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LECTURES ON GAS FLOW IN POROUS MEDIA

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The idea behind these lectures is to present in a relatively simple setting, that of solutions to porous media in one space dimension, several of the main ideas and the main techniques that are at the center of the regularity theory for nonlinear evolution equations and phase transitions. These include exploiting the invariances of the equation to obtain infinitesimal relations and geometric control of the solutions, the role of particular solutions to guide us in our theory and provide us with barriers and asymptotic profiles, the idea of viscosity solutions to a free boundary problem to deduce the geometric properties of the free boundary and the methods of blowing up solutions and classifying the global profiles to obtain the differentiability properties of a free boundary.

1. Introduction

The traditional way of modeling phenomena in continuum mechanics is through the description of conservation laws (of mass, energy, etc.) and constitutive relations among the different unknowns, due to the properties of the media or material at hand.

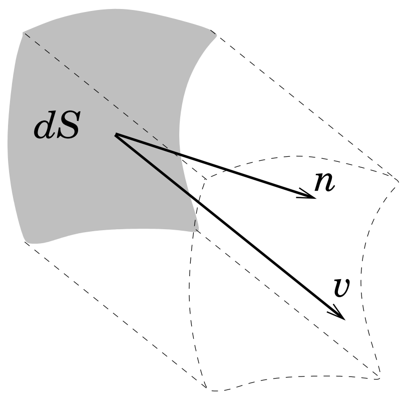
Conservation laws are many times introduced as additive set functions and it is a consequence of the fact that their validity in a very small set implies by superposition their validity in the large, that conservation laws end up as infinitesimal relations on one hand while their being originally set functions implies in turn their divergence structure. The model we are going to consider is described in terms of the gas density $\rho(x,t)$, the velocity field $v(x,t)$ and a pressure $p(x,t)$. The first relation that we will discuss is the conservation of mass: it says that as time evolves the amount of mass of the flowing gas in a domain G changes proportionally to the gas flowing through the boundary of G.

Let us consider some given volume G, then the mass (amount) of the gas occupying G at time t is

$$\int_G \rho(x,t)dx.$$

Through the elementary area dS on the boundary of G, the amount of the gas that crosses it per unit of time is $\rho(v \cdot n)dS$, where n is the outward unit normal of ∂G. $v \cdot n$ is positive if the gas flows out of G and negative when it flows in G. The total mass of the gas crossing through ∂G per unit of time is

$$\int_{\partial G} \rho(v \cdot n)dS.$$



On the other hand the rate of change of the gas in volume G per unit of time is equal to

$$\frac{\partial}{\partial t}\int_G \rho(x,t)dx.$$

Therefore we may write the conservation of mass as

$$-\frac{\partial}{\partial t}\int_G \rho = \int_{\partial G} \rho v n dS. \tag{1.1}$$

Hence after applying the divergence theorem to the right hand side of this identity and in view of the fact that G is arbitrary we get $\rho_t + \mathrm{div}\rho v = 0$. This is the equation of conservation of mass.

Next equation comes from a constitutive relation for flow in porous media, known as Darcy's law (named after H.Darcy) stating that v is the gradient of a potential function (the pressure) $v = -Dp$.

Finally we introduce the *equations of state* $p = \rho^{m-1}, m > 1$ and we get the porous medium equation

$$\rho_t = \mathrm{div}(\rho D\rho^{m-1})$$

or explicitly

$$\rho_t = m\rho^{m-1}\Delta\rho + m(m-1)\rho^{m-2}|D\rho|^2. \tag{1.2}$$

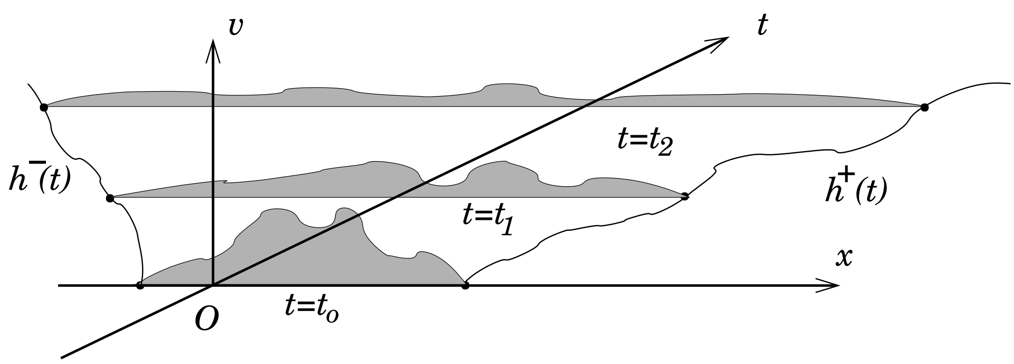
This is a parabolic quasilinear divergence type equation. One can define the weak solution of the initial value problem

$$\begin{cases} \rho_t = \Delta(\rho^m) & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+ \\ \rho(x,0) = \rho_0(x) & x \in \mathbb{R} \end{cases} \tag{1.3}$$

in a standard manner: ρ is said to be a weak solution of (1.3) if $D(\rho^m)$ is a distribution and for any $T > 0$ and any smooth $\phi(x,t), \mathrm{supp}\,\phi(x,t) \subset B_R \times [0,T]$ one has

$$\iint_{\mathbb{R}^n \times [0,T]} [\rho(x,t)\phi_t(x,t) - D\phi(x,t)D(\rho^m(x,t))]dxdt + \int_{\mathbb{R}^n} \rho_0(x)\phi(x,0)dx = 0,$$

where B_R is the ball centered at the origin with radius R, for some $R > 0$.



Theorem 1. *There exists unique weak solution to Cauchy problem provided that* $D(\rho_0^m)$ *is bounded. Moreover comparison principle holds: if* $\rho_{01}(x) \leq \rho_{02}(x)$ *then* $\rho_1(x,t) \leq \rho_2(x,t)$*. If the initial data has a compact support then* $\rho(x,t)$ *has a compact support for every time* t*.*

The proof of the existence and uniqueness of the weak solutions can be found in [O], [OKC].

It is helpful to understand many features of the problem to write the equation satisfied by the pressure p. One of the main reasons is that the particles as the edge of the support of the region occupied by the gas are material points, i.e. they always remain on the moving front and therefore the speed of the interface separating gas from vacuum is equal to the speed of the flow Dp. If we consider the normalize pressure

$$v = \frac{m}{m-1}\rho^{m-1} \tag{1.4}$$

then v verifies

$$v_t = (m-1)v\Delta v + |Dv|^2. \tag{1.5}$$

This can be seen logarithmically since $p_t/p = (m-1)\rho_t/\rho$ and

$$\frac{Dp}{p} = (m-1)\frac{D\rho}{\rho}$$

from the equation

$$\rho_t = \operatorname{div}(\rho Dp) = \rho\Delta p + D\rho Dp$$

dividing by ρ we obtain

$$\frac{p_t}{p} = \frac{1}{m-1}\Delta p + \frac{|Dp|^2}{p}$$

or

$$p_t = |Dp|^2 + \frac{1}{m-1}p\Delta p.$$

Notice that along the interface the speed of the material point $x(t)$ is $|Dp| = |\frac{\partial p}{\partial n}|$, therefore the speed of the interface being the same as that of the material point becomes

$$\frac{p_t}{p_n} = |\frac{\partial p}{\partial n}|$$

or $p_t = |\frac{\partial p}{\partial n}|^2$, a Hamilton-Jacobi type relation. Formally this means that the term $p\Delta p$ should go to zero at the interface.

Using these computations and changing p with $\frac{m-1}{m}v$ we obtain

$$v_t = (m-1)v\Delta v + |Dv|^2. \tag{1.6}$$

In what follows we refer to (1.5) as the porous medium equation [A], [C2].

