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ABSTRACT: We study the center symmetry of $\mathrm{SU}(N)$ gauge theories with fermions in the two–index representations, by computing the effective potential of the Polyakov loop in the large–mass expansion on the lattice. In the large–N limit and at non–zero temperature, we find that the center symmetry is Z_N for fermions in the adjoint representation and just Z_2 for fermions in the (anti)symmetric representation. We discuss the fact that our results do not contradict the orientifold planar equivalence, which relates a common sector defined by the bosonic gauge–invariant C–even states of theories with fermions in different two–index representations. Our results complement the work of Armoni *et al.* (2007), who showed how at zero temperature a Z_N center symmetry is dynamically recovered also for fermions in the (anti)symmetric representation, by considering the theories at finite temperature.

KEYWORDS: Lattice Gauge Field Theories, Large N.

Contents

1. Introduction

Confinement of colour is related to center symmetry: where center symmetry is present and not broken, it implies confinement. Its spontaneous breaking leads to the deconfinement transition, and the critical behaviour at the phase transition is described by a three–dimensional effective theory which encodes the relevant pattern of symmetry breaking [1].

On the other hand, orientifold planar equivalence is the equivalence in the large–N limit of the SU(N) gauge theory with a Dirac fermion in the symmetric/antisymmetric two–index representations (OrientiQCD) and the SU(N) gauge theory with a Majorana fermion in the adjoint representation (AdjQCD) in a common sector [2, 3, 4]. The validity of this equivalence was discussed in detail in Refs. [5, 6]. A proof of the equivalence on the lattice in the strong–coupling and large–mass phase can be found in Ref. [7] (a general

setup for descussing planar equivalences between theories with two–index representations in the strong–coupling and large–mass phase was also presented in [8, 9]).

A potential inconsistency was pointed out in Ref. [10]: the two theories involved in the equivalence have different symmetries and in particular different center symmetries. The action of the adjoint theory is invariant under the full center Z_N of the gauge group at every value of N. In the orientifold theories the center symmetry is explicitly broken by the matter to a Z_2 if N is even and to nothing if N is odd. The different symmetry pattern in the two theories suggests a different dynamics at the deconfinement phase transition. It is therefore interesting to investigate to which extent the orientifold planar equivalence can describe correctly the dynamics at the deconfinement phase transition.

A solution to this puzzle in the case of zero temperature was presented in Ref. [11]: in the orientifold theories the action can be separated in a part that is invariant under Z_N and a part that is not. In the large–N limit, the non–invariant part decouples from the expectation values of the Wilson loops. The Z_N symmetry is therefore dynamically recovered and the apparent contradiction is resolved. One of the physical consequences is the stability of the k–strings.

In this work we consider the orientifold theories at finite temperature, both in the confining and deconfining phases. The main difference with the zero temperature case is that now the theory contains also loops wrapping around the thermal direction. We show by explicit computations that the non–invariant part of the action decouples only from the expectation values of the loops with zero winding number around the thermal direction. In the general case of wrapped loops, the non–invariant part of the action does not decouple, thus the Z_N symmetry is not restored in all the sectors of the theory. In particular, we will show that matter in the orientifold theories explicitly breaks the center symmetry to the Z_2 subgroup in the large–N limit at finite temperature.

In the deconfined phase, this result is well known. It can be obtained in the high–temperature regime, by analysing the one–loop effective action for the Polyakov loop. The invariance under Z_2 but not Z_N implies that in the deconfined phase (where the Z_2 is spontaneously broken), only two degenerate vacua exist.

In the large–N limit, the gauge theories discretised on the lattice exhibit two confined phases as the 't Hooft coupling, $\lambda = g^2 N$, is varied: an unphysical strong–coupling phase and a physical weak–coupling one [12, 13]. In both these phases, we investigate the symmetry content by computing the functional generator (W) of the connected expectation values of the Polyakov loop in the framework of the large–mass expansion. The generator W is a legitimate tool to study the center symmetry, since it encodes the same amount of information as the effective potential of the Polyakov loop. For finite N, W can be expanded in powers of $1/\lambda$:

$$W(\lambda, N) = \sum_n a_n(N)\lambda^{-n} \ . \tag{1.1}$$

At strong coupling the large–N limit of W is obtained by taking the limit of each coefficient $a_n(N)$ in the above expansion. Properties of Polyakov loops correlators can be inferred

from W. In particular we will show that:

$$\lim_{N \to \infty} \langle (\text{tr } \Omega)^2 \rangle_c \neq 0 \ , \tag{1.2}$$

in the orientifold theories, where $(\text{tr } \Omega)$ is the Polyakov loop; the above result shows clearly that the full Z_N center is not a symmetry. We may add that it is explicitly broken (and not spontaneously), since the vacuum is not degenerate in the strong-coupling regime. Under some mild assumptions, this result remains valid in the physical (weak–coupling) confined phase.

Let us now anticipate some comments on the original question. How can the orientifold planar equivalence be valid if the two theories that should be equivalent have got different symmetries? It was already pointed out that the equivalence holds only in a common *neutral* sector of the various theories. The neutral sector was identified in Ref. [5] to be the set of all the single–trace gauge–invariant observables that are also invariant under charge–conjugation (C) symmetry. The Polyakov loop does not belong to the neutral sector; indeed under C–symmetry $(\text{tr } \Omega) \to (\text{tr } \Omega)^\dagger$, therefore the real part is C–even and belongs to neutral sector, while the imaginary part is C–odd (it belongs to the *twisted* sector). Under the action of an element $e^{\frac{2\pi i k}{N}}$ of Z_N, the neutral and twisted sectors are in general mixed:

$$\text{Re}\,[\text{tr } \Omega] \to \cos\left(\frac{2\pi k}{N}\right) \text{Re}\,[\text{tr } \Omega] - \sin\left(\frac{2\pi k}{N}\right) \text{Im}\,[\text{tr } \Omega] \ , \tag{1.3a}$$

$$\text{Im}\,[\text{tr } \Omega] \to \sin\left(\frac{2\pi k}{N}\right) \text{Re}\,[\text{tr } \Omega] + \cos\left(\frac{2\pi k}{N}\right) \text{Im}\,[\text{tr } \Omega] \ . \tag{1.3b}$$

The adjoint theory has a Z_N symmetry, but the neutral and twisted sectors are mixed by the center symmetry. Only the Z_2 subgroup (which corresponds to $e^{\frac{2\pi i k}{N}} = -1$ in Eqs. (1.3)) maps the neutral sector into itself. Therefore the orientifold planar equivalence requires just a Z_2 symmetry in the neutral sector of the orientifold theories. The fact that Z_2 (and not Z_N) is the center symmetry for the full orientifold theories does not follow from the equivalence and is not in contradiction with it.

The paper is organised as follows. In Sect. 2 we give a brief review of center symmetry. In Sect. 3, 4, and 5 both the strong–coupling and weak–coupling confining phases on the lattice are analysed. Although the general setting is valid for both phases, explicit analytical computations can be performed only in the strong–coupling phase. In particular, the effective potential in the large–mass phase is computed. Even though this phase does not describe continuum physics it is interesting because it provides a testing bed where the planar equivalence holds and computations can be performed explicitly.

The possibility of extrapolating some of the results from the strong–coupling regime to the continuum limit is discussed in Sect. 4, together with the necessary underlying assumptions. A brief analysis of the center symmetry in the high–temperature regime is presented in Sect. 6.

Finally in Sect. 7, we introduce the effective potential in the Hamiltonian formalism and investigate the center symmetry by means of the formalism of the coherent states [14, 5].

2. Center symmetry

Whenever a gauge group has non–trivial center, the center symmetry plays a central role in understanding the dynamics at the deconfinement phase transition. We briefly summarise in this section some relevant properties of gauge theories under transformations that belong to the center of the gauge group. The same notation is used in the rest of the paper. For a gauge theory discretised on the lattice, a transformation belonging to the center of the gauge group acts on the link variables according to:

$$U_0(x_0 = 0, \mathbf{x}) \rightarrow u U_0(x_0 = 0, \mathbf{x}) \ , \tag{2.1}$$

where x_0 is the coordinate along the time direction, and u is an element of the center. All the other link variables are left unchanged by the center transformation. The center transformation counts the winding number of Wilson loops around the time direction. If the closed path Γ wraps w times around the time direction, the transformation rule is:

$$\text{tr } U(\Gamma) \rightarrow u^w \text{tr } U(\Gamma) \ , \tag{2.2}$$

where $U(\Gamma)$ is the path–ordered product of the link variables along Γ. Since the gauge action is given by a sum of plaquettes, which have zero winding number, it is invariant under the action of the center.

A generic representation R of the gauge group induces a representation of the center subgroup, that can be labelled by an integer N_R (with $0 \leq N_R < N$), the N–ality of the representation:

$$R[u] = u^{N_R} \ , \quad R[uU] = u^{N_R} R[U] \ . \tag{2.3}$$

Clearly, if matter in a representation with zero N–ality is present, the theory is still invariant under the action of the center. Otherwise, the theory is invariant only under some subgroup. The center symmetry of the theories investigated in this work are summarised in the following table.

gauge group	matter	center symmetry
SO(2N)	none/adjoint	Z_2
SU(N)	none/adjoint	Z_N
SU(N)	fundamental	-
SU(N) with N even	AS/S	Z_2
SU(N) with N odd	AS/S	-
U(N)	none/adjoint	U(1)
U(N)	fundamental	-
U(N)	AS/S	Z_2

3. Potentials for the Polyakov loop

In this section, we introduce effective potentials that describe the dynamics of Polyakov loops for the gauge theory discretised on the lattice. We shall first discuss the case of a pure

gauge theory, while fermions in arbitrary representations are introduced in the next section within the large–mass expansion framework. In what follows, it is useful to distinguish the temporal coordinate x_0 from the spatial ones, i.e. $x = (x_0, \mathbf{x})$, and the temporal link variable $U_0(x_0, \mathbf{x})$ from the spatial ones $U_k(x_0, \mathbf{x})$.

