

$$1<\frac{65}{24}\hat{\Delta}\cdot H=\frac{65e}{48}.$$

However,

$$e\leqslant\frac{1}{4}(d+4e)\leqslant\frac{3c}{4}\leqslant\frac{3}{4}(b+c)\leqslant\frac{3(3+55a)}{4\cdot 17}<\frac{48}{65}.$$

Therefore, the point A must be either in $\hat{F}$ or in $\hat{E}$.

Suppose that $A\in\hat{F}$. Then the log pair $\left(\hat{X},\frac{65}{24}\hat{\Delta}+\theta_1\hat{F}+\theta_3H\right)$ is not log canonical at the point A. Applying Lemma 1.3.4, we get

$$1<\left(\frac{65}{24}\hat{\Delta}+\theta_3H\right)\cdot\hat{F}=\frac{65}{24}\left(\frac{c}{4}-\frac{d}{12}-\frac{e}{3}\right)+\theta_3=\frac{2592+65(12a+44b+44c)}{3\cdot 32\cdot 41}.$$

However,

$$\frac{2592+65(12a+44b+44c)}{3\cdot 32\cdot 41}\leqslant\frac{321+1040a}{3\cdot 8\cdot 17}<1.$$

Therefore, the point A is the intersection point of H and $\hat{E}$. Then the log pair $\left(\hat{X},\frac{65}{24}\hat{\Delta}+\theta_2\hat{E}+\theta_3H\right)$ is not log canonical at the point A. From Lemma 1.3.4, we obtain

$$1<2\left(\frac{65}{24}\hat{\Delta}+\theta_3H\right)\cdot\hat{E}=\frac{65}{24}\left(\frac{2d}{3}-\frac{e}{3}\right)+\theta_3=\frac{2592+65(12a+44b+3c)}{3\cdot 32\cdot 41}+\frac{65d}{32}$$

However,

$$\frac{2592+65(12a+44b+3c)}{3\cdot 32\cdot 41}+\frac{65d}{32}=\frac{648+715b}{3\cdot 8\cdot 41}+\frac{65(4a+c+41d)}{32\cdot 41}\leqslant\frac{12186+21515b}{17\cdot 24\cdot 41}<1$$

since $b\leqslant\frac{1}{18}$ and $4a+c+41d\leqslant\frac{4}{13}(6+48b)$. The obtained contradiction completes the proof. □

Lemma 3.3.6. Let X be a quasismooth hypersurface of degree 196 in $\mathbb{P}(13,27,61,98)$. Then $\mathrm{lct}(X)=\frac{91}{30}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2+y^5z+xz^3+x^{13}y=0.$$

The surface X is singular at the points O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{91}{30}=\mathrm{lct}\left(X,\frac{3}{13}C_x\right)<\mathrm{lct}\left(X,\frac{3}{27}C_y\right)=\frac{15}{2}.$$

Therefore, $\mathrm{lct}(X)\leqslant\frac{91}{30}$.

Suppose that $\mathrm{lct}(X)<\frac{91}{30}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{91}{30}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither the curve C_x nor the curve C_y. Then the inequalities

$$61D\cdot C_x=\frac{2}{9}<\frac{30}{91},\quad 13D\cdot C_y=\frac{6}{61}<\frac{30}{91}$$

show that the point P is a smooth point in the outside of C_x. However, $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(793))$ contains x^{61}, $y^{26}x^7$, $y^{13}x^{34}$ and z^{13}, it follows from Lemma 1.3.9 that the point P must be a singular point of X or a point on C_x. This is a contradiction. □

Lemma 3.3.7. Let X be a quasismooth hypersurface of degree 148 in $\mathbb{P}(15, 19, 43, 74)$. Then $\mathrm{lct}(X) = \frac{57}{14}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + yz^3 + xy^7 + x^7z = 0.$$

The surface X is singular at the points O_x, O_y and O_z. The curves C_x, C_y and C_z are irreducible. We can see that

$$\mathrm{lct}\left(X, \frac{3}{19}C_y\right) = \frac{57}{14} < \mathrm{lct}\left(X, \frac{3}{15}C_x\right) = \frac{25}{6} < \mathrm{lct}\left(X, \frac{3}{43}C_z\right) = \frac{129}{14}.$$

Therefore, $\mathrm{lct}(X) \leqslant \frac{57}{14}$.

Suppose that $\mathrm{lct}(X) < \frac{57}{14}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{57}{14}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains none of C_x, C_y, C_z. Note that the curve C_y is singular at the point O_z. The inequalities

$$19D \cdot C_x = \frac{6}{43} < \frac{14}{57}, \quad \frac{43}{2}D \cdot C_y = \frac{1}{5} < \frac{14}{57}, \quad D \cdot C_z = \frac{2}{95} < \frac{14}{57}$$

show that the point P is located in the outside of $C_x \cup C_y \cup C_z$.

Now we consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda z^3 + \mu xy^6 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. Then there is a unique member C in $\mathcal{L}$ passing through the point P. Since the point P is located in the outside of $C_x \cup C_y \cup C_z$, the curve C is cut out by the equation of the form $xy^6 + \alpha z^3 = 0$, where α is a non-zero constant. Since the curve C is a double cover of the curve defined by the equation $xy^6 + \alpha z^3 = 0$ in $\mathbb{P}(15, 19, 43)$, we have $\mathrm{mult}_P(C) \leqslant 2$. Therefore, we may assume that the support of D does not contain at least one irreducible component. If $\alpha \neq 1$, then the curve C is irreducible, and hence the inequality

$$\mathrm{mult}_P(D) \leqslant D \cdot C = \frac{6}{5 \cdot 19} < \frac{14}{57}$$

is a contradiction. If $\alpha = 1$, then the curve C consists of two distinct irreducible and reduced curve C_1 and C_2. We have

$$D \cdot C_1 = D \cdot C_2 = \frac{3}{5 \cdot 19}, \quad C_1^2 = C_2^2 = \frac{11}{19}.$$

Put $D = a_1C_1 + a_2C_2 + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither C_1 nor C_2. Since the pair $(X, \frac{57}{14}D)$ is log canonical at O_x, both a_1 and a_2 are at most $\frac{14}{57}$. Then a contradiction follows from Lemma 1.3.8 since

$$(D - a_iC_i) \cdot C_i \leqslant D \cdot C_i = \frac{3}{5 \cdot 19} < \frac{14}{57}$$

for each i. □

3.4. Sporadic cases with $I = 4$

Lemma 3.4.1. Let X be a quasismooth hypersurface of degree 24 in $\mathbb{P}(5, 6, 8, 9)$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3 + yt^2 - y^4 + \epsilon x^2yz + x^3t = 0,$$

where $\epsilon \in \mathbb{C}$. The surface X is singular at the points O_x, O_t, $Q_2 = [0 : 1 : 1 : 0]$ and $Q_3 = [0 : 1 : 0 : 1]$.

The curves C_x, C_y, C_z and C_t are all irreducible. We have

$$1 = \mathrm{lct}\Big(X, \frac{4}{6}C_y\Big) < \mathrm{lct}\Big(X, \frac{4}{5}C_x\Big) = \frac{5}{4} < \mathrm{lct}\Big(X, \frac{4}{8}C_z\Big) = 2$$

and $\mathrm{lct}\big(X, \frac{4}{9}C_t\big) > 1$. Therefore, $\mathrm{lct}(X) \leqslant 1$.

Suppose that $\mathrm{lct}(X) < 1$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair (X, D) is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains none of the curves C_x, C_y, C_z and C_t. Also, the curve C_y is singular at the point O_t with multiplicity 3 and the curve C_t is singular at the point O_x. Then the following intersection numbers show that the point P is located in the outside of the set $C_x \cup C_y \cup C_z \cup C_t$:

$$3D \cdot C_x = \frac{2}{3} < 1, \quad \frac{9}{3}D \cdot C_y = \frac{4}{5} < 1, \quad D \cdot C_z = \frac{16}{45} < 1, \quad \frac{5}{2}D \cdot C_t = 1.$$

Now we consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda xt + \mu yz = 0$, where $[\lambda : \mu] \in \mathbb{P}^1$. There is a unique member Z in the pencil $\mathcal{L}$ passing through the point P. Since $P \notin C_x \cup C_y \cup C_z \cup C_t$, the divisor Z is defined by an equation of the form

$$xt = \alpha yz,$$

where α is non-zero constant. Note that the curve C_x is not contained in the support of Z. The open subset $Z \setminus C_x$ of the curve Z is a $\mathbb{Z}_5$-quotient of the affine curve

$$t - \alpha yz = z^3 + yt^2 + y^4 + \epsilon yz + t = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big),$$

that is isomorphic to the plane affine quintic curve Z' given by the equation

$$z^3 + \alpha^2y^3z^2 + y^4 + (\epsilon + \alpha)yz = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big).$$

This affine plane curve Z' is irreducible and hence the curve Z is also irreducible. The multiplicity of Z at the point P is at most 3 since the quintic Z' is singular at the origin. This implies that the log pair $(X, \frac{4}{14}Z)$ is log canonical at the point P. Thus, we may assume that $\mathrm{Supp}(D)$ does not contain the curve Z by Lemma 1.3.6. Then we obtain a contradictory inequality

$$\frac{28}{45} = D \cdot Z \geqslant \mathrm{mult}_P\big(D\big) > 1.$$

□

Lemma 3.4.2. Let X be a quasismooth hypersurface of degree 30 in $\mathbb{P}(5, 6, 8, 15)$. Then $\mathrm{lct}(X) = 1$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t(t-x^3)-y^5+yz^3+\epsilon x^2y^2z=0.$$

The surface X is singular at the points O_x, O_z, $Q_5=[1:0:0:1]$, $Q_3=[0:1:0:1]$ and $Q_2=[0:1:1:0]$.

The curve C_x is irreducible. However, the curve C_y consists of two irreducible curves L_{yt} and $L=\{y=t-x^3=0\}$. It is easy to check

$$1=\mathrm{lct}\left(X,\frac{4}{6}C_y\right)<\mathrm{lct}\left(X,\frac{4}{5}C_x\right)=\frac{5}{4}.$$

Therefore, $\mathrm{lct}(X)\leqslant 1$.

Suppose that $\mathrm{lct}(X)<1$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair (X,D) is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D does not contain the curve C_x. Similarly, we may assume that the support of D does not contain either L_{yt} or L.

We have the following intersection numbers for L_{yt} and L:

$$L_{yt}^2=L^2=-\frac{9}{40},\quad L_{yt}\cdot L=\frac{3}{8}.$$

Since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(30))$ contains the monomials y^5, yz^3 and t^2, it follows from Lemma 1.3.9 that the point P is either a singular point of X or a point on C_y. However, since $3D\cdot C_x=\frac{1}{2}<1$, the point P must belong to the curve C_y.

Since the support of D does not contain either L_{yt} or L, one of the inequalities

$$\mathrm{mult}_{O_z}(D)\leqslant 8D\cdot L_{yt}=\frac{4}{5}<1,\quad \mathrm{mult}_{O_z}(D)\leqslant 8D\cdot L=\frac{4}{5}<1$$

must hold, and hence the point P cannot be the point O_z.

We put $D=kL+mL_{yt}+\Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither L nor L_{yt}. If $k\neq 0$, then $m=0$ and

$$\frac{1}{10}=D\cdot L_{yt}\geqslant kL\cdot L_{yt}=\frac{3k}{8}.$$

Therefor, $k\leqslant\frac{4}{15}$. By the same way, we can also obtain $m\leqslant\frac{4}{15}$. Then, by Lemma 1.3.8, the inequalities

$$5(D-kL)\cdot L=\frac{4+9k}{8}<1,\quad 5(D-mL_{yt})\cdot L_{yt}=\frac{4+9m}{8}<1$$

show that the point P cannot belong to the curve C_y. This is a contradiction. □

Lemma 3.4.3. Let X be a quasismooth hypersurface of degree 45 in $\mathbb{P}(9,11,12,17)$. Then $\mathrm{lct}(X)=\frac{77}{60}$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2y+y^3z+xz^3+x^5=0.$$

It is singular at the points O_y, O_z, O_t, and the point $Q_3=[1:0:-1:0]$. The curve C_x consists of two irreducible and reduced curves L_{xy} and $R_x=\{x=t^2+y^2z=0\}$. The curve C_y consists of two irreducible and reduced curves L_{xy} and $R_y=\{y=z^3+x^4=0\}$. The curves C_z and C_t

are irreducible and reduced. It is easy to check that $\mathrm{lct}(X, \frac{4}{11}C_y) = \frac{77}{60}$ is less than each of the numbers $\mathrm{lct}(X, \frac{4}{9}C_x)$, $\mathrm{lct}(X, \frac{4}{12}C_z)$ and $\mathrm{lct}(X, \frac{4}{17}C_t)$.

Suppose that $\mathrm{lct}(X) < \frac{77}{60}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{77}{60}D)$ is not log canonical at some point P. By Lemma 1.3.6 we may assume that the support of D contains neither C_z nor C_t. Similarly, we may assume that the support of D does not contain either L_{xy} or R_x. Also, we may assume that the support of D does not contain either L_{xy} or R_y. Then in each of the following pairs of inequalities, at least one of two must hold:

$$\mathrm{mult}_{O_z}(D) \leqslant 12D \cdot L_{xy} = \frac{4}{17} < \frac{60}{77}, \quad \mathrm{mult}_{O_z}(D) \leqslant 12D \cdot R_x = \frac{8}{11} < \frac{60}{77};$$
$$\mathrm{mult}_{O_t}(D) \leqslant 17D \cdot L_{xy} = \frac{1}{3} < \frac{60}{77}, \quad \mathrm{mult}_{O_t}(D) \leqslant \frac{17}{3}D \cdot R_y = \frac{4}{9} < \frac{60}{77}.$$

Therefore, the point P can be neither O_z nor O_t. The curve C_z is singular at the point O_y. Then the inequalities

$$\frac{11}{2}D \cdot C_z = \frac{10}{17} < \frac{60}{77}, \quad 3D \cdot C_t = \frac{5}{11} < \frac{60}{77}$$

imply that the point P cannot belong to $C_z \cup C_t$.

We can see that

$$L_{xy} \cdot D = \frac{1}{17 \cdot 3}, \quad R_x \cdot D = \frac{2}{33}, \quad R_y \cdot D = \frac{4}{3 \cdot 17}, \quad L_{xy} \cdot R_x = \frac{1}{6},$$
$$L_{xy} \cdot R_y = \frac{3}{17}, \quad L_{xy}^2 = -\frac{25}{12 \cdot 17}, \quad R_x^2 = -\frac{1}{33}, \quad R_y^2 = \frac{2}{3 \cdot 17}.$$

If we write $D = nL_{xy} + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{xy}, then we can see that $n \leqslant \frac{4}{11}$ since $D \cdot R_x \geqslant nR_x \cdot L_{xy}$ for $n \neq 0$. By Lemma 1.3.8 the inequality

$$(L_{xy} \cdot D - nL_{xy}^2) = \frac{4 + 25n}{12 \cdot 17} < \frac{60}{77}$$

implies that the point P cannot belong to the curve L_{xy}. By the same method, we see that the point P must be in the outside of R_x.

If we write $D = mR_y + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve R_y, then we can see that $0 \leqslant m \leqslant \frac{1}{9}$ since $D \cdot L_{xy} \geqslant mR_y \cdot L_{xy}$ for $m \neq 0$. By Lemma 1.3.8 the inequality

$$(R_y \cdot D - mR_y^2) \leqslant R_y \cdot D < \frac{60}{77}$$

implies that the point P cannot belong to the curve R_y.

Now we consider the pencil $\mathcal{L}$ on X cut out by $\lambda t^2 + \mu y^2 z = 0$. The base locus of the pencil $\mathcal{L}$ consists of three points O_y, O_z, and Q. Let F be the member in $\mathcal{L}$ defined by $t^2 + y^2 z = 0$. The divisor F consists of two irreducible and reduced curves R_x and $E = \{t^2 + y^2 z = x^4 + z^3 = 0\}$. The curve E is smooth in the outside of the base points. We have

$$E \cdot D = \frac{8}{33}.$$

Since

$$E^2 = F \cdot E - R_x \cdot E \geqslant F \cdot E - (L_{xy} + R_x) \cdot E = \frac{25}{4}D \cdot E,$$

the self-intersection E^2 is positive. We write $D = kE + \Gamma$, where Γ is an effective $\mathbb{Q}$-divisor whose support does not contain the curve E. Since $(X, \frac{77}{60}D)$ is log canonical at the point O_y, the non-negative number k is at most $\frac{60}{77}$. By Lemma 1.3.8, the inequality

$$(E \cdot D - kE^2) \leqslant E \cdot D = \frac{8}{33} < \frac{60}{77}$$

implies that the point P cannot belong to the curve E.

So far we have seen that the point P must lie in the outside of $C_x \cup C_y \cup C_z \cup C_t \cup E$. In particular, it is a smooth point. There is a unique member C in $\mathcal{L}$ which passes through the point P. Then the curve C is cut out by $t^2 = \alpha y^2 z$ where α is a constant different from 0 and -1. The curve C is isomorphic to the curve defined by $y^3z + xz^3 + x^5 = 0$ and $t^2 = y^2z$. The curve C is smooth in the outside of the base points and the singular locus of X by the Bertini theorem, since it is isomorphic to a general curve in the pencil $\mathcal{L}$. We claim that the curve C is irreducible. If so then we may assume that the support of D does not contain the curve C and hence we obtain

$$\mathrm{mult}_P(D) \leqslant C \cdot D = \frac{10}{33} < \frac{60}{77}.$$

This is a contradiction.

For the irreducibility of the curve C, we may consider the curve C as a surface in $\mathbb{C}^4$ defined by the equations $y^3z + xz^3 + x^5 = 0$ and $t^2 = y^2z$. Then, we consider the surface in $\mathbb{P}^4$ defined by the equations $y^3zw + xz^3w + x^5 = 0$ and $t^2w = y^2z$. We take the affine piece defined by $t \neq 0$. This affine piece is isomorphic to the surface defined by the equation $y^3zw + xz^3w + x^5 = 0$ and $w = y^2z$ in $\mathbb{C}^4$. It is isomorphic to the irreducible hypersurface $y^5z^2 + xy^2z^5 + x^5 = 0$ in $\mathbb{C}^3$. Therefore, the curve C is irreducible. □

Lemma 3.4.4. Let X be a quasismooth hypersurface of degree 75 in $\mathbb{P}(10, 13, 25, 31)$. Then $\mathrm{lct}(X) = \frac{91}{60}$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2y + z^3 + xy^5 + x^5z = 0.$$

It has singular points at O_x, O_y, O_t and $Q = [-1 : 0 : 1 : 0]$. The curve C_x and C_t are irreducible and reduced. The curve C_y (resp. C_z) consists of two irreducible reduced curves L_{yz} and $R_y = \{y = z^2 + x^5 = 0\}$ (resp. $R_z = \{y = t^2 + xy^4 = 0\}$). It is easy to see that

$$\mathrm{lct}(X, \frac{4}{13}C_y) = \frac{91}{60} < \mathrm{lct}(X, \frac{4}{10}C_x) < \mathrm{lct}(X, \frac{4}{25}C_z) < \mathrm{lct}(X, \frac{4}{31}C_t).$$

Also, we have the following intersection numbers:

$$-K_X \cdot L_{yz} = \frac{2}{5 \cdot 31}, \quad -K_X \cdot R_y = \frac{4}{5 \cdot 31}, \quad -K_X \cdot R_z = \frac{4}{5 \cdot 13},$$

$$L_{yz} \cdot R_y = \frac{5}{31}, \quad L_{yz} \cdot R_z = \frac{1}{5}, \quad L_{yz}^2 = -\frac{37}{10 \cdot 31}, \quad R_y^2 = -\frac{12}{5 \cdot 31}, \quad R_z^2 = \frac{12}{5 \cdot 13}.$$

Suppose that $\mathrm{lct}(X) < \frac{91}{60}$. Then, there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{91}{60}D)$ is not log canonical at some point $P \in X$. Since the curves C_x and C_t are irreducible we may assume that the support of D contains none of them. The inequalities

$$13D \cdot C_x < \frac{60}{91}, \quad 5D \cdot C_t < \frac{60}{91}$$

show that the point P must lie in the outside of $C_x \cup C_t \setminus \{O_x, O_t\}$.

By Lemma 1.3.6, we may assume that the support of D does not contain either L_{yz} or R_y. If the support of D does not contain L_{yz}, then the inequality

$$31D \cdot L_{yz} = \frac{2}{5} < \frac{60}{91}$$

shows that the point P cannot be O_t. On the other hand, if the support of D does not contain R_y, then the inequality

$$\frac{31}{2}D \cdot R_y = \frac{2}{5} < \frac{60}{91}$$

shows that the point P cannot be O_t. Note that the curve R_y is singular at the point O_t. We use the same method for $C_z = R_z + L_{yz}$ so that we can see that the point P cannot be O_x.

We write $D = mR_y + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support does not contain the curve R_y. Then we see $m \leqslant \frac{2}{25}$ since the support of D does not contain either L_{yz} or R_y and $D \cdot L_{yz} \geqslant mR_y \cdot L_{yz}$. Since $R_y \cdot D - mR_y^2 < \frac{60}{91}$, Lemma 1.3.8 implies that the point P is located in the outside of R_y. Using the same argument for L_{yz} , we can also see that the point P is located in the outside of L_{yz}. Also, the same method shows that the point P is located in the outside of R_z. Consequently, the point P must lie in the outside of $C_x \cup C_y \cup C_z \cup C_t$.

