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Compactly generated domain theory†

INGO BATTENFELD, MATTHIAS SCHRÖDER and
ALEX SIMPSON

LFCS, School of Informatics, University of Edinburgh, UK

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Dedicated to Klaus Keimel on the occasion of his 65th birthday

We propose compactly generated monotone convergence spaces as a well-behaved topological generalisation of directed-complete partial orders (dcpos). The category of such spaces enjoys the usual properties of categories of 'predomains' in denotational semantics. Moreover, such properties are retained if one restricts to spaces with a countable pseudobase in the sense of E. Michael, a fact that permits connections to be made with computability theory, realizability semantics and recent work on the closure properties of topological quotients of countably based spaces (qcb spaces). We compare the standard domain-theoretic constructions of products and function spaces on dcpos with their compactly generated counterparts, showing that these agree in important cases, though not in general.

1. Introduction

Domain theory was originally developed by Dana Scott in order to build mathematical models of recursion, datatypes and other programming language features. It has since blossomed into a rich mathematical theory, centred around the study of directed-complete partial orders (dcpos), and their Scott topologies, see, for example, Abramsky and Jung (1994) and Gierz *et al.* (2003) for overviews.

One would like domain theory to provide a flexible toolkit for modelling different aspects of computation. However, in spite of its many achievements, traditional dcpo-based domain theory has limitations in this regard. For example, Gordon Plotkin has pointed out that, although traditional domain theory can be used to model higher-order types (using cartesian-closed categories), computability for non-discrete datatypes (using ω-continuous dcpos), and computational effects (as free algebras for inequational theories), it is not capable of modelling all three in combination. Similarly, although one can build domain-theoretic models of the Girard–Reynolds' polymorphic lambda-calculus, it is not known how to combine such models with computational effects, or how to build models satisfying the parametricity properties that are vital for proving program equivalences.

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In order to address these weaknesses, it seems necessary to leave the familiar dcpo-based world of traditional domain theory. One possible alternative is to identify subcategories of domain-like structures within suitable 'realizability' categories, see, for example, Phoa (1990), Longley and Simpson (1997) and Reus and Streicher (1999). However, while such categories do indeed resolve the problems identified above (though no comprehensive account of this has ever been published), important properties of traditional domain theory, such as the connections with topology (Gierz *et al.* 2003), are generally lost.

Fortunately, the situation is not always so bad. In Simpson (2003), the third author observed that, in one particular realizability category built over Scott's combinatory algebra $\mathscr{P}\omega$ (Scott 1976), a natural category of 'predomains' (the *complete extensional objects*) can equivalently be presented as a category of topological spaces, called *topological predomains* in *op. cit.* The importance of this coincidence, a proof of which appears in the first author's Diploma thesis (Battenfeld 2004), is that it opens up the possibility of obtaining the benefits of realizability-based notions of domain within an orthodox topological framework. Furthermore, topological predomains extend the scope of traditional domain theory by including familiar topological spaces such as the Euclidean reals and Cantor space, which make sense as datatypes, but whose topology is not the Scott topology on any underlying partial order.

The objective of the present paper is to provide a self-contained introduction to topological predomains aimed specifically at readers familiar with traditional domain theory and its topological connections. We thus ignore the realizability side of topological predomains entirely, for which the interested reader is referred to Simpson (2003) and Battenfeld (2004). Instead, we derive topological predomains from first principles, taking cartesian closedness as our starting point.

For dcpos, cartesian closedness is a consequence of the order-theoretic setting. Once more general topological spaces are allowed as predomains, more inclusive cartesian-closed categories of topological spaces are required. There is thus a natural connection with the realm of so-called 'convenient topology', introduced by Ronnie Brown in Brown (1963; 1964) and popularised by Norman Steenrod (Steenrod 1967). Convenient topology is the study of categories of spaces enjoying additional useful properties, in particular, cartesian closedness, that are not possessed by the category **Top** of all topological spaces. In Brown (1963; 1964) and Steenrod (1967), the category of compactly generated Hausdorff spaces is shown to be one such cartesian-closed category. Many other cartesian-closed subcategories (and supercategories) of **Top**, have since been studied for similar reasons: see Preuss (2002) and Escardó *et al.* (2004) for recent overviews. The idea behind the present paper is to take such a cartesian-closed category of topological spaces as the basis for developing a generalised domain theory.

In Section 3 we develop notions of predomain and domain within the cartesian-closed category of compactly generated spaces (of course we do not restrict to Hausdorff spaces, because interesting domains are never Hausdorff). *Compactly generated predomains* are simply the compactly generated 'monotone convergence spaces' in the sense of Gierz *et al.* (2003), and *domains* are predomains with least element in the specialisation order. The main results of the section establish the fact that the compactly generated and monotone convergence space properties combine nicely with each other. Indeed, we obtain cartesian-closed categories of predomains and domains that are exponential ideals of the category of compactly generated spaces, with the former subcategory being reflective.

In Section 4, we show that the good properties of compactly generated (pre)domains are retained under the imposition of a smallness condition on the topology. Here, the appropriate condition is to require a countable *pseudobase* in the sense of Michael (1966), which generalises the standard notion of base for a topology. In fact, the countably pseudobased compactly generated spaces have a simple characterisation as the topological quotients of countably-based spaces (*qcb spaces*), which themselves form a cartesian-closed category of topological spaces (Schröder 2003; Menni and Simpson 2002). In Section 4, we study the subcategory of monotone convergence qcb spaces, which coincides with the category of *topological predomains* introduced in Simpson (2003), as discussed above. We show that this category is a full reflective exponential ideal of the category of qcb spaces (this result was stated but not proved in *op. cit.*), hence it too is cartesian closed with countable limits and colimits.

One advantage of considering the larger category of *all* compactly generated predomains is that, by a result due to Jimmie Lawson (Escardó *et al.* 2004, Theorem 4.7), it contains the category of dcpos as a subcategory. Thus, compactly generated predomains extend the world of traditional domain theory. In Section 5, we address the question of whether the traditional domain-theoretic constructions on dcpos, such as products and function spaces, agree with their compactly generated counterparts. Although products of domains always agree, function spaces differ in general. Nevertheless, in important cases where the domain-theoretic construction is known to be well behaved, we show that function spaces do coincide. In other cases, we suggest that it is the compactly generated topology that is the more reasonable choice.

The topic of this paper lies on the boundary between domain theory and general topology, two subjects that have enjoyed an extremely rich interaction ever since the inception of domain theory: see Gierz *et al.* (2003) for a survey. It is a pleasure to dedicate this paper to Klaus Keimel, who has been one of the main contributors to the development of this interaction.

2. Preliminaries

Our notation and terminology is mainly standard. For a topological space X, we write $\mathcal{O}(X)$ for the family of open sets of X. For arbitrary (possibly non-Hausdorff) spaces we use *compact* to mean the Heine–Borel property. We write $\mathbb{S}$ for the Sierpinski space, which has underlying set $\{\bot, \top\}$ with $\{\top\}$ open but $\{\bot\}$ not.

In domain theory, we consider directed-complete partial orders (dcpos) and directed-complete pointed partial orders (dcppos), that is, dcpos with least element. We use $\sqsubseteq$ for the partial order structure in dcpos, $\bigsqcup$ for suprema, $\bigsqcap$ for infima, and $\downarrow X$ and $\uparrow X$ for the down- and up-closure, respectively, of a subset in the order. For continuous dcpos, we write $\ll$ for the way-below relation, and use $\twoheaddownarrow x$ and $\twoheaduparrow x$, respectively, for the sets of elements way-below and way-above an element x.

3. Compactly generated predomains

We begin with an overview of compactly generated spaces, *cf.* Brown (1963; 1964) and Steenrod (1967) (but without the Hausdorff restriction). We give full definitions, but state many standard properties without proof. For a recent comprehensive treatment, see Escardó *et al.* (2004).

Definition 3.1 (Compactly generated topology). A subset V of a topological space X is open in the *compactly generated topology* on X if, for every compact Hausdorff space K and continuous $p: K \to X$, the preimage $p^{-1}V$ is open in K. We write $k(X)$ for X with the compactly generated topology, and we say that X is *compactly generated* if $X = k(X)$.