The parallel transport Ω around the temporal dimension and the Polyakov loop P are defined as usual by:

$$\Omega(\mathbf{x}) = U_0(0, \mathbf{x})U_0(1, \mathbf{x}) \cdots U_0(T-1, \mathbf{x}) , \tag{3.1}$$

$$P(\mathbf{x}) = \mathrm{tr}\ \Omega(\mathbf{x}) , \tag{3.2}$$

where T is the extent of the lattice in the temporal direction in units of the lattice spacing. Gauge invariance is used to fix the temporal gauge, where we have:

$$U_0(x_0, \mathbf{x}) = \begin{cases} 1 & \text{if } x_0 = 0, \ldots, T-2 \\ \Omega(\mathbf{x}) & \text{if } x_0 = T-1 \end{cases} . \tag{3.3}$$

After such a gauge fixing, the gauge action $S(\Omega, U_k)$ is invariant under time–independent gauge transformations. The Fadeev–Popov determinant being trivial, the measure of the functional integral is simply:

$$\exp\left\{-S(\Omega, U_k)\right\} \prod_{k,x_0,\mathbf{x}} dU_k(x_0, \mathbf{x}) \prod_{\mathbf{x}} d\Omega(\mathbf{x}) . \tag{3.4}$$

Let us now define several functionals of the field $\Omega(\mathbf{x})$; they all encode information about the dynamics of the Polyakov loops, and provide useful information on the structure of the deconfinement phase transition.

The probability distribution of the parallel transport $\Omega(\mathbf{x})$ is obtained by integrating out all the link variables in the spatial directions:

$$e^{-S_\Omega(\Omega)} = \frac{1}{Z} \int e^{-S(\Omega,U_k)} \prod_{k,x_0,\mathbf{x}} dU_k(x_0, \mathbf{x}) . \tag{3.5}$$

The functional $S_\Omega(\Omega)$ defines an effective action for the Polyakov loop, and the vacuum expectation value of a generic functional $f(\Omega)$ can be computed as:

$$\langle f(\Omega) \rangle = \frac{1}{Z} \int f(\Omega)\, e^{-S_\Omega(\Omega)} \prod_{\mathbf{x}} d\Omega(\mathbf{x}) . \tag{3.6}$$

The functional $S_\Omega(\Omega)$ induces a natural definition for the probability distribution of the Polyakov loop as the expectation value of the delta function:

$$e^{-N^2 V(P)} = \left\langle \prod_{\mathbf{x}} \delta\left(P(\mathbf{x}) - \mathrm{tr}\ \Omega(\mathbf{x})\right) \right\rangle =$$

$$= \frac{1}{Z} \int d\Omega(\mathbf{x}) \prod_{\mathbf{x}} \delta\left(P(\mathbf{x}) - \mathrm{tr}\ \Omega(\mathbf{x})\right) e^{-S_\Omega(\Omega)} . \tag{3.7}$$

The function $V(P)$ is sometimes referred to as the effective potential for the Polyakov loop.

In this work, we shall concentrate instead on the quantum action for the Polyakov loop, which is defined as the Legendre transform of the free energy of the system. First we construct the generator of connected n–point functions by coupling the Polyakov loop to a complex external source:

$$e^{-N^2 W(\alpha, \bar{\alpha})} = \int d\Omega(\mathbf{x}) \, e^{-S_\Omega(\Omega)} \, \exp\left\{ -N \sum_{\mathbf{x}} \left[\bar{\alpha}(\mathbf{x}) P(\mathbf{x}) + \alpha(\mathbf{x}) P^\dagger(\mathbf{x}) \right] \right\} \ . \qquad (3.8)$$

A Taylor expansion of the exponential in Eq. (3.8) yields:

$$W(\alpha, \bar{\alpha}) = 1 - \sum_{p,q=0}^{\infty} (-1)^{p+q} \frac{1}{p!q!} \sum_{\substack{\mathbf{x}_1, \ldots, \mathbf{x}_p \\ \mathbf{y}_1, \ldots, \mathbf{y}_q}} \bar{\alpha}(\mathbf{x}_1) \ldots \bar{\alpha}(\mathbf{x}_n) \, \alpha(\mathbf{y}_1) \ldots \alpha(\mathbf{y}_q)$$

$$\times \frac{\left\langle P(\mathbf{x}_1) \ldots P(\mathbf{x}_p) \, P^\dagger(\mathbf{y}_1) \ldots P^\dagger(\mathbf{y}_q) \right\rangle_c}{N^{2-p-q}} \ , \qquad (3.9)$$

where the (properly normalised) connected expectation values are defined as:

$$\frac{\left\langle P(\mathbf{x}_1) \ldots P(\mathbf{x}_p) \, P^\dagger(\mathbf{y}_1) \ldots P^\dagger(\mathbf{y}_q) \right\rangle_c}{N^{2-p-q}} \equiv$$

$$\equiv (-1)^{p+q+1} \frac{\delta^{p+q}}{\delta\bar{\alpha}(\mathbf{x}_1) \cdots \delta\bar{\alpha}(\mathbf{x}_p) \delta\alpha(\mathbf{y}_1) \cdots \delta\alpha(\mathbf{y}_q)} W(\alpha, \bar{\alpha}) \bigg|_{\alpha=\bar{\alpha}=0} \ . \qquad (3.10)$$

Finally, the Legendre transform of the generator of connected Polyakov correlators yields the generator of 1PI diagrams:

$$z(\mathbf{x}) = \frac{\delta}{\bar{\alpha}(\mathbf{x})} W(\alpha, \bar{\alpha}) = \frac{\left\langle P(\mathbf{x}) \exp\left\{ -(\bar{\alpha} P + \alpha P^\dagger) \right\} \right\rangle}{\left\langle \exp\left\{ -(\bar{\alpha} P + \alpha P^\dagger) \right\} \right\rangle} \ , \qquad (3.11\text{a})$$

$$\Gamma(z, \bar{z}) = W(\alpha, \bar{\alpha}) - (\bar{\alpha} z + \alpha \bar{z}) \ , \qquad (3.11\text{b})$$

where we have omitted the dependence on the spatial coordinates in order to simplify the notation. The functions $z(\bar{\alpha}, \alpha), \bar{z}(\bar{\alpha}, \alpha)$ yield respectively the expectation values of the Polyakov loop and its complex conjugate in the presence of the external sources $\alpha, \bar{\alpha}$. We shall refer to the function $\Gamma(z, \bar{z})$ as the quantum action for the Polyakov loop. In general, it is different from the function $V(z)$; however it can be shown that the two functions coincide for space–independent Polyakov loops in the limit of infinite spatial volume[1].

The center is a symmetry group of the quantized theory if and only if the effective potential $\Gamma(z, \bar{z})$ is invariant[2]. Because of the Legendre transform, the effective potential

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[1]The provided definition of the quantum action and the decomposition of $W(\alpha, \bar{\alpha})$ in connected expectation values are correct as long as the vacuum is non–degenerate. Otherwise the relationship between $(\alpha, \bar{\alpha})$ and $(z, \bar{z})$ is not invertible and the definition of the quantum action is more involved; while a sum over all the vacua is required in the Taylor expansion of $W(\alpha, \bar{\alpha})$. The non-degeneracy condition is satisfied in the strong-coupling phase, in which we perform explicit computation.

[2]In principle, one should consider the effective potential of a generic loop arbitrarily wrapping along the time direction. This can be done using the same formalism developed here. However we shall assume that the Polyakov loop completely characterises the center symmetry.

is not easily computed. However, since $z(u\alpha, \bar{u}\bar{\alpha}) = uz(\alpha, \bar{\alpha})$, we can readily prove that, for any element of the center u:

$$\Gamma(uz, \bar{u}\bar{z}) = W(u\alpha, \bar{u}\bar{\alpha}) - \bar{\alpha}z(\alpha, \bar{\alpha}) + \alpha\bar{z}(\alpha, \bar{\alpha}) , \qquad (3.12)$$

and the following implications hold:

$$\text{center is a symmetry} \Leftrightarrow \Gamma(uz, \bar{u}\bar{z}) = \Gamma(z, \bar{z}) \Leftrightarrow W(u\alpha, \bar{u}\bar{\alpha}) = W(\alpha, \bar{\alpha}) . \qquad (3.13)$$

Therefore, the functional $W(\alpha, \bar{\alpha})$ encodes all the information that we need about the center symmetry. The terms of the expansion in Eq. (3.9) are classified in Tab. 1 according to their symmetry properties.

Term in W	Symmetry
$p = q \mod 2$	Z_2
$p = q \mod N$	Z_N
$p = q$	U(1)

Table 1: Symmetry properties of the Polyakov connected correlators $\langle (P)^p (P^\dagger)^q \rangle_c$. We say that G is a symmetry group for the Polyakov connected correlator if the latter is invariant under the transformation $\Omega(\mathbf{x}) \to u\Omega(\mathbf{x})$, with $u \in G$. Note that G is not necessarily a subgroup of the center.

The connected correlators can be computed analytically for the pure gauge theory in the strong coupling regime, where a topological classification of diagrams can be setup which closely follows the one obtained in the continuum by 't Hooft [15, 16]. The details relevant for our computations are summarised in App. A. The large–N behaviours of the correlators are:

$$\frac{\langle (P)^p (P^\dagger)^q \rangle_{c,\text{YM}}}{N^{2-p-q}} = O(1) , \qquad \text{if } p = q ,$$

$$\frac{\langle (P)^p (P^\dagger)^q \rangle_{c,\text{YM}}}{N^{2-p-q}} = O\left(\frac{1}{N}\right) , \qquad \text{if } p = q \mod N , \quad p \neq q ,$$

$$\frac{\langle (P)^p (P^\dagger)^q \rangle_{c,\text{YM}}}{N^{2-p-q}} = 0 , \qquad \text{if } p \neq q \mod N . \qquad (3.14)$$

4. The connected graphs in the large–mass expansion

Let us now discuss the introduction of fermions in the theory. Since we are interested in the orientifold planar equiavalence, we shall consider fermions in arbitrary representations. The fermionic effective action can be written as a sum of Wilson loops in the large–mass phase [17]:

$$S_f(U_k, \Omega) = -\sum_{\omega \in \mathcal{C}} c(\omega) \, \text{tr } R[U(\omega)] , \qquad (4.1)$$

where ω indicates a generic closed path on the lattice of length $L(\omega)$, $U(\omega)$ is the parallel transport along ω, and $R[U]$ is the matrix representing U in the representation R. The coefficients $c(\omega)$ depend on the spin structure of the chosen discretisation of the Dirac

operator and are independent of the colour structure. They are of order of $m^{-L(\omega)}$, where m is the bare fermion mass. It will be convenient to split the set $\mathcal{C}$ of all the closed paths into the union of the sets $\mathcal{C}(w)$ of closed paths with winding number w around the thermal dimension.