Now we consider the pencil $\mathcal{L}$ on X cut out by $\lambda t^2 + \mu xy^4 = 0$. The base locus of the pencil $\mathcal{L}$ consists of three points O_x, O_y, and Q. Let F be the member of $\mathcal{L}$ defined by $t^2 + xy^4 = 0$. The divisor F consists of two irreducible and reduced curves R_z and $E = \{t^2 + xy^4 = z^2 + x^5 = 0\}$. The curve E is smooth in the outside of $\mathrm{Sing}(X)$. We have

$$E \cdot D = \frac{8}{5 \cdot 13}.$$

Since

$$E^2 = F \cdot E - R_z \cdot E \geqslant F \cdot E - (L_{yz} + R_z) \cdot E = \frac{37}{4} D \cdot E,$$

the self-intersection E^2 is positive. We write $D = kE + \Gamma$, where Γ is an effective $\mathbb{Q}$-divisor whose support does not contain the curve E. Since $(X, \frac{91}{60}D)$ is log canonical at the point O_y, the non-negative number k is at most $\frac{60}{91}$. By Lemma 1.3.8, the inequality

$$(E \cdot D - kE^2) \leqslant E \cdot D = \frac{8}{5 \cdot 13} < \frac{60}{91}$$

implies that the point P cannot belong to the curve E.

So far we have seen that the point P must lie in the outside of $C_x \cup C_y \cup C_z \cup C_t \cup E$. In particular, it is a smooth point. There is a unique member C in $\mathcal{L}$ which passes through the point P. Then the curve C is cut out by $t^2 = \alpha xy^4$ where α is a constant different from 0 and -1. The curve C is isomorphic to the curve defined by $xy^5 + z^3 + x^5 z = 0$ and $t^2 = xy^4$. The curve C is smooth in the outside of the base points and the singular locus of X by Bertini theorem, since it is isomorphic to a general curve in the pencil $\mathcal{L}$. We claim that the curve C is irreducible. If so then we may assume that the support of D does not contain the curve C and hence we obtain

$$\operatorname{mult}_P(D) \leqslant C \cdot D = \frac{12}{5 \cdot 13} < \frac{60}{91}.$$

This is a contradiction.

For the irreducibility of the curve C, we may consider the curve C as a surface in $\mathbb{C}^4$ defined by the equations $xy^5 + z^3 + x^5z = 0$ and $t^2 = xy^4$. Then, we consider the surface in $\mathbb{P}^4$ defined by the equations $xy^5 + w^3z^3 + x^5z = 0$ and $t^2w^3 = xy^4$. We then take the affine piece defined by $y \neq 0$. This affine piece is isomorphic to the surface defined by the equation $x + w^3z^3 + x^5z = 0$ and $t^2w^3 = x$ in $\mathbb{C}^4$. It is isomorphic to the hypersurface defined by $t^2w^3 + w^3z^3 + t^{10}w^{15}z = 0$ in $\mathbb{C}^3$. It has two irreducible components $w = 0$ and $t^2+z^3+t^{10}w^{12}z = 0$. The former component originates from the hyperplane at infinity in $\mathbb{P}^4$. Therefore, the curve C must be irreducible. □

Lemma 3.4.5. Let X be a quasismooth hypersurface of degree 71 in $\mathbb{P}(11, 17, 20, 27)$. Then $\operatorname{lct}(X) = \frac{11}{6}$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2y + y^3z + xz^3 + x^4t = 0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xy} (resp. L_{xy}, L_{zt}, L_{zt}) and $R_x = \{x = y^2z + t^2 = 0\}$ (resp. $R_y = \{y = x^3t + z^3 = 0\}$, $R_z = \{z = x^4 + yt = 0\}$, $R_t = \{t = y^3 + xz^2 = 0\}$). Also, we see that

$$L_{xy} \cap R_x = \{O_z\},\ L_{xy} \cap R_y = \{O_t\},\ L_{zt} \cap R_z = \{O_y\},\ L_{zt} \cap R_t = \{O_x\}.$$

One can easily check that $\operatorname{lct}(X, \frac{11}{4}C_x) = \frac{11}{6}$ is less than each of the numbers $\operatorname{lct}(X, \frac{17}{4}C_y)$, $\operatorname{lct}(X, \frac{20}{4}C_z)$ and $\operatorname{lct}(X, \frac{27}{4}C_t)$. Therefore, $\operatorname{lct}(X) \leqslant \frac{11}{6}$. Suppose $\operatorname{lct}(X) < \frac{11}{6}$. Then, there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{11}{6}D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xy}, L_{zt}, R_x, R_y, R_z, R_t are as follows:

$$D \cdot L_{xy} = \frac{1}{5 \cdot 27},\quad D \cdot R_x = \frac{2}{5 \cdot 17},\quad D \cdot R_y = \frac{4}{9 \cdot 11},$$
$$D \cdot L_{zt} = \frac{4}{11 \cdot 17},\quad D \cdot R_z = \frac{16}{17 \cdot 27},\quad D \cdot R_t = \frac{3}{5 \cdot 11},$$
$$L_{xy} \cdot R_x = \frac{1}{10},\quad L_{xy} \cdot R_y = \frac{1}{9},\quad L_{zt} \cdot R_z = \frac{4}{17},\quad L_{zt} \cdot R_t = \frac{3}{11},$$
$$L_{xy}^2 = -\frac{43}{20 \cdot 27},\quad R_x^2 = -\frac{3}{5 \cdot 17},\quad R_y^2 = \frac{2}{3 \cdot 11},$$
$$L_{zt}^2 = -\frac{24}{11 \cdot 17},\quad R_z^2 = -\frac{28}{17 \cdot 27},\quad R_t^2 = \frac{21}{20 \cdot 11}.$$

By Lemma 1.3.6 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. Since the curve R_t is singular at the point O_x and the curve R_y

is singular at the point O_t with multiplicity 3, in each of the following pairs of inequalities, at least one of two must hold:

$$\mathrm{mult}_{O_x}(D) \leqslant 11D\cdot L_{zt} = \frac{4}{17} < \frac{6}{11}, \quad \mathrm{mult}_{O_x}(D) \leqslant \frac{11}{2}D\cdot R_t = \frac{3}{10} < \frac{6}{11};$$

$$\mathrm{mult}_{O_z}(D) \leqslant 20D\cdot L_{xy} = \frac{4}{27} < \frac{6}{11}, \quad \mathrm{mult}_{O_z}(D) \leqslant 20D\cdot R_x = \frac{8}{17} < \frac{6}{11};$$

$$\mathrm{mult}_{O_t}(D) \leqslant 27D\cdot L_{xy} = \frac{1}{5} < \frac{6}{11}, \quad \mathrm{mult}_{O_t}(D) \leqslant \frac{27}{3}D\cdot R_y = \frac{4}{11} < \frac{6}{11}.$$

Therefore, the point P can be none of O_x, O_z, O_t.

Suppose that the point P is the point O_y. We then put $D = mL_{zt} + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support does not contain the curve L_{zt}. If $m = 0$, then

$$\mathrm{mult}_{O_y}(D) \leqslant 17D\cdot L_{zt} = \frac{4}{11} < \frac{6}{11}.$$

This is a contradiction. Therefore, $m > 0$, and hence the support of D does not contain the curve R_z. Since

$$\frac{16}{17\cdot 27} = D\cdot R_z \geqslant \frac{4m}{17} + \frac{\mathrm{mult}_{O_y}(D) - m}{17} > \frac{3m}{17} + \frac{6}{11\cdot 17}$$

we obtain $m < \frac{14}{3\cdot 11\cdot 27}$. However, we obtain

$$17(D - mL_{zt})\cdot L_{zt} = \frac{4+24m}{11} > \frac{6}{11}$$

from Lemma 1.3.8. This is a contradiction. Therefore, the point P is a smooth point of X.

We write $D = a_0L_{xy} + a_1L_{zt} + a_2R_x + a_3R_y + a_4R_z + a_5R_t + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of the curves L_{xy}, L_{zt}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{11}{6}D)$ is log canonical at the points O_x, O_z, O_t, the numbers a_i are at most $\frac{6}{11}$. Then by Lemma 1.3.8 the following inequalities enable us to conclude that the point P is in the outside of $C_x \cup C_y \cup C_z \cup C_t$:

$$(D - a_0L_{xy})\cdot L_{xy} = \frac{4+43a_0}{20\cdot 27} \leqslant \frac{6}{11}, \quad (D - a_1L_{zt})\cdot L_{zt} = \frac{4+24a_1}{11\cdot 17} \leqslant \frac{6}{11},$$

$$(D - a_2R_x)\cdot R_x = \frac{2+3a_2}{5\cdot 17} \leqslant \frac{6}{11}, \quad (D - a_3R_y)\cdot R_y = \frac{4-6a_3}{9\cdot 11} \leqslant \frac{6}{11},$$

$$(D - a_4R_z)\cdot R_z = \frac{16+28a_4}{17\cdot 27} \leqslant \frac{6}{11}, \quad (D - a_5R_t)\cdot R_t = \frac{12-21a_5}{20\cdot 11} \leqslant \frac{6}{11}.$$

We consider the pencil $\mathcal{L}$ defined by $\lambda ty + \mu x^4 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. The base locus of the pencil $\mathcal{L}$ consists of the curve L_{xy} and the point O_y. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P \notin C_x \cup C_y \cup C_z \cup C_t$, the divisor E is defined by the equation $ty = \alpha x^4$, where $\alpha \neq 0$.

Suppose that $\alpha \neq -1$. Then the curve E is isomorphic to the curve defined by the equations $ty = x^4$ and $x^4t + y^3z + xz^3 = 0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t \neq 0$ is the curve given by

$x(x^2 + x^{11}z + z^3) = 0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{xy} and C. We have

$$D \cdot C = D \cdot E - D \cdot L_{xy} = \frac{267}{5 \cdot 17 \cdot 27},$$

$$C^2 = E \cdot C - L_{xy} \cdot C \geqslant E \cdot C - L_{xy} \cdot C - R_x \cdot C = \frac{33}{4} D \cdot C > 0.$$

By Lemma 1.3.8 the inequality $D \cdot C < \frac{6}{11}$ gives us a contradiction.

Suppose that $\alpha = -1$. Then divisor E consists of three irreducible and reduced curves L_{xy}, R_z, and M. Note that the curve M is different from the curves R_x and L_{zt}. Also, it is smooth at the point P. We have

$$D \cdot M = D \cdot E - D \cdot L_{xy} - D \cdot R_z = \frac{11}{5 \cdot 27},$$

$$M^2 = E \cdot M - L_{xy} \cdot M - R_z \cdot M \geqslant E \cdot M - C_x \cdot M - C_z \cdot M = \frac{13}{4} D \cdot M > 0.$$

By Lemma 1.3.8 the inequality $D \cdot M < \frac{6}{11}$ gives us a contradiction. □

Lemma 3.4.6. Let X be a quasismooth hypersurface of degree 79 in $\mathbb{P}(11, 17, 24, 31)$. Then $\mathrm{lct}(X) = \frac{33}{16}$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$t^2y + tz^2 + xy^4 + x^5z = 0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xt} (resp. L_{yz}, L_{yz}, L_{xt}) and $R_x = \{x = yt + z^2 = 0\}$ (resp. $R_y = \{y = zt + x^5 = 0\}$, $R_z = \{z = xy^3 + t^2 = 0\}$, $R_t = \{t = y^4 + x^4z = 0\}$). Also, we see that

$$L_{xt} \cap R_x = \{O_y\},\ L_{yz} \cap R_y = \{O_t\},\ L_{yz} \cap R_z = \{O_x\},\ L_{xt} \cap R_t = \{O_z\}.$$

One can easily check that $\mathrm{lct}(X, \frac{4}{11}C_x) = \frac{33}{16}$ is less than each of the numbers $\mathrm{lct}(X, \frac{4}{17}C_y)$, $\mathrm{lct}(X, \frac{4}{24}C_z)$ and $\mathrm{lct}(X, \frac{4}{31}C_t)$. Therefore, $\mathrm{lct}(X) \leqslant \frac{33}{16}$. Suppose $\mathrm{lct}(X) < \frac{33}{16}$. Then, there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{33}{16}D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xt}, L_{yz}, R_x, R_y, R_z, R_t are as follows:

$$D \cdot L_{xt} = \frac{1}{6 \cdot 17},\quad D \cdot R_x = \frac{8}{17 \cdot 31},\quad D \cdot R_y = \frac{5}{6 \cdot 31},$$

$$D \cdot L_{yz} = \frac{4}{11 \cdot 31},\quad D \cdot R_z = \frac{8}{11 \cdot 17},\quad D \cdot R_t = \frac{2}{3 \cdot 11},$$

$$L_{xt} \cdot R_x = \frac{2}{17},\quad L_{yz} \cdot R_y = \frac{5}{31},\quad L_{yz} \cdot R_z = \frac{2}{11},\quad L_{xt} \cdot R_t = \frac{1}{6},$$

$$L_{xt}^2 = -\frac{37}{17 \cdot 24},\quad R_x^2 = -\frac{40}{17 \cdot 31},\quad R_y^2 = -\frac{35}{24 \cdot 31},$$

$$L_{yz}^2 = -\frac{38}{11 \cdot 31},\quad R_z^2 = \frac{14}{11 \cdot 17},\quad R_t^2 = \frac{10}{3 \cdot 11}.$$

By Lemma 1.3.6 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. The inequalities

$$17D\cdot L_{xt}=\frac{1}{6}<\frac{16}{33},\quad 17D\cdot R_x=\frac{8}{31}<\frac{16}{33}$$

imply that $P\neq O_y$. The inequalities

$$11D\cdot L_{yz}=\frac{4}{31}<\frac{16}{33},\quad 11D\cdot R_z=\frac{8}{17}<\frac{16}{33}$$

imply that $P\neq O_x$. Since the curve R_t is singular at the point O_z with multiplicity 4 the inequalities

$$24D\cdot L_{xt}=\frac{24}{6\cdot 17}<\frac{16}{33},\quad \frac{24}{4}D\cdot R_t=\frac{4}{11}<\frac{16}{33}$$

imply that $P\neq O_z$.

We write $D=a_1L_{xt}+a_2L_{yz}+a_3R_x+a_4R_y+a_5R_z+a_6R_t+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of the curves L_{xt}, L_{yz}, R_x, R_y, R_z, R_t. Since the pair $(X,\frac{33}{16}D)$ is log canonical at the points O_x, O_y, O_z, the numbers a_i are at most $\frac{16}{33}$. Then by Lemma 1.3.8 the following inequalities enable us to conclude that either the point P is in the outside of $C_x\cup C_y\cup C_z\cup C_t$ or $P=O_t$:

$$\frac{33}{16}D\cdot L_{xt}-L_{xt}^2=\frac{181}{3\cdot 17\cdot 32}<1,\quad \frac{33}{16}D\cdot R_x-R_x^2=\frac{113}{2\cdot 17\cdot 31}<1,\quad \frac{33}{16}D\cdot R_y-R_y^2=\frac{25}{3\cdot 31}<1,$$

$$\frac{33}{16}D\cdot L_{yz}-L_{xt}^2=\frac{185}{4\cdot 11\cdot 31}<1,\quad \frac{33}{16}D\cdot R_z-R_z^2=\frac{5}{2\cdot 11\cdot 17}<1,\quad \frac{33}{16}D\cdot R_t-R_t^2=\frac{-47}{3\cdot 8\cdot 11}<1.$$

Suppose that $P\neq O_t$. Then we consider the pencil $\mathcal{L}$ defined by $\lambda yt+\mu z^2=0$, $[\lambda:\mu]\in\mathbb{P}^1$. The base locus of the pencil $\mathcal{L}$ consists of the curve L_{yz} and the point O_y. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P\notin C_x\cup C_y\cup C_z\cup C_t$, the divisor E is defined by the equation $z^2=\alpha yt$, where $\alpha\neq 0$.

Suppose that $\alpha\neq -1$. Then the curve E is isomorphic to the curve defined by the equations $yt=z^2$ and $t^2y+xy^4+x^5z=0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t\neq 0$ is the curve given by $z(z+xz^7+x^5)=0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{yz} and C. We have the intersection numbers

$$D\cdot C=D\cdot E-D\cdot L_{yz}=\frac{564}{11\cdot 17\cdot 31},\quad C\cdot L_{yz}=E\cdot L_{yz}-L_{yz}^2=\frac{2}{11}.$$

Also, we see

$$C^2=E\cdot C-C\cdot L_{yz}>0.$$

By Lemma 1.3.8 the inequality $D\cdot C<\frac{16}{33}$ gives us a contradiction.

Suppose that $\alpha=-1$. Then divisor E consists of three irreducible and reduced curves L_{yz}, R_x, and M. Note that the curve M is different from the curves R_y and L_{xt}. Also, it is smooth at the point P. We have

$$D\cdot M=D\cdot E-D\cdot L_{yz}-D\cdot R_x=\frac{4\cdot 119}{11\cdot 17\cdot 31},$$

$$M^2=E\cdot M-L_{yz}\cdot M-R_x\cdot M\geqslant E\cdot M-C_y\cdot M-C_x\cdot M=5D\cdot M>0.$$

By Lemma 1.3.8 the inequality $D \cdot M < \frac{16}{33}$ gives us a contradiction. Therefore, $P = O_t$.

We write $D = aL_{yz} + bR_x + \Delta$, where Δ is an effective divisor whose support contains neither L_{yz} nor R_x. Note that we already assumed that the support of D cannot contain either L_{yz} or R_y. If the support of D contains R_y, then it does not contain L_{yz}. However, the inequality $31D \cdot L_{yz} = \frac{4}{11} < \frac{16}{33}$ shows that $P \neq O_t$. Therefore, the support of D does not contain the curve R_y. The inequality $D \cdot L_{xt} \geqslant bR_x \cdot L_{xt}$ implies $b \leqslant \frac{1}{12}$. On the other hand, we have

$$\frac{5}{6 \cdot 31} = D \cdot R_y \geqslant \frac{5a}{31} + \frac{b}{31} + \frac{\mathrm{mult}_{O_t}(D) - a - b}{31} > \frac{4a}{31} + \frac{16}{31 \cdot 33},$$

and hence $a < \frac{23}{4 \cdot 66}$.

Let $\pi\colon \bar{X} \to X$ be the weighted blow up of O_t with weights $(7, 4)$ and let F be the exceptional curve of π. Then

$$K_{\bar{X}} \sim_{\mathbb{Q}} \pi^*(K_X) - \frac{20}{31}F,\ \bar{L}_{yz} \sim_{\mathbb{Q}} \pi^*(L_{yz}) - \frac{4}{31}F,\ \bar{R}_x \sim_{\mathbb{Q}} \pi^*(R_x) - \frac{7}{31}F,\ \bar{\Delta} \sim_{\mathbb{Q}} \pi^*(\Delta) - \frac{c}{31}F,$$

where $\bar{\Delta}$, $\bar{L}_{yz}$, $\bar{R}_x$ are the proper transforms of Δ, L_{yz}, R_x, respectively, and c is a non-negative rational number. The curve F contains two singular points Q_7 and Q_4 of $\bar{X}$. The point Q_7 is a singular point of type $\frac{1}{7}(1, 1)$ and the point Q_4 is of type $\frac{1}{4}(1, 3)$. Note that the curve $\bar{R}_x$ passes through the point Q_4 but not the point Q_7. The curve $\bar{L}_{yz}$ passes through the point Q_7 but not the point Q_4.

The log pull-back of the log pair $(X, \frac{33}{16}D)$ by π is the log pair

$$\left(\bar{X},\ \frac{33a}{16}\bar{L}_{yz} + \frac{33b}{16}\bar{R}_x + \frac{33}{16}\bar{\Delta} + \theta_1 F\right),$$

where

$$\theta_1 = \frac{33(4a + 7b + c) + 320}{16 \cdot 31}.$$

This pair is not log canonical at some point $Q \in F$. We have

$$0 \leqslant \bar{\Delta} \cdot \bar{R}_x = \frac{8 + 40b}{17 \cdot 31} - \frac{a}{31} - \frac{c}{4 \cdot 31}.$$

This inequality shows $4a + c \leqslant \frac{4}{17}(8 + 40b)$. Then

$$\theta_1 = \frac{33(4a + c) + 231b + 320}{16 \cdot 31} \leqslant \frac{6496 + 9207b}{16 \cdot 17 \cdot 31} < 1$$

since $b \leqslant \frac{1}{12}$.

Suppose that the point Q is neither the point Q_7 nor the point Q_4. Then the log pair $\left(\bar{X}, \frac{33}{16}\bar{\Delta} + F\right)$ is not log canonical at the point Q. Then

$$\frac{33c}{16 \cdot 28} = \frac{33}{16}\bar{\Delta} \cdot F > 1$$

by Lemma 1.3.4. However, $c \leqslant 4a + c \leqslant \frac{4}{17}(8 + 40b)$. This is a contradiction since $b \leqslant \frac{1}{12}$. Therefore, the point Q is either the point Q_7 or the point Q_4.

Suppose that the point Q is the point Q_4. This point is the intersection point of F and $\bar{R}_x$. Then the log pair $\big(\bar{X}, \frac{33b}{16}\bar{R}_x + \frac{33}{16}\bar{\Delta} + \theta_1 F\big)$ is not log canonical at the point Q. It then follows from Lemma 1.3.4 that

$$1 < 4\left(\frac{33}{16}\bar{\Delta} + \theta_1 F\right)\cdot \bar{R}_x = \frac{33\cdot 4}{16}\left(\frac{8+40b}{17\cdot 31} - \frac{a}{31} - \frac{c}{4\cdot 31}\right) + \theta_1.$$

However,

$$\frac{33\cdot 4}{16}\left(\frac{8+40b}{17\cdot 31} - \frac{a}{31} - \frac{c}{4\cdot 31}\right) + \theta_1 = \frac{6496+9207b}{16\cdot 17\cdot 31} < 1.$$

Therefore, the point Q is the point Q_7. This point is the intersection point of F and $\bar{L}_{yz}$.

