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GCA in 2d

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Arjun Bagchi[1], **Rajesh Gopakumar**[1], **Ipsita Mandal**[1,2], **Akitsugu Miwa**[1]

[1]*Harish-Chandra Research Institute,*
Chhatnag Road, Jhusi,
Allahabad 211019, India

[2]*LPTHE, Universite Pierre et Marie Curie,*
Paris 6, 4 Place Jussieu,
75252 Paris Cedex 05, France

`arjun, gopakumr, ipsita, akitsugu@hri.res.in`

ABSTRACT: We make a detailed study of the infinite dimensional Galilean Conformal Algebra (GCA) in the case of two spacetime dimensions. Classically, this algebra is precisely obtained from a contraction of the generators of the relativistic conformal symmetry in $2d$. Here we find quantum mechanical realisations of the (centrally extended) GCA by considering scaling limits of certain 2d CFTs. These parent CFTs are non-unitary and have their left and right central charges become large in magnitude and opposite in sign. We therefore develop, in parallel to the usual machinery for $2d$ CFT, many of the tools for the analysis of the quantum mechanical GCA. These include the representation theory based on GCA primaries, Ward identities for their correlation functions and a nonrelativistic Kac table. In particular, the null vectors of the GCA lead to differential equations for the four point function. The solution to these equations in the simplest case is explicitly obtained and checked to be consistent with various requirements.

Contents

1. Introduction

1.1 Non-relativistic Conformal Symmetries

Conformal symmetry has gradually moved from being a mere mathematical curiosity (e.g. as an invariance of certain classical equations of motion like the Maxwell and more generally Yang-Mills equations) to playing a central dynamical role governing the quantum behaviour of field theories through RG fixed points. Thus an understanding of CFTs in various dimensions has had a major impact on physics, starting with the study of critical phenomena all the way through to the more recent AdS/CFT correspondence.

In all this, the focus has largely been on relativistic conformal field theories. Non relativistic versions of conformal symmetry have been somewhat studied in the context of the so-called Schrodinger symmetry [1, 2, 3, 4] which is an enhanced symmetry that arises in taking the nonrelativistic limit of the massive Klein-Gordon equation. In d spacetime dimensions, this group has, in addition to the $\frac{d(d+1)}{2}$ parameters of the Galilean group, a nonrelativistic dilatation $x_i \to \lambda x_i, t \to \lambda^2 t$ and *one* more conformal generator. Much less explored are the nonrelativistic limits of the full $SO(d, 2)$ conformal invariance. In fact, a non-relativistic group contraction of $SO(d, 2)$ gives rise to a $\frac{(d+1)(d+2)}{2}$ parameter group (see for e.g. [5, 6] and in an *AdS* context [7]). This contains the Galilean symmetries together with a uniform dilatation $x_i \to \lambda x_i, t \to \lambda t$ and d other generators which are the analogues of the special conformal transformations.

Quite remarkably, the actual set of conformal isometries of nonrelativistic spacetime is much larger than the above contracted version of $SO(d, 2)$ [8]. In *every*

spacetime dimension it is actually an infinite dimensional algebra which was dubbed the Galilean Conformal Algebra (GCA). This algebra consists of local conformal transformations acting on time (generating a single copy of the Virasoro algebra) together with a current algebra for rotations as well as arbitrary time dependent boosts in the spatial directions. It was shown in [19] that this infinite dimensional algebra can also be obtained by considering the natural nonrelativistic limit of the relativistic conformal killing equations and is the maximal set of such nonrelativistic conformal isometries.[1] The situation is very analogous to that in the relativistic $d = 2$ case where one has two copies of the Virasoro algebra generating the maximal set of local conformal isometries.

1.2 The role of the GCA in physical systems

At this point it is worthwhile to remind the reader why one should be intersted in the GCA. In [8], the motivation for looking at the contracted algebra was to construct a systematic non-relativistic limit of the AdS/CFT conjecture[9]. The non-relativistic AdS/CFT correspondence is one of the avenues of exploration of the recent studies of condensed matter systems using the gauge-gravity duality (for a review and a list of relevant papers, see [11]). By taking a parametric limit of the conformal algebra in [8], we wished to isolate a sector of AdS/CFT in the spirit of the BMN limit[10]. Though the realization of the bulk symmetry was very different from BMN, there were novel structures discovered on both sides of the duality. We hope that further explorations of the GCA would lead to a better understanding of the AdS/CFT correspondence which even today is not properly understood.

On the other hand, the GCA may play a role in real life physical systems. It was shown in [17] (see also [15, 16]) that the non-relativistic Navier-Stokes equation has an invariance under the generators of boosts and constant accelerations. In fact, the algebra of invariances found there is precisely the algebra of the finite GCA (with dilatations also being an invariance when the viscous term is dropped) [8]. In fact, all the time dependent boosts *are* invariances of the local equations (both the Navier-Stokes and the Euler equations)[14][2]. The Navier-Stokes equation is believed to be realized in the non-relativistic limit of the hydrodynamic regime of all quantum field theories. The realization of a part of the infinite GCA as its symmetries possibly indicates a universality of the existence of the GCA.

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[1]For related work on various aspects of the GCA, see [19]–[27].

[2]This is so despite the interpretational claim to the contrary in [18].

Our focus in this paper would be on the two dimensional GCA. It is worthwhile to mention that a mathematically identical infinite dimensional algebra (called alt_1) in the case of one spatial dimension had also appeared independently in [13] in the context of statistical mechanics [3]. So we hope that our explorations in the present paper would also be useful in the study of non-equilibrium statistical mechanical systems.

It would be satisfying if one also had a Lagrangian description of systems with GCA. This has eluded us so far, and would continue to do so in our study of the two-dimensional case. But we hasten to remind the reader that Lagrangians, though useful, are not an essential characteristic for this symmetry to be realised in some context. It can very well be that the GCA has no Lagrangian realisation and in fact here we can draw analogies with the BMN limit where too the sector in $\mathcal{N} = 4$ Supersymmetric Yang-Mills theory does not admit a lagrangian description.

A word about the original motivation of non-relativistic AdS/CFT and the case of GCA in two dimensions: recently, after the first version of this paper, in [28], it was found that the GCA in 2d could be realized as an asymptotic symmetry algebra of Cosmological Topologically Massive Gravity when the coefficient of the gravitational Chern-Simons term is scaled in a particular way. The central charges, that we would go on to describe in the 2D GCA have also been realized in the same scaling limit.

1.3 Quantum GCA and the role of $D = 2$

The entire analysis in [8] was classical and one would like to understand the quantum mechanical realisation of these symmetries better. This is specially true if one would like to exploit this symmetry in the context of a nonrelativistic limit of AdS/CFT as proposed in [8]. A first step was taken in [21] (see also [20, 22]) where two and three point correlation functions (of primary fields) were obtained as solutions of the Ward identities for the finite part of the GCA (which arises as the contraction of $SO(d, 2)$).[4] These were found to be completely determined in form just as in the relativistic case. One would therefore imagine that the infinite dimensional GCA should give much stronger constraints when realised on the quantum dynamics just as in the case of relativistic $2d$ conformal invariance.

With this in mind and given the centrality of $2d$ CFTs it is natural to specialise to two spacetime dimensions and study realisations of the infinite dimensional GCA

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[3]It was also noted in [12] that alt_1 arises mathematically as a contraction of (two copies of) the Virasoro algebra though the contraction is slightly different from what we consider below.

[4]The expression for the two point function is actually contained in [12] as part of more general expressions derived for arbitrary dynamical exponents.

in this case. In this process we might hope to extract dynamical information from the GCA comparable to that obtained from the Virasoro algebra. The main aim of this paper is to show that this hope can be largely realised. We find, in fact, that there is a very tight relation between the GCA symmetry in 2d and the relativistic Virasoro symmetry. Classically, we see that the infinite dimensional GCA in 2d arises as a contraction of the generators of the usual holomorphic and antiholomorphic transformations. Quantum mechanically, in a 2d CFT, the latter get promoted to left and right moving Virasoro generators each with its central extension. We will find evidence that taking a scaling limit on a suitable family of 2d CFTs gives consistent quantum mechanical realisations of the GCA (now centrally extended), in what might be termed as 2d Galilean conformal field theories (or 2d GCFTs, for short).

The families of 2d CFTs we will need to consider are rather unusual in that their left and right central charges c and $\bar{c}$ are scaled (as we take the nonrelativistic limit) such that their magnitudes go to infinity but are opposite in sign. The parent theories are thus necessarily non-unitary and, not unsurprisingly, this non-unitariness is inherited by the daughter GCFTs. Since non-unitary 2d CFTs arise in a number of contexts in statistical mechanics as well as string theory, one might expect that the 2d GCFTs realised here would also be interesting objects to study.

1.4 Strategy and outline of this paper

Our study of 2d GCFTs in this paper proceeds along two parallel lines. The first line of development is as described above and consists of taking carefully the non-relativistic scaling limit of the parent 2d CFT. We find that this limit, while unusual, appears to give sensible answers. Specifically, we will study in this way, the representation theory (including null vectors determined by a nonrelativistic Kac table), the Ward identities, fusion rules and finally the equations for the four point function following from the existence of level two null states. In all these cases we find that a non-trivial scaling limit of the 2d CFT's exists. This is not, *a priori*, obvious since, as we will see, the limit involves keeping terms both of $\mathcal{O}(\frac{1}{\epsilon})$ as well as of $\mathcal{O}(1)$ (where ϵ is the scaling parameter which is taken to zero).

The second line of development obtains many of these same results by carrying out an autonomous analysis of the GCA i.e. independent of the above limiting procedure. In some cases we will see that the constraints from the GCA are slightly weaker than those arising from the scaling limit of parent relativistic CFTs. This is understandable since the limit of 2d CFTs is presumably only a particular way

to realise GCFT's and there is no reason to expect it to be the only way to realise such GCFTs. Thus our dual considerations help to distinguish between some of the general results for any 2d GCFT from that obtained by taking the nonrelativistic limit. It is also an important consistency check of our investigation that these two strands of development agree whenever they do and that it is always the GCA that gives weaker constraints.

The structure of the paper is as follows: In the next section we discuss how, at the classical level, the generators of the 2d GCA arise from a group contraction of (combinations of) the usual holomorphic and anti-holomorphic vector fields. We also discuss the scaling limit one should take in the quantum theory thus relating the central charges of the GCA to the Virasoro central charges c and $\bar{c}$. In Sec. 3 we proceed to construct representations of the 2d GCA in a manner analogous to the Virasoro representation theory, defining primaries and descendants. The primaries are labelled by a conformal weight Δ and a boost eigenvalue ξ. We also show that the state space is generically non-unitary. Sec. 4 deals with the non-relativistic Ward identities focussing on the case of two and three point functions. We briefly review the derivation of [21] and show how the answers there may also be obtained from the nonrelativistic scaling limit of the Virasoro algebra discussed in Sec. 2.

We go back to the representation theory in Sec. 5 to consider null vectors of the GCA. We explicitly find the conditions for having null states at level two and check that the resulting conditions are precisely those obtained from the scaling limit of the Virasoro algebra. We go on to take the scaling limit of the Kac table for null states at arbitrary level and find a sensible nonrelativistic Kac table. From Sec. 6 onwards we focus on GCA primaries taking values in the nonrelativistic Kac table. We first derive the general differential equations for an n-point correlator which follow from the existence of level two null states and check that it is consistent with the form of the two point function of [21]. We proceed in Sec. 7 to derive the GCA fusion rules which follow from the GCA three point function using the differential equations of Sec. 6. These turn out to be only slightly weaker than what one would obtain from the scaling of the corresponding Virasoro fusion rules.

Finally, in Sec. 8 we consider the four point function which satisfies second order differential equations when one of the primaries has a null descendant at level two. We can find the solution in an explicit form both from this equation as well as from considering the limit of the solution of the corresponding Virasoro four point functions. The solution obeys various nontrivial conditions and seems to be consistent with the factorization into the three point function and the fusion rules.

There is a final section with remarks of a general nature. Appendix A discusses some issues regarding GCA descendants and their conformal blocks. Appendix B discusses a specific fusion channel in which the GCA gives a weaker constraint than the Virasoro. Appendix C contains various technical details relevant to the four point function in Sec. 8.

2. 2d GCA from Group Contraction

The maximal set of conformal isometries of Galilean spacetime generates the infinite dimensional Galilean Conformal Algebra [8]. The notion of Galilean spacetime is a little subtle since the spacetime metric degenerates into a spatial part and a temporal piece. Nevertheless there is a definite limiting sense (of the relativistic spacetime) in which one can define the conformal isometries (see [19]) of the nonrelativistic geometry. Algebraically, the set of vector fields generating these symmetries are given by

$$L^{(n)} = -(n+1)t^n x_i \partial_i - t^{n+1}\partial_t \,,$$
$$M_i^{(n)} = t^{n+1}\partial_i \,,$$
$$J_a^{(n)} \equiv J_{ij}^{(n)} = -t^n(x_i\partial_j - x_j\partial_i) \,, \tag{2.1}$$

for integer values of n. Here $i = 1 \ldots (d-1)$ range over the spatial directions. These vector fields obey the algebra

$$[L^{(m)}, L^{(n)}] = (m-n)L^{(m+n)}, \qquad [L^{(m)}, J_a^{(n)}] = -nJ_a^{(m+n)},$$
$$[J_a^{(n)}, J_b^{(m)}] = f_{abc}J_c^{(n+m)}, \qquad [L^{(m)}, M_i^{(n)}] = (m-n)M_i^{(m+n)}. \tag{2.2}$$

We expect that there would be a central extension to the Virasoro and Current algebras in the quantum theory. In fact, we will see that in two dimensions the GCA with central charges can be realised by taking a special limit of a relativistic 2d CFT (albeit non-unitary).

2.1 GCA from Virasoro in 2d

There is a finite dimensional subalgebra of the GCA (also sometimes referred to as the GCA) which consists of taking $n = 0, \pm 1$ for the $L^{(n)}, M_i^{(n)}$ together with $J_a^{(0)}$. This algebra is obtained by considering the nonrelativistic contraction of the usual (finite dimensional) global conformal algebra $SO(d, 2)$ (in $d > 2$ spacetime dimensions) (see for example [5]–[8]).

However, in two spacetime dimensions, as is well known, the situation is special. The relativistic conformal algebra is infinite dimensional and consists of two copies

of the Virasoro algebra. One expects this to be also related, now to the infinite dimensional GCA algebra. Indeed in two dimensions the non-trivial generators in (2.2) are the L_n and the M_n (where we have dropped the spatial index from the latter since there is only one spatial direction and instead restored the mode number n to the conventional subscript) :

$$
\begin{aligned}
L_n &= -(n+1)t^n x \partial_x - t^{n+1} \partial_t \,, \\
M_n &= t^{n+1} \partial_x \,,
\end{aligned}
\tag{2.3}
$$

which obey

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} \,, \qquad [M_m, M_n] = 0 \,, \\
[L_m, M_n] &= (m-n)M_{m+n} \,.
\end{aligned}
\tag{2.4}
$$

We will now show that the generators in (2.3) arise precisely from a nonrelativistic contraction of the two copies of the Virasoro algebra of the relativistic theory.[5] The non-relativistic contraction consists of taking the scaling

$$
t \to t \,, \qquad x \to \epsilon x \,,
\tag{2.5}
$$

with $\epsilon \to 0$. This is equivalent to taking the velocities $v \sim \epsilon$ to zero (in units where $c = 1$).

Consider the vector fields which generate (two copies of) the centre-less Virasoro Algebra (or Witt algebra as it is often called) in two dimensions :

$$
\mathcal{L}_n = -z^{n+1} \partial_z \,, \quad \bar{\mathcal{L}}_n = -\bar{z}^{n+1} \partial_{\bar{z}} \,.
\tag{2.6}
$$

In terms of space and time coordinates, $z = t + x$, $\bar{z} = t - x$. Hence $\partial_z = \frac{1}{2}(\partial_t + \partial_x)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_t - \partial_x)$. Expressing $\mathcal{L}_n, \bar{\mathcal{L}}_n$ in terms of t, x and taking the above scaling (2.5) reveals that in the limit the combinations

$$
\begin{aligned}
\mathcal{L}_n + \bar{\mathcal{L}}_n &= -t^{n+1} \partial_t - (n+1)t^n x \partial_x + \mathcal{O}(\epsilon^2) \,, \\
\mathcal{L}_n - \bar{\mathcal{L}}_n &= -\frac{1}{\epsilon} t^{n+1} \partial_x + \mathcal{O}(\epsilon) \,.
\end{aligned}
\tag{2.7}
$$

Therefore we see that as $\epsilon \to 0$

$$
\mathcal{L}_n + \bar{\mathcal{L}}_n \longrightarrow L_n \,, \quad \epsilon(\mathcal{L}_n - \bar{\mathcal{L}}_n) \longrightarrow -M_n \,.
\tag{2.8}
$$

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[5]This observation has also been independently made in [27]. As mentioned in footnote 1, a slightly different contraction of the Virasoro algebra was also made in [13] to obtain the same result.