The connected expectation value of Polyakov loops can be formally thought as a function of the coefficients $c(\omega)$. In the large–mass phase, it can be expanded as a power series around $c(\omega) = 0$:

$$\frac{1}{N^{2-p-q}} \langle (P)^p (P^\dagger)^q \rangle_c = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\omega_1 \ldots \omega_n} \frac{c(\omega_1) \cdots c(\omega_n)}{N^{2-p-q}} \left. \frac{d^n \langle (P)^p (P^\dagger)^q \rangle_c}{dc(\omega_1) \cdots dc(\omega_n)} \right|_{S_f=0} . \qquad (4.2)$$

Each derivative with respect to $c(\omega)$ corresponds to the insertion of a vertex $\mathrm{tr}\, R[U(\omega)]$ in the expectation value. Since the derivative must be computed at $S_f = 0$, each term in the expansion above can be written as combinations of connected expectation values in the pure Yang–Mills, as shown in some explicit examples later on. Therefore the asymptotic behaviour of the connected expectation values in the theory with dynamical fermions can be reconstructed from the asymptotic behaviour of the connected expectation values in the pure Yang–Mills.

Let $\omega_1, \ldots \omega_n$ be some closed paths and $w_1, \ldots w_n$ be their respective winding numbers around the thermal direction. Consider the generic normalised connected expectation value in the pure Yang–Mills theory:

$$\frac{\langle \prod_i \mathrm{tr}\, U(\omega_i) \rangle_{c,\mathrm{YM}}}{N^{2-n}} . \qquad (4.3)$$

This quantity is always finite in the large–N limit (it could be also 0 and in this case we say that the leading order is not saturated). This fact is proved in the strong–coupling, and in the perturbative expansion, and it is consistent with results from numerical simulations. Moreover, it is equivalent to the N^2 scaling of the free energy of the system, when external sources of order N are coupled to the operators $\mathrm{tr}\, U(\omega_i)$ in analogy to the definition (3.8), and consequently it is equivalent to the N^2 scaling of the quantum action for the expectation values of the operators $\mathrm{tr}\, U(\omega_i)/N$. The following cases can arise:

- $\sum_i w_i \neq 0 \mod N$. The expression in Eq. (4.3) transforms non–trivially under center transformations. Since the center symmetry is not broken in the confined phase, the expression is exactly zero for every value of N.

- $\sum_i w_i = 0 \mod N$ but $\sum_i w_i \neq 0$ (see Figs. 2, 3). The expression in Eq. (4.3) is invariant under Z_N but not U(1). Using the strong–coupling expansion (see App. A) it is possible to prove that the correlator vanishes in the large–N limit. This result can be extended beyond the strong–coupling phase, under the commonly accepted assumption that the U(N) and SU(N) Yang–Mills theories differ in the large–N limit only for subleading contributions. If the gauge group were U(N), the expression in Eq. (4.3) would be exactly zero for every value of N. Since we are interested in the SU(N) gauge group, it must be zero in the large–N limit.

- The loops with winding number different from zero appear in pairs with opposite winding number, as shown in Fig. 1. In particular $\sum_i w_i = 0$. From the strong–coupling expansion (see App. A) we get that the expression in Eq. (4.3) in this case is different from zero in the large–N limit. We will assume that this result extends beyond the strong–coupling phase.

- $\sum_i w_i = 0$ but some of the loops with winding number $w \neq 0$ cannot be paired with a loop with winding number $-w$. Although the expression in Eq. (4.3) is invariant under the center (both Z_N and U(1)), it vanishes in the strong–coupling phase (see App. A) because of the topology of the torus. We cannot say if this result extends beyond the strong–coupling expansion, but anyway we will not use it in this work.
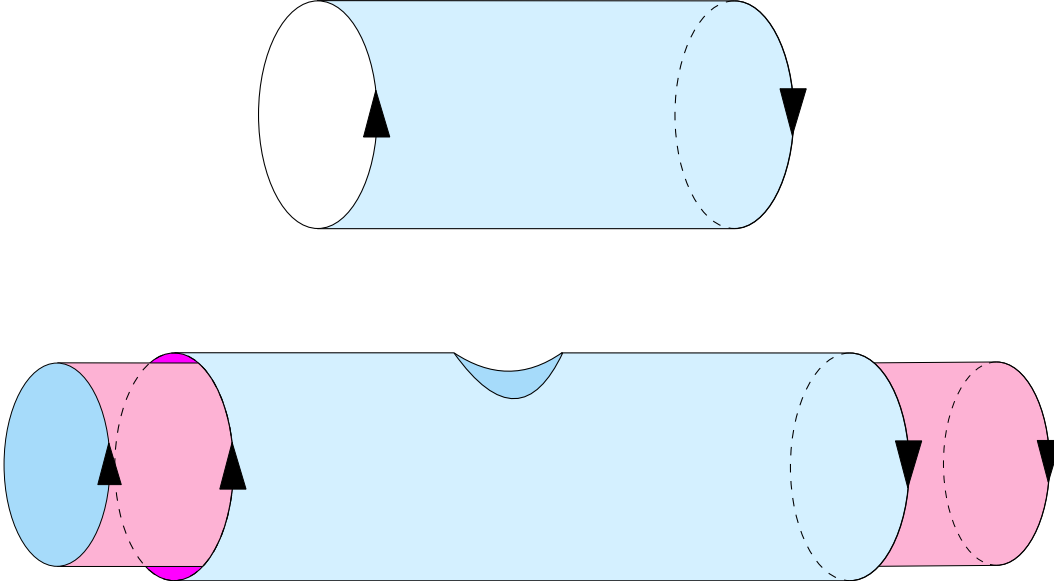

Figure 1: Representations of some connected expectation values of products of loops with net winding number equal to zero, in pure Yang–Mills. The first one is $\langle \operatorname{tr} \Omega \operatorname{tr} \Omega^\dagger \rangle_{c,\mathrm{YM}}$; the Polyakov loops are connected by an oriented surface that wraps around the thermal dimension. In the strong–coupling expansion, the surface is tiled by plaquettes coming from the Wilson action; products of two (or more) group elements associated to each link of the lattice are integrated with respect to the Haar measure. This graph yields a contribution $O(N^0)$. The second graph represents $\langle (\operatorname{tr} \Omega \operatorname{tr} \Omega^\dagger)^2 \rangle_{c,\mathrm{YM}}$; the two tubes are glued together through a hole. This graph is $O(N^{-2})$.

It is worth to remind that in the theories with fermions the expansion of the fermionic effective action in Eq. (4.2) is convergent only in the limit where the bare fermion mass is large, and therefore yields only limited information on the continuum limit of the lattice theory. It provides nonetheless a framework where analytical calculations can be performed and the planar equivalence can be tested explicitly. Note that for fermions in the adjoint representation, the full theory is still invariant under the center group Z_N and the large–N behaviour of the connected correlators is the same as the one we obtained above for the pure gauge theory.
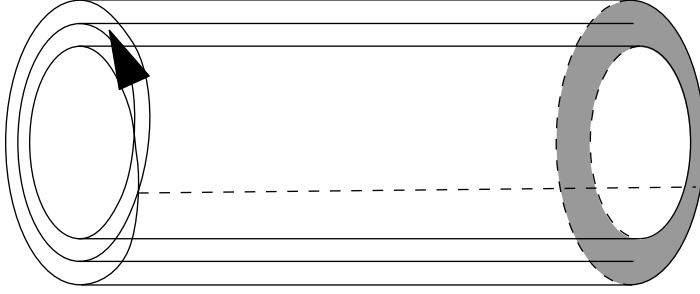
Figure 2: Graphical representation of $\langle \mathrm{tr}\,(\Omega^N)\rangle_{c,\mathrm{YM}}$ for $N = 3$. The surface wraps N times around the thermal direction and ends in the gray area, which represents the integration of the product of N group elements, all oriented in the same direction. This integration yields a non–zero value, because the product of N fundamental representations of $\mathrm{SU}(N)$ contains a singlet, given by the contraction with the completely skew–symmetric tensor. This graph yields a contribution $O(N^0)$.
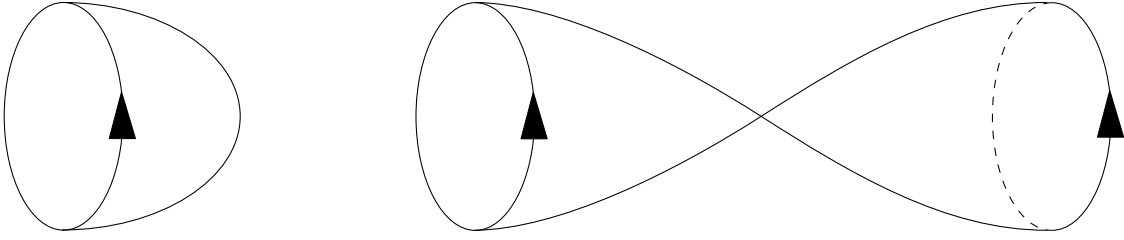

Figure 3: These graphs are the would–be leading contributions of respectively $\langle \mathrm{tr}\,\Omega\rangle_{\mathrm{YM}}$ and $\langle (\mathrm{tr}\,\Omega)^2\rangle_{c,\mathrm{YM}}$. They are forbidden by the topology of the thermal dimension.

