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Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes

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ABSTRACT: We study the factorization of soft and collinear singularities in dimensionally-regularized fixed-angle scattering amplitudes in massless gauge theories. Our factorization is based on replacing the hard massless partons by light-like Wilson lines, and defining gauge-invariant jet and soft functions in dimensional regularization. In this scheme the factorized amplitude admits a powerful symmetry: it is invariant under rescaling of individual Wilson-line velocities. This symmetry is broken by cusp singularities in both the soft and the eikonal jet functions. We show that the cancellation of these cusp anomalies in any multi-leg amplitude imposes all-order constraints on the kinematic dependence of the corresponding soft anomalous dimension, relating it to the cusp anomalous dimension. For amplitudes with two or three hard partons the solution is unique: the constraints fully determine the kinematic dependence of the soft function. For amplitudes with four or more hard partons we present a minimal solution where the soft anomalous dimension is a sum over colour dipoles, multiplied by the cusp anomalous dimension. In this case additional contributions to the soft anomalous dimension at three loops or beyond are not excluded, but they are constrained to be functions of conformal cross ratios of kinematic variables.

Contents

1. Introduction

Studies of infrared and collinear singularities of fixed-angle scattering amplitudes in massless gauge theories have a long history (for early results, see for example [1] and [2]), and they have led to remarkable insights into the all-order structure of the perturbative expansion.

These studies are not motivated by a purely theoretical interest: in fact, a detailed understanding of the long-distance singularity structure of QCD amplitudes is a crucial element in predicting high-energy collider cross sections. Indeed, the calculation of observable cross sections involves intricate cancellations of soft and collinear singularities between real and virtual corrections (see for example [3]). Furthermore, with a precise knowledge of singularities one can predict dominant higher–order corrections, and in many occasions resum certain classes of logarithmically enhanced contributions to all orders [4, 5].

Our understanding of long–distance singularities is based on the ideas of factorization and universality [6]. Fixed-angle scattering amplitudes are functions of Lorentz-invariant combinations of external momenta, which are assumed to be uniformly much larger than the relevant infrared cutoff, typically given by the scale of confinement; one expects then that exchanges of virtual particles with vanishingly small energies, or with vanishing transverse

momenta with respect to given external legs, should decouple from hard exchanges, which happen at much shorter distances. Such a decoupling is far from apparent in Feynman diagram calculations, but it can indeed be proven to all orders in perturbation theory, once gauge invariance is enforced by means of appropriate Ward identities [4].

The result has a simple structure, explained in detail in Section 2. Briefly, multi-leg amplitudes can be organized as vectors, in a vector space spanned by the irreducible representations of the gauge group that can be constructed with the given external particles; such a vector can be shown to have a factorized structure: each external leg is dressed by virtual collinear emissions, building up a colour-singlet 'jet' function; soft gluons exchanged at wide angles are assigned to a separate factor, which is a matrix, mixing the available colour representations; this matrix of 'soft' functions is then contracted with a vector of hard scattering coefficients, which contain no infrared or collinear singularities.

Given this factorized structure, one may immediately deduce that the various factors in the amplitude obey simple evolution equations, embodying the consequences of renormalization group as well as gauge invariance [7] (for reviews of this viewpoint, see [8,9]). Evolution equations of this type were derived for the first time for the form factors of elementary fields, with a variety of methods [10–12], and were later extended to Wilson loops [13–20] and to cross sections and amplitudes of phenomenological interest [21–24]. Solving these equations leads to the exponentiation of all infrared and collinear singularities. The singularities in the exponent are generated by integrals over the scale of the running coupling of specific anomalous dimensions, which can be computed order-by-order in perturbation theory. A significant step was taken in [25], where the evolution equation for the Sudakov form factor was solved in dimensional regularization. Within this framework, infrared and collinear poles are generated by integration over the scale of the D-dimensional version of the running coupling; the results of exponentiation can then be directly compared with finite-order Feynman diagram calculations; the Landau pole is also regulated by dimensional continuation, so that resummed amplitudes can be computed as analytic functions of the coupling at a fixed scale and of the dimension of space-time [26]. This approach was extended to multi-leg amplitudes in [27], confirming earlier predictions [28] on the structure of singularities at NNLO.

In recent years, the development of novel and advanced techniques for high-order calculations in QCD and in general gauge theories has stimulated further investigation of the exponentiation of infrared and collinear singularities. In particular, great theoretical effort has been made to further our understanding of amplitudes in supersymmetric gauge theories, and most notably in the maximally supersymmetric $\mathcal{N} = 4$ Yang-Mills theory. This theory is of special interest for several reasons: it is quantum conformal invariant; in the planar limit, it is expected to have a simple, theoretically accessible strong-coupling limit, because of its connection with string theory through the AdS-CFT correspondence [29]; finally, its amplitudes and anomalous dimensions are of practical relevance, since they have nontrivial relations with the corresponding quantities in QCD (see for example Refs. [30,31], and recent studies of the Regge limit [32,33]). Explicit calculations for the four-point function in $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory have led to an all-order conjecture [34], suggesting that non-singular terms exponentiate together with

infrared and collinear poles, at least for the class of maximally helicity violating (MHV) amplitudes. While this conjecture has now been shown to fail starting with the two-loop six-point function [35], it is clear that $\mathcal{N} = 4$ SYM perturbative amplitudes must have a remarkably simple all-order structure, which may well be brought under full theoretical control in the near future. A step in this direction was taken with the discovery of a surprising duality between scattering amplitudes in momentum space and expectation values of Wilson loops taken in an auxiliary coordinate space [36]. Further, remarkable progress was made at strong coupling in Ref. [37], where a calculation of the four-point amplitude was performed, by adapting string techniques to dimensional regularization. This allowed a direct comparison with resummed perturbative calculations, finding an exact matching in the structure of long-distance singularities. Recent results in this fast-developing field are reviewed in Ref. [38].

Most of the calculations just described have been carried out in the planar limit[1], which has special simplifying properties. In this limit, soft contributions to multi-leg amplitudes can be further factorized into a product of 'wedges', each one proportional to a form factor, since soft exchanges can only take place between adjacent external legs. In essence, in the planar limit amplitudes can have only a single colour structure, so that the soft anomalous dimension matrix must be proportional to the unit matrix. All soft and collinear singularities are then determined by just two colour-diagonal functions: the cusp anomalous dimension $\gamma_K(\alpha_s)$ [16–20], and a subleading function $G(\alpha_s)$ [40], responsible for single soft or collinear poles.

It is of great interest to push our understanding of infrared singularities in terms of a limited set of anomalous dimensions beyond the planar limit. Indeed, from a theoretical point of view, only at non-planar level one begins to see the intricate pattern of colour correlations that are characteristic of non-abelian gauge theories: only at this level space-time and colour degrees of freedom become explicitly correlated. Furthermore, colour-subleading contributions in QCD have important phenomenological effects on resummed hadronic cross sections, beginning at the next-to-leading logarithmic order, and the understanding of subleading poles would also play an important role in the development of infrared and collinear subtraction schemes at higher orders in perturbation theory. Finally, recent work [40, 43–48] has highlighted new properties of the functions that generate infrared and collinear enhancements in gauge theory amplitudes and cross sections in the case of two hard partons, leading to a better understanding of the process–dependence of soft radiation, to the discovery of all-order connections between different physical processes, and to the possibility of performing internal resummations of running–coupling corrections within the Sudakov exponent. It would be very interesting to extend these studies to general colour configurations.

Soft anomalous dimension matrices for multi-particle scattering have also been intensively studied in recent years. A complete one-loop calculation for the simplest non-trivial case of $2 \rightarrow 2$ scattering was carried out originally in [49]. More recently, the calculation was reproduced in a physically motivated, dipole-based formalism in [50]: an interesting obser-

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[1]An exception is Ref. [55], which studies the leading infrared singularities in subleading colour components of the $\mathcal{N} = 4$ SYM gluon-gluon scattering amplitude to three-loop order.

vation there was that the anomalous dimension matrix for gluon-gluon scattering displays an unexplained symmetry relating kinematic invariants with the number of colours N_c. A different symmetry property was observed by [51], where it was noted that all one-loop anomalous dimension matrices are complex symmetric matrices in a suitably chosen orthonormal basis. This property was later explicitly verified with the calculation of the matrices for all $2 \to 3$ processes at one loop [52, 53], and very recently proven [54].

Finally, a remarkable result was derived in [56], where it was shown that soft anomalous dimension matrices at two loops, with any number of external legs, are proportional to their one-loop value, with the proportionality constant given by the two-loop coefficient of the cusp anomalous dimension. This is of course a great reduction in the number of possible degrees of freedom, since a priori each matrix element could have acquired an independent two-loop correction. The fact that the correlation between colour and kinematic dependence in the soft function does not get more complex at two loops as compared to one loop, calls for a deeper explanation. At present it is not known whether this remarkable property remains valid at higher orders.

In this paper, we begin to tackle this question. In Section 2 we develop in detail the factorization of soft and collinear singularities for fixed–angle scattering amplitudes, following the approach of Ref. [40]. There are two main differences between our factorization and earlier calculations of soft matrices. First, we employ dimensional regularization as the unique infrared and collinear regulator: thus, for example, in contrast with Ref. [56] we do not tilt the Wilson lines off the light cone to regulate collinear poles. While this approach makes explicit loop calculations slightly more delicate, it has the advantage that Wilson line correlators are given by pure counterterms to all orders in perturbation theory, and they do not depend on any mass scales. Second, instead of using the jet definition as the square-root of the Sudakov form factor as in Refs. [27, 56], we define each jet J_i by introducing a separate auxiliary vector n_i, as suggested in early work on Sudakov factorization [57]. This will allow us to conveniently trace the effect of rescaling of the Wilson–line velocities.

Section 3 studies the kinematic dependence of the eikonal functions that enter the factorization of multi-parton amplitudes. First, in Section 3.1, we consider eikonal jets, and we determine their kinematic dependence to all orders in perturbation theory in terms of the cusp anomalous dimension. This simple result follows from the fact that the eikonal jet is defined as a correlator of semi-infinite Wilson lines (see (2.5) below) one of which goes along the light-like direction defined by the momentum of an external hard parton. Any such correlator of semi-infinite Wilson lines is classically invariant under rescaling of any of the corresponding velocity vectors (independently of whether they are light-like or not): this invariance is a property of the eikonal Feynman rules. In the presence of cusps with light-like rays, however, the renormalization procedure breaks this invariance: the counter terms include double poles, corresponding to overlapping ultraviolet and collinear singularities, along with single poles that carry explicit dependence on the normalization of the light-like velocity vectors. Thus, the renormalized correlators, which do retain their invariance under rescaling of any non-light-like Wilson-line velocity vector, acquire a dependence on the normalization of the light-like ones. The origin of this anomaly is well understood [13–16]:

the violation of classical rescaling invariance is governed by the cusp anomalous dimension, and we will refer to it as the 'cusp anomaly'.

In Section 3.2 we extend the analysis to soft gluon functions. To deal with the general multi-leg case, we examine combinations of soft and jet correlators where the cusp anomaly cancels, so that rescaling invariance must be recovered. We find that this strongly constrains the dependence of the soft anomalous dimension on the kinematics of the scattering process, and eventually also on the colour degrees of freedom.

Section 4 deals with the simple case of amplitudes with only two hard coloured partons. In Section 4.1 we develop the consequences of the new constraints for the case of the Sudakov form factor, and show that the complete dependence of the corresponding eikonal function on the kinematics is indeed governed by the cusp anomalous dimension. In Section 4.2 we analyse the kinematic dependence of the partonic jet function and contrast it with the eikonal case.

In Section 5 we return to the case of generic multi-parton fixed-angle scattering amplitudes and study the impact of the new constraints on the soft function. We show that while these constraints are insufficient to fully determine the functional dependence on the kinematic variables, they admit a remarkably simple solution, where the soft anomalous dimension matrix at any order in perturbation theory is proportional to the one-loop result. The solution corresponds to a sum over all colour dipoles, which correlate the kinematic dependence to the colour degrees of freedom, multiplied by the cusp anomalous dimension. This formula is consistent with the result of Ref. [56] at two loops and generalizes it to all orders. We also discuss possible sources of further corrections. We conclude in Section 6 by summarizing our results, while two appendices discuss concrete examples. In Appendix A we describe the special case of amplitudes with three hard partons, where the sum-over-dipoles formula is the unique solution to the new constraints. Appendix B studies $2 \to 2$ scattering of quarks at one loop, describing the way in which conformal cross ratios are formed through a sum over diagrams.