To try understand an evolution problem one of the first things we should explore are the invariances of the equation and particular solutions. We start by exploring classes of particular solutions. We use the pressure equations. There are three standard type of solutions that we may try.

- Travelling profiles i.e. solutions that depend only on the variable $x_1 - \alpha t$, α a constant,
- Separation of variables,
- If we have conservation of mass we can put a Dirac δ (a mass) at the origin and let it go.

1.1. **Traveling fronts.** Let α be a constant and $(\cdot)_+ = \max(\cdot, 0)$. Then

$$v_\alpha = (\alpha^2 t + \alpha x)_+ \tag{1.7}$$

is a solution to (1.5) in the whole space. The free boundary is the line $x = h(t) \equiv -\alpha t$. Note that on the free boundary $x = h(t)$ the Darcy's law is satisfied

$$h'(t) = -Dv_\alpha. \tag{1.8}$$

In the N-dimensional case one can consider

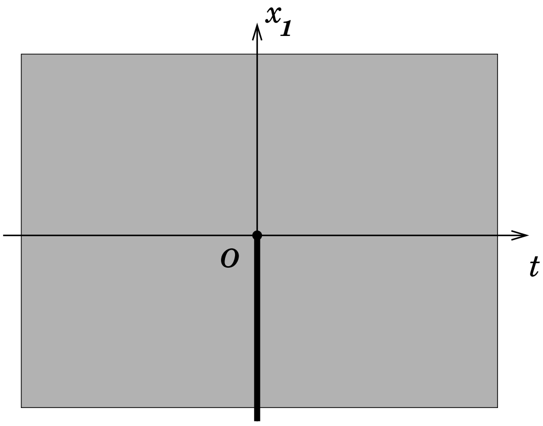
$$v_\alpha = (\alpha^2 t + \alpha(e \cdot x))_+, \ |e| = 1 \tag{1.9}$$

as a generalization of $(\alpha^2 t + \alpha x)_+$.

1.2. **Quadratic solution (Separation of variables).** If we try for the solutions of the form $f(x)g(t)$ we find another explicit solution of (1.5) in $\mathbb{R}^N \times \mathbb{R}$ is

$$v_p = \frac{1}{2(m+1)} \frac{(x_1^+)^2}{t_0 - t}. \tag{1.10}$$

This example shows that the free boundary may stay stagnant for a quadratic initial data, (see section 5.2).



1.3. **Fundamental solution.** Let ρ be the Dirac delta at time zero. We expect such a solution to be radially symmetric and selfsimilar due to the homogeneity of the equation. That reduces the equation to an ODE. More precisely we must have $\rho(x,t) = W\rho(\frac{x}{M}, 1)$ for some W, M depending on t. But ρ preserves the mass so

$$\int \rho(x,t)dx = \int \rho(x,1)dx = W \int \rho(\frac{x}{M}, 1)dx = WM^N \int \rho(y,1)dy$$

hence $W = M^{-N}$. Next, we want ρ to be self similar, that is for some constants γ, δ, M

$$\rho(x,t) = M^\delta \rho(Mx, M^\gamma t) = \frac{M^\gamma t}{M^2} M^\delta \rho(\frac{x}{M}, 1)$$

implying that M is a power of t, so we seek a solution in the following form

$$\rho(x,t) = \frac{1}{t^\alpha} F(\frac{x}{t^\beta}).$$

Recall that $p = \rho^{m-1}$. Since ρ is self-similar then after plugging in $\rho(x,t) = \frac{1}{t^\beta} F(\frac{x}{t^\beta})$ into $\rho_t = \mathrm{div} \rho \nabla p$ all the powers of t will cancel each other giving

$$\mathrm{div}(F(z)\nabla F^{m-1}(z)) = -N\beta F(z) - \beta \nabla F(z) \cdot z = -\beta \mathrm{div}(zF(z)).$$

Since F is a common factor then it is enough to make

$$\frac{d}{dz}(F^{m-1} - \frac{\beta}{2} z^2) = const.$$

This gives the following solution

$$v_\delta = \frac{m}{m-1} \frac{1}{t^{\alpha(m-1)}} (a - b\frac{|x|^2}{t^{2\beta}})_+, \tag{1.11}$$

$$\alpha = N\beta, \ \beta = \frac{1}{2 + N(m-1)}, \ b = \beta \frac{m-1}{2m}, \tag{1.12}$$

a is an arbitrary constant. If ρ_δ is the density corresponding to v_δ then

$$\rho_\delta = \frac{1}{t^\alpha}(a - b\frac{|x|^2}{t^{2\beta}})_+^{\frac{1}{m-1}}.$$

This is the "*fundamental solution* " for porous medium equation. Note that ρ_δ converges to Gaussian kernel $t^{-N/2}e^{-\frac{|x|^2}{4t}}$, the fundamental solution of the heat equation, when $m \to 1$ and $a = 1$. Indeed it is easy to check that the mass of the gas is

$$\begin{aligned} \text{mass} &= \int_{\mathbb{R}^N} \frac{1}{t^\alpha}(a - b\frac{|x|^2}{t^{2\beta}})_+^{\frac{1}{m-1}} dx \\ &= \frac{1}{t^\alpha}\int_{S^N}\int_0^{\sqrt{a}} [a - R^2]_+^{\frac{1}{m-1}} \left[\frac{t^\beta R}{\sqrt{b}}\right]^{N-1} \frac{t^\beta}{\sqrt{b}} dR \\ &= \omega_N b^{-\frac{N}{2}} \int_0^{\sqrt{a}} [a - R^2]_+^{\frac{1}{m-1}} R^{N-1} dR \\ &= \frac{\omega_N}{2} b^{-N/2} B(\frac{m}{m-1}, \frac{N}{2}), \end{aligned} \tag{1.13}$$

where ω_N is the area of unit sphere and $B(\cdot,\cdot)$ is the Euler's beta function. Then using

$$B(\frac{m}{m-1}, \frac{N}{2}) \sim \Gamma(\frac{N}{2})(\frac{m}{m-1})^{-\frac{N}{2}}$$

in conjunction with $\omega_N = \frac{2\pi^{N/2}}{\Gamma(\frac{N}{2})}$ we conclude

$$\begin{aligned} \text{mass} &= \frac{\omega_N}{2} b^{-N/2} B(\frac{m}{m-1}, \frac{N}{2}) \\ &\sim \frac{\omega_N}{2} b^{-N/2} \Gamma(\frac{N}{2})(\frac{m}{m-1})^{-\frac{N}{2}} \\ &\sim (\pi\frac{2}{\beta})^{\frac{N}{2}}. \end{aligned}$$

As $\beta \to 1/2$ when $m \to 1$, we get mass$= (2\sqrt{\pi})^N = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4}} dx$. For $m = 1$ the heat equation takes the form $\rho_t = \mathrm{div}\rho\nabla\log\rho$.