Let $\phi\colon \tilde{X} \to \bar{X}$ be the blow up at the point Q_7. Let G be the exceptional divisor of the morphism ϕ. The surface $\tilde{X}$ is smooth along the exceptional divisor G. Let $\tilde{L}_{yz}$, $\tilde{R}_x$, $\tilde{\Delta}$ and $\tilde{F}$ be the proper transforms of L_{yz}, R_x, Δ and F by ϕ, respectively. We have

$$K_{\tilde{X}} \sim_{\mathbb{Q}} \phi^*(K_{\bar{X}}) - \frac{5}{7}G,\ \tilde{L}_{yz} \sim_{\mathbb{Q}} \phi^*(\bar{L}_{yz}) - \frac{1}{7}G,\ \tilde{F} \sim_{\mathbb{Q}} \phi^*(F) - \frac{1}{7}G,\ \tilde{\Delta} \sim_{\mathbb{Q}} \phi^*(\bar{\Delta}) - \frac{d}{7}G,$$

where d is a non-negative rational number. The log pull-back of the log pair $(X, \frac{33}{16}D)$ via $\pi\circ\phi$ is

$$\left(\tilde{X}, \frac{33a}{16}\tilde{L}_{yz} + \frac{33b}{16}\tilde{R}_x + \frac{33}{16}\tilde{\Delta} + \theta_1\tilde{F} + \theta_2 G\right),$$

where

$$\theta_2 = \frac{33}{7\cdot 16}(a+d) + \frac{\theta_1}{7} + \frac{5}{7} = \frac{2800+33(35a+7b+c+31d)}{7\cdot 16\cdot 31}.$$

This log pair is not log canonical at some point $O\in G$. We have

$$0 \leqslant \tilde{\Delta}\cdot\tilde{L}_{yz} = \frac{4+38a}{11\cdot 31} - \frac{b}{31} - \frac{c}{7\cdot 31} - \frac{d}{7}.$$

We then obtain $7b+c+31d \leqslant \frac{7}{11}(4+38a)$. Since $a\leqslant\frac{23}{264}$, we see

$$\theta_2 = \frac{2800+33(35a+7b+c+31d)}{7\cdot 16\cdot 31} \leqslant \frac{4532+3069a}{11\cdot 16\cdot 31} < 1.$$

Suppose that $O\notin \tilde{F}\cup\tilde{L}_{yz}$. The log pair $\left(\tilde{X}, \frac{13}{8}\tilde{\Delta} + G\right)$ is not log canonical at the point O. Applying Lemma 1.3.4, we get

$$1 < \frac{33}{16}\tilde{\Delta}\cdot G = \frac{33d}{16},$$

and hence $d > \frac{16}{33}$. However, $d \leqslant \frac{1}{31}(7b+c+31d) \leqslant \frac{7}{11\cdot 31}(4+38a)$. This is a contradiction since $a\leqslant\frac{23}{264}$. Therefore, the point O is either the intersection point of G and $\tilde{F}$ or the intersection point of G and $\tilde{L}_{yz}$. In the latter case, the pair $\left(\tilde{X}, \frac{33a}{16}\tilde{L}_{yz} + \frac{33}{16}\tilde{\Delta} + \theta_2 G\right)$ is not log canonical at the point O. Then, applying Lemma 1.3.4, we get

$$1 < \left(\frac{33}{16}\tilde{\Delta} + \theta_2 G\right)\cdot\tilde{L}_{yz} = \frac{33}{16}\left(\frac{4+38a}{11\cdot 31} - \frac{b}{31} - \frac{c}{7\cdot 31} - \frac{d}{7}\right) + \theta_2.$$

However,

$$\frac{33}{16}\left(\frac{4+38a}{11\cdot 31} - \frac{b}{31} - \frac{c}{7\cdot 31} - \frac{d}{7}\right) + \theta_2 = \frac{4532+3069a}{11\cdot 16\cdot 31} < 1.$$

Therefore, the point O must be the intersection point of G and $\tilde{F}$.

Let $\xi\colon \hat{X} \to \tilde{X}$ be the blow up at the point O and let H be the exceptional divisor of ξ. We also let $\hat{L}_{yz}$, $\hat{R}_x$, $\hat{\Delta}$, $\hat{G}$, and $\hat{F}$ be the proper transforms of $\tilde{L}_{yz}$, $\tilde{R}_x$, $\tilde{\Delta}$, G and $\tilde{F}$ by ξ, respectively. Then $\hat{X}$ is smooth along the exceptional divisor H. We have

$$K_{\hat{X}} \sim_{\mathbb{Q}} \xi^*(K_{\tilde{X}}) - H,\ \hat{G} \sim_{\mathbb{Q}} \xi^*(G) - H,\ \hat{F} \sim_{\mathbb{Q}} \xi^*(\tilde{F}) - H,\ \hat{\Delta} \sim_{\mathbb{Q}} \xi^*(\tilde{\Delta}) - eH,$$

where e is a non-negative rational number. The log pull-back of the log pair $(X, \frac{33}{16}D)$ via $\pi\circ\phi\circ\xi$ is

$$\left(\hat{X}, \frac{33a}{16}\hat{L}_{yz} + \frac{33b}{16}\hat{R}_x + \frac{33}{16}\hat{\Delta} + \theta_1\hat{F} + \theta_2\hat{G} + \theta_3 H\right),$$

where

$$\theta_3 = \theta_1 + \theta_2 + \frac{33e}{16} - 1 = \frac{1568 + 33(63a + 56b + 8c + 31d + 217e)}{7\cdot 16\cdot 31}.$$

This log pair is not log canonical at some point $A \in H$. We have

$$\frac{c}{28} - \frac{d}{7} - e = \hat{\Delta}\cdot\hat{F} \geqslant 0.$$

Therefore, $4d + 28e \leqslant c$.

Then

$$\begin{aligned}\theta_3 &= \frac{1568 + 33(63a + 56b + 8c)}{7\cdot 16\cdot 31} + \frac{33\cdot 31(d + 7e)}{7\cdot 16\cdot 31} \leqslant \\ &\leqslant \frac{6272 + 33(252a + 224b + 63c)}{4\cdot 7\cdot 16\cdot 31} = \\ &= \frac{6272 + 7392b}{4\cdot 7\cdot 16\cdot 31} + \frac{33\cdot 63(4a + c)}{4\cdot 7\cdot 16\cdot 31} \leqslant \\ &\leqslant \frac{28 + 33b}{2\cdot 31} + \frac{9\cdot 33(1 + 5b)}{2\cdot 17\cdot 31} = \frac{773 + 2046b}{2\cdot 17\cdot 31} < 1\end{aligned}$$

since $b \leqslant \frac{1}{12}$ and $4a + c \leqslant \frac{4}{17}(8 + 40b)$. In particular, θ_3 is a positive number.

Suppose that $A \notin \hat{F}\cup\hat{G}$. Then the log pair $\left(\hat{X}, \frac{33}{16}\hat{\Delta} + \theta_3 H\right)$ is not log canonical at the point A. Applying Lemma 1.3.4, we get

$$1 < \frac{33}{16}\hat{\Delta}\cdot H = \frac{33e}{16}.$$

However,

$$e \leqslant \frac{1}{28}(4d + 28e) \leqslant \frac{c}{28} \leqslant \frac{1}{28}(4a + c) \leqslant \frac{4(8 + 40b)}{17\cdot 28} \leqslant \frac{4}{11}.$$

Therefore, the point A must be either in $\hat{F}$ or in $\hat{G}$.

Suppose that $A \in \hat{F}$. Then the log pair $\left(\hat{X}, \frac{33}{16}\hat{\Delta} + \theta_1\hat{F} + \theta_3 H\right)$ is not log canonical at the point A. Applying Lemma 1.3.4, we get

$$1 < \left(\frac{33}{16}\hat{\Delta} + \theta_3 H\right)\cdot\hat{F} = \frac{33}{16}\left(\frac{c}{28} - \frac{d}{7} - e\right) + \theta_3 = \frac{6272 + 33(252a + 224b + 63c)}{4\cdot 7\cdot 16\cdot 31}.$$

However,

$$\frac{6272+33(252a+224b+63c)}{4\cdot 7\cdot 16\cdot 31}\leqslant\frac{773+2046b}{2\cdot 17\cdot 31}<1.$$

Therefore, the point A is the intersection point of H and $\hat{G}$. Then the log pair $\left(\hat{X}, \frac{33}{16}\hat{\Delta}+\theta_2\hat{G}+\theta_3 H\right)$ is not log canonical at the point A. From Lemma 1.3.4, we obtain

$$1<\left(\frac{33}{16}\hat{\Delta}+\theta_3 H\right)\cdot\hat{G}=\frac{33}{16}(d-e)+\theta_3=\frac{1568+33(63a+56b+8c+248d)}{7\cdot 16\cdot 31}.$$

However,

$$\frac{1568+33(63a+56b+8c+248d)}{7\cdot 16\cdot 31}=\frac{224+297a}{16\cdot 31}+\frac{33(7b+c+31d)}{2\cdot 7\cdot 31}\leqslant\frac{320+1209a}{16\cdot 31}<1$$

since $a<\frac{23}{4\cdot 66}$ and $7b+c+31d\leqslant\frac{7}{11}(4+38a)$. The obtained contradiction completes the proof. □

Lemma 3.4.7. Let X be a quasismooth hypersurface of degree 166 in $\mathbb{P}(11,31,45,83)$. Then $\mathrm{lct}(X)=\frac{55}{24}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2+yz^3+xy^5+x^{11}z=0.$$

The surface X is singular only at the points O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{55}{24}=\mathrm{lct}\left(X,\frac{4}{11}C_x\right)<\mathrm{lct}\left(X,\frac{4}{31}C_y\right)=\frac{13\cdot 31}{88}.$$

Therefore, $\mathrm{lct}(X)\leqslant\frac{55}{24}$.

Suppose that $\mathrm{lct}(X)<\frac{55}{24}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{55}{24}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither C_x nor C_y. Then the inequalities

$$45D\cdot C_x=\frac{8}{31}<\frac{24}{55},\quad 11D\cdot C_y=\frac{8}{45}<\frac{24}{55}$$

show that the point P is a smooth point in the outside of C_x. However, $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(495))$ contains the monomials x^{45}, $y^{11}x^{14}$ and z^{11}, it follows from Lemma 1.3.9 that the point P is either a singular point of X or a point on C_x. This is a contradiction. □

Lemma 3.4.8. Let X be a quasismooth hypersurface of degree 71 in $\mathbb{P}(13,14,19,29)$. Then $\mathrm{lct}(X)=\frac{65}{36}$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$ty^3+yz^3+xt^2+x^4z=0.$$

The surface X is singular at the points O_x, O_y, O_z, O_t. Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xy} (resp. L_{xy}, L_{zt}, L_{zt}) and $R_x = \{x = z^3 + ty^2 = 0\}$ (resp. $R_y = \{y = x^3z + t^2 = 0\}$, $R_z = \{z = y^3 + xt = 0\}$, $R_t = \{t = x^4 + yz^2 = 0\}$). Also, we see that

$$L_{xy} \cap R_x = \{O_t\},\ L_{xy} \cap R_y = \{O_z\},\ L_{zt} \cap R_z = \{O_x\},\ L_{zt} \cap R_t = \{O_y\}.$$

One can easily check that $\mathrm{lct}(X, \frac{4}{13}C_x) = \frac{65}{36}$ is less than each of the numbers $\mathrm{lct}(X, \frac{4}{14}C_y)$, $\mathrm{lct}(X, \frac{4}{19}C_z)$ and $\mathrm{lct}(X, \frac{4}{29}C_t)$. Therefore, $\mathrm{lct}(X) \leqslant \frac{65}{36}$. Suppose $\mathrm{lct}(X) < \frac{65}{36}$. Then, there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{65}{36}D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xy}, L_{zt}, R_x, R_y, R_z, R_t are as follows:

$$D \cdot L_{xy} = \frac{4}{19 \cdot 29},\quad D \cdot R_x = \frac{6}{7 \cdot 29},\quad D \cdot R_y = \frac{8}{13 \cdot 19},$$
$$D \cdot L_{zt} = \frac{2}{7 \cdot 13},\quad D \cdot R_z = \frac{12}{13 \cdot 29},\quad D \cdot R_t = \frac{8}{7 \cdot 19},$$
$$L_{xy} \cdot R_x = \frac{3}{29},\quad L_{xy} \cdot R_y = \frac{2}{19},\quad L_{zt} \cdot R_z = \frac{3}{13},\quad L_{zt} \cdot R_t = \frac{2}{7},$$
$$L_{xy}^2 = -\frac{44}{19 \cdot 29},\quad R_x^2 = -\frac{3}{14 \cdot 29},\quad R_y^2 = \frac{2}{13 \cdot 19},$$
$$L_{zt}^2 = -\frac{23}{13 \cdot 14},\quad R_z^2 = -\frac{30}{13 \cdot 29},\quad R_t^2 = \frac{20}{7 \cdot 19}.$$

By Lemma 1.3.6 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. Since the curve R_t is singular at the point O_y and the curve R_y is singular at the point O_z, in each of the following pairs of inequalities, at least one of two must hold:

$$\mathrm{mult}_{O_x}(D) \leqslant 13D \cdot L_{zt} = \frac{2}{7} < \frac{36}{65},\quad \mathrm{mult}_{O_x}(D) \leqslant 13D \cdot R_z = \frac{12}{29} < \frac{36}{65};$$
$$\mathrm{mult}_{O_y}(D) \leqslant 14D \cdot L_{zt} = \frac{4}{13} < \frac{36}{65},\quad \mathrm{mult}_{O_y}(D) \leqslant \frac{14}{2}D \cdot R_t = \frac{8}{19} < \frac{36}{65};$$
$$\mathrm{mult}_{O_z}(D) \leqslant 19D \cdot L_{xy} = \frac{4}{29} < \frac{36}{65},\quad \mathrm{mult}_{O_z}(D) \leqslant \frac{19}{2}D \cdot R_y = \frac{4}{13} < \frac{36}{65};$$
$$\mathrm{mult}_{O_t}(D) \leqslant 29D \cdot L_{xy} = \frac{4}{19} < \frac{36}{65},\quad \mathrm{mult}_{O_t}(D) \leqslant \frac{29}{2}D \cdot R_x = \frac{3}{7} < \frac{36}{65}.$$

Therefore, the point P can be none of O_x, O_y, O_z, O_t.

We write $D = a_0L_{xy} + a_1L_{zt} + a_2R_x + a_3R_y + a_4R_z + a_5R_t + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of the curves L_{xy}, L_{zt}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{65}{36}D)$ is log canonical at the points O_x, O_y, O_z, O_t, the numbers a_i are at most $\frac{36}{65}$. Then by Lemma 1.3.8 the following inequalities enable us to conclude that the point P must be located in the outside of $C_x \cup C_y \cup C_z \cup C_t$:

$$(D - a_0L_{xy}) \cdot L_{xy} = \frac{4 + 44a_0}{19 \cdot 29} \leqslant \frac{36}{65},\quad (D - a_1L_{zt}) \cdot L_{zt} = \frac{4 + 23a_1}{13 \cdot 14} \leqslant \frac{36}{65},$$
$$(D - a_2R_x) \cdot R_x = \frac{12 + 3a_2}{14 \cdot 29} \leqslant \frac{36}{65},\quad (D - a_3R_y) \cdot R_y = \frac{8 - 2a_3}{13 \cdot 19} \leqslant \frac{36}{65},$$
$$(D - a_4R_z) \cdot R_z = \frac{12 + 30a_4}{13 \cdot 29} \leqslant \frac{36}{65},\quad (D - a_5R_t) \cdot R_t = \frac{8 - 20a_5}{7 \cdot 19} \leqslant \frac{36}{65}.$$

We consider the pencil $\mathcal{L}$ defined by $\lambda tx + \mu y^3 = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. The base locus of the pencil consists of the curve L_{xy} and the point O_x. Let E be the unique divisor in $\mathcal{L}$ that passes through the point P. Since $P \notin C_x \cup C_y \cup C_z \cup C_t$, the divisor E is defined by the equation $tx = \alpha y^3$, where $\alpha \neq 0$.

Suppose that $\alpha \neq -1$. Then the curve E is isomorphic to the curve defined by the equations $tx = y^3$ and $xt^2 + yz^3 + x^4 z = 0$. Since the curve E is isomorphic to a general curve in $\mathcal{L}$, it is smooth at the point P. The affine piece of E defined by $t \neq 0$ is the curve given by $y(y^2 + y^{11}z + z^3) = 0$. Therefore, the divisor E consists of two irreducible and reduced curves L_{xy} and C. We have

$$D \cdot C = D \cdot E - D \cdot L_{xy} = \frac{800}{13 \cdot 19 \cdot 29}.$$

Also, we see

$$C^2 = E \cdot C - C \cdot L_{xy} \geqslant E \cdot C - C_x \cdot C > 0.$$

By Lemma 1.3.8 the inequality $D \cdot C < \frac{36}{65}$ gives us a contradiction.

Suppose that $\alpha = -1$. Then divisor E consists of three irreducible and reduced curves L_{xy}, R_z, and M. Note that the curve M is different from the curves R_x and L_{zt}. Also, it is smooth at the point P. We have

$$D \cdot M = D \cdot E - D \cdot L_{xy} - D \cdot R_z = \frac{572}{13 \cdot 19 \cdot 29},$$

$$M^2 = E \cdot M - L_{xy} \cdot M - R_z \cdot M \geqslant E \cdot M - C_x \cdot M - C_z \cdot M = \frac{5}{2} D \cdot M > 0.$$

By Lemma 1.3.8 the inequality $D \cdot M < \frac{36}{65}$ gives us a contradiction. □

Lemma 3.4.9. Let X be a quasismooth hypersurface of degree 79 in $\mathbb{P}(13, 14, 23, 33)$. Then $\mathrm{lct}(X) = \frac{65}{32}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^2 t + y^4 z + xt^2 + x^5 y = 0.$$

The surface X is singular at O_x, O_y, O_z and O_t. We have

$$\mathrm{lct}\left(X, \frac{4}{13} C_x\right) = \frac{65}{32} < \mathrm{lct}\left(X, \frac{4}{13} C_x\right) = \frac{21}{8} < \mathrm{lct}\left(X, \frac{5}{25} C_t\right) = \frac{33}{10} < \mathrm{lct}\left(X, \frac{4}{23} C_z\right) = \frac{69}{20}.$$

In particular, $\mathrm{lct}(X) \leqslant \frac{65}{32}$.

Each of the divisors C_x, C_y, C_z, and C_t consists of two irreducible and reduced components. The divisor C_x (resp. C_y, C_z, C_t) consists of L_{xz} (resp. L_{yt}, L_{xz}, L_{yt}) and $R_x = \{x = y^4 + zt = 0\}$ (resp. $R_y = \{y = z^2 + xt = 0\}$, $R_z = \{z = x^4 y + t^2 = 0\}$, $R_t = \{t = x^5 + y^3 z = 0\}$). The curve L_{xz} intersects R_x (resp. R_z) only at the point O_t (resp. O_y). The curve L_{yt} intersects R_y (resp. R_t) only at the point O_x (resp. O_z).

We suppose that $\mathrm{lct}(X) < \frac{65}{32}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the log pair $(X, \frac{65}{32} D)$ is not log canonical at some point $P \in X$.

The intersection numbers among the divisors D, L_{xz}, L_{yt}, R_x, R_y, R_z, R_t are as follows:

$$L_{xz}^2 = -\frac{43}{14 \cdot 33},\ R_x^2 = -\frac{40}{23 \cdot 33},\ L_{xz} \cdot R_x = \frac{4}{33},\ D \cdot L_{xz} = \frac{4}{14 \cdot 33},\ D \cdot R_x = \frac{16}{23 \cdot 33},$$

$$L_{yt}^2=-\frac{32}{13\cdot 23},\ R_y^2=-\frac{38}{13\cdot 33},\quad L_{yt}\cdot R_y=\frac{2}{13},\ D\cdot L_{yt}=\frac{4}{13\cdot 23},\ D\cdot R_y=\frac{8}{13\cdot 33},$$
$$R_z^2=\frac{20}{13\cdot 14},\ L_{xz}\cdot R_z=\frac{2}{14},\ D\cdot R_z=\frac{8}{13\cdot 14},$$
$$R_t^2=\frac{95}{14\cdot 13},\ L_{yt}\cdot R_t=\frac{5}{23},\ D\cdot R_t=\frac{20}{14\cdot 23}.$$

By Lemma 1.3.6 we may assume that the support of D does not contain at least one component of each divisor C_x, C_y, C_z, C_t. Since the curve R_t is singular at the point O_z with multiplicity 3 and the curve R_z is singular at the point O_y, in each of the following pairs of inequalities, at least one of two must hold:

$$\mathrm{mult}_{O_x}(D)\leqslant 13D\cdot L_{yt}=\frac{4}{23}<\frac{32}{65},\quad \mathrm{mult}_{O_x}(D)\leqslant 13D\cdot R_y=\frac{8}{33}<\frac{32}{65};$$
$$\mathrm{mult}_{O_y}(D)\leqslant 14D\cdot L_{xz}=\frac{4}{33}<\frac{32}{65},\quad \mathrm{mult}_{O_y}(D)\leqslant \frac{14}{2}D\cdot R_z=\frac{4}{13}<\frac{32}{65};$$
$$\mathrm{mult}_{O_z}(D)\leqslant 23D\cdot L_{yt}=\frac{4}{13}<\frac{32}{65},\quad \mathrm{mult}_{O_z}(D)\leqslant \frac{23}{3}D\cdot R_t=\frac{10}{21}<\frac{32}{65}.$$

Therefore, the point P can be none of O_x, O_y, O_z.