2. Factorization of fixed–angle scattering amplitudes

We begin by describing the factorization of a general fixed–angle massless gauge theory amplitude into soft, hard and jet functions. We follow the notations of Ref. [40] and generalize the definition of the soft function given there to the case of multileg amplitudes. The amplitude $\mathcal{M}$ describes the scattering of n hard massless gauge particles (plus any number of colour–singlet particles) so it is characterized by n colour indices $\{\alpha_i\}$, $i = 1, \ldots n$, belonging to arbitrary representations of the gauge group. The representation content of the amplitude is collectively denoted by $[f]$. Such a coloured object can be decomposed into components by picking a basis of independent colour tensors with the same index structure. We denote these tensors by $(c_L)_{\{\alpha_i\}}$, where $L = 1, \ldots N^{[f]}$ and $N^{[f]}$ is the number of irreducible representations of the gauge group that can be constructed

with the given particles. We write then

$$\mathcal{M}^{[\mathrm{f}]}_{\{\alpha_i\}}\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) = \sum_{L=1}^{N^{[\mathrm{f}]}} \mathcal{M}^{[\mathrm{f}]}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right)\, (c_L)_{\{\alpha_i\}}\ , \tag{2.1}$$

with μ being the renormalization scale and $\epsilon = 2 - D/2$, where D is the dimension of space-time. General factorization arguments guarantee that the colour components $\mathcal{M}_L$ of the amplitude may be written in a factorized form. Following [2, 27, 40, 49, 56], we write

$$\mathcal{M}^{[\mathrm{f}]}_L\left(p_i/\mu, \alpha_s(\mu^2), \epsilon\right) = \mathcal{S}^{[\mathrm{f}]}_{LK}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)\, H^{[\mathrm{f}]}_K\left(\frac{2p_i \cdot p_j}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2)\right)$$
$$\times \prod_{i=1}^{n} \frac{J_i\left(\dfrac{(2p_i \cdot n_i)^2}{n_i^2\mu^2}, \alpha_s(\mu^2), \epsilon\right)}{\mathcal{J}_i\left(\dfrac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}\ , \tag{2.2}$$

where the hard function $H^{[\mathrm{f}]}_K$, like the amplitude $\mathcal{M}^{[\mathrm{f}]}_L$ itself, is a vector in the colour space described above; the soft function $\mathcal{S}^{[\mathrm{f}]}_{LK}$ is a matrix in this space, while the jet functions J_i and $\mathcal{J}_i$ do not carry any colour index. A sum over K is assumed on the *r.h.s.* The soft matrix $\mathcal{S}$ and the jet functions J and $\mathcal{J}$ contain all infrared and collinear singularities of the amplitude, while the hard functions H_K are independent of ϵ. Each of the functions appearing in Eq. (2.2) is separately gauge invariant and admits an operator definition given below. These definitions also clarify the choice of the arguments of each function. In particular, we have exhibited here the fact that the eikonal functions $\mathcal{S}$ and $\mathcal{J}$ depend on the dimensionless four-velocities β_i associated with external particles, rather than the particle momenta p_i. The velocities are defined by scaling the momenta p_i according to $p_i = \beta_i Q_0/\sqrt{2}$, where the magnitude of Q_0 is unimportant so long as this substitution[2] is restricted to the eikonal functions. The fixed-angle assumption implies that all scalar products $\beta_i \cdot \beta_j$ $(i \neq j)$ are of order 1.

The definitions of the soft and jet functions all involve Wilson lines, which we write as

$$\Phi_n(\lambda_2, \lambda_1) = P \exp\left[\mathrm{i} g \int_{\lambda_1}^{\lambda_2} d\lambda\, n \cdot A(\lambda n)\right]\ . \tag{2.3}$$

In terms of these Wilson–line operators, one may then define the 'partonic jet' functions (for, say, an outgoing quark with momentum p) as

$$\overline{u}(p)\, J\left(\frac{(2p \cdot n)^2}{n^2\mu^2}, \alpha_s(\mu^2), \epsilon\right) = \langle p\, |\, \overline{\psi}(0)\, \Phi_n(0, -\infty)\, |0\rangle\ . \tag{2.4}$$

The function J represents a transition amplitude connecting the vacuum and a one-particle state. The eikonal line Φ_n simulates interactions with fast partons moving in different

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[2]As soon as the β_i variables are used in the hard functions H or in the partonic jet functions J, which do depend on physical scales, Q_0 needs to be specified. We shall avoid that, except in Section 4.2 where we relate the normalization of the partonic jet to the Sudakov form factor.

directions: the direction n^μ is arbitrary, but off the light-cone (in order to avoid spurious collinear singularities). Since eikonal Feynman rules are invariant under rescalings of the eikonal vector n^μ, and this invariance is not broken by the cusp anomaly for $n^2 \neq 0$, J can depend on the vectors p and n only through the argument given in Eq. (2.4)[3]. To avoid any ambiguity with respect to unitarity phases associated with the first argument of J we shall choose n such that $p \cdot n > 0$. Note that this can be done so long as one retains the vectors n corresponding to different partons independent of each other.

The factorization formula (2.2) also requires to introduction of the eikonal approximation to the partonic jet J, which we call the 'eikonal jet'. It is defined by

$$\mathcal{J}\left(\frac{2(\beta \cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon\right) = \langle 0| \, \Phi_\beta(\infty, 0) \, \Phi_n(0, -\infty) \, |0\rangle \ . \tag{2.5}$$

Both the partonic jet (2.4) and the eikonal jet (2.5) have infrared divergences, as well as collinear divergences associated to their light-like leg; thus, they display double poles order-by-order in perturbation theory. The double-pole singularities are however the same, since in the infrared region $\mathcal{J}$ correctly approximates J: singular contributions to the two functions differ only by hard collinear radiation.

It is important to note that the eikonal jet $\mathcal{J}$ depends on the renormalization scale only through the coupling: indeed, diagram by diagram $\mathcal{J}$ is given by integrals with no dimensionful parameter. Such integrals vanish identically in dimensional regularization, but since this trivial result involves cancellations between ultraviolet and infrared singularities, upon renormalization $\mathcal{J}$ becomes non-trivial: the contribution of each graph equals minus the corresponding ultraviolet counterterm. As a consequence, using a minimal subtraction scheme, the result for $\mathcal{J}$ at each order in α_s is a sum of poles in ϵ, without any non-negative powers. These properties are not special to the jet function, but apply to any eikonal function not involving dimensionful parameters, provided it is defined in dimensional regularization and in a minimal subtraction scheme.

The final ingredient in Eq. (2.2) is the soft matrix. It is constructed by taking the eikonal approximation for all soft exchanges. Since soft gluons are insensitive to the structure of hard collinear emissions, they couple effectively to Wilson lines in the colour representation of the corresponding hard external parton. Such exchanges mix the colour components of the amplitude, so one is led to define

$$(c_L)_{\{\alpha_k\}} \, \mathcal{S}_{LK}^{[f]}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right) = \sum_{\{\eta_k\}} \langle 0| \prod_{i=1}^{n} \left[\Phi_{\beta_i}(\infty, 0)_{\alpha_k, \eta_k}\right] |0\rangle \, (c_K)_{\{\eta_k\}} \ , \tag{2.6}$$

where for simplicity of notation we have defined all eikonal lines as outgoing. Note that in our definition we keep all Wilson lines on the light-cone. As a consequence, also the soft matrix is a pure counterterm in dimensional regularization and it depends on the renormalization scale only through the coupling; furthermore, the homogeneity of the eikonal Feynman rules with respect to rescalings of the eikonal vectors β_i would suggest that $\mathcal{S}$

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[3]For later convenience factors of 2 have been introduced into the arguments of the jet functions.

can depend on β_i only through homogeneous ratios invariant under such rescalings. As described in [40], this is not true: indeed, rescaling invariance is broken by the cusp anomaly, so that the soft matrix acquires nontrivial dependence on the scalar products $\beta_i \cdot \beta_j$. This observation will be central to our arguments in the rest of the paper.

The soft matrix, Eq. (2.6), displays both infrared and collinear poles. One must then correct the factorization formula in order to avoid double counting of the infrared-collinear region for each external leg. This is achieved in Eq. (2.2) by dividing by an eikonal jet $\mathcal{J}_i$ for each external leg, thus removing from J_i its eikonal part, which is already accounted for in $\mathcal{S}_{LK}^{[f]}$. One may then observe that the ratios $J_i/\mathcal{J}_i$ are free of infrared poles, and thus contain only single collinear poles at each order in perturbation theory. Similarly, the 'reduced' soft matrix

$$\overline{\mathcal{S}}_{LK}^{[f]}\left(\rho_{ij}, \alpha_s(\mu^2), \epsilon\right) = \frac{\mathcal{S}_{LK}^{[f]}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)}{\displaystyle\prod_{i=1}^{n} \mathcal{J}_i\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)} \tag{2.7}$$

is free of collinear poles, and contains only infrared singularities originating from soft gluon radiation at large angles with respect to all external legs. This means that the effects of the cusp anomaly, which is the source of double infrared-collinear poles, must cancel in $\overline{\mathcal{S}}$. More generally, invariance under rescaling of each individual light-like eikonal velocity,

$$\beta_i \rightarrow \kappa_i \beta_i, \tag{2.8}$$

which is broken separately in $\mathcal{S}$ and in $\mathcal{J}$, must be recovered in their ratio, Eq. (2.7). Indeed, in the factorized amplitude (2.2) the dependence on the normalizations of the vectors β_i appears only though the eikonal functions contained in Eq. (2.7), so the invariance of the amplitude as a whole with respect to such rescaling amounts to invariance of $\overline{\mathcal{S}}$. The immediate consequence is that $\overline{\mathcal{S}}$ can only depend on arguments that are simultaneously homogeneus in β_i and in n_i. Given the different functional dependencies of $\mathcal{S}$ and $\mathcal{J}$, this can be achieved only if $\overline{\mathcal{S}}$ depends on kinematics only through the variables[4]

$$\rho_{ij} \equiv \frac{n_i^2 \, n_j^2 \, (\beta_i \cdot \beta_j)^2 \, e^{2\pi i \, \lambda_{ij}}}{4 \, (\beta_i \cdot n_i)^2 (\beta_j \cdot n_j)^2} \,. \tag{2.9}$$

In Section 3.2 we will explore further consequences of this constraint on the functional dependence of the reduced soft matrix.

Finally, it is important to control the ultraviolet behavior of the jet and soft functions thus introduced. All these functions are multiplicatively renormalizable [14, 15]; there is however an important difference between eikonal operators involving light-like Wilson lines and partonic amplitudes. First, as already mentioned, the ultraviolet divergence of eikonal operators is directly related to their infrared singularities. Moreover, anomalous dimensions of operators involving cusped Wilson lines with light-like segments are themselves divergent, due to the overlapping of collinear and ultraviolet poles. These divergences are controlled

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[4]Following [28] we keep track of the unitarity phase by writing $-\beta_i \cdot \beta_j = |\beta_i \cdot \beta_j| \, e^{i\pi\lambda_{ij}}$ where $\lambda_{ij} = 1$ if i and j are both initial-state partons or are both final–state partons, and $\lambda_{ij} = 0$ otherwise.

by the the cusp anomalous dimension [16–20]. Let us then write down renormalization group equations for the various functions defined above.

The partonic jet J does not involve any light-like Wilson line, and therefore does not have a cusp anomaly. Its anomalous dimension is finite, and one may write

$$\mu \frac{d}{d\mu} \ln J_i \left(\frac{(2p \cdot n)^2}{n^2 \mu^2}, \alpha_s(\mu^2), \epsilon \right) = -\gamma_{J_i}(\alpha_s(\mu^2)) \,. \tag{2.10}$$

In contrast, for the eikonal jet $\mathcal{J}_i$ we write

$$\mu \frac{d}{d\mu} \ln \mathcal{J}_i \left(\frac{2(\beta \cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon \right) = -\gamma_{\mathcal{J}_i} \left(\frac{2(\beta \cdot n)^2}{n^2}, \alpha_s(\mu^2), \epsilon \right) \,. \tag{2.11}$$

In both cases the index i is kept as a reminder that the jet function J for a given parton i carries information not only on the kinematics, but also on the parton spin, flavor and colour, while the eikonal jet $\mathcal{J}$ depends on the colour representation only.

For the soft matrix $\mathcal{S}$ multiplicative renormalizability must be understood in the sense of matrix multiplication [15], and one writes

$$\mu \frac{d}{d\mu} \mathcal{S}_{IK}^{[f]} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) = -\Gamma_{IJ}^{\mathcal{S}} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) \mathcal{S}_{JK}^{[f]} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon \right) \,. \tag{2.12}$$

where $\Gamma_{IJ}^{\mathcal{S}}$ will be referred to as the 'soft anomalous dimension'; it is similar to the 'cross anomalous dimension' of Ref. [15, 22], taken in the limit where all the lines are light-like. In the following sections we shall examine the dependence of the anomalous dimensions defined above on the kinematic variables as well as on the colour degrees of freedom.

3. On the kinematic dependence of eikonal functions

We now discuss the properties of the eikonal functions $\mathcal{J}$ and $\mathcal{S}$ taking into account their gauge invariance, their renormalization group evolution and their independence of any dimensionful kinematic scale. By considering the effect of velocity rescaling we deduce that the kinematic dependence of these functions is tightly connected with cusp singularities. We first illustrate this for the eikonal jet function $\mathcal{J}$, and then we move on to consider the central object of our work, the soft anomalous dimension matrix introduced in Eq. (2.6).

3.1 Explicit solution for the eikonal jet

Let us consider first the anomalous dimension of the eikonal jet in Eq. (2.11). One observes that the homogeneity of eikonal Feynman rules under the rescaling in Eq. (2.8) would forbid any dependence on $w_i \equiv 2(\beta_i \cdot n_i)^2/n_i^2$, were it not for the cusp singularity. One expects then that the full w_i dependence of $\gamma_{\mathcal{J}_i}$ should be proportional to the cusp anomalous dimension, and this is indeed the case as we now explicitly show.

Our starting point is Eq. (2.11); in dimensional regularization, the statement that $\mathcal{J}_i$ is a pure counterterm implies that it can depend on μ only through the running coupling; one may then solve Eq. (2.11) as

$$\mathcal{J}_i \left(w_i, \alpha_s(\mu^2), \epsilon \right) = \exp \left[-\frac{1}{2} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \gamma_{\mathcal{J}_i} \left(w_i, \alpha_s \left(\xi^2, \epsilon \right), \epsilon \right) \right] \,. \tag{3.1}$$

Next, we observe that the eikonal jet must obey an evolution equation of the same form as the Sudakov form factor itself, a so-called 'K+G' equation; a similar observation was made in Ref. [40] concerning the eikonal approximation to the form factor. Following the standard reasoning, one rewrites the anomalous dimension $\gamma_{\mathcal{J}_i}$ as a sum of a singular term, generated by the cusp singularity, and a residual finite function that contains the kinematic dependence. We write then

$$\gamma_{\mathcal{J}_i}\left(w_i, \alpha_s\left(\mu^2, \epsilon\right), \epsilon\right) = -\frac{1}{2}\, G_{\mathcal{J}_i}\left(w_i, \alpha_s(\mu^2, \epsilon)\right) \,+\, \frac{1}{4}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\gamma_K^{(i)}\left(\alpha_s(\lambda^2, \epsilon)\right)\,. \qquad (3.2)$$

Here we have introduced the cusp anomalous dimension $\gamma_K^{(i)}(\alpha_s)$, for an eikonal line in the representation of parton i, and a remainder function $G_{\mathcal{J}_i}(w_i, \alpha_s)$. The normalization of the singular term on the *r.h.s.* of Eq. (3.2) is one half of the corresponding term in the Sudakov form factor, since the form factor is comprised of two jets[5].