Compactly generated spaces include all locally compact Hausdorff spaces. They also include a rich collection of non-Hausdorff spaces. For example, every sequential space is compactly generated. Hence, as special cases, all first-countable spaces are compactly generated, and so are ω-cpos with the ω-Scott topology. A far less straightforward fact, due to Jimmie Lawson, is that compactly generated spaces also include all dcpos (with Scott topology). This result is pivotal to the development of this paper, so we state it as a proposition.

Proposition 3.2. Every dcpo with its Scott topology is compactly generated.

For a proof, see Escardó *et al.* (2004, Theorem 4.7).

We write **kTop** for the category of compactly generated spaces and continuous functions. As is well known (and easily seen), **kTop** is a coreflective subcategory of **Top**, where the coreflection functor maps X to $k(X)$. It follows that **kTop** is cocomplete with colimits calculated as in **Top**, and complete with limits obtained by coreflecting limits in **Top**. In particular, the cartesian product in **kTop** of a family $\{X_i\}_{i \in I}$ of compactly generated spaces is given by $k(\prod_{i \in I} X_i)$, where $\prod_{i \in I} X_i$ is the topological product. We write $\prod_{i \in I}^k X_i$ and $X \times_k Y$ for the products $k(\prod_{i \in I} X_i)$ and $k(X \times Y)$ in **kTop**. In certain cases, this description of $X \times_k Y$ can be simplified. For example, if X and Y are countably based, then so is $X \times Y$, and hence $X \times_k Y = X \times Y$. The proposition below gives another important case in which the topologies coincide. A topological space X is said to be *core compact* if its open sets form a continuous lattice under inclusion. Core compactness is a mild generalisation of local compactness: every locally compact space is core compact, and every core compact sober space is locally compact. Core compact spaces arise naturally as the exponentiable objects in **Top**.

Proposition 3.3. If X, Y are compactly generated spaces and X is core compact, then $X \times_k Y = X \times Y$.

For a proof, see Escardó *et al.* (2004, Theorem 5.4).

For topological spaces X, Y, we write $C(X, Y)$ for the set of continuous functions from X to Y.

Definition 3.4 (Compact open topology). For topological spaces X, Y, the *compact open topology* on $C(X, Y)$ is generated by the subbasic opens

$$\langle K, V\rangle = \{f \mid f(K) \subseteq V\},$$

where $K \subseteq X$ is compact and $V \subseteq Y$ is open. We write $C_{\mathrm{co}}(X, Y)$ for $C(X, Y)$ with the compact open topology.

It is well known that every locally compact space X is exponentiable in **Top**, with the exponential Y^X given by $C_{\mathrm{co}}(X, Y)$.

Proposition 3.5. The category **kTop** is cartesian closed with exponential $X \Rightarrow_k Y$ given by $k(C_{\mathrm{co}}(X, Y))$.

This is a standard result, see Escardó *et al.* (2004) for references and for a recent exposition of the general machinery underlying the construction of exponentials in cartesian-closed subcategories of **Top**. (Although it does not appear explicitly in Escardó *et al.* (2004), the coincidence of $X \Rightarrow_k Y$ and $k(C_{\mathrm{co}}(X, Y))$ follows easily from Theorem 5.15 and Remark 5.20 of *op. cit.*, using the fact that the Isbell topology refines the compact open topology.)

The next two results give special cases in which the exponential in **kTop** is easily calculated.

Proposition 3.6. If X, Y are countably based and X is locally compact, then $C_{\mathrm{co}}(X, Y)$ is countably based, and hence $X \Rightarrow_k Y \ = \ C_{\mathrm{co}}(X, Y)$.

Proposition 3.7. If X is compactly generated, $X \Rightarrow_k \mathbb{S}$ has the Scott topology.

A proof of the first proposition can be found in Lambrinos and Papadopoulos (1985). The second, which appears as Escardó *et al.* (2004, Corollary 5.16), is again due to Jimmie Lawson.

In traditional domain theory, the Scott topology is derived from the partial order. To define our notion of a predomain, we also work with order-theoretic properties, but we take the topology as primary and the order as derived. Recall that the *specialisation order* on a topological space is defined as follows.

Definition 3.8 (Specialisation order). The *specialisation order* $\sqsubseteq$ on a topological space X is defined by $x \sqsubseteq y$ if, for all open $U \subseteq X$, $x \in U$ implies $y \in U$.

Trivially, every open set $U \subseteq X$ is upper-closed in the specialisation order, which is, in general, a preorder on X. The space X is said to be T_0 if $\sqsubseteq$ is a partial order.

Definition 3.9 (Monotone convergence space). A topological space X is a *monotone convergence space* if the specialisation order on X is a dcpo (in particular, X is T_0), and every open subset of X is Scott-open with respect to the order.

Monotone convergence spaces include all T_1 spaces, all sober spaces, and all dcpos with the Scott topology. Monotone convergence spaces were introduced in Wyler (1981), where they were called *d-spaces*. We take our terminology from Gierz *et al.* (2003).

We now come to the central definition in this paper.

Definition 3.10 (Compactly generated predomain). A *compactly generated predomain* is a topological space X that is both compactly generated and a monotone convergence space.

We write **kP** for the category of compactly generated predomains and continuous functions. By combining previous remarks, one sees that compactly generated predomains include all locally compact Hausdorff spaces and all dcpos with the Scott topology.

In order to state the main theorem of this section, we recall that a full subcategory $\mathscr{C}'$ of a cartesian-closed category $\mathscr{C}$ is said to be an *exponential ideal* if it is closed under finite products and isomorphisms in $\mathscr{C}$ and, for all objects X of $\mathscr{C}$ and Y of $\mathscr{C}'$, the $\mathscr{C}$-exponential Y^X is an object of $\mathscr{C}'$. Obviously, exponential ideals are themselves cartesian closed.

Theorem 3.11. The category **kP** is a full reflective exponential ideal of **kTop**.

It follows that **kP** is complete and cocomplete. Limits are calculated as in **kTop**. Colimits are calculated by reflecting colimits from **kTop**. Thus, in **kP**, neither limits nor colimits are, in general, calculated as in **Top** (though it is easy to see that coproducts in **kP** *are* calculated as in **Top**).

We prove the theorem in stages. First, we observe that the coreflection from **Top** to **kTop** cuts down to monotone convergence spaces.

Proposition 3.12. If X is a monotone convergence space, then so is $k(X)$.

Proof. Since Sierpinski space $\mathbb{S}$ is compactly generated, $C(\mathbb{S}, k(X)) = C(\mathbb{S}, X)$, thus $k(X)$ and X have the same specialisation order, which is a dcpo. It is easy to see that $k(X)$ is the coarsest compactly generated topology on the set X that refines the topology on X. Since X is a monotone convergence space, the Scott topology on the specialisation order refines the topology on X. By Proposition 3.2, the Scott topology is compactly generated. Hence every open in $k(X)$ is Scott open. □

Lemma 3.13. If X is a topological space and Y is a monotone convergence space, then the pointwise order on $C(X, Y)$ is a dcpo with directed suprema constructed pointwise.

For the straightforward proof see Gierz *et al.* (2003, Lemma II.3.14).

Proposition 3.14. The category **kP** is an exponential ideal of **kTop**.

Proof. For closure under finite products, it is easy to show that topological products preserve monotone convergence spaces, from which the result follows by Proposition 3.12.

For the exponential property, suppose X is compactly generated and Y is a compactly generated predomain. By Propositions 3.5 and 3.12, it suffices to show that $C_{\text{co}}(X, Y)$ is a monotone convergence space. It is easy to check that the specialisation order on $C_{\text{co}}(X, Y)$ is pointwise, and this is a dcpo by Lemma 3.13. It remains to show that every subbasic open $\langle K, V\rangle$ of $C_{\text{co}}(X, Y)$ is Scott open. But suppose $(\bigsqcup_{i\in I} f_i) \in \langle K, V\rangle$, where $\{f_i\}_{i\in I}$ is directed. Because V is Scott open, $\{f_i^{-1}V\}_{i\in I}$ is an open cover of K. By directedness, there exists $j \in I$ such that f_j is an upper bound for finitely many f_i determining a finite subcover. Then $f_j \in \langle K, V\rangle$. Thus $\langle K, V\rangle$ is indeed Scott open. □

One way of obtaining the reflection functor from **kTop** to **kP** would be to apply the Special Adjoint Functor Theorem. Instead, we provide a more informative direct construction. We show that the reflection from **Top** to the category of monotone convergence spaces, as described in Wyler (1981), cuts down to a reflection from **kTop** to **kP**. This nicely mirrors the symmetric property established in Proposition 3.12 for the corefelection k.