Thus the GCA in 2d arises as the non-relativistic limit of the relativistic algebra. This was at the classical level of vector fields. At the quantum level the two copies of the Virasoro get respective central extensions

$$[\mathcal{L}_m, \mathcal{L}_n] = (m - n)\mathcal{L}_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0}\,,$$

$$[\bar{\mathcal{L}}_m, \bar{\mathcal{L}}_n] = (m - n)\bar{\mathcal{L}}_{m+n} + \frac{\bar{c}}{12}m(m^2 - 1)\delta_{m+n,0}\,. \tag{2.9}$$

Considering the linear combinations (2.7) which give rise to the GCA generators as in (2.8), we find

$$[L_m, L_n] = (m - n)L_{m+n} + C_1 m(m^2 - 1)\delta_{m+n,0}\,,$$
$$[L_m, M_n] = (m - n)M_{m+n} + C_2 m(m^2 - 1)\delta_{m+n,0}\,,$$
$$[M_m, M_n] = 0\,. \tag{2.10}$$

This is the centrally extended GCA in 2d.[6] Note that the relation between central charges is

$$C_1 = \frac{c + \bar{c}}{12}\,, \qquad \frac{C_2}{\epsilon} = \frac{\bar{c} - c}{12}\,. \tag{2.11}$$

Thus, for a non-zero C_2 in the limit $\epsilon \to 0$ we see that we need $\bar{c} - c \propto \mathcal{O}(\frac{1}{\epsilon})$. At the same time requiring C_1 to be finite we find that $c + \bar{c}$ should be $\mathcal{O}(1)$. (Motivated by the second equation in (2.7), we will make the slightly stronger assumption that $\bar{c} - c = \mathcal{O}(1/\epsilon) + \mathcal{O}(\epsilon)$.) Thus (2.11) can hold only if c and $\bar{c}$ are large (in the limit $\epsilon \to 0$) and opposite in sign. This immediately implies that the original 2d CFT on which we take the non-relativistic limit cannot be unitary. This is, of course, not a problem since there are many statistical mechanical models which are described at a fixed point by non-unitary CFTs.[7]

3. Representations of the 2d GCA

With these requirements in mind we now turn to the representations of the 2d GCA. We will be guided in this by the representation theory of the Virasoro algebra.

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[6]One can check that a central extension in the commutator $[M_m, M_n]$ of the form $C_3(m)\delta_{m+n,0}$ is not allowed by the Jacobi identity.

[7]We note that modular invariance of a 2d CFT implies that $\bar{c} - c \equiv 0 (\mathrm{mod} 24)$. We will implicitly assume that this is true for the CFT's we are considering. In other words, we will take $\frac{C_2}{\epsilon}$ to be an even integer.

3.1 Primary States and Descendants

We will construct the representations by considering the states having definite scaling dimensions :

$$L_0|\Delta\rangle = \Delta|\Delta\rangle\,. \tag{3.1}$$

Using the commutation relations (2.10), we obtain

$$L_0 L_n|\Delta\rangle = (\Delta - n)L_n|\Delta\rangle, \quad L_0 M_n|\Delta\rangle = (\Delta - n)M_n|\Delta\rangle. \tag{3.2}$$

Then the L_n, M_n with $n > 0$ lower the value of the scaling dimension, while those with $n < 0$ raise it. If we demand that the dimension of the states be bounded from below then we are led to defining primary states in the theory having the following properties :

$$L_n|\Delta\rangle_p = 0\,, \quad M_n|\Delta\rangle_p = 0\,, \tag{3.3}$$

for all $n > 0$. Since the conditions (3.3) are compatible with M_0 in the sense

$$L_n M_0|\Delta\rangle_p = 0\,, \quad M_n M_0|\Delta\rangle_p = 0\,, \tag{3.4}$$

and also since L_0 and M_0 commute, we may introduce an additional label, which we will call "rapidity" ξ:

$$M_0|\Delta,\xi\rangle_p = \xi|\Delta,\xi\rangle_p\,. \tag{3.5}$$

Starting with a primary state $|\Delta,\xi\rangle_p$, one can build up a tower of operators by the action of L_{-n} and M_{-n} with $n > 0$. These will be called the GCA descendants of the primary. The primary state together with its GCA descendants form a representation of GCA. As in the Virasoro case, we have to be careful about the presence of null states. We will look at these in some detail later in Sec.5. An interesting property of the representation constructed above is that a generic secondary state will not be an eigenstate of M_0. Examples of the secondary states, which are eigenstates of M_0 are the ones which are constructed only with M_{-n}'s acting on the primary state. In the rest of the paper, we omit the subscript "p" for the primary state.

The above construction is quite analogous to that of the relativistic 2d CFT. In fact, from the viewpoint of the limit (2.8) we see that the two labels Δ and ξ are related to the conformal weights in the 2d CFT as

$$\Delta = \lim_{\epsilon \to 0}(h + \bar{h})\,, \qquad \xi = \lim_{\epsilon \to 0}\epsilon(\bar{h} - h)\,, \tag{3.6}$$

where h and $\bar{h}$ are the eigenvalues of $\mathcal{L}_0$ and $\bar{\mathcal{L}}_0$, respectively. We will proceed to assume that such a scaling limit (as $\epsilon \to 0$) of the 2d CFT exists. In particular, we

will assume that the operator state correspondence in the 2d CFT gives a similar correspondence between the states and the operators in the GCA :

$$\mathcal{O}(t, x) \leftrightarrow \mathcal{O}(0)|0\rangle \,, \tag{3.7}$$

where $|0\rangle$ would be the vacuum state which is invariant under the generators $L_0, L_{\pm 1}$, $M_0, M_{\pm 1}$. Indeed in the rest of the paper, we will offer several pieces of evidence that the scaling limit gives a consistent quantum mechanical system.

3.2 Unitarity

However already at level one of the GCA (using the usual terminology of CFT) we can see that the representation is generically non-unitary. For this we have to define hermiticity and an inner product. We will assume that this is what would be naturally inherited from the parent relativistic CFT. In other words, we take $L_n^\dagger = L_{-n}$ and $M_n^\dagger = M_{-n}$.[8]

Let us now consider the tower above a primary state $|\Delta, \xi\rangle$. The states at level one are $L_{-1}|\Delta, \xi\rangle$ and $M_{-1}|\Delta, \xi\rangle$. The (hermitian) matrix of inner products of these states can be evaluated and the determinant is easily seen to be given by $-\xi^2$. Thus, for non-zero ξ, at least one of the eigenvalues must be negative. In such a case the Hilbert space clearly has negative norm states. Thus only for $\xi = 0$ might one have the possibility of a unitary representation though there are clearly null states even in this case. This is not too surprising given that the parent CFT is generically non-unitary as we argued above.

4. Non-Relativistic Ward Identities and Two and Three Point Functions

In [21] the GCA transformation laws for primary operators were written down as (See eq. (4.3) and (4.4) of [21]) :

$$\begin{aligned}
\delta_{L_n} \mathcal{O}_p(t, x) &= [L_n, \mathcal{O}_p(t, x)] \\
&= [t^{n+1}\partial_t + (n+1)t^n x \partial_x + (n+1)(\Delta t^n - n\xi t^{n-1} x)]\mathcal{O}(t, x) \,, \\
\delta_{M_n} \mathcal{O}_p(t, x) &= [M_n, \mathcal{O}_p(t, x)] = [-t^{n+1}\partial_x + (n+1)\xi t^n]\mathcal{O}(t, x) \,.
\end{aligned} \tag{4.1}$$

In [21] these transformation laws, in the special case of the global transformations (i.e. for $L_{0, \pm 1}, M_{0, \pm 1}$), were used to derive Ward Identities (see also [20, 22]) for two

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[8]Although the operator M_0 is Hermitian, its action on the states is, in general, not diagonalizable. This occurs for a special class of operators when the inner product is not positive definite. See also the discussion around (5.5).

and three point functions. The functional dependence of these correlators was then completely fixed by the resulting equations (see below for the explicit expressions). The analysis in [21] is based on the argument directly coming from GCA. We will now see that all these statements can be also derived from the non-relativistic limit of the relativistic 2d CFT.

4.1 Transformation Laws

Consider first how the transformation laws (4.1) of GCA primaries arise from the transformation laws of primary operators in 2d CFT, which are given by

$$\tilde{\mathcal{O}}(\tilde{z}, \tilde{\bar{z}}) = \mathcal{O}(z, \bar{z})(\frac{dz}{d\tilde{z}})^h (\frac{d\bar{z}}{d\tilde{\bar{z}}})^{\bar{h}} \,, \tag{4.2}$$

or infinitesimally under $z \to \tilde{z} = z + a(z)$ as

$$\delta\mathcal{O}(z, \bar{z}) = -(h\partial_z a(z) + a(z)\partial_z)\mathcal{O}(z, \bar{z}) \,, \tag{4.3}$$

and similarly for $\bar{z} \to \tilde{\bar{z}} = \bar{z} + \bar{a}(\bar{z})$. We take $a(z) = -z^{n+1}$ for $\delta_{\mathcal{L}_n}$ and $\bar{a}(\bar{z}) = -\bar{z}^{n+1}$ for $\delta_{\bar{\mathcal{L}}_n}$.

Motivated from the relation (2.8), we may define the infinitesimal transformation δ_{L_n} and δ_{M_n} as

$$\delta_{L_n}\mathcal{O} = \lim_{\epsilon \to 0}(\delta_{\mathcal{L}_n} + \delta_{\bar{\mathcal{L}}_n})\mathcal{O} \,, \qquad \delta_{M_n}\mathcal{O} = -\lim_{\epsilon \to 0}\epsilon(\delta_{\mathcal{L}_n} - \delta_{\bar{\mathcal{L}}_n})\mathcal{O} \,. \tag{4.4}$$

Then by taking the limits on the right hand sides (RHS's), we obtain

$$\begin{aligned}
\delta_{L_n}\mathcal{O} &= \lim_{\epsilon \to 0}[(n+1)h(t+\epsilon x)^n + \frac{1}{2}(t+\epsilon x)^{n+1}(\partial_t + \frac{1}{\epsilon}\partial_x) \\
&\quad + (n+1)\bar{h}(t-\epsilon x)^n + \frac{1}{2}(t-\epsilon x)^{n+1}(\partial_t - \frac{1}{\epsilon}\partial_x)]\mathcal{O} \\
&= [t^{n+1}\partial_t + (n+1)t^n x\partial_x + (n+1)(\Delta t^n - n\xi t^{n-1}x)]\mathcal{O} \,, \tag{4.5}
\end{aligned}$$

and similarly,

$$\delta_{M_n}\mathcal{O} = [-t^{n+1}\partial_x + (n+1)t^n\xi]\mathcal{O} \,. \tag{4.6}$$

where recall that Δ and ξ are defined as (3.6). These are exactly those given in (4.1).

4.2 Two and Three Point Functions

The constraints from the Ward identities for the global transformations $L_{0,\pm 1}$, $M_{0,\pm 1}$ apply to primary GCA operators.

Therefore consider the two point function of primary operators $\mathcal{O}_1(t_1, x_1)$ and $\mathcal{O}_2(t_2, x_2)$ of conformal and rapidity weights (Δ_1, ξ_1) and (Δ_2, ξ_2) respectively.

$$G_{\text{GCA}}^{(2)}(t_1, x_1, t_2, x_2) = \langle \mathcal{O}_1(t_1, x_1) \mathcal{O}_2(t_2, x_2) \rangle. \tag{4.7}$$

The correlation functions only depend on differences of the coordinates $t_{12} = t_1 - t_2$ and $x_{12} = x_1 - x_2$ because of the translation symmetries L_{-1} and M_{-1}. The remaining symmetries give four more differential equations which constrain the answer to be [21]

$$G_{\text{GCA}}^{(2)}(\{t_i, x_i\}) = C_{12} \delta_{\Delta_1, \Delta_2} \delta_{\xi_1, \xi_2} t_{12}^{-2\Delta_1} \exp\left(\frac{2\xi_1 x_{12}}{t_{12}}\right). \tag{4.8}$$

Here C_{12} is an arbitrary constant, which we can always take to be one by choosing the normalization of the operators.

Similarly, the three point function of primary operators is given by

$$G_{\text{GCA}}^{(3)}(\{t_i, x_i\}) = C_{123} t_{12}^{-(\Delta_1 + \Delta_2 - \Delta_3)} t_{23}^{-(\Delta_2 + \Delta_3 - \Delta_1)} t_{13}^{-(\Delta_1 + \Delta_3 - \Delta_2)}$$
$$\times \exp\left(\frac{(\xi_1 + \xi_2 - \xi_3) x_{12}}{t_{12}} + \frac{(\xi_2 + \xi_3 - \xi_1) x_{23}}{t_{23}} + \frac{(\xi_1 + \xi_3 - \xi_2) x_{13}}{t_{13}}\right), \tag{4.9}$$

where C_{123} is an arbitrary constant. So we see that like in the case of relativistic CFTs, the three point function is fixed upto a constant.[9]

4.3 GCA Correlation Functions from 2d CFT

We now show that these expressions for the GCA two and three point functions can also be obtained by taking an appropriate scaling limit of the usual 2d CFT answers. This limit requires scaling the quantum numbers of the operators as (3.6), along with the non-relativistic limit for the coordinates (2.5).

Let us first study the scaling limit of the two point correlator.[10]

$$G_{\text{2d CFT}}^{(2)} = \delta_{h_1, h_2} \delta_{\bar{h}_1, \bar{h}_2} z_{12}^{-2h_1} \bar{z}_{12}^{-2\bar{h}_1}$$
$$= \delta_{h_1, h_2} \delta_{\bar{h}_1, \bar{h}_2} t_{12}^{-2h_1}\left(1 + \epsilon \frac{x_{12}}{t_{12}}\right)^{-2h_1} t_{12}^{-2\bar{h}_1}\left(1 - \epsilon \frac{x_{12}}{t_{12}}\right)^{-2\bar{h}_1}$$
$$= \delta_{h_1, h_2} \delta_{\bar{h}_1, \bar{h}_2} t_{12}^{-2(h_1 + \bar{h}_1)} \exp\left(-2(h_1 - \bar{h}_1)\left(\epsilon \frac{x_{12}}{t_{12}} + \mathcal{O}(\epsilon^2)\right)\right). \tag{4.10}$$

Now by taking the scaling limit as (3.6), we obtain the GCA two point function

$$\lim_{\epsilon \to 0} G_{\text{2d CFT}}^{(2)} = \delta_{\Delta_1, \Delta_2} \delta_{\xi_1, \xi_2} t_{12}^{-2\Delta_1} \exp\left(\frac{2\xi_1 x_{12}}{t_{12}}\right) = G_{\text{GCA}}^{(2)}. \tag{4.11}$$

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[9]See, however, the explanation in the footnote 11.

[10]This was obtained in discussion with S. Minwalla.

A similar analysis yields the three point function of the GCA from the relativistic three point function. The relativistic three point function is written as

$$G^{(3)}_{\text{2d CFT}} = C_{123} z_{12}^{-(h_1+h_2-h_3)} z_{23}^{-(h_2+h_3-h_1)} z_{13}^{-(h_1+h_3-h_2)} \times \text{ (anti-holomorphic)} \qquad (4.12)$$

$$= C_{123} t_{12}^{-(h_1+h_2-h_3)} t_{12}^{-(\bar{h}_1+\bar{h}_2-\bar{h}_3)} e^{-(h_1+h_2-h_3)(\epsilon\frac{x_{12}}{t_{12}}+\mathcal{O}(\epsilon^2))} e^{(\bar{h}_1+\bar{h}_2-\bar{h}_3)(\epsilon\frac{x_{12}}{t_{12}}+\mathcal{O}(\epsilon^2))}$$

$$\times \text{ (product of two permutations)} . \qquad (4.13)$$

Then taking the non-relativistic limit, we obtain the GCA three point function (4.9). Note that the constant factor in (4.9) would be given by taking the limit of the constant in (4.12).

We should also mention here the issue of singlevaluedness of the correlation functions. The holomorphic part, or the anti-holomorphic part, of the correlation functions in the 2d CFT have branch cuts for generic values of the conformal dimensions h_i and $\bar{h}_i$. Then the requirement for the singlevaluedness of the two point functions gives rise to a condition that the difference $h_i - \bar{h}_i$ of every operator must be an integer or a half integer. This is the usual spin statistics theorem.[11] We see that this theorem need not hold in a generic GCFT since there is no such singlevaluedness requirement. However, GCFT's arising as limits of 2d CFTs would inherit this relation.