Upon inserting Eq. (3.2) into Eq. (3.1), and changing the order of integration, one readily arrives at

$$\mathcal{J}_i\left(w_i, \alpha_s(\mu^2), \epsilon\right) = \exp\left[\frac{1}{2}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\left(\frac{1}{2}G_{\mathcal{J}_i}\left(w_i, \alpha_s(\xi^2, \epsilon)\right) \,-\, \frac{1}{4}\gamma_K^{(i)}\left(\alpha_s\left(\xi^2, \epsilon\right)\right)\ln\frac{\mu^2}{\xi^2}\right)\right],$$
$$(3.3)$$

which is analogous to the expression for the Sudakov form factor, as given in Eq. (4.22) below (or in Eq. (2.11) in Ref. [40]), with the physical scale of the form factor, $-Q^2$, replaced here by the renormalization point μ^2. The finite function $G_{\mathcal{J}_i}$ has no explicit ϵ dependence in a minimal subtraction scheme, since $\mathcal{J}_i$ is a pure counterterm.

We are now going to show that Eq. (3.3) can be further simplified, since the dependence on the kinematic variable w_i in the function $G_{\mathcal{J}_i}$ can be completely solved for. In order to do that, we use the results of Ref. [40] for the w_i dependence of the eikonal jet, which is given by

$$w_i\frac{\partial}{\partial w_i}\ln\mathcal{J}_i\left(w_i, \alpha_s(\mu^2), \epsilon\right) \,=\, -\frac{1}{8}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\,\gamma_K^{(i)}\left(\alpha_s\left(\xi^2, \epsilon\right)\right)\,. \qquad (3.4)$$

Clearly, Eq. (3.3) and Eq. (3.4) are compatible only if $G_{\mathcal{J}_i}(w_i, \alpha_s)$ is a linear function of $\ln w_i$. Indeed, by taking the derivative of Eq. (3.3) with respect to $\ln w_i$, and using Eq. (3.4), one gets

$$w_i\frac{\partial}{\partial w_i}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}G_{\mathcal{J}_i}\left(w_i, \alpha_s(\xi^2, \epsilon)\right) \,=\, -\frac{1}{2}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\,\gamma_K^{(i)}\left(\alpha_s\left(\xi^2, \epsilon\right)\right)\,, \qquad (3.5)$$

for any μ^2. Therefore

$$w_i\frac{\partial}{\partial w_i}G_{\mathcal{J}_i}\left(w_i, \alpha_s(\xi^2, \epsilon)\right) \,=\, -\frac{1}{2}\gamma_K^{(i)}\left(\alpha_s\left(\xi^2, \epsilon\right)\right)\,, \qquad (3.6)$$

which we integrate to get

$$G_{\mathcal{J}_i}\left(w_i, \alpha_s\right) = -\frac{1}{2}\gamma_K^{(i)}\left(\alpha_s\right)\ln(w_i) \,+\, \delta_{\mathcal{J}_i}(\alpha_s), \qquad (3.7)$$

---

[5]Indeed an alternative definition of the partonic jet function, which was used for example in Refs. [27,56], is based on taking the square root of the Sudakov form factor.

where $\delta_{\mathcal{J}_i}$ is a constant of integration, free of any kinematic dependence. Using Eq. (3.7) in Eq. (3.2) we finally get

$$\gamma_{\mathcal{J}_i}\left(w_i, \alpha_s(\mu^2, \epsilon), \epsilon\right) = -\frac{1}{2}\delta_{\mathcal{J}_i}\left(\alpha_s(\mu^2, \epsilon)\right) + \frac{1}{4}\gamma_K^{(i)}\left(\alpha_s(\mu^2, \epsilon)\right)\ln(w_i)$$
$$+\frac{1}{4}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\gamma_K^{(i)}\left(\alpha_s(\xi^2, \epsilon)\right). \tag{3.8}$$

We can now write down our final expression for the eikonal jet, using Eq. (3.3). We obtain

$$\mathcal{J}_i\left(w_i, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left[\frac{1}{2}\delta_{\mathcal{J}_i}\left(\alpha_s(\lambda^2, \epsilon)\right) - \frac{1}{4}\gamma_K^{(i)}\left(\alpha_s(\lambda^2, \epsilon)\right)\ln\left(\frac{w_i\mu^2}{\lambda^2}\right)\right]\right\}, \tag{3.9}$$

where, as anticipated, the kinematic dependence of the eikonal jet is explicitly written, to all orders in perturbation theory, in terms of the cusp anomalous dimension. We observe that the cusp anomalous dimension simultaneously controls the double poles and the kinematic dependence of the single poles. In the following sections we will see that this property holds also in the more complex soft functions.

Returning to the comparison with the Sudakov form factor case (see Eq. (4.22)), we now see that the physical scale $-Q^2$ is replaced here by $\mu^2 w_i = 2\mu^2(\beta_i \cdot n_i)^2/n_i^2$. It is important to note that Eq. (3.9) can also be expressed as

$$\mathcal{J}_i(w_i, \alpha_s(\mu^2), \epsilon) = \exp\left\{\frac{1}{2}\int_0^1\frac{d\theta}{\theta}\left[\frac{1}{2}\delta_{\mathcal{J}_i}(\alpha_s(\mu^2\theta, \epsilon)) - \frac{1}{4}\gamma_K^{(i)}(\alpha_s(\mu^2\theta, \epsilon))\ln\left(\frac{w_i}{\theta}\right)\right]\right\}, \tag{3.10}$$

exhibiting the fact that μ dependence appears only through the D-dimensional running coupling.

Finally we emphasize that the above result for the eikonal jet holds for quarks as well as for gluons. In fact, the dependence of Eq. (3.9) on the colour representation of the parton i appear only though the two functions $\gamma_K^{(i)}$ and $\delta_{\mathcal{J}_i}$. Moreover, the non-Abelian exponentiation theorem [41] implies that the colour structure of these functions is 'maximally non-Abelian'. Up to three loops, this implies, in particular (see e.g. Refs. [18, 20]) that the cusp anomalous dimension depends on the representation only through an overall multiplicative factor, the total colour charge, given by the quadratic Casimir C_i in the representation of parton i,

$$C_i\,\mathrm{I} = \sum_a \mathrm{T}_i^{(a)}\,\mathrm{T}_i^{(a)}, \tag{3.11}$$

where I is the unit matrix and $\mathrm{T}_i^{(a)}$ is a generator in the corresponding representation[6]. Casimir scaling, namely the universality of $\gamma_K^{(i)}\left(\alpha_s\right)/C_i$ between quarks and gluons, has been explicitly verified in recent years by three-loop calculation of the QCD splitting functions in Ref. [58]. Starting at four loops, however, the colour structure in the exponent may

---

[6]$\mathrm{T}^{(a)}$ should be interpreted as follows: for a final–state quark or an initial–state antiquark: $t_{\alpha\beta}^a$; for a final–state antiquark or an initial–state quark: $-t_{\beta\alpha}^a$; for a gluon: $\mathrm{i}\,f_{cab}$. For SU(N_c) the index a runs from 1 to $N_c^2 - 1$, and specifically $C_F = T_R(N_c^2 - 1)/N_c$ for quarks and $C_A = N_c$ for gluons. In our normalization $T_R = 1/2$.

not be expressible in terms of quadratic Casimirs. In general, higher Casimir contributions do appear in QCD calculations at this order, for example in the QCD beta function [59,60], where one finds colour-singlet contributions constructed of traces of products of four generators. The potential appearance of such higher Casimir terms in the exponent, despite the non-Abelian exponentiation theorem [41], was first observed in Ref. [42], where it was proposed to describe the colour structure of the exponent by 'colour connected webs', giving a more precise meaning to the notion 'maximally non-Abelian'. Recently it has been argued [39], based on different theoretical considerations[7], that such terms may indeed appear in the cusp anomalous dimension starting at four loops.

Let us therefore write in full generality,

$$\gamma_K^{(i)}\left(\alpha_s\right) \equiv C_i\,\widehat{\gamma}_K\left(\alpha_s\right) + \widetilde{\gamma}_K^{(i)}\,, \tag{3.12}$$

where C_i is given by (3.11), $\widehat{\gamma}_K\left(\alpha_s\right) = 2\,\alpha_s/\pi + \ldots$, and $\widetilde{\gamma}_K^{(i)} = \mathcal{O}(\alpha_s^4)$. Note that $\widehat{\gamma}_K\left(\alpha_s\right)$ is a universal function of the coupling, strictly independent of the representation of the parton i. This function is known [58] up to three loops in QCD. In contrast, the residual term $\widetilde{\gamma}_K^{(i)}$ represents (yet unknown) potential contributions which violate Casimir scaling; it depends on the representation in a more complicated way, for example through terms that involve irreducible combinations of four colour generators. The particular way in which the cusp anomalous dimension depends on the representation will not be important for most of what follows, but it will be used in Section 5 for constructing an explicit expression for the soft anomalous dimension.

In a similar way one expects that $\delta_{\mathcal{J}_i}$ of Eq. (3.7) would be proportional to the quadratic Casimir at least up to three loops, so we write

$$\delta_{\mathcal{J}}^{(i)}\left(\alpha_s\right) \equiv C_i\,\widehat{\delta}_{\mathcal{J}}\left(\alpha_s\right) + \widetilde{\delta}_{\mathcal{J}}^{(i)}\,, \tag{3.13}$$

where $\widehat{\delta}_{\mathcal{J}}\left(\alpha_s\right) = \alpha_s/\pi + \ldots$, $\widetilde{\delta}_{\mathcal{J}}^{(i)} = \mathcal{O}(\alpha_s^4)$, and C_i is the Casimir operator defined in Eq. (3.11). The one-loop result quoted here can be deduced from the calculation in Ref. [40].

3.2 Factorization constraints for soft anomalous dimension matrices

Having established the simple result in Eq. (3.9) for the eikonal jet, where the kinematic dependence is determined by the cusp anomalous dimension, one may wonder if the same is true for the soft function. In other words, one may ask whether the full dependence of $\Gamma_{IJ}^{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)$ in Eq. (2.12) on $\beta_i \cdot \beta_j$ is associated with the cusps.

One may try to apply the rescaling argument, arguing that if not for the cusp singularities $\Gamma_{IJ}^{\mathcal{S}}$ should have been invariant with respect to independent rescalings of each β_i. One realises however that if the number of hard external lines is $n \geq 4$, it is possible to construct homogeneous conformal cross–ratios such as

$$\rho_{ijkl} \equiv \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)} \tag{3.14}$$

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[7]We thank Juan Maldacena for pointing this out to us.

which are inherently invariant with respect to rescalings of each individual velocity, thus evading this argument. Kinematic dependence does not necessarily lead to violation of the rescaling–invariance property, and therefore might not be associated with the cusp singularities. It is important to note, though, that for $n = 2, 3$ there are no such ratios and the argument does hold. One should expect, therefore, that at least for $n = 2, 3$ the full kinematic dependence should be controlled by γ_K to all orders. We shall see below that this is indeed the case.

The observation that allows us to make a step forward is that the soft function, for any number of legs, can be indirectly constrained by considering the kinematic dependence of the reduced soft function, defined in Eq. (2.7). Here the cusp singularity itself cancels out, and yet, as we will see, it leaves its trace through the dependence on the kinematics. To proceed, consider the renormalization group equation for the reduced soft matrix $\overline{\mathcal{S}}$, which reads

$$\mu \frac{d}{d\mu} \overline{\mathcal{S}}_{IK}^{[f]} \left(\rho_{ij}, \alpha_s(\mu^2), \epsilon \right) = -\Gamma_{IJ}^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s(\mu^2) \right) \overline{\mathcal{S}}_{JK}^{[f]} \left(\rho_{ij}, \alpha_s(\mu^2), \epsilon \right) , \qquad (3.15)$$

where $\Gamma_{IJ}^{\overline{\mathcal{S}}}$, in contrast to $\Gamma_{IJ}^{\mathcal{S}}$ and $\gamma_{\mathcal{J}}$, is free of singularities. Its invariance with respect to scaling of each individual velocity β_i is manifest in its functional dependence on the velocities only through the ratios ρ_{ij}, defined in Eq. (2.9).

