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Reply to "Comment on 'Entropy, energy, and proximity to criticality in global earthquake populations'" by Chien-chih Chen and Chun-Ling Chang

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[1] We are grateful to *Chen and Chang* [2004] for their comment on our first paper [*Main and Al-Kindy*, 2002]. We share their concern regarding the definition of self-organised criticality as a concept, and welcome their brief review of earlier work on the suggestion that far-from-equilibrium systems may behave in similar ways to equilibrium ones [see also *Egolf*, 2000]. However, we disagree with their main conclusion, for reasons stated below.

[2] *Chen and Chang* [2004] correctly point out an algebraic error in our appendix derivation of the relation between entropy and energy by integrating the probability density function (see Appendix A). The change of variables from dE to $d\ln E$ in the integral in our appendix does not fundamentally change the result

$$S = S_0 + (B+1)\langle \ln E \rangle \tag{1}$$

that would be expected at the critical point for accurate sampling of the *p.d.f.* for the entropy S by arbitrarily narrow linear increments of energy δE. This alternative form of the relationship, for linear bins, was also presented (but not tested for reasons given below) in a subsequent paper where we analysed the global earthquake catalogue using the spatial ensemble [*Al-Kindy and Main*, 2003].

[3] The reason equation (1) cannot be tested on real data at present is the requirement of obtaining a stable estimate of

$$S = -\Sigma p_i \ln p_i \tag{2}$$

by summing over the individual discrete energy intervals i. We adopted logarithmic bins in energy in the discrete sum for entropy specifically to avoid terms such as $\ln p_i =$

$-\infty$ at large energies that would inevitably result from linear sampling of the current database where many discrete intervals would have zero entry $p_i = 0$. It is this method of data analysis that results in the effective criterion for criticality

$$S = S_0 + B\langle \ln E \rangle, \tag{3}$$

rather than the ideal theoretical relation (1). We had previously shown numerically that the synthetic ideal calculations of S as a function of $\langle \ln E \rangle$ shown in *Al-Kindy and Main* [2003] respected equation (3) at the critical point if the analysis was conducted in logarithmic bins. In appendix B we now present a closed analytical solution for equations (1) and (3) for linear and logarithmic bins respectively. This analysis confirms the validity of using equation (3) in the criticality test for the coarse-grained analysis of real data using logarithmic increments, while pointing out the desirability of testing the fundamental form of equation (1) in numerical models for comparison to other physical systems. The analytical nature of the two solutions implies that an entropy defined by logarithmic increments can be converted to the standard form used in statistical mechanics once the model parameters are known. At the time of writing, we are specifically testing hypothesis (1) by analysing numerical models of a far-from equilibrium non-conservative cellular automaton model for earthquakes where we can generate arbitrarily large numbers of data points to sample the *p.d.f.* adequately.

[4] We could not disagree more with their final statement: "the developed strategy from equilibrium thermodynamics of attacking far-from-equilibrium threshold systems seems no longer as promising...". We now know that at least some real far-from-equilibrium systems, such as the fluidised granular medium experiment of *D'Anna et al.* [2003] and the gas-fluidised particle experiment of *Ojha et al.* [2004], do exhibit Brownian fluctuations that can define an effective temperature. *Dewar* [2003] also showed theoretically that self-organised criticality was one of several classes of steady-state non-equilibrium systems that exhibit equilibrium-like properties. Based on this work, as well as our own, we would argue instead that 'some threshold systems, driven to a steady state with small fluctuations, can be treatable by equilibrium-like concepts'. After all, the fundamental distribution $p(E) \sim E^{-B-1}e^{-E/\theta}$, which is currently the best description of global seismicity in terms of conventional frequency analysis [*Leonard et al.*, 2001], was originally derived using the principle of maximum entropy first applied in

equilibrium thermodynamics [*Shen and Mansinha*, 1983; *Main and Burton*, 1984].

Appendix A: Corrigendum to Integral Solution

[5] We note an error in the derivation in the appendix of *Main and Al-Kindy* [2002] for the integral definition of entropy and the relevant expectation values. Equation (A5) in the notation of that paper should read

$$S = -Z^{-1} \int_{E_{\min}}^{\max} \left(\frac{E}{E_0}\right)^{-B} \exp\left(-\frac{E}{\theta}\right) \ln\left[\frac{E^{-B-1}}{E_0^{-B}} \frac{\exp(-E/\theta)}{Z}\right] d\ln E$$

(A5)

whence equation (A6) should read

$$S = \ln Z - B\ln E_0 + (B+1)\langle \ln E\rangle + \langle E\rangle/\theta.$$ (A6)

Appendix B: Discrete Summation Solution

[6] A general expression for the incremental probability p_i for the i'th energy interval E_i of linear width δE is

$$p_i = p(E_i)\delta E,$$ (B1)

where in the limit $\delta E \to 0$ the probability density is defined by

$$p(E) = dp_i/dE.$$ (B2)

The current best-fitting probability density function for earthquakes takes the form of the density distribution for the generalised gamma function

$$p(E) = AE_i^{-B-1}e^{-E_i/\theta},$$ (B3)

where A is a constant to be determined for the summation solution below. For logarithmic intervals $\delta\ln E$ we have the relation

$$\delta\ln E = \delta E/E.$$ (B4)

Combining equations (B1), (B3), and (B4) we have

$$p_i = AE_i^{-B}e^{-E_i/\theta}\delta\ln E$$ (B5)

or

$$p_i = \frac{E_i^{-B}e^{-E_i/\theta}}{Z},$$ (B6)

where $Z = (A\delta\ln E)^{-1}$. This highlights the fact that the partition function Z itself depends on the size of the increment - smaller increments will have a greater number n of energy states for a continuous probability density function. At this stage we note that B is the slope on a plot of log incremental probability (counted in logarithmic increments of energy) versus log energy at low energies $E \ll \theta$. At higher energies $E \approx \theta$ the incremental probability

distribution has an exponential tail with a characteristic energy θ.