5. GCA Null Vectors

Just as in the representation of the Virasoro algebra, we will find that there are null states in the GCA tower built on a primary $|\Delta, \xi\rangle$ for special values of (Δ, ξ). These are states which are orthogonal to all states in the tower including itself. We can find the null states at a given level by writing the most general state at that level as a combination of the L_{-n}, M_{-n}'s $(n > 0)$ acting on the GCA primary and then imposing the condition that all the positive modes L_n, M_n (with $n > 0$) annihilate this state. Actually one needs to only impose this condition for $n = 1$ and $n = 2$ since the others are given as the commutators of these modes. This will give conditions

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[11] A word regarding the overall phase of correlation functions: If we pay attention to the overall phase in the derivation (4.10), it would be like ($x = 0$ for simplicity) $z^{-2h}\bar{z}^{-2\bar{h}} \rightarrow (\pm 1)^{2(h-\bar{h})}|t|^{-2(h+\bar{h})}$. Here the plus (the minus) sign is for $t > 0$ ($t < 0$), and the exponent $2(h - \bar{h})$ is the twice of the spin as mentioned in the main text. Note that in the derivation of (4.8), the differential equation is solved separately for the two segments $t < 0$ and $t > 0$. So far we have no argument, from the GCA side, to fix the relative coefficient of the solutions for these segments. However, when we take the non-relativistic limit from 2d CFT, this ambiguity is fixed as we saw above. This relative phase plays an important role in the discussion of the four point function.

that fix the relative coefficients in the linear combination as well as give a relation between Δ, ξ and the central charges C_1, C_2.

Thus at level one we have only the states $L_{-1}|\Delta, \xi\rangle$ and $M_{-1}|\Delta, \xi\rangle$. It is easy to check that one has a null state only if ξ is zero. At level two things are a little more non-trivial. Let us consider the most general level two state of the form

$$|\chi\rangle = (a_1 L_{-2} + a_2 L^2_{-1} + b_1 L_{-1} M_{-1} + d_1 M^2_{-1} + d_2 M_{-2})|\Delta, \xi\rangle. \tag{5.1}$$

We now impose the condition that $L_{1,2}, M_{1,2}$ annihilate this state.

A little algebra using (2.10) gives us the conditions :

$$3a_1 + 2(2\Delta + 1)a_2 + 2\xi b_1 = 0; \quad (4\Delta + 6C_1)a_1 + 6\Delta a_2 + 6\xi b_1 + (6C_2 + 4\xi)d_2 = 0;$$
$$2(\Delta + 1)b_1 + 4\xi d_1 + 3d_2 = 0; \quad (4\xi + 6C_2)a_1 = 0; \quad 3a_1 + 2a_2 + 2\xi b_1 = 0; \quad \xi a_2 = 0. \tag{5.2}$$

We will now separately consider the two cases where $C_2 \neq 0$ and $C_2 = 0$.

5.1 The Case of $C_2 \neq 0$

Here we will first consider the case where $\xi \neq 0$. In this case we have two further options. Either $a_1 = 0$ or $a_1 \neq 0$. In the former case, $b_1 = 0$ as well. We then find that for a nontrivial solution, $\xi = -\frac{3C_2}{2}$ and $d_1 = -\frac{3}{4\xi}d_2$. Thus there is a null state of the form

$$|\chi^{(1)}\rangle = (M_{-2} - \frac{3}{4\xi}M^2_{-1})|\Delta, \xi\rangle. \tag{5.3}$$

We can also consider solutions for which $a_1 \neq 0$. In this case $b_1 = -\frac{3}{2\xi}a_1$. We have once again $\xi = -\frac{3C_2}{2}$. But then we also have to satisfy the consistency condition $\Delta = \frac{(9 - 6C_1)}{4}$. For $\Delta \neq -1$, we also must have at least one of d_1, d_2 nonzero for such a solution. By taking a suitable linear combination with $|\chi^{(1)}\rangle$ we can choose $d_2 = 0$ and then we get another null state of the form

$$|\chi^{(2)}\rangle = (L_{-2} - \frac{3}{2\xi}L_{-1}M_{-1} + \frac{3(\Delta + 1)}{4\xi^2}M^2_{-1})|\Delta, \xi\rangle. \tag{5.4}$$

We can also have the case where $\xi = 0$. Then we must have $a_1 = a_2 = d_2 = 0$ and d_1 is again undetermined corresponding again to $|\chi^{(1)}\rangle = M^2_{-1}|\Delta, 0\rangle$. However, it is only for $\Delta = -1$ that one gets a second null state. Here b_1 is undetermined and corresponds to $|\chi^{(2)}\rangle = L_{-1}M_{-1}|\Delta = -1, 0\rangle$. Note, however, that both these states are descendants of a level *one* null state $M_{-1}|\Delta, 0\rangle$.

In any case, for both $\xi = 0$ as well as $\xi \neq 0$, the null states obey

$$M_0|\chi^{(2)}\rangle = \xi|\chi^{(2)}\rangle + \alpha|\chi^{(1)}\rangle, \qquad M_0|\chi^{(1)}\rangle = \xi|\chi^{(1)}\rangle, \tag{5.5}$$

where α is either of $1, 2$. That is, they are eigenstates of M_0 (upto null states). More generally, this is a reflection of the fact that the operator M_0 is not diagonal on the states at a given level. In fact, it can be easily seen to take an upper triangular form in terms of a suitable ordering of the basis elements at a given level. This is reminiscent of the behaviour in logarithmic CFTs.

At a general level K one will find in parallel with $|\chi^{(1)}\rangle$ a null state formed only from the M_{-n}'s taking the form

$$|\chi\rangle = (M_{-K} + \eta_1 M_{-K+1} M_{-1} + \ldots) |\Delta, \xi\rangle. \tag{5.6}$$

The action of L_K gives the constraint as

$$L_K |\chi\rangle = ([L_K, M_{-K}] + \ldots) |\Delta, \xi\rangle \quad \rightarrow \quad 2K\xi + C_2 K(K^2 - 1) = 0. \tag{5.7}$$

Here we have used the fact that all other commutators generate M_m with $m > 0$ and these go through and annihilate the state $|\Delta, \xi\rangle$. Thus there are null states at level K if ξ obeys the above relation with respect to C_2.

5.2 The Case of $C_2 = 0$

In this case, it is easy to see that we have a non-zero null state only if $\xi = 0$. Then from (5.2) we find that for $(\Delta \neq -\frac{3}{2}C_1)$ that both a_2 and $a_1 = 0$ and we have a relation $d_2 = -\frac{2(\Delta+1)}{3} b_1$ and d_1 is undetermined. Therefore the two independent null states are now

$$|\chi^{(1)}\rangle = M_{-1}^2 |\Delta, 0\rangle, \tag{5.8}$$

(which is just a descendant of the level one null state) and

$$|\chi^{(2)}\rangle = (L_{-1} M_{-1} - \frac{2(\Delta + 1)}{3} M_{-2}) |\Delta, 0\rangle. \tag{5.9}$$

Note that there is no constraint on the values of C_1, Δ apart from the fact that $(\Delta \neq -\frac{3}{2}C_1)$. In fact, in the case of $(\Delta = -\frac{3}{2}C_1)$ we find that we can have $a_1, a_2 \neq 0$ only if $\Delta = C_1 = 0$, which is a trivial case. So we continue to take $a_1 = a_2 = 0$ even when $(\Delta = -\frac{3}{2}C_1)$ and we have the two null states $|\chi^{(1,2)}\rangle$ for all values of Δ, C_1.

We can actually say more about these states. Note that the states $M_{-1}^2 |\Delta, 0\rangle$ and $L_{-1} M_{-1} |\Delta, 0\rangle$ are descendants of the level one null state $M_{-1} |\Delta, \xi\rangle$. Thus if we consistently set the null state at level one to zero together with its descendants, then the new null state at level two is given by $M_{-2} |\Delta, 0\rangle$. Continuing this way it is easy to see that we have a new null state given by $M_{-K} |\Delta, 0\rangle$ at level K if we set all the null states (and their descendants) at lower levels to zero. In this case, the GCA

tower precisely reduces to the Virasoro tower given by the Virasoro descendants of the primary. As noted for level two, there is generically no condition on C_1, Δ : we only require $\xi = C_2 = 0$. Thus we can consider a truncation of the Hilbert space to this Virasoro module. We can, of course, have the Virasoro tower reducible by having Virasoro null vectors. These are analysed in the usual way. Note that we can have unitary representations if Δ and C_1 obey the conditions familiar from the study of the Virasoro algebra.

Unfortunately, this sector is relatively uninteresting from the point of view of its spacetime dependence. The correlation functions of operators are ultralocal depending only on time. This is because all x dependence arises in combination with the ξ dependence so as to survive the nonrelativistic limit. Setting $\xi = 0$ thus removes the spatial dependence of correlators.

5.3 GCA Null Vectors from 2d CFT

Here we will show that the level two GCA null vectors can alternatively be obtained by taking the nonrelativistic scaling limit of the familiar level two null vectors of 2d CFT.

The null vector at level two in a Virasoro tower is given by

$$|\chi_L\rangle = (\mathcal{L}_{-2} + \eta \mathcal{L}_{-1}^2)|h\rangle \otimes |\bar{h}\rangle \,, \tag{5.10}$$

with

$$\eta = -\frac{3}{2(2h+1)}, \tag{5.11}$$

$$h = \frac{1}{16}(5 - c \pm \sqrt{(1-c)(25-c)}). \tag{5.12}$$

One has a similar null state for the antiholomorphic Virasoro obtained by replacing $\mathcal{L}_n \to \bar{\mathcal{L}}_n$, $h \to \bar{h}$ and $c \to \bar{c}$.

Using these expressions let us first take the limit of the relation in (5.12) (together with its antiholomorphic counterpart). Recall that the nonrelativistic scaling limit for the central charges and conformal weights are given by (2.11) and (3.6). Since the central charges $c, \bar{c}$ are opposite in sign as $\epsilon \to 0$, on taking the positive sign for the negative central charge part,[12] and vice versa, for the square root in (5.12), we get

$$\xi = -\frac{3C_2}{2}, \qquad \Delta = \frac{(9 - 6C_1)}{4} \,. \tag{5.13}$$

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[12]If we take the negative sign in the square root we get in the same limit where $C_2 \neq 0$ that $\xi = 0$ and $\Delta = -1$. This is precisely what we obtained in Sec. 5.1.

These are precisely the relations we obtained in the previous section if we require the existence of both the GCA null states $|\chi^{(1)}\rangle, |\chi^{(2)}\rangle$ at level two.

These states themselves can be obtained by taking the nonrelativistic limit on appropriate combinations of the relativistic null vectors $|\chi_L\rangle$ and its antiholomorphic counterpart $|\chi_R\rangle$. Consider

$$|\chi^{(1)}\rangle = \lim_{\epsilon \to 0} \epsilon(-|\chi_L\rangle + |\chi_R\rangle), \qquad |\chi^{(2)}\rangle = \lim_{\epsilon \to 0}(|\chi_L\rangle + |\chi_R\rangle). \qquad (5.14)$$

From the expressions (3.6), we obtain $\eta = \frac{3\epsilon}{2\xi}(1 + \frac{(\Delta+1)\epsilon}{\xi})$ and $\bar\eta = -\frac{3\epsilon}{2\xi}(1 - \frac{(\Delta+1)\epsilon}{\xi})$ upto terms of order ϵ^2. Substituting this into (5.14), using the relations (2.8) and taking the limit $\epsilon \to 0$, we obtain

$$|\chi^{(1)}\rangle = (M_{-2} - \frac{3}{4\xi}M_{-1}^2)|\Delta, \xi\rangle,$$
$$|\chi^{(2)}\rangle = (L_{-2} - \frac{3}{2\xi}L_{-1}M_{-1} + \frac{3(\Delta+1)}{4\xi^2}M_{-1}^2)|\Delta, \xi\rangle, \qquad (5.15)$$

which are exactly what we found from the intrinsic GCA analysis in (5.3) and (5.4).

Similarly the case mentioned in footnote 12 is also easily seen to correspond to the null states constructed in Sec. 5.1 for $C_2 \neq 0$. Finally, there is the case when $C_2 = 0$ and hence $c = \bar c$. Therefore it follows from (5.12) and its antiholomorphic counterpart that $h = \bar h$, i.e., $\xi = 0$ and $\Delta = 2h$. It is then easy to verify that the pair of states in (5.10) and its antiholomorphic counterpart reduce in the non-relativistic limit to the states constructed in Sec. 5.2 (see (5.8) and (5.9)). It is satisfying that the limiting process gives answers consistent with the intrinsic GCA analysis.

5.4 Non-Relativisitic Limit of the Kac Formula

More generally if we want to examine the GCA null states at a general level, we would have to perform an analysis similar to that in the Virasoro representation theory. A cornerstone of this analysis is the Kac determinant which gives the values of the weights of the Virasoro Primaries $h(\bar h)$ for which the matrix of inner products at a given level has a zero eigenvalue :

$$\det M^{(l)} = \alpha_l \prod_{1 \leq r,s;rs \leq l} (h - h_{rs}(c))^{p(l-rs)}, \qquad (5.16)$$

where α_l is a constant independent of (h, c); $p(l - rs)$ is the number of partitions of the integer $l - rs$. The functions $h_{r,s}(c)$ are expressed in a variety of ways. One

convenient representation is :

$$h_{r,s}(c) = h_0 + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2 \,, \tag{5.17}$$

$$h_0 = \frac{1}{24}(c-1) \,, \tag{5.18}$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \,. \tag{5.19}$$

One can write a similar expression for the antiholomorphic sector. The values $h_{r,s}$ are the ones for which we have zeroes of the determinant and hence null vectors (and their descendants).

One could presumably generalise our analysis for GCA null vectors at level two and directly obtain the GCA determinant at a general level. This would give us a relation for Δ and ξ in terms of C_1, C_2 for which there are null states, generalising the result

$$\xi = -\frac{3C_2}{2} \,, \qquad \Delta = \frac{(9-6C_1)}{4} \,, \tag{5.20}$$

at level two. However, here instead of a direct analysis we will simply take the non-relativistic limit of the Kac formula and see that one obtains sensible expressions for the Δ and ξ at which the GCA determinant would vanish.

In taking the non-relativistic limit, let us first consider the case where $C_2 \neq 0$ and chosen to be positive. Therefore (from (2.11)) we need to take $c \ll -1$ and $\bar{c} \gg 1$ as $\epsilon \to 0$. We then find

$$h_{r,s} = \frac{1}{24}c(1-r^2) + \frac{1}{24}(13r^2 - 12rs - 1) + \mathcal{O}(\epsilon), \tag{5.21}$$

$$\bar{h}_{r',s'} = \frac{1}{24}\bar{c}(1-r'^2) + \frac{1}{24}(13r'^2 - 12r's' - 1) + \mathcal{O}(\epsilon). \tag{5.22}$$

Then eq. (3.6) gives the values of Δ and ξ in terms of the RHS of (5.21) and (5.22) which in turn can be expressed in terms of C_1 and C_2. In the simple case where we take $(r,s) = (r',s')$[13]

$$\Delta_{r,s} = \lim_{\epsilon \to 0}(h_{r,s} + \bar{h}_{r,s}) = \frac{1}{2}C_1(1-r^2) + \frac{1}{12}(13r^2 - 12rs - 1) \,, \tag{5.23}$$

$$\xi_{r,s} = -\lim_{\epsilon \to 0}\epsilon(h_{r,s} - \bar{h}_{r,s}) = \frac{1}{2}C_2(1-r^2) \,. \tag{5.24}$$

In the case of $r = 2$, $s = 1$ we have two null states at level two built on the primary $|\Delta_{2,1}, \xi_{2,1}\rangle$. We see from (5.23) and (5.24) that the values of $\Delta_{2,1}$ and $\xi_{2,1}$

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[13]Requiring that Δ should not have a $\frac{1}{\epsilon}$ piece immediately implies that $r = r'$. The choice $s = s'$ is merely to simplify expressions.

are exactly those given in (5.20). This is also what we explicitly constructed in the previous subsection.

We can also construct a pair of null states at level two for $r = 1, s = 2$. In this case, we see that $\Delta_{1,2} = -1$ and $\xi_{1,2} = 0$. Such a null state for $C_2 \neq 0$ was found in Sec. 5.1 and we see that it corresponds to the case mentioned in footnote 12.

Finally, as a last consistency check, we note that at level K, we can have a null state with $r = K, s = 1$. The condition on $\xi_{K,1}$ given by (5.24) is exactly the same as given in (5.7).