4.1 Fundamental fermions

When fermions in the fundamental representation are considered, the generic term of the r.h.s. of the Eq. (4.2) is:

$$\frac{1}{N^{2-p-q}}\frac{d^n\langle (P)^p(P^\dagger)^q\rangle_{c,\mathrm{F}}}{dc(\omega_1)\cdots dc(\omega_n)}\bigg|_{S_f=0} = \frac{\langle (P)^p(P^\dagger)^q\,\mathrm{tr}\,U(\omega_1)\ldots\mathrm{tr}\,U(\omega_n)\rangle_{c,\mathrm{YM}}}{N^{2-p-q}}\ . \tag{4.4}$$

The expectation value that appears in the numerator of the r.h.s. of Eq. (4.4) is at most of order $N^{2-p-q-n}$, since it contains $p + q + n$ loops. Thus for $n \neq 0$, the whole term is at most of order N^{-1}. Now, if $p = q$ the Yang–Mills contribution $(n = 0)$ is $O(1)$ and dominates the sum. Otherwise, the leading contribution is at most of order N^{-1}:

$$\frac{\langle (P)^p(P^\dagger)^q\rangle_{c,\mathrm{F}}}{N^{2-p-q}} \simeq \frac{\langle (P)^p(P^\dagger)^q\rangle_{c,\mathrm{YM}}}{N^{2-p-q}} = O(1)\ , \qquad \text{if } p = q$$

$$\frac{\langle (P)^p(P^\dagger)^q\rangle_{c,\mathrm{F}}}{N^{2-p-q}} = O\left(\frac{1}{N}\right)\ , \qquad \text{if } p \neq q\ . \tag{4.5}$$

For $p \neq q$, the $1/N$ scaling is only an upper limit for the asymptotic behaviour. However in the case of the expectation value of the Polyakov loop, the $O(N^{-1})$ is saturated

by the contribution:

$$\frac{\langle P \rangle}{N} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\omega_1 \dots \omega_n} \frac{c(\omega_1) \cdots c(\omega_n)}{N^{2-p-q}} \left. \frac{d^n \langle P \rangle_c}{dc(\omega_1) \cdots dc(\omega_n)} \right|_{S_f=0} =$$

$$= \frac{1}{N} \sum_{\omega \in \mathcal{C}(-1)} c(\omega) \langle P \text{ tr } U(\omega) \rangle_{c,\text{YM}} + O\left(\frac{1}{N^2}\right) . \tag{4.6}$$

We recall that $\mathcal{C}(-1)$ is the set of the closed paths on the lattice with winding number equal to -1 around the thermal dimension.

From the large–N behaviour of the coefficients of the Taylor expansion, we see that, in the large–N limit, the functional W is the same for Yang–Mills and for the theory with fundamental quarks:

$$\lim_{N \to \infty} W_F(\alpha, \bar{\alpha}) = \lim_{N \to \infty} W_{YM}(\alpha, \bar{\alpha}) =$$

$$= 1 - \sum_{p=0}^{\infty} \frac{|\alpha|^{2p}}{(p!2^p)^2} \lim_{N \to \infty} \frac{\langle |\text{tr } \Omega|^{2p} \rangle_c}{N^{2-2p}} . \tag{4.7}$$

This is the manifestation in this particular sector of the theory of the usual subleading contribution from the fermion determinant when fermions are in the fundamental representation.

4.2 S/AS fermions

Again, in the generic term of the r.h.s. of the Eq. (4.2):

$$\frac{1}{N^{2-p-q}} \left. \frac{d^n \langle (P)^p (P^\dagger)^q \rangle_{c,\text{S/AS}}}{dc(\omega_1) \cdots dc(\omega_n)} \right|_{S_f=0} , \tag{4.8}$$

each derivative in $c(\omega)$ produces the insertion of a vertex $\text{tr } R[U(\omega)]$. When the fermions are in the (anti)symmetric representation, we can use the algebraic relationships:

$$\text{tr S/AS}[U] = \frac{(\text{tr } U)^2 \pm \text{tr } (U^2)}{2} . \tag{4.9}$$

In general, the insertion of some term of the form $(\text{tr } U)^2/2$ will disconnect the expectation value $\langle (P)^p (P^\dagger)^q \rangle_{c,\text{S/AS}}$. In order to gain some confidence with these computations, we report in the following subsections the explicit computation of the leading orders of $\langle (P)^2 \rangle_{c,\text{S/AS}}$, $\langle P P^\dagger \rangle_{c,\text{S/AS}}$ and $\langle P \rangle_{c,\text{S/AS}}$. The end of the section is devoted to a discussion of the general case.

Before entering into the details of the computations, we summarise here the results:

$$\frac{\langle (P)^p (P^\dagger)^q \rangle_{c,\text{S/AS}}}{N^{2-p-q}} = O(1) , \qquad \text{if } p - q \text{ even },$$

$$\frac{\langle (P)^p (P^\dagger)^q \rangle_{c,\text{S/AS}}}{N^{2-p-q}} = O\left(\frac{1}{N}\right) , \qquad \text{if } p - q \text{ odd and } N \text{ odd },$$

$$\frac{\langle (P)^p (P^\dagger)^q \rangle_{c,\text{S/AS}}}{N^{2-p-q}} = 0 , \qquad \text{if } p - q \text{ odd and } N \text{ even }. \tag{4.10}$$

Hence the large–N limit for the functional W yields:

$$\lim_{N\to\infty} W_{S/AS}(\alpha, \bar{\alpha}) =$$

$$= 1 - \sum_{\substack{p,q=0 \\ p-q \text{ even}}}^{\infty} \frac{1}{p!q!} \bar{\alpha}^p \alpha^q \lim_{N\to\infty} \frac{\langle (P)^p (P^\dagger)^q \rangle_{c,S/AS}}{N^{2-p-q}} . \tag{4.11}$$

Since all the terms with even $(p-q)$ contribute to the sum, we get

$$W_{S/AS}(\alpha, \bar{\alpha}) = W_{S/AS}(u\alpha, \bar{u}\bar{\alpha})$$

in the planar limit if and only if $u = \pm 1$. Therefore a Z_2 symmetry is recovered in the large–N limit.

4.2.1 Computation of $\langle P \rangle_{S/AS}$

Let us start with the single–derivative term in the expansion (4.2):

$$\frac{1}{N} \frac{d\langle P \rangle_{S/AS}}{dc(\omega)}\bigg|_{S_f=0} = \frac{1}{N} \left\{ \langle P \text{ tr } R[U(\omega)] \rangle_{YM} - \langle P \rangle_{YM} \langle \text{tr } R[U(\omega)] \rangle_{YM} \right\} =$$

$$= \frac{1}{2N} \left\{ \langle P [\text{tr } U(\omega)]^2 \rangle_{YM} \pm \langle P \text{ tr } [U(\omega)^2] \rangle_{YM} \right\} =$$

$$= \frac{1}{2N} \left\{ \langle P [\text{tr } U(\omega)]^2 \rangle_{c,YM} + 2\langle P \text{ tr } U(\omega) \rangle_{c,YM} \langle \text{tr } U(\omega) \rangle_{c,YM} \right.$$

$$\left. \pm \langle P \text{ tr } [U(\omega)^2] \rangle_{c,YM} \right\} . \tag{4.12}$$

Let w be the winding number around the thermal dimension of the closed path ω. We shall analyse each term in turn. The term $\langle P [\text{tr } U(\omega)]^2 \rangle_{c,YM}$ saturates its N^{-1} behaviour only if $1+2w = 0$, corresponding to correlators that are invariant under the action of U(1). But this equation has no integer solution, therefore the highest order is never saturated. The only non–vanishing contributions come from loops that satisfy $1+2w = kN$ (both k and N must be odd). In this case, the first term goes like N^{-2}.

The second term $\langle P \text{ tr } U(\omega) \rangle_{c,YM} \langle \text{tr } U(\omega) \rangle_{c,YM}$ saturates its N asymptotic behaviour only if each factor is separately invariant under the action of U(1). This corresponds to the equations $1 + w = 0$ and $w = 0$. Again, this equation admits no solution and the leading order is not saturated. We can look for subleading contributions. Since we want $\langle \text{tr } U(\omega) \rangle_{c,YM}$ not vanishing, it must be $w = kN$. Requiring that $\langle P \text{ tr } U(\omega) \rangle_{c,YM}$ is not vanishing, we get $1 + w = 1 + kN = k'N$ which implies $(k'-k)N = 1$. This equation has no solution, and therefore the second term is always zero.

The condition for the last term $\langle P \text{ tr } [U(\omega)^2] \rangle_{c,YM}$ to saturate its N^0 behaviour is again $1 + 2w = 0$. Looking for subleading contributions, we have to request $1 + 2w = kN$. This equation admits a solution if both k and N are odd, as for the first term. But unlike the first term, the last one goes like N^{-1}.

Putting all these contributions together, the single–derivative in Eq. (4.2) is of order N^{-2} and this behaviour is summarised by the formula:

$$\frac{1}{N} \frac{d\langle P \rangle_{S/AS}}{dc(\omega)}\bigg|_{S_f=0} = \pm \frac{1}{2N} \langle P \text{ tr } [U(\omega)^2] \rangle_{c,YM} + O\left(\frac{1}{N^3}\right) . \tag{4.13}$$

Consider now a generic term in the expansion (4.2). We will not write the explicit expression, but it should be clear that it can be written as a sum of products of connected expectation values in the pure Yang–Mills theory. Let w_i be the winding number of the path ω_i. Only products that are invariant under the action of U(1) saturate the leading contribution; hence the sum of all the winding numbers must vanish:

$$1 + 2 \sum_i w_i = 0 \ . \tag{4.14}$$

The factor 2 is due to the two–index representation. For each closed path ω_i with winding number w_i, a loop tr $R[U(\omega)]$ with N–ality equal to $2w_i$ is inserted. The equation above has no solution, therefore the leading order for $\langle P \rangle_{c,\mathrm{S/AS}}$ is not saturated. This argument lets us to conclude that:

$$\lim_{N \to \infty} \frac{\langle P \rangle_{\mathrm{S/AS}}}{N} = 0 \ , \tag{4.15}$$

but we still cannot say if it is a $O(1/N)$ or rather a $O(1/N^2)$ as the single–derivative term suggests. Since this is not essential to our discussion of the center symmetry, we state simply the result as:

$$\frac{\langle P \rangle}{N} = \pm \frac{1}{N} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k \ \mathrm{odd}} \sum_{\bar{\omega} \in \mathcal{C}\left(\frac{kN-1}{2}\right)} \sum_{\omega_1 \dots \omega_n}$$
$$c(\bar{\omega})c(\omega_1) \cdots c(\omega_n) \left. \frac{d^n \langle P \mathrm{tr}\ [U(\bar{\omega})^2] \rangle_c}{dc(\omega_1) \cdots dc(\omega_n)} \right|_{S_f=0} + O\left(\frac{1}{N^3}\right) \ , \tag{4.16}$$

which means that $\langle P \rangle / N$ is of order $O(1/N^2)$.