First, we present the reflection from **Top** to the category of monotone convergence spaces, *cf*. Wyler (1981). Recall that a filter $\mathscr{F} \subseteq \mathcal{O}(X)$ is said to be *completely prime* if whenever $(\bigcup_{i \in I} U_i) \in \mathscr{F}$ we have $U_i \in \mathscr{F}$ for some $i \in I$. (This implies that $\varnothing \notin \mathscr{F}$.) For any point $x \in X$, the filter $\eta(x)$ of open neighbourhoods of x is always completely prime. A topological space is said to be *sober* if every completely prime filter is the filter of open neighbourhoods of a unique point. The *sobrification* $\mathfrak{S}(X)$ of a topological space X has the set of completely prime filters of $\mathcal{O}(X)$ as its underlying set with open sets

$$\{\mathscr{F} \in \mathfrak{S}(X) \mid U \in \mathscr{F}\},$$

where $U \in \mathcal{O}(X)$. It is easy to see that the specialisation order on $\mathfrak{S}(X)$ is inclusion, with least upper bounds of directed subsets $D \subseteq \mathfrak{S}(X)$ given by $\bigcup D$, which is indeed a completely prime filter. Define $\mathfrak{M}(X)$ to be the smallest subspace of $\mathfrak{S}(X)$ that contains all neighbourhood filters and is closed under directed lubs in the specialisation order. It will be useful to have an explicit description of the topology on $\mathfrak{M}(X)$.

Lemma 3.15. The following are equivalent for a subset $V \subseteq \mathfrak{M}(X)$:

1 V is open.
2 $\eta^{-1}V$ is an open subset of X and $V = \uparrow(V \cap \eta(X))$ in the specialisation order on $\mathfrak{M}(X)$.
3 $\eta^{-1}V$ is an open subset of X and V is Scott-open in the specialisation order on $\mathfrak{M}(X)$.

Proof.

$2 \Rightarrow 1$: Suppose that $\eta^{-1}V$ is open in X and $V = \uparrow(V \cap \eta(X))$. We show that $V = \{\mathscr{F} \in \mathfrak{M}(X) \mid (\eta^{-1}V) \in \mathscr{F}\}$, and hence V is open by the definition of the topology on $\mathfrak{S}(X)$. If $\mathscr{F} \supseteq \eta(x)$ for $x \in \eta^{-1}V$, then, trivially, $(\eta^{-1}V) \in \mathscr{F}$. Thus $\{\mathscr{F} \in \mathfrak{M}(X) \mid (\eta^{-1}V) \in \mathscr{F}\} \supseteq \uparrow(V \cap \eta(X)) = V$. Conversely, suppose that $(\eta^{-1}V) \in \mathscr{F} \in \mathfrak{M}(X)$. For any $x \in \eta^{-1}V$ for which $\eta(x) \nsubseteq \mathscr{F}$, there exists open $U_x \ni x$ in X such that $U_x \notin \mathscr{F}$. Suppose, to show a contradiction, that such U_x exists for every $x \in \eta^{-1}V$. Then $\bigcup_{x \in \eta^{-1}V} U_x \supseteq \eta^{-1}V$, so $\bigcup_{x \in \eta^{-1}V} U_x \in \mathscr{F}$. Hence, because $\mathscr{F}$ is completely prime, $U_x \in \mathscr{F}$ for some $x \in \eta^{-1}V$, which is a contradiction. Thus there exists $x \in \eta^{-1}V$ with $\eta(x) \subseteq \mathscr{F}$. So, indeed, $\{\mathscr{F} \in \mathfrak{M}(X) \mid (\eta^{-1}V) \in \mathscr{F}\} \subseteq \uparrow(V \cap \eta(X)) = V$.

$1 \Rightarrow 3$: Suppose $V \subseteq \mathfrak{M}(X)$ is open, that is, there exists open $U \subseteq X$ such that $V = \{\mathscr{F} \in \mathfrak{M}(X) \mid U \in \mathscr{F}\}$. Then $\eta^{-1}V = U$, so $\eta^{-1}V$ is indeed an open subset of X. Also, V is obviously upwards closed in the specialisation order. To show that V is inaccessible by directed suprema, suppose that $D \subseteq \mathfrak{M}(X)$ is directed with $\bigsqcup D \in V$, that is, $\bigcup D \in V$. Then $U \in \bigcup D$, so there exists $\mathscr{F} \in D$ with $U \in \mathscr{F}$, and thus $\mathscr{F} \in V$, as required.

$3 \Rightarrow 2$: Suppose that $\eta^{-1}V$ is open in X and V is Scott-open in $\mathfrak{M}(X)$. It is obvious that $V \supseteq \uparrow(V \cap \eta(X))$, because Scott-open sets are upper closed. It remains to show that $V \subseteq \uparrow(V \cap \eta(X))$. Because V is Scott-open, it is contained in the Scott-interior of $\mathfrak{M}(X)\backslash\eta(X\backslash\eta^{-1}V)$. It is enough to show that the Scott-interior of $\mathfrak{M}(X)\backslash\eta(X\backslash\eta^{-1}V)$ is $\uparrow(V \cap \eta(X))$. We prove the equivalent statement that the Scott-closure of $\eta(X\backslash\eta^{-1}V)$ is $\mathfrak{M}(X)\backslash\uparrow(V \cap \eta(X))$. Let S be the Scott-closure of $\eta(X\backslash\eta^{-1}V)$. Then

$$S \cup \uparrow(V \cap \eta(X)) = \mathfrak{M}(X), \tag{1}$$

because the left-hand side contains $\eta(X)$ and is closed under suprema of directed sets. Also, $S \cap V = \varnothing$ because $\eta(X\backslash\eta^{-1}V) \cap V = \varnothing$ and V is Scott-open. Hence

$$S \cap \uparrow(V \cap \eta(X)) = \varnothing, \tag{2}$$

because $\uparrow(V \cap \eta(X)) \subseteq V$. Thus, by (1) and (2), $S = \mathfrak{M}(X)\backslash\uparrow(V \cap \eta(X))$, as required. □

Note that the equivalence of 1 and 2 above is inherited by $\mathfrak{M}(X)$ from an analogous characterisation of open sets in $\mathfrak{S}(X)$. That Property 3 characterises open sets is, however, a feature specific to $\mathfrak{M}(X)$.

It follows from Lemma 3.15 that $\mathfrak{M}(X)$ is a monotone convergence space. In fact it is the free monotone convergence space over X.

Proposition 3.16. For any topological space X, the space $\mathfrak{M}(X)$ is a monotone convergence space. Moreover, for any monotone convergence space Y and continuous function $f : X \to Y$ there exists a unique continuous $g : \mathfrak{M}(X) \to Y$ such that $g \circ \eta = f$.

Proof. This is Wyler (1981, Theorem 2.7). □

Proposition 3.17. If X is compactly generated, so is $\mathfrak{M}(X)$.

Proof. Suppose $V \subseteq \mathfrak{M}(X)$ is such that for every compact Hausdorff K and continuous $p : K \to \mathfrak{M}(X)$ we have that $p^{-1}V$ is open in K. We use Lemma 3.15 to show that V is open in $\mathfrak{M}(X)$, establishing Condition 3. To show that $\eta^{-1}V$ is open in X, suppose that K is compact Hausdorff and $q : K \to X$ is continuous. Then $q^{-1}(\eta^{-1}V) = (\eta \circ q)^{-1}V$, and thus $q^{-1}(\eta^{-1}V)$ is open in K. Thus $\eta^{-1}V$ is indeed open in X, because X is compactly generated. It remains to show that V is Scott-open in $\mathfrak{M}(X)$. By assumption, V is open in $k(\mathfrak{M}(X))$, which has the same specialisation order as $\mathfrak{M}(X)$, and is a monotone convergence space by Proposition 3.12. Thus V is indeed Scott-open in $\mathfrak{M}(X)$. □

In combination, Propositions 3.14, 3.16 and 3.17 prove Theorem 3.11.