2. Scaling

All three particular solutions: travelling front, quadratic and Barenblatt solutions are self-similar, that is they are invariant under a family of scalings. Let v be a pressure solution to porous medium equation, then for any A, B, positive constants

$$v_{A,B} = \frac{B}{A^2} v(Ax, Bt) \tag{2.1}$$

is also a solution. If $A = B$ then we call the scaling *hyperbolic*, for $B = A^2$ we call it to be *parabolic*. Note that the porous medium equation has in some sense as reach a family of scalings than the heat equation. Although nonlinear, it still has two free parameters. The comparison principle with the semigroup generated by the invariant scalings is very useful to obtain global a priori estimates for dilations of the solution. For instance

Lemma 2. *If v is the solution to (1.5) then*

$$v_t \geq -\frac{v}{t}. \tag{2.2}$$

Proof. We will compare $v(x,t)$ with $(1+\epsilon)v(x,(1+\epsilon)t)$. Indeed let $v_{01}(x) = v(x,0), v_{02}(x) = (1+\epsilon)v(x,0)$ for some positive constant ϵ. Then if v_i is the solution to

$$\left\{ \begin{array}{l} v_{i,t}(x,t) = (m-1)v_i \Delta v_i + |Dv_i|^2, i = 1,2 \\ v_i(x,0) = v_{0i}(x) \end{array} \right. \tag{2.3}$$

then comparison principle implies

$$v_1 \leq v_2.$$

But $v_2(x,t) = (1+\epsilon)v_1(x,(1+\epsilon)t)$, since we can take $A = 1, B = 1+\epsilon$ as the scaling constants so that

$$v(x,t) \leq (1+\epsilon)v(x,(1+\epsilon)t) = v(x,(1+\epsilon)t) + \epsilon v(x,(1+\epsilon)t) \tag{2.4}$$

hence

$$\frac{v(x,(1+\epsilon)t) - v(x,t)}{\epsilon} + v(x,(1+\epsilon)t) \geq 0. \tag{2.5}$$

Letting $\epsilon \to 0$ the result follows. □

This type of argument can be used in many cases for radial symmetry (using infinitesimal rotations) or for monotonicity of solutions (see later the reflexion method in section 5.5)

An important corollary of this lemma is the expansion of the support.

Corollary 3. *For $t > t_0$ we have*

$$\frac{v(x_0,t)}{v(x_0,t_0)} \geq e^{ct/t_0}. \tag{2.6}$$

Hence if for some point (x_0,t_0) v is positive then it remains so for any instant of time $t > t_0$.

Proof. By integrating (2.2) the result follows. □

A much more delicate and beautiful estimate is due to Benilan:

Lemma 4. *If v is a solution to porous medium equation then one has Benilan's estimate*

$$\Delta v \geq -\frac{1}{(m-1)t}. \tag{2.7}$$

Note that this estimate implies the previous one, except for the constant.

Proof. Let us assume for a moment that v is smooth, then applying the Laplacian to both sides of the equation (1.5) we obtain

$$\begin{aligned} \frac{\partial}{\partial t}(\Delta v) &= (m-1)\Delta(v\Delta v) + \Delta(|Dv|^2) \\ &= (m-1)v\Delta(\Delta v) + 2mDv\Delta Dv + (m-1)(\Delta v)^2 + 2\sum_{ik} v_{ik}^2. \end{aligned} \tag{2.8}$$

Set $w = \Delta v$, then w satisfies to the partial differential inequality

$$\mathcal{L}(w) = 2\sum_{ik} v_{ik}^2 \geq 0 \tag{2.9}$$

where

$$\mathcal{L}(w) = \frac{\partial w}{\partial t} - [(m-1)v\Delta w + 2mDvDw + (m-1)w^2].$$

Due to the presence of v in the equation, the only obvious barrier one can built should be a function of t only so we want to compare w to a function $-c/t$ for some constant c such that $\mathcal{L}(-c/t) = 0$, in fact this requires that $c = 1/(m-1)$ hence

$$\mathcal{L}(w) \geq 0 = \mathcal{L}(-\frac{1}{t(m-1)})$$

while on the boundary we have that

$$\Delta v \geq -\infty.$$

Using comparison principle the result follows. In the general case one can approximate (1.5) by a family of uniformly elliptic equations and then pass to the limit. □

Remark. The constant $-1/(m-1)$ is not optimal. Indeed if we estimate the trace of Hessian D^2v more carefully then

$$\frac{(TrD^2v)^2}{N} \leq \sum_{i=1}^{N} v_{ii}^2 \leq \sum_{ik} v_{ik}^2$$

thus introducing

$$\mathcal{L}_1(w) = \frac{\partial w}{\partial t} - [(m-1)v\Delta w + 2mDvDw + (m-1-\frac{2}{N})w^2]$$

and comparing w with $-\frac{1}{t(m-1+\frac{2}{N})}$ we get the sharp form of Benilan's estimate

$$\Delta v \geq -\frac{1}{t(m-1+\frac{2}{N})} = -\frac{\beta N}{t}. \tag{2.10}$$

One can check that for the Barenblatt solution this inequality becomes equality.

Then the immediate consequence of this is

Corollary 5. *In the one dimensional case*

$$v_x + \frac{x}{(m-1)t} \tag{2.11}$$

is nondecreasing, so v_x has one-sided limits everywhere. Furthermore v is semi-convex so it is locally Lipschitz.

Theorem 6. *Let v be a solution to (1.5). If v is Lipschitz in space, then v is also Lipschitz in time.*

The idea of the proof is very general and can be applied to a more general class of equations. It is again a combination of the scaling invariance of solutions of (1.5) and maximum principle. First we illustrate the underlying idea for the solutions of the heat equation. Let u be a solution of $\Delta u - u_t = 0$ in a cylinder $Q_\lambda(x_0, t_0)$ and assume that the modulus of continuity of u with respect to x is σ i.e. $\mathrm{osc}_{x \in B_\lambda(x_0)} u(x,t) \leq \sigma(\lambda)$ independently of t, then the function

$$u_\lambda(x,t) = \frac{u(x_0 + \lambda x, t_0 + \lambda^2 t)}{\sigma(\lambda)}$$

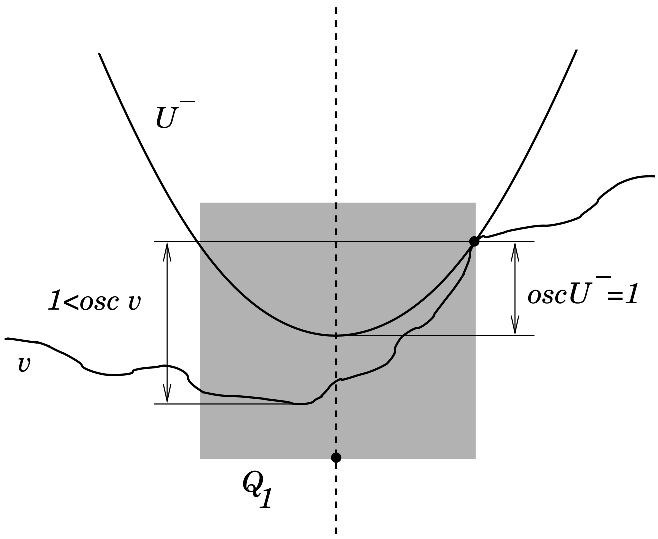
solves heat equation in the unit cylinder $Q_1(0,0) = B_1 \times (0,1)$ for any $\lambda > 0$. Let us show that then $u_\lambda(0,1) - u_\lambda(0,0) \leq c_1$, c_1 depending on σ. If $\epsilon > 0$ and C is a large constant then $h(x,t) = |x|^2 + 2NCt + 1 + u_\lambda(0,0)$ is a supersolution to the heat equation in the unit cylinder Q_1. Since $\mathrm{osc}_{B_1} u_\lambda \leq 1$, we conclude that $u_\lambda(x,0) < h(x,0)$. Assume that the first contact of u_λ and h happens at the point (x_1, t_1). By the maximum principle $(x_1, t_1) \in \partial B_1 \times (0,1)$. But then one has

$$1 \geq u_\lambda(x_1, t_1) - u_\lambda(0,0) \geq h(x_1, t_1) - h(0,0) = 1 + 2CNt_1. \tag{2.12}$$

Hence u_λ never catches-up with h and $u_\lambda < h$ in Q_1 (see figure 1). Scaling back we get that $u(x_0, t_0 + \lambda^2) - u(x_0, t_0) \le c_1\sigma(\lambda)$, thus setting $\delta = \lambda^2$ we have

$$u(x_0, t_0 + \delta) - u(x_0, t_0) \le c_1\sigma(\sqrt{\delta}).$$

Using the function $-h$ as a subsolution we can prove also the lower estimate. In particular if u is Lipschitz continuous in space then u is $1/2$ Hölder continuous in time.