Given the definition of the reduced soft matrix in Eq. (2.7), one easily sees that the various eikonal anomalous dimensions are related by

$$\Gamma_{IJ}^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s \right) = \Gamma_{IJ}^{\mathcal{S}} \left(\beta_i \cdot \beta_j, \alpha_s, \epsilon \right) - \delta_{IJ} \sum_{k=1}^{n} \gamma_{\mathcal{J}_k} \left(w_k, \alpha_s, \epsilon \right) , \qquad (3.16)$$

where, as above, $w_k \equiv 2(\beta_k \cdot n_k)^2/n_k^2$. In words, pole terms must cancel on the right-hand side of Eq. (3.16), and the functional dependence on eikonal vectors must be arranged so as to reconstruct functions of ρ_{ij} in order to be consistent with the left-hand side. Substituting Eq. (3.8) into Eq. (3.16) we get an explicit expression for $\Gamma_{IJ}^{\overline{\mathcal{S}}}$. In terms of the D-dimensional running coupling, we can write

$$\begin{aligned} \Gamma_{IJ}^{\overline{\mathcal{S}}} \left(\rho_{ij}, \alpha_s(\mu^2, \epsilon) \right) &= \Gamma_{IJ}^{\mathcal{S}} \left(\beta_i \cdot \beta_j, \alpha_s(\mu^2, \epsilon), \epsilon \right) - \delta_{IJ} \sum_{k=1}^{n} \left[-\frac{1}{2} \delta_{\mathcal{J}_k} \left(\alpha_s(\mu^2, \epsilon) \right) \right. \\ &\left. + \frac{1}{4} \gamma_K^{(k)} \left(\alpha_s(\mu^2, \epsilon) \right) \ln(w_i) + \frac{1}{4} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \gamma_K^{(k)} \left(\alpha_s(\xi^2, \epsilon) \right) \right] . \end{aligned} \qquad (3.17)$$

This immediately implies that

- off-diagonal terms in $\Gamma^{\mathcal{S}}$ must be *finite*, and must depend only on conformal cross ratios of the form of ρ_{ijkl} in Eq. (3.14), which indeed can readily be turned into ratios of ρ_{ij}'s, as defined in Eq. (2.9), for example

$$\rho_{ijkl} \equiv \frac{(\beta_i \cdot \beta_j)(\beta_k \cdot \beta_l)}{(\beta_i \cdot \beta_k)(\beta_j \cdot \beta_l)} = \left(\frac{\rho_{ij} \, \rho_{kl}}{\rho_{ik} \, \rho_{jl}} \right)^{1/2} e^{-i\pi(\lambda_{ij} + \lambda_{kl} - \lambda_{ik} - \lambda_{jl})} ; \qquad (3.18)$$

- singular terms in $\Gamma^{\mathcal{S}}$ must be confined to diagonal matrix elements, and must be determined by the cusp anomalous dimension according to

$$\Gamma^{\mathcal{S}}_{IJ}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2, \epsilon), \epsilon\right) = \delta_{IJ} \sum_{k=1}^{n} \frac{1}{4} \int_0^{\mu^2} \frac{d\xi^2}{\xi^2} \gamma_K^{(k)}\left(\alpha_s(\xi^2, \epsilon)\right) + \mathcal{O}(\epsilon^0)\,; \qquad (3.19)$$

- finite terms in the diagonal matrix elements must include — in addition to terms that depend exclusively on conformal cross-ratios, as in Eq. (3.18) — terms with definite kinematic dependence on $\beta_i \cdot \beta_j$, which are proportional to γ_K, so as to properly combine with the $\ln(w_i)$ terms in Eq. (3.17).

To illustrate how these features arise, in Appendix B we perform an explicit one-loop calculation of a $2 \to 2$ quark-antiquark scattering amplitude. Note in particular that a given diagram violates rescaling invariance also in off-diagonal terms, but this violation is eliminated upon taking the sum of all diagrams, which is where conformal cross ratios like ρ_{ijkl} are formed. This is a consequence of gauge invariance.

Returning to the general case, in Section 5 we will give an explicit formula that satisfies the requirements outlined above. Our goal here is to first formulate the requirements in a compact and general way. To this end, let us consider the derivative of Eq. (3.16) (or Eq. (3.17)) with respect to $\ln(w_i)$. Noting that the w_i dependence appears only through the eikonal jet functions, and using the results of Section 3.1, which imply

$$w_i \frac{\partial}{\partial w_i} \gamma_{\mathcal{J}_i}\left(w_i, \alpha_s, \epsilon\right) = \frac{1}{4} \gamma_K^{(i)}\left(\alpha_s\right)\,, \qquad (3.20)$$

we obtain a simple result for the w_i-dependence of $\Gamma^{\overline{\mathcal{S}}}_{IJ}$,

$$w_i \frac{\partial}{\partial w_i} \Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij}, \alpha_s\right) = -\delta_{IJ}\, w_i \frac{\partial}{\partial w_i} \gamma_{\mathcal{J}}\left(w_i, \alpha_s, \epsilon\right) = -\frac{1}{4} \gamma_K^{(i)}\left(\alpha_s\right) \delta_{IJ}\,. \qquad (3.21)$$

This result can be turned into an equation for the dependence of the anomalous dimension matrix on its proper arguments, ρ_{ij}, just using the chain rule. Indeed, for any function F depending on w_i only through ρ_{ij}, one finds

$$\frac{\partial}{\partial \ln w_i} F\left(\rho_{ij}\right) = -\sum_{j \neq i} \frac{\partial}{\partial \ln \rho_{ij}} F\left(\rho_{ij}\right)\,. \qquad (3.22)$$

We conclude that

$$\sum_{j \neq i} \frac{\partial}{\partial \ln \rho_{ij}} \Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij}, \alpha_s\right) = 0\,, \qquad \forall i\,, \qquad I \neq J\,,$$

$$\sum_{j \neq i} \frac{\partial}{\partial \ln \rho_{ij}} \Gamma^{\overline{\mathcal{S}}}_{IJ}\left(\rho_{ij}, \alpha_s\right) = \frac{1}{4} \gamma_K^{(i)}\left(\alpha_s\right)\,, \qquad \forall i\,, \qquad I = J\,. \qquad (3.23)$$

As expected, the first equation in (3.23) states that off-diagonal matrix elements of the soft anomalous dimension matrix should be logarithmic functions of homogeneous conformal cross–ratios of ρ_{ij}'s, such as ρ_{ijkl}. For diagonal terms, the second equation in (3.23) states that inhomogeneous terms are allowed, but they must be proportional to the cusp anomalous dimension in the colour representation of parton i. We will explore the consequences of these constraints in the following sections, beginning with the case of two-parton amplitudes.

4. Two-parton amplitudes

In this section we consider in some detail the consequences of the new constraints in the simplest case of amplitudes with two hard coloured partons. We choose to analyse in Section 4.1 the special case of the spacelike Sudakov form factor of a quark, but the results apply, with minor modifications, to any amplitude with two hard partons. In Section 4.2 we consider the partonic jet function, which is an important building block in the factorization formula, Eq. (2.2), for any amplitude. We use there the results of Section 4.1 to constrain the kinematic dependence of the partonic jet, which is significantly more involved than that of the eikonal jet considered above.

4.1 The case of the Sudakov form factor

Let us consider the implications of the factorization constraints derived above on the simplest fixed-angle scattering amplitude, the Sudakov form factor. We will see that the constraints of Eq. (3.23) lead to a refinement of the results of Ref. [40], since the kinematic dependence of the Sudakov soft function can be explicitly determined in terms of the cusp anomalous dimension.

As for any amplitude, our starting point is the factorization formula of Eq. (2.2), which here takes the form

$$
\Gamma\left(\frac{Q^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) = H\left(\frac{Q^2}{\mu^2}, \frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2)\right) \times \mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right)
$$

$$
\times \prod_{i=1}^{2} \frac{J\left(\frac{(2p_i \cdot n_i)^2}{n_i^2 \mu^2}, \alpha_s(\mu^2), \epsilon\right)}{\mathcal{J}\left(\frac{2(\beta_i \cdot n_i)^2}{n_i^2}, \alpha_s(\mu^2), \epsilon\right)}, \tag{4.1}
$$

where $Q^2 = (p_1 + p_2)^2 = 2p_1 \cdot p_2$. For definiteness, we will consider the space-like form factor, $Q^2 < 0$.

In the case of the form factor, the soft function $\mathcal{S}$ is simply the eikonal correlator defined by two Wilson lines running along the classical light-like parton trajectories, with velocities given by β_1 and β_2. Thus

$$
\mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right) = \langle 0|\Phi_{\beta_2}(\infty, 0)\Phi_{\beta_1}(0, -\infty)|0\rangle . \tag{4.2}
$$

To determine the kinematic dependence of $\mathcal{S}$, we consider the reduced soft function $\overline{\mathcal{S}}$, which is given by

$$
\overline{\mathcal{S}}\left(\rho_{12}, \alpha_s(\mu^2), \epsilon\right) = \frac{\mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2), \epsilon\right)}{\Pi_{i=1}^{2} \mathcal{J}\left(w_i, \alpha_s(\mu^2), \epsilon\right)}, \tag{4.3}
$$

where, as before, $w_i \equiv 2(\beta_i \cdot n_i)^2/n_i^2$ and $\rho_{12} = (\beta_1 \cdot \beta_2)^2/(w_1 w_2)$; the latter is specific to the spacelike momentum configuration where $\lambda_{12} = 0$ so the phase in Eq. (2.9) is absent.

The reduced soft function $\overline{\mathcal{S}}$ obeys the renormalization group equation

$$
\frac{d \ln \overline{\mathcal{S}}\left(\rho_{12}, \alpha_s(\mu^2), \epsilon\right)}{d \ln \mu} = -\gamma_{\overline{\mathcal{S}}}\left(\rho_{12}, \alpha_s(\mu^2)\right), \tag{4.4}
$$

which leads to exponentiation. Since $\overline{\mathcal{S}}$ is a pure counterterm, one simply gets

$$\overline{\mathcal{S}}\left(\rho_{12}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{-\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\gamma_{\overline{\mathcal{S}}}\left(\rho_{12}, \alpha_s(\lambda^2, \epsilon)\right)\right\},\qquad (4.5)$$

in analogy with Eq. (3.1) for the eikonal jet.

Factorization now requires the anomalous dimension $\gamma_{\overline{\mathcal{S}}}$ to be a linear function of $\ln\rho_{12}$. Indeed, Eq. (3.23) in this case reads

$$\frac{\partial\gamma_{\overline{\mathcal{S}}}\left(\rho_{12}, \alpha_s\right)}{\partial\ln\rho_{12}} = \frac{1}{4}\gamma_K(\alpha_s),\qquad (4.6)$$

which integrates to

$$\gamma_{\overline{\mathcal{S}}}\left(\rho_{12}, \alpha_s\right) = \frac{1}{4}\gamma_K(\alpha_s)\,\ln\rho_{12} + \delta_{\overline{\mathcal{S}}}(\alpha_s),\qquad (4.7)$$

where $\delta_{\overline{\mathcal{S}}}(\alpha_s)$ is introduced as a constant of integration, and does not depend on ρ_{12}. As expected, the dependence of $\gamma_{\overline{\mathcal{S}}}$ on the kinematic variable ρ_{12} is very simple, and is fully determined by the cusp anomalous dimension.

Note that, similarly to what was done for the jet function in Eq. (3.13), we may extract from the anomalous dimension $\delta_{\overline{\mathcal{S}}}$ a factor of the Casimir operator of the relevant representation, defining

$$\delta_{\overline{\mathcal{S}}}^{(i)}(\alpha_s) \equiv C_i\,\widehat{\delta}_{\mathcal{S}}(\alpha_s) + \widetilde{\delta}_{\overline{\mathcal{S}}}^{(i)},\qquad (4.8)$$

where as usual $\widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s) = \alpha_s/\pi + \ldots$, $\widetilde{\delta}_{\overline{\mathcal{S}}}^{(i)} = \mathcal{O}(\alpha_s^4)$, and $\widehat{\delta}_{\overline{\mathcal{S}}}$ is a universal function of the coupling, independent of the colour representation. The one-loop result quoted here will be determined in Eq. (4.21) below.

Using Eq. (4.7) and Eq. (4.5), we can now write down an explicit expression for $\overline{\mathcal{S}}$, where the kinematic dependence is completely solved. We find

$$\overline{\mathcal{S}}\left(\rho_{12}, \alpha_s(\mu^2), \epsilon\right) = \exp\left\{-\frac{\ln(\rho_{12})}{8}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\gamma_K\left(\alpha_s(\lambda^2, \epsilon)\right) - \frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\delta_{\overline{\mathcal{S}}}\left(\alpha_s(\lambda^2, \epsilon)\right)\right\}.\qquad (4.9)$$

Finally, using the definition of $\overline{\mathcal{S}}$ in Eq. (4.3) we obtain an explicit result for the original soft function $\mathcal{S}$,

$$\begin{aligned}\mathcal{S}\left(\beta_1\cdot\beta_2, \alpha_s(\mu^2), \epsilon\right) &= \overline{\mathcal{S}}\left(\rho_{12}, \alpha_s(\mu^2), \epsilon\right)\times\Pi_{i=1}^2\,\mathcal{J}\left(w_i, \alpha_s(\mu^2), \epsilon\right)\\ &= \exp\left\{\frac{1}{2}\int_0^{\mu^2}\frac{d\xi^2}{\xi^2}\left[\delta_{\mathcal{S}}\left(\alpha_s(\xi^2, \epsilon)\right) - \frac{1}{2}\gamma_K\left(\alpha_s(\xi^2, \epsilon)\right)\ln\left(\frac{-\beta_1\cdot\beta_2\mu^2}{\xi^2}\right)\right]\right\},\end{aligned}\qquad (4.10)$$

where we defined

$$\delta_{\mathcal{S}}(\alpha_s) = \delta_{\mathcal{J}}(\alpha_s) - \delta_{\overline{\mathcal{S}}}(\alpha_s),\qquad (4.11)$$

combining the two constants of integration introduced in Eqs. (3.7) and (4.7). Eq. (4.10) is intuitively appealing: the spacelike or timelike nature of the eikonal form factor is associated with the explicit logarithm multiplying the cusp anomalous dimension, just as is the case for the full form factor.

We conclude this section by briefly comparing our results with those reported in Ref. [40]. In order to do so, consider the anomalous dimension of the soft function $\mathcal{S}$, defined by

$$\frac{d \ln \mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s, \epsilon\right)}{d \ln \mu} = -\gamma_{\mathcal{S}}\left(\beta_1 \cdot \beta_2, \alpha_s, \epsilon\right) \tag{4.12}$$

Using Eq. (4.10), we obtain an explicit expression for $\gamma_{\mathcal{S}}$. At the scale μ^2, it reads

$$\gamma_{\mathcal{S}}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2, \epsilon), \epsilon\right) = -\delta_{\mathcal{S}}\left(\alpha_s(\mu^2, \epsilon)\right) + \frac{1}{2}\gamma_K\left(\alpha_s(\mu^2, \epsilon)\right) \ln\left(-\beta_1 \cdot \beta_2\right)$$
$$+ \frac{1}{2}\int_0^{\mu^2} \frac{d\xi^2}{\xi^2}\gamma_K\left(\alpha_s(\xi^2, \epsilon)\right) , \tag{4.13}$$

which is analogous to the result for the anomalous dimension of the eikonal jet in Eq. (3.8). One may of course verify the consistency of the various renormalization group equations corresponding to Eq. (4.3), observing that

$$\gamma_{\overline{\mathcal{S}}}\left(\rho_{12}, \alpha_s\right) = \gamma_{\mathcal{S}}\left(\beta_1 \cdot \beta_2, \alpha_s, \epsilon\right) - \gamma_{\mathcal{J}_1}\left(w_1, \alpha_s, \epsilon\right) - \gamma_{\mathcal{J}_2}\left(w_2, \alpha_s, \epsilon\right) , \tag{4.14}$$

where logarithms of different arguments on the right-hand side nicely combine to form a logarithm of the scale–invariant ratio ρ_{12}, as expected.