Put $D=m_0L_{xz}+m_1L_{yt}+m_2R_x+m_3R_y+m_4R_z+m_5R_t+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of L_{xz}, L_{yt}, R_x, R_y, R_z, R_t. Since the pair $(X,\frac{65}{32}D)$ is log canonical at the points O_x, O_y, O_z, we have $m_i\leqslant\frac{32}{65}$ for each i. Since

$$(D-m_0L_{xz})\cdot L_{xz}=\frac{4+43m_0}{14\cdot 33}\leqslant\frac{32}{65},\quad (D-m_1L_{yt})\cdot L_{yt}=\frac{4+32m_1}{13\cdot 23}\leqslant\frac{32}{65},$$
$$(D-m_2R_x)\cdot R_x=\frac{16+40m_2}{23\cdot 33}\leqslant\frac{32}{65},\quad (D-m_3R_y)\cdot R_y=\frac{8+38m_3}{13\cdot 33}\leqslant\frac{32}{65},$$
$$(D-m_4R_z)\cdot R_z=\frac{8-20m_4}{13\cdot 14}\leqslant\frac{32}{65},\quad (D-m_5R_t)\cdot R_t=\frac{20-95m_5}{14\cdot 23}\leqslant\frac{32}{65}$$

Lemma 1.3.8 implies that the point P cannot be a smooth point of X on $C_x\cup C_y\cup C_z\cup C_t$. Therefore, the point P is either a point in the outside of $C_x\cup C_y\cup C_z\cup C_t$ or the point O_t.

Suppose that $P\notin C_x\cup C_y\cup C_z\cup C_t$. Then we consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda xt+\mu z^2=0$, $[\lambda:\mu]\in\mathbb{P}^1$. There is a unique curve Z_α in the pencil passing through the point P. This curve is cut out by

$$xt+\alpha z^2=0,$$

where α is a non-zero constant.

The curve Z_α is reduced. But it is always reducible. Indeed, one can easily check that

$$Z_\alpha=C_\alpha+L_{xz}$$

where C_α is a reduced curve whose support contains no L_{xy}. Let us prove that C_α is irreducible if $\alpha\neq 1$.

Any component of the curve C_t is not contained in the curve Z_α. The open subset $Z_\alpha\setminus C_t$ of the curve Z_α is a $\mathbb{Z}_{33}$-quotient of the affine curve

$$x+\alpha z^2=z^2+y^4z+x+x^5y=0\subset\mathbb{C}^3\cong\mathrm{Spec}\Big(\mathbb{C}\big[x,y,z\big]\Big),$$

that is isomorphic to a plane affine curve defined by the equation

$$z\left((\alpha-1)z+y^4-\alpha^5yz^9\right)=0\subset\mathbb{C}^2\cong\mathrm{Spec}\Big(\mathbb{C}\big[y,z\big]\Big).$$

Thus, if $\alpha\neq 1$, then the curve Z_α consists of two irreducible and reduced curves L_{xz} and C_α. If $\alpha=1$, then the curve Z_α consists of three irreducible and reduced curves L_{xz}, R_y, and C_1. In both cases, the curve C_α (including $\alpha=1$) is smooth at the point P. By Lemma 1.3.6, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve Z_α.

If $\alpha\neq 1$, then

$$D\cdot C_\alpha=\frac{8}{13\cdot 14},$$

$$C_\alpha^2=Z_\alpha\cdot C_\alpha-L_{xz}\cdot C_\alpha\geqslant Z_\alpha\cdot C_\alpha-(R_x+L_{xz})\cdot C_\alpha=\frac{33}{4}D\cdot C_\alpha>0.$$

If $\alpha=1$, then

$$D\cdot C_1=\frac{152}{13\cdot 14\cdot 33},$$

$$C_1^2=Z_1\cdot C_1-(L_{xz}+R_y)\cdot C_1\geqslant Z_1\cdot C_1-(R_x+L_{xz})\cdot C_1-(L_{yt}+R_y)\cdot C_1=\frac{19}{4}D\cdot C_1>0.$$

We put $D=mC_\alpha+\Delta_\alpha$, where Δ_α is an effective $\mathbb{Q}$-divisor such that $C_\alpha\not\subset\mathrm{Supp}(\Delta_\alpha)$. Since C_α intersects the curve C_t and the pair $(X,\frac{65}{32}D)$ is log canonical along the curve C_t, we obtain $m\leqslant\frac{32}{65}$. Then, the inequality

$$(D-mC_\alpha)\cdot C_\alpha\leqslant D\cdot C_\alpha<\frac{32}{65}$$

implies that the pair $(X,\frac{65}{32}D)$ is log canonical at the point P by Lemma 1.3.8. The obtained contradiction conclude that the point P must be the point O_t.

If L_{xz} is not contained in the support of D, then the inequality

$$\mathrm{mult}_{O_t}(D)\leqslant 33D\cdot L_{xz}=\frac{2}{7}<\frac{32}{65}$$

is a contradiction. Therefore, the curve L_{xz} must be contained in the support of D, and hence the curve R_x is not contained in the support of D. Put $D=aL_{xz}+bR_y+\Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither L_{xz} nor R_y. Then

$$\frac{16}{23\cdot 33}=D\cdot R_x\geqslant aL_{xz}\cdot R_x+\frac{\mathrm{mult}_{O_t}(D)-a}{33}>\frac{3a}{33}+\frac{32}{33\cdot 65}$$

and hence $a<\frac{304}{3\cdot 23\cdot 65}$. If $b\neq 0$, then L_{yt} is not contained in the support of D. Therefore,

$$\frac{4}{13\cdot 23}=D\cdot L_{yt}\geqslant bR_y\cdot L_{yt}=\frac{2b}{13},$$

and hence $b\leqslant\frac{2}{23}$.

Let $\pi\colon\bar{X}\to X$ be the weighted blow up at the point O_t with weights $(13,19)$ and let F be the exceptional curve of the morphism π. Then F contains two singular points Q_{13} and Q_{19} of $\bar{X}$ such that Q_{13} is a singular point of type $\frac{1}{13}(1,1)$, and Q_{19} is a singular point of type $\frac{1}{19}(3,7)$. Then

$$K_{\bar{X}}\sim_{\mathbb{Q}}\pi^*(K_X)-\frac{1}{33}F,\quad \bar{L}_{xz}\sim_{\mathbb{Q}}\pi^*(L_{xz})-\frac{19}{33}F,\quad \bar{R}_y\sim_{\mathbb{Q}}\pi^*(R_y)-\frac{13}{33}F,\quad \bar{\Delta}\sim_{\mathbb{Q}}\pi^*(\Delta)-\frac{c}{33}F,$$

where $\bar{L}_{xz}$, $\bar{R}_y$ and $\bar{\Delta}$ are the proper transforms of L_{xz}, R_y and Δ by π, respectively, and c is a non-negative rational number. Note that $F \cap \bar{R}_y = \{Q_{19}\}$ and $F \cap \bar{L}_{xz} = \{Q_{13}\}$.

The log pull-back of the log pair $(X, \frac{65}{32}D)$ by π is the log pair

$$\left(\bar{X},\ \frac{65a}{32}\bar{L}_{xz} + \frac{65b}{32}\bar{R}_y + \frac{65}{32}\bar{\Delta} + \theta_1 F\right),$$

where

$$\theta_1 = \frac{32 + 65(19a + 13b + c)}{32 \cdot 33}.$$

This is not log canonical at some point $Q \in F$. We have

$$0 \leqslant \bar{\Delta} \cdot \bar{L}_{xz} = \frac{4 + 43a}{14 \cdot 33} - \frac{b}{33} - \frac{c}{13 \cdot 33}.$$

This inequality shows $13b + c \leqslant \frac{13}{14}(4 + 43a)$. Since $a \leqslant \frac{304}{3 \cdot 23 \cdot 65}$, we obtain

$$\theta_1 = \frac{32 + 1235a}{32 \cdot 33} + \frac{65(13b + c)}{32 \cdot 33} \leqslant \frac{32 + 1235a}{32 \cdot 33} + \frac{13 \cdot 65(4 + 43a)}{14 \cdot 32 \cdot 33} < 1.$$

Suppose that the point Q is neither Q_{13} nor Q_{19}. Then, the point Q is not in $\bar{L}_{xz} \cup \bar{R}_y$. Therefore, the pair $\left(\bar{X}, \frac{65}{32}\bar{\Delta} + F\right)$ is not log canonical at the point Q, and hence

$$1 < \frac{65}{32}\bar{\Delta} \cdot F = \frac{65c}{13 \cdot 19 \cdot 32}.$$

But $c \leqslant 13b + c \leqslant \frac{13}{14}(4 + 43a) < \frac{13 \cdot 19 \cdot 32}{65}$ since $a \leqslant \frac{304}{3 \cdot 23 \cdot 65}$. Therefore, the point Q is either Q_{13} or Q_{19}.

Suppose that the point Q is Q_{13}. Then the point Q is in $\bar{L}_{xz}$ but not in $\bar{R}_y$. Therefore, the pair $\left(\bar{X}, \bar{L}_{xz} + \frac{65}{32}\bar{\Delta} + \theta_1 F\right)$ is not log canonical at the point Q. However, this is impossible since

$$13\left(\frac{65}{32}\bar{\Delta} + \theta_1 F\right) \cdot \bar{L}_{xz} = \frac{13 \cdot 65}{32}\left(\frac{4 + 43a}{14 \cdot 33} - \frac{b}{33} - \frac{c}{13 \cdot 33}\right) + \theta_1 = $$
$$= \frac{32 + 1235a}{32 \cdot 33} + \frac{13 \cdot 65(4 + 43a)}{14 \cdot 32 \cdot 33} < 1.$$

Therefore, the point Q must be the point Q_{19}.

Let $\psi\colon \tilde{X} \to \bar{X}$ be the weighted blow up at the point Q_{19} with weights $(3, 7)$ and let E be the exceptional curve of the morphism ψ. The exceptional curve E contains two singular points O_3 and O_7 of $\tilde{X}$. The point O_3 is of type $\frac{1}{3}(1, 2)$ and the point O_7 is of type $\frac{1}{7}(4, 5)$. Then

$$K_{\tilde{X}} \sim_{\mathbb{Q}} \psi^*(K_{\bar{X}}) - \frac{9}{19}E, \quad \tilde{R}_y \sim_{\mathbb{Q}} \psi^*(\bar{R}_y) - \frac{3}{19}E, \quad \tilde{F} \sim_{\mathbb{Q}} \psi^*(F) - \frac{7}{19}E, \quad \tilde{\Delta} \sim_{\mathbb{Q}} \psi^*(\bar{\Delta}) - \frac{d}{19}E,$$

where $\tilde{R}_y$, $\tilde{F}$ and $\tilde{\Delta}$ are the proper transforms of $\bar{R}_y$, F and $\bar{\Delta}$ by ψ, respectively, and d is a non-negative rational number.

The log pull-back of the log pair $(X, \frac{65}{32}D)$ by $\pi \circ \psi$ is the log pair

$$\left(\tilde{X},\ \frac{65a}{32}\tilde{L}_{xz} + \frac{65b}{32}\tilde{R}_y + \frac{65}{32}\tilde{\Delta} + \theta_1 \tilde{F} + \theta_2 E\right),$$

where $\tilde{L}_{xz}$ is the proper transform of $\bar{L}_{xz}$ by ψ and

$$\theta_2 = \frac{65(3b+d)}{19\cdot 32} + \frac{7}{19}\theta_1 + \frac{9}{19} = \frac{9728+65(133a+190b+7c+33d)}{19\cdot 32\cdot 33}.$$

This is not log canonical at some point $O \in E$.

We have

$$0 \leqslant \tilde{\Delta}\cdot\tilde{R}_y = \bar{\Delta}\cdot\bar{R}_y - \frac{d}{7\cdot 19} = \frac{8+38b}{13\cdot 33} - \frac{19a+c}{19\cdot 33} - \frac{d}{7\cdot 19},$$

and hence $133a+7c+33d \leqslant \frac{133}{13}(8+38b)$. Therefore, this inequality together with $b<\frac{2}{23}$ gives us

$$\theta_2 = \frac{9728+65\cdot 190b}{19\cdot 32\cdot 33} + \frac{65(133a+7c+33d)}{19\cdot 32\cdot 33} \leqslant$$
$$\leqslant \frac{9728+65\cdot 190b}{19\cdot 32\cdot 33} + \frac{65\cdot 7(8+38b)}{13\cdot 32\cdot 33} < 1.$$

Suppose that the point O is in the outside of $\tilde{R}_y$ and $\tilde{F}$. Then the log pair $(E, \frac{65}{32}\tilde{\Delta}|_E)$ is not log canonical at the point O and hence

$$1 < \frac{65}{32}\tilde{\Delta}\cdot E = \frac{65d}{3\cdot 7\cdot 32}.$$

However,

$$d \leqslant \frac{1}{33}(133a+7c+33d) \leqslant \frac{133}{13\cdot 33}(8+38b) < \frac{3\cdot 7\cdot 32}{65}$$

since $b \leqslant \frac{2}{23}$. This is a contradiction.

Suppose that the point O belongs to $\tilde{R}_y$. Then the log pair $\left(\tilde{X}, \frac{65b}{32}\tilde{R}_y + \frac{65}{32}\tilde{\Delta} + \theta_2 E\right)$ is not log canonical at the point O and hence

$$1 < 7\left(\frac{65}{32}\tilde{\Delta} + \theta_2 E\right)\cdot\tilde{R}_x = \frac{7\cdot 65}{32}\left(\frac{8+38b}{13\cdot 33} - \frac{19a+c}{19\cdot 33} - \frac{d}{7\cdot 19}\right) + \theta_2.$$

However,

$$\frac{7\cdot 65}{32}\left(\frac{8+38b}{13\cdot 33} - \frac{19a+c}{19\cdot 33} - \frac{d}{7\cdot 19}\right) + \theta_2 = \frac{9728+65\cdot 190b}{19\cdot 32\cdot 33} + \frac{65\cdot 7(8+38b)}{13\cdot 32\cdot 33} < 1.$$

This is a contradiction. Therefore, the point O is the point O_3.

Suppose that the point O belongs to $\tilde{F}$. Then the log pair $\left(\tilde{X}, \frac{65}{32}\tilde{\Delta} + \theta_1\tilde{F} + \theta_2 E\right)$ is not log canonical at the point O and hence

$$1 < 3\left(\frac{65}{32}\tilde{\Delta} + \theta_2 E\right)\cdot\tilde{F} = \frac{3\cdot 65}{32}\left(\frac{c}{13\cdot 19} - \frac{d}{3\cdot 19}\right) + \theta_2.$$

However,

$$\frac{3\cdot 65}{32}\left(\frac{c}{13\cdot 19}-\frac{d}{3\cdot 19}\right)+\theta_2=\frac{3\cdot 65c}{13\cdot 19\cdot 32}+\frac{9728+65(133a+190b+7c)}{19\cdot 32\cdot 33}=$$
$$=\frac{512+455a}{32\cdot 33}+\frac{65\cdot 190(13b+c)}{13\cdot 19\cdot 32\cdot 33}\leqslant$$
$$\leqslant\frac{512+455a}{32\cdot 33}+\frac{65\cdot 190(4+43a)}{14\cdot 19\cdot 32\cdot 33}<1$$

since $13b+c\leqslant\frac{13}{14}(4+43a)$ and $a\leqslant\frac{304}{3\cdot 23\cdot 65}$. This is a contradiction. □

Lemma 3.4.10. Let X be a quasismooth hypersurface of degree 166 in $\mathbb{P}(13,23,51,83)$. Then $\mathrm{lct}(X)=\frac{91}{40}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2+y^5z+xz^3+x^{11}y=0.$$

The surface X is singular only at the points O_x, O_y and O_z. The curves C_x and C_y are irreducible. We have

$$\frac{91}{40}=\mathrm{lct}\left(X,\frac{4}{13}C_x\right)<\mathrm{lct}\left(X,\frac{4}{23}C_y\right)=\frac{115}{24},$$

and hence $\mathrm{lct}(X)\leqslant\frac{91}{40}$.

Suppose that $\mathrm{lct}(X)<\frac{91}{40}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{91}{40}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither C_x nor C_y. Then the inequalities

$$51D\cdot C_x=\frac{8}{23}<\frac{40}{91},\quad 13D\cdot C_y=\frac{8}{51}<\frac{40}{91}$$

show that the point P is a smooth point of X in the outside of C_x. However, $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(663))$ contains x^{51}, $y^{13}x^{28}$, $y^{26}x^5$ and z^{13}, and hence it follows from Lemma 1.3.9 that the point P is either a singular point of X or a point on C_x. This is a contradiction. □

3.5. Sporadic cases with $I=5$

Lemma 3.5.1. Let X be a quasismooth hypersurface of degree 63 in $\mathbb{P}(11,13,19,25)$. Then $\mathrm{lct}(X)=\frac{13}{8}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^2t+yt^2+xy^4+x^4z=0,$$

and X is singular at O_x, O_y, O_z and O_t.

The curve C_x (resp. C_y, C_z, C_t) consists of two irreducible and reduced curves L_{xt} (resp L_{yz}, L_{yz}, L_{xt}) and $R_x=\{x=z^2+yt=0\}$ (resp. $R_y=\{y=x^4+zt=0\}$, $R_z=\{z=t^2+xy^3=0\}$, $R_t=\{t=y^4+x^3z=0\}$). The curve L_{xt} intersects R_x (resp. R_t) only at the point O_y (resp. O_z). The curve L_{yz} intersects R_y (resp. R_z) only at the point O_t (resp. O_x).

We have the following intersection numbers

$$D\cdot L_{xt}=\frac{5}{13\cdot 19},\quad D\cdot L_{yz}=\frac{1}{5\cdot 11},\quad D\cdot R_x=\frac{2}{5\cdot 13},\quad D\cdot R_y=\frac{4}{5\cdot 19},\quad D\cdot R_z=\frac{10}{11\cdot 13},$$

$$D\cdot R_t=\frac{20}{11\cdot 19},\quad L_{xt}\cdot R_x=\frac{2}{13},\quad L_{xt}\cdot R_t=\frac{4}{19},\quad L_{yz}\cdot R_y=\frac{4}{25},\quad L_{yz}\cdot R_z=\frac{2}{11},$$

$$L_{xt}^2=-\frac{27}{13\cdot 19},\quad L_{yz}^2=-\frac{31}{11\cdot 25},\quad R_x^2=-\frac{28}{13\cdot 25},\quad R_y^2=-\frac{24}{19\cdot 25},\quad R_z^2=\frac{12}{11\cdot 13},\quad R_t^2=\frac{56}{11\cdot 19}.$$

We have

$$\mathrm{lct}\left(X,\frac{5}{13}C_y\right)=\frac{13}{8}<\mathrm{lct}\left(X,\frac{5}{11}C_x\right)=\frac{33}{20}<\mathrm{lct}\left(X,\frac{5}{25}C_t\right)=\frac{35}{16}<\mathrm{lct}\left(X,\frac{5}{19}C_z\right)=\frac{19}{8}.$$

In particular, we have $\mathrm{lct}(X)\leqslant\frac{13}{8}$.

We suppose that $\mathrm{lct}(X)<\frac{13}{8}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the log pair $(X,\frac{13}{8}D)$ is not log canonical at some point $P\in X$.

Suppose that the point P is located in the outside of $C_x\cup C_y\cup C_z\cup C_t$. We consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda x^4+\mu zt=0$, where $[\lambda:\mu]\in\mathbb{P}^1$. The curve L_{xt} is the unique base component of the pencil $\mathcal{L}$. There is a unique member Z in the pencil $\mathcal{L}$ passing through the point P. Since the point P is in the outside of $C_x\cup C_y\cup C_z\cup C_t$, the curve Z is defined by an equation of the form

$$\alpha x^4+zt=0,$$

where α is a non-zero constant.

The open subset $Z\setminus C_z$ of the curve Z is a $\mathbb{Z}_{19}$-quotient of the affine curve

$$\alpha x^4+t=t+yt^2+xy^4+x^4=0\subset\mathbb{C}^3\cong\mathrm{Spec}\Big(\mathbb{C}\big[x,y,z\big]\Big),$$

that is isomorphic to the affine curve given by the equation

$$x\left((1-\alpha)x^3+\alpha^2x^7y+y^4\right)=0\subset\mathbb{C}^2\cong\mathrm{Spec}\Big(\mathbb{C}\big[y,z\big]\Big).$$

If $\alpha\neq 1$, the divisor Z consists of two irreducible and reduced curves L_{xt} and Z_α. On the other hand, if $\alpha=1$, then the divisor Z consists of three irreducible and reduced curves L_{xt}, R_y and Z_1. Since $P\notin C_x\cup C_y\cup C_z\cup C_t$, the point P must be contained in Z_α (including $\alpha=1$). Also, the curve Z_α is smooth at the point P.

Write $D=nZ_\alpha+\Gamma$, where Γ is an effective $\mathbb{Q}$-divisor whose support contains Z_α. Since Z_α passing through the point O_t and the pair $(X,\frac{13}{8}D)$ is log canonical at the point O_z, we have $n\leqslant\frac{8}{13}$. We can easily check

$$D\cdot Z_\alpha=\begin{cases}D\cdot(Z-L_{xt})=\dfrac{227}{5\cdot 13\cdot 19}\ \text{if}\ \alpha\neq 1,\\ D\cdot(Z-L_{xt}-R_y)=\dfrac{35}{13\cdot 19}\ \text{if}\ \alpha=1.\end{cases}$$

Also, if $\alpha\neq 1$, then

$$Z_\alpha^2=Z\cdot Z_\alpha-L_{xt}\cdot Z_\alpha\geqslant Z\cdot Z_\alpha-(L_{xt}+R_x)\cdot Z_\alpha=\frac{33}{5}D\cdot Z_\alpha.$$

If $\alpha = 1$,

$$Z_\alpha^2 = Z \cdot Z_\alpha - (L_{xt} + R_y) \cdot Z_\alpha \geqslant Z \cdot Z_\alpha - (L_{xt} + R_x + L_{yz} + R_y) \cdot Z_\alpha = 4D \cdot Z_\alpha.$$

In both cases, we have $Z_\alpha^2 > 0$. Since

$$(D - nZ_\alpha) \cdot Z_\alpha \leqslant D \cdot Z_\alpha < \frac{8}{13}$$

Lemma 1.3.8 shows that the pair $(X, \frac{13}{8}D)$ is log canonical at the point P. This is a contradiction. Therefore, the point P must belong to the set $C_x \cup C_y \cup C_z \cup C_t$.