The similar argument applies to the solutions of (1.5) though with a hyperbolic scaling. First we need the following lemma

Lemma 7. *If $v(x_0, t_0) = \alpha$, then $v(x_0, t_0 + h) \le C_1\alpha$ for any $h \le \frac{\alpha}{M}$, where M is a large positive number and C_1 is a positive universal constant.*

Proof. To fix the ideas let's assume that $(x_0, t_0) = (0, 0)$. Introduce

$$S^- = c\frac{|x|^2 + 2\alpha^2}{\frac{2\alpha}{M} - t}. \tag{2.13}$$

By a direct computation one can see that S^- is a supersolution to (1.5) in $\{|x| \le \alpha\} \times (0, 1)$

$$S^-_t \ge (m-1)S^-\Delta S^- + |DS^-|^2$$

provided $c > 0$ is large enough. Indeed by a direct computation one can see that it is enough to prove $(1 - 4c)|x|^2 < 2\alpha^2(2Nc(m-1) - 1)$ for $|x| \le \alpha$. Hence it suffices to assume that $c > 1/(2 + N(m-1))$.

Then

$$S^-(x,0) = c\frac{|x|^2 + 2\alpha^2}{\frac{2\alpha}{M}} \geq c\alpha M.$$

Since v is Lipschitz in space we conclude that in $|x| \leq \alpha$

$$\begin{aligned} v(x,0) &\leq v(0,0) + L|x| = (1+L)\alpha \leq c\alpha M \\ &\leq S^-(x,0) \end{aligned} \tag{2.14}$$

provided $M > (1+L)/c$ (this is the relation between M, c and L-Lipschitz constant). Let t_1 be the first time when S^- touches v at (x_1, t_1). this cannot happen in the interior of cylinder $\{|x| \leq \alpha\} \times [0, \alpha/M]$. From strong maximum principle we conclude that $|x_1| = \alpha$, and

$$v(x_1, t_1) = S^-(x_1, t_1) = c\frac{3\alpha^2}{\frac{2\alpha}{M} - t_1}. \tag{2.15}$$

Furthermore,

$$\begin{aligned} v(x_1, t_1) - v(0, t_1) &\geq S^-(x_1, t_1) - S^-(0, t_1) \\ &= c\frac{3\alpha^2}{\frac{2\alpha}{M} - t_1} - c\frac{\alpha^2}{\frac{2\alpha}{M} - t_1} \\ &\geq c\alpha M. \end{aligned} \tag{2.16}$$

This contradicts to the Lipschitz regularity in space if M is large. Hence $v < S^- \leq C\alpha$. Note that using hyperbolic scaling one can assume that $\alpha = 1$. □

In the same way using the Barenblatt solution S^+ as a subsolution one can obtain $v \geq S^+$.

Lemma 8. *If $v(x_0, t_0) = \alpha$, then $v(x_0, t_0 + h) - \alpha \geq -C_2\alpha$ for any $h \leq \frac{\alpha}{M}$, where M is a large positive number and C_2 is positive universal constant.*

Combining this two lemmas the theorem follows. Next using the scaling and the Lipschitz regularity we also can prove that Schauder estimates hold in the positivity set.

Theorem 9. *Let v be a solution to (1.5). If $v \sim \alpha$ in $B_\alpha(x_0) \times (t_0, t_0 + \frac{\alpha}{M})$ then*

$$|D^k v| \leq \frac{C(k)}{\alpha^{|k|+1}}, \text{ in } B_{\alpha/2}(x_0) \times (t_0, t_0 + \frac{\alpha}{2M}).$$

Proof. After scaling $v_\alpha = v(x_0 + \alpha x, t_0 + \alpha t)/\alpha \sim 1$ in $B_1 \times (0, \frac{1}{M})$ and the equation becomes uniformly parabolic. Then using parabolic Schauder estimates for v_α and scaling back the result follows. □

3. Regularity of the free boundary

We will now illustrate the main steps in proving the free boundary regularity for our problem. That is: the (increasing) boundary of the support of v may stay stationary for a while but as soon as it starts to move it will always have positive speed, in fact its speed will satisfy a differential inequality and it will be a C^1 curve. The two main ingredients that reappear in much more complex problems are present already here: an asymptotic convexity of the free boundary under dilations and the possibility to classify global profiles. The two main barriers we will use are the pressure form of the fundamental solution and the travelling fronts.

First we observe the following property of the Barenblatt solutions. Let

$$v_1(x) = \frac{m}{m-1}(A - B|x|^2)_+.$$

Recall that the Barenblatt solution in N-dimension is

$$v(x,t) = \frac{m}{m-1}\frac{1}{t^{\alpha(m-1)}}(A - B\frac{|x|^2}{t^{2\beta}})_+ \tag{3.1}$$

$$\alpha = N\beta, \qquad \beta = \frac{1}{2+(m-1)N}, \qquad B = \frac{(m-1)\beta}{2m}$$

and $A > 0$ is the constant which determines the total mass. Hence $v(x,t)$ is the solution to

$$\begin{cases} v_t = (m-1)v\Delta v + |Dv|^2, & t > 1 \\ v(x,1) = v_1(x). \end{cases} \tag{3.2}$$

A direct computation ($N = 1$) then shows that on the free boundary $x = h(t)$

$$\begin{aligned} h'(1) &= \beta\sqrt{\frac{A}{B}} \\ h''(1) &= -(1-\beta)h'(1) \end{aligned} \tag{3.3}$$

We now consider a solution $v(x,t)$, with initial data $v_0 = (x, t_0)$ supported in the interval $[a,b]$, then the free boundary for $t \geq t_0 > 0$ consists of two monotone, Lipschitz curves $h^+(t), h^-(t)$. More precisely we have

Lemma 10. *Let $h(t) \equiv h^+(t) = \sup\{x, v(x,t) > 0\}$. Then $h(t)$ is monotone and Lipschitz.*

Proof. $h(t)$ is monotone since by (2.6) we have

$$v(\bar{x}, t) \geq v(\bar{x}, \bar{t})e^{-ct/\bar{t}} > 0$$

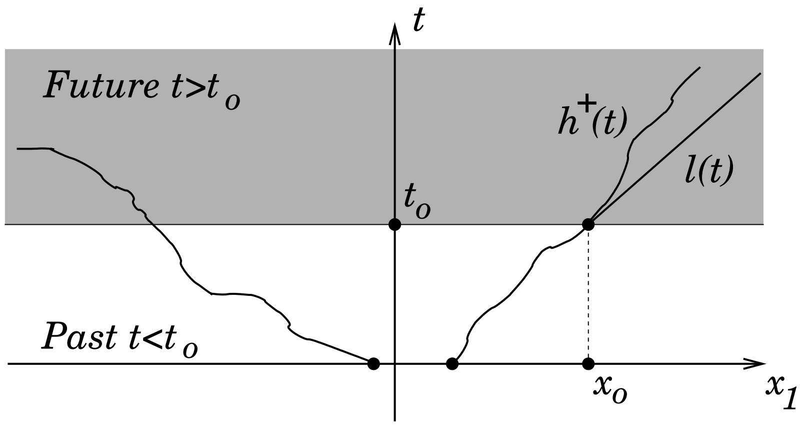
provided $v(\bar{x}, \bar{t}) > 0$. To prove that $h(t)$ is Lipschitz we compare v with a travelling front solution. Recall that v is Lipschitz for $N = 1$. Let x_0, t_0 be a free boundary point. From the mean value theorem

$$v(x, t_0) = -v_x(\cdot, t_0)(h(t_0) - x) \leq C(h(t_0) - x), \quad x < x_0 = h(t_0).$$

Now consider the wave solution

$$v_\alpha = \alpha(\alpha(t - t_0) + (x_0 - x))_+$$

then if $\alpha = C$, the Lipschitz constant, and applying the comparison principle we conclude that $v(x, t) \leq v_\alpha(x, t), t > t_0, x > x_0$ hence the free boundary of v is inside of the free boundary of v_α, so the slope of h is controlled by Lipschitz constant of v. □



Remark. v can be controlled from above by a travelling front. Next we shall see that v can be controlled from below by a Barenblatt solution. In its turn this will imply a formula for a speed $h'(t)$ of the free boundary.