6. Differential Equations for GCA Correlators from Null States

The presence of the null states gives additional relations between correlation functions which is at the heart of the solvability of relativistic (rational) conformal field theories. To obtain these relations one starts with differential operator realisations $\hat{\mathcal{L}}_{-k}$ of the $\mathcal{L}_{-k}$ with ($k \geq 1$). Thus one has

$$\langle (\mathcal{L}_{-k}\phi(z,\bar{z}))\phi_1(z_1,\bar{z}_1) \cdots \phi_p(z_n,\bar{z}_n) \rangle = \hat{\mathcal{L}}_{-k}\langle \phi(z,\bar{z})\phi_1(z_1,\bar{z}_1) \cdots \phi_p(z_n,\bar{z}_n) \rangle, \quad (6.1)$$

where

$$\hat{\mathcal{L}}_{-1} = \partial_z, \quad \hat{\mathcal{L}}_{-k} = \sum_{i=1}^{n} \left\{ \frac{(k-1)h_i}{(z_i - z)^k} - \frac{1}{(z_i - z)^{k-1}}\partial_{z_i} \right\} \quad \text{(for } k \geq 2\text{).} \quad (6.2)$$

We can obtain similar differential operators for the GCA generators by taking appropriate non-relativistic limits (2.5) of these relativistic expressions. Expanding the operators $\hat{\mathcal{L}}_{-k}$ as

$$\hat{\mathcal{L}}_{-k} = \epsilon^{-1}\hat{\mathcal{L}}_{-k}^{(-1)} + \hat{\mathcal{L}}_{-k}^{(0)} + \mathcal{O}(\epsilon), \quad (6.3)$$

and similarly for the anti-holomorphic part. Using (2.8), we obtain the expressions for the differential operators $\hat{M}_{-k}$ and $\hat{L}_{-k}$ as :

$$\hat{M}_{-k} = \sum_{i=1}^{n} \left[\frac{(k-1)\xi_i}{t_{i0}^k} + \frac{1}{t_{i0}^{k-1}}\partial_{x_i} \right],$$

$$\hat{L}_{-k} = \sum_{i=1}^{n} \left[\frac{(k-1)\Delta_i}{t_{i0}^k} + \frac{k(k-1)\xi_i}{t_{i0}^{k+1}}x_{i0} - \frac{1}{t_{i0}^{k-1}}\partial_{t_i} + \frac{k-1}{t_{i0}^k}x_{i0}\partial_{x_i} \right], \quad (6.4)$$

where $x_{i0} = x_i - x$ and $t_{i0} = t_i - t$. For $k = 1$ we have the simpler expressions $\hat{M}_{-1} = -\partial_x$ and $\hat{L}_{-1} = \partial_t$.

Therefore, correlation functions of GCA descendants of a primary field are given in terms of the correlators of the primaries by the action of the corresponding differential operators $\hat{M}_{-k}$ and $\hat{L}_{-k}$. One can derive the same result from intrinsic

GCA analysis by following the method used to derive eq (13) in [29]. In appendix A we illustrate how this works and give the analogue of the conformal blocks for the non-relativistic case.

Now we will study the consequences of having null states at level two. We will consider the two null states $|\chi^{(1)}\rangle$, $|\chi^{(2)}\rangle$ of Sec. 5.1, or rather correlators involving the corresponding fields $\chi^{(1,2)}(t,x)$. Setting the null state and thus its correlators to zero gives rise to differential equations for the correlators involving the primary $\phi_{\Delta,\xi}(t,x)$ with other fields. Using the forms (5.3) and (5.4), we find that the differential equations take the form

$$(\hat{M}_{-2} - \frac{3}{4\xi}\hat{M}^2_{-1})\langle\phi_{\Delta,\xi}(t,x)\phi_1(t_1,x_1)\cdots\phi_n(t_n,x_n)\rangle = 0\,, \tag{6.5}$$

$$(\hat{L}_{-2} - \frac{3}{2\xi}\hat{L}_{-1}\hat{M}_{-1} + \frac{3(\Delta+1)}{4\xi^2}\hat{M}^2_{-1})\langle\phi_{\Delta,\xi}(t,x)\phi_1(t_1,x_1)\cdots\phi_n(t_n,x_n)\rangle = 0\,, \tag{6.6}$$

with $\hat{L}_{-2}$ and $\hat{M}_{-2}$ as given in equation (6.4).

Acting on a two point function

$$G^{(2)}_{\text{GCA}}(t,x) = \langle\phi_{\Delta,\xi}(t,x)\phi_{\Delta',\xi'}(0,0)\rangle\,, \tag{6.7}$$

we have the simple differential equations

$$[\frac{\xi'}{t^2} + \frac{1}{t}\partial_x - \frac{3}{4\xi}\partial^2_x]G^{(2)}_{\text{GCA}}(t,x) = 0\,, \tag{6.8}$$

$$[\frac{\Delta'}{t^2} + 2\frac{\xi'x}{t^3} - \frac{1}{t}\partial_t + \frac{x}{t^2}\partial_x + \frac{3}{2\xi}\partial_t\partial_x + \frac{3(\Delta+1)}{4\xi^2}\partial^2_x]G^{(2)}_{\text{GCA}}(t,x) = 0\,. \tag{6.9}$$

It is not difficult to check that the GCA two point function [21] given in (4.8) $G^{(2)}_{\text{GCA}}(t,x) \propto t^{-2\Delta}e^{\frac{2\xi x}{t}}$ (together with $\xi = \xi'$ and $\Delta = \Delta'$) identically satisfies, as it should, both these differential equations.

7. GCA Fusion Rules

Analogous to the relativistic case [30], we can derive "*Fusion rules*",

$$[\phi_1] \times [\phi_2] \simeq \sum_p [\phi_p]\,,$$

for the GCA conformal families, that determine which families $[\phi_p]$ have their primaries and descendants occurring in an OPE of any two members of the families $[\phi_1]$ and $[\phi_2]$. Here we have denoted a family $[\phi_i]$ by the corresponding primary ϕ_i.

We illustrate how the fusion rules can be obtained for the families $[\phi_{\Delta,\xi}]$ and $[\phi_{\Delta_1,\xi_1}]$, where both fields are members of the GCA Kac table as specified by (5.21) (5.22). As mentioned in footnote 13, we need to take $r = r'$. The resulting Δ, ξ are thus labelled by a triple $\{r(s, s')\}$. In particular, we will consider below the case of $\Delta = \Delta_{2(1,1)}$ and $\xi = \xi_{2(1,1)}$. (In Appendix B, we consider the case of the fusion rule following from the case where $\Delta = \Delta_{1(2,2)}$ and $\xi = \xi_{1(2,2)}$. This case is interesting in that the GCA limit generally gives a weaker constraint than that following from the nonrelativistic limit of the 2d CFT.)

The fusion rules are derived from applying the condition that $\phi_{\Delta,\xi}$ has a null descendant at level two. For (Δ_1, ξ_1) we will consider a general member $r(s, s')$ of the GCA Kac table. Thus we have from (5.13), (5.21) and (5.22) :

$$\Delta = \Delta_{2(1,1)} = \frac{1}{4}(9 - 6C_1), \quad \xi = \xi_{2(1,1)} = -\frac{3C_2}{2}; \tag{7.1}$$

$$\Delta_1 = \Delta_{r(s,s')} = \frac{C_1}{2}(1 - r^2) + \frac{1}{12}\{13r^2 - 6r(s + s') - 1\}, \tag{7.2}$$

$$\xi_1 = \xi_{r(s,s')} = \frac{C_2}{2}(1 - r^2). \tag{7.3}$$

We need to consider the conditions (6.5) and (6.6) for the case of the three point function, i.e., $n = 2$. With $G_{\text{GCA}}^{(3)}(t, x, \{t_i, x_i\}) = \langle \phi_{\Delta,\xi}(t, x)\phi_{\Delta_1,\xi_1}(t_1, x_1)\phi_{\Delta_2,\xi_2}(t_2, x_2)\rangle$, these give the constraints :

$$[\sum_{i=1}^{2}(\frac{\xi_i}{t_{i0}^2} + \frac{1}{t_{i0}}\partial_{x_i}) - \frac{3}{4\xi}\partial_x^2] \, G_{\text{GCA}}^{(3)} = 0, \tag{7.4}$$

$$[\sum_{i=1}^{2}(\frac{\Delta_i}{t_{i0}^2} + \frac{2\xi_i}{t_{i0}^3}x_{i0} - \frac{1}{t_{i0}}\partial_{t_i} + \frac{1}{t_{i0}^2}x_{i0}\partial_{x_i}) + \frac{3}{2\xi}\partial_x\partial_t + \frac{3}{4}\frac{\Delta + 1}{\xi^2}\partial_x^2]G_{\text{GCA}}^{(3)} = 0, \tag{7.5}$$

respectively. Now by using (4.9), these translate into

$$\frac{1}{2}\xi + (\xi_1 + \xi_2) - \frac{3}{2\xi}(\xi_1 - \xi_2)^2 = 0,$$

$$4(2\Delta_1 - \Delta_2 + \Delta) + \frac{3(\xi_2 - \xi - \xi_1)^2}{\xi^2}(\Delta + 1) - \frac{6(\xi_2 - \xi - \xi_1)}{\xi}(\Delta_2 - \Delta_1 - \Delta - 1) = 0.$$

Solving the above equations, we get two simple sets of solutions :

$$\xi_2 = \frac{C_2}{2}[1 - (r \pm 1)^2], \quad \Delta_2 = \frac{1}{2}C_1\{1 - (r \pm 1)^2\} + \frac{1}{12}[13(r \pm 1)^2 - 6(r \pm 1)(s + s') - 1]. \tag{7.6}$$

Comparing with (7.2) and (7.3), we see that

$$\Delta_2 = \Delta_{r \pm 1(s,s')}, \quad \xi_2 = \xi_{r \pm 1(s,s')}, \tag{7.7}$$

which is exactly what the relativistic fusion rules imply, namely

$$[\phi_{2(1,1)}] \times [\phi_{r(s,s')}] = [\phi_{r+1(s,s')}] + [\phi_{r-1(s,s')}]. \qquad (7.8)$$

Thus once again we see evidence for the consistency of the GCA limit of the 2d CFT. In this case the GCA analysis gives as strong a constraint as the relativistic CFT. However, as mentioned above, in Appendix B we will give an example where the GCA analysis gives a weaker constraint than what can be extracted from the 2d CFT.

8. The Four Point Function

In this section we make the most nontrivial check, as yet, of the consistency of the scaling limit on the 2d CFT. First we consider the GCA differential equation for the four point function of GCA primaries one of which is $\phi_{2(1,1)}$ which has a level two null descendant. The general solution of this equation consistent with the crossing symmetry of the problem is discussed. We then proceed to rederive this result by taking the scaling limit of the corresponding four point function in the 2d CFT. One finds a solution in the scaling limit which is of the general form inferred from the GCA but now further constrained by the requirements of monodromy invariance present in the parent CFT. Finally, as a zeroth order consistency check of the full theory we briefly examine the factorisation of the four point answer into three point functions and find results in agreement with the fusion rules derived in the previous section.

8.1 GCA Four Point Function

We study the correlation function of four GCA primary fields (in the nonrelativistic Kac table)

$$G_{\text{GCA}}^{(4)}(\{t_i, x_i\}) = \langle \phi_{r_0(s_0,s_0')}(t_0, x_0)\phi_{r_1(s_1,s_1')}(t_1, x_1)\phi_{r_2(s_2,s_2')}(t_2, x_2)\phi_{r_3(s_3,s_3')}(t_3, x_3)\rangle. \tag{8.1}$$

By solving the Ward identities coming from the symmetries $L_{0,\pm 1}$, $M_{0,\pm 1}$, the form of the four point function is restricted to

$$G_{\text{GCA}}^{(4)}(\{t_i, x_i\}) = \prod_{0 \le i < j \le 3} t_{ij}^{\frac{1}{3}\Sigma_{k=0}^3 \Delta_k - \Delta_i - \Delta_j} \, e^{-\frac{x_{ij}}{t_{ij}}(\frac{1}{3}\Sigma_{k=0}^3 \xi_k - \xi_i - \xi_j)} \mathcal{G}_{\text{GCA}}(t, x). \tag{8.2}$$

Here Δ_i and ξ_i are defined by (7.2) and (7.3) with replacing $\{r(s,s')\} \to \{r_i(s_i,s_i')\}$.

The non-relativistic analogues of the cross ratio t and x, which are defined by

$$t = \frac{t_{01}t_{23}}{t_{03}t_{21}}, \qquad \frac{x}{t} = \frac{x_{01}}{t_{01}} + \frac{x_{23}}{t_{23}} - \frac{x_{03}}{t_{03}} - \frac{x_{21}}{t_{21}}, \tag{8.3}$$

are invariant under the coordinate transformation $L_{0,\pm1}$, $M_{0,\pm1}$. Hence the function $\mathcal{G}_{\mathrm{GCA}}(t,x)$ is not determined from these symmetries. As explained in Sec.6, for null states we have differential equations which can be used to further restrict the four point function. In the following we consider the differential equations coming from the primary $\{r_0(s_0, s_0')\} = \{2(1,1)\}$. So, in the following $(\Delta_0, \xi_0) = (\Delta, \xi)$ given in (7.1).

The differential equations coming from the null states $|\chi^{(1)}\rangle$ and $|\chi^{(2)}\rangle$ are given by

$$[\frac{1}{2}\partial_{x_0}^2 + \sum_{i=1}^{3} \kappa^{(-1)}(\frac{\xi_i}{t_{i0}^2} + \frac{1}{t_{i0}}\partial_{x_i})]G_{\mathrm{GCA}}^{(4)} = 0, \tag{8.4}$$

$$[\partial_{x_0}\partial_{t_0} - \frac{1}{2}\frac{\kappa^{(0)}}{\kappa^{(-1)}}\partial_{x_0}^2 - \sum_{i=1}^{3} \kappa^{(-1)}(\frac{2\xi_i x_{i0}}{t_{i0}^3} + \frac{\Delta_i}{t_{i0}^2} + \frac{x_{i0}}{t_{i0}^2}\partial_{x_i} - \frac{1}{t_{i0}}\partial_{t_i})]G_{\mathrm{GCA}}^{(4)} = 0, \tag{8.5}$$

respectively. Here we have introduced the notation

$$\kappa^{(-1)} = -\frac{2}{3}\xi_0, \qquad \kappa^{(0)} = \frac{2}{3}\Delta_0 + \frac{2}{3}. \tag{8.6}$$

By substituting (8.2) into (8.4) and (8.5), we obtain differential equations for $\mathcal{G}_{\mathrm{GCA}}(t,x)$. We can set $t_1 = 0$, $t_2 = 1$, $t_3 = \infty$ and $x_1 = x_2 = x_3 = 0$, by using the finite part of GCA (we can use the $L_{0,\pm1}$ to fix the t's and then the $M_{0,\pm1}$ to fix the x's), so that we have $t_0 = t$, $x_0 = x$. Further introducing $H(t,x)$ as

$$|t|^{\Sigma_{i=0}^3 \frac{\Delta_i}{3} - \Delta_0 - \Delta_1}|1-t|^{\Sigma_{i=0}^3 \frac{\Delta_i}{3} - \Delta_0 - \Delta_2}\mathcal{G}_{\mathrm{GCA}}(t,x) = \mathrm{e}^{\frac{x}{t}(\Sigma_{i=0}^3 \frac{\xi_i}{3} - \xi_0 - \xi_1) - \frac{x}{1-t}(\Sigma_{i=0}^3 \frac{\xi_i}{3} - \xi_0 - \xi_2)}H(t,x),$$

the differential equations for $H(t,x)$ are given by

$$\partial_x^2 H + 2\kappa^{(-1)}\frac{1-2t}{t(1-t)}\partial_x H + 2\kappa^{(-1)}\left(\frac{\xi_1}{t^2} + \frac{\xi_2}{(1-t)^2} + \frac{\xi_0 + \xi_1 + \xi_2 - \xi_3}{t(1-t)}\right)H = 0, \tag{8.7}$$

$$\partial_t\partial_x H - \frac{1}{2}\frac{\kappa^{(0)}}{\kappa^{(-1)}}\partial_x^2 H + \kappa^{(-1)}\left\{\frac{1-2t}{t(1-t)}\partial_t H - x\frac{1-2t+2t^2}{t^2(1-t)^2}\partial_x H\right.$$
$$- \left(\frac{\Delta_1}{t^2} + \frac{\Delta_2}{(1-t)^2} + \frac{\Delta_0 + \Delta_1 + \Delta_2 - \Delta_3}{t(1-t)}\right)H$$
$$\left. - x\left(2\frac{\xi_1}{t^3} - 2\frac{\xi_2}{(1-t)^3} + (\xi_0 + \xi_1 + \xi_2 - \xi_3)\frac{1-2t}{t^2(1-t)^2}\right)H\right\} = 0. \tag{8.8}$$

The first equation can be easily solved to give the following two independent solutions:

$$H_\pm(t,x) = (D(t))^{-\frac{1}{2}}|t(1-t)|^{-\kappa^{(0)}+1}\mathcal{H}_\pm(t)\exp\left\{\frac{xC_2}{t(1-t)}\left(-1+2t\pm\sqrt{D(t)}\right)\right\}, \quad (8.9)$$

where $\mathcal{H}_\pm(t)$ are undetermined functions and we have extracted some t-dependence just for later convenience. The function $D(t)$ is defined by

$$D(t) = r_1^2(1-t) + r_2^2 t - r_3^2 t(1-t), \quad (8.10)$$

and it satisfies $D(t) \geq 0$ because of the triangle inequality for (r_1, r_2, r_3), which follows from the fusion rule (see appendix C.2).