It follows from Lemma 1.3.6 that we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curves C_x, C_y, C_z, C_t. Since the curve R_t is singular at the point O_z with multiplicity 3 and the support of D does not contain either L_{xt} or R_t, one of the inequalities

$$\mathrm{mult}_{O_z}(D) \leqslant 19D \cdot L_{xt} = \frac{5}{13} < \frac{8}{13}, \quad \mathrm{mult}_{O_z}(D) \leqslant \frac{19}{3} D \cdot R_t = \frac{20}{3 \cdot 11} < \frac{8}{13}$$

must hold, and hence the point P cannot be the point O_z. Similarly, we see that the point P can be neither O_x nor O_y.

Now we write $D = m_0 L_{xt} + m_1 L_{yz} + m_2 R_x + m_3 R_y + m_4 R_z + m_5 R_t + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor whose support contains none of L_{xt}, L_{yz}, R_x, R_y, R_z, R_t. Since the pair $(X, \frac{13}{8}D)$ is log canonical at the points O_x, O_y, O_z, we must have $m_i \leqslant \frac{8}{13}$. Then the inequalities

$$\left.\begin{array}{l}
(D - m_0 L_{xt}) \cdot L_{xt} = \dfrac{5 + 27m_0}{13 \cdot 19} \\
(D - m_1 L_{yz}) \cdot L_{yz} = \dfrac{5 + 31m_1}{11 \cdot 25} \\
(D - m_2 R_x) \cdot R_x = \dfrac{10 + 28m_2}{25 \cdot 13} \\
(D - m_3 R_y) \cdot R_y = \dfrac{20 + 24m_3}{25 \cdot 19} \\
(D - m_4 R_z) \cdot R_z = \dfrac{10 - 12m_4}{11 \cdot 13} \\
(D - m_5 R_t) \cdot R_t = \dfrac{20 - 56m_5}{11 \cdot 19}
\end{array}\right\} \leqslant \frac{8}{13}$$

imply that the point P must be the point O_t.

Put $D = aL_{yz} + bR_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither the curve L_{yz} nor R_x. If $a = 0$, then we obtain

$$\mathrm{mult}_{O_t}(D) \leqslant 25D \cdot L_{yz} = \frac{5}{11} < \frac{8}{13}.$$

This is a contradiction. Therefore, $a > 0$, and hence the support of D dose not contain the curve R_y. Since

$$\frac{4}{5 \cdot 19} = D \cdot R_y \geqslant aL_{yz} \cdot R_y + \frac{\mathrm{mult}_{O_t}(D) - a}{25} > \frac{3a}{25} + \frac{8}{13 \cdot 25},$$

and hence $a \leqslant \frac{36}{247}$. If $b > 0$, then

$$\frac{5}{13\cdot 19} = D\cdot L_{xt} \geqslant bR_x\cdot L_{xt} = \frac{2b}{13},$$

and hence $b \leqslant \frac{5}{38}$.

Let $\pi\colon \bar{X}\to X$ be the weighted blow up of O_t with weights $(7,3)$ and let F be the exceptional curve of π. Then

$$K_{\bar{X}} \sim_{\mathbb{Q}} \pi^*(K_X) - \frac{15}{25}F,\ \bar{L}_{yz} \sim_{\mathbb{Q}} \pi^*(L_{yz}) - \frac{3}{25}F,\ \bar{R}_x \sim_{\mathbb{Q}} \pi^*(R_x) - \frac{7}{25}F,\ \bar{\Delta} \sim_{\mathbb{Q}} \pi^*(\Delta) - \frac{c}{25}F,$$

where $\bar{\Delta}$, $\bar{L}_{yz}$, $\bar{R}_x$ are the proper transforms of Δ, L_{yz}, R_x, respectively, and c is a non-negative rational number. The curve F contains two singular points Q_7 and Q_3 of $\bar{X}$. The point Q_7 is a singular point of type $\frac{1}{7}(1,1)$ and the point Q_3 is of type $\frac{1}{3}(2,1)$. Note that the curve $\bar{R}_x$ passes through the point Q_3 but not the point Q_7. The curve $\bar{L}_{yz}$ passes through the point Q_7 but not the point Q_3.

The log pull-back of the log pair $(X, \frac{13}{8}D)$ by π is the log pair

$$\left(\bar{X},\ \frac{13a}{8}\bar{L}_{yz} + \frac{13b}{8}\bar{R}_x + \frac{13}{8}\bar{\Delta} + \theta_1 F\right),$$

where

$$\theta_1 = \frac{13(3a+7b+c)+120}{8\cdot 25}.$$

This pair is not log canonical at some point $Q\in F$. We have

$$0 \leqslant \bar{\Delta}\cdot\bar{R}_x = \frac{10+28b}{13\cdot 25} - \frac{a}{25} - \frac{c}{3\cdot 25}.$$

This inequality shows $3a + c \leqslant \frac{3}{13}(10+28b)$. Then

$$\theta_1 = \frac{13(3a+c)+91b+120}{8\cdot 25} \leqslant \frac{6+7b}{8} < 1$$

since $b \leqslant \frac{5}{38}$.

Suppose that the point Q is neither the point Q_7 nor the point Q_3. Then the log pair $\left(\bar{X}, \frac{13}{8}\bar{\Delta} + F\right)$ is not log canonical at the point Q. Then

$$\frac{13c}{8\cdot 21} = \frac{13}{8}\bar{\Delta}\cdot F > 1$$

by Lemma 1.3.4. However, $c \leqslant 3a + c \leqslant \frac{3}{13}(10+28b)$. This is a contradiction since $b \leqslant \frac{5}{38}$. Therefore, the point Q is either the point Q_7 or the point Q_3.

Suppose that the point Q is the point Q_3. This point is the intersection point of F and $\bar{R}_x$. Then the log pair $\left(\bar{X}, \frac{13b}{8}\bar{R}_x + \frac{13}{8}\bar{\Delta} + \theta_1 F\right)$ is not log canonical at the point Q. It then follows from Lemma 1.3.4 that

$$1 < 3\left(\frac{13}{8}\bar{\Delta} + \theta_1 F\right)\cdot\bar{R}_x = \frac{13\cdot 3}{8}\left(\frac{10+28b}{13\cdot 25} - \frac{a}{25} - \frac{c}{3\cdot 25}\right) + \theta_1.$$

However,

$$\frac{13\cdot 3}{8}\left(\frac{10+28b}{13\cdot 25} - \frac{a}{25} - \frac{c}{3\cdot 25}\right) + \theta_1 = \frac{6+7b}{8} < 1.$$

Therefore, the point Q is the point Q_7. This point is the intersection point of F and $\bar{L}_{yz}$.

Let $\phi\colon \tilde{X} \to \bar{X}$ be the blow up at the point Q_7. Let G be the exceptional divisor of the morphism ϕ. The surface $\tilde{X}$ is smooth along the exceptional divisor G. Let $\tilde{L}_{yz}$, $\tilde{R}_x$, $\tilde{\Delta}$ and $\tilde{F}$ be the proper transforms of L_{yz}, R_x, Δ and F by $\pi\circ\phi$, respectively. We have

$$K_{\tilde{X}} \sim_{\mathbb{Q}} \phi^*(K_{\bar{X}}) - \frac{5}{7}G,\ \tilde{L}_{yz} \sim_{\mathbb{Q}} \phi^*(\bar{L}_{yz}) - \frac{1}{7}G,\ \tilde{F} \sim_{\mathbb{Q}} \phi^*(F) - \frac{1}{7}G,\ \tilde{\Delta} \sim_{\mathbb{Q}} \phi^*(\bar{\Delta}) - \frac{d}{7}G,$$

where d is a non-negative rational number. The log pull-back of the log pair $(X, \frac{13}{8}D)$ via $\pi\circ\phi$ is

$$\left(\tilde{X}, \frac{13a}{8}\tilde{L}_{yz} + \frac{13b}{8}\tilde{R}_x + \frac{13}{8}\tilde{\Delta} + \theta_1\tilde{F} + \theta_2 G\right),$$

where

$$\theta_2 = \frac{13}{7\cdot 8}(a+d) + \frac{\theta_1}{7} + \frac{5}{7} = \frac{1120 + 13(28a + 7b + c + 25d)}{7\cdot 8\cdot 25}.$$

This log pair is not log canonical at some point $O \in G$. We have

$$0 \leqslant \tilde{\Delta}\cdot\tilde{L}_{yz} = \frac{5+31a}{11\cdot 25} - \frac{b}{25} - \frac{c}{7\cdot 25} - \frac{d}{7}.$$

We then obtain $7b + c + 25d \leqslant \frac{7}{11}(5+31a)$. Since $a \leqslant \frac{36}{247}$, we see

$$\theta_2 = \frac{1120 + 13(28a + 7b + c + 25d)}{7\cdot 8\cdot 25} \leqslant \frac{511 + 273a}{7\cdot 8\cdot 11} < 1.$$

Suppose that $O \notin \tilde{F}\cup\tilde{L}_{yz}$. The log pair $\left(\tilde{X}, \frac{13}{8}\tilde{\Delta} + G\right)$ is not log canonical at the point O. Applying Lemma 1.3.4, we get

$$1 < \frac{13}{8}\tilde{\Delta}\cdot G = \frac{13d}{8},$$

and hence $d > \frac{8}{13}$. However, $d \leqslant \frac{1}{25}(7b + c + 25d) \leqslant \frac{7}{11\cdot 25}(5+31a)$. This is a contradiction since $a \leqslant \frac{36}{247}$. Therefore, the point O is either the intersection point of G and $\tilde{F}$ or the intersection point of G and $\tilde{L}_{yz}$. In the latter case, the pair $\left(\tilde{X}, \frac{13a}{8}\tilde{L}_{yz} + \frac{13}{8}\tilde{\Delta} + \theta_2 G\right)$ is not log canonical at the point O. Then, applying Lemma 1.3.4, we get

$$1 < \left(\frac{13}{8}\tilde{\Delta} + \theta_2 G\right)\cdot\tilde{L}_{yz} = \frac{13}{8}\left(\frac{5+31a}{11\cdot 25} - \frac{b}{25} - \frac{c}{7\cdot 25} - \frac{d}{7}\right) + \theta_2.$$

However,

$$\frac{13}{8}\left(\frac{5+31a}{11\cdot 25} - \frac{b}{25} - \frac{c}{7\cdot 25} - \frac{d}{7}\right) + \theta_2 = \frac{511 + 273a}{7\cdot 8\cdot 11} < 1.$$

Therefore, the point O must be the intersection point of G and $\tilde{F}$.

Let $\xi\colon \hat{X} \to \tilde{X}$ be the blow up at the point O and let H be the exceptional divisor of ξ. We also let $\hat{L}_{yz}$, $\hat{R}_x$, $\hat{\Delta}$, $\hat{G}$, and $\hat{F}$ be the proper transforms of $\tilde{L}_{yz}$, $\tilde{R}_x$, $\tilde{\Delta}$, G and $\tilde{F}$ by ξ, respectively. Then $\hat{X}$ is smooth along the exceptional divisor H. We have

$$K_{\hat{X}} \sim_{\mathbb{Q}} \xi^*(K_{\tilde{X}}) - H,\ \hat{G} \sim_{\mathbb{Q}} \xi^*(G) - H,\ \hat{F} \sim_{\mathbb{Q}} \xi^*(\tilde{F}) - H,\ \hat{\Delta} \sim_{\mathbb{Q}} \xi^*(\tilde{\Delta}) - eH,$$

where e is a non-negative rational number. The log pull-back of the log pair $(X, \frac{13}{8}D)$ via $\pi\circ\phi\circ\xi$ is

$$\left(\hat{X}, \frac{13a}{8}\hat{L}_{yz} + \frac{13b}{8}\hat{R}_x + \frac{13}{8}\hat{\Delta} + \theta_1\hat{F} + \theta_2\hat{G} + \theta_3 H\right),$$

where

$$\theta_3 = \theta_1 + \theta_2 + \frac{13e}{8} - 1 = \frac{560 + 13(49a + 56b + 8c + 25d + 175e)}{7\cdot 8\cdot 25}.$$

This log pair is not log canonical at some point $A \in H$. We have

$$\frac{c}{21} - \frac{d}{7} - e\hat{\Delta}\cdot\hat{F} \geqslant 0.$$

Therefore, $3d + 21e \leqslant c$.

Then

$$\begin{aligned}\theta_3 &= \frac{560 + 13(49a + 56b + 8c)}{7\cdot 8\cdot 25} + \frac{13(d + 7e)}{7\cdot 8} \leqslant \\ &\leqslant \frac{1680 + 13(147a + 168b + 49c)}{3\cdot 7\cdot 8\cdot 25} = \\ &= \frac{1680 + 1284b}{3\cdot 7\cdot 8\cdot 25} + \frac{13\cdot 49(3a + c)}{3\cdot 7\cdot 8\cdot 25} \leqslant \\ &\leqslant \frac{140 + 107b}{2\cdot 7\cdot 25} + \frac{7(5 + 14b)}{4\cdot 25} = \frac{21 + 36b}{28} < 1\end{aligned}$$

since $b \leqslant \frac{5}{38}$ and $3a + c \leqslant \frac{3}{13}(10 + 28b)$. In particular, θ_3 is a positive number.

Suppose that $A \notin \hat{F}\cup\hat{G}$. Then the log pair $\left(\hat{X}, \frac{13}{8}\hat{\Delta} + \theta_3 H\right)$ is not log canonical at the point A. Applying Lemma 1.3.4, we get

$$1 < \frac{13}{8}\hat{\Delta}\cdot H = \frac{13e}{8}.$$

However,

$$e \leqslant \frac{1}{21}(3d + 21e) \leqslant \frac{c}{21} \leqslant \frac{1}{21}(3a + c) \leqslant \frac{3(10 + 28b)}{13\cdot 21} \leqslant \frac{8}{13}.$$

Therefore, the point A must be either in $\hat{F}$ or in $\hat{G}$.

Suppose that $A \in \hat{F}$. Then the log pair $\left(\hat{X}, \frac{13}{8}\hat{\Delta} + \theta_1\hat{F} + \theta_3 H\right)$ is not log canonical at the point A. Applying Lemma 1.3.4, we get

$$1 < \left(\frac{13}{8}\hat{\Delta} + \theta_3 H\right)\cdot\hat{F} = \frac{13}{8}\left(\frac{c}{21} - \frac{d}{7} - e\right) + \theta_3 = \frac{1680 + 13(147a + 168b + 49c)}{3\cdot 7\cdot 8\cdot 25}.$$

However,

$$\frac{1680 + 13(147a + 168b + 49c)}{3\cdot 7\cdot 8\cdot 25} \leqslant \frac{21 + 36b}{28} < 1.$$

Therefore, the point A is the intersection point of H and $\hat{G}$. Then the log pair $\left(\hat{X}, \frac{13}{8}\hat{\Delta} + \theta_2\hat{G} + \theta_3 H\right)$ is not log canonical at the point A. From Lemma 1.3.4, we obtain

$$1 < \left(\frac{13}{8}\hat{\Delta} + \theta_3 H\right)\cdot\hat{G} = \frac{13}{8}(d - e) + \theta_3 = \frac{560 + 13(49a + 56b + 8c + 200d)}{7\cdot 8\cdot 25}.$$

However,

$$\frac{560+13(49a+56b+8c+200d)}{7\cdot 8\cdot 25}=\frac{80+91a}{8\cdot 25}+\frac{13(7b+c+25d)}{7\cdot 25}\leqslant\frac{56+169a}{8\cdot 11}<1$$

since $a<\frac{36}{247}$ and $7b+c+25d\leqslant\frac{7}{11}(5+31a)$. The obtained contradiction completes the proof. □

Lemma 3.5.2. Let X be a quasismooth hypersurface of degree 136 in $\mathbb{P}(11,25,37,68)$. Then $\mathrm{lct}(X)=\frac{11}{6}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$xy^5+x^9z+yz^3+t^2=0.$$

The surface X is singular at the points O_x, O_y and O_z.

The curves C_x and C_y are reduced and irreducible. We have

$$\frac{11}{6}=\mathrm{lct}\left(X,\frac{5}{11}C_x\right)<\mathrm{lct}\left(X,\frac{5}{25}C_y\right)=\frac{55}{18}.$$

Thus, $\mathrm{lct}(X)\leqslant\frac{11}{6}$.

Suppose that $\mathrm{lct}(X)<\frac{11}{6}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{11}{6}D)$ is not log canonical at some point P. By Lemma 1.3.6 we may assume that the support of D contains neither C_x nor C_y. Then two inequalities

$$37D\cdot C_x=\frac{2}{5}<\frac{6}{11},\quad 11D\cdot C_y=\frac{10}{37}<\frac{6}{11}$$

imply that the point P is neither a singular point of X nor a point on C_x. Since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(407))$ contains x^{37}, z^{11} and $x^{12}y^{11}$, we see that this cannot happen by Lemma 1.3.9. □

Lemma 3.5.3. Let X be a quasismooth hypersurface of degree 136 in $\mathbb{P}(13,19,41,68)$. Then $\mathrm{lct}(X)=\frac{91}{50}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$x^9y+xz^3+y^5z+t^2=0.$$

The surface X is singular only at the points O_x, O_y and O_z.

The curves C_x and C_y are reduced and irreducible. Also, it is easy to check

$$\frac{91}{50}=\mathrm{lct}\left(X,\frac{5}{13}C_x\right)<\mathrm{lct}(X,\frac{5}{19}C_y)=\frac{19}{6}.$$

Therefore, $\mathrm{lct}(X)\leqslant\frac{50}{91}$.

Suppose that $\mathrm{lct}(X)<\frac{91}{50}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{91}{50}D)$ is not log canonical at some point P. By Lemma 1.3.6 we may assume that the support of D contains neither C_x nor C_y. Then two inequalities

$$41D\cdot C_x=\frac{10}{19}<\frac{50}{91},\quad 13D\cdot C_y=\frac{10}{41}<\frac{50}{91}$$

imply that the point P is neither a singular point of X nor a point on C_x. However, by Lemma 1.3.9 this is impossible since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(533))$ contains x^{41}, z^{13} and x^3y^{26}. □

3.6. Sporadic cases with $I = 6$

Lemma 3.6.1. Let X be a quasismooth hypersurface of degree 45 in $\mathbb{P}(7, 10, 15, 19)$. Then $\mathrm{lct}(X) = \frac{35}{54}$.

Proof. The surface X can be defined by the equation $z^3 - y^3z + xt^2 + x^5y = 0$. It is singular at the points O_x, O_y, O_t and $Q = [0 : 1 : 1 : 0]$.

The curve C_x consists of two irreducible and reduced curves L_{xz} and $R_x = \{x = z^2 - y^3 = 0\}$. These two curves L_{xz} and R_x meets each other at the point O_t. Also,

$$L_{xz}^2 = -\frac{23}{10 \cdot 19}, \quad R_x^2 = -\frac{8}{5 \cdot 19}, \quad L_{xz} \cdot R_x = \frac{3}{19}.$$

The curve R_x is singular at the point O_t. The curve C_y is irreducible and

$$\frac{35}{54} = \mathrm{lct}\left(X, \frac{6}{7}C_x\right) < \mathrm{lct}\left(X, \frac{6}{10}C_y\right) = \frac{25}{18}.$$

Therefore, $\mathrm{lct}(X) \leqslant \frac{35}{54}$.

Suppose that $\mathrm{lct}(X) < \frac{35}{54}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{35}{54}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D does not contain the curve C_y. Similarly, we may assume that either $L_{xz} \not\subseteq \mathrm{Supp}(D)$ or $R_x \not\subseteq \mathrm{Supp}(D)$.

Since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(105))$ contains the monomials x^{15}, y^7x^5 and z^7, it follows from Lemma 1.3.9 that the point P is either a point on C_x or the singular point O_x.

Since either $L_{xz} \not\subseteq \mathrm{Supp}(D)$ or $R_x \not\subseteq \mathrm{Supp}(D)$, one of the inequalities

$$\mathrm{mult}_{O_t}(D) \leqslant 19D \cdot L_{xz} = \frac{3}{5} < \frac{54}{35}, \quad \mathrm{mult}_{O_t}(D) \leqslant \frac{19}{2}D \cdot R_x = \frac{3}{5} < \frac{54}{35}$$

must hold, and hence the point P cannot be the point O_t. On the other hand, the inequality $7D \cdot C_y = \frac{18}{19} < \frac{54}{35}$ shows that the point P cannot be the point O_x.

Put $D = mL_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz} \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{6}{5 \cdot 19} = D \cdot R_x \geqslant mL_{xz} \cdot R_x = \frac{3m}{19},$$

and hence $m \leqslant \frac{2}{5}$. Then,

$$10(D - mL_{xz}) \cdot L_{xz} = \frac{6 + 23m}{19} \leqslant \frac{54}{35}.$$

Thus it follows from Lemma 1.3.8 that the point P cannot belong to L_{xz}.

Now we write $D = \epsilon R_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $R_x \not\subset \mathrm{Supp}(\Delta)$. If $\epsilon \neq 0$, then

$$\frac{3}{5 \cdot 19} = D \cdot L_{xz} \geqslant \epsilon R_x \cdot L_{xz} = \frac{3\epsilon}{19},$$

and hence $\epsilon \leqslant \frac{1}{5}$. Then

$$5(D - \epsilon R_x) \cdot R_x = \frac{3 + 8\epsilon}{19} \leqslant \frac{54}{35}.$$

By Lemma 1.3.8 the point P cannot be contained in R_x either. Therefore, the point P is located nowhere. □

Lemma 3.6.2. Let X be a quasismooth hypersurface of degree 106 in $\mathbb{P}(11,19,29,53)$. Then $\mathrm{lct}(X)=\frac{55}{36}$.