In domain theory, one is often interested in domains (that is, dcppos) as opposed to predomains (that is, dcpos). We make the analogous definition for compactly generated spaces.

Definition 3.18 (Compactly generated domain). A *compactly generated domain* is a compactly generated predomain with a least element in the specialisation order.

We write **kD** for the category of compactly generated domains and continuous functions.

Proposition 3.19. The category **kD** is an exponential ideal of **kTop** that is closed under arbitrary products.

Proof. Given Theorem 3.11, all that remains to check is that the required constructions preserve the property of having a least element. This is straightforward. □

The category of compactly generated domains is a category that properly extends the category of dcppos and enjoys all the usual properties of a category of domains. Indeed, one can show that the category **kD** enjoys the usual interrelationship with its subcategory of *strict* (that is, bottom-preserving) maps, that the expected strict type constructors (smash product, strict function space and coalesced sum) are available, that recursive domain equations have solutions, and so on. The constructions, which are routine modifications of the familiar domain-theoretic ones, are omitted from the current paper.

4. Countably pseudobased spaces

In traditional domain theory, countable (domain-theoretic) bases allow a theory of computability for domains to be developed. Such bases exist for all ω-continuous dcpos. Although the categories of ω-continuous dcpos and dcppos are not cartesian closed, they have cartesian-closed subcategories that, for many purposes (modulo the limitations discussed in Section 1), do provide workable categories of (pre)domains.

In our more general topological setting, a natural first attempt at doing something similar is to restrict to compactly generated spaces with countable (topological) bases. As with the category of ω-continuous dcpos, this category is not cartesian closed. In this case, it seems that the most natural remedy is to enlarge the category rather than to shrink it. This is done by weakening the requirement of a countable base to a countable *pseudobase* in the sense of E. Michael (Michael 1966).

Definition 4.1 (Pseudobase). A *pseudobase* for a topological space X is a family $\mathscr{B}$ of (not necessarily open) subsets of X satisfying, whenever $K \subseteq U$ with K compact and U open subsets of X, there exist finitely many $B_1, \ldots, B_k \in \mathscr{B}$ such that $K \subseteq \bigcup_{i=1}^{k} B_k \subseteq U$. A *pseudosubbase* is a family of subsets whose closure under finite intersection forms a pseudobase.

Obviously, any (sub)base for the topology on X is also a pseudo(sub)base. Conversely, whenever a pseudo(sub)base $\mathscr{B}$ consists of open sets, it is itself a (sub)base.

The requirements on a pseudobase are weak enough that it need have very little to do with the topology. For example, the powerset of X is always a pseudobase for X. However, as the results below demonstrate, pseudobases do become interesting when cardinality restrictions are placed upon them.

We say that a topological space X is a *qcb (quotient of countably based) space*, if it can be exhibited as a topological quotient $q : A \twoheadrightarrow X$, where A is a countably based space. We write **QCB** for the full subcategory of **Top** consisting of such spaces. In his Ph.D. thesis, the second author established that qcb spaces are exactly the sequential spaces with countable pseudobase (Schröder 2003). The proposition below, which follows from Escardó *et al.* (2004, Theorem 6.10), generalises this result to compactly generated spaces.

Proposition 4.2. The following are equivalent for a topological space X:

1 X is compactly generated and has a countable pseudobase.
2 X is a qcb space.

(To obtain this result as consequence of Escardó *et al.* (2004, Theorem 6.10), let $\mathscr{C}$ be the category of compact Hausdorff spaces. Then, every $\ll$-pseudobase in the sense of *loc. cit.* is a pseudobase as defined above, and every pseudobase as defined above is a $\ll_{\mathscr{C}}$-pseudosubbase in the sense of *loc. cit.*)

Trivially, every countably based space is itself a qcb space. However, not every qcb space is countably based. The next result gives a useful sufficient condition under which qcb spaces are countably based.

Proposition 4.3. If a locally compact space has a countable pseudobase, it has a countable base.

Proof. The interiors of pseudobase sets form a base when the topology is locally compact, *cf.* Escardó *et al.* (2004, Corollary 6.11). □

The next proposition reviews some of the useful properties of countably based spaces that are shared by the more general class of qcb spaces.

Proposition 4.4. If X is a qcb space, then:

1 X is a sequential space.
2 X is hereditarily Lindelöf (that is, for every family of opens $\{U_i\}_{i\in I}$ there exists countable $J \subseteq I$ such that $\bigcup_{i\in I} U_i = \bigcup_{j\in J} U_j$).
3 X is hereditarily separable (that is, for any $Y \subseteq X$ there exists a countable $Y' \subseteq Y$ such that Y' is dense in the subspace topology on Y).

Proof. Properties 1 and 2 hold for countably based spaces and are preserved under quotienting. For 3, every space with countable pseudobase is separable, and pseudobases restrict to subspaces. (Note that the subspace topology on Y need not itself be compactly generated.) □

Quite unexpectedly, the category **QCB** has a very rich structure.

Proposition 4.5. The category **QCB** has all countable limits and colimits and is cartesian closed. Moreover, this structure is preserved by the inclusion **QCB** $\hookrightarrow$ **kTop**.

This result is a special case of Escardó *et al.* (2004, Corollary 7.3), where a full proof is given. Earlier proofs of cartesian closedness appear in Schröder (2003) and Menni and Simpson (2002). Here we simply state that if $\mathscr{A}$ and $\mathscr{B}$ are countable pseudosubbases for qcb spaces X and Y, respectively, then the family of all sets of the form

$$\{f \in C(X,Y) \mid f(\bigcap \mathscr{A}') \subseteq \bigcup \mathscr{B}'\},$$

where $\mathscr{A}'$ is a finite subset of $\mathscr{A}$ and $\mathscr{B}'$ is a finite subset of $\mathscr{B}$, form a countable pseudosubbase for $X \Rightarrow_k Y$.

The goal of this section is to show that countably pseudobased compactly generated spaces (that is, qcb spaces) form a good environment for restricting the notions of predomain and domain from the previous sections. We write $\omega\mathbf{P}$ and $\omega\mathbf{D}$ for the full subcategories of $\mathbf{kP}$ and $\mathbf{kD}$, respectively, whose objects are qcb spaces. Clearly, $\omega\mathbf{P}$ contains every ω-continuous dcpo and $\omega\mathbf{D}$ contains every ω-continuous dcppo.

As is well known, ω-continuous dcpos can be equivalently defined using ω-completeness rather than directed-completeness. The proposition below shows that countable pseudo-bases permit an analogous flexibility in the definition of a compactly generated predomain.

As is standard, we call an ascending sequence $x_0 \sqsubseteq x_1 \sqsubseteq x_2 \sqsubseteq \dots$ in a partial order an *ω-chain*. An *ω-complete partial order (ω-cpo)* is a partial order in which every ω-chain has a lub. A subset X of an ω-cpo D is said to be *ω-Scott open* if it is upper closed and, whenever $(\bigsqcup_i x_i) \in X$, for an ω-chain (x_i), we have $x_i \in X$ for some i.

Definition 4.6 (Monotone ω-convergence space). A topological space X is a *monotone ω-convergence space* if the specialisation order on X is an ω-complete partial order, and every open subset of X is ω-Scott-open with respect to the order.

Proposition 4.7. A qcb space is a monotone convergence space if and only if it is a monotone ω-convergence space.