Now in terms of $\mathcal{H}_\pm(t)$, the second equation for $H(t,x)$ is simplified as

$$\partial_t \log \mathcal{H}_\pm = \pm\frac{1}{\sqrt{D}}\left\{\frac{C_1}{t} + \frac{C_2}{1-t} - C_3\right\}. \quad (8.11)$$

Here t-independent constants C_i are defined by

$$C_i = \frac{1}{3}r_i^2\Delta_0 + \frac{1}{3}r_i^2 + \Delta_i + \frac{1}{3}\Delta_0 - \frac{2}{3} \quad (8.12)$$

$$= \pm r_i(\Delta_{r_i\pm1(s_i,s_i')} - \Delta_0 - \Delta_i + \kappa^{(0)} - 1). \quad (8.13)$$

The second line is quite a remarkable simplification. In fact it has a simple physical interpretation with $\Delta_{r_i\pm1(s_i,s_i')}$ corresponding to the conformal dimension of the allowed intermediate states. We will return to this point later in Sec. 8.4.

Finally the differential equation (8.11) can be solved if we notice the relation

$$\partial_t \log|r_1 + r_3 t + \sqrt{D}| = \frac{1}{2\sqrt{D}}\frac{1}{t}(-r_1 + r_3 t + \sqrt{D}). \quad (8.14)$$

Notice that this equation still holds after a flip of the signs in front of r_1 and/or r_3, and it also gives an additional relation by replacing $r_1 \leftrightarrow r_2$ and $t \to 1 - t$. By taking linear combinations of these equations, we obtain the solutions of the above differential equations which are given by

$$\mathcal{H}_+(t) = \mathcal{I}_{1,2,3}(t), \quad \mathcal{H}_-(t) = \mathcal{I}_{2,1,3}(1-t), \quad (8.15)$$

with

$$\mathcal{I}_{1,2,3}(t) = \left|\frac{r_1+r_2+r_3}{r_1+r_2-r_3}\frac{-r_1+r_3 t+\sqrt{D}}{r_1+r_3 t+\sqrt{D}}\right|^{\frac{C_1}{r_1}}\left|\frac{r_1+r_2+r_3}{r_1+r_2-r_3}\frac{r_2-r_3(1-t)+\sqrt{D}}{r_2+r_3(1-t)-\sqrt{D}}\right|^{\frac{C_2}{r_2}}$$

$$\times\left|\frac{r_1+r_2-r_3}{-r_1+r_2+r_3}\frac{r_2+r_3(1-t)+\sqrt{D}}{r_2-r_3(1-t)+\sqrt{D}}\right|^{\frac{C_3}{r_3}}. \quad (8.16)$$

The overall constant factor is chosen for later convenience.

Thus the four point function is given by a general linear combination of the two solutions $H_\pm(t, x)$, which are defined by (8.9) with the functions (8.15). In particular, we may allow different linear combinations for the different segments $t < 0$, $0 < t < 1$ and $1 < t$, since the differential equations are solved independently for each segment. It may be worth pointing out that the term in each absolute value sign takes a definite signature for the each segment. In the relativistic 2d CFT, we would determine the particular linear combination by requiring the four point function to be singlevalued on the complex z plane. In the case of the GCA such an argument is not available. Thus we will discuss only the constraint coming from crossing symmetry.

First of all, it is clear that the following linear combination is invariant under the exchange $1 \leftrightarrow 2$ and $(t, x) \leftrightarrow (1 - t, -x)$:

$$H(t, x) = H_+(t, x) + H_-(t, x). \tag{8.17}$$

Next, by using the following property of the function $\mathcal{I}_{1,2,3}(t)$:

$$\mathcal{I}_{3,2,1}(1/t) = \begin{cases} \mathcal{I}_{1,2,3}(t), \\ \mathcal{I}_{2,1,3}(1 - t), \end{cases} \qquad \mathcal{I}_{2,3,1}(1 - 1/t) = \begin{cases} \mathcal{I}_{2,1,3}(1 - t), & (t > 0), \\ \mathcal{I}_{1,2,3}(t), & (t < 0), \end{cases} \tag{8.18}$$

it is easy to show that $H(t, x)$ behaves correctly under the exchange $1 \leftrightarrow 3$ and $(t, x) \leftrightarrow (1/t, -x/t^2)$, namely,

$$H(t, x) \to |t|^{2\Delta_0} e^{-2\xi_0 \frac{x}{t}} H(t, x). \tag{8.19}$$

This would be the GCA analogue of the transformation $G(z, \bar{z}) \to z^{2h} \bar{z}^{2\bar{h}} G(z, \bar{z})$ in 2d CFT.

Then we may ask whether (8.17) is the unique linear combination which has these properties. In fact since we should allow independent coefficients for different segments (see footnote 11), we have infinitely many combinations which have the correct property of crossing symmetry. Here we will not pursuit the most general form of such possibilities, but just write down a class of such combinations:

$$H(t, x) = \begin{cases} f_{123} H_+(t, x) + f_{321} H_-(t, x), & t < 0, \\ f_{132} H_+(t, x) + f_{231} H_-(t, x), & 0 < t < 1, \\ f_{312} H_+(t, x) + f_{213} H_-(t, x), & 1 < t. \end{cases} \tag{8.20}$$

Here f_{123} is an arbitrary function of the quantum numbers of the three fields $\phi_{r_i(s_i, s'_i)}$ $(i = 1, 2, 3)$. It is obvious that this combination has the same property as (8.17).

Also, by changing the relative sign of H_+ and H_-, we would obtain the combination which flips the overall sign under the exchange $1 \leftrightarrow 2$, $(t,x) \leftrightarrow (1-t, -x)$ or/and $1 \leftrightarrow 3$, $(t,x) \leftrightarrow (1/t, -x/t^2)$. We see no reason to forbid these combinations.

Thus, we find that the requirement of crossing symmetry does not fix the GCA four point function uniquely. In the next section, we consider the corresponding four point function in the 2d CFT and take the non-relativistic limit. In this case, the original four point function is fixed uniquely by the requirement of singlevaluedness. It would be interesting to study whether further conditions coming directly from GCA context would fix the four point function completely or not. We will leave this issue as a problem for the future.

8.2 GCA Four Point Function from 2d CFT

Next we discuss the four point function starting from the 2d CFT and taking the limit. We take the holomorphic part of the four point function in 2d CFT as[14]

$$G^{(4)}_{\text{2d CFT}}(\{z_i\}) = \langle \phi_0(z_0) \phi_1(z_1) \phi_2(z_2) \phi_3(z_3) \rangle = \prod_{0 \le i < j \le 3} z_{ij}^{\frac{1}{3} \sum_{k=0}^3 h_k - h_i - h_j} \mathcal{G}_{\text{2d CFT}}(z), \tag{8.21}$$

where the cross ratio is defined by $z = (z_{01} z_{23})/(z_{03} z_{21})$. By taking $(r_0, s_0) = (2, 1)$, we obtain the differential equation of the form

$$(\hat{\mathcal{L}}_{-2} + \eta \hat{\mathcal{L}}_{-1}^2) G^{(4)}_{\text{2d CFT}}(\{z_i\}) = 0, \tag{8.22}$$

where η is introduced in (5.11) and the differential operators $\hat{\mathcal{L}}_{-n}$ are defined in (6.2).

We introduce the function $K(z)$ as

$$z^{\beta_1} (1-z)^{\beta_2} K(z) = z^{\Sigma_{i=0}^3 \frac{h_i}{3} - h_0 - h_1} (1-z)^{\Sigma_{i=0}^3 \frac{h_i}{3} - h_0 - h_2} \mathcal{G}_{\text{2d CFT}}(z) \tag{8.23}$$

$$= \lim_{z_3 \to \infty} z_3^{2h_3} G^{(4)}_{\text{2d CFT}}(z_0 = z, z_1 = 0, z_2 = 1, z_3), \tag{8.24}$$

with β_i defined as any one of the solutions of the quadratic equation

$$\beta_i(\beta_i - 1) + \kappa \beta_i - \kappa h_i = 0, \qquad (\kappa = -\eta^{-1}). \tag{8.25}$$

The second expression (8.24) is upto a constant phase. Then the differential equation is rewritten into the standard form of the Hypergeometric differential equation:

$$z(1-z) \partial_z^2 K(z) + (\gamma - (\alpha + \beta + 1)z) \partial_z K(z) - \alpha \beta K(z) = 0. \tag{8.26}$$

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[14]Hereafter, to avoid complexity, we use the notation such as ϕ_i and h_i for the objects corresponding to the member (r_i, s_i) on the 2d CFT Kac table.

The parameters α, β and γ are given by

$$\gamma = 2\beta_1 + \kappa, \quad \alpha + \beta + 1 = 2(\beta_1 + \beta_2 + \kappa), \tag{8.27}$$

$$\alpha\beta = \beta_1(\beta_1 - 1) + 2\beta_1\beta_2 + \beta_2(\beta_2 - 1) + \kappa(2\beta_1 + 2\beta_2 + h_0 - h_3). \tag{8.28}$$

The two independent solutions of (8.26) which diagonalize the Monodromy transformation $z \to e^{2\pi i}z$ around $z = 0$ are given by $K_1(\alpha, \beta, \gamma, z) = F(\alpha, \beta, \gamma, z)$ and $K_2(\alpha, \beta, \gamma, z) = z^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z)$, where $F(\alpha, \beta, \gamma, z)$ is defined as the Hypergeometric series and its analytic continuation. The argument so far would be applied in a parallel way to the anti-holomorphic part with the replacement of $h_i \to \bar{h}_i$, $\beta_i \to \bar{\beta}_i$, $\kappa \to \bar{\kappa}$ and $(\alpha, \beta, \gamma) \to (\bar{\alpha}, \bar{\beta}, \bar{\gamma})$, along with the replacement of the coordinate $z \to \bar{z}$.

Now we construct the full four point function by multiplying the holomorphic and the anti-holomorphic contributions and taking an appropriate linear combination. A standard argument (see for e.g. [30]) for choosing the combination is based on the singlevaluedness of the full four point function on the complex plane. In fact, by considering the behaviour under the Monodromy transformations $z \to e^{2\pi i}z$ ($1 - z \to e^{2\pi i}(1-z)$) around $z = 0$ ($z = 1$), we find that the following combination:

$$I(z, \bar{z}) = K_1(\alpha, \beta, \gamma, z)K_1(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{z}) + AK_2(\alpha, \beta, \gamma, z)K_2(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{z}), \tag{8.29}$$

with

$$A = -\frac{\Gamma(\gamma)\Gamma(1-\beta)\Gamma(1-\alpha)}{\Gamma(2-\gamma)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\frac{\Gamma(\bar{\gamma})\Gamma(\bar{\alpha}-\bar{\gamma}+1)\Gamma(\bar{\beta}-\bar{\gamma}+1)}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta})\Gamma(2-\bar{\gamma})}, \tag{8.30}$$

is the unique singlevalued combination, provided $\alpha - \bar{\alpha}$, $\beta - \bar{\beta}$ and $\gamma - \bar{\gamma}$ are all integers. To see this one may use the formula (C.32).

Next we take the non-relativistic limit. For the purpose, we use the asymptotic form of the Hypergeometric functions for large values of α, β and γ, which can be derived by the saddle point analysis of the integral formula:

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 dw\, w^{\beta-1}(1-w)^{\gamma-\beta-1}(1-zw)^{-\alpha}, \tag{8.31}$$

where the conditions $\gamma > \beta > 0$ and that z does not take a real value greater than 1 are assumed. We expand every parameter as

$$\beta_i = \frac{1}{\epsilon}\beta_i^{(-1)} + \beta_i^{(0)} + \cdots, \quad \alpha = \frac{1}{\epsilon}\alpha^{(-1)} + \alpha^{(0)} + \cdots, \quad \text{etc.} \tag{8.32}$$

Then, the quadratic equations for β_i's are solved by

$$\beta_i^{(-1)} = -\frac{1}{2}C_2(1 + \delta_i r_i), \quad \beta_i^{(0)} = -\frac{1}{2}\delta_i \ell_i - \frac{1}{2}(\kappa^{(0)} - 1), \quad (\delta_i = \pm 1), \tag{8.33}$$

with

$$\ell_i = \frac{(\kappa^{(0)} - 1)\kappa^{(-1)} + 2\kappa^{(-1)}h_i^{(0)} + 2\kappa^{(0)}h_i^{(-1)}}{C_2 r_i} = \left(\frac{13}{6} - C_1\right)r_i - s_i. \qquad (8.34)$$

Since β_i are auxiliary parameters introduced in order to change the form of the differential equation, we take $(\delta_1, \delta_2) = (-1, +1)$ without loss of generality. Also, since the Hypergeometric function is symmetric under the exchange $\alpha \leftrightarrow \beta$, we take $\alpha^{(-1)} < \beta^{(-1)}$. Then the solutions (α, β, γ) of Eqs.(8.27) (8.28) are expanded as

$$\alpha^{(-1)} = \frac{C_2}{2}(r_1 - r_2 - r_3), \qquad \beta^{(-1)} = \frac{C_2}{2}(r_1 - r_2 + r_3), \qquad \gamma^{(-1)} = C_2 r_1,$$
$$\alpha^{(0)} = \frac{1}{2}(\ell_1 - \ell_2 - \ell_3 + 1), \quad \beta^{(0)} = \frac{1}{2}(\ell_1 - \ell_2 + \ell_3 + 1), \quad \gamma^{(0)} = \ell_1 + 1, \qquad (8.35)$$

where ℓ_3 is defined by (8.34). As for the expansion of the parameters of the anti-holomorphic part, we first choose the leading term of $\bar{\beta}_i$ as $\bar{\beta}_i^{(-1)} = -\beta_i^{(-1)}$. This can be achieved by taking appropriate branch of the quadratic equation for $\bar{\beta}_i$. Then the leading order terms of the rest of the parameters are given just by flipping the sign as $(\bar{\alpha}^{(-1)}, \bar{\beta}^{(-1)}, \bar{\gamma}^{(-1)}) = -(\alpha^{(-1)}, \beta^{(-1)}, \gamma^{(-1)})$, while $\mathcal{O}(1)$ term is independent from those of the holomorphic part, since we consider different s and s' as (5.21) and (5.22). So, we introduce the notation $\bar{\ell}_i$, which are defined by (8.34) with changing $s_i \to s_i'$. Then the $\mathcal{O}(1)$ terms are given by replacing ℓ_i with $\bar{\ell}_i$ as $\bar{\gamma}^{(0)} = \bar{\ell}_1 + 1$, and the same for $\bar{\alpha}^{(0)}$ and $\bar{\beta}^{(0)}$.

For this parameter choice, the integral (8.31) can be evaluated by taking account of a single saddle point located on the segment $0 < w < 1$, and the result is given by

$$K_1(\alpha, \beta, \gamma, z) \to \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(\gamma-\alpha)} K_+(z), \quad (z = t + \epsilon x, \quad t < 1), \qquad (8.36)$$

where the function K_+ is defined by (C.13) and (C.14) (δ_{C_2} in (C.13) is the sign of C_2, which we take $+1$ in the main text). See the appendix C for a detailed discussion on the saddle point analysis, where the asymptotic form of the rest of the region $1 < t$, the other solution $K_2 = z^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z)$, and the anti-holomorphic counterparts of them are also discussed.