Proof. We may assume that the surface X is defined by the quasihomogeneous equation

$$x^7z+xy^5+yz^3+t^2=0.$$

The surface X is singular at O_x, O_y and O_z. The curves C_x and C_y are irreducible. It is easy to see

$$\mathrm{lct}(X,\frac{6}{11}C_x)=\frac{55}{36}<\mathrm{lct}(X,\frac{6}{19}C_y)=\frac{57}{28}.$$

Suppose that $\mathrm{lct}(X)<\frac{55}{36}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{55}{36}D)$ is not log canonical. For a smooth point $P\in X\setminus C_x$, we have

$$\mathrm{mult}_P(D)\leqslant\frac{6\cdot 319\cdot 106}{11\cdot 19\cdot 29\cdot 53}<\frac{36}{55}$$

by Lemma 1.3.9 since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(319))$ contains the monomials x^{29}, z^{11} and $x^{10}y^{11}$. Therefore, either there is a point $P\in C_x$ such that $\mathrm{mult}_P(D)>\frac{36}{55}$ or we have $\mathrm{mult}_{O_x}(D)>\frac{36}{55}$. Since the pairs $(X,\frac{6\cdot 55}{11\cdot 36}C_x)$ and $(X,\frac{6\cdot 55}{19\cdot 36}C_y)$ are log canonical and the curves C_x and C_y are irreducible, we may assume that the support of D contains neither the curve C_x nor the curve C_y. Then we can obtain

$$\mathrm{mult}_{O_x}(D)\leqslant 11C_y\cdot D\leqslant\frac{11\cdot 19\cdot 106\cdot 6}{11\cdot 19\cdot 29\cdot 53}<\frac{36}{55}$$

and for any point $P\in C_x$

$$\mathrm{mult}_P(D)\leqslant 29C_x\cdot D\leqslant\frac{29\cdot 11\cdot 106\cdot 6}{11\cdot 19\cdot 29\cdot 53}<\frac{36}{55}.$$

This is a contradiction. Therefore, $\mathrm{lct}(X)=\frac{55}{36}$. □

Lemma 3.6.3. Let X be a quasismooth hypersurface of degree 106 in $\mathbb{P}(13,15,31,53)$. Then $\mathrm{lct}(X)=\frac{91}{60}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$x^7y+xz^3+y^5z+t^2=0.$$

The surface X is singular at the points O_x, O_y and O_z.

The curves C_x, C_y and C_z are reduced and irreducible. We have

$$\mathrm{lct}\left(X,\frac{6}{13}C_x\right)=\frac{91}{60}<\mathrm{lct}\left(X,\frac{6}{15}C_y\right)=\frac{25}{12}<\mathrm{lct}\left(X,\frac{6}{31}C_z\right)=\frac{93}{28}.$$

Therefore, $\mathrm{lct}(X)\leqslant\frac{91}{60}$.

Suppose that $\mathrm{lct}(X)<\frac{91}{60}$. Then there is an effective $\mathbb{Q}$-divisor $D\sim_{\mathbb{Q}}-K_X$ such that the pair $(X,\frac{91}{60}D)$ is not log canonical at some point P. By Lemma 1.3.6 we may assume that the support of D contains none of C_x, C_y, C_z. Since C_y is singular at the point O_z and $\frac{31}{2}D\cdot C_y=\frac{6}{13}<\frac{60}{91}$, the point P must be in the outside of C_y. Furthermore, the point P is in the outside of $C_x\cup C_z$ since $15D\cdot C_x=\frac{12}{31}<\frac{60}{91}$ and $D\cdot C_z=\frac{4}{65}<\frac{60}{91}$.

Now we consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda z^3+\mu x^6y=0$, $[\lambda:\mu]\in\mathbb{P}^1$. Then there is a unique member C in $\mathcal{L}$ passing through the point P. Since the point P is located in the outside of $C_x \cup C_y \cup C_z$, the curve C is cut out by the equation of the form $x^6y + \alpha z^3 = 0$, where α is a non-zero constant. Since the curve C is a double cover of the curve defined by the equation $x^6y + \alpha z^3 = 0$ in $\mathbb{P}(13, 15, 31)$, we have $\mathrm{mult}_P(C) \leqslant 2$. Therefore, we may assume that the support of D does not contain at least one irreducible component. If $\alpha \neq 1$, then the curve C is irreducible, and hence the inequality

$$\mathrm{mult}_P(D) \leqslant D \cdot C = \frac{12}{65} < \frac{60}{91}$$

is a contradiction. If $\alpha = 1$, then the curve C consists of two distinct irreducible and reduced curve C_1 and C_2. We have

$$D \cdot C_1 = D \cdot C_2 = \frac{6}{65}, \quad C_1^2 = C_2^2 = \frac{8}{13}.$$

Put $D = a_1C_1 + a_2C_2 + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor whose support contains neither C_1 nor C_2. Since the pair $(X, \frac{91}{60}D)$ is log canonical at O_x, both a_1 and a_2 are at most $\frac{60}{91}$. Then a contradiction follows from Lemma 1.3.8 since

$$(D - a_iC_i) \cdot C_i \leqslant D \cdot C_i = \frac{12}{65} < \frac{60}{91}$$

for each i. □

3.7. Sporadic cases with $I = 7$

Lemma 3.7.1. Let X be a quasismooth hypersurface of degree 76 in $\mathbb{P}(11, 13, 21, 38)$. Then $\mathrm{lct}(X) = \frac{13}{10}$.

Proof. We may assume that the surface X is defined by the equation $t^2 + yz^3 + xy^5 + x^5z = 0$. The surface X is singular at O_x, O_y and O_z. The curves C_x, C_y and C_z are irreducible. We have

$$\frac{21}{10} = \mathrm{lct}(X, \frac{7}{21}C_z) > \frac{55}{42} = \mathrm{lct}(X, \frac{7}{11}C_x) > \mathrm{lct}(X, \frac{7}{13}C_y) = \frac{13}{10}.$$

Therefore, $\mathrm{lct}(X) \leqslant \frac{13}{10}$.

Suppose that $\mathrm{lct}(X) < \frac{13}{10}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{13}{10}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of D contains none of the curves C_x, C_y and C_z.

Since the curve C_y is singular at the point O_z, the inequality $11D \cdot C_y = \frac{2}{3} < \frac{10}{13}$ shows that the point P does not belong to the curve C_y. Also, the inequality $13D \cdot C_x = \frac{2}{3} < \frac{10}{13}$ implies that the point P cannot belong to C_x either. The inequality $D \cdot C_z = \frac{14}{11 \cdot 13} < \frac{10}{13}$ shows that the point P cannot belong to C_z

Consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda y^5 + \mu x^4 z = 0$, $[\lambda : \mu] \in \mathbb{P}^1$. There is a unique member Z in $\mathcal{L}$ passing through the point P. Since $P \notin C_x \cup C_y \cup C_z$, the curve Z is defined by an equation of the form $x^4z = \alpha y^5$, where α is a non-zero constant. The open subset $Z \setminus C_x$ of the curve Z is a $\mathbb{Z}_{11}$-quotient of the affine curve

$$z - \alpha y^5 = t^2 + yz^3 + y^5 + z = 0 \subset \mathbb{C}^3 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z, t\big]\Big)$$

that is isomorphic to the plane affine curve $C \subset \mathbb{C}^2$ defined by the equation

$$t^2 + \alpha^3 y^{16} + (1+\alpha)y^5 = 0 \subset \mathbb{C}^2 \cong \mathrm{Spec}\Big(\mathbb{C}\big[y, z\big]\Big).$$

The curve C is irreducible if $\alpha \neq -1$ and reducible if $\alpha = 1$. Since the C_x is not contained in the support of Z, the curve Z is irreducible if $\alpha \neq -1$ and reducible if $\alpha = 1$. From the equation of C, we can see that the log pair $(X, \frac{7}{50}Z)$ is log canonical at the point P. By Lemma 1.3.6, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve Z.

Suppose that $\alpha \neq -1$. Then $Z \not\subseteq \mathrm{Supp}(D)$ and

$$\frac{10}{33} = D \cdot Z \geqslant \mathrm{mult}_P\big(D\big) > \frac{10}{13}.$$

This is a contradiction. Thus, $\alpha = -1$. Then it follows from the equation of C that the curve Z consists of two irreducible and reduced curves Z_1 and Z_2. Without loss of generality we may assume that the point P belongs to the curve Z_1.

Put $D = mZ_1 + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $Z_1 \not\subset \mathrm{Supp}(\Omega)$. Since the pair $(X, \frac{13}{10}D)$ is log canonical at the point O_x, one has $m \leqslant \frac{10}{13}$. Then

$$\big(D - mZ_1\big) \cdot Z_1 < D \cdot Z_1 = \frac{5}{33} < \frac{10}{13}.$$

since $Z_1^2 > 0$. By Lemma 1.3.8, the log pair $(X, \frac{13}{10}D)$ is log canonical at the point P. This is a contradiction. $\square$

3.8. Sporadic cases with $I = 8$

Lemma 3.8.1. Let X be a quasismooth hypersurface of degree 46 in $\mathbb{P}(7, 11, 13, 23)$. Then $\mathrm{lct}(X) = \frac{35}{48}$.

Proof. The surface X can be defined by the equation $t^2 + y^3z + xz^3 + x^5y = 0$. The surface X is singular at the points O_x, O_y and O_z. The curves C_x, C_y and C_z are irreducible. We have

$$\frac{35}{48} = \mathrm{lct}\left(X, \frac{8}{7}C_x\right) < \mathrm{lct}\left(X, \frac{8}{13}C_z\right) = \frac{91}{80} < \mathrm{lct}\left(X, \frac{8}{11}C_y\right) = \frac{55}{48}.$$

In particular, $\mathrm{lct}(X) \leqslant \frac{35}{48}$. Suppose that $\mathrm{lct}(X) < \frac{35}{48}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{35}{48}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains none of the curves C_x, C_y and C_z.

Since the curve C_x is singular at the point O_z, the inequality

$$11D \cdot C_x = \frac{16}{13} < \frac{48}{35}$$

shows that the point P cannot belong to C_x. Also, the inequality

$$7D \cdot C_y = \frac{16}{13} < \frac{48}{35}$$

implies that the point P is not in C_y. Since

$$D \cdot C_z = \frac{16}{7 \cdot 11} < \frac{48}{35},$$

the point P cannot be in C_z either.

Consider the pencil $\mathcal{L}$ on X defined by the equations $\lambda x^4y+\mu z^3=0$, $[\lambda:\mu]\in\mathbb{P}^1$. There is a unique member Z in $\mathcal{L}$ passing through the point P. Since $P\notin C_x\cup C_y\cup C_z$, the curve Z is defined by an equation of the form $x^4y=\alpha z^3$, where α is a non-zero constant. The open subset $Z\setminus C_x$ of the curve Z is a $\mathbb{Z}_7$-quotient of the affine curve

$$y-\alpha z^3=t^2+y^3z+z^3+y=0\subset\mathbb{C}^3\cong\mathrm{Spec}\Big(\mathbb{C}\big[y,z,t\big]\Big)$$

that is isomorphic to the plane affine curve $C\subset\mathbb{C}^2$ defined by the equation

$$t^2+\alpha^3z^{10}+(1+\alpha)z^3=0\subset\mathbb{C}^2\cong\mathrm{Spec}\Big(\mathbb{C}\big[y,z\big]\Big).$$

The curve C is irreducible if $\alpha\neq-1$ and reducible if $\alpha=-1$. Since the C_x is not contained in the support of Z, the curve Z is irreducible if $\alpha\neq-1$ and reducible if $\alpha=-1$. From the equation of C, we can see that the log pair $(X,\frac{35}{234}Z)$ is log canonical at the point P. By Lemma 1.3.6, we may assume that $\mathrm{Supp}(D)$ does not contain at least one irreducible component of the curve Z.

Suppose that $\alpha\neq-1$. Then $Z\not\subseteq\mathrm{Supp}(D)$ and

$$\frac{48}{77}=D\cdot Z\geqslant\mathrm{mult}_P\big(D\big)>\frac{48}{35}.$$

This is a contradiction. Thus, $\alpha=-1$. Then it follows from the equation of C that the curve Z consists of two irreducible and reduced curves Z_1 and Z_2. Without loss of generality we may assume that the point P belongs to the curve Z_1.

Put $D=mZ_1+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $Z_1\not\subset\mathrm{Supp}(\Omega)$. Since the pair $(X,\frac{48}{35}D)$ is log canonical at the point O_x, one has $m\leqslant\frac{35}{48}$. Then

$$\big(D-mZ_1\big)\cdot Z_1<D\cdot Z_1=\frac{24}{77}<\frac{48}{35}.$$

since $Z_1^2>0$. By Lemma 1.3.8, the log pair $(X,\frac{48}{35}D)$ is log canonical at the point P. This is a contradiction. □

Lemma 3.8.2. Let X be a quasismooth hypersurface of degree 81 in $\mathbb{P}(7,18,27,37)$. Then $\mathrm{lct}(X)=\frac{35}{72}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$z^3-y^3z+xt^2+x^9y=0.$$

The surface X is singular at the points O_x, O_y, O_t and $Q=[0:1:1:0]$.

The curve C_x consists of two irreducible and reduced curves L_{xz} and $R_x=\{x=z^2-y^3=0\}$. These two curves intersect each other only at the point O_t. Also,

$$L_{xz}^2=-\frac{47}{18\cdot 37},\quad R_x^2=-\frac{20}{9\cdot 37},\quad L_{xz}\cdot R_x=\frac{3}{37}.$$

The curve C_y is irreducible and

$$\frac{35}{72}=\mathrm{lct}\left(X,\frac{8}{7}C_x\right)<\mathrm{lct}\left(X,\frac{8}{18}C_y\right)=\frac{15}{8}.$$

Suppose that $\mathrm{lct}(X) < \frac{35}{72}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{35}{72}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D does not contain the curve C_y. Similarly, we may assume that either $L_{xz} \not\subseteq \mathrm{Supp}(D)$ or $R_x \not\subseteq \mathrm{Supp}(D)$.

Since either $L_{xz} \not\subseteq \mathrm{Supp}(D)$ or $R_x \not\subseteq \mathrm{Supp}(D)$, one of the inequalities

$$\mathrm{mult}_{O_t}(D) \leqslant 37D \cdot L_{xz} = \frac{4}{9} < \frac{72}{35}, \quad \mathrm{mult}_{O_t}(D) \leqslant 37D \cdot R_x = \frac{8}{9} < \frac{72}{35}$$

must hold, and hence the point P cannot be O_t. Since $\mathrm{mult}_{O_x}(D) \leqslant 7D \cdot C_y = \frac{24}{37} < \frac{72}{35}$, the point P cannot be the point O_x.

Put $D = mL_{xz} + \Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz} \not\subset \mathrm{Supp}(\Omega)$. If $m \neq 0$, then

$$\frac{16}{18 \cdot 37} = D \cdot R_x \geqslant mL_{xz} \cdot R_x = \frac{3m}{37},$$

and hence $m \leqslant \frac{8}{27}$. Since

$$18\big(D - mL_{xz}\big) \cdot L_{xz} = \frac{8 + 47m}{37} \leqslant \frac{72}{35}$$

it follows from Lemma 1.3.8 that the point P cannot belong to L_{xz}.

Now we write $D = \epsilon Z_x + \Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $Z_x \not\subset \mathrm{Supp}(\Delta)$. If $\epsilon \neq 0$, then

$$\frac{8}{18 \cdot 37} = D \cdot L_{xz} \geqslant \epsilon R_x \cdot L_{xz} = \frac{3\epsilon}{37},$$

and hence $\epsilon \leqslant \frac{4}{27}$. Since

$$9\big(D - \epsilon R_x\big) \cdot R_x = \frac{8 + 20\epsilon}{37} \leqslant \frac{72}{35}$$

it follows from Lemma 1.3.8 that the point P cannot belong to R_x. Consequently, the point P must be a smooth point in the outside of C_x. However, since $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(189))$ contains the monomials x^{27}, y^7x^9 and z^7, it follows from Lemma 1.3.9 that P must be either a singular point of X or a point on C_x. This is a contradiction. □

3.9. Sporadic cases with $I = 9$

Lemma 3.9.1. Let X be a quasismooth hypersurface of degree 64 in $\mathbb{P}(7, 15, 19, 32)$. Then $\mathrm{lct}(X) = \frac{35}{54}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + y^3z + xz^3 + x^7y = 0.$$

The surface X is singular only at the points O_x, O_y and O_z. The curves C_x and C_y are irreducible, and

$$\frac{35}{54} = \mathrm{lct}\left(X, \frac{9}{7}C_x\right) < \mathrm{lct}\left(X, \frac{9}{15}C_y\right) = \frac{25}{18}.$$

In particular, $\mathrm{lct}(X) \leqslant \frac{35}{54}$.

Suppose that $\mathrm{lct}(X) < \frac{35}{54}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{35}{54}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither the curve C_x nor the curve C_y. Then two inequalities $19D \cdot C_x = \frac{6}{5} < \frac{54}{35}$, $7D \cdot C_y = \frac{18}{19} < \frac{54}{35}$ show that the point P must be a smooth point in the outside of C_x.

Note that $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(133))$ contains the monomials x^{19}, y^7x^4 and z^7 and hence it follows from Lemma 1.3.9 that the point P is either a singular point of X or a point on C_x. This is a contradiction. $\square$

3.10. Sporadic cases with $I = 10$

Lemma 3.10.1. Let X be a quasismooth hypersurface of degree 82 in $\mathbb{P}(7, 19, 25, 41)$. Then $\mathrm{lct}(X) = \frac{7}{12}$.

Proof. The surface X can be defined by the quasihomogeneous equation

$$t^2 + y^3z + xz^3 + x^9y = 0.$$

It is singular at the points O_x, O_y and O_z.

The curves C_x and C_y are irreducible. We have

$$\frac{7}{12} = \mathrm{lct}\left(X, \frac{10}{7}C_x\right) < \mathrm{lct}\left(X, \frac{10}{19}C_y\right) = \frac{19}{12},$$

and hence $\mathrm{lct}(X) \leqslant \frac{7}{12}$.

Suppose that $\mathrm{lct}(X) < \frac{7}{12}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{7}{12}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D contains neither the curve C_x nor the curve C_y. Since $25D \cdot C_x = \frac{20}{19} < \frac{12}{7}$ and $7D \cdot C_y = \frac{4}{5} < \frac{12}{7}$, the point P must be a smooth point in the outside of the curve C_x. Note that $H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(175))$ contains the monomials x^{25}, x^6y^7 and z^7, and hence the point P cannot be a smooth point in the outside of C_x by Lemma 1.3.9. Consequently, $\mathrm{lct}(X) = \frac{7}{12}$. $\square$

Lemma 3.10.2. Let X be a quasismooth hypersurface of degree 117 in $\mathbb{P}(7, 26, 39, 55)$. Then $\mathrm{lct}(X) = \frac{7}{18}$.

Proof. The surface X can be defined by the equation $z^3 - y^3z + xt^2 + x^{13}y = 0$. It is singular at the points O_x, O_y, O_t and $Q = [0 : 1 : 1 : 0]$.

The curve C_x consists of two irreducible curves L_{xz} and $R_x = \{x = z^2 - y^3 = 0\}$. These two curves intersect each other only at the point O_t. It is easy to check

$$L_{xz}^2 = -\frac{71}{26 \cdot 55}, \quad R_x^2 = -\frac{32}{13 \cdot 55}, \quad L_{xz} \cdot R_x = \frac{3}{55}.$$

On the other hand, the curve C_y is irreducible. We have

$$\frac{7}{18} = \mathrm{lct}\left(X, \frac{10}{7}C_x\right) < \mathrm{lct}\left(X, \frac{10}{26}C_y\right) = \frac{13}{6}.$$

In particular, $\mathrm{lct}(X) \leqslant \frac{7}{18}$.

Suppose that $\mathrm{lct}(X) < \frac{7}{18}$. Then there is an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ such that the pair $(X, \frac{7}{18}D)$ is not log canonical at some point P. By Lemma 1.3.6, we may assume that the support of the divisor D does not contain the curve C_y. Similarly, we may assume that either $L_{xz} \not\subset \mathrm{Supp}(D)$ or $R_x \not\subset \mathrm{Supp}(D)$.

Since $7D\cdot C_y=\frac{6}{11}<\frac{18}{7}$, the point P cannot be the point O_x. Meanwhile, since the support of D does not contain at least one components of C_x, one of the inequalities

$$\mathrm{mult}_{O_t}(D)\leqslant 55D\cdot L_{xz}=\frac{5}{13}<\frac{18}{7},$$
$$\mathrm{mult}_{O_t}(D)\leqslant 55D\cdot R_x=\frac{10}{13}<\frac{18}{7}$$

must hold, and hence the point P cannot be the point O_t.

Put $D=mL_{xz}+\Omega$, where Ω is an effective $\mathbb{Q}$-divisor such that $L_{xz}\not\subset\mathrm{Supp}(\Omega)$. If $m\neq 0$, then

$$\frac{10}{13\cdot 55}=D\cdot R_x\geqslant mL_{xz}\cdot R_x=\frac{3m}{55},$$

and hence $m\leqslant\frac{10}{39}$. Then

$$26\big(D-mL_{xz}\big)\cdot L_{xz}=\frac{10+71m}{55}<\frac{18}{7},$$

and hence Lemma 1.3.8 implies that the point P cannot belong to L_{xz}.

Now we write $D=\epsilon R_x+\Delta$, where Δ is an effective $\mathbb{Q}$-divisor such that $R_x\not\subset\mathrm{Supp}(\Delta)$. If $\epsilon\neq 0$, then

$$\frac{10}{26\cdot 55}=D\cdot L_{xz}\geqslant\epsilon R_x\cdot L_{xz}=\frac{3\epsilon}{55},$$

and hence $\epsilon\leqslant\frac{5}{39}$. Then

$$13(D-\epsilon R_x\big)\cdot R_x=\frac{10+32\epsilon}{55}<\frac{18}{7}.$$

Thus, Lemma 1.3.4 shows that the point P is not on R_x.