4.2.2 Computation of $\langle (P)^2 \rangle_{c,\mathrm{S/AS}}$

We want to check here that the leading term of $\langle (P)^2 \rangle_{c,\mathrm{S/AS}}$, which is expected to be N^0, is actually different from zero. This can be easily shown by computing the single–derivative term in the expansion Eq. (4.2). Starting from the definition of the connected correlator: $\langle P^2 \rangle_{c,\mathrm{S/AS}} = \langle P^2 \rangle_{\mathrm{S/AS}} - \langle P \rangle_{\mathrm{S/AS}}^2$, and assuming that ω is a generic closed path with winding number -1 around the thermal dimension, we can rewrite the single–derivative coefficient of the expansion of $\langle P^2 \rangle_{c,\mathrm{S/AS}}$ as:

$$\left. \frac{d\langle P^2 \rangle_{c,\mathrm{S/AS}}}{dc(\omega)} \right|_{S_f=0} = \langle P^2 R[U(\omega)] \rangle_{\mathrm{YM}} - \langle P^2 \rangle_{\mathrm{YM}} \langle R[U(\omega)] \rangle_{\mathrm{YM}} =$$
$$= \frac{1}{2} \langle P^2 [\mathrm{tr}\ U(\omega)]^2 \rangle_{\mathrm{YM}} =$$
$$= \frac{1}{2} \left\{ \langle P^2 [\mathrm{tr}\ U(\omega)]^2 \rangle_{c,\mathrm{YM}} + 2 \langle P\ \mathrm{tr}\ U(\omega) \rangle_{c,\mathrm{YM}} \langle P\ \mathrm{tr}\ U(\omega) \rangle_{c,\mathrm{YM}} \right\} =$$
$$= \langle P\ \mathrm{tr}\ U(\omega) \rangle_{c,\mathrm{YM}} \langle P\ \mathrm{tr}\ U(\omega) \rangle_{c,\mathrm{YM}} + O\left(\frac{1}{N^2}\right) \ , \tag{4.17}$$

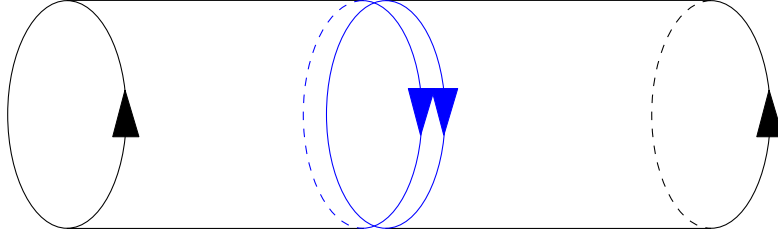as illustrated in Fig. 4. This is exactly a $O(N^0)$ term.

Figure 4: This graph represents a leading contribution of $\langle (\mathrm{tr}\,\Omega)^2 \rangle_{c,\mathrm{S/AS}}$. The pair of blue loops represent the insertion of a vertex $\mathrm{tr}\,R[U(\omega)]$, due to the derivative with respect to $c(\omega)$.

4.2.3 Computation of $\langle PP^\dagger \rangle_{c,\mathrm{S/AS}}$

This is the easiest example: since $PP^\dagger$ is invariant under the action of U(1), the zero–derivative term $\langle PP^\dagger \rangle_{c,\mathrm{YM}}$ in the expansion (4.2) trivially saturates the N^0 behaviour of $\langle PP^\dagger \rangle_{c,\mathrm{S/AS}}$.

4.2.4 The generic $\langle P^p (P^\dagger)^q \rangle_{c,\mathrm{S/AS}}$

As already discussed in the case of $\langle P \rangle_{c,\mathrm{S/AS}}$, each term in the expansion (4.2) of the generic connected expectation value can be written as a sum of products of connected expectation values with respect the YM vacuum. Each of these connected expectation values contains some P's and $P^\dagger$'s from the original expectation value, and some $\mathrm{tr}\,U(\omega)$ from the derivatives with respect to the coefficient $c(\omega)$. When the derivative with respect to $c(\omega_i)$ is computed, a $[\mathrm{tr}\,U(\omega_i)]^2$ or a $\mathrm{tr}\,[U(\omega_i)^2]$ is inserted. In both cases, if w_i is the winding number of the path ω_i, a parallel transport with winding number $2w_i$ is inserted. A necessary condition for the leading order to be saturated is that the overall winding number, that is the sum of all the winding numbers of the operators involved, must be zero (this corresponds to the invariance under action of U(1)):

$$p - q + 2\sum_i w_i = 0 \ . \tag{4.18}$$

This argument implies that $p - q$ must be even. If it is odd, the leading behaviour cannot be saturated.

Now we want to show that this is also a sufficient condition, by explicitly constructing a term that saturates the leading behaviour. The $p = q$ case is trivial since the leading behaviour is saturated by the pure YM term:

$$\frac{\langle (P)^p (P^\dagger)^p \rangle_{c,\mathrm{S/AS}}}{N^{2-2p}} = \frac{\langle (P)^p (P^\dagger)^p \rangle_{c,\mathrm{YM}}}{N^{2-2p}} + \dots \tag{4.19}$$

Consider now the $p > q$ case (the opposite can be obtained by charge–conjugation). If ω is a path with winding number -1, a leading contribution comes for instance from (see

fig. 5):

$$\frac{1}{N^{2-p-q}} \frac{d^n \langle (P)^p (P^\dagger)^q \rangle_c}{dc(\omega)^n}\bigg|_{S_f=0} =$$

$$= \frac{p!(p-q)!}{\left(\frac{p+q}{2}\right)! \left[\left(\frac{p-q}{2}\right)!\right]^2} \frac{\langle (P)^{\frac{p+q}{2}} [\text{tr } U(\omega)]^{\frac{p-q}{2}} (P^\dagger)^q \rangle_{c,\text{YM}}}{N^{2-p-q}} (\langle P \text{ tr } U(\omega) \rangle_{c,\text{YM}})^{\frac{p-q}{2}} + \dots$$
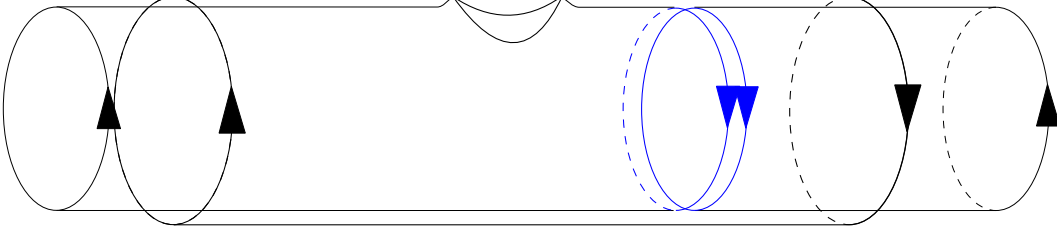
$$\text{(4.20)}$$

Figure 5: This graph is a leading contribution of $\langle (\text{tr } \Omega)^3 \text{tr } \Omega^\dagger \rangle_{c,\text{S/AS}}$. The pair of blue loops represent the insertion of a vertex $\text{tr } R[U(\omega)]$, due to the derivative with respect to $c(\omega)$.

5. The effective potentials in the large–mass expansion

The analysis of the center symmetry of the effective potential can be deduced entirely from the results presented in the previous section. Nonetheless, it is interesting to compare those results with the prediction in Ref. [18] about the effective potential of the Polyakov loop, bearing in mind that in Ref. [18] only the lowest–dimensional terms (relevant operators) are kept in the effective potential for the Polyakov loop $\Gamma(z, \bar{z})$. They can be summarised as follows.

Pure gauge or adjoint fermions.

$$\Gamma(z, \bar{z}) = a_2 z\bar{z} + a_4 (z\bar{z})^2 + a_N (z^N + \bar{z}^N) \tag{5.1}$$

Fundamental fermions.

$$\Gamma(z, \bar{z}) = b_1 (z + \bar{z}) + a_2 z\bar{z} + a_4 (z\bar{z})^2 + a_N (z^N + \bar{z}^N) \tag{5.2}$$

S/AS fermions with N even.

$$\Gamma(z, \bar{z}) = b_2 (z^2 + \bar{z}^2) + a_2 z\bar{z} + a_4 (z\bar{z})^2 + a_N (z^N + \bar{z}^N) \tag{5.3}$$

S/AS fermions with N odd.

$$\Gamma(z, \bar{z}) = b_1 (z + \bar{z}) + b_2 (z^2 + \bar{z}^2) + a_2 z\bar{z} + a_4 (z\bar{z})^2 + a_N (z^N + \bar{z}^N) \tag{5.4}$$

In the large–N limit not all the coefficients survive. In particular:

$$\lim_{N \to \infty} a_N(N) = 0 \;, \tag{5.5}$$

for all the theories; moreover, in the case of fundamental fermions:

$$\lim_{N \to \infty} b_1(N) = 0 \;, \tag{5.6}$$

while in the case of S/AS fermions:

$$\lim_{N \to \infty} b_1(N) = 0 \;, \qquad\qquad \lim_{N \to \infty} b_2(N) \neq 0 \;, \tag{5.7}$$

As shown in the previous Section, this behaviour is constrained by the symmetry properties of each term in the potential. It is nonetheless intersting to check explicitly the last two equations.

5.1 The b_1 coefficient

Since $\alpha(z, \bar{z}) = -2\bar{\partial}_z \Gamma(z, \bar{z})$, from the Taylor expansion of the effective potential:

$$b_1 = \partial_z \Gamma(0,0) = -\frac{1}{2}\alpha_0 \;, \tag{5.8}$$

where α_0 is the real source, defined by the condition $z(\alpha_0, \alpha_0) = \bar{z}(\alpha_0, \alpha_0) = 0$, which is the same as:

$$(\partial_\alpha + \bar{\partial}_\alpha)W(\alpha_0, \alpha_0) = 0 \;. \tag{5.9}$$

Using the series expansion of $W(\alpha, \bar{\alpha})$, the above condition becomes:

$$\frac{\langle \mathrm{tr}\ \Omega \rangle}{N} - \alpha_0 \sum_{n=0}^{\infty} \frac{(-\alpha_0)^n}{(n+1)!} \frac{\langle (\mathrm{Re\, tr}\ \Omega)^{n+2} \rangle_c}{N^{-n}} = 0 \;. \tag{5.10}$$

Since $\frac{\langle \mathrm{tr}\ \Omega \rangle}{N}$ vanishes in the large–N limit, the equation can be solved iteratively. At the leading order:

$$\alpha_0 = \frac{\langle \mathrm{tr}\ \Omega \rangle}{N \langle (\mathrm{Re\, tr}\ \Omega)^2 \rangle_c} + \text{ subleadings }\;. \tag{5.11}$$

Expanding the denominator, the b_1 coefficient is:

$$b_1 = -\frac{\langle \mathrm{tr}\ \Omega \rangle}{N \left(\langle |\mathrm{tr}\ \Omega|^2 \rangle_c + \langle (\mathrm{tr}\ \Omega)^2 \rangle_c \right)} + \text{ subleadings }\;, \tag{5.12}$$

and it vanishes in the large–N limit, with both fundamental and S/AS fermions.