Proof. It is immediate that any space that is a monotone convergence space is a monotone ω-convergence space. For the converse, suppose that X is a monotone ω-convergence qcb space. To show that the specialisation order is a dcpo, suppose $D \subseteq X$ is directed. We must show that $\bigsqcup D$ exists. By Proposition 4.4, X is hereditarily separable, so D considered as a subspace of X has a countable dense subset $\{d_i \mid i \in \mathbb{N}\} \subseteq D$. Because D is directed, we can construct $\{e_i \mid i \in \mathbb{N}\} \subseteq D$ such that each e_i is an upper bound for the finite set $\{d_i\} \cup \{e_j \mid j < i\}$. Obviously $e_o \sqsubseteq e_1 \sqsubseteq e_2 \dots$ is an ascending sequence. Define $e_\infty = \bigsqcup_i e_i$. We claim that $e_\infty = \bigsqcup D$. To see it is an upper bound, suppose $d \in D$. To show that $d \sqsubseteq e_\infty$, suppose that $d \in U \subseteq X$ where U is open. We must show that $e_\infty \in U$. Because $\{d_i \mid i \in \mathbb{N}\} \subseteq D$ is dense, there exists $d_i \in U$. Hence, indeed, $e_\infty \in U$, because $d_i \sqsubseteq e_i \sqsubseteq e_\infty$. For leastness, suppose e is any upper bound for D. To show that $e_\infty \sqsubseteq e$, suppose $e_\infty \in U \subseteq X$ where U is open. Because X is an ω-convergence space there exists i such that $e_i \in U$. But $e_i \sqsubseteq e$ because $e_i \in D$. So, indeed, $e \in U$.

It remains to show that every open is Scott-open. Suppose $U \subseteq X$ is open $D \subseteq X$ is directed and $\bigsqcup D \in U$. We must show that $d \in U$ for some $d \in D$. But, as above, $\bigsqcup D = \bigsqcup_i e_i$, so $e_i \in U$ for some i. Thus $d = e_i$ is the required element of D. □

Theorem 4.8. The category $\omega\mathbf{P}$ is a full reflective exponential ideal of $\mathbf{QCB}$.

It follows that $\omega\mathbf{P}$ is cartesian closed with countable limits and colimits, where limits are calculated as in $\mathbf{QCB}$.

The proof of Theorem 4.8 follows similar lines to that of the analogous Theorem 3.11.

Proposition 4.9. The category $\omega\mathbf{P}$ is an exponential ideal of $\mathbf{QCB}$.

Proof. This is immediate from Propositions 3.14 and 4.5. □

To establish the reflection part of Theorem 4.8, we show that the reflection $\mathfrak{M}$ of monotone convergence spaces in $\mathbf{Top}$ cuts down to $\mathbf{QCB}$.

Proposition 4.10. If X is a qcb space, so is $\mathfrak{M}(X)$.

Proof. We first make the following observations. If Y is compactly generated, then, by the cartesian closedness of **kTop**, the function $i\colon Y \to ((Y \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S})$, defined by $i(y) = \lambda U.\, U(y)$, is continuous. Also by cartesian closedness, for any open $U \subseteq Y$, the function $p_U : ((Y \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}) \to \mathbb{S}$ defined by $p_U(F) = F(U)$ (here we are identifying U with its characteristic function in $Y \Rightarrow_k \mathbb{S}$) is continuous. An easy calculation shows that, for any $y \in Y$ and open $U \subseteq Y$, we have $i(y) \in {p_U}^{-1}\{\top\}$ iff $y \in U$. Thus $i\colon Y \to ((Y \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S})$ is a topological pre-embedding. Furthermore, if Y is T_0, then i is an injective function, and hence an embedding.

To prove the proposition, suppose that X is a qcb space. By Proposition 3.17, $\mathfrak{M}(X)$ is compactly generated. We must show that it also has a countable pseudobase. By the observations above, there is a topological embedding of $\mathfrak{M}(X)$ in $(\mathfrak{M}(X) \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}$. By Lemma 3.15, the function mapping $V \in \mathcal{O}(\mathfrak{M}(X))$ to $\eta^{-1}V$ gives a lattice isomorphism $\mathcal{O}(\mathfrak{M}(X)) \cong \mathcal{O}(X)$, and thus, by Proposition 3.7, there is an induced homeomorphism $(\mathfrak{M}(X) \Rightarrow_k \mathbb{S}) \cong (X \Rightarrow_k \mathbb{S})$, and hence $((\mathfrak{M}(X) \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}) \cong ((X \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S})$. Thus there is a topological embedding of $\mathfrak{M}(X)$ in $(X \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}$. However, $(X \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}$ is a qcb space because, by Theorem 4.8, qcb spaces are closed under function spaces in **kTop**. Thus $(X \Rightarrow_k \mathbb{S}) \Rightarrow_k \mathbb{S}$ has a countable pseudobase, and thus $\mathfrak{M}(X)$ does too, since pseudobases restrict to subspaces. $\square$

Theorem 4.8 now follows from Propositions 4.9, 3.16 and 4.10.

Proposition 4.11. The category $\omega\mathbf{D}$ is an exponential ideal of **QCB**, closed under countable products.

Proof. This is immediate from Propositions 3.19, 4.5 and Theorem 4.8. $\square$

The categories $\omega\mathbf{P}$ and $\omega\mathbf{D}$ were introduced in Simpson (2003), where their objects were called *topological predomains* and *topological domains*, respectively. The countable pseudobase requirement is sufficient for the development of a computability theory, due to the connections, established in the second author's Ph.D. thesis (Schröder 2003), between qcb spaces and Klaus Weihrauch's theory of *type two effectivity* (Weihrauch 2000). As outlined in Simpson (2003), the categories of topological predomains and domains support the usual constructions of traditional domain theory, and also overcome the limitations discussed in Section 1. The details of this will appear elsewhere.

5. Comparison with traditional domain theory

Traditional domain theory is concerned with the categories **dcpo** and **dcppo** of continuous functions between dcpos and dcppos, and subcategories of them. By Proposition 3.2, **dcpo** and **dcppo** are full subcategories of **kP** and **kD**, respectively. As is well known, **dcpo** and **dcppo** are themselves cartesian closed. The products $\prod_{i\in I}^{s} D_i$ and $D \times_s E$ of dc(p)pos are given by the product partial order. The exponential $D \Rightarrow_s E$ is given by the set of Scott continuous functions ordered pointwise.

In this section, we investigate the extent to which the cartesian-closed structures on **dcpo** and **kP** agree. For finite products, there is no difference.

Proposition 5.1. For dcpos D, E, we have $D \times_s E = D \times_k E$.

Proof. Let D, E be dcpos. Then:

$$\begin{aligned}
\mathcal{O}(D \times_s E) &\cong C(D \times_s E,\ \mathbb{S}) & \\
&\cong C(D,\ E \Rightarrow_s \mathbb{S}) && \text{cartesian closedness of } \mathbf{dcpo} \\
&\cong C(D,\ E \Rightarrow_k \mathbb{S}) && \text{by Proposition 3.7} \\
&\cong C(D \times_k E,\ \mathbb{S}) && \text{cartesian closedness of } \mathbf{kP} \\
&\cong \mathcal{O}(D \times_k E).
\end{aligned}$$

So, $D \times_s E$ and $D \times_k E$ carry the same topology. □

Note that Martín Escardó has independently obtained the same result (Escardó 2005).

The above proposition shows that the inclusion $\mathbf{dcpo} \hookrightarrow \mathbf{kP}$ preserves finite products. It does not preserve infinite products. For example, the countable power of the two point discrete space is discrete in **dcpo**, but has the topology of Cantor space in **kP**. This counterexample makes essential use of a non-pointed space.

Proposition 5.2. For any family $\{D_i\}_{i \in I}$ of dcppos, the Scott product $\prod^s_{i \in I} D_i$ and compactly generated product $\prod^k_{i \in I} D_i$ coincide.