Therefore, the point P must be a smooth point in the outside of the curve C_x. Since $H^0(\mathbb{P},\mathcal{O}_{\mathbb{P}}(273))$ contains the monomials x^{39}, y^7x^{13} and z^7, it follows from Lemma 1.3.9 that P is either a point on C_x or a singular point of X. This is a contradiction. □

Part 4. The Big Table

The tables contains the following information on del Pezzo surfaces.

- The first column: the weights (a_0,a_1,a_2,a_3) of the weighted projective space $\mathbb{P}$.
- The second column: the degree of the surface $X\subset\mathbb{P}$.
- The third column: the self-intersection number K_X^2 of an anticanonical divisor of X.
- The fourth column: the rank ρ of the Picard group of the surface X.
- The fifth column: the global log canonical theeshold $\mathrm{lct}(X)$ of X.
- The sixth column: the possible monomials in x, y, z, t in the defining equation $f(x,y,z,t)=0$ of the surface X.
- The seventh column: the information on the singular points of X. We use the standard notation for cyclic quotient singularities along with the following convention: when we write, for instance, $O_xO_y=n\times\frac{1}{r}(a,b)$, we mean that there are n cyclic quotient singularities of type $\frac{1}{r}(a,b)$ cut out on X by the equations $z=t=0$ that are different from the point O_x in the case when $O_x\in X$ and O_x is not of type $\frac{1}{r}(a,b)$, and that are different from the point O_y in the case when $O_y\in X$ and O_y is not of type $\frac{1}{r}(a,b)$.

Log del Pezzo surfaces with $I = 1$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(2, 2n+1, 2n+1, 4n+1)$	$8n+4$	$\frac{2}{(2n+1)(4n+1)}$	8	1	y^4, y^3z, y^2z^2, yz^3, z^4, xt^2, $x^{n+1}yt$, $x^{n+1}zt$, $x^{2n+1}y^2$, $x^{2n+1}yz$, $x^{2n+1}z^2$, x^{4n+2}	$O_t = \frac{1}{4n+1}(1,1)$ $O_yO_z =$ $= 4 \times \frac{1}{2n+1}(1,n)$
$(1, 2, 3, 5)$	10	$\frac{1}{3}$	9	1 [a] $\frac{7}{10}$ [b]	t^2, yzt, y^2z^2, y^5, xz^3, xy^2t, xy^3z, x^2zt, x^2yz^2, x^2y^4, x^3yt, x^3y^2z, x^4z^2, x^4y^3, x^5t, x^5yz, x^6y^2, x^7z, x^8y, x^{10}	$O_z = \frac{1}{3}(1,1)$
$(1, 3, 5, 7)$	15	$\frac{1}{7}$	9	1 [c] $\frac{8}{15}$ [d]	z^3, yzt, y^5, xt^2, xy^3z, x^2yz^2, x^2y^2t, x^3zt, x^3y^4, x^4y^2z, x^5z^2, x^5yt, x^6y^3, x^7yz, x^8t, x^9y^2, $x^{10}z$, $x^{12}y$, x^{15}	$O_t = \frac{1}{7}(3,5)$
$(1, 3, 5, 8)$	16	$\frac{2}{15}$	10	1	t^2, yzt, y^2z^2, xz^3, xy^5, x^2y^2t, x^2y^3z, x^3zt, x^3yz^2, x^4y^4, x^5yt, x^5y^2z, x^6z^2, x^7y^3, x^8t, x^8yz, $x^{10}y^2$, $x^{11}z$, $x^{13}y$, x^{16}	$O_y = \frac{1}{3}(1,1)$ $O_z = \frac{1}{5}(1,1)$

a: if C_x has an ordinary double point, b: if C_x has a non-ordinary double point,
c: if the defining equation of X contains yzt, d: if the defining equation of X does not contain yzt.

Log del Pezzo surfaces with $I = 1$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(2,3,5,9)$	18	$\frac{1}{15}$	7	2[a] $\frac{11}{6}$[b]	t^2, yz^3, y^3t, y^6, xy^2z^2, x^2zt, x^2y^3z, x^3yt, x^3y^4, x^4z^2, x^5yz, x^6y^2, x^9	$O_z = \frac{1}{5}(1,2)$ $O_yO_t = 2 \times \frac{1}{3}(1,1)$
$(3,3,5,5)$	15	$\frac{1}{15}$	5	2	t^3, zt^2, z^2t, z^3, y^5, xy^4, x^2y^3, x^3y^2, x^4y, x^5	$O_xO_y = 5 \times \frac{1}{3}(1,1)$ $O_zO_t = 3 \times \frac{1}{5}(1,1)$
$(3,5,7,11)$	25	$\frac{5}{231}$	5	$\frac{21}{10}$	z^2t, y^5, xt^2, xy^3z, x^2yz^2, x^3yt, x^5y^2, x^6z	$O_x = \frac{1}{3}(1,1)$ $O_z = \frac{1}{7}(3,5)$ $O_t = \frac{1}{11}(5,7)$
$(3,5,7,14)$	28	$\frac{2}{105}$	6	$\frac{9}{4}$	t^2, z^2t, z^4, xy^5, x^2y^3z, x^3yt, x^3yz^2, x^6y^2, x^7z	$O_x = \frac{1}{3}(1,1)$ $O_y = \frac{1}{5}(1,2)$ $O_zO_t = 2 \times \frac{1}{7}(3,5)$
$(3,5,11,18)$	36	$\frac{2}{165}$	6	$\frac{21}{10}$	t^2, y^5z, xz^3, xy^3t, x^2y^6, x^3yz^2, x^5y^2z, x^6t, x^7y^3, x^{12}	$O_y = \frac{1}{5}(1,1)$ $O_z = \frac{1}{11}(5,7)$ $O_xO_t = 2 \times \frac{1}{3}(1,1)$
$(5,14,17,21)$	56	$\frac{4}{1785}$	4	$\frac{25}{8}$	yt^2, y^4, xz^3, x^5yz, x^7t	$O_x = \frac{1}{5}(2,1)$ $O_z = \frac{1}{17}(7,2)$ $O_t = \frac{1}{21}(5,17)$ $O_yO_t = 1 \times \frac{1}{7}(5,3)$
$(5,19,27,31)$	81	$\frac{3}{2945}$	3	$\frac{25}{6}$	z^3, yt^2, xy^4, x^7yz, $x^{10}t$	$O_x = \frac{1}{5}(2,1)$ $O_y = \frac{1}{19}(2,3)$ $O_t = \frac{1}{31}(5,27)$
$(5,19,27,50)$	100	$\frac{2}{2565}$	4	$\frac{25}{6}$	t^2, yz^3, xy^5, x^7y^2z, $x^{10}t$, x^{20}	$O_y = \frac{1}{19}(2,3)$ $O_z = \frac{1}{27}(5,23)$ $O_xO_t = 2 \times \frac{1}{5}(2,1)$
$(7,11,27,37)$	81	$\frac{3}{2849}$	3	$\frac{49}{12}$	z^3, y^4t, xt^2, x^3y^3z, $x^{10}y$	$O_x = \frac{1}{7}(3,1)$ $O_y = \frac{1}{11}(7,5)$ $O_t = \frac{1}{37}(11,27)$

a: if C_y has a tacnodal point, b: if C_y has no tacnodal points.

Log del Pezzo surfaces with $I = 1$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(7, 11, 27, 44)$	88	$\frac{2}{2079}$	4	$\frac{35}{8}$	t^2, y^4t, y^8, xz^3, x^4y^3z, $x^{11}y$	$O_x = \frac{1}{7}(3, 1)$ $O_z = \frac{1}{27}(11, 17)$ $O_yO_t = 2 \times \frac{1}{11}(7, 5)$
$(9, 15, 17, 20)$	60	$\frac{1}{765}$	3	$\frac{21}{4}$	t^3, y^4, xz^3, x^5y	$O_x = \frac{1}{9}(4, 1)$ $O_z = \frac{1}{17}(5, 1)$ $O_xO_y = 1 \times \frac{1}{3}(1, 1)$ $O_yO_t = 1 \times \frac{1}{5}(2, 1)$
$(9, 15, 23, 23)$	69	$\frac{1}{1035}$	5	6	t^3, zt^2, z^2t, z^3, xy^4, x^6y	$O_x = \frac{1}{9}(1, 1)$ $O_y = \frac{1}{15}(1, 1)$ $O_xO_y = 1 \times \frac{1}{3}(1, 1)$ $O_zO_t = 3 \times \frac{1}{23}(3, 5)$
$(11, 29, 39, 49)$	127	$\frac{127}{609609}$	3	$\frac{33}{4}$	z^2t, yt^2, xy^4, x^8z	$O_x = \frac{1}{11}(7, 5)$ $O_y = \frac{1}{29}(1, 2)$ $O_z = \frac{1}{39}(11, 29)$ $O_t = \frac{1}{49}(11, 39)$
$(11, 49, 69, 128)$	256	$\frac{2}{37191}$	2	$\frac{55}{6}$	t^2, yz^3, xy^5, $x^{17}z$	$O_x = \frac{1}{11}(5, 7)$ $O_y = \frac{1}{49}(2, 3)$ $O_z = \frac{1}{69}(11, 59)$
$(13, 23, 35, 57)$	127	$\frac{127}{596505}$	3	$\frac{65}{8}$	z^2t, y^4z, xt^2, x^8y	$O_x = \frac{1}{13}(9, 5)$ $O_y = \frac{1}{23}(13, 11)$ $O_z = \frac{1}{35}(13, 23)$ $O_t = \frac{1}{57}(23, 35)$
$(13, 35, 81, 128)$	256	$\frac{2}{36855}$	2	$\frac{91}{10}$	t^2, y^5z, xz^3, $x^{17}y$	$O_x = \frac{1}{13}(3, 11)$ $O_y = \frac{1}{35}(13, 23)$ $O_z = \frac{1}{81}(35, 47)$

Log del Pezzo surfaces with $I = 2$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(3, 3n, 3n+1, 3n+1)$	$9n+3$	$\frac{4}{3n(3n+1)}$	7	1	t^3, zt^2, z^2t, z^3, xy^3, $x^{n+1}y^2$, $x^{2n+1}y$, x^{3n+1}	$O_y = \frac{1}{3n}(1,1)$ $O_xO_y = 3 \times \frac{1}{3}(1,1)$ $O_zO_t = 3 \times \frac{1}{3n+1}(1,n)$
$(3, 3n+1, 3n+2, 3n+2)$	$9n+6$	$\frac{4}{(3n+1)(3n+2)}$	5	1	t^3, zt^2, z^2t, z^3, xy^3, $x^{n+1}yt$, $x^{n+1}yz$, x^{3n+2}	$O_y = \frac{1}{3n+1}(1,1)$ $O_zO_t =$ $= 3 \times \frac{1}{3n+2}(3, 3n+1)$
$(3, 3n+1, 3n+2, 6n+1)$	$12n+5$	$\frac{4(12n+5)}{3(3n+1)(3n+2)(6n+1)}$	5	1	$x^{3n+1}z$, y^3z, z^2t, t^2x, x^nyz^2, $x^{n+1}yt$, $x^{2n+1}y^2$	$O_x = \frac{1}{3}(1,1)$ $O_y = \frac{1}{3n+1}(3, 3n)$ $O_z = \frac{1}{3n+2}(3, 3n+1)$ $O_t = \frac{1}{6n+1}(3n+1, 3n+2)$
$(3, 3n+1, 6n+1, 9n)$	$18n+3$	$\frac{4}{9n(3n+1)}$	5	1	z^3, y^3t, xt^2 x^nyz^2, $x^{2n}y^2z$, $x^{3n}y^3$, $x^{3n+1}t$, x^{6n+1}	$O_y = \frac{1}{3n+1}(1,n)$ $O_t = \frac{1}{9n}(3n+1, 6n+1)$ $O_xO_t = 2 \times \frac{1}{3}(1,1)$
$(3, 3n+1, 6n+1, 9n+3)$	$18n+6$	$\frac{8}{3(3n+1)(6n+1)}$	6	1	t^2, y^3t, y^6, xz^3, $x^{n+1}yz^2$, $x^{2n+1}y^2z$, $x^{3n+1}t$, $x^{3n+1}y^3$, x^{6n+2}	$O_z = \frac{1}{6n+1}(3n+1, 3n+2)$ $O_xO_t = 2 \times \frac{1}{3}(1,1)$ $O_yO_t = 2 \times \frac{1}{3n+1}(1,n)$
$(4, 2n+1, 4n+2, 6n+1)$	$12n+6$	$\frac{3}{(2n+1)(6n+1)}$	6	1	z^3, y^2z^2, y^4z, y^6, xt^2, $x^{n+1}yt$, $x^{2n+1}z$, $x^{2n+1}y^2$	$O_x = \frac{1}{4}(1,1)$ $O_t = \frac{1}{6n+1}(2n+1, 4n+2)$ $O_xO_z = 1 \times \frac{1}{2}(1,1)$ $O_yO_z = 3 \times \frac{1}{2n+1}(1,n)$
$(4, 2n+3, 2n+3, 4n+4)$	$8n+12$	$\frac{1}{(n+1)(2n+3)}$	7	1	y^4, y^3z, y^2z^2, yz^3, z^4, xt^2, $x^{n+2}t$, x^{2n+3}	$O_t = \frac{1}{4n+4}(2,2)$ $O_xO_t = 2 \times \frac{1}{4}(1,1)$ $O_yO_z =$ $= 4 \times \frac{1}{2n+3}(4, 2n+1)$

Log del Pezzo surfaces with $I=2$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(2,3,4,5)$	12	$\frac{2}{5}$	5	1^{a} $\frac{7}{12}^{\mathrm{b}}$	z^3, yzt, y^4, xt^2, xy^2z, x^2z^2, x^2yt, x^3y^2, x^4z, x^6	$O_t=\frac{1}{5}(3,4)$ $O_xO_z=3\times\frac{1}{2}(1,1)$
$(2,3,4,7)$	14	$\frac{1}{3}$	6	1	t^2, yzt, y^2z^2, xz^3, xy^4, x^2yt, x^2y^2z, x^3z^2, x^4y^2, x^5z, x^7	$O_y=\frac{1}{3}(1,1)$ $O_z=\frac{1}{4}(1,1)$ $O_xO_z=3\times\frac{1}{2}(1,1)$
$(3,4,5,10)$	20	$\frac{2}{15}$	5	$\frac{3}{2}$	t^2, z^2t, z^4, y^5, xy^3z, x^2yt, x^2yz^2, x^4y^2, x^5z	$O_x=\frac{1}{3}(1,1)$ $O_yO_t=1\times\frac{1}{2}(1,1)$ $O_zO_t=2\times\frac{1}{5}(3,4)$
$(3,4,10,15)$	30	$\frac{1}{15}$	7	$\frac{3}{2}$	t^2, z^3, y^5z, xy^3t, x^2yz^2, x^2y^6, x^4y^2z, x^5t, x^6y^3, x^{10}	$O_y=\frac{1}{4}(1,1)$ $O_yO_z=1\times\frac{1}{2}(1,1)$ $O_zO_t=1\times\frac{1}{5}(3,4)$ $O_xO_t=2\times\frac{1}{3}(1,1)$
$(5,13,19,22)$	57	$\frac{6}{715}$	3	$\frac{25}{12}$	t^2y, z^3, xy^4, x^5yz, x^7t	$O_x=\frac{1}{5}(3,4)$ $O_y=\frac{1}{13}(2,3)$ $O_t=\frac{1}{22}(5,19)$
$(5,13,19,35)$	70	$\frac{8}{1235}$	3	$\frac{25}{12}$	t^2, yz^3, xy^5, x^5y^2z, x^7t, x^{14}	$O_y=\frac{1}{13}(2,3)$ $O_z=\frac{1}{19}(5,16)$ $O_xO_t=2\times\frac{1}{5}(3,4)$
$(6,9,10,13)$	36	$\frac{4}{195}$	4	$\frac{25}{12}$	t^2z, y^4, xz^3, x^3y^2, x^6	$O_z=\frac{1}{10}(3,1)$ $O_t=\frac{1}{13}(2,3)$ $O_xO_y=2\times\frac{1}{3}(1,1)$ $O_xO_z=1\times\frac{1}{2}(1,1)$

a: if the defining equation of X contains yzt, b: if the defining equation of X contains no yzt.

Log del Pezzo surfaces with $I = 2$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(7,8,19,25)$	57	$\frac{57}{6650}$	3	$\frac{49}{24}$	$ty^4,\ z^3,\ xt^2,\ x^2y^3z,$ x^7y	$O_x = \frac{1}{7}(5,4)$ $O_y = \frac{1}{8}(7,3)$ $O_t = \frac{1}{25}(8,19)$
$(7,8,19,32)$	64	$\frac{1}{133}$	4	$\frac{35}{16}$	$t^2,\ ty^4,\ y^8,\ xz^3,$ $x^3y^3z,\ x^8y$	$O_x = \frac{1}{7}(5,4)$ $O_z = \frac{1}{19}(8,13)$ $O_yO_t = 2 \times \frac{1}{8}(7,3)$
$(9,12,13,16)$	48	$\frac{1}{117}$	3	$\frac{63}{24}$	$t^3,\ y^4,\ xz^3,\ x^4y$	$O_x = \frac{1}{9}(4,7)$ $O_z = \frac{1}{13}(4,1)$ $O_xO_y = 1 \times \frac{1}{3}(1,1)$ $O_yO_t = 1 \times \frac{1}{4}(1,1)$
$(9,12,19,19)$	57	$\frac{1}{171}$	5	3	$t^3,\ t^2z,\ tz^2,\ z^3,\ xy^4,$ x^5y	$O_x = \frac{1}{9}(1,1)$ $O_y = \frac{1}{12}(1,1)$ $O_xO_y = 1 \times \frac{1}{3}(1,1)$ $O_zO_t = 3 \times \frac{1}{19}(3,4)$
$(9,19,24,31)$	81	$\frac{3}{1178}$	3	3	$t^2y,\ y^3z,\ xz^3,\ x^9$	$O_y = \frac{1}{19}(3,4)$ $O_z = \frac{1}{24}(19,7)$ $O_t = \frac{1}{31}(3,8)$ $O_xO_z = 1 \times \frac{1}{3}(1,1)$
$(10,19,35,43)$	105	$\frac{6}{4085}$	3	$\frac{57}{14}$	$t^2y,\ z^3,\ xy^5,\ x^7z$	$O_x = \frac{1}{10}(3,1)$ $O_y = \frac{1}{19}(16,5)$ $O_t = \frac{1}{43}(2,7)$ $O_xO_z = 1 \times \frac{1}{5}(4,3)$
$(11,21,28,47)$	105	$\frac{5}{3619}$	3	$\frac{77}{30}$	$y^5,\ yz^3,\ xt^2,\ x^7z$	$O_x = \frac{1}{11}(10,3)$ $O_y = \frac{1}{28}(11,19)$ $O_t = \frac{1}{47}(3,4)$ $O_yO_z = 1 \times \frac{1}{7}(4,5)$

Log del Pezzo surfaces with $I = 2$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(11, 25, 32, 41)$	107	$\frac{107}{90200}$	3	$\frac{11}{3}$	$t^2y,\ y^3z,\ xz^3,\ x^6t$	$O_x = \frac{1}{11}(3, 10)$ $O_y = \frac{1}{25}(11, 16)$ $O_z = \frac{1}{32}(25, 9)$ $O_t = \frac{1}{41}(11, 32)$
$(11, 25, 34, 43)$	111	$\frac{222}{201025}$	3	$\frac{33}{8}$	$t^2y,\ z^2t,\ xy^4,\ x^7z$	$O_x = \frac{1}{11}(3, 10)$ $O_y = \frac{1}{25}(1, 2)$ $O_z = \frac{1}{34}(11, 25)$ $O_t = \frac{1}{43}(11, 34)$
$(11, 43, 61, 113)$	226	$\frac{8}{28853}$	2	$\frac{55}{12}$	$t^2,\ yz^3,\ xy^5,\ x^{15}z$	$O_x = \frac{1}{11}(10, 3)$ $O_y = \frac{1}{43}(2, 3)$ $O_z = \frac{1}{61}(11, 52)$
$(13, 18, 45, 61)$	135	$\frac{2}{2379}$	3	$\frac{91}{30}$	$z^3,\ y^5z,\ xt^2,\ x^9y$	$O_x = \frac{1}{13}(2, 3)$ $O_y = \frac{1}{18}(13, 7)$ $O_t = \frac{1}{61}(2, 5)$ $O_yO_z = 1 \times \frac{1}{9}(4, 7)$
$(13, 20, 29, 47)$	107	$\frac{107}{88595}$	3	$\frac{65}{18}$	$y^3t,\ yz^3,\ xt^2,\ x^6z$	$O_x = \frac{1}{13}(7, 8)$ $O_y = \frac{1}{20}(13, 9)$ $O_z = \frac{1}{29}(13, 18)$ $O_t = \frac{1}{47}(20, 29)$
$(13, 20, 31, 49)$	111	$\frac{111}{98735}$	3	$\frac{65}{16}$	$z^3t,\ y^4z,\ xt^2,\ x^7y$	$O_x = \frac{1}{13}(1, 2)$ $O_y = \frac{1}{20}(13, 9)$ $O_z = \frac{1}{31}(13, 20)$ $O_t = \frac{1}{49}(20, 31)$
$(13, 31, 71, 113)$	226	$\frac{8}{28613}$	2	$\frac{91}{20}$	$t^2,\ y^5z,\ xz^3,\ x^{15}y$	$O_x = \frac{1}{13}(6, 9)$ $O_y = \frac{1}{31}(13, 20)$ $O_z = \frac{1}{71}(31, 42)$
$(14, 17, 29, 41)$	99	$\frac{198}{141491}$	3	$\frac{51}{10}$	$t^2y,\ z^2t,\ xy^5,\ x^5z$	$O_x = \frac{1}{14}(3, 13)$ $O_y = \frac{1}{17}(12, 7)$ $O_z = \frac{1}{29}(14, 17)$ $O_t = \frac{1}{41}(14, 29)$