5.2 The b_2 coefficient

The second order in the Taylor expansion of the effective potential is obtained by inverting the Hessian matrix of the functional W:

$$\begin{pmatrix} 2b_2 & a_2 \\ a_2 & 2b_2 \end{pmatrix}(0) = -\frac{1}{4}\begin{pmatrix} \bar{\partial}_\alpha^2 W & \partial_\alpha\bar{\partial}_\alpha W \\ \partial_\alpha\bar{\partial}_\alpha W & \partial_\alpha^2 W \end{pmatrix}^{-1}(\alpha_0) \; . \tag{5.13}$$

The entries of the Hessian matrix in the large–N limit are:

$$\begin{aligned}
\partial_\alpha^2 W(\alpha_0) = & -\frac{1}{4}\frac{\langle (\mathrm{tr}\ \Omega)^2\ \exp\{-N\alpha_0\mathrm{Retr}\ \Omega\}\rangle}{\langle \exp\{-N\alpha_0\mathrm{Retr}\ \Omega\}\rangle} + \\
& + \frac{1}{4}\left(\frac{\langle \mathrm{tr}\ \Omega\ \exp\{-N\alpha_0\mathrm{Retr}\ \Omega\}\rangle}{\langle \exp\{-N\alpha_0\mathrm{Retr}\ \Omega\}\rangle}\right)^2 = \\
= & -\frac{1}{4}\langle (\mathrm{tr}\ \Omega)^2\rangle_c + \dots
\end{aligned} \tag{5.14}$$

$$\begin{aligned}
\partial_\alpha\bar{\partial}_\alpha W(\alpha_0) = & -\frac{1}{4}\frac{\langle |\mathrm{tr}\ \Omega|^2\ \exp\{-N\alpha_0\mathrm{Retr}\ \Omega\}\rangle}{\langle \exp\{-N\alpha_0\mathrm{Retr}\ \Omega\}\rangle} + \\
& + \frac{1}{4}\left|\frac{\langle \mathrm{tr}\ \Omega\ \exp\{-N\alpha_0\mathrm{Retr}\ \Omega\}\rangle}{\langle \exp\{-N\alpha_0\mathrm{Retr}\ \Omega\}\rangle}\right|^2 = \\
= & -\frac{1}{4}\langle |\mathrm{tr}\ \Omega|^2\rangle_c + \dots
\end{aligned} \tag{5.15}$$

Computing the inverse of the Hessian yields:

$$a_2 = -\frac{\langle |\mathrm{tr}\ \Omega|^2\rangle_c}{[\langle (\mathrm{tr}\ \Omega)^2\rangle_c]^2 - [\langle |\mathrm{tr}\ \Omega|^2\rangle_c]^2} \; , \tag{5.16}$$

$$b_2 = \frac{\langle (\mathrm{tr}\ \Omega)^2\rangle_c}{2\,[\langle (\mathrm{tr}\ \Omega)^2\rangle_c]^2 - [\langle |\mathrm{tr}\ \Omega|^2\rangle_c]^2} \; . \tag{5.17}$$

In particular, the a_2 coefficient is always of order 1; the b_2 coefficient is of order N^{-1} for fundamental fermions, while it is of order 1 for S/AS fermions.

6. The effective potential in the deconfined phase

We consider here the high–temperature regime. In other words, the space is taken to be $\mathbb{R}^3 \times S_1$, where the extension L of the compact dimension is much smaller than $\Lambda_{\mathrm{QCD}}^{-1}$. Antiperiodic boundary conditions conditions are imposed for the fermion fields, so that the path integral can be interpreted as the partition function for a quantum field theory at finite temperature.

If $e^{iv_1},\dots e^{iv_N}$ are the eigenvalues of the parallel transport Ω around the compact dimension, the effective potential for $v_1,\dots v_N$ can be computed in the one–loop approximation. We refer to Ref. [19] for the details of the computation. At fixed N, the effective

potential for respectively OrientiQCD and AdjQCD are:

$$V_{\text{Or}}(v) = \frac{1}{L^4} \left[\sum_{i,j=1}^{N} f(0, v_i - v_j) - 2N_f \sum_{i<j=1}^{N} f(mL, v_i + v_j + \pi) \right] , \qquad (6.1)$$

$$V_{\text{Adj}}(v) = \frac{1}{L^4} \left[\sum_{i,j=1}^{N} f(0, v_i - v_j) - N_f \sum_{i,j=1}^{N} f(mL, v_i - v_j + \pi) \right] , \qquad (6.2)$$

where m is the mass of the fermions and the function f is defined in terms of the modified Bessel function K_2:

$$f(x, v) = \frac{1}{\pi^2} \sum_{g=1}^{\infty} \frac{2 + (gx)^2 K_2(gx) \cos(gv)}{g^4} . \qquad (6.3)$$

In the large–N limit, the function $v(i/N) = v_i$ can be defined. It is related to the eigenvalue density $\rho(v(x)) = [v'(x)]^{-1}$. Thus, in the large–$N$ limit, the effective potential is a functional of $v(x)$:

$$\frac{V_{\text{Or}}(v)}{N^2} = \frac{1}{L^4} \int_0^1 [f(0, v(x) - v(y)) - N_f f(mL, v(x) + v(y) + \pi)] \; dxdy , \qquad (6.4)$$

$$\frac{V_{\text{Adj}}(v)}{N^2} = \frac{1}{L^4} \int_0^1 [f(0, v(x) - v(y)) - N_f f(mL, v(x) - v(y) + \pi)] \; dxdy . \qquad (6.5)$$

In the large–N limit, the center Z_N acts on the function $v(x)$ as:

$$v(x) \to v(x) + \alpha . \qquad (6.6)$$

Clearly, the effective potential for the AdjQCD is invariant under the full center action. Instead, the effective potential for OrientiQCD transform as:

$$\frac{V_{\text{Or}}(v)}{N^2} \to \frac{1}{L^4} \int_0^1 [f(0, v(x) - v(y)) - N_f f(mL, v(x) + v(y) + 2\alpha + \pi)] \; dxdy . \qquad (6.7)$$

Since $f(x, v)$ has a 2π period in v, the effective potential is invariant if and only if $\alpha = k\pi$.

This fact has consequences on the vacuum structure of the two theories. Since in both theories, the terms depending on $v(x) - v(y)$ generate attraction between the eigenvalues, the minima are characterised by a $v(x) = \bar{v}$ constant. In the AdjQCD $\bar{v}$ can take any value, instead in the OrientiQCD the fermionic term generates only two minima at $\bar{v} = 0, \pi$, which correspond to $P = \pm 1$.

Let us now discuss the implications for the orientifold planar equivalence. The charge conjugation symmetry acts on the gauge fields as:

$$A_\mu(x_0, \mathbf{x}) \to -A_\mu^T(x_0, \mathbf{x}) . \qquad (6.8)$$

The vacua of the OrientiQCD are invariant under charge conjugation. Instead, only two of the infinite vacua of AdjQCD are invariant under charge conjugation. These are precisely the two corresponding to $P = \pm 1$, and the orientifold planar equivalent must be valid

in this two vacua. It can be objected that, since all the vacua of AdjQCD are unitarily equivalent, the orientifold planar equivalence should be valid in all the vacua. However the matching between the observables in the two theories is more involved. Consider for instance a Wilson loop $W(\Gamma) = \text{tr } U(\Gamma)/N$ along the closed path Γ, wrapping w times around the compact dimension. Let $\langle W(\Gamma)\rangle_{\alpha,\text{Adj}}$ be its expectation value with respect to the vacua of AdjQCD identified by $\bar{v} = \alpha$, and let $\langle W(\Gamma)\rangle_{\text{Or}}$ be its expectation value with respect to the vacua of OrientiQCD identified by $\bar{v} = 0$. The following equalities hold:

$$\langle W(\Gamma)\rangle_{\alpha,\text{Adj}} = e^{i\alpha w}\langle W(\Gamma)\rangle_{0,\text{Adj}} = e^{i\alpha w}\langle W(\Gamma)\rangle_{\text{Or}} , \tag{6.9}$$

in the planar limit. Therefore the observable $W(\Gamma)$ in the α–vacuum of AdjQCD correspond to the observable $e^{i\alpha w}W(\Gamma)$ in the 0–vacuum of OrientiQCD. In the α–vacuum of AdjQCD an unbroken charge conjugation can be defined, by properly composing the naive one with the center symmetry. On the gauge field, it acts like:

$$A_0(x_0,\mathbf{x}) \rightarrow -A_0^T(x_0,\mathbf{x}) + \frac{2\alpha}{L} , \tag{6.10a}$$

$$A_k(x_0,\mathbf{x}) \rightarrow -A_k^T(x_0,\mathbf{x}) . \tag{6.10b}$$

It acts on the Polyakov loop like:

$$P(\mathbf{x}) = \text{tr Pexp}\left[i\int_0^L A_0(x_0,,\mathbf{x})\ dx_0\right] \rightarrow$$

$$\rightarrow e^{2i\alpha}\text{tr Pexp}\left[i\int_0^L A_0(x_0,,\mathbf{x})\ dx_0\right]^* = e^{2i\alpha}P(\mathbf{x})^* , \tag{6.11}$$

and in particular it does not change the expectation value of the Polyakov loop.

7. The center symmetry in the Hamiltonian formalism

It is instructive to reproduce the results of the previous sections using the Hamiltonian and coherent states formalism. This approach uncovers a deeper picture of the center symmetry, and its remnants in the large–N limit.

The large–N limit of gauge theories via coherent states was introduced in [14]. This formalism was used in [5] to prove orientifold planar equivalence, by defining an orientifold projection on the parent $\text{SO}(N)$ gauge theory with fermions in the adjoint representation. A review of these concepts is beyond the aims of this paper, therefore the reader should refer to the bibliography for details.

The center symmetry is not a symmetry of the Hamiltonian. Indeed the center acts only on the temporal component of the gauge field, which is not a real degree of freedom of the theory: it is the Langrange multiplier for the Gauss constraint. Let us review some of the details here.