Proof. It is obvious that $\prod^s_{i \in I} D_i$ refines $\prod^k_{i \in I} D_i$, so we establish the converse. For any finite $J \subseteq I$, consider the set-theoretic function $\rho_J : \prod_{i \in I} D_i \to \prod_{i \in I} D_i$ defined by

$$(\rho_j(\pi))i \;=\; \begin{cases} \pi_i & \text{if } i \in J \\ \bot_{D_j} & \text{otherwise.} \end{cases}$$

Using the universal property of products in **kTop** and **dcppo**, respectively, the two functions $\rho_J : \prod^k_{i \in I} D_i \to \prod^k_{i \in I} D_i$ and $\rho_J : \prod^s_{i \in I} D_i \to \prod^s_{i \in I} D_i$ are continuous idempotents, whose splittings are the retracts

$$\prod_{i \in I}^{k} D_i \xrightarrow{r_k} \prod_{j \in J}^{k} D_j \xrightarrow{s_k} \prod_{i \in I}^{k} D_i$$
$$\prod_{i \in I}^{s} D_i \xrightarrow{r_s} \prod_{j \in J}^{s} D_j \xrightarrow{s_s} \prod_{i \in I}^{s} D_i.$$

By Proposition 5.1, the finite products $\prod^k_{j \in J} D_j$ and $\prod^s_{j \in J} D_j$ are homeomorphic. Thus the function $\rho_J = s_s \circ r_k : \prod^k_{i \in I} D_i \to \prod^s_{i \in I} D_i$ is continuous. Hence, $\{\rho_J\}_J$ forms a directed family of continuous functions from $\prod^k_{i \in I} D_i$ to $\prod^s_{i \in I} D_i$, indexed by finite subsets $J \subseteq I$. Clearly, the pointwise supremum of this family is the identity function from $\prod^k_{i \in I} D_i$ to $\prod^s_{i \in I} D_i$. By Lemma 3.13, this is continuous, as required. □

The above argument is adapted from Reinhold Heckmann's proof of his analogous Theorem 7.8 in Heckmann (2003).

The countable power of the two point discrete space again demonstrates a disagreement between function spaces in **dcpo** and **kP**: the function space $\mathbb{N} \Rightarrow_s \{0, 1\}$ is discrete, whereas

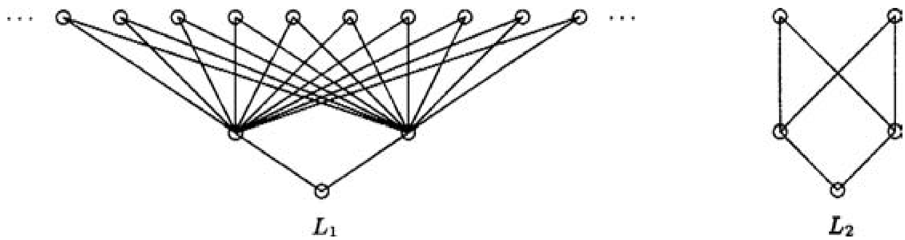

Fig. 1.

$\mathbb{N} \Rightarrow_k \{0,1\}$ is Cantor space. However, traditional domain theory largely concerns pointed dcpos. In the remainder of this section, we investigate the relationship between function spaces in **dcppo** and **kD**. We shall see that the inclusion **dcppo** $\hookrightarrow$ **kD** does not always preserve function spaces, but it does preserve them in many interesting cases. In fact, our claim is that the inclusion preserves function spaces in exactly those cases in which dcpo function spaces are 'well behaved'. In other cases, **kD** defines a more reasonable function space than **dcppo**.

We begin with a counterexample to show that function spaces are not in general preserved by the inclusion **dcppo** $\hookrightarrow$ **kD**. Consider the two (ω-algebraic) dcppos L_1 and L_2 presented in Figure 1.

Proposition 5.3. The function space $L_1 \Rightarrow_k L_2$ in **kP** does not have the Scott topology.

Proof. Both L_1 and L_2 are countably based spaces, hence $L_1 \Rightarrow_k L_2$ is a qcb space. The Scott topology on the function space (that is, $L_1 \Rightarrow_s L_2$) has been calculated by Achim Jung (Jung 1989). The resulting dcppo is algebraic, but not countably based (it has $2^{\aleph_0}$ compact elements). Because it is algebraic, the topology on $L_1 \Rightarrow_s L_2$ is locally compact. But then $L_1 \Rightarrow_s L_2$ cannot be a qcb space, since this would contradict Proposition 4.3. $\square$

An identical argument shows that the function space $L_1 \Rightarrow_k L_1$ in **kP** does not carry the Scott topology. The reason for choosing L_2 above was to give a counterexample with a finite poset as codomain.

The dcppos L_1 and L_2 are both algebraic L-domains in the sense of Jung (1989) (that is, algebraic dcppos in which every principal ideal is a complete lattice). It was shown in *op. cit.* that the category of algebraic (respectively, continuous) L-domains forms one of the two maximal cartesian-closed categories of algebraic (respectively, continuous) dcppos. One might thus be tempted to think of L-domains as belonging to the 'well-behaved' part of traditional domain theory. But this disregards the fact that the ω-algebraic and ω-continuous L-domains do not form cartesian-closed categories, due to the loss of countable base in the construction of function spaces. On the other hand, by Proposition 3.6, the function space $L_1 \Rightarrow_k L_2$ in **kD** is $C_{co}(L_1, L_2)$, which is countably based. In our view, it is the compactly generated function space that is the better behaved of the two.

When D is a continuous dcpo, the compact open topology $C_{co}(D, Y)$ can be given a simpler description.

Definition 5.4 (Point open topology). For topological spaces X, Y, the *point open topology* on $C(X, Y)$ is generated by the subbasic opens

$$\langle x, V\rangle = \{f \mid f(x) \subseteq V\},$$

where $x \in X$ and $V \subseteq Y$ is open. We write $C_{\mathrm{po}}(X, Y)$ for $C(X, Y)$ with the compact open topology.

The point open topology is equivalently characterised as the topology of pointwise convergence, or as the relative topology on $C(X, Y)$ as a subspace of the product topology on the power Y^X. Trivially, the compact open topology $C_{\mathrm{co}}(X, Y)$ always refines the point open topology $C_{\mathrm{po}}(X, Y)$. When X is a continuous dcpo, the two agree.

Lemma 5.5. If D is a continuous dcpo and Y a topological space, then $C_{\mathrm{po}}(D, Y)$ and $C_{\mathrm{co}}(D, Y)$ coincide.

Proof. Suppose $\langle K, V\rangle$ is a subbasic compact open neighbourhood of f. Then $K \subseteq f^{-1}(V)$ and thus, as D is a continuous dcpo, $K \subseteq \bigcup_{x\in f^{-1}(V)} \twoheaduparrow x$. As K is compact, there exists a finite $F \subseteq f^{-1}(V)$ such that $K \subseteq \bigcup_{x\in F} \twoheaduparrow x$. But now we have $\bigcap_{x\in F}\langle x, V\rangle \subseteq \langle K, V\rangle$, and therefore $f \in \bigcap_{x\in F}\langle x, V\rangle \subseteq \langle K, V\rangle$, showing that $\langle K, V\rangle$ is point open, as required. □

Together with Proposition 3.6, the above lemma, which is part of domain-theoretic folklore, implies that for ω-continuous dcpos D, E, we always have $D \Rightarrow_k E = C_{\mathrm{co}}(D, E) = C_{\mathrm{po}}(D, E)$, and this is countably based.

We have seen that $D \Rightarrow_k E$ does not always carry the Scott topology for ω-continuous D, E, even when D, E are L-domains. We next switch attention to the other of the two maximal cartesian-closed categories of continuous dcppos, the category of FS-domains, introduced by Achim Jung (Jung 1990).

Definition 5.6 (FS-domain). An *FS-domain* is a dcpo D for which there exists a directed family $(f_i)_{i\in I}$ of continuous endofunctions on D, each *strongly finitely separated* from id_D, that is, for each f_i there exists a finite *separating set* M_{f_i} such that for each $x \in D$ there exists $m \in M_{f_i}$ with $f_i(x) \sqsubseteq m \ll x$, and $\bigsqcup_{i\in I} f_i = \mathsf{id}_D$.

This definition, which differs from the original Jung (1990) in the use of *strong* finite separation, is nonetheless equivalent to it by Lemma 2 of *op. cit*. Also, as in Gierz *et al.* (2003), we do not require FS-domains to be pointed. This allows Theorem 5.7 below to be formulated as generally as possible.

Any FS-domain is automatically a continuous dcppo. If D and E are FS-domains, then $D \times_s E$ and $D \Rightarrow_s E$ are also FS-domains. Furthermore, if D and E are countably based, then so are $D \times_s E$ and $D \Rightarrow_s E$. Thus the categories **FS** and ω**FS** of FS-domains and countably based FS-domains are both cartesian closed subcategories of **dcpo**. These results are proved in Jung (1990) and Gierz *et al.* (2003).

In contrast to the situation for L-domains, compactly generated function spaces of FS-domains do carry the Scott topology.