We can compare our final result for $\gamma_{\mathcal{S}}$ in Eq. (4.13) to the result of Ref. [40], where the same anomalous dimension is written as

$$\gamma_{\mathcal{S}}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2, \epsilon), \epsilon\right) = -G_{\text{eik}}\left(\beta_1 \cdot \beta_2, \alpha_s(\mu^2, \epsilon)\right) + \frac{1}{2}\int_0^{\mu^2} \frac{d\xi^2}{\xi^2}\gamma_K\left(\alpha_s(\xi^2, \epsilon)\right) , \tag{4.15}$$

defining the eikonal function G_{eik}. This allows us to relate G_{eik} to $\delta_{\mathcal{S}}$, while solving the dependence of G_{eik} on the kinematical vectors β_i. We find

$$G_{\text{eik}}\left(\beta_1 \cdot \beta_2, \alpha_s\right) = -\frac{1}{2}\gamma_K(\alpha_s) \ln\left(-\beta_1 \cdot \beta_2\right) + \delta_{\mathcal{S}}(\alpha_s) . \tag{4.16}$$

Finally, let us collect the one-loop expressions for the different eikonal functions discussed here[8], showing that the kinematic dependence found in explicit calculations is consistent with the general statements we have made. The soft function $\mathcal{S}$ was computed at one loop in [23], with the result

$$\mathcal{S}\left(\beta_1 \cdot \beta_2, \alpha_s, \epsilon\right) = 1 - \frac{\alpha_s}{4\pi}C_F\left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\ln(-\beta_1 \cdot \beta_2)\right] + \mathcal{O}(\alpha_s^2) . \tag{4.17}$$

The eikonal jet at this order can be computed by combining Eq. (3.9) with Eq. (3.13), obtaining

$$\mathcal{J}_i\left(w_i, \alpha_s, \epsilon\right) = 1 - \frac{\alpha_s}{4\pi}C_F\left[\frac{1}{\epsilon^2} + \frac{1}{\epsilon}\left(1 - \ln\left(w_i\right)\right)\right] + \mathcal{O}(\alpha_s^2) . \tag{4.18}$$

According to Eq. (4.3), the reduced soft function is then given by

$$\overline{\mathcal{S}}\left(\rho_{12}, \alpha_s, \epsilon\right) = 1 + \frac{\alpha_s}{2\pi}C_F\frac{1}{\epsilon}\left[1 + \frac{1}{2}\ln(\rho_{12})\right] + \mathcal{O}(\alpha_s^2) , \tag{4.19}$$

---

[8]We choose to work with the quark form factor; in case of the gluon one the overall colour factor C_F should simply be replaced by C_A.

which is indeed a function of ρ_{12}, consistent with the general expression in Eq. (4.9). Taking the logarithmic derivative of Eq. (4.19) one computes the anomalous dimension

$$\gamma_{\overline{S}}(\rho_{12}, \alpha_s) = \frac{\alpha_s}{\pi} C_F \left(1 + \frac{1}{2} \ln(\rho_{12}) \right) + \mathcal{O}(\alpha_s^2). \tag{4.20}$$

The ρ_{12} dependence of $\gamma_{\overline{S}}$, as expected, is given by Eq. (4.7), when the one-loop result for $\gamma_K^{(1)} = 2 C_F \alpha_s/\pi$ is used. The leading-order term of the function $\delta_{\overline{S}}$ is then given by

$$\delta_{\overline{S}}(\alpha_s) = C_F \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2), \tag{4.21}$$

while, using Eq. (4.11) and Eq. (3.13), we verify that $\delta_S = \mathcal{O}(\alpha_s^2)$, in agreement with the results of Ref. [40].

To summarize, we have shown that the kinematic dependence of all purely eikonal functions entering the form factor can be reconstructed using constraints arising from factorization, together with general properties of pure counterterms in dimensional regularization. The result is that the entire kinematic dependence of these functions is proportional to the cusp anomalous dimension. This confirms our initial expectation, based on the fact that violation of rescaling invariance in such functions can only be introduced by the cusp anomaly. In Section 5 we will return to the generic case of multi-leg fixed-angle scattering amplitudes, and work out the consequences of our constraints. Before that, however, let us briefly contrast our findings concerning eikonal functions with the case of partonic amplitudes. Again, we choose the simplest example, that of the partonic jet.

4.2 Constraining the partonic jet function

We have seen that the kinematic dependence of eikonal functions is remarkably simple, to all-order in perturbation theory. The singularity structure of a partonic amplitude resembles that of the corresponding eikonal function. Nevertheless, because partonic amplitudes do depend on dimensionful kinematic variables and have infrared singularities as well as distinct ultraviolet renormalization properties, they do not admit similarly simple all-order expressions.

An exception is the Sudakov form factor $\Gamma(Q^2, \epsilon)$ itself, which does not get renormalized ($d\Gamma(Q^2, \epsilon)/\ln \mu = 0$), so that its entire perturbative expansion in dimensional regularization is determined by its infrared singularities [25, 40], according to

$$\Gamma(Q^2, \epsilon) = \exp \left\{ \frac{1}{2} \int_0^{-Q^2} \frac{d\xi^2}{\xi^2} \left[G\left(-1, \alpha_s(\xi^2, \epsilon), \epsilon\right) - \frac{1}{2}\gamma_K\left(\alpha_s(\xi^2, \epsilon)\right) \ln\left(\frac{-Q^2}{\xi^2}\right) \right] \right\}. \tag{4.22}$$

Indeed, in this case the eikonal function in Eq. (4.10) is very similar to the full form factor, and apart from the different scales, the only qualitative difference is the appearance of non-negative powers of ϵ in the function $G(-1, \alpha_s, \epsilon)$.

In a general scattering amplitude, which does get renormalized, the dependence on the renormalization point μ^2 and on the kinematic variables is of course distinct, and neither of them is associated exclusively with the cusp anomalous dimension. The simplest example is that of the partonic jet, which we now consider.

Using the factorization formula for the Sudakov form factor, Eq. (4.1), together with Eqs. (4.22), (3.9) and (4.10), one can directly determine the product

$$H\left(\frac{Q^2}{\mu^2},\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s\right)\prod_{i=1}^{2}J\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s,\epsilon\right)\equiv\prod_{i=1}^{2}\widetilde{J}\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s,\epsilon\right)\ ,\quad(4.23)$$

where we defined

$$\widetilde{J}\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\mu^2},\alpha_s,\epsilon\right)=$$
$$\exp\left\{\frac{1}{4}\int_{\mu^2}^{-Q^2}\frac{d\lambda^2}{\lambda^2}\left[-\frac{1}{2}\gamma_K\left(\alpha_s(\lambda^2,\epsilon)\right)\ln\left(\frac{-Q^2}{\lambda^2}\right)+G(-1,\alpha_s(\lambda^2,\epsilon),\epsilon)\right]\right.\quad(4.24)$$
$$\left.+\frac{1}{4}\int_{0}^{\mu^2}\frac{d\lambda^2}{\lambda^2}\left[-\frac{1}{2}\gamma_K\left(\alpha_s(\lambda^2,\epsilon)\right)\ln\left(\frac{(2p_i\cdot n_i)^2}{n_i^2\lambda^2}\right)+\delta_J(\alpha_s(\lambda^2,\epsilon),\epsilon)\right]\right\},$$

with

$$\delta_J(\alpha_s,\epsilon)\equiv\delta_{\overline{S}}(\alpha_s)+G(-1,\alpha_s,\epsilon)\qquad(4.25)$$

where $\delta_{\overline{S}}$ is the function introduced in Eq. (4.7). Note that $\delta_J(\alpha_s,\epsilon)$, in contrast to $\delta_{\mathcal{J}}(\alpha_s)$ in Eq. (3.7), depends on ϵ explicitly, and it does have non-negative powers of ϵ coming from $G(-1,\alpha_s,\epsilon)$.

We now observe that in Eq. (4.24) all infrared singularities emerge from the $\lambda^2\to0$ limit of the integral over the running coupling in the last line. These singularities must all reside in the partonic jet function J, while the renormalized hard coefficient function H is finite. This observation, together with the renormalization group equation for J, Eq. (2.10), and the fact that J does not depend explicitly on Q^2, implies that J takes the form

$$J\left(\frac{p_n^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right)=\exp\left\{h_J\left(\alpha_s(p_n^2)\right)+\frac{1}{2}\int_{\mu^2}^{p_n^2}\frac{d\lambda^2}{\lambda^2}\gamma_J\left(\alpha_s(\lambda^2)\right)\right.\qquad(4.26)$$
$$\left.+\int_{0}^{p_n^2}\frac{d\lambda^2}{\lambda^2}\left[-\frac{1}{8}\gamma_K\left(\alpha_s(\lambda^2,\epsilon)\right)\ln\left(\frac{p_n^2}{\lambda^2}\right)+\frac{1}{4}\delta_J\left(\alpha_s(\lambda^2,\epsilon),\epsilon\right)\right]\right\},$$

where we defined $p_n^2\equiv(2p\cdot n)^2/n^2$ for brevity. All the functions appearing here are finite as $\epsilon\to0$, so also here singularities are generated only through the integration over the running coupling in the last line. The function $\delta_J(\alpha_s(\lambda^2),\epsilon)$ depends on ϵ explicitly, while γ_K, γ_J and h_J do not. As far as h_J is concerned, this last statement is not obvious a priori, and it will be proven to all orders below.

The structure of the result in Eq. (4.26) for the partonic jet is intuitively clear: infrared and collinear singularities are generated by the integration over the scale of the running coupling in the second line. These terms in the exponent are similar to the expression found for the eikonal jet, Eq. (3.9), and indeed the double poles of the two expressions match, as they must. Single poles, on the other hand, are different, on account of hard collinear radiation, which is not correctly approximated by $\mathcal{J}$. Further dependence on the hard

scale p_n^2 arises in the first line of Eq. (4.26), due to the non-trivial ultraviolet behaviour of J, which is dictated by γ_J. It is natural in this case to start the renormalization group evolution at $\mu^2 = p_n^2$, where the function h_J gives the initial condition.

Once again, we may compare our all-order expression with one-loop results. One may start with the well-known result [25] for the function G,

$$
G\left(\frac{Q^2}{\mu^2}, \alpha_s, \epsilon\right) = \frac{\alpha_s}{\pi} C_F \left[\left(\frac{\mu^2}{-Q^2}\right)^\epsilon \left(\frac{1}{\epsilon} + \frac{3}{2} - \epsilon\left(\frac{\pi^2}{12} - 4\right) + \mathcal{O}(\epsilon^2)\right) - \frac{1}{\epsilon} \right] + \mathcal{O}(\alpha_s^2) .
\tag{4.27}
$$

Using Eq. (4.25) and Eq. (4.7) one then finds

$$
\delta_J\left(\alpha_s, \epsilon\right) = \frac{\alpha_s}{\pi} C_F \left(\frac{5}{2} - \epsilon\left(\frac{\pi^2}{12} - 4\right) + \mathcal{O}(\epsilon^2)\right) + \mathcal{O}(\alpha_s^2) .
\tag{4.28}
$$

The renormalization of the jet, on the other hand, yields

$$
\gamma_J(\alpha_s) = -\frac{3}{4} C_F \frac{\alpha_s}{\pi} + \mathcal{O}(\alpha_s^2) .
\tag{4.29}
$$

Inserting the one-loop results of Eqs. (3.12), (4.28) and (4.29) into our general expression, Eq. (4.26), and expanding to $\mathcal{O}(\alpha_s)$, we obtain finally

$$
\begin{aligned}
J\left(\frac{p_n^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) &= 1 + h_J^{(1)}(\alpha_s) + \frac{\alpha_s}{4\pi} C_F \left\{ -\frac{1}{\epsilon^2} - \frac{1}{\epsilon}\left[\frac{5}{2} - \ln\left(\frac{p_n^2}{\mu^2}\right)\right] \right. \\
&\quad \left. -\frac{1}{2}\ln^2\left(\frac{p_n^2}{\mu^2}\right) + \ln\left(\frac{p_n^2}{\mu^2}\right) + 4 - \frac{\pi^2}{12} + \mathcal{O}(\epsilon) \right\} + \mathcal{O}(\alpha_s^2) .
\end{aligned}
\tag{4.30}
$$

Comparing this result to the explicit one-loop calculation, reported in Section 3 of Ref. [40], we find full consistency and determine the finite coefficient h_J,

$$
h_J(\alpha_s) = -\frac{\alpha_s}{\pi} C_F \left(\frac{3}{2} + \frac{\pi^2}{12}\right) + \mathcal{O}(\alpha_s^2) .
\tag{4.31}
$$

We conclude this section by comparing our parametrization of the partonic jet, Eq. (4.26), with the results of Ref. [40]. This will then be used to relate the function $h_J(\alpha_s)$, which sets the normalization of the jet function to the hard function appearing in the factorized the Sudakov form factor. We begin by considering the differential equation that controls the dependence of J on p_n^2. This is, once again, a '$K + G$' equation, and can be written as [40]

$$
\frac{\partial \ln J\left(p_n^2/\mu^2, \alpha_s(\mu^2), \epsilon\right)}{\partial \ln p_n^2} = \frac{1}{2} \mathcal{G}\left(\frac{p_n^2}{\mu^2}, \alpha_s(\mu^2), \epsilon\right) - \frac{1}{8}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \gamma_K\left(\alpha_s(\lambda^2, \epsilon)\right) ,
\tag{4.32}
$$

where the cusp–related singularity was explicitly extracted. The renormalization group equation for J, Eq. (2.10), implies that

$$
\frac{d}{d\ln\mu^2} \frac{\partial \ln J\left(p_n^2/\mu^2, \alpha_s, \epsilon\right)}{\partial \ln p_n^2} = \frac{\partial}{\partial \ln p_n^2} \frac{d\ln J\left(p_n^2/\mu^2, \alpha_s, \epsilon\right)}{d\ln\mu^2} = 0 .
\tag{4.33}
$$