If H is the Hamiltonian of the gauge theory on the lattice, the partition function is given by:

$$Z = \text{tr } (e^{-\beta H}\mathbb{P}) , \tag{7.1}$$

where $\mathbb{P}$ is the projector on the gauge–invariant states. If g is an element of the gauge group, let $\mathcal{G}_{\mathbf{x}}[g]$ denote the unitary operator, which acts on the Hilbert space by producing the gauge transformation g in the point $\mathbf{x}$. As a consequence of the invariance of the Haar measure, the projector $\mathbb{P}$ and the partition function can be written as:

$$\mathbb{P} = \int \prod_{\mathbf{x}} \{\mathcal{G}_{\mathbf{x}}[\Omega(\mathbf{x})]d\Omega(\mathbf{x})\} \ , \tag{7.2}$$

$$Z = \int \mathrm{tr} \left(e^{-\beta H} \prod_{\mathbf{x}} \mathcal{G}_{\mathbf{x}}[\Omega(\mathbf{x})] \right) \prod_{\mathbf{x}} d\Omega(\mathbf{x}) \ . \tag{7.3}$$

If $\{\psi_n\}$ is a basis for the Hilbert space, the trace in the integral can be written as:

$$\mathrm{tr} \left(e^{-\beta H} \prod_{\mathbf{x}} \mathcal{G}_{\mathbf{x}}[\Omega(\mathbf{x})] \right) = \sum_n \langle \psi_n | e^{-\beta H} | \psi_n^{(\Omega)} \rangle \ , \tag{7.4}$$

where $\psi_n^{(\Omega)}$ is obtained by applying the gauge transformation Ω to the state ψ_n. From the equation above, $\Omega(\mathbf{x})$ is the $\mathrm{SU}(N)$ phase that the state ψ_n acquires after a translation around the temporal direction. By writing the matrix element of $e^{-\beta H}$ as a functional integral, one sees that $\Omega(\mathbf{x})$ is the parallel transport around the time direction and $\mathrm{tr}\,\Omega(\mathbf{x})$ is the Polyakov loop.

All the potentials of Sect. 5 can be written using the Hamiltonian formalism. For instance, the probability distribution of the parallel transport $\Omega(\mathbf{x})$ is:

$$e^{-S_\Omega(\Omega)} = \frac{1}{Z} \mathrm{tr} \left\{ e^{-\beta H} \prod_{\mathbf{x}} \mathcal{G}_{\mathbf{x}}[\Omega(\mathbf{x})] \right\} \ , \tag{7.5}$$

If u is an element of the center, a center transformation is defined as $\Omega(\mathbf{x}) \rightarrow u\Omega(\mathbf{x})$. It does not affect the degrees of freedom in the Hamiltonian, but only the Gauss constraint:

$$e^{-S_\Omega(u\Omega)} = \frac{1}{Z} \mathrm{tr} \left\{ e^{-\beta H} \prod_{\mathbf{x}} \mathcal{G}_{\mathbf{x}}[u\Omega(\mathbf{x})] \right\} \ . \tag{7.6}$$

When a center symmetry is present, it is not a symmetry for the quantum system in a proper sense: it is not implemented by a unitary operator on the Hilbert space. It is a symmetry only of the potential $S_\Omega(\Omega)$.

This fact implies that an analysis of the center symmetry in the large–N limit using the coherent states formalism cannot be developed in a straightforward way. Indeed, in the coherent state formalism, the Gauss constraint is completely solved by taking only gauge–invariant observables as degrees of freedom. Thus, no analog of the Polyakov loop exists.

This problem can be circumvented as follows. Rotate the system (in Euclidean space–time) and interpret the gauge theory on an infinite space and at finite temperature, as the same theory at zero temperature and on a space $S_1 \times \mathbf{R}^2$ with antiperiodic boundary conditions for the fermions. After the rotation, the Polyakov loop becomes the parallel transport around the spatial compact dimension, and the center acts now on the physical

degrees of freedom of the theory. If Σ is a plane orthogonal to the compact dimension, and $\mathcal{L}^+_\rightarrow(\Sigma)$ is the set of all the positive links departing from sites of Σ and orthogonal to it, the center acts as:

$$U_\ell \rightarrow u U_\ell \quad \text{if } \ell \in \mathcal{L}^+_\rightarrow(\Sigma) . \tag{7.7}$$

Consider a Wilson loop $W(\Gamma)$ in the spatial lattice. The center symmetry counts the winding number $w(\Gamma)$ of the Wilson loop:

$$W(\Gamma) \rightarrow u^{w(\Gamma)} W(\Gamma) . \tag{7.8}$$

It is clear that, unless $u \in Z_2$, the center mixes the real and imaginary parts of the Wilson loops, therefore it does not commute with the charge conjugation symmetry.

Let us come now to the orientifold planar equivalence. The $SO(N)$ parent theory has a Z_2 symmetry in the large–N limit. This symmetry is mapped through the orientifold projection in the Z_2 symmetry of the OrientiQCD in the large–N limit. Instead, the AdjQCD has a Z_N symmetry, but only the Z_2 subgroup in the large–N limit maps the neutral sector into itself. We conclude once more, that Z_2 (and not Z_N) is the only symmetry of OrientiQCD that we can deduce from the orientifold planar equivalence.

8. Conclusions

In this work we addressed the issue of the center symmetry of orientifold theories, by considering SU(N) gauge theories with fermions in the symmetric/antisymmetric two–index representations in the large–N limit. Our approach is based on the idea that the right tool for an exhaustive analysis of the center symmetry is the quantum action (or equivalently the generator of the connected correlators) for the Polyakov loop in the theory at finite temperature. We investigated the quantum action in both the confined and the deconfined phases and we conclude that the orientifold theory is invariant under the Z_2 subgroup of the center.

In the deconfined phase the invariance under (the spontaneously broken) Z_2 implies the well–known existence of only two degenerate vacua. The analytic computation can be carried out in the high–temperature regime, where the one–loop approximation holds. In this case the effective potential of the Polyakov loop has two degenerate minima corresponding to $\langle P \rangle / N = \pm 1$.

In the confined phase the generator of the connected correlators of Polyakov loops can be analytically computed in the large–mass expansion of the lattice theory. We showed that all the terms, that are not invariant under Z_2, vanish in the large–N limit. On the other hand the Z_2–invariant terms are different from zero, showing that actually the center symmetry group is explicitly broken to the Z_2 subgroup in the large–N limit.

A previous analysis of the center symmetry of the orientifold theories can be found in Ref. [11], where the authors concluded that the Z_N center symmetry is dynamically recovered in the large–N limit at zero temperature: the part of the fermionic action that is not invariant under the action of Z_N decouples from the expectation values in the large–N limit in the confined phase. We want to point out that this result was obtained for

observables with trivial topology with respect to the temporal compact dimension. Only these observables are relevant at zero temperature. In particular the dynamics of Wilson loops is determined by the Z_N invariant part of the fermionic action and this fact implies the stability of the k–strings. However at non–zero temperature, the theory contains also observables with non–trivial topology with respect to the temporal compact dimension. We showed explicitly that the Polyakov loop couples to the part of the fermionic action with non–trivial winding number and this is in general a leading effect in the large–N limit. Therefore, although the Z_N symmetry is dynamically recovered in a subsector, the whole orientifold theory is invariant only under the action of Z_2.

This picture fits well with the predictions of the orientifold planar equivalence. In the large–N limit the orientifold theory is equivalent to an $SU(N)$ theory with Majorana fermions in the adjoint representation in a neutral sector, defined by all the single–trace gauge–invariant C–even observables. The latter theory is invariant under the full Z_N center symmetry. Although the two theories have different symmetry contents, we showed that this fact is not in contradiction with the orientifold planar equivalence. Indeed the equivalence is valid only between neutral sectors, thus we can expect the same symmetry in the two theories only after removing all the states and the observables outside of these neutral sectors. In other word, the orientifold planar equivalence implies the equality in the two theories of the symmetry subgroups that map the neutral sector into itself. In the case of the center symmetry, this subgroup is Z_2 for both theories.

A. Lattice theory and large–N counting

In this work we are interested in computing connected expectation values of products of closed loops in pure Yang–Mills theories. The Boltzmann weight e^{-S} in the path integral can be expanded in a series in $1/\lambda$. The expansion of the Wilson gauge action produces a series of monomials of elementary plaquettes. The expectation values are computed by integrating each link variable over the group manifold, and non–vanishing results are obtained only when the plaquettes from the expansion of the action produce a tiling of a surface whose boundary is given by the closed loops. Each plaquette in this expansion appears with a factor of N. The rules for the $SU(N)$ group integration are known in detail, and the result of integrating over the group manifold a generic product of matrix elements

can be found e.g. in Refs. [20, 21]. The integration contributes further factors of $1/N$:

$$\int dU U_{i_1 j_1} U^\dagger_{l_1 m_1} = \frac{1}{N} \delta_{i_1, m_1} \delta_{j_1, l_1} , \tag{A.1}$$

$$\int dU U_{i_1 j_1} U_{i_2 j_2} U^\dagger_{l_1 m_1} U^\dagger_{l_2 m_2} = \frac{1}{N^2 - 1} \left(\delta_{i_1, m_1} \delta_{j_1, l_1} \delta_{i_2, m_2} \delta_{j_2, l_2} + \delta_{i_1, m_2} \delta_{j_1, l_2} \delta_{i_2, m_1} \delta_{j_2, l_1} \right)$$
$$- \frac{1}{N(N^2 - 1)} \left(\delta_{i_1, m_1} \delta_{j_2, l_2} \delta_{i_2, m_2} \delta_{j_1, l_1} + \delta_{i_1, m_2} \delta_{j_2, l_1} \delta_{i_2, m_1} \delta_{j_1, l_2} \right) ,$$
$$\tag{A.2}$$

$$\int dU U_{i_1 j_1} \cdots U_{i_N j_N} = \frac{1}{N!} \epsilon_{i_1 \dots i_N} \epsilon_{j_1 \dots j_N} . \tag{A.3}$$

Each trace over colour indices (i, j, l, m) contributes a factor of N. Collecting all contributions one obtains that each diagram is proportional to N^χ, where χ is the Euler characteristics of the surface spanned by the tiling, see e.g. Ref. [16].

For the diagrams considered in this work:

$$\chi = 2 - 2H - 2B , \tag{A.4}$$

where H is the number of handles, and B the number of boundaries of the surface. The number of boundaries is given by the number of closed loops. One can readily see that the planar limit is obtained by considering surfaces with $H = 0$.