Theorem 5.7. If D and E are FS-domains, then $D \Rightarrow_k E = D \Rightarrow_s E = C_{\mathrm{co}}(D,E) = C_{\mathrm{po}}(D,E)$.

In particular, the inclusions $\mathbf{FS} \hookrightarrow \mathbf{kP}$ and $\omega\mathbf{FS} \hookrightarrow \omega\mathbf{P}$ both preserve the cartesian-closed structure.

The first lemma needed in the proof of the theorem is a mild generalisation (with an identical proof) of Jung (1989, Corollary 1.36).

Lemma 5.8. If D is a dcpo and E a continuous dcpo such that $D \Rightarrow_s E$ is a continuous dcpo, then $f \ll g$ implies $f(x) \ll g(x)$ for all $x \in D$.

Proof. Suppose $\{e_i\}_{i\in I}$ is a directed family of elements in E with $\bigsqcup_i e_i = g(x)$. We have to show that there exists $i_0 \in I$ such that $e_{i_0} \sqsupseteq f(x)$. It is easy to show that $(\downarrow x)$ and $(\downarrow g(x))$ are continuous retracts of D and E, hence $(\downarrow x) \Rightarrow_s (\downarrow g(x))$ is a continuous retract of $D \Rightarrow_s E$, and therefore a continuous dcpo. For each $h\colon D \to E$, let $h' : (\downarrow x) \to (\downarrow g(x))$ denote the image of h under the retraction. Then $f' \ll g'$, since if $\{\psi_j\}_{j\in J}$ is a directed family of functions in $(\downarrow x) \Rightarrow_s (\downarrow g(x))$ with $\bigsqcup_j \psi_j = g'$, then $\{\Psi_j\}_{j\in J}$, defined as

$$\Psi_j(y) = \begin{cases} \psi_j(y) & \text{if } y \sqsubseteq x \\ g(y) & \text{otherwise} \end{cases}$$

is a directed family of functions in $D \Rightarrow_s E$ with $\bigsqcup_j \Psi_j = g$. Thus there exists $j_0 \in J$ such that $\Psi_{j_0} \sqsupseteq f$, and hence $\psi_{j_0} \sqsupseteq f'$.

Now for each $i \in I$, let $c_{e_i} : (\downarrow x) \to (\downarrow g(x))$ denote the constant function with value e_i. Then $\bigsqcup_i c_{e_i} \sqsupseteq g'$, so there exists $i_0 \in I$ such that $c_{e_{i_0}} \sqsupseteq f'$, giving

$$e_{i_0} = c_{e_{i_0}}(x) \sqsupseteq f'(x) = f(x),$$

as desired. □

Lemma 5.9. If D is an FS-domain and E a continuous dcpo such that $D \Rightarrow_s E$ is a continuous dcpo, then $D \Rightarrow_s E$ carries the point open topology.

Proof. Since D is an FS-domain, there exists a directed set $\{f_i\}_{i\in I}$ of endofunctions that are each strongly finitely separated from id_D, with finite separating sets M_{f_i}, and $\bigsqcup_i f_i = \mathrm{id}_D$. Furthermore, since $D \Rightarrow_s E$ is a continuous dcpo, there exists a directed set $\{\psi_j\}_{j\in J}$ of endofunctions such that each $\psi_j \ll h$ and $\bigsqcup_j \psi_j = h$. Then $\{\psi_j \circ f_i\}_{i\in I, j\in J}$ is also a directed set with $\psi_j \circ f_i \ll h$ and $\bigsqcup_j \psi_j \circ f_i = h$. So $\{\Uparrow(\psi_j \circ f_i)\}_{i\in I, j\in J}$ is a Scott-open neighbourhood basis for $h\colon D \to E$. Thus, for each Scott-open neighbourhood $\mathscr{U}$ of h, there exist $i_0 \in I$ and $j_0 \in J$ such that $h \in \uparrow(\psi_{j_0} \circ f_{i_0}) \subseteq \mathscr{U}$. Set $\mathscr{V} = \bigcap_{m\in M_{f_{i_0}}} \langle m, \Uparrow\psi_{j_0}(m)\rangle$, and hence $h \in \mathscr{V}$, by Lemma 5.8. We claim that $\mathscr{V} \subseteq \mathscr{U}$. To see this, let $x \in D$ and

$h' \in \mathscr{V}$. Then there exists $m \in M_{f_{i_0}}$ with $f_{i_0}(x) \sqsubseteq m \ll x$, so

$$(\psi_{j_0} \circ f_{i_0})(x) \sqsubseteq \psi_{j_0}(m) \ll h'(m) \sqsubseteq h'(x)\,.$$

Thus $\mathscr{V} \subseteq \uparrow(\psi_{j_0} \circ f_{i_0}) \subseteq \mathscr{U}$, showing the claim. □

Proof of Theorem 5.7. If D and E are FS-domains, then the conditions of Lemma 5.9 are satisfied, so $D \Rightarrow_s E$ carries the point open topology, which, by Lemma 5.5, coincides with the compact open topology. So $C_{\mathrm{co}}(D, E)$ carries the Scott topology, and is thus compactly generated. Therefore, $D \Rightarrow_k E = C_{\mathrm{co}}(D, E) = D \Rightarrow_s E$. □

Theorem 5.7 requires both domain and codomain of the function space to be FS-domains. In contrast, Proposition 3.7 asserts that the compactly generated exponential $X \Rightarrow_k \mathbb{S}$ carries the Scott topology for every compactly generated space X. We conclude the paper by considering to what extent this property generalises to continuous dcpos other than Sierpinski space $\mathbb{S}$. Clearly, it does not always hold since, by Proposition 5.3, the property fails when $\mathbb{S}$ is replaced by the five element pointed poset L_2 of Figure 1.

Theorem 5.10. If X is compactly generated and E is a continuous dcpo with binary infima, then $X \Rightarrow_k E = X \Rightarrow_s E$.

For the proof, we need a lemma, which is part of the domain-theoretic folklore.

Lemma 5.11. If a continuous dcpo has binary infima, it has infima for all non-empty compact subsets.

As we could only find an indirect proof in the literature (Schalk 1993, Lemma 7.14), it seems worth giving a direct argument.

Proof. Suppose D has binary, and therefore non-empty, finite meets. Let $K \subseteq D$ be non-empty and compact. Then $\Downarrow K = \{x \in D \mid \forall y \in K.\, x \ll y\}$ is non-empty, as $K \subseteq \bigcup_{x \in D} \Uparrow x$, so there exists a non-empty finite $F \subseteq D$ such that $K \subseteq \bigcup_{x \in F} \Uparrow x$, hence $\bigsqcap F \ll K$. We claim that $\Downarrow K$ is directed.

To see this, let $a, b \ll K$. Then for each $x \in K$, there exists $c_x \in D$ such that $a, b \sqsubseteq c_x \ll x$. Thus $K \subseteq \bigcup_{x \in K} \Uparrow c_x$, so there exists a non-empty finite $F \subseteq K$ such that $K \subseteq \bigcup_{x \in F} \Uparrow c_x$. But then $\bigsqcap_{x \in F} c_x$ fulfills $a, b \sqsubseteq \bigsqcap_{x \in F} c_x \ll K$, showing the claim.

Now let $x \sqsubseteq K$. Then $x = \bigsqcup \Downarrow x = \bigsqcup (\Downarrow x \cap \Downarrow K)$. Thus $x \sqsubseteq \bigsqcup \Downarrow K$, so $\bigsqcup \Downarrow K$ is the greatest lower bound of K, as required. □

Proof of Theorem 5.10. By Theorem 3.11, $X \Rightarrow_k E$ is a monotone convergence space. Thus we just need to verify that every Scott-open subset of $X \Rightarrow_k E$ is indeed open.

Let $W \subseteq X \Rightarrow_k E$ be Scott-open. Consider any continuous $p : K \to (X \Rightarrow_k E)$ with K compact Hausdorff. We must show that $p^{-1}(W) \subseteq K$ is open. Suppose we have $k \in K$ with $p(k) \in W$. We must find a neighbourhood of k contained in $p^{-1}(W)$.