Applying this to Eq. (4.32) in turn implies that the scale dependence of $\mathcal{G}$ is controlled by the cusp anomalous dimension, according to

$$\frac{d}{d\ln\mu^2}\,\mathcal{G}\left(\frac{p_n^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = \frac{1}{4}\,\gamma_K\left(\alpha_s\right),\tag{4.34}$$

or, upon integration,

$$\mathcal{G}\left(\frac{p_n^2}{\mu^2},\alpha_s(\mu^2),\epsilon\right) = \mathcal{G}\left(1,\alpha_s(p_n^2),\epsilon\right) - \frac{1}{4}\int_{\mu^2}^{p_n^2}\frac{d\lambda^2}{\lambda^2}\gamma_K\left(\alpha_s(\lambda^2,\epsilon)\right).\tag{4.35}$$

Now, using our general expression for the partonic jet in Eq. (4.26), we can readily interpret the function $\mathcal{G}$ in (4.32) in terms of the anomalous dimensions γ_J and δ_J, and of the finite function h_J. We find

$$\mathcal{G}\left(1,\alpha_s,\epsilon\right) = \beta(\alpha_s,\epsilon)\frac{d}{d\alpha_s}h_J(\alpha_s) + \gamma_J(\alpha_s) + \frac{1}{2}\delta_J(\alpha_s,\epsilon),\tag{4.36}$$

where the first term is proportional to the β function, $\beta(\alpha_s,\epsilon) = d\alpha_s(\mu^2,\epsilon)/d\ln\mu$, so that it starts contributing at $\mathcal{O}(\alpha_s^2)$.

Using the definition of $\delta_J(\alpha_s,\epsilon)$, given in Eq. (4.25), and setting[9] $\mu^2 = p_n^2$, we may now obtain an expression for the function G appearing in the Sudakov form factor, Eq. (4.22), which we can compare to the results of Ref. [40]. We find

$$G\left(-1,\alpha_s,\epsilon\right) = -2\,\beta(\alpha_s,\epsilon)\frac{d}{d\alpha_s}h_J(\alpha_s) - \delta_{\overline{\mathcal{S}}}(\alpha_s) - 2\,\gamma_J(\alpha_s) + 2\,\mathcal{G}(1,\alpha_s,\epsilon).\tag{4.37}$$

Eq. (4.37) can be compared with the result of Section 4 in Ref. [40], which reads

$$G(-1,\alpha_s,\epsilon) = \beta(\alpha_s,\epsilon)\frac{\partial}{\partial\alpha_s}\ln H(-1,w_i,\alpha_s) - \gamma_{\overline{\mathcal{S}}}(\rho_{12},\alpha_s) - 2\gamma_J(\alpha_s) + \sum_{i=1}^{2}\mathcal{G}_i(w_i,\alpha_s,\epsilon),\tag{4.38}$$

where we have chosen a specific common normalization for the velocities, $p_i = \beta_i Q_0/\sqrt{2}$, with $Q_0 = \sqrt{-Q^2}$, corresponding to $(2p_i\cdot n_i)^2/n_i^2 = -Q^2\,w_i$, and $\beta_1\cdot\beta_2 = -1$, so that $\rho_{12} = 1/(w_1\,w_2)$. Using now our result for $\gamma_{\overline{\mathcal{S}}}(\rho_{12},\alpha_s)$, Eq. (4.7), we see that Eq. (4.38) takes the form

$$\begin{aligned}G(-1,\alpha_s,\epsilon) = {}& \beta(\alpha_s,\epsilon)\frac{\partial}{\partial\alpha_s}\ln H(-1,w_i,\alpha_s) + \frac{1}{4}\,\gamma_K(\alpha_s)\,\ln(w_1\,w_2)\\ & - \delta_{\overline{\mathcal{S}}}(\alpha_s) - 2\,\gamma_J(\alpha_s) + \sum_{i=1}^{2}\mathcal{G}_i(w_i,\alpha_s,\epsilon).\end{aligned}\tag{4.39}$$

Eq. (4.39) holds for arbitrary w_1 and w_2, implying that the dependence of the *r.h.s.* on these parameters through the explicit γ_K term, and through the functions H and $\mathcal{G}$, must cancel out.

---

[9]Note that in setting $(2p\cdot n)^2/n^2 = \mu^2$ we assume that the auxiliary vector n is timelike, $n^2 > 0$. A spacelike n is also possible of course, and this choice would be reflected in replacing $\mathcal{G}(1,\alpha_s,\epsilon)$ below by $\mathcal{G}(-1,\alpha_s,\epsilon)$.

Comparing Eqs. (4.39) and (4.37) we obtain

$$-2\,\beta(\alpha_s,\epsilon)\,\frac{d}{d\alpha_s}h_J(\alpha_s) + 2\,\mathcal{G}(1,\alpha_s,\epsilon) = \frac{1}{4}\,\gamma_K(\alpha_s)\,\ln(w_1\,w_2)$$
$$+\,\beta(\alpha_s,\epsilon)\frac{\partial}{\partial\alpha_s}\ln H(-1,w_i,\alpha_s) + \sum_{i=1}^{2}\mathcal{G}_i(w_i,\alpha_s,\epsilon)\,. \tag{4.40}$$

We already know that the dependence of the *r.h.s.* on the w_i cancels out, so we are allowed to use this equality for any w_i. Picking $w_1 = w_2 = 1$, we obtain

$$h_J(\alpha_s) = -\frac{1}{2}\ln H(-1,w_i = 1,\alpha_s)\,, \tag{4.41}$$

which determines the function h_J in terms of the hard function H of the Sudakov form factor. Eq. (4.41) implies in particular that $h_J(\alpha_s)$ does not carry any explicit dependence on ϵ, but depends only on the coupling, as anticipated.

5. Multi-parton amplitudes: the soft anomalous dimension matrix

Let us now consider the general case of multi-parton amplitudes and examine the consequences of our new constraints on the all-order structure of the soft anomalous dimension matrix. We first note that the constraint of Eq. (3.23), which can be written as

$$\sum_{j,j\neq i}\frac{\partial}{\partial\ln(\rho_{ij})}\Gamma^{\overline{\mathcal{S}}}_{MN}(\rho_{ij},\alpha_s) = \frac{1}{4}\,\gamma_K^{(i)}(\alpha_s)\,\delta_{MN}\,, \qquad \forall i\,, \tag{5.1}$$

is an equality between colour matrices, valid in any basis. We will then proceed to work with the colour generator notation, as in [28], without specifying a basis; consequently, we will drop the explicit matrix indices M and N in the following.

Next, we observe that Eq. (5.1) effectively relates the colour structure of the soft matrix, which is a priori very complicated, to the much simpler colour structure on the right hand side. Clearly the particular way in which the cusp anomalous dimension $\gamma_K^{(i)}$ depends on the colour representation of parton i becomes important. Following Eq. (3.12) we can write

$$\sum_{j,j\neq i}\frac{\partial}{\partial\ln(\rho_{ij})}\Gamma^{\overline{\mathcal{S}}}(\rho_{ij},\alpha_s) = \frac{1}{4}\left[C_i\,\widehat{\gamma}_K(\alpha_s) + \widetilde{\gamma}_K^{(i)}(\alpha_s)\right]\,, \qquad \forall i\,, \tag{5.2}$$

where in the first term the dependence on the representation is explicit, while in the second it is implicit. Using the linearity of these equations we can obviously write the general solution as a superposition of two functions

$$\Gamma^{\overline{\mathcal{S}}} = \Gamma^{\overline{\mathcal{S}}}_{\text{Q.C.}} + \Gamma^{\overline{\mathcal{S}}}_{\text{H.C.}} \tag{5.3}$$

which are, respectively, solutions of the equations

$$\sum_{j,j\neq i}\frac{\partial}{\partial\ln(\rho_{ij})}\Gamma^{\overline{\mathcal{S}}}_{\text{Q.C.}}(\rho_{ij},\alpha_s) = \frac{1}{4}\left(\sum_a \mathrm{T}_i^{(a)}\,\mathrm{T}_i^{(a)}\right)\widehat{\gamma}_K(\alpha_s)\,, \qquad \forall i\,, \tag{5.4}$$

$$\sum_{j,j\neq i}\frac{\partial}{\partial\ln(\rho_{ij})}\Gamma^{\overline{\mathcal{S}}}_{\text{H.C.}}(\rho_{ij},\alpha_s) = \frac{1}{4}\widetilde{\gamma}_K^{(i)}(\alpha_s)\,, \qquad \forall i\,. \tag{5.5}$$

Here Q.C. and H.C. stand for Quadratic Casimir and Higher (order) Casimir, respectively, reflecting the different group theoretical structure of the two contributions . In the first equation we exhibited the dependence on the colour representation of parton i, using Eq. (3.11). In the following we will focus on determining $\Gamma^{\overline{\mathcal{S}}}_{\text{Q.C.}}$ leaving aside $\Gamma^{\overline{\mathcal{S}}}_{\text{H.C.}} = \mathcal{O}(\alpha_s^4)$, which is driven by yet unknown corrections to the cusp anomalous dimension, that are not proportional to C_i. For simplicity of the notation we henceforth drop the subscript Q.C.

A solution for $\Gamma^{\overline{\mathcal{S}}}$, obeying Eq. (5.4), is given by

$$
\begin{aligned}
\Gamma^{\overline{\mathcal{S}}}\left(\rho_{ij}, \alpha_s\right)\Big|_{\text{ansatz}} &= -\frac{1}{8}\,\widehat{\gamma}_K\left(\alpha_s\right) \sum_{i=1}^{n} \sum_{j,\, j \neq i} \ln(\rho_{ij}) \sum_a \mathrm{T}_i^{(a)}\mathrm{T}_j^{(a)} \\
&\quad + \frac{1}{2}\,\widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s) \sum_{i=1}^{n} \sum_a \mathrm{T}_i^{(a)}\mathrm{T}_i^{(a)}\,,
\end{aligned}
\tag{5.6}
$$

where $\widehat{\gamma}_K$ and $\widehat{\delta}_{\overline{\mathcal{S}}}$ were defined in Eq. (3.12) and in Eq. (4.8), respectively. Note that in Eq. (5.6) the first term, which couples each pair of partons into a colour dipole, carries the entire dependence on kinematics, correlating it with the colour structure, while the second term is independent of kinematics and is proportional to the unit matrix in colour space. Such a term, in a different factorization scheme, such as the one adopted in Refs. [27, 56], could be associated with jet functions. In our case, since we start with specific operator definitions for the jets, we may find leftover colour-diagonal contributions in $\Gamma^{\overline{\mathcal{S}}}$. Note also that Eq. (5.6), which is valid for a general n-parton amplitude, reduces to Eq. (4.7) in the $n = 2$ case of the Sudakov form factor. This relation was used in determining the second term in Eq. (5.6), which is obviously not constrained by Eq. (5.4).

The explicit calculation of Ref. [56] at two-loops has established that Eq. (5.6) is the full answer to this order. In particular, the soft anomalous dimension does not contain correlations involving generators of three different partons, such as $f_{abc}\,\mathrm{T}_i^{(a)}\mathrm{T}_j^{(b)}\mathrm{T}_k^{(c)}$ (see Eq. (5.13) below), despite the fact that single poles in ϵ at two loops are known to contain such terms. Ref. [56] has demonstrated that these terms are generated at two loops only upon expansion of the product of the soft and hard functions.