In particular, for the region $t < 0$, the asymptotic form of the basis functions are given by Eqs.(C.20)–(C.22) in terms of the functions $K_\pm(z)$ and $\bar{K}_\pm(\bar{z})$. Inserting these into (8.29), it is easy to show that the "cross term" $K_+(z) \times \bar{K}_-(\bar{z})$ cancels and we obtain the following asymptotic form of the function $I(z, \bar{z})$ for $z = t + \epsilon x$ and $\bar{z} = t - \epsilon x$:

$$I(z, \bar{z}) \to K_+(z)\bar{K}_+(\bar{z})$$
$$+ (-1)^{n_\beta + n_\gamma} z^{-\frac{C_2}{\epsilon}r_1 - \ell_1} \bar{z}^{\frac{C_2}{\epsilon}r_1 - \bar{\ell}_1}(1-z)^{\frac{C_2}{\epsilon}r_2 + \ell_2}(1-\bar{z})^{-\frac{C_2}{\epsilon}r_2 + \bar{\ell}_2} K_-(z)\bar{K}_-(\bar{z}). \qquad (8.37)$$

Here we have omitted the overall (t, x)-independent constant and $n_\gamma = \bar{\gamma} - \gamma$, $n_\beta = \bar{\beta} - \beta$ are integers (see the next subsection). The asymptotic form of the other regions are also derived in the same way and we obtain the same result (8.37) (See appendix C).

Now remember that the four point function is given by multiplying the additional factor $z^{\beta_1}(1 - z)^{\beta_2}$ of (8.23) (and also the anti-holomorphic factor) with $I(z, \bar{z})$. By further expanding with respect to ϵ which appears in $z = t + \epsilon x$ and $\bar{z} = t - \epsilon x$, all singular exponents cancel among the holomorphic and the anti-holomorphic contribution. Finally we obtain the following form of the four point function:[15]

$$H(t, x) = \begin{cases} (-1)^{n_1} H_+(t, x) + (-1)^{n_2+n_3} H_-(t, x)\,, & t < 0\,, \\ H_+(t, x) + (-1)^{n_1+n_2+n_3} H_-(t, x)\,, & 0 < t < 1\,, \\ (-1)^{n_2} H_+(t, x) + (-1)^{n_1+n_3} H_-(t, x)\,, & 1 < t\,. \end{cases} \tag{8.38}$$

Here the functions $H_\pm(t, x)$ are the ones introduced in the previous subsection and n_i are defined by

$$n_i = \frac{C_2}{\epsilon} r_i + \frac{1}{2}(s'_i - s_i). \tag{8.39}$$

As we will explain in the next subsection, the differences $s'_i - s_i$ are now even integers. Hence the factors like $(-1)^{n_1}$ in (8.38) are just signs.

The behaviour of the function (8.38) under the exchange $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$ can be easily found as

$$H(t, x) \rightarrow \begin{cases} (-1)^{n_1+n_2+n_3} H(t, x)\,, & (1 \leftrightarrow 2, (t, x) \rightarrow (1 - t, -x))\,, \\ (-1)^{n_2} |t|^{2\Delta_0} e^{-2\xi_0 \frac{x}{t}} H(t, x)\,, & (1 \leftrightarrow 3, (t, x) \rightarrow (1/t, -x/t^2))\,. \end{cases} \tag{8.40}$$

Here we should notice that the above monodromy argument does not fix the overall factor $C^{(4)}_{1,2,3}$ of the four point function, which should depend on the quantum numbers of the fields ϕ_i $(i = 1, 2, 3)$. For example, if it satisfies the conditions $C^{(4)}_{2,1,3} = (-1)^{n_1+n_2+n_3} C^{(4)}_{1,2,3}$ and $C^{(4)}_{3,2,1} = (-1)^{n_2} C^{(4)}_{1,2,3}$, then the four point function including the overall constant belongs to the class which is written down in (8.20), with $f_{123} = (-1)^{n_1} C^{(4)}_{1,2,3}$. Deriving the explicit form of the overall constant requires the three point functions of the theory, whose overall factor we have also not determined in the present paper.

In summary, contrary to the discussion in the previous section, the asymptotic form of the four point function (8.38) is determined upto an overall constant factor.

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[15] We have used the assumption that C_2/ϵ is an even integer (see footnote 7).

8.3 Singlevaluedness Condition

Now we study the issue of the singlevaluedness in detail. It contains two aspects. As we mentioned below the equation (8.30), the singlevaluedness of the function $I(z, \bar{z})$ requires $\bar{\alpha} - \alpha$, $\bar{\beta} - \beta$ and $\bar{\gamma} - \gamma$ to be integers. Another condition comes from the singlevaluedness of the factor $z^{\beta_1} \bar{z}^{\bar{\beta}_1} (1 - z)^{\beta_2} (1 - \bar{z})^{\bar{\beta}_2}$, which requires $\bar{\beta}_1 - \beta_1$ and $\bar{\beta}_2 - \beta_2$ to be integers.

Before studying these conditions, let us mention the singlevaluedness of the two point functions of the members on the Kac table, which is given by

$$h_{r,s} - \bar{h}_{r,s'} = \frac{1}{2} \times \text{integer} . \tag{8.41}$$

From (5.21) and (5.22), we have

$$h_{r,s} - \bar{h}_{r,s'} = -\frac{1}{2} \frac{C_2}{\epsilon} (1 - r^2) - \frac{1}{2} r(s - s') + \mathcal{O}(\epsilon) . \tag{8.42}$$

Then the above condition requires that $-\frac{C_2}{2\epsilon}(1 - r^2) + \mathcal{O}(\epsilon)$ is an integer or a half integer. Although it is still not very clear in this parameterization, more detailed study[16] tells us that it is satisfied only in the strict limit $\epsilon = 0$ with a "large" integer C_2/ϵ. This final point is always satisfied in the modular invariant 2d CFT, in which case C_2/ϵ is required to be an even integer (see the footnote 7).

Now from (8.34) and (8.35), we first notice that $\bar{\gamma} - \gamma$ is an integer because of the same reason as above. Then in order for the differences $\bar{\beta} - \beta$ and $\bar{\alpha} - \alpha$ to be integers, $\Sigma_{i=1}^{3}(s_i' - s_i)$ must be an even integer. On the other hand the requirement for $\bar{\beta}_i - \beta_i$ to be integers give rise to the condition that each $s_i' - s_i$ $(i = 1, 2)$ is an even integer. In fact, the latter condition is directly related to the singlevaluedness of the three point function which includes the primary field $\phi_{2(1,1)}$, namely, the single valuedness of the three point function $\langle \phi_{2(1,1)} \phi_{r(s,s')} \phi_{r \pm 1(s,s')} \rangle$ gives rise to the condition that $s' - s$ is an even integer.[17] This just means that the primary field $\phi_{2(1,1)}$ interacts only with the primary field having an even $s' - s$. For an odd $s' - s$, the three point function, and hence the four point function, vanishes. Then for the non-vanishing four point function, $s_i' - s_i$ $(i = 1, 2, 3)$ are automatically even integers and the singlevaluedness condition is satisfied. A relation between the parameter β_i and the three point function is further clarified in the next section.

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[16]If we parameterize as $c = 13 - 6(\kappa + 1/\kappa)$ and $\bar{c} = 13 - 6(\bar{\kappa} + 1/\bar{\kappa})$ with $\kappa \gg 1$ and $\bar{\kappa} \ll -1$, then the singlevaluedness requires both $\kappa - \bar{\kappa}$ and $1/\kappa - 1/\bar{\kappa}$ to be integers, which is satisfied only in the case with $1/\kappa = 1/\bar{\kappa} = 0$.

[17]It can be shown explicitly by using (5.21) and (5.22) and requiring that the summation of the spins of the three fields is an integer.

8.4 A Quick Look at Factorization and the Fusion Rule

In 2d CFT, the four point function is written as the summation over the various intermediate fields. Let us consider the four point function (8.21), with its anti-holomorphic part, and set $z_0 = z$, $z_1 = 0$, $z_2 = 1$ and $z_3 = \infty$. Then for small z, the four point function is given by

$$\sum_{p,\{\vec{k},\vec{\bar{k}}\}} C_{01}^{p,\{\vec{k},\vec{\bar{k}}\}} z^{h_p - h_0 - h_1 + K} \bar{z}^{\bar{h}_p - \bar{h}_0 - \bar{h}_1 + \bar{K}} \langle \phi_p^{\{\vec{k},\vec{\bar{k}}\}}(0) \phi_2(1) \phi_3(\infty) \rangle. \tag{8.43}$$

Here $\phi_p^{\{\vec{k},\vec{\bar{k}}\}}$ is the intermediate primary field (for $\vec{k} = \vec{\bar{k}} = 0$), and its descendants (for nontrivial $\vec{k}$ and $\vec{\bar{k}}$), which are given by acting, on the primary, series of the operators $\mathcal{L}_{-k_n} \cdots \mathcal{L}_{-k_1}$ and the same for the anti-holomorphic part. These intermediate states appear in the OPE of ϕ_0 and ϕ_1. This relation shows that the four point function is essentially written in terms of the three point functions. It would be interesting to study the similar property in detail in the case of the GCA. Here, as a first step in such an analysis, we study the contribution coming from the intermediate primary fields.

In the previous subsections, we studied the four point function of the primary fields with setting one of the field to have $\{r_0(s_0, s_0')\} = \{2(1,1)\}$ on the Kac table. In this case, the possible primary fields $\phi_p^{\{0,0\}}$ appearing in (8.43) are the ones with $r_p(s_p, s_p') = r_1 \pm 1(s_1, s_1')$. Let us see this from the equation (8.25). Remember that the holomorphic part of the four point function is given in terms of the Hypergeometric functions $K_1(z)$ and $K_2(z)$ as

$$\lim_{z_3 \to \infty} z_3^{2h_3} G_{\text{2d CFT}}^{(4)}(\{z_i\}) = z^{\beta_1}(1-z)^{\beta_2}\left(a_1 K_1(z) + a_2 K_2(z)\right), \tag{8.44}$$

where a_i are coefficients. Hence for small z, we have two channels contributing to the four point function, namely, one is $z^{\beta_1} K_1(z) \sim z^{\beta_1}$ and the other is $z^{\beta_1} K_2(z) \sim z^{\beta_1 - \gamma + 1} = z^{-\beta_1 - \kappa + 1}$. In terms of the intermediate states of (8.43), this means there are two intermediate conformal families. The conformal dimensions of the primary fields of these families are given by $h_{p_+} = h_0 + h_1 + \beta_1$ and $h_{p_-} = h_0 + h_1 - \beta_1 - \kappa + 1$. Now we may notice that these two exponents β_1 and $-\beta_1 - \kappa + 1$ are the two solutions of (8.25), which means that $h_{p_\pm}$ are indeed the conformal dimensions which are allowed from the relativistic version of the fusion rule, namely, $h_{p_\pm} = h_{r_1 \pm 1}$ (or $h_{p_\pm} = h_{r_1 \mp 1}$ depending on the choice of the branch of the quadratic equation for β_i).[18]

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[18]In 2d CFT, the differential equations for the three point function $\langle \phi_{2,(1,1)} \phi_{r_1(s_1,s_1')} \phi_{r_p(s_p,s_p')} \rangle$ give rise to the following constraint: $2(2h_{2,1}+1)(h_{2,1}+2h_{r_p,s_p}-h_{r_1,s_1}) = 3(h_{2,1}-h_{r_1,s_1}+h_{r_p,s_p})(h_{2,1}-h_{r_1,s_1}+h_{r_p,s_p}+1)$ and the same for the anti-holomorphic part. This is the same as equation (8.25).

Next let us study the GCA four point function. In subsection 8.1, we have seen that the GCA four point function is composed of, again, two solutions of the differential equation, which we call $H_\pm(t, x)$. The solution is given by (8.9), with $\mathcal{H}_\pm(t)$ satisfying the differential equation (8.11). In fact, the small t behaviour of the four point function can be derived by using Eqs.(8.9)–(8.11) as

$$H_\pm(t,x) \sim t^{-\kappa^{(0)}+1} \cdot t^{\pm\frac{c_1}{r_1}} \cdot \exp\left(\frac{x}{t}C_2(-1\pm r_1)\right)$$
$$= t^{\Delta_{r_1\pm1(s_1,s_1')}-\Delta_0-\Delta_1} \exp\left(-(\xi_{r_1\pm1}-\xi_0-\xi_1)\frac{x}{t}\right). \qquad (8.45)$$

These are indeed the two leading behaviours which appear for the operator product expansion of the primary fields $\phi_{2(1,1)}$ and $\phi_{r_1(s_1,s_1')}$ in GCA (see appendix A for a preliminary examination of the GCA OPE). Now it is clear from (8.11) and (8.13), that the other ordering, i.e., first taking the OPE between $\phi_0(z)$ and $\phi_2(1)$, gives two intermediate states which are again allowed from the fusion rule.

9. Concluding Remarks

This concludes our present study of the quantum mechanical realisation of the GCA in two dimensions. We found that 2d GCFT's with nonzero central charges C_1, C_2 can be readily obtained by considering a somewhat unusual limit of a non-unitary 2d CFT. While the resulting Hilbert space of the GCFT is again non-unitary, the theory seems to be otherwise well defined. We found that many of the structures parallel those in the Virasoro algebra and indeed arise from them when we realise the GCA by means of the scaling limit. But in most cases we could also obtain many of the same results autonomously from the definition of the GCA itself, showing that these are features of any realisation of this symmetry. Along these lines, one of the nice things to independently compute would be the form of the GCA Kac Table which we obtained in Sec. 5 from the scaling limit of the relativistic table. It would be interesting if one can find an analogue of minimal models for the GCA for which there are a finite number of GCA primaries and for which the solution of the theory can be completely given. Though we must hasten to remind the reader that such theories (if they at all exist) would not, in any sense, arise from the Virasoro minimal models which have $0 < c < 1$. Another interesting direction would be to investigate possible generalizations of our results to higher dimensions where we do have the GCA but no analogue of the Virasoro Algebra. In this context it must be clear from the explicit solutions in Sec. 8 that the lack of holomorphicity, while it

makes the expressions more cumbersome, is not a real barrier to solving the theory provided one has GCA null vectors.

As far as the scaling limit of the 2d CFT is concerned, it is important to make some further checks on its consistency with the requirements from the GCA. From the point of view of the 2d CFT it would be nice to get a better understanding of this scaling limit on the central charges. Sending their magnitude to infinity seems like some kind of large N or classical limit. While it is somewhat dismaying that the theory is non-unitary, could there perhaps be an interesting "physical" subsector which is unitary? It is amusing to note in this context that in perturbative string theories we do have 2d CFTs with large (26 or even 15!) positive and negative values of c and $\bar{c}$ in the matter and ghost sectors.

In this paper we have only studied the limit of generic 2d CFTs. It would be interesting to carry out a similar scaling limit on CFT's with supersymmetry and/or Kac-Moody symmetries.[19] More generally, it is important to understand whether there are concrete physical situations where such a scaling limit is actually realised. That would make the further study of these symmetries all the more exciting.

Acknowledgements

Appendix

A. Descendants and GCA Conformal Blocks

A.1 GCA Descendants

By means of the differential operators $\hat{L}_{-k}, \hat{M}_{-k}$ (with $k > 0$) in (6.4), we may express the correlation function including a general GCA descendant with the correlation

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[19]Supersymmetric versions of the GCA have been recently studied in [23, 24, 25]

function of its primary $\phi_{\Delta\xi}(t,x)$. We had in fact already used this in Sec. 6 for the simple cases of these descendants corresponding to null states as (6.5) and (6.6). The general expression can be written as

$$\langle\phi_{\Delta\xi}^{\{\vec{k},\vec{q}\}}(t,x)\phi_1(t_1,x_1)\cdots\phi_n(t_n,x_n)\rangle$$
$$= \hat{L}_{-k_i}\cdots\hat{L}_{-k_1}\hat{M}_{-q_j}\cdots\hat{M}_{-q_1}\langle\phi_{\Delta\xi}(t,x)\phi_1(t_1,x_1)\cdots\phi_n(t_n,x_n)\rangle, \quad \text{(A.1)}$$

where $\vec{k}=(k_1,k_2,\cdots,k_i)$ and $\vec{q}=(q_1,q_2,\cdots,q_j)$ are sequences of positive integers such that $k_1\leq k_2\cdots\leq k_i$ and similarly for the q's. Also note that $\phi_{\Delta\xi}^{\{0,0\}}(t,x)$ denotes the primary $\phi_{\Delta\xi}(t,x)$ itself.