Corollary 11. *If $v(x_0, t_0) > 0$ and $v_x(x_0, t_0) = -\alpha$, then there is a parabola $P(x)$ such that $v \geq P(x), t \geq t_0$ and $P'' = \beta/2, P'(x_0) = -\alpha, P(x_0) = v(x_0, t_0)$.*

Proof. If it is necessary we may consider the scaled function $\bar{v}(x, t) = \frac{1}{t_0} v(xt_0, tt_0)$ and we may assume that $t_0 = 1$. Then by (2.10)

$$\Delta \bar{v} = \bar{v}_{xx} \geq -\beta$$

and therefore

$$\bar{v}(x,t) + \beta\frac{|x-\bar{x}_0|^2}{2}$$

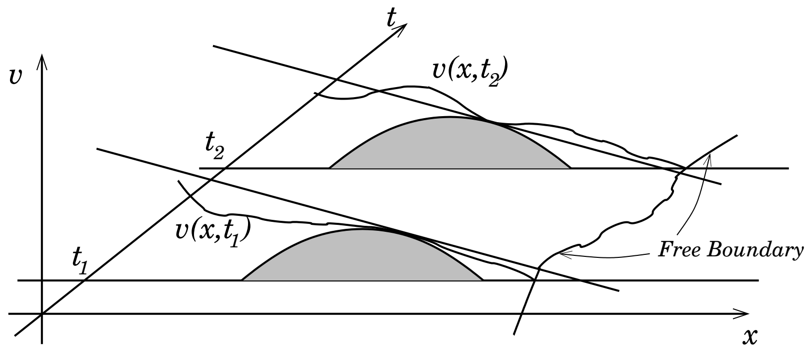
is convex. Then if $\ell(x)$ is the support plane at the point $\bar{x}_0 = t_0 x_0$, then $\bar{v} \geq P(x)$ where

$$\begin{aligned}
(3.4) \qquad P(x) &= -\beta\frac{|x-\bar{x}_0|^2}{2} + \ell(x-\bar{x}_0) \\
&= -\beta\frac{|x-\bar{x}_0|^2}{2} - \alpha(x-\bar{x}_0) + \bar{v}(\bar{x}_0,\bar{t}_0) \\
&= \frac{m}{m-1}(-b|x-\bar{x}_0|^2 - 2Nb(x-\bar{x}_0)) + \bar{v}(\bar{x}_0,\bar{t}_0) \\
&= \frac{m}{m-1}(bN^2 - b|x-\bar{x}_0+N|^2) + \bar{v}(\bar{x}_0,\bar{t}_0).
\end{aligned}$$

This is the Barenblatt solution truncated at $t=1$. Note that the free boundary condition (3.3) $h' = \alpha$ is satisfied. Scaling back to the original variables the result follows. $\square$

Corollary 12. *Let (x_0,t_0) be a free boundary point. For $t \geq t_0$ v is above the corresponding Barenblatt solution.*

Proof. Without loss of generality we can assume that $t_0 = 1$. From the previous corollary we have $P(x)$ is the Barenblatt solution truncates at $t=1$. Since all this parabolas are below v, have the same second derivatives and v_x is semicontinuous then the conclusion of the corollary holds for free boundary points as well (see the figure). Indeed we can approach to the free boundary point (x_0,t_0) a little bit from the future or a little bit from the past, since everything is continuous then we can pass to the limit and get a desired limit parabola $P(x)$, which is the truncated Barenblatt, touching v from below at the free boundary point. Then let v_P be the Barenblatt corresponding to the initial condition $P(x)$. Thus by comparison principle $v_P \leq v, t > 1$. $\square$



Next lemma makes precise the free boundary condition, which heuristically is the Darcy's law. Notice that the lim sup below is taken for both (x,t) converging to (x_0,t_0), both from the past and the future.

Lemma 13. *Let (x_0,t_0) be a free boundary point and let*

$$\alpha = \limsup_{(x,t)\to(x_0,t_0)} (-v_x). \tag{3.5}$$

Then for $t \geq t_0$ we have

$$h(t) = x_0 + \alpha(t-t_0) + \omega(t-t_0). \tag{3.6}$$

Further, from above we only have that $\omega = o(t-t_0)$, but from below we have the stronger inequality

$$\omega(t-t_0) \geq -\alpha C(t-t_0)^2 + o((t-t_0)^2).$$

Proof. From the previous result it follows that

$$h(t) \geq h(t_0) + \alpha(t-t_0) - C\alpha(t-t_0)^2.$$

To show that the reversed inequality is satisfied we take $\epsilon > 0$ and use the definition of α,

$$-v_x \leq (\alpha+\epsilon), \qquad x_0 - \delta < x < x_0$$

then from the Lipschitz continuity and the mean value theorem we get

$$v(x,t_0) = -v_x(\cdot)(x_0 - x) \leq -(\alpha+\epsilon)(x-x_0)$$

in the future.

Lemma 14. *Coming from the past $h(t) \geq x_0 + \alpha(t-t_0) - o(t-t_0)$.*

Proof. Assume that for a sequence $t_k \uparrow t_0$

$$x_k = h(t_k) \leq x_0 + (\alpha + \delta)(t_k - t_0).$$

Since

$$\limsup(-v_x) = \alpha$$

we have that $(-v_x) \leq \alpha + \delta/4$ in a small enough neighborhood of (x_0, t_0), $N_s(x_0, t_0)$, in space and time. We will compare v with the traveling front solution $w(x,t)$, going trough (x_k, t_k) with speed $\alpha + \delta/2$. From the estimate by above of x_k, this wave goes through the left of x_0 at t_0 and thus crosses the free boundary. But if we go backwards in x from x_k at time t_k we have that

$$w(x, t_k) \geq v(x, t_k) + \frac{\delta}{4}|x - x_k|.$$

So for $x = x_k - s$, $w(x_k - s, t_k) \geq v(x_k - s, t_k) + \frac{\delta}{4}h$. This is enough room to go into the future starting at $x_k - s, t_k$ to use w as a barrier for v in the region $\{x \geq x_k - s, t_k \leq t \leq t_0\}$ and get a contradiction. □

We will now get a differential inequality for h. We start by proving that $h(t)$ is a "viscosity subsolution" of $h'' \geq Ch'$.

Lemma 15. *If $h(t)$ has at $t = t_0$ a tangent parabola $x(t) = \ell(t) + a(t - t_0)^2$ by above then a must be $a \geq -C\ell'$.*

Proof. At t_0, h has a tangent line with a slope α, from lemmas 14 and 13. Therefore ℓ' must be equal to α. If $a \leq -C\ell' = -C\alpha$ going into future we have a contradiction to Lemma 13.

□

We are now in the final step. In this section we want to illustrate how to show the regularity of a free boundary by classifying global "blow-outs" of a solution. We have already an important fact. We know that the free boundary of the blow out must be convex. We will now show that every blow out is a travelling front solution and go back and deduce that the free boundary was indeed C^1.