To verify that Eq. (5.6) satisfies Eq. (5.4), let us take a derivative with respect to $\ln(\rho_{ij})$, for specific partons i and j, with $i \neq j$. We find

$$
\frac{\partial \Gamma^{\overline{\mathcal{S}}}\left(\rho_{ij}, \alpha_s\right)}{\partial \ln(\rho_{ij})} = -\frac{1}{4}\,\widehat{\gamma}_K\left(\alpha_s\right) \sum_a \mathrm{T}_i^{(a)}\mathrm{T}_j^{(a)}\,,
\tag{5.7}
$$

where we used the fact that $\sum_a \mathrm{T}_j^{(a)}\mathrm{T}_i^{(a)} = \sum_a \mathrm{T}_i^{(a)}\mathrm{T}_j^{(a)}$. Next we sum over j for fixed i, as in the *l.h.s.* of (5.4), and we get

$$
\begin{aligned}
\sum_{j,\, j \neq i} \frac{\partial \Gamma^{\overline{\mathcal{S}}}\left(\rho_{ij}, \alpha_s\right)}{\partial \ln(\rho_{ij})} &= -\frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right) \sum_{j,\, j \neq i} \sum_a \mathrm{T}_i^{(a)}\mathrm{T}_j^{(a)} \\
&= -\frac{1}{4}\widehat{\gamma}_K\left(\alpha_s\right) \sum_a \mathrm{T}_i^{(a)}\left(-\mathrm{T}_i^{(a)}\right)\,,
\end{aligned}
\tag{5.8}
$$

which coincides with the *r.h.s.* of Eq. (5.4), as required. The second line of Eq. (5.8) follows from colour conservation, which is expressed in the colour generator notation simply by

$$\sum_{i=1}^{n} \mathbf{T}_i^{(a)} = 0 \,. \tag{5.9}$$

Clearly, our ansatz does not in general provide a unique solution. Indeed, Eq. (5.4) is a set of n linear differential equations in the variables $\ln(\rho_{ij})$, while the number of variables is quadratic in n, so there may be contributions beyond Eq. (5.6). To clarify the nature of possible corrections to our ansatz, let us define

$$\Gamma_{MN}^{\overline{S}}(\rho_{ij}, \alpha_s) = \Gamma_{MN}^{\overline{S}}(\rho_{ij}, \alpha_s)\Big|_{\text{ansatz}} + \Delta_{MN}^{\overline{S}}(\rho_{ij}, \alpha_s) \tag{5.10}$$

and summarize our knowledge of $\Delta_{MN}^{\overline{S}}$. First, as discussed in Section 3.2, and shown explicitly in Section 4.1 and Appendix A, $\Delta_{MN}^{\overline{S}}$ vanishes identically for $n = 2, 3$. Thus it may contribute only to $n \geq 4$ parton amplitudes, starting at three loops. Second, according to Eq. (5.4), it should satisfy the homogeneous equation:

$$\sum_{j, j \neq i} \frac{\partial}{\partial \ln(\rho_{ij})} \Delta_{MN}^{\overline{S}}(\rho_{ij}, \alpha_s) = 0 \qquad \forall i, \, M, \, N \,. \tag{5.11}$$

Eq. (5.11) is solved by any function that depends on ρ_{ij} only through conformal cross ratios such as ρ_{ijkl}, defined in Eq. (3.14)[10]. As an example, for four external partons there are only two independent conformal cross ratios (note that $1/\rho_{1423} = \rho_{1234}\,\rho_{1342}$), and the general solution can be written as

$$\Delta_{MN}^{\overline{S}} = \Delta_{MN}^{\overline{S}}(\rho_{1234}, \rho_{1342}, \alpha_s) \,. \tag{5.12}$$

Note that Eq. (5.11) does not restrict the functional dependence of $\Delta_{MN}^{\overline{S}}$ on conformal cross ratios, allowing in particular non-linear dependence. Interesting examples, still for four partons, are

$$\hat{\mathbf{H}}_{[f]}^{(2)} = \sum_{j,k,l} \sum_{a,b,c} i\, f_{abc}\, \mathbf{T}_j^a \mathbf{T}_k^b \mathbf{T}_l^c \, \ln(\rho_{ijkl}) \ln(\rho_{iklj}) \ln(\rho_{iljk}) \,, \tag{5.13}$$

and

$$\widetilde{\mathbf{H}}_{[f]} = \sum_{j,k,l} \sum_{a,b,c} d_{abc}\, \mathbf{T}_j^a \mathbf{T}_k^b \mathbf{T}_l^c \, \ln^2(\rho_{ijkl}) \ln^2(\rho_{iklj}) \ln^2(\rho_{iljk}) \,, \tag{5.14}$$

where in both equations the sum over partons is understood to exclude identical indices. As already mentioned, $\hat{\mathbf{H}}_{[f]}^{(2)}$ is known to appear[11] as part of the $\mathcal{O}(1/\epsilon)$ coefficient in two

---

[10] As explained in Section 3.2 and illustrated in Appendix B, conformal cross ratios appear in $\Gamma^{\overline{S}}$ also in (5.6), starting at one-loop.

[11] In Ref. [56] $\hat{\mathbf{H}}_f^{(2)}$ is written in terms of the three kinematic invariants of a four-leg amplitude, assuming momentum conservation. It is straightforward to show that the same expression can be written in terms of conformal cross ratios, where momentum conservation is not enforced.

loops amplitudes. Nevertheless, as was shown in Ref. [56], it does not appear in the soft anomalous dimension at this order.

For multi-leg amplitudes with more than four legs, the space of possible solutions to the available constraints increases further, as there is an increasing number of conformal cross ratios. It remains for future work to decide whether contributions beyond the ansatz of Eq. (5.6) do indeed appear.

We conclude by working out the consequences of Eq. (5.6) for the anomalous dimension of the original soft matrix $\mathcal{S}$. Using Eq. (3.16), we write

$$\Gamma_{IJ}^{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)\Big|_{\text{ansatz}} = \left.\Gamma_{IJ}^{\overline{\mathcal{S}}}\left(\rho_{ij}, \alpha_s(\mu^2)\right)\right|_{\text{ansatz}} + \delta_{IJ} \sum_{i=1}^{n} \gamma_{\mathcal{J}_i}\left(w_i, \alpha_s(\mu^2), \epsilon\right),$$

(5.15)

where, as before, $w_i = 2(\beta_i \cdot n_i)^2 / n_i^2$. Substituting here Eq. (5.6), together with the expression in Eq. (3.8) for the eikonal jet anomalous dimension, we obtain, after some algebra

$$\Gamma^{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s(\mu^2), \epsilon\right)\Big|_{\text{ansatz}} = -\frac{1}{4}\widehat{\gamma}_K\left(\alpha_s(\mu^2)\right) \sum_{i=1}^{n} \sum_{j, j \neq i} \ln\left(\beta_i \cdot \beta_j \, e^{i\pi\lambda_{ij}}\right) \sum_a \mathrm{T}_i^{(a)} \mathrm{T}_j^{(a)}$$

$$+ \left[-\frac{1}{2}\widehat{\delta}_{\mathcal{S}}\left(\alpha_s(\mu^2)\right) + \frac{1}{4}\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2}\widehat{\gamma}_K\left(\alpha_s(\lambda^2, \epsilon)\right)\right] \sum_{i=1}^{n} \sum_a \mathrm{T}_i^{(a)} \mathrm{T}_i^{(a)},$$

(5.16)

where, as in Eq. (4.11),

$$\widehat{\delta}_{\mathcal{S}}(\alpha_s) = \widehat{\delta}_{\mathcal{J}}(\alpha_s) - \widehat{\delta}_{\overline{\mathcal{S}}}(\alpha_s).$$

(5.17)

The manipulation we applied in order to write Eq. (5.15) in its final form is similar to the one we used in Eq. (5.8), but in the reverse order: here one first rewrites the colour–diagonal terms that depend on $\ln(w_i)$, which originate in the eikonal jet contributions, in terms of a sum over all other partons j, using $\mathrm{T}_i^{(a)} = -\sum_{j \neq i} \mathrm{T}_j^{(a)}$; then one combines these terms with the $\ln(\rho_{ij})$ terms in $\Gamma^{\overline{\mathcal{S}}}$; finally, using Eq. (2.9), one observes that $\Gamma^{\mathcal{S}}$ depends only on $\beta_i \cdot \beta_j$, as it must. It is straightforward to check that Eq. (5.16) reduces to Eq. (4.13) for the Sudakov form factor. Once again, the second line in Eq. (5.16) is proportional to the identity matrix in colour space, and could be associated with jets in a different factorization scheme. Indeed, the square bracket gives just one half of the kinematics-independent terms in the soft anomalous dimension $\gamma_{\mathcal{S}}$ for the Sudakov form factor, Eq. (4.13). This is the contribution that needs to be combined with our jets in order to reconstruct the choice of Refs. [27,56], where jets are defined as square roots of the form factor.

Eq. (5.16) can be compared with the direct computation of the soft anomalous dimension at one loop. For the latter we express the sum over all diagrams as

$$\mathcal{S}\left(\beta_i \cdot \beta_j, \alpha_s, \epsilon\right) = 1 + \frac{\alpha_s}{4\pi} \sum_{i=1}^{n} \sum_{j, j \neq i} I_{ij}^{(1)}\left(\beta_i \cdot \beta_j, \epsilon\right) \sum_a \mathrm{T}_i^{(a)} \mathrm{T}_j^{(a)} + \mathcal{O}(\alpha_s^2),$$

(5.18)

where the basic one-loop integral, stripped of colour factors, is

$$I_{ij}^{(1)}\left(\beta_i \cdot \beta_j, \epsilon\right) = \frac{1}{\epsilon^2} - \frac{1}{\epsilon}\ln(\beta_i \cdot \beta_j \, e^{i\pi\lambda_{ij}}).$$

(5.19)

The anomalous dimension corresponding to Eq. (5.18) is

$$\Gamma^{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s, \epsilon\right) = \sum_{i=1}^{n} \sum_{j,\, j \neq i} \frac{\alpha_s}{2\pi} \left[\frac{1}{\epsilon} - \ln(\beta_i \cdot \beta_j \, \mathrm{e}^{\mathrm{i}\pi\lambda_{ij}})\right] \sum_a \mathrm{T}_i^{(a)} \mathrm{T}_j^{(a)} + \mathcal{O}(\alpha_s^2)\,, \qquad (5.20)$$

which coincides with Eq. (5.16) upon substituting there the one-loop values of the anomalous dimensions, and upon using again colour conservation[12] for the $1/\epsilon$ term.

6. Conclusions

We have explored the singularity structure of fixed–angle scattering amplitudes in massless gauge theories using their factorization into gauge–invariant hard, soft and jet functions. The factorization formula we use, Eq. (2.2), is based exclusively on dimensional regularisation. It has the advantage that Wilson line correlators, such as the soft function $\mathcal{S}$, are pure counterterms to all orders in perturbation theory, and they do not depend on any mass scales.

Our central observation is the presence of a symmetry under rescaling of the velocities that define the eikonal functions $\mathcal{S}$ and $\mathcal{J}$, Eq. (2.8); this symmetry is built into the eikonal Feynman rules, but it is broken for any eikonal function involving light-like segments, due to the cusp anomaly. Multi-parton scattering amplitudes depend on a specific combination of eikonal soft and jet functions, the reduced soft function of Eq. (2.7), where the cusp anomaly cancels and the rescaling symmetry is restored. We have shown that the specific way in which this symmetry is broken and then restored imposes tight constraints on the functional dependence of these functions on the kinematic variables, as well as on the colour variables. The all-order constraints on the reduced soft function are summarized by Eq. (3.23).

For purely eikonal functions which do not depend on any kinematic scale and are affected by the cusp anomaly, we observe that the complete kinematic dependence of the single pole terms in the exponent is tightly connected to the double poles, and they are both associated with the cusp anomaly. This is easy to see for eikonal functions involving two rays, such as the eikonal jet or the eikonal version of the Sudakov form factor, but the result turns out to be more general, and through Eq. (3.23) it carries over to the multi-leg case. This is contrasted with partonic amplitudes, such as the partonic jet function, Eq. (4.26), that have more complicated dependence on the kinematic variables, since they depend on dimensionful scales and have non-trivial renormalization properties.

Our conclusions concerning large–angle soft singularities based on Eq. (3.23) can be summarized as follows: for amplitudes involving two or three hard external partons (and

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[12]Note that in explicit calculations in a given colour basis the fact that

$$\left[\sum_{j,\, j \neq i} \sum_a \mathrm{T}_i^{(a)} \mathrm{T}_j^{(a)}\right]_{MN} = -C_i \delta_{MN}\,,$$

which is a statement of gauge invariance, provides a useful consistency check on the calculation of the colour factors. A simple example is provided in Eq. (B.2): one can verify that the sum of the three matrices there yields $-C_F$ times the unit matrix, as it should.

any number of non-coloured particles) the kinematic dependence of the soft function is completely determined, and it is governed, to all orders in perturbation theory, by the cusp anomalous dimension. We emphasize that this conclusion does not depend on the way in which the cusp anomalous dimension itself depends on the colour representation of a given parton. The results for the anomalous dimensions of the reduced soft matrices in these cases are given in Eq. (4.7) and Eq. (A.3).

For amplitudes involving four or more hard external partons the available constraints on the soft function are insufficient to uniquely determine the kinematic dependence. They nevertheless relate it to the cusp anomalous dimension. Considering the component of the cusp anomalous dimension that is proportional to the quadratic Casimir, we have found a minimal solution, Eq. (5.6), for the reduced soft anomalous dimension or, equivalently, Eq. (5.16) for the original soft anomalous dimension. This solution, which is valid for any number of legs, satisfies the all-order constraints and is consistent with all explicit calculations available to date. Eq. (5.6) is written as a sum over colour dipoles: it correlates the kinematic dependence with the dependence on the colour variables for each pair of hard partons in the very same way as one-loop diagrams do. In contrast with the two- and three-leg cases, for four or more legs the solution is not unique: the soft anomalous dimension may receive additional contributions (at three loops or beyond) which however depend on kinematic variables exclusively through conformal cross ratios. We point out that the absence of such contributions at two loops is non trivial: terms that satisfy this requirement, displayed in Eq. (5.13), do appear in two-loop diagrams at $\mathcal{O}(1/\epsilon)$, but the calculation of Ref. [56] proves that they do not contribute to the soft anomalous dimension at this order. This suggests that Eq. (5.6) is in fact the full answer to any loop order, aside from corrections (which in turn satisfy Eq. (5.5)) that are induced by possible contributions of higher-order Casimir operators to the cusp anomalous dimension itself, which may appear at four loops or beyond.

To summarize, we took here a step towards the understanding of the infrared singularity structure of gauge theory amplitudes to all orders in perturbation theory. We did so by identifying a rescaling symmetry of eikonal functions appearing in the factorization of the amplitudes, which is broken by the cusp anomaly, and then restored in a specific way. Using this observation, we derived all-order constraints on soft anomalous dimension matrices, that are valid for any number of external partons. We then studied the consequences of these constraints and established the complete solution for the soft anomalous dimension for amplitudes involving two or three partons, and a minimal solution for four partons or more. By formulating our results for the jet and soft functions in terms of a few universal anomalous dimensions, which depend only on the dimensionally–regularized coupling, and not on kinematics, or on a colour basis, we significantly increase the predictive power of the factorization formula, providing powerful checks on multi-loop calculations and a better starting point for soft gluon resummation.

Note added

A few hours before the submission of our paper to the arXiv, T. Becher and M. Neubert published an independent study [61], proposing an ansatz for the soft anomalous dimension matrix which is essentially equivalent to our Eq. (5.16).

Acknowledgments

A. The case of amplitudes with three partons

It is well known that the colour structure of the soft function for amplitudes with three hard partons is trivial: the soft matrix is proportional to the identity matrix. Here we briefly explain this property and observe another special property of the soft function in this case: similarly to the $n = 2$ case, discussed in Section 4.1, for $n = 3$ the contraints of Eq. (3.23) completely determine the dependence on the kinematics.