Let D be the set of compact neighbourhoods of k, ordered by reverse inclusion. K is compact Hausdorff, and hence locally compact, so D is directed. For every $L \in D$, define

$h_L : X \to E$, by

$$h_L(x) = \bigsqcap_{z \in L} p(z)(x).$$

Then h_L is continuous because it arises as a composite of continuous functions

$$X \xrightarrow{\tilde{p}} C_{\mathrm{co}}(K, E) \longrightarrow C_{\mathrm{co}}(L, E) \xrightarrow{M_L} E,$$

the components of which we now describe. The map $\tilde{p}$ is obtained by the following manipulations:

$p : K \longrightarrow (X \Rightarrow_k E)$	
$K \times_k X \longrightarrow E$	exponential transpose in **kTop**
$K \times X \longrightarrow E$	by Proposition 3.3, using K core compact
$\tilde{p} : X \longrightarrow C_{\mathrm{co}}(K, E)$	exponential transpose in **Top**.

The map $C_{\mathrm{co}}(K, E) \to C_{\mathrm{co}}(L, E)$ is induced by the inclusion $L \subseteq K$. Finally, M_L is the function $M_L(f) = \bigsqcap_{z \in L} f(z)$. This is continuous because if $y \ll M_L(f)$, there exists y' with $y \ll y' \ll M_L(f)$. Hence $\langle L, \Uparrow y' \rangle$ is a neighbourhood of f in $C_{\mathrm{co}}(L, E)$ satisfying $y \ll M_L(g)$ for all $g \in \langle L, \Uparrow y' \rangle$.

It is easy to show that $L' \supseteq L \in D$ implies $h_{L'} \sqsubseteq h_L \in X \Rightarrow_s E$, so $H = \{h_L \mid L \in D\}$ is a directed subset of $X \Rightarrow_s E$. We show that $\bigsqcup H = p(k)$.

For every $L \in D$, we have $k \in L$, so $h_L(x) = \bigsqcap_{z \in L} p(z)(x) \sqsubseteq p(k)(x)$. Therefore $\bigsqcup H \sqsubseteq p(k)$.

Conversely, we show that $p(k) \sqsubseteq \bigsqcup H$, that is, $p(k)(x) \sqsubseteq (\bigsqcup H)(x)$, for all $x \in X$. Take any $x \in X$ and $y \ll p(k)(x)$. By the continuity of p, the set $U = \{z \mid y \ll p(z)(x)\} \subseteq K$ is open. Also, $k \in U$. By local compactness, there exists a compact neighbourhood $L \ni k$ with $L \subseteq U$. Then $h_L(x) = \bigsqcap_{z \in L} p(z)(x) \sqsupseteq y$. Thus, indeed, $y \sqsubseteq \bigsqcup(H(x)) = (\bigsqcup H)(x)$.

Summarising, we have directed $H \subseteq X \Rightarrow_k E$ with $\bigsqcup H = p(k) \in W$. As W is Scott-open, there exists $L \in D$ such that $h_L \in W$. Since L is a neighbourhood of k, we just need to show that $L \subseteq p^{-1} W$. So, consider any $z \in L$. Then we have $p(z) \sqsupseteq h_L \in W$. Thus, indeed, $p(z) \in W$, since W is upper-closed. □

Note that a special case of Theorem 5.10 follows more easily from existing results in the domain-theoretic literature. It is known that if X is a locally compact topological space and E is a bounded-complete continuous dcppo, then $C_{\mathrm{co}}(X, E)$ carries the Scott topology and is itself a bounded-complete continuous dcppo, see Gierz *et al.* (2003, Proposition II-4.6). So, in this case, the coincidence of the spaces $X \Rightarrow_k E$ and $X \Rightarrow_s E$ follows as a consequence of Propositions 3.2 and 3.5. We do not know if, more generally, $C_{\mathrm{co}}(X, E)$ carries the Scott topology, for locally compact X, when E merely has binary infima. It is certainly not true in general that $X \Rightarrow_s E$ is a continuous dcpo in this case. A counterexample is the space $U_\omega \Rightarrow_s U_\omega$, where U_ω is the well-known non-bifinite ω-algebraic dcppo from Figure 2. Although U_ω has binary infima, the dcppo $U_\omega \Rightarrow_s U_\omega$ is not continuous. Thus, from a traditional domain-theoretic viewpoint, the Scott topology on the function space $U_\omega \Rightarrow_s U_\omega$ is poorly behaved. In contrast, from a compactly generated viewpoint, the Scott topology is well behaved in this case. By Theorem 5.10, the

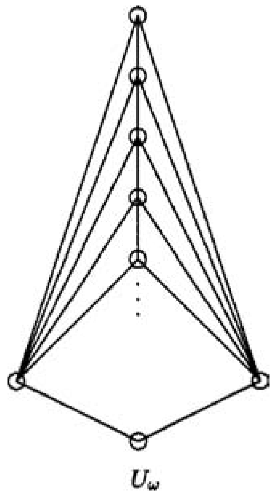

Fig. 2.

function spaces $U_\omega \Rightarrow_k U_\omega$ and $U_\omega \Rightarrow_s U_\omega$ coincide. Moreover, by Proposition 3.6 (and Lemma 5.5), they have the compact open (equivalently point open) topology, and this is countably based.

6. Discussion

In this paper we have demonstrated that the topological (pre)domains of Simpson (2003) fit naturally into the world of convenient topology. As argued in *op. cit*., topological predomains overcome the limitations, discussed in Section 1, of traditional domain theory. Moreover, the larger collection of compactly generated (pre)domains investigated in the present paper provides a natural topological generalisation of the dcpo-based world of traditional domain theory.

It is appropriate to question the use of compactly generated spaces in this paper. Even in algebraic topology, it is hard to give an *a priori* justification for taking the notion of compactly generated space as basic. From a domain-theoretic perspective, the motivation is even less clear. In particular, the choice of compact Hausdorff spaces as the generating spaces in Definition 3.1 seems utterly arbitrary.

In fact, compactly generated spaces form just one of many cartesian-closed subcategories of **Top**. Arguably, a more natural subcategory is the category of *core compactly generated spaces* introduced in Day (1972), which is the largest cartesian-closed subcategory of **Top** obtainable using the general theory of Day (1972) and Escardó *et al*. (2004). This category properly includes the category of compactly generated spaces (Isbell 1987). It seems likely that the results of the present paper should generalise to taking core compactly generated monotone convergence space as a notion of predomain, and other variants should be possible too.

Alternatively, some might prefer to carry out an analogous generalisation of domain theory within a cartesian-closed supercategory of **Top**, such as Scott's category of equilogical spaces (Bauer *et al.* 2004), or one of the categories of 'convergence' or 'filter' spaces, see, for example, Wyler (1974) and Hyland (1979). In fact, to some extent, Reinhold Heckmann has already embarked upon such a programme. In Heckmann (2003), he develops a convergence space variant of the notion of monotone convergence space, and establishes results analogous to our Propositions 3.14, 5.1 and 5.2 for that notion.

It is a pleasing fact that apparent differences between the subcategory and supercategory approaches disappear if attention is restricted to qcb spaces. As shown in Menni and Simpson (2002) and Escardó *et al.* (2004), the category **QCB** lives, via structure-preserving embeddings, in all the principal cartesian-closed subcategories of **Top**, and also in Scott's category of equilogical spaces. (It is the latter embedding that forms the basis of the connections with realizability semantics mentioned in Section 1.) Analogous embeddings of **QCB** in categories of convergence spaces have not been established, but are expected.

Seemingly, **QCB** is an inevitable category, ocurring within any sufficiently general approach to combining cartesian closedness and topology. In the authors' view, it is the category of paramount interest for the semantics of computation. For example, the size restriction naturally expresses the requirement that data should be representable by a sequence of discrete approximations (Weihrauch 2000; Schröder 2003).

Each of the larger categories embedding **QCB** offers its own valuable perspective on qcb spaces. In particular, as demonstrated in this paper, the approach via compactly generated spaces provides a good framework for relating topological (pre)domains and traditional domain theory. Thus, even though the notion of a compactly generated space seems to lack intrinsic motivation, such spaces do, nonetheless, provide a useful bridge between traditional topological domain theory and its topological generalisations.

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