Consider the travelling front $v_\alpha = (\alpha+\epsilon)[(\alpha+\epsilon)(t-t_0)-(x-x_0)]_+$ then from the comparison principle $v \le v_\alpha,\ t > t_0$, therefore

$$h(t) \le h(t_0) + (\alpha+\epsilon)(t-t_0) + o((t-t_0)^2).$$

At this point, at least coming from the future we seem to have the differential inequality

$$h'' \ge -Ch'$$

that heuristically would imply that h is "quasi convex", i.e. $h(t)+Ct^2$ should be convex in the neighborhood of t_0. In this opportunity, we introduce a new idea, the idea of "viscosity solution", i.e. using comparison with smooth super and subsolution.

In one dimension the idea is straightforward, as we will see below. In more dimensions it has become very fruitful to show that very weak solutions of an equations are actually smooth.

Corollary 16. *There exists a large positive constant C depending on Lipschitz norm of v such that*

$$\phi(t) = h(t) + Ct^2 \tag{3.7}$$

is convex.

Proof. If not find a parabola touching h with $a \le -Ch''$

Corollary 17. *$h(t)$, satisfies to*

$$h''(t) \ge -Ch'(t), \tag{3.8}$$

in the viscosity sense. Hence

$$h'(t) \ge h'(t_0)e^{-c(t-t_0)}.$$

4. Differentiability of the free boundary

We want to show that h is actually differentiable. Since $h(t)+Ct^2$ is convex, it has left and right differentials at every point, and for $t < s$

$$(h'(t))^- \le (h'(t))^+ \le (h'(s))^- \le (h'(s))^+.$$

To fix the ideas we assume that origin is on the free boundary.

4.1. **Blow-up.** For $\lambda > 0$ consider function

$$v_\lambda(x,t) = \frac{v(\lambda x, \lambda t)}{\lambda}.$$

It follows that v_λ is a solution to porous medium equation. Moreover, v_λ is Lipschitz, therefore $\lim_{\lambda\to 0} v_\lambda = v^\infty$ exists and it is called the blow-up of v. Note that

- second derivative

  $$(v_\lambda)_{xx} = \lambda v_{xx}(\lambda x, \lambda t) \geq -\frac{\lambda}{(m-1)t} \to 0$$

  so v^∞ is convex.

- free boundary $h(t)$ is convex and consists of two lines

$$h(t) = \begin{cases} At, t > 0 \\ Bt, t < 0 \end{cases} \tag{4.1}$$

  with $A \geq B \geq 0$.

Note that

$$\begin{aligned} v^\infty(x,0) &= \lim \frac{v(x_0 + \lambda x, t_0 + \lambda t)}{\lambda} = 0, x > 0, \\ v^\infty_x(x,0) &= \lim v_x(x_0 + \lambda x, t_0 + \lambda t) = -A \end{aligned} \tag{4.2}$$

i.e. $v^\infty(x,0) = (-Ax)_+$ therefore we conclude from the uniqueness theorem that $v^\infty(x,t) = A(At - x)_+$ for $t > 0$.

We want to show now that the travelling front cannot "break" going into the past. To do this we will go far to the left for $t = t_0$ and get a contradiction. We start with an estimate for the decay of v_{tt}.

Lemma 18. *There exists a constant $C > 0$ such that for $x_0 < 0$ large we have*

$$v^\infty_{tt}(x,t) \leq \frac{C}{|x_0|}, \forall (x,t) \in \{|x - x_0| \leq \frac{|x_0|}{2}, |t| \leq \frac{|x_0|}{M}\} \equiv D. \tag{4.3}$$

Proof. Let us consider the scaled function

$$v_R(x,t) = \frac{v^\infty(Rx, Rt)}{R}, R = \frac{1}{|x_0|}.$$

Then v_R is Lipschitz in $|t| \leq \frac{1}{M}, |x - x_0| \leq 1/2$. By parabolic Schauder estimates we have that

$$|(v_R)_{tt}| \leq C \tag{4.4}$$

and returning to v^∞ the result follows. $\square$

Corollary 19.

$$\lim_{x \to -\infty} [v^\infty(x, t) - A(At - x)_+] = 0.$$

Proof. Take $x < 0$ large, then at $(x, 0)$ v^∞ is C^1 smooth, therefore using Taylor's formula

$$\begin{aligned} v^\infty(x,t) - A(At - x) &= v^\infty(x,0) + tv^\infty(x,0) + \frac{t^2}{2} v^\infty_{tt}(\cdot) - A(At - x) \\ &= \frac{t^2}{2} v^\infty_{tt}(\cdot) \to 0, \end{aligned} \tag{4.5}$$

when $x \to -\infty$. $\square$

4.2. **Classification of the global solutions.** Next, we want to show that $v^\infty = A(At - x)$. An important step to prove this, is to show that at any point

$$v_x \geq -A.$$

Assume that for some (x_0, t_0) we have $-v^\infty_x(x_0, t_0) = -A - \delta < -A$, then we can put under v a travelling front with speed $A + \delta$ that will catch up with the free boundary. If $x < x_0 < 0$ we have

$$\begin{aligned} v^\infty(x,t) &= v^\infty(x_0,t) - v^\infty_x(\cdot)(x_0 - x) \\ &\geq v^\infty(x_0,t) + (A + \delta)(x_0 - x). \end{aligned} \tag{4.6}$$

We used $v^\infty_x(\cdot) \leq v^\infty_x(x_0, t)$ since $v^\infty_{xx} \geq 0$.

Thus

$$\begin{aligned} 0 \leftarrow v^\infty(x,t) - A(At - x) &\geq v^\infty(x_0,t) + (A + \delta)(x_0 - x) - At^2 + Ax \\ &= v^\infty(x_0,t) - A^2 t + Ax_0 + \delta(x_0 - x) \to +\infty \end{aligned} \tag{4.7}$$

provided $\delta > 0$, this is a contradiction, hence

$$v^\infty_x \geq -A.$$

This implies that for any (x,t)

$$v^\infty(x,t) \geq A(At - x).$$

Finally let us show that v^∞ is the wave function $A(At-x)$. Take a point $(\bar{x},\bar{t})$ and assume that $\bar{x} \leq A\bar{t}$ such that $v^\infty(\bar{x},\bar{t}) > A(A\bar{t}-\bar{x})$ which contradicts to the strong maximum principle. Next assume that $\bar{x} > A\bar{t}$. But then for $t > \bar{t}$ we know that $v^\infty(\bar{x},t) > 0$ by (2.6). Contradiction. $\square$

5. Remarks

5.1. **N-dimensional results.** In the N dimensional case v may not be Lipschitz though it is always Hölder continuous. In [CVW] the authors proved Lipschitz continuity for large times. More precisely if T_0 is the time when the support of $v(x,t)$ overflows the smallest ball, where the initial support is contained then v is Lipschitz in $\mathbb{R}^N \times (\tau, \infty)$ for any $\tau > T_0$, with bounds depending on the initial data and τ. Also supp$v$ is bounded for any t but eventually it covers the whole space. As a consequence the free boundary is Lipschitz. Furthermore it is also $C^{1,\alpha}$ [CW]. However there is an example constructed by J. Graveleau showing that if suppv_0 has holes then Dv may blow up. Therefore the result in [CVW] is optimal.

5.2. **Waiting time.** As the example of quadratic solution indicates the free boundary may stay stagnant. If there exists a $t^\star \in [0,T]$ so that $h(t)$ does not move for $t \in (0,t^\star)$ and $h(t)$ moves for $t > t^\star$ then $t^\star$ is called waiting time. Note that when h starts moving it never stops. The value of $t^\star$ depends on the initial condition. Next theorem is due to Knerr [K].

Theorem 20. *If initial data $v_0(x) \geq c(-x)^\gamma, -\delta < x < 0$ for some $\gamma \in (0,2)$ then $t^\star = 0$. If $v_0(x) \leq cx^2, -\delta < x < 0$ then $t^\star > 0$.*

If $t \in (t^\star, T)$ then $h \in C^1(t^\star, T)$ [CF], and hence the free boundary condition is satisfied in the classical sense.