A.2 The OPE and GCA Blocks

Just as in the relativistic case, the OPE of two GCA primaries can be expressed in terms of the GCA primaries and their descendants as

$$\phi_1(t,x)\phi_2(0,0) = \sum_p\sum_{\{\vec{k},\vec{q}\}} C_{12}^{p\{\vec{k},\vec{q}\}}(t,x)\phi_p^{\{\vec{k},\vec{q}\}}(0,0). \quad \text{(A.2)}$$

We should mention that, unlike in the case of a 2d CFT such an expansion is not analytic (see (A.7) below). The form of the two and three point function clearly exhibit essential singularities. Nevertheless we will go ahead with the expansion assuming it makes sense in individual segments such as $x,t>0$. One can find the first few coefficients $C_{12}^{p\{\vec{k},\vec{q}\}}(t,x)$ by considering the three point function of the primary fields $\langle\phi_3\phi_1\phi_2\rangle$ in the situation where ϕ_1 approaches ϕ_2. In such a situation one can replace $\phi_1\phi_2$, in the three point function, with the RHS of (A.2) and obtain

$$\langle\phi_3(t',x')\phi_1(t,x)\phi_2(0,0)\rangle = \sum_{p,\{\vec{k},\vec{q}\}} C_{12}^{p\{\vec{k},\vec{q}\}}(t,x)\langle\phi_3(t',x')\phi_p^{\{\vec{k},\vec{q}\}}(0,0)\rangle. \quad \text{(A.3)}$$

We can find $C_{12}^{p\{0,0\}}$, $C_{12}^{p\{1,0\}}$ and $C_{12}^{p\{0,1\}}$ by expanding the left hand side (LHS) of (A.3) with respect to the small parameter $\frac{t}{t'}$ while keeping $\frac{x'}{t'}$ and $\frac{x}{t}$ finite, and comparing the (t',x')-dependence of the both sides. To make the final formulae simple, we concentrate on the case with $\Delta_1=\Delta_2=\Delta$ and $\xi_1=\xi_2=\xi$.

The expansion of the LHS is given as

$$\langle\phi_3(t',x')\phi_1(t,x)\phi_2(0,0)\rangle$$
$$= C_{312}t^{-2\Delta+\Delta_3}t'^{-\Delta_3}(t'-t)^{-\Delta_3}\exp[\xi_3\frac{x'-x}{t'-t}+(2\xi-\xi_3)\frac{x}{t}+\xi_3\frac{x'}{t'}]$$
$$= C_{312}t'^{-2\Delta_3}e^{2\xi_3\frac{x'}{t'}}\cdot t^{-2\Delta+\Delta_3}e^{(2\xi-\xi_3)\frac{x}{t}}[1+\{\Delta_3-\xi_3(\frac{x}{t}-\frac{x'}{t'})\}\frac{t}{t'}+\mathcal{O}((t/t')^2)], \quad \text{(A.4)}$$

while the RHS is given by

$$\sum_{p,\{\vec{k},\vec{q}\}} C_{12}^{p\{\vec{k},\vec{q}\}}(t,x)\langle\phi_3(t',x')\phi_p^{\{\vec{k},\vec{q}\}}(0,0)\rangle$$

$$= [C_{12}^{3\{0,0\}}(t,x) + C_{12}^{3\{1,0\}}(t,x)\hat{L}_{-1} + C_{12}^{3\{0,1\}}(t,x)\hat{M}_{-1} + \ldots](t'^{-2\Delta_3}e^{2\xi_3\frac{x'}{t'}})$$

$$= t'^{-2\Delta_3}e^{2\xi_3\frac{x'}{t'}}[C_{12}^{3\{0,0\}} + C_{12}^{3\{1,0\}}(2\Delta_3 + 2\xi_3\frac{x'}{t'})t'^{-1} + C_{12}^{3\{0,1\}}(2\xi_3)t'^{-1} + \cdots].\text{(A.5)}$$

One can easily read off the coefficients by comparing (A.4) and (A.5) :

$$C_{12}^{3\{0,0\}} = C_{312}t^{-2\Delta+\Delta_3}e^{(2\xi-\xi_3)\frac{x}{t}},$$

$$C_{12}^{3\{1,0\}} = \frac{1}{2}C_{312}t^{-2\Delta+\Delta_3+1}e^{(2\xi-\xi_3)\frac{x}{t}}, \tag{A.6}$$

$$C_{12}^{3\{0,1\}} = -\frac{1}{2}C_{312}xt^{-2\Delta+\Delta_3}e^{(2\xi-\xi_3)\frac{x}{t}}.$$

So, in this case we have the GCA OPE as:

$$\phi_1(t,x)\phi_2(0,0)$$

$$= \sum_p C_{p12}t^{-2\Delta+\Delta_p}e^{(2\xi-\xi_p)\frac{x}{t}}\left(\phi_p(0,0) + \frac{t}{2}\phi_p^{\{1,0\}}(0,0) - \frac{x}{2}\phi_p^{\{0,1\}}(0,0) + \ldots\right). \tag{A.7}$$

B. Fusion Rule for $\phi_{1(2,2)}$

Here we consider the level two null state corresponding to the primary

$$\Delta = \Delta_{1(2,2)} = -1, \quad \xi = \xi_{1(2,2)} = 0, \tag{B.1}$$

and study the fusion rule for the product $[\phi_{\Delta,\xi}] \times [\phi_{\Delta_1,\xi_1}]$, where Δ_1, ξ_1 are given by (7.2) and (7.3) with a generic triple $r(s,s')$.

From the GCA in this case we have level two null states (see below (5.4)) given by $|\chi^{(1)}\rangle = M_{-1}^2|\Delta,0\rangle$ and $|\chi^{(2)}\rangle = L_{-1}M_{-1}|\Delta = -1,0\rangle$. However, these null states are actually just descendants of the level one null state $M_{-1}|\Delta = -1,0\rangle$. The imposition of the GCA constraint

$$\hat{M}_{-1}\langle\phi_{\Delta,\xi}(t,x)\phi_{\Delta_1,\xi_1}(t_1,x_1)\phi_{\Delta_2,\xi_2}(t_2,x_2)\rangle$$

$$= -\partial_x\langle\phi_{\Delta,\xi}(t,x)\phi_{\Delta_1,\xi_1}(t_1,x_1)\phi_{\Delta_2,\xi_2}(t_2,x_2)\rangle = 0, \tag{B.2}$$

simply gives $\xi_1 = \xi_2$. Examination of (7.2) and (7.3) immediately implies that the r value of $\phi_{\Delta_2,\xi_2}(t_2,x_2)$ is the same as that of $\phi_{\Delta_1,\xi_1}(t_1,x_1)$ but its (s,s') values are undetermined. As we will see below, in taking the nonrelativistic limit of the 2d CFT we can extract the fusion rule obeyed in the relativistic CFT.

Thus, let us take the nonrelativistic limit of the relativistic fusion rule involving the primary $\phi_{2(1,1)}$. The conformal weights corresponding to $2(1,1)$ and $r(s,s')$ are expanded as

$$h = h_{1,2} = -\frac{1}{2} + \frac{3\epsilon}{4C_2} + \mathcal{O}(\epsilon^2)\,, \quad \bar{h} = \bar{h}_{1,2} = -\frac{1}{2} + \frac{3\epsilon}{4C_2} + \mathcal{O}(\epsilon^2)\,;$$

$$h_1 = h_{r,s} = \frac{C_2}{4\epsilon}(r^2 - 1) + (\frac{C_1}{4}(1 - r^2) + \frac{13}{24}r^2 - \frac{rs}{2} - \frac{1}{24}) - \frac{\epsilon}{4C_2}(r^2 - s^2) + \mathcal{O}(\epsilon^2)\,,$$

$$\bar{h}_1 = \bar{h}_{r,s'} = \frac{C_2}{4\epsilon}(1 - r^2) + (\frac{C_1}{4}(1 - r^2) + \frac{13}{24}r^2 - \frac{rs'}{2} - \frac{1}{24}) + \frac{\epsilon}{4C_2}(r^2 - s'^2) + \mathcal{O}(\epsilon^2)\,,$$

respectively. Here in addition to the terms in (5.21) and (5.22), if we maintain the terms of order $\mathcal{O}(\epsilon)$ then we can extract the information in the relativistic fusion rule.

The differential equation for the three point function $\langle \phi_{1(2,2)} \phi_1 \phi_2 \rangle$ now implies that the h's obey the equation

$$\frac{3}{2}(h_2 - h - h_1)(h_2 - h - h_1 - 1) + (h_2 - h - 2h_1)(2h + 1) = 0\,, \qquad \text{(B.3)}$$

and the same for $\bar{h}$'s. Solving these equations, keeping the $\mathcal{O}(\epsilon)$ terms, we get the solutions as

$$h_2 = h_{r,s\pm1}\,, \quad \bar{h}_2 = \bar{h}_{r,s'\pm1}\,,$$

which tell us that we have four possible families appearing in the OPE between $[\phi_{1(2,2)}]$ and $[\phi_{r(s,s')}]$. Note that the (s,s') values are now constrained. In terms of the GCA weights, the four families are specified as

$$\Delta_2 = \Delta_{r(s\pm1,s'\pm1)}\,, \qquad \xi_2 = \xi_{r(s\pm1,s'\pm1)}\,, \qquad \text{(B.4)}$$

where both the $\pm$ signs are taken independently.

C. Asymptotic Form of the Hypergeometric Function

C.1 Saddle Point Analysis

In this section we perform the saddle point analysis of the integral formula for the Hypergeometric function:

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 dw\, w^{\beta-1}(1 - w)^{\gamma-\beta-1}(1 - zw)^{-\alpha}. \qquad \text{(C.1)}$$

This formula is valid for the parameter $\text{Re}(\gamma) > \text{Re}(\beta) > 0$, and we will use it for $\text{Re}(z) < 0$. The parameters α, β and γ are related to the physical parameters as

$$\alpha + \beta + 1 = 2(\beta_1 + \beta_2 + \kappa)\,, \quad \gamma = 2\beta_1 + \kappa\,, \qquad \text{(C.2)}$$

$$\alpha\beta = \beta_1(\beta_1 - 1) + 2\beta_1\beta_2 + \beta_2(\beta_2 - 1) + \kappa(2\beta_1 + 2\beta_2 + h_0 - h_3)\,, \qquad \text{(C.3)}$$

with $\kappa = (2/3)(2h_0 + 1)$ and β_i defined as solutions of the equations

$$\beta_i(\beta_i - 1) + \kappa\beta_i - \kappa h_i = 0, \quad (i = 1, 2). \tag{C.4}$$

We expand parameters as

$$\beta_i = \frac{1}{\epsilon}\beta_i^{(-1)} + \beta_i^{(0)} + \cdots, \quad \alpha = \frac{1}{\epsilon}\alpha^{(-1)} + \alpha^{(0)} + \cdots, \tag{C.5}$$

etc.. Then the quadratic equations for β_i are solved as

$$\beta_i^{(-1)} = -\frac{1}{2}C_2(1 + \delta_i r_i), \quad \beta_i^{(0)} = -\frac{\delta_i}{2}\ell_i - \frac{1}{2}(\kappa^{(0)} - 1), \tag{C.6}$$

with

$$\ell_i = \frac{(\kappa^{(0)} - 1)\kappa^{(-1)} + 2\kappa^{(-1)}h_i^{(0)} + 2\kappa^{(0)}h_i^{(-1)}}{C_2 r_i}$$

$$= \frac{1}{r_i}\left\{\frac{1}{3}\Delta_0 - \frac{2}{3} + 2h_i^{(0)} + \frac{1}{3}\Delta_0 r_i^2 + \frac{1}{3}r_i^2\right\}. \tag{C.7}$$

Since β_i are auxiliary parameters introduced in order to change the form of the differential equation (see the explanation around (8.25)), we take $(\delta_1, \delta_2) = \delta_{C_2}(-1, +1)$ without loss of generality, where δ_{C_2} is the sign of C_2. Also, since the Hypergeometric function is symmetric under the exchange $\alpha \leftrightarrow \beta$, we take $\alpha^{(-1)} < \beta^{(-1)}$, and then α, β and γ are expanded as

$$\alpha^{(-1)} = \frac{|C_2|}{2}(r_1 - r_2 - r_3), \quad \beta^{(-1)} = \frac{|C_2|}{2}(r_1 - r_2 + r_3), \quad \gamma^{(-1)} = |C_2|r_1,$$

$$\alpha^{(0)} = \frac{\delta_{C_2}}{2}(\ell_1 - \ell_2 - \ell_3) + \frac{1}{2}, \quad \beta^{(0)} = \frac{\delta_{C_2}}{2}(\ell_1 - \ell_2 + \ell_3) + \frac{1}{2}, \quad \gamma^{(0)} = \delta_{C_2}\ell_1 + 1. \tag{C.8}$$

Here ℓ_3 is defined in the same way as $(C.7)$.

Now the integral $(C.1)$ is written as

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma - \beta)} \int_0^1 dw e^{\frac{1}{\epsilon}g^{(-1)}(w,t) + x\partial_t g^{(-1)}(w,t) + g^{(0)}(w,t) + \mathcal{O}(\epsilon)}, \tag{C.9}$$

where $g^{(n)}(w, z)$'s are defined as the functions appearing in the expansion of the exponent of the integrand as

$$(\beta - 1)\log w + (\gamma - \beta - 1)\log(1 - w) - \alpha\log(1 - zw) = \sum_n \epsilon^n g^{(n)}(w, z). \tag{C.10}$$

In $(C.9)$, we have further expanded based on $z = t + \epsilon x$. The saddle points of the integrand are given by $\partial_w g^{(-1)}(w_*, t) = 0$ and they are located at

$$w_*^{\pm} = \frac{\gamma^{(-1)} - (\alpha^{(-1)} - \beta^{(-1)})t \mp |C_2|\sqrt{D(t)}}{2(\gamma^{(-1)} - \alpha^{(-1)})t}. \tag{C.11}$$

By using the fact that r_i's satisfy the triangle inequality, which follows from the fusion rule (see the next section), we can show that both the saddle points are on the real axis, namely $D(t) \geq 0$. In fact, for the parameterization explained above, only one $(w = w_*^+)$ of the saddle points is located on the segment $0 < w < 1$, and the integral is evaluated by taking it. By taking the saddle point value and doing the Gaussian integral, we obtain the following asymptotic form of the Hypergeometric function for small ϵ:

$$F(\alpha, \beta, \gamma, z) \to \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\gamma)\Gamma(1-\alpha)}{\Gamma(\gamma-\alpha)} \mathcal{K}_+(t, x), \quad (t < 1), \tag{C.12}$$

where the function $\mathcal{K}_+$ is defined by

$$\mathcal{K}_+(t, x) = |C_2|^{-\frac{1}{2}} \big(D(t)\big)^{-\frac{1}{4}} \left\{ \frac{r_1 + r_2 + r_3}{r_1 + r_2 - r_3} \frac{-r_1 + r_3 t + \sqrt{D(t)}}{r_1 + r_3 t + \sqrt{D(t)}} \frac{1}{t} \right\}^{\frac{1}{\epsilon} \frac{|C_2|}{2} r_1 + \frac{\delta_{C_2}}{2} \ell_1}$$

$$\times \left\{ \frac{r_1 + r_2 + r_3}{r_1 + r_2 - r_3} \frac{r_2 - r_3(1-t) + \sqrt{D(t)}}{r_2 + r_3(1-t) - \sqrt{D(t)}}(1-t) \right\}^{\frac{1}{\epsilon} \frac{|C_2|}{2} r_2 + \frac{\delta_{C_2}}{2} \ell_2}$$

$$\times \left\{ \frac{r_1 + r_2 - r_3}{-r_1 + r_2 + r_3} \frac{r_2 + r_3(1-t) + \sqrt{D(t)}}{r_2 - r_3(1-t) + \sqrt{D(t)}} \right\}^{\frac{1}{\epsilon} \frac{|C_2|}{2} r_3 + \frac{\delta_{C_2}}{2} \ell_3}$$

$$\times \exp\left\{ \frac{x|C_2|}{2t(1-t)}\big(-r_1(1-t) - r_2 t + \sqrt{D(t)}\big) \right\} \tag{C.13}$$

$$= K_+(t) \exp\left\{ \frac{x|C_2|}{2t(1-t)}\big(-r_1(1-t) - r_2 t + \sqrt{D(t)}\big) \right\}. \tag{C.14}$$

The function $K_+(t) = \mathcal{K}_+(t, 0)$ is introduced for convenience. The x-dependence appears only in the exponential form, which can be derived by expanding $K_+(z = t + \epsilon x)$ with respect to ϵ, as it can be seen from (C.9). The overall factor $\epsilon^{\frac{1}{2}}$ on the RHS of (C.12) is cancelled by the factor $\epsilon^{-\frac{1}{2}}$ when we expand the Gamma functions.