From the constraint

$$\sum_{i=1}^{n} p_i = 1,$$ (B7)

where again n is the number of increments, we have

$$Z = \sum_{i=1}^{n} E_i^{-B}e^{-E_i/\theta}.$$ (B8)

For an arbitrary function $f(E)$, its expectation value in summation is defined by

$$\langle f(E)\rangle = \sum_{i=1}^{n} p_i f(E_i).$$ (B9)

The definition of the entropy in summation is

$$S = -K\sum_{i=1}^{n} p_i \ln p_i.$$ (B10)

For $K = 1$, we have from equations (B6) and (B10)

$$S = -\sum_{i=1}^{n} p_i \ln\left[\frac{E_i^{-B}e^{-E_i/\theta}}{Z}\right],$$ (B11)

or equivalently

$$S = -\sum_{i=1}^{n} p_i \left[\ln\left(E_i^{-B}\right) + \ln\left(e^{-E_i/\theta}\right) - \ln Z\right].$$ (B12)

After expanding out the brackets,

$$S = -\sum_{i=1}^{n} p_i \ln\left(E_i^{-B}\right) - \sum_{i=1}^{n} p_i \ln\left(e^{-E_i/\theta}\right) + \sum_{i=1}^{n} p_i \ln Z.$$ (B13)

From the definition (B9), some algebra of logarithms, and using the fact that B, θ and $Z = f(B, \theta)$ are constants, we have finally

$$S = B\langle\ln E\rangle + \langle E\rangle/\theta + \ln Z.$$ (B14)

For $\theta \to \infty$

$$S = S_0 + B\langle\ln E\rangle,$$ (B15)

where $S_0 = \ln Z$ is a constant for a given value of B. This equation is identical to equation (5) of *Main and Al-Kindy* [2002]. It then follows that their analysis, undertaken by summing logarithmic increments, is correct.

[7] To calculate the effect of linear increments δE we would use equations (B1) and (B3) to replace equation (B6) above with

$$p_i = \frac{E_i^{-B-1}e^{-E_i/\theta}}{Z},$$ (B16)

and repeat the exercise equations (B7)–(B14). We then obtain the result

$$S = (B + 1)\langle \ln E \rangle + \langle E \rangle / \theta + \ln Z. \qquad \text{(B17)}$$

This has exactly the form of the integral solution (A6) above, allowing for the change in notation in incorporating the scaling term E_0 into the partition function when introducing equation (B17).

To summarise, as $\theta \to \infty$, from equation (B14)

$$\partial S / \partial \langle \ln E \rangle = B \quad \text{(logarithmic increments)} \qquad \text{(B18)}$$

and from equation (B17)

$$\partial S / \partial \langle \ln E \rangle = B + 1 \quad \text{(linear increments)}. \qquad \text{(B19)}$$

References

Al-Kindy, F. H., and I. G. Main (2003), Testing self-organized criticality in the crust using entropy: A regionalized study of the CMT global earthquake catalogue, *J. Geophys. Res.*, *108*(B11), 2521, doi:10.1029/2002JB002230.

Chen, C.-c., and C.-L. Chang (2004), Comment on "Entropy, energy, and proximity to criticality in global earthquake populations" by Ian G. Main and Fahad H. Al-Kindy, *Geophys. Res.*, *31*, L06608, doi:10.1029/2003GL019008.

D'Anna, G., P. Mayor, A. Barrat, V. Loreto, and F. Nori (2003), Observing Brownian motion in vibration-fluidized granular matter, *Nature*, *424*, 909–912.

Dewar, R. (2003), Information theory explanation of the fluctuation theorem, maximum entropy production and self-organized criticality in nonequilibrium stationary states, *J. Phys. A Math. Gen.*, *36*, 631–641.

Egolf, D. A. (2000), Equilibrium regained: From nonequilibrium chaos to statistical mechanics, *Science*, *287*, 101–104.

Leonard, T., O. Papasouliotis, and I. G. Main (2001), A Poisson model for identifying characteristic size effects in frequency data: Application to frequency-size distributions for global earthquakes, "starquakes", and fault lengths, *J. Geophys. Res.*, *106*, 13,473–13,484.

Main, I. G., and P. W. Burton (1984), Information theory and the earthquake frequency magnitude distribution, *Bull. Seismol. Soc. Am.*, *74*, 1409–1426.

Main, I. G., and F. H. Al-Kindy (2002), Entropy, energy, and proximity to criticality in global earthquake populations, *Geophys. Res. Lett.*, *29*(7), 1121, doi:10.1029/2001GL014078.

Ojha, R. P., P.-A. Lemieux, P. K. Dixon, A. J. Liu, and D. J. Durian (2004), Statistical mechanics of a gas-fluidized particle, *Nature*, *427*, 521–523.

Shen, P. Y., and L. Mansinha (1983), On the principle of maximum entropy and the earthquake frequency-magnitude relation, *Geophys. J. R. Astron Soc.*, *74*, 777–785.

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