The reduced soft matrix, defined in Eq. (2.7), in this case is given by

$$\overline{\mathcal{S}}_{MN}^{[f]}\left(\rho_{12},\,\rho_{23},\,\rho_{31},\,\alpha_s,\epsilon\right) = \frac{\mathcal{S}_{MN}^{[f]}\left(\beta_1\cdot\beta_2,\,\beta_2\cdot\beta_3,\,\beta_3\cdot\beta_1,\,\alpha_s,\epsilon\right)}{\mathcal{J}_1\left(\frac{2(\beta_1\cdot n_1)^2}{n_1^2},\alpha_s,\epsilon\right)\,\mathcal{J}_2\left(\frac{2(\beta_2\cdot n_2)^2}{n_2^2},\alpha_s,\epsilon\right)\,\mathcal{J}_3\left(\frac{2(\beta_3\cdot n_3)^2}{n_3^2},\alpha_s,\epsilon\right)}\,.$$

(A.1)

We now note that Eq. (3.23) yields three independent differential equations, which are linear in $\ln(\rho_{ij})$. Explicitly, they are

$$\left[\frac{\partial}{\partial\ln(\rho_{12})}\,+\,\frac{\partial}{\partial\ln(\rho_{31})}\right]\Gamma_{MN}^{\overline{\mathcal{S}}}\left(\rho_{12},\,\rho_{23},\,\rho_{31},\alpha_s\right) = \frac{1}{4}\delta_{MN}\,\gamma_K^{(1)}\left(\alpha_s\right)\,,$$

$$\left[\frac{\partial}{\partial\ln(\rho_{23})}\,+\,\frac{\partial}{\partial\ln(\rho_{12})}\right]\Gamma_{MN}^{\overline{\mathcal{S}}}\left(\rho_{12},\,\rho_{23},\,\rho_{31},\alpha_s\right) = \frac{1}{4}\delta_{MN}\,\gamma_K^{(2)}\left(\alpha_s\right)\,,$$

(A.2)

$$\left[\frac{\partial}{\partial\ln(\rho_{23})}\,+\,\frac{\partial}{\partial\ln(\rho_{31})}\right]\Gamma_{MN}^{\overline{\mathcal{S}}}\left(\rho_{12},\,\rho_{23},\,\rho_{31},\alpha_s\right) = \frac{1}{4}\delta_{MN}\,\gamma_K^{(3)}\left(\alpha_s\right)\,.$$

Having three independent linear equations in the three variables there is a unique solution. It is given by

$$\Gamma_{MN}^{\overline{\mathcal{S}}}\left(\rho_{ij},\alpha_s\right) = \left\{-\frac{1}{8}\left[\left(\gamma_K^{(3)}-\gamma_K^{(1)}-\gamma_K^{(2)}\right)\ln(\rho_{12}) + \left(\gamma_K^{(1)}-\gamma_K^{(2)}-\gamma_K^{(3)}\right)\ln(\rho_{23})\right.\right.$$

$$\left.\left.+\left(\gamma_K^{(2)}-\gamma_K^{(3)}-\gamma_K^{(1)}\right)\ln(\rho_{13})\right] + \frac{1}{2}\left[\delta_{\overline{\mathcal{S}}}^{(1)}+\delta_{\overline{\mathcal{S}}}^{(2)}+\delta_{\overline{\mathcal{S}}}^{(3)}\right]\right\}\delta_{MN}\,,$$

(A.3)

where a constant term was added, as in Eq. (5.6). Note that we have shown that $\Gamma_{MN}^{\overline{S}}$ is proportional to the unit matrix in colour space without specifying the representation of the partons. This result is completely general, and it generalizes the previously known two-loop result to all orders in perturbation theory. Just as in the case of $n = 2$, the entire kinematic dependence is controlled by the cusp anomalous dimension. As explained in Sections 3.2 and 5, the fact that uniqueness can be established for $n = 2, 3$ but not for $n \geq 4$ is related to the absence of conformal cross ratios of the form of Eq. (3.18).

In Eq. (A.3), the dependence on the representation of the various partons is implicit, appearing through the functions $\gamma_K^{(i)}(\alpha_s)$ and $\delta_{\overline{S}}^{(i)}(\alpha_s)$. As discussed in Section 3.1, these may include higher–order corrections that are not proportional to the quadratic Casimir. In this respect the result of Eq. (A.3) goes beyond the ansatz of Eq. (5.6), which considers only terms that are associated with the quadratic Casimir contributions to $\gamma_K^{(i)}$ in Eq. (3.12). Of course, ignoring $\widetilde{\gamma}_K$ in Eq. (3.12) one recovers Eq. (5.6). Upon substituting Eq. (5.6) into Eq. (A.2), these three equations yield

$$\sum_a \mathrm{T}_1^{(a)} \left(\mathrm{T}_2^{(a)} + \mathrm{T}_3^{(a)} \right) = - C_1 \,,$$
$$\sum_a \mathrm{T}_2^{(a)} \left(\mathrm{T}_3^{(a)} + \mathrm{T}_1^{(a)} \right) = - C_2 \,, \tag{A.4}$$
$$\sum_a \mathrm{T}_3^{(a)} \left(\mathrm{T}_1^{(a)} + \mathrm{T}_2^{(a)} \right) = - C_3 \,,$$

which are satisfied owing to colour conservation, Eq. (5.9). Eq. (A.4) also implies that

$$2 \sum_a \mathrm{T}_1^{(a)} \mathrm{T}_2^{(a)} = C_3 - C_1 - C_2 \,,$$
$$2 \sum_a \mathrm{T}_2^{(a)} \mathrm{T}_3^{(a)} = C_1 - C_2 - C_3 \,, \tag{A.5}$$
$$2 \sum_a \mathrm{T}_3^{(a)} \mathrm{T}_1^{(a)} = C_2 - C_1 - C_3 \,,$$

which is consistent with the observation that all colour factors entering Eq. (5.6) in this case are proportional to the unit matrix. Finally, the explicit sum-over-dipoles solution to Eq. (A.2) takes the form

$$\Gamma_{\mathrm{Q.C.}}^{\overline{S}} (\rho_{ij}, \alpha_s) = - \frac{1}{8} \widehat{\gamma}_K (\alpha_s) \Bigg[(C_3 - C_1 - C_2) \ln(\rho_{12}) + (C_1 - C_2 - C_3) \ln(\rho_{23})$$
$$+ (C_2 - C_3 - C_1) \ln(\rho_{13}) \Bigg] + \frac{1}{2} \widehat{\delta}_{\overline{S}}(\alpha_s) \left(C_1 + C_2 + C_3 \right) . \tag{A.6}$$

B. The case of $q\bar{q} \to q\bar{q}$ scattering at one loop

The four-parton amplitude $q\bar{q} \to q\bar{q}$ provides a simple example where the colour matrix structure is non-trivial. We perform the calculation along the lines of Sec. IV of Ref. [56], but factorize the amplitude as in Eq. (2.2), using light-like Wilson lines.

The velocities are defined by:

$$q(\beta_1) + \bar{q}(\beta_2) \to q(\beta_3) + \bar{q}(\beta_4) \,.$$

It is convenient to set $\beta_1 = -v_1, \beta_2 = -v_2, \beta_3 = v_3$ and $\beta_4 = v_4$, so that all the scalar products $v_i \cdot v_j > 0$. Note that below we will formally treat the four velocities as independent variables, choosing not to enforce explicitly momentum conservation.

Following Ref. [56], we pick the colour basis

$$c_1 = \delta_{12}\, \delta_{34} \,; \qquad c_2 = \delta_{13}\, \delta_{24} \,, \qquad (B.1)$$

and we use the convention of Eq. (2.6) to write the result in a matrix form. There are six one-loop diagrams altogether, and for each one of them the loop integral yields Eq. (5.19). Computing the colour factors in the chosen basis and summing up the contributions of the six diagrams according to Eq. (5.18) we get

$$\mathcal{S}\left(v_i \cdot v_j, \alpha_s, \epsilon\right) = 1 + \frac{\alpha_s}{2\pi} \left\{ \begin{pmatrix} \frac{1}{2N_c} & 0 \\ -\frac{1}{2} & -C_F \end{pmatrix} \left[\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\ln(v_1 \cdot v_3) - \frac{1}{\epsilon}\ln(v_2 \cdot v_4) \right] \right.$$

$$+ \begin{pmatrix} -\frac{1}{2N_c} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2N_c} \end{pmatrix} \left[\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\ln(v_1 \cdot v_4) - \frac{1}{\epsilon}\ln(v_2 \cdot v_3) \right] \qquad (B.2)$$

$$\left. + \begin{pmatrix} -C_F & -\frac{1}{2} \\ 0 & \frac{1}{2N_c} \end{pmatrix} \left[\frac{2}{\epsilon^2} - \frac{1}{\epsilon}\left(\ln(v_1 \cdot v_2) + \mathrm{i}\pi\right) - \frac{1}{\epsilon}\left(\ln(v_3 \cdot v_4) + \mathrm{i}\pi\right) \right] \right\} + \mathcal{O}(\alpha_s^2) \,.$$

The corresponding one-loop soft anomalous dimension matrix is then

$$\Gamma^S(v_i \cdot v_j, \alpha_s, \epsilon) = \frac{\alpha_s}{\pi} C_F \begin{pmatrix} -\frac{2}{\epsilon} + \ln\left((v_1 \cdot v_2)(v_3 \cdot v_4)\right) & 0 \\ 0 & -\frac{2}{\epsilon} + \ln\left((v_1 \cdot v_3)(v_2 \cdot v_4)\right) \end{pmatrix}$$

$$+ \frac{\alpha_s}{2\pi} \begin{pmatrix} \frac{1}{N_c}\ln\left(\frac{(v_1 \cdot v_4)(v_2 \cdot v_3)}{(v_1 \cdot v_3)(v_2 \cdot v_4)}\right) & \ln\left(\frac{(v_1 \cdot v_2)(v_3 \cdot v_4)}{(v_1 \cdot v_4)(v_2 \cdot v_3)}\right) \\ \ln\left(\frac{(v_1 \cdot v_3)(v_2 \cdot v_4)}{(v_1 \cdot v_4)(v_2 \cdot v_3)}\right) & \frac{1}{N_c}\ln\left(\frac{(v_1 \cdot v_4)(v_2 \cdot v_3)}{(v_1 \cdot v_2)(v_3 \cdot v_4)}\right) \end{pmatrix} - 2\pi\mathrm{i}\,\frac{\alpha_s}{\pi} \begin{pmatrix} -C_F & -\frac{1}{2} \\ 0 & \frac{1}{2N_c} \end{pmatrix} + \mathcal{O}(\alpha_s^2) \,.$$

$$(B.3)$$

Finite terms agree with Eq. (4.21) in Ref. [56]. Note that in that paper there are no poles in Γ^S since the regularization used takes the Wilson lines off the light cone.

In order to compute the anomalous dimension matrix for the reduced soft function, we should subtract the jet anomalous dimensions, according to Eq. (3.16),

$$\Gamma_{IJ}^{\overline{\mathcal{S}}}\left(\rho_{ij}, \alpha_s\right) = \Gamma_{IJ}^{\mathcal{S}}\left(\beta_i \cdot \beta_j, \alpha_s, \epsilon\right) - \delta_{IJ}\sum_{k=1}^{4} \gamma_{\mathcal{J}_k}\left(w_k, \alpha_s, \epsilon\right) \,. \qquad (B.4)$$

The jet anomalous dimensions $\gamma_{\mathcal{J}_k}$ can be computed using Eq. (3.8) in the fundamental representation. At one loop this yields

$$\gamma_{\mathcal{J}_k}\left(w_k, \alpha_s, \epsilon\right) = \frac{\alpha_s}{2\pi} C_F \left[-1 + \ln\left(\frac{2(v_k \cdot n_k)^2}{n_k^2}\right) - \frac{1}{\epsilon} \right] + \mathcal{O}(\alpha_s^2) \,. \qquad (B.5)$$

The subtracted terms are proportional to the unit matrix in colour space, so they affect only the diagonal elements of the anomalous dimension matrix. We end up with the following anomalous dimension for $\overline{\mathcal{S}}$, which is of course finite,

$$\Gamma^{\overline{\mathcal{S}}}(\rho_{ij}, \alpha_s) = \frac{\alpha_s}{\pi} C_F \begin{pmatrix} 2 + \frac{1}{2}\ln(\rho_{12}\,\rho_{34}) & 0 \\ 0 & 2 + \frac{1}{2}\ln(\rho_{13}\,\rho_{24}) \end{pmatrix} \tag{B.6}$$
$$+ \frac{\alpha_s}{4\pi} \begin{pmatrix} \frac{1}{N_c}\ln\left(\frac{\rho_{14}\,\rho_{23}}{\rho_{13}\,\rho_{24}}\right) & \ln\left(\frac{\rho_{12}\,\rho_{34}}{\rho_{14}\,\rho_{23}}\right) \\ \ln\left(\frac{\rho_{13}\,\rho_{24}}{\rho_{14}\,\rho_{23}}\right) & \frac{1}{N_c}\ln\left(\frac{\rho_{14}\,\rho_{23}}{\rho_{12}\,\rho_{34}}\right) \end{pmatrix} - 2\pi\mathrm{i}\,\frac{\alpha_s}{\pi} \begin{pmatrix} -C_F & -\frac{1}{2} \\ 0 & \frac{1}{2N_c} \end{pmatrix} + \mathcal{O}(\alpha_s^2)\,.$$

It is straightforward to check that the general expressions in Eq. (5.16) and Eq. (5.6) indeed reduce to Eq. (B.6) and Eq. (B.3), respectively, upon evaluating the colour factors in the chosen basis and substituting the one-loop values for γ_K, $\widehat{\delta}_{\overline{\mathcal{S}}}$ and $\widehat{\delta}_{\mathcal{S}}$.

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