Theorem 21. *Let $t_m = 1/2(m+1)$ and v is the solution of*

$$\begin{cases} v_t = (m-1)v\Delta v + |Dv|^2 & (x,t) \in \mathbb{R}^N \times \mathbb{R}^+, \\ v(x,0) = v_0(x) & x \in \mathbb{R}. \end{cases} \tag{5.1}$$

If for some $\alpha, \beta > 0$

$$v_0(x) \le \alpha x^2 + o(x^2) \ \textit{as } x \uparrow 0 \tag{5.2}$$

$$v_0(x) = 0, \forall x \in \mathbb{R}^+$$

$$v_0(x) \le \beta x^2 \ \textit{in } x \in \mathbb{R}^+$$

then

$$\frac{t_m}{\beta} \le t^\star \le \frac{t_m}{\alpha},$$

in particular if $\alpha = \beta$ $t^\star = t_m/\alpha$.

5.3. **Viscosity solutions.** Viscosity solutions were introduced by M. Crandall and P. Lions in the context of the first order equations of Hamilton-Jacobi type.

For instance if we are given in the interval $[-1, 1]$ the equation

$$\begin{cases} |w_x| = 1 \\ w(-1) = w(1) = 0 \end{cases}$$

any zig-zag with slopes 1 and -1 would be a candidate for a weak solution. But there are two natural ones: $w = 1 - |x|$ and $-w = |x| - 1$. The solution w is selected by the "vanishing viscosity" method i.e. it is the limit of w^ϵ, solutions to

$$\epsilon w^\epsilon_{xx} + (1 - |w^\epsilon_x|) = 0,$$

(thus the name of viscosity solution). They realized that w would be "touched by below" by a smooth function ϕ only if $|\phi'| \ge 1$, while by above only if $|\phi'| \le 1$ (i.e. it is the most concave solution).

It was soon realized that this was an excellent way to define weak solutions for equations in non-divergence form, i.e. defined by a comparison with a "specific profiles" (quadratic polynomials for second order PDE's, global profiles for the phase transition problems, etc.)

Here we sketch how the theory works for the Laplacian [CC].

Definition 22. *A function* $u : \mathbb{R}^N \to \mathbb{R}$*, continuous in* Ω*, is said to be a subsolution (supersolution) to* $\Delta u = 0$*, and we write* $\Delta u \ge 0$ $(\Delta u \le 0)$*, if the following holds: if* $x_0 \in \Omega, \phi \in C^2(\Omega)$ *and* $u - \phi$ *has local maximum (minimum) at* x_0 *then* $\Delta\phi \ge 0 (\Delta\phi \le 0)$*. A solution is a function* u *which is both a subsolution and a supersolution.*

Heuristically this definition tells that a subsolution cannot touch a solution from below. Indeed assume that ϕ is a C^2 *strict* subsolution touching the solution u from below at x_0, then $u(x) \le \phi(x), u(x_0) = \phi(x_0)$. Since ϕ touches u from below then $D^2u \ge D^2\phi$. On the other hand $0 < \Delta\phi \le \Delta u = 0$, contradiction. To make this argument work for subsolutions one needs to consider $\phi + \epsilon|x|^2$ and let $\epsilon \downarrow 0$. Another way of checking this is to use the maximum principle, for we have from previous computation that $\Delta\phi = \Delta u = 0$ and $\phi - u$ has a local minimum at x_0, thus buy maximum principle $u = \phi$.

If in the definition one changes Δu with $F(D^2u)$ then the definition of viscosity solutions for elliptic operator F follows.

As an example let us show that any continuous viscosity solution of $\Delta u = 0$ is a classical harmonic function. In 1-dimensional case the classical solution is a line $\ell(x)$. Now if u is above of this line then bringing the parabola $P(x) = \ell(x) - \varepsilon x^2$ from infinity will touch u at some point which will contradict to $P_{xx} \ge 0$.

In N-dimensional case let B_ρ be a ball of radius ρ, and let $\Delta u = 0$ in B_ρ in viscosity sense. Let v be harmonic in B_ρ and $u = v$ on ∂B_ρ. Then v is the Poisson integral of the continuous function u. Thus $v \in C^2(B_\rho)$. We want to compare u with v in B_ρ choosing ρ to be sufficiently small.

Denote $M = \max_{B_\rho}(u - v)$ and suppose that $M > 0$. Then for some x_0, $M = u(x_0) - v(x_0)$ and x_0 is an interior point, since $u = v$ on the boundary of B_ρ. Consider $u_\varepsilon(x) = v(x) + \varepsilon(x - x_0)^2$ where ε is a small positive number. Let $M_1 = \max_{B_\rho}(u - v + \varepsilon(x - x_0)^2)$ and it is attained at some x_1. Then if we choose ε to be very small we have that x_1 should be close to x_0, that is x_1 is an interior point, then we have

$$
\begin{aligned}
u(x) &\le v(x) + \varepsilon(x - x_0)^2 + M_1, \\
u(x_1) &= v(x_1) + \varepsilon(x_1 - x_0)^2 + M_1,
\end{aligned}
\tag{5.3}
$$

hence $\phi(x) = v(x) + \varepsilon(x - x_0)^2 + M_1$, which is C^2 in B_ρ, touches u at x_1 from above. From definition of the viscosity solutions $0 \le \Delta\phi = \Delta(v - \varepsilon(x - x_0)^2) = -2N\varepsilon$ contradiction. Thus x_0 is not an interior point and $u \le v$ in B_ρ.

In the same way one can show that $m = \min_{B_\rho}(u - v) \geq 0$. Otherwise if $m < 0$ and the minimum is attained at some point y_0 we will compare u to the function $\psi(x) = v - \varepsilon(x - y_0)^2 + m_1$ where $m_1 = \min_{B_rho}(u - v - \varepsilon(x - y_0)^2)$ and the inequality $u \geq v$ follows .

In this section, we give an idea of how the techniques described in the lectures surface in the theory of minimal surfaces and free boundary problems.

5.4. **Global profiles and regularity.** A minimal surface is as surface which has smallest area among all surfaces with prescribed boundary condition. Classical solutions to minimal surface problem do not always exist. Therefore one has to seek the solution in a weak sense, that is to define the area in some generalized way. This is given in a weak fashion through the divergence theorem, by means of the *perimeter*. Ω is said to be a set of finite perimeter if for any smooth vectorfield $\psi, \sup_{x\in\Omega} |\psi| \leq 1$ compactly supported in Ω

$$|\int_\Omega \mathrm{div} v| \leq C_0.$$

The best constant C_0 is called the perimeter of set $\partial\Omega$. Then perimeter is semicontinuous under L^1 convergence of characteristic functions χ_Ω. Note that heuristically using the divergence theorem

$$|\int_\Omega \mathrm{div} v| = |\int_{\partial\Omega} v \cdot \nu| \leq area(\partial\Omega).$$

Sets of finite perimeter can also be thought as L^1 limits of polyhedra with a uniformly finite area. Then we can look at the following problem:

Among all sets of finite perimeter $\Omega \subset B_1$ find one which has minimum perimeter.

The existence of a set with minimal perimeter is immediate by compactness.

Having defined the generalized area and generalized minimal surface one tries to explore how "classical" it can be. In other words to show that except an unavoidable singular set Σ it is smooth hypersurface satisfying to equation of mean curvature.