Log del Pezzo surfaces with $I = 3$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(5, 7, 11, 13)$	33	$\frac{27}{455}$	3	$\frac{49}{36}$	t^2y, z^3, xy^4, x^3yz, x^4t	$O_x = \frac{1}{5}(2, 1)$ $O_y = \frac{1}{7}(2, 3)$ $O_t = \frac{1}{13}(5, 11)$
$(5, 7, 11, 20)$	40	$\frac{18}{385}$	4	$\frac{25}{18}$	t^2, yz^3, xy^5, x^3y^2z, x^4t, x^8	$O_y = \frac{1}{7}(2, 3)$ $O_z = \frac{1}{11}(1, 4)$ $O_xO_t = 2 \times \frac{1}{5}(2, 1)$
$(11, 21, 29, 37)$	95	$\frac{285}{82621}$	3	$\frac{11}{4}$	t^2y, z^2t, xy^4, x^6z	$O_x = \frac{1}{11}(5, 2)$ $O_y = \frac{1}{21}(1, 2)$ $O_z = \frac{1}{29}(11, 21)$ $O_t = \frac{1}{37}(11, 29)$
$(11, 37, 53, 98)$	196	$\frac{18}{21571}$	2	$\frac{55}{18}$	t^2, yz^3, xy^5, $x^{13}z$	$O_x = \frac{1}{11}(2, 5)$ $O_y = \frac{1}{37}(2, 3)$ $O_z = \frac{1}{53}(11, 45)$
$(13, 17, 27, 41)$	95	$\frac{95}{27183}$	3	$\frac{65}{24}$	z^2t, y^4z, xt^2, x^6y	$O_x = \frac{1}{13}(1, 2)$ $O_y = \frac{1}{17}(13, 7)$ $O_z = \frac{1}{27}(13, 17)$ $O_t = \frac{1}{41}(17, 27)$
$(13, 27, 61, 98)$	196	$\frac{2}{2379}$	2	$\frac{91}{30}$	t^2, y^5z, xz^3, $x^{13}y$	$O_x = \frac{1}{13}(9, 7)$ $O_y = \frac{1}{27}(13, 17)$ $O_z = \frac{1}{61}(27, 37)$
$(15, 19, 43, 74)$	148	$\frac{6}{4085}$	2	$\frac{57}{14}$	t^2, yz^3, xy^7, x^7z	$O_x = \frac{1}{15}(2, 7)$ $O_y = \frac{1}{19}(5, 17)$ $O_z = \frac{1}{43}(15, 31)$

Log del Pezzo surfaces with $I = 4$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(6, 6n+3, 6n+5, 6n+5)$	$18n+15$	$\frac{8}{(6n+3)(6n+5)}$	5	1	t^3, zt^2, z^2t, z^3, xy^3, $x^{2n+2}y$	$O_x = \frac{1}{6}(1,1)$ $O_y = \frac{1}{6n+3}(1,1)$ $O_xO_y = 1 \times \frac{1}{3}(1,1)$ $O_zO_t =$ $= 3 \times \frac{1}{6n+5}(2, 2n+1)$
$(6, 6n+5, 12n+8, 18n+9)$	$36n+24$	$\frac{8}{3(6n+3)(6n+5)}$	3	1	z^3, y^3t, xt^2, $x^{2n+1}y^2z$, x^{6n+4}	$O_y = \frac{1}{6n+5}(2, 2n+1)$ $O_t =$ $= \frac{1}{18n+9}(6n+5, 12n+8)$ $O_xO_z = 1 \times \frac{1}{2}(1,1)$ $O_xO_t = 1 \times \frac{1}{3}(1,1)$
$(6, 6n+5, 12n+8, 18n+15)$	$36n+30$	$\frac{4}{3(3n+2)(6n+5)}$	4	1	t^2, y^3t, y^6, xz^3, $x^{2n+2}y^2z$, x^{6n+5}	$O_z = \frac{1}{12n+8}(6n+5, 6n+7)$ $O_xO_z = 1 \times \frac{1}{2}(1,1)$ $O_xO_t = 1 \times \frac{1}{3}(1,1)$ $O_yO_t =$ $= 2 \times \frac{1}{6n+5}(2, 2n+1)$
$(5, 6, 8, 9)$	24	$\frac{8}{45}$	3	1	t^2y, y^4, z^3, x^2yz, x^3t	$O_x = \frac{1}{5}(1,3)$ $O_t = \frac{1}{9}(5,8)$ $O_yO_z = 1 \times \frac{1}{2}(1,1)$ $O_yO_t = 1 \times \frac{1}{3}(1,1)$
$(5, 6, 8, 15)$	30	$\frac{2}{15}$	4	1	t^2, y^5, yz^3, x^2y^2z, x^3t, x^6	$O_z = \frac{1}{8}(5,7)$ $O_xO_t = 2 \times \frac{1}{5}(1,3)$ $O_yO_t = 1 \times \frac{1}{3}(1,1)$ $O_yO_z = 1 \times \frac{1}{2}(1,1)$
$(9, 11, 12, 17)$	45	$\frac{20}{561}$	3	$\frac{77}{60}$	t^2y, y^3z, xz^3, x^5	$O_y = \frac{1}{11}(3,2)$ $O_z = \frac{1}{12}(11,5)$ $O_t = \frac{1}{17}(3,4)$ $O_xO_z = 1 \times \frac{1}{3}(1,1)$

Log del Pezzo surfaces with $I = 4$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(10, 13, 25, 31)$	75	$\frac{24}{2015}$	3	$\frac{91}{60}$	$t^2y,\ z^3,\ xy^5,\ x^5z$	$O_x = \frac{1}{10}(3, 1)$ $O_y = \frac{1}{13}(12, 5)$ $O_t = \frac{1}{31}(2, 5)$ $O_xO_z = 1 \times \frac{1}{5}(3, 1)$
$(11, 17, 20, 27)$	71	$\frac{284}{25245}$	3	$\frac{11}{6}$	$t^2y,\ y^3z,\ xz^3,\ x^4t$	$O_x = \frac{1}{11}(2, 3)$ $O_y = \frac{1}{17}(11, 10)$ $O_z = \frac{1}{20}(17, 7)$ $O_t = \frac{1}{27}(11, 20)$
$(11, 17, 24, 31)$	79	$\frac{158}{17391}$	3	$\frac{33}{16}$	$t^2y,\ tz^2,\ xy^4,\ x^5z$	$O_x = \frac{1}{11}(2, 3)$ $O_y = \frac{1}{17}(1, 2)$ $O_z = \frac{1}{24}(11, 17)$ $O_t = \frac{1}{31}(11, 24)$
$(11, 31, 45, 83)$	166	$\frac{32}{15345}$	2	$\frac{55}{24}$	$t^2,\ yz^3,\ xy^5,\ x^{11}z$	$O_x = \frac{1}{11}(3, 2)$ $O_y = \frac{1}{31}(2, 3)$ $O_z = \frac{1}{45}(11, 38)$
$(13, 14, 19, 29)$	71	$\frac{568}{50141}$	3	$\frac{65}{36}$	$ty^3,\ yz^3,\ xt^2,\ x^4z$	$O_x = \frac{1}{13}(1, 3)$ $O_y = \frac{1}{14}(13, 5)$ $O_z = \frac{1}{19}(13, 10)$ $O_t = \frac{1}{29}(14, 19)$
$(13, 14, 23, 33)$	79	$\frac{632}{69069}$	3	$\frac{65}{32}$	$tz^2,\ y^4z,\ xt^2,\ x^5y$	$O_x = \frac{1}{13}(1, 2)$ $O_y = \frac{1}{14}(13, 5)$ $O_z = \frac{1}{23}(13, 14)$ $O_t = \frac{1}{33}(14, 23)$
$(13, 23, 51, 83)$	166	$\frac{32}{15249}$	2	$\frac{91}{40}$	$t^2,\ y^5z,\ xz^3,\ x^{11}y$	$O_x = \frac{1}{11}(7, 6)$ $O_y = \frac{1}{23}(13, 14)$ $O_z = \frac{1}{51}(23, 32)$

Log del Pezzo surfaces with $I = 5$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(11, 13, 19, 25)$	63	$\frac{63}{2717}$	3	$\frac{13}{8}$	$t^2y,\ tz^2,\ xy^4,\ x^4z$	$O_x = \frac{1}{11}(2, 3)$ $O_y = \frac{1}{13}(1, 2)$ $O_z = \frac{1}{19}(11, 13)$ $O_t = \frac{1}{25}(11, 19)$
$(11, 25, 37, 68)$	136	$\frac{2}{407}$	2	$\frac{11}{6}$	$t^2,\ yz^3,\ xy^5,\ x^9z$	$O_x = \frac{1}{11}(3, 2)$ $O_y = \frac{1}{25}(2, 3)$ $O_z = \frac{1}{37}(11, 31)$
$(13, 19, 41, 68)$	136	$\frac{50}{10127}$	2	$\frac{91}{50}$	$t^2,\ y^5z,\ xz^3,\ x^9y$	$O_x = \frac{1}{13}(2, 3)$ $O_y = \frac{1}{19}(13, 11)$ $O_z = \frac{1}{41}(19, 27)$

Log del Pezzo surfaces with $I = 6$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(8, 4n+5, 4n+7, 4n+9)$	$12n+23$	$\frac{9(12n+23)}{2(4n+5)(4n+7)(4n+9)}$	3	1	$z^2t,\ yt^2,\ xy^3,$ $x^{n+2}z$	$O_x = \frac{1}{8}(4n+5, 4n+9)$ $O_y = \frac{1}{4n+5}(1, 2)$ $O_z = \frac{1}{4n+7}(8, 4n+5)$ $O_t = \frac{1}{4n+9}(8, 4n+7)$
$(9, 3n+8, 3n+11, 6n+13)$	$12n+35$	$\frac{4(12n+35)}{(3n+8)(3n+11)(6n+13)}$	3	1	$z^2t,\ y^3z,\ xt^2,$ $x^{n+3}y$	$O_x = \frac{1}{9}(3n+11, 6n+13)$ $O_y = \frac{1}{3n+8}(9, 6n+13)$ $O_z = \frac{1}{3n+11}(9, 3n+8)$ $O_t =$ $= \frac{1}{6n+13}(3n+8, 3n+11)$
$(7, 10, 15, 19)$	45	$\frac{54}{665}$	3	$\frac{35}{54}$	$z^3,\ y^3z,\ xt^2,$ x^5y	$O_x = \frac{1}{7}(1, 5)$ $O_y = \frac{1}{10}(7, 9)$ $O_t = \frac{1}{19}(2, 3)$ $O_yO_z = 1 \times \frac{1}{5}(1, 2)$
$(11, 19, 29, 53)$	106	$\frac{72}{6061}$	2	$\frac{55}{36}$	$t^2,\ yz^3,\ xy^5,$ x^7z	$O_x = \frac{1}{11}(8, 9)$ $O_y = \frac{1}{19}(2, 3)$ $O_z = \frac{1}{29}(11, 24)$
$(13, 15, 31, 53)$	106	$\frac{24}{2015}$	2	$\frac{91}{60}$	$t^2,\ y^5z,\ xz^3,$ x^7y	$O_x = \frac{1}{13}(5, 1)$ $O_y = \frac{1}{15}(13, 8)$ $O_z = \frac{1}{31}(15, 22)$

Log del Pezzo surfaces with $I = 7$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(11, 13, 21, 38)$	76	$\frac{14}{429}$	2	$\frac{13}{10}$	$t^2,\ yz^3,\ xy^5,\ x^5z$	$O_x = \frac{1}{11}(2, 5)$ $O_y = \frac{1}{13}(2, 3)$ $O_z = \frac{1}{21}(11, 17)$

Log del Pezzo surfaces with $I = 8$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(7, 11, 13, 23)$	46	$\frac{128}{1001}$	2	$\frac{35}{48}$	$t^2,\ y^3z,\ xz^3,\ x^5y$	$O_x = \frac{1}{7}(3, 1)$ $O_y = \frac{1}{11}(7, 1)$ $O_z = \frac{1}{13}(11, 10)$
$(7, 18, 27, 37)$	81	$\frac{32}{777}$	3	$\frac{35}{72}$	$y^3z,\ z^3,\ xt^2,\ x^9y$	$O_x = \frac{1}{7}(3, 1)$ $O_y = \frac{1}{18}(7, 1)$ $O_t = \frac{1}{37}(2, 3)$ $O_yO_z = 1 \times \frac{1}{9}(7, 1)$

Log del Pezzo surfaces with $I = 9$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(7, 15, 19, 32)$	64	$\frac{54}{665}$	2	$\frac{35}{54}$	$t^2,\ y^3z,\ xz^3,\ x^7y$	$O_x = \frac{1}{7}(5, 4)$ $O_y = \frac{1}{15}(7, 2)$ $O_z = \frac{1}{19}(15, 13)$

Log del Pezzo surfaces with $I = 10$

Weights	Degree	K_X^2	ρ	lct	Monomials	Singular Points
$(7, 19, 25, 41)$	82	$\frac{8}{133}$	2	$\frac{7}{12}$	$t^2,\ y^3z,\ xz^3,\ x^9y$	$O_x = \frac{1}{7}(2, 3)$ $O_y = \frac{1}{19}(7, 3)$ $O_z = \frac{1}{25}(19, 16)$
$(7, 26, 39, 55)$	117	$\frac{30}{1001}$	3	$\frac{7}{18}$	$y^3z,\ z^3,\ xt^2,\ x^{13}y$	$O_x = \frac{1}{7}(2, 3)$ $O_y = \frac{1}{26}(7, 3)$ $O_t = \frac{1}{55}(26, 39)$ $O_yO_z = 1 \times \frac{1}{13}(7, 3)$

References

[1] C. Araujo, *Kähler–Einstein metrics for some quasi-smooth log del Pezzo surfaces* Transactions of the American Mathematical Society **354** (2002), 4303–3312

[2] C. Boyer, *Sasakian geometry: the recent work of Krzysztof Galicki* arXiv:0806.0373 (2008)

[3] C. Boyer, K. Galicki, M. Nakamaye, *Sasakian-Einstein structures on* $9\#(S^2 \times S^3)$ Transactions of the American Mathematical Society **354** (2002), 2983–2996

[4] C. Boyer, K. Galicki, M. Nakamaye, *On the geometry of Sasakian–Einstein* 5*-manifolds* Mathematische Annalen **325** (2003), 485–524

[5] J. W. Bruce, C. T. C. Wall, *On the classification of cubic surfaces* Journal of the London Mathematical Society **19** (1979), 245–256

[6] I. Cheltsov, *Log canonical thresholds on hypersurfaces* Sbornik: Mathematics **192** (2001), 1241–1257

[7] I. Cheltsov, *Fano varieties with many selfmaps* Advances in Mathematics **217** (2008), 97–124

[8] I. Cheltsov, *Double spaces with isolated singularities* Sbornik: Mathematics **199** (2008), 291–306

[9] I. Cheltsov, *Log canonical thresholds and Kähler–Einstein metrics on Fano threefold hypersurfaces* Izvestiya: Mathematics, to appear

[10] I. Cheltsov, *Extremal metrics on two Fano varieties* Sbornik: Mathematics, to appear

[11] I. Cheltsov, *Log canonical thresholds of del Pezzo surfaces* Geometric and Functional Analysis, **18** (2008), 1118–1144

[12] I. Cheltsov, *On singular cubic surfaces* arXiv:0706.2666 (2007)

[13] I. Cheltsov, J. Park, J. Won, *Log canonical thresholds of certain Fano hypersurfaces* arXiv:math.AG/0706.0751 (2007)

[14] I. Cheltsov, C. Shramov, *Log canonical thresholds of smooth Fano threefolds. With an appendix by Jean-Pierre Demailly* Russian Mathematical Surveys **63** (2008), 73–180

[15] I. Cheltsov, C. Shramov, *Del Pezzo zoo* arXiv:0904.0114 (2009)

[16] J.-P. Demailly, J. Kollár, *Semi-continuity of complex singularity exponents and Kähler–Einstein metrics on Fano orbifolds* Annales Scientifiques de l'École Normale Supérieure **34** (2001), 525–556

[17] S. Donaldson, *Scalar curvature and stability of toric varieties* Journal of Differential Geometry **62** (2002), 289–349

[18] R. Elkik, *Rationalitè des singularitès canoniques* Inventiones Mathematicae **64** (1981), 1–6

[19] A. Futaki, *An obstruction to the existence of Einstein–Kähler metrics* Inventiones Mathematicae **73** (1983), 437–443

[20] J. Gauntlett, D. Martelli, J. Sparks, S.-T. Yau, *Obstructions to the existence of Sasaki-Einstein metrics* Communications in Mathematical Physics **273** (2007) 803–827

[21] J.-M. Hwang, *Log canonical thresholds of divisors on Fano manifolds of Picard rank* 1 Compositio Mathematica **143** (2007), 89–94

[22] A. R. Iano-Fletcher, *Working with weighted complete intersections* L.M.S. Lecture Note Series **281** (2000), 101–173

[23] S. Ishii, Yu. Prokhorov, *Hypersurface exceptional singularities*
International Journal of Mathematics **12** (2001), 661–687

[24] J. Johnson, J. Kollár, *Fano hypersurfaces in weighted projective* 4*-spaces*
Experimental Mathematics **10** (2001), 151–158

[25] J. Johnson, J. Kollár, *Kähler–Einstein metrics on log del Pezzo surfaces in weighted projective* 3*-spaces*
Annales de l'Institut Fourier **51** (2001), 69–79

[26] J. Kollár, *Singularities of pairs*
Proceedings of Symposia in Pure Mathematics **62** (1997), 221–287

[27] S. Keel, J. McKernan, *Rational curves on quasi-projective surfaces*
Memoirs of the American Mathematical Society **669** (1999)

[28] S. Kudryavtsev, *On purely log terminal blow ups*
Mathematical Notes **69** (2002), 814–819

[29] S. Kudryavtsev, *Classification of three-dimensional exceptional log-canonical hypersurface singularities. I*
Izvestiya: Mathematics **66** (2002), 949–1034

[30] T. Kuwata, *On log canonical thresholds of reducible plane curves*
American Journal of Mathematics **121** (1999), 701–721

[31] M. Lübke, *Stability of Einstein-Hermitian vector bundles*
Manuscripta Mathematica **42** (1983), 245–257

[32] D. Markushevich, Yu. Prokhorov, *Exceptional quotient singularities*
American Journal of Mathematics **121** (1999), 1179–1189

[33] Y. Matsushima, *Sur la structure du groupe d'homéomorphismes analytiques d'une certaine variété kählérienne*
Nagoya Mathematical Journal **11** (1957), 145–150

[34] A. Nadel, *Multiplier ideal sheaves and Kähler–Einstein metrics of positive scalar curvature*
Annals of Mathematics **132** (1990), 549–596

[35] D. Phong, N. Sesum, J. Sturm, *Multiplier ideal sheaves and the Kähler-Ricci flow*
Communications in Analysis and Geometry **15** (2007), 613–632

[36] Yu. Prokhorov, *Blow-ups of canonical singularities*
arXiv:math/9810097 (1998)

[37] Yu. Prokhorov, *Lectures on complements on log surfaces*
MSJ Memoirs **10** (2001)

[38] A. Pukhlikov, *Birational geometry of Fano direct products*
Izvestiya: Mathematics **69** (2005), 1225–1255

[39] M. Reid, *Canonical* 3*-folds*
Journes de Gèometrie Algèbrique d'Angers (1980), 273–310

[40] J. Ross, R. Thomas, *An obstruction to the existence of constant scalar curvature Kähler metrics*
Journal of Differential Geometry **72** (2006), 429–466

[41] Y. Rubinstein, *Some discretizations of geometric evolution equations and the Ricci iteration on the space of Kähler metrics*
Advances in Mathematics **218** (2008), 1526–1565

[42] V. Shokurov, *Complements on surfaces*
Journal of Mathematical Sciences **102** (2000), 3876–3932

[43] J. Sparks, *New results in Sasaki-Einstein geometry*
arXiv:math/0701518 (2007)

[44] G. Tian, *On Kähler–Einstein metrics on certain Kähler manifolds with* $c_1(M) > 0$
Inventiones Mathematicae **89** (1987), 225–246

[45] G. Tian, *On Calabi's conjecture for complex surfaces with positive first Chern class* Inventiones Mathematicae **101** (1990), 101–172

[46] G. Tian, *On a set of polarized Kähler metrics on algebraic manifolds* Journal of Differential Geometry **32** (1990), 99–130

[47] G. Tian, *Kähler–Einstein metrics with positive scalar curvature* Inventiones Mathematicae **130** (1997), 1–37

[48] G. Tian, S.-T. Yau, *Kähler–Einstein metrics metrics on complex surfaces with* $C_1 > 0$ Communications in Mathematical Physics **112** (1987), 175–203

[49] S. S.-T. Yau, Y. Yu, *Classification of 3-dimensional isolated rational hypersurface singularities with* $\mathbb{C}^*$*-action* arXiv:math/0303302 (2003)

[50] Q. Zhang, *Rational connectedness of log* $\mathbb{Q}$*-Fano varietiess* Journal fur die Reine und Angewandte Mathematik **590** (2006), 131–142

Ivan Cheltsov
School of Mathematics, The University of Edinburgh, Edinburgh, EH9 3JZ, UK; cheltsov@yahoo.com

Jihun Park
Department of Mathematics, POSTECH, Pohang, Kyungbuk 790-784, Korea; wlog@postech.ac.kr

Constantin Shramov
School of Mathematics, The University of Edinburgh, Edinburgh, EH9 3JZ, UK; shramov@mccme.ru