Let us now describe in detail the computations that appear in the derivation of the results in Sect. 5.

Consider the connected expectation values:

$$\langle \prod_{i=1}^{n} \text{tr } U(\omega_i) \rangle_{c, \text{YM}} . \tag{A.5}$$

We will prove that the leading N^{2-n} behaviour is saturated if loops with winding number different from zero only appear in pairs with opposite winding numbers.

The easiest case is $\langle \text{tr } U(\omega_1) \text{tr } U(\omega_2) \rangle_{\text{YM}, c}$, with $w_1 = -w_2$. In this case the surface connecting the two loops is a cylinder, wrapping $|w_1|$ times around the thermal dimension (see Fig. 1). Notice that it is essential that the loops have opposite directions, since the cylinder cannot be twisted because of the topology of the torus. Since the surface is planar, this diagram saturates the leading behaviour of the connected expectation value:

$$\langle \text{tr } U(\omega_1) \text{tr } U(\omega_2) \rangle_{\text{YM}, c} = O(N^0) \qquad \text{with } w_1 = -w_2 . \tag{A.6}$$

Consider now the more complex case of $\langle \text{tr } U(\omega_1) \text{tr } U(\omega_2) \text{tr } U(\omega_3) \text{tr } U(\omega_4) \rangle_{\text{YM}, c}$ with $w_1 = -w_2$ and $w_3 = -w_4$. The planar surface is built by connecting each pair of loops with a cylinder with an hole, and than gluing the two cylinders through the boundary of the holes (see Fig. 1). The Euler characteristic of this surface is $\chi = -2$ therefore the corresponding diagram behaves like N^{-2}.

The above procedure can be iteratively generalised to the case of an arbitrary number of pairs of loops. Moreover an arbitrary number of loops with zero winding number can be included simply adding holes to the surface. The resulting surface is always planar.

Consider again the generic connected expectation value in Eq. (A.5). If some loop ω_i with winding number w_i different from zero is not paired to a loop with winding number $-w_i$ then the leading behaviour cannot be saturated. Since this fact is not crucial for our work, we only illustrate it in some simple cases.

For $n = 3$ and $w_1 = w_2 = 1$, $w_3 = -2$, the connected expectation value

$$\langle \text{tr } U(\omega_1) \text{tr } U(\omega_2) \text{tr } U(\omega_3) \rangle_{\text{YM},c}$$

has zero net winding number. However it is not possible to build a surface connecting this three loops because of the topology of the torus. One can ask if it is possible to use the four–link integration to get a leading contribution. We shall proceed in steps to prove that this is not the case.

First of all, we can deform each loop by taking a contiguous plaquette from the expansion of the Bolzmann weight and integrating the common link, as depicted in Fig. 6. If $\text{tr } (AU)$ is schematically the loop and $N\lambda^{-1}\text{tr } (U^\dagger B)$ is the relevant term of the action:

$$N\lambda^{-1} \int dU \text{tr } (AU) \text{tr } (U^\dagger B) = \lambda^{-1}\text{tr } (AB) . \tag{A.7}$$

In this way, the loop is replaced by the deformed one times a factor $1/\lambda$. We can use iteratively this procedure for each loop until the glued plaquettes shape three cylinders that end in three loops in the same spatial point (Fig. 7). Now we start to integrate one of the links of these three loops, by using the four–link formula in Eq. (A.2). If we want to integrate the link U, we can write the three loops schematically as $\text{tr } (AU)$, $\text{tr } (AU)$, $\text{tr } (U^\dagger A^\dagger U^\dagger A^\dagger)$ (remember that they all are coincident). The result of the integration is:

$$\int dU \text{tr } (AU) \text{tr } (AU) \text{tr } (U^\dagger A^\dagger U^\dagger A^\dagger) =$$
$$= \frac{2}{N^2 - 1}\text{tr } (AA^\dagger AA^\dagger) - \frac{2}{N(N^2 - 1)}[\text{tr } (AA^\dagger)]^2 =$$
$$= \frac{2N}{N^2 - 1} - \frac{2N^2}{N(N^2 - 1)} = 0 . \tag{A.8}$$

In the last line we used the fact that A is the product of all the links around the loop but U, and therefore it is a unitary matrix. We conclude that this kind of diagrams does not contribute to the connected expectation value.

The last non–trivial case we want to illustrate is $n = 1$ and $w_1 = N$. In this case the connected expectation value $\langle \text{tr } U(\omega_1) \rangle_{\text{YM},c}$ is invariant under the center Z_N, but not U(1). Therefore a non–zero contribution can be constructed only using the N–link integration formula for SU(N) in Eq. (A.3). By using the same construction as above, the loop $\text{tr } U(\omega_1)$ can be deformed (without introducing extra N factors) into a loop wrapping straight in the thermal direction (see Fig. 2), that can be written schematically as $\text{tr } [(AU)^N]$. Integrating the link U:

$$\int dU \text{tr } [(UA)^N] = \frac{1}{N!}\epsilon_{i_1...i_N} A_{i_1 j_1} \cdots A_{i_N j_N}\epsilon_{j_1...j_N} =$$
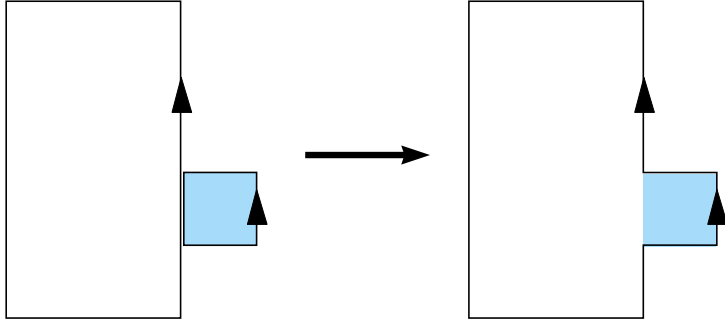$$= \det A = 1 . \tag{A.9}$$

Figure 6: The Eq. (A.7) is represented. In the left side, the big loop tr (AU) and the plaquette tr $(U^\dagger B)$; in the right side, the deformed loop tr (AB).
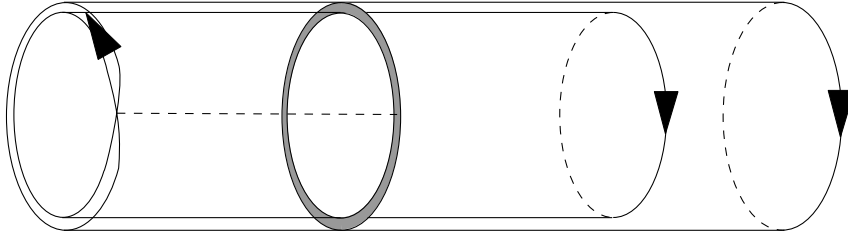

Figure 7: The connected expectation value $\langle \text{tr } U(\omega_1)\text{tr } U(\omega_2)\text{tr } U(\omega_3)\rangle_{\text{YM},c}$ with $w_1 = w_2 = 1$, $w_3 = -2$. The four–link integration in the gray area makes vanish this contribution, as explained in Eq. (A.8).

This is a subleading contribution since the leading term of the expectation value of a single loop is expected to be proportional to N.

References

[1] B. Svetitsky and L. G. Yaffe. Critical behavior at finite temperature confinement transitions. *Nucl. Phys.*, B210:423, 1982.

[2] A. Armoni, M. Shifman, and G. Veneziano. SUSY relics in one-flavor QCD from a new 1/N expansion. *Phys. Rev. Lett.*, 91:191601, 2003.

[3] A. Armoni, M. Shifman, and G. Veneziano. Exact results in non-supersymmetric large N orientifold field theories. *Nucl. Phys.*, B667:170–182, 2003.

[4] A. Armoni, M. Shifman, and G. Veneziano. Refining the proof of planar equivalence. *Phys. Rev.*, D71:045015, 2005.

[5] M. Unsal and L. G. Yaffe. (In)validity of large N orientifold equivalence. *Phys. Rev.*, D74:105019, 2006.

[6] A. Armoni, M. Shifman, and G. Veneziano. A note on C-parity conservation and the validity of orientifold planar equivalence. *Phys. Lett.*, B647:515–518, 2007.

[7] A. Patella. An insight on the proof of orientifold planar equivalence on the lattice. *Phys. Rev.*, D74:034506, 2006.

[8] P. Kovtun, M. Unsal, and L. G. Yaffe. Non-perturbative equivalences among large N(c) gauge theories with adjoint and bifundamental matter fields. *JHEP*, 12:034, 2003.

[9] P. Kovtun, M. Unsal, and L. G. Yaffe. Necessary and sufficient conditions for non-perturbative equivalences of large N(c) orbifold gauge theories. *JHEP*, 07:008, 2005.

[10] F. Sannino and K. Tuominen. Orientifold theory dynamics and symmetry breaking. *Phys. Rev.*, D71:051901, 2005.

[11] A. Armoni, M. Shifman, and M. Unsal. Planar limit of orientifold field theories and emergent center symmetry. 2007.

[12] D. J. Gross and E. Witten. Possible third order phase transition in the large N lattice gauge theory. *Phys. Rev.*, D21:446–453, 1980.

[13] J. Kiskis, R. Narayanan, and H. Neuberger. Does the crossover from perturbative to nonperturbative physics in QCD become a phase transition at infinite N? *Phys. Lett.*, B574:65–74, 2003.

[14] L. G. Yaffe. Large N limits as classical mechanics. *Rev. Mod. Phys.*, 54:407, 1982.

[15] G. 't Hooft. A planar diagram theory for strong interactions. *Nucl. Phys.*, B72:461, 1974.

[16] G. 't Hooft. Large N. 2002.

[17] K. G. Wilson. Confinement of quarks. *Phys. Rev.*, D10:2445–2459, 1974.

[18] F. Sannino. Higher representations: Confinement and large N. *Phys. Rev.*, D72:125006, 2005.

[19] J. L. F. Barbon and Carlos Hoyos-Badajoz. Small volume expansion of almost supersymmetric large N theories. *JHEP*, 01:114, 2006.

[20] I. Bars and F. Green. Complete integration of U(N) lattice gauge theory in a large N limit. *Phys. Rev.*, D20:3311, 1979.

[21] A. Gonzalez-Arroyo and C. Pena. Large N corrections to the strong coupling behaviour of SU(N)/Z(N) lattice gauge theories. *JHEP*, 09:007, 1999.