One of the ways of doing so is to exploit the invariance of area minimizing surfaces (such as scaling!) and a monotonicity formula. The latter one is the following: if S is an area minimizing surface and $0 \in S$ in $\mathbb{R}^{N+1}$ then

$$A(r) = \frac{area(S \cap B_r)}{r^N}$$

is a monotone function of r. Moreover, if $A(r)$ is identically constant S has to be a cone, i.e. the defining function is homogeneous. One then considers the sequence of dilations $S_k = \{x, r_k x \in S\}, r_k \downarrow 0$. The "limiting blow-up" object is a surface S_0, also called global solution, for which $A(r) = const. = A(0^+)$. Hence if one can classify all possible minimal cones S_0 which are alternatives to a hyperplane a regularity theorem can be deduced. For instance if $N < 8$ the only such cones are hyperplanes and the generalized minimal surface S is really an analytic graph.

Many free boundary problems can be treated parallel to the theory of minimal surfaces [CS]. For instance consider the classical two phase problem [ACF].

Let u be Lipschitz function in unit ball B_1, such that

$$\begin{gathered}\Delta u = 0, \text{ in } \{u > 0\} \cup \{u < 0\}, \\ (u_\nu^+)^2 - (u_\nu^-)^2 = 1 \text{ on } \mathcal{F} = \partial\{u > 0\}.\end{gathered} \tag{5.4}$$

The free boundary here is $\mathcal{F}$ and the extra gradient jump condition $(u_\nu^+)^2 - (u_\nu^-)^2 = 1$ on $\mathcal{F}$ is satisfied in some weak sense. Weak solution of this problem can be obtained by minimizing the functional

$$J(u) = \int_{B_1} |Du|^2 + \lambda_+^2 \chi_{\{u>0\}} + \lambda_-^2 \chi_{\{u \le 0\}}$$

for some positive constants λ_+, λ_-. Note that if $\Lambda = \lambda_+^2 - \lambda_-^2 > 0$ then

$$J(u) = \int_{B_1} |Du|^2 + \Lambda \chi_{\{u>0\}} + \lambda_-^2 |B_1|$$

so $J(u)$ is the sum of Dirichlet energy and $\Lambda \text{meas}\{u > 0\}$. This suggests the fact that u is a minimizer imposes some minimality on the volume of positivity set. It turns out that $\partial\{u > 0\}$ is a generalized surface of positive mean curvature [C1] i.e. if $\partial\{u > 0\}$ is perturbed inside of positivity set $\{u > 0\}$ near B_r then for perturbed surface S', $H^{n-1}(S') \ge H^{n-1}(\partial\{u > 0\})$.

Let us illustrate how the ideas from minimal surface theory can be applied to classify the global solutions of (5.4) in two dimensional case. Assume that u is a minimizer of J so that it solves (5.4) in some weak sense. First note that Lipschitz is the best possible regularity for u one can expect in view of the gradient jump along the free boundary $\mathcal{F}$. Using a monotonicity formula one can show that u is Lipschitz [ACF]. For $r_k \downarrow 0$ and $0 \in \partial\{u > 0\}$ let us consider $u_k(x) = u(r_k x)/r_k$. This function is well-defined for Lipschitz function u. Then $S_k = \partial\{u_k > 0\}$

and $S_0 = \partial\{u_0 > 0\}$ where $u_0 = \lim u_k$ must be homogeneous global solution. If the free boundary $\partial\{u_0 > 0\}$ forms an angle with aperture θ at zero then $\partial\{u_0 > 0\}$ is a cone Γ_θ with aperture θ. We want to show that there are no alternative to u_0 of being a linear function, i.e. Γ_θ is a half-plane.

By rotation of coordinate system we may assume that $\Gamma_\theta = \{x \in \mathbb{R}^2 : 0 < x_2 < x_1 \tan\theta\}$. Let us write the Laplacian in polar coordinates

$$\Delta u = \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_r) + \frac{1}{r}u_{\phi\phi}\right).$$

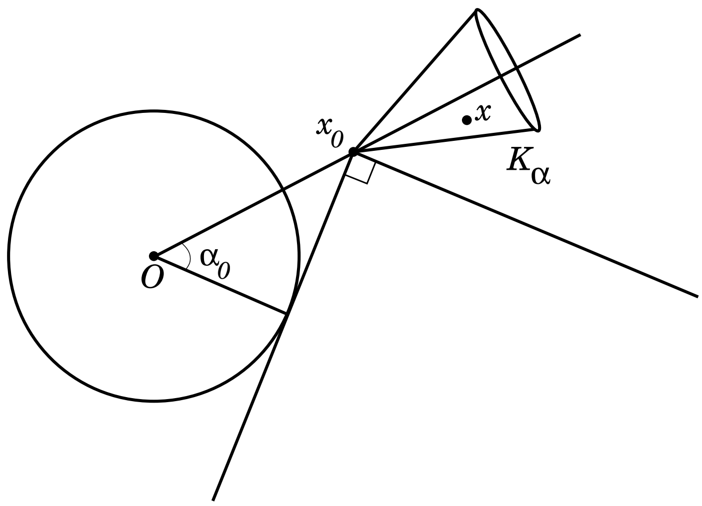
Since $u = rg(\phi)$, $g(\phi)$ verifies to Cauchy problem

$$\begin{cases} g''(\phi) + g(\phi) = 0 \\ g(0) = g(\theta) = 0, \end{cases} \tag{5.5}$$

which has a unique solution $g(\phi) = \sin\phi$. Therefore $\theta = \pi$ and Γ_θ is a the upper half-plane. Hence in two dimensional case all the global solutions are linear functions and $\mathcal{F}$ is differentiable.

5.5. **Moving plane method.** As a final example of the power of symmetries we describe the moving plane method created by A.D. Aleksandrov in his study of the surfaces of constant curvature. The well-known theorem of A.D. Aleksandrov states: if S is a surface of constant nonzero mean curvature then S is a sphere. Let's take a one parameter family of planes and move it in some constant direction. Let S_t be the surface which is reflection of S with respect to the plane corresponding to t. Then at some point, S_t and S would be tangent to each other hence by Hopf lemma $S = S_t$, so S is symmetric with respect to any plane thus it is a sphere.

This technique can be used to prove Lipschitz continuity of the free boundary in N-dimensional case when $\operatorname{supp} u_0 \subset B_1$ and the free boundary is strictly outside of B_1. Let $\Omega = \operatorname{supp} u_0(x)$ and let $a = \inf_{x\in\Omega} x_1, b = \sup_{x\in\Omega} x_1$, where $x = (x_1, x'), x' \in \mathbb{R}^{N-1}$. Then for any $\lambda \in (a, b)$ consider $x_\lambda = 2\lambda - x_1$. So x_λ is the reflection of x with respect to the plane $x_1 = \lambda$. Our goal is to show that $u(x,t) \geq u(x_\lambda, t), t > 0$ when $a > 0$ or $b < 0$. Indeed $u_0(x) \geq u_0(x_\lambda)$ and $u_0(x) = u_0(x_\lambda)$ for $x_1 = \lambda$ hence the comparison principle applies. In particular this implies monotonicity of u in the x_1 direction since λ is arbitrary number in (a, b). Clearly this reflection argument applies to any plane $\Sigma = \{x \in \mathbb{R}^N, (x - y_0) \cdot \ell = 0\}$ for some fixed point y_0 and unit direction ℓ provided that Ω has a positive distance from Σ. Now to prove that the boundary of supp u is Lipschitz it is enough to show that there exists a uniform cone of monotonicity at each point on the boundary of the support of $u(x,t)$ when the free boundary lies outside B_1.



Now take $x_0, |x_0| > 1$ and let $K_\alpha = \{x, \angle(x - x_0, x_0) \leq \alpha\}, \alpha < \alpha_0$ with $\cos \alpha_0 = 1/|x_0|$. Then for any plane Σ reflecting x to x_0 we can apply Aleksandrov's idea and conclude that in K_α u is monotone.

Notice that we did not assume Lipschitz regularity for u.

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