C.2 Fusion Rule and Relation Between $r_i(s_i, s_i')$'s

In the main text and this appendix, relations between three primary fields ϕ_1, ϕ_2 and ϕ_3 in the four point function $\langle \phi_0 \phi_1 \phi_2 \phi_3 \rangle$ with $\phi_0 = \phi_{2(1,1)}$ are used in several places. These relations are derived from the generic fusion rule in the relativistic 2d CFT:

$$[\phi_{r_i(s_i, s_i')}] \times [\phi_{r_j(s_j, s_j')}] = \sum_{r_p=|r_i-r_j|+1}^{r_i+r_j-1} \sum_{s_p=|s_i-s_j|+1}^{s_i+s_j-1} \sum_{s_p'=|s_i'-s_j'|+1}^{s_i'+s_j'-1} [\phi_{r_p(s_p, s_p')}]. \tag{C.15}$$

The variables r_p, s_p and s_p' are incremented by 2. Let us discuss only the channel in which we take the OPE between ϕ_0 and ϕ_1 first, then we take the OPE between the

intermediate field ϕ_p and ϕ_2 to give the remaining field ϕ_3. Symbolically, we consider the order $\langle\phi_0\phi_1\phi_2\phi_3\rangle = \Sigma_p C_{01p}\langle\phi_p\phi_2\phi_3\rangle = \Sigma_{p,q} C_{01p}C_{p2q}\langle\phi_q\phi_3\rangle$. Here in the last step, because of the orthogonality of the conformal families, we have $r_q(s_q, s_q') = r_3(s_3, s_3')$. The other cases can be discussed similarly.

In this case, the possible intermediate state is given by $r_p(s_p, s_p') = (r_1 \pm 1)(s_1, s_1')$ and the fusion rule for the OPE between ϕ_p and ϕ_2 is given by

$$r_3 = |r_1 \pm 1 - r_2| + 1, \ |r_1 \pm 1 - r_2| + 3, \ \cdots, \ r_1 \pm 1 + r_2 - 1, \tag{C.16}$$

$$s_3 = |s_1 - s_2| + 1, \ |s_1 - s_2| + 3, \ \cdots, \ s_1 + s_2 - 1, \tag{C.17}$$

$$s_3' = |s_1' - s_2'| + 1, \ |s_1' - s_2'| + 3, \ \cdots, \ s_1' + s_2' - 1. \tag{C.18}$$

From the first equation (C.16), we see that the three r_i's satisfy the triangle inequality, which is important in the saddle point analysis in appendix C.1. Also we notice that the possible values for the summation of three r_i's are even, while the summation of three s_i's and s_i''s are always odd numbers. It is consistent with the property that $s_i' - s_i$ are even integers (see Sec. 8.3).

C.3 Validity of the Analysis

Let us now clarify the validity of the analysis in appendix C.1. As we mentioned, the integral formula (C.1) is valid for the parameter satisfying $\gamma > \beta > 0$. From the expansion (C.8) with (C.7), it is easy to see that for the case with $r_1 - r_2 + r_3 = 0$ or $r_1 + r_2 - r_3 = 0$, namely, $\beta_{-1} = 0$ or $\gamma_{-1} = \beta_{-1}$, the condition is violated for general s_i's. Hence for these cases, the simple integral formula (C.1) is not available and we need to consider more generic formula, which we do not pursuit in the present paper. Examples of such situations include the case when one of the fields $\phi_{1,2,3}$ is the identity operator. On the other hand, in the case with $r_1 - r_2 - r_3 = 0$, the above analysis itself is valid. However, for deriving the asymptotic form for the rest of the region, namely, $t > 1$ and also for the other basis function in the next subsection, the case $r_1 - r_2 - r_3 = 0$ violates the conditions which are equivalent to the ones explained above. Hence our derivation of the asymptotic form of the four point function is applicable only for the cases with $r_1 < r_2 + r_3$, $r_2 < r_1 + r_3$ and $r_3 < r_1 + r_2$, i.e. the strict triangle inequality is satisfied.

The validity of the saddle point analysis also requires the second derivative

$$\partial_w^2 g^{(-1)}(w_*^+, t) = -\frac{|C_2|\sqrt{D}}{(1 - tw_*^+)^2}\frac{(2\alpha_{-1}\beta_{-1} - \gamma_{-1}(\alpha_{-1} + \beta_{-1}))t + \gamma_{-1}^2 + \gamma_{-1}|C_2|\sqrt{D}}{2\beta_{-1}(\gamma_{-1} - \beta_{-1})},$$
$$\tag{C.19}$$

to be negative definite (with large $1/\epsilon$ in front of it). This condition is perfectly satisfied with the above assumption for the parameters.

C.4 A Complete Set of the Solutions and Useful Formulae

Next we derive the other solution $(-z)^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z)$ for the region $t < 0$ by using (C.12) and the formula (C.31) (formula (I)). By changing the parameters of the formula (I) as $\alpha \to \alpha - \gamma + 1$, $\beta \to \beta - \gamma + 1$ and $\gamma \to 2 - \gamma$, and multiplying the factor $(-z)^{1-\gamma}$ to the both sides of the resulting equation, we obtain the similar formula (formula (II)) for $(-z)^{1-\gamma}F(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma, z)$ expressed in terms of the same two functions on the RHS of (C.31). Regarding the second function on the RHS, i.e., $(-1)^{-\beta}F(\beta, \beta-\gamma+1, \beta-\alpha+1, 1/z)$, the condition $\alpha < 0 < \beta < \gamma$, which we are now assuming to hold, assures $\alpha' < 0 < \beta' < \gamma'$ with $\alpha' = \beta - \gamma + 1$, $\beta' = \beta$ and $\gamma' = \beta-\alpha+1$. Then the asymptotic form (C.12) is applicable for this case with the parameter change $\alpha \to \alpha'$, $\beta \to \beta'$ and $\gamma \to \gamma'$ along with the coordinate change $z \to 1/z$. Note that the condition $\alpha^{(-1)} = \frac{|C_2|}{2}(r_1 - r_2 - r_3) < 0$ is necessary here in order for the condition $\beta' < \gamma'$ to be satisfied in general, as we mentioned in the previous subsection.

Then by using the two formulae (I) and (II), which are mentioned above, we can write down the asymptotic form of the function $(-z)^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z)$ in terms of the asymptotic forms of the functions $F(\alpha, \beta, \gamma, z)$ and $(-z)^{-\beta}F(\alpha', \beta', \gamma', 1/z)$. The explicit form is given by

$$
(-z)^{1-\gamma}F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z) \to \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}}\left\{\frac{\Gamma(\beta)\Gamma(2 - \gamma)}{\Gamma(\beta - \gamma + 1)}K_+(z)\right.
$$
$$
\left. +\frac{\Gamma(\beta)\Gamma(\gamma - \beta)}{\Gamma(\gamma - 1)}(-z)^{-\frac{1}{\epsilon}|C_2|r_1-\delta_{C_2}\ell_1}(1 - z)^{\frac{1}{\epsilon}|C_2|r_2+\delta_{C_2}\ell_2}K_-(z)\right\}. \quad \text{(C.20)}
$$

Here again $z = t + \epsilon x$ with small ϵ is assumed. The function $K_-(z)$ can be obtained from $K_+(z)$ by replacing as $1 \leftrightarrow 2$ and $(t, x) \leftrightarrow (1-t, -x)$, or equivalently $z \leftrightarrow 1-z$.

The asymptotic form of the anti-holomorphic functions for the region $t < 0$ can be derived in the same way. In our choice of parameters (see the explanation after (8.35) for the parameter choice of the anti-holomorphic part), the parameters in the anti-holomorphic part satisfy $\bar{\gamma} < \bar{\beta} < 0 < \bar{\alpha}$. By finding appropriate basis functions to which the asymptotic form (C.12) is applicable, and also by using the formula

(C.31), we obtain the following result:

$$F(\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{z}) \rightarrow (2\pi\epsilon)^{\frac{1}{2}} \left\{ \frac{\Gamma(\bar{\alpha} - \bar{\gamma} + 1)}{\Gamma(1 - \bar{\gamma})\Gamma(\bar{\alpha})} \bar{K}_+(\bar{z}) \right.$$

$$\left. + \frac{\Gamma(\bar{\alpha} - \bar{\gamma} + 1)\Gamma(\bar{\gamma})}{\Gamma(\bar{\alpha})\Gamma(\bar{\gamma} - \bar{\beta})\Gamma(\bar{\beta} - \bar{\gamma} + 1)} (-\bar{z})^{\frac{1}{\epsilon}|C_2|r_1 - \delta_{C_2}\bar{\ell}_1}(1 - \bar{z})^{-\frac{1}{\epsilon}|C_2|r_2 + \delta_{C_2}\bar{\ell}_2} \bar{K}_-(\bar{z}) \right\}, \quad (C.21)$$

$$(-\bar{z})^{1 - \bar{\gamma}} F(\bar{\alpha} - \bar{\gamma} + 1, \bar{\beta} - \bar{\gamma} + 1, 2 - \gamma, \bar{z})$$

$$\rightarrow (2\pi\epsilon)^{\frac{1}{2}} \frac{\Gamma(2 - \bar{\gamma})}{\Gamma(1 - \bar{\beta})\Gamma(\bar{\beta} - \bar{\gamma} + 1)} (-\bar{z})^{\frac{1}{\epsilon}|C_2|r_1 - \delta_{C_2}\bar{\ell}_1}(1 - \bar{z})^{-\frac{1}{\epsilon}|C_2|r_2 + \delta_{C_2}\bar{\ell}_2} \bar{K}_-(\bar{z}),$$

$$(C.22)$$

where $\bar{K}_\pm(\bar{z})$ are obtained from $K_\pm(z)$ with replacements $\epsilon \rightarrow -\epsilon$ and $\ell_i \rightarrow \bar{\ell}_i$. In the main text, we use Eqs.(C.20), (C.21) and (C.22) for deriving the asymptotic form of the four point function in the region $t < 0$.

In fact, by means of (C.12), we can derive the asymptotic forms of the sets of two independent solutions for the remaining segments. Here is the list of the asymptotic forms:

1. The functions whose asymptotic forms are given in terms of $K_+(z)$:

   - $t < 1$
   $$F(\alpha, \beta, \gamma, z) \rightarrow \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\gamma)\Gamma(1 - \alpha)}{\Gamma(\gamma - \alpha)} K_+(z) \qquad (C.23)$$

   - $1 < t$
   $$z^{-\beta} F(\beta - \gamma + 1, \beta, \beta - \alpha + 1, 1/z) \rightarrow \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\beta - \alpha + 1)\Gamma(\gamma - \beta)}{\Gamma(\gamma - \alpha)} K_+(z)$$
   $$(C.24)$$

2. The functions whose asymptotic forms are given in terms of $K_-(z)$:

   - $t < 0$
   $$(-z)^{-\beta} F(\beta - \gamma + 1, \beta, \beta - \alpha + 1, 1/z)$$
   $$\rightarrow \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\beta - \alpha + 1)\Gamma(\gamma - \beta)}{\Gamma(\gamma - \alpha)} (-z)^{-\frac{1}{\epsilon}|C_2|r_1 - \delta_{C_2}\ell_1}(1 - z)^{\frac{1}{\epsilon}|C_2|r_2 + \delta_{C_2}\ell_2} K_-(z)$$
   $$(C.25)$$

   - $0 < t$
   $$z^{1 - \gamma}(1 - z)^{\gamma - \alpha - \beta} F(1 - \beta, 1 - \alpha, \gamma - \alpha - \beta + 1, 1 - z)$$
   $$\rightarrow \left(\frac{\epsilon}{2\pi}\right)^{\frac{1}{2}} \frac{\Gamma(\gamma - \alpha - \beta + 1)\Gamma(\beta)}{\Gamma(\gamma - \alpha)} z^{-\frac{1}{\epsilon}|C_2|r_1 - \delta_{C_2}\ell_1}(1 - z)^{\frac{1}{\epsilon}|C_2|r_2 + \delta_{C_2}\ell_2} K_-(z)$$
   $$(C.26)$$

3. The functions whose asymptotic forms are given in terms of $\bar{K}_+(\bar{z})$:

- $t < 0$

$$(-\bar{z})^{\bar{\beta}-\bar{\gamma}}(1-\bar{z})^{\bar{\gamma}-\bar{\alpha}-\bar{\beta}}F(\bar{\gamma}-\bar{\beta},1-\bar{\beta},\bar{\alpha}-\bar{\beta}+1,1/\bar{z})$$
$$\rightarrow (2\pi\epsilon)^{\frac{1}{2}}\frac{\Gamma(\bar{\alpha}-\bar{\beta}+1)}{\Gamma(1-\bar{\beta})\Gamma(\bar{\alpha})}\bar{K}_+(\bar{z}) \qquad (\text{C.27})$$

- $0 < t$

$$F(\bar{\beta},\bar{\alpha},\bar{\alpha}+\bar{\beta}-\bar{\gamma}+1,1-\bar{z}) \rightarrow (2\pi\epsilon)^{\frac{1}{2}}\frac{\Gamma(\bar{\alpha}+\bar{\beta}-\bar{\gamma}+1)}{\Gamma(\bar{\alpha})\Gamma(\bar{\beta}-\bar{\gamma}+1)}\bar{K}_+(\bar{z}) \quad (\text{C.28})$$

4. The functions whose asymptotic forms are given in terms of $\bar{K}_-(\bar{z})$:

- $t < 1$

$$\bar{z}^{1-\bar{\gamma}}(1-\bar{z})^{\bar{\gamma}-\bar{\alpha}-\bar{\beta}}F(1-\bar{\alpha},1-\bar{\beta},2-\bar{\gamma},\bar{z})$$
$$\rightarrow (2\pi\epsilon)^{\frac{1}{2}}\frac{\Gamma(2-\bar{\gamma})}{\Gamma(1-\bar{\beta})\Gamma(\bar{\beta}-\bar{\gamma}+1)}\bar{z}^{\frac{1}{\epsilon}|C_2|r_1-\delta_{C_2}\bar{\ell}_1}(1-\bar{z})^{-\frac{1}{\epsilon}|C_2|r_2+\delta_{C_2}\bar{\ell}_2}\bar{K}_-(\bar{z})$$
$$(\text{C.29})$$

- $1 < t$

$$\bar{z}^{\bar{\beta}-\bar{\gamma}}(1-\bar{z})^{\bar{\gamma}-\bar{\alpha}-\bar{\beta}}F(\bar{\gamma}-\bar{\beta},1-\bar{\beta},\bar{\alpha}-\bar{\beta}+1,1/\bar{z})$$
$$\rightarrow (2\pi\epsilon)^{\frac{1}{2}}\frac{\Gamma(\bar{\alpha}-\bar{\beta}+1)}{\Gamma(\bar{\alpha})\Gamma(1-\bar{\beta})}\bar{z}^{\frac{1}{\epsilon}|C_2|r_1-\delta_{C_2}\bar{\ell}_1}(1-\bar{z})^{-\frac{1}{\epsilon}|C_2|r_2+\delta_{C_2}\bar{\ell}_2}\bar{K}_-(\bar{z}) \quad (\text{C.30})$$

The functions on the LHS's are the solutions of the same Hypergeometric differential equation (8.26). This basis covers all the segments $t < 0$, $0 < t < 1$ and $t > 1$. Note that these functions satisfy the parameter condition which is required for the application of (C.12). On the RHS's, $z = t + \epsilon x$ and $\bar{z} = t - \epsilon x$ with the small parameter ϵ are assumed. Further expansion with respect to this ϵ gives the x-dependent exponential factors.

Finally, we summarize some useful formulae (see for example [31]):

$$F(\alpha,\beta,\gamma,z) = \frac{\Gamma(\beta-\alpha)\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\alpha)}(-z)^{-\alpha}F(\alpha,\alpha-\gamma+1,\alpha-\beta+1,1/z)$$
$$+ \frac{\Gamma(\alpha-\beta)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\beta)}(-z)^{-\beta}F(\beta,\beta-\gamma+1,\beta-\alpha+1,1/z), \qquad (\text{C.31})$$
$$= \frac{\Gamma(\alpha+\beta-\gamma)\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-z)^{\gamma-\alpha-\beta}F(\gamma-\alpha,\gamma-\beta,\gamma-\alpha-\beta+1,1-z)$$
$$+ \frac{\Gamma(\gamma-\alpha-\beta)\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta,\alpha+\beta-\gamma+1,1-z). \qquad (\text{C.32})$$

The functions appearing in (C.31) and (C.32) satisfy the Hypergeometric differential equation independently. The first form (C.31) is valid for any z except for the positive real number, while the second (C.32) is valid for any z except on the segment $[0,1]$ of the real axis. By using the following formula:

$$F(\alpha, \beta, \gamma, z) = (1-z)^{\gamma-\alpha-\beta} F(\gamma - \alpha, \gamma - \beta, \gamma, z), \qquad \text{(C.33)}$$

the functions on the LHS of (C.23)–(C.30) can be converted into the functions appearing in (C.31) and (C.32) or $z^{1-z} F(\alpha - \gamma + 1, \beta - \gamma + 1, 2 - \gamma, z)$.

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