The geometry of solutions to a segregation problem for nondivergence systems

Citation for published version:
Karakhanyan, A, Caffarelli, L & Lin, F 2009, 'The geometry of solutions to a segregation problem for nondivergence systems' Journal of Fixed Point Theory and Applications, vol. 5, no. 2, pp. 319-351. DOI: 10.1007/s11784-009-0110-0

Digital Object Identifier (DOI):
10.1007/s11784-009-0110-0

Link:
Link to publication record in Edinburgh Research Explorer

Document Version:
Peer reviewed version

Published In:
Journal of Fixed Point Theory and Applications

Karakhanyan, A, Caffarelli, L & Lin, F 2009, 'The Geometry of solutions to a segregation problem for nondivergence systems' Journal of Fixed Point Theory and Applications, vol 5, no. 2, pp. 319-351.

THE GEOMETRY OF SOLUTIONS TO A SEGREGATION PROBLEM FOR NON-DIVERGENCE SYSTEMS

L. A. CAFFARELLI, A. L. KARAKHANYAN, AND FANG-HUA LIN

ABSTRACT. Segregation systems and their singular perturbations arise in different areas: particle annihilation, population dynamics, material sciences. In this article we study the elliptic and parabolic limits of a non variational singularly perturbed problem. Existence and regularity properties of solutions and their limits are obtained.

One of the simplest models for the segregation of species (or systems of particles that annihilate on contact) consists of setting a system of equations for the (vector) of nonnegative species densities $\vec{u}^\varepsilon = (u_1^\varepsilon, \dots, u_k^\varepsilon)$, of the form

$$L_j(u_j^\varepsilon) = \frac{1}{\varepsilon} F_j(\vec{u}^\varepsilon\ )$$

where L_j is a second order differential operator, F_j vanishes if $u_j^\varepsilon u_k^\varepsilon = 0$ for $k \neq j$ and it is strictly positive otherwise, forcing u_j^ε to segregate ($u_j^\varepsilon u_k^\varepsilon$ converge to zero) as ε goes to zero.

In some applications, the system has a variational (or divergence) structure. For instance: (see [CLLL], [CTV])

$$\Delta u_j^\varepsilon = \sum_{k \neq j} \frac{1}{\varepsilon} u_j^\varepsilon (u_k^\varepsilon)^2 \ ,$$

the Euler-Lagrange equations for vectors $\vec{u}$, stationary points of the functional

$$\mathcal{E}(\vec{u}\ ) = \int \sum_j (\nabla u_j)^2 + \frac{1}{\varepsilon} \sum_{j,k} [u_j^2 u_k^2] \ .$$

In others, i.e., the case in this article, (and of particle annihilation) the system is symmetric,

$$\Delta u_j^\varepsilon = \sum_{k \neq j} \frac{1}{\varepsilon} u_j^\varepsilon u_k^\varepsilon$$

and although it may appear to be a minimal change, its lack of variational structure imposes a different approach.

The final result is, though, very similar to those attained in [CL2], [CL1] for the variational case, mainly that the interphase between each two components is smooth (the level set of a harmonic function),

1991 *Mathematics Subject Classification.* 35B25, 35K55, 35B65, 35K55.

Key words and phrases. segregation system, singular perturbation, parabolic.

except in a "filament" (a set of Hausdorff dimension $n-2$), where three or more species may concur, mirroring the basic two dimensional example given by

$$w(x) = r^{3/2}\Big|\cos\frac{3}{2}\theta\Big|$$

where each connected component of $\{w>0\}$ represents the support of a different species, and the three components concur at the origin.

This problem has received a considerable attention. See [CTV], [CTV2], [CTV3] and [CTV4] for the discussion of the variational solutions and [CL1], [CL2] and [CLLL] for optimal partition problems. The system with a singular limit also appears in the combustion theory related to flame propagations [CR],[BS].

The parabolic version is not treated in the literature. In this paper we give a full description of the problem for the heat equation as a model case.

For the elliptic case we prove an improvement of the regularity result. We discuss the elliptic and parabolic versions separately. The paper is organized as follows: in the first section we show that the solutions u^ε are uniformly Hölder continuous in ε, giving rise to a Hölder continuous vector $\vec{u}$ as a uniform limit as ε goes to zero. Our approach works for more general classes of nonlinear uniformly elliptic and parabolic equations.

The vector $\vec{u}$ inherits several properties from $\vec{u}_\varepsilon$ that compose the starting hypotheses of the regularity theory. In the next section we prove several properties of the limit function $\vec{u} = \lim_{\varepsilon\to 0} \vec{u^\varepsilon}$, such as the harmonicity across the free boundary, the regularity of $|\nabla u|^2$ across interphases, and the Lipschitz regularity. The latter one is an application of a monotonicity formula introduced in [ACF].

The third section contains the geometric description of free boundary and the proof of the clean-up lemma which states that a "flatness" implies the regularity property of the free boundary near a point where only two components concur.

Next we introduce Almgren's monotonicity formula [A] in order to find out the structure of the free boundary near a singular point. The proof of Almgren's monotonicity formula for the heat equation is given in the Appendix.

1. Uniform Hölder continuity for the system u^ε

We consider, in the ball B_1 of $\mathbb{R}^n$, a nonnegative solution, $u_j^\varepsilon \geq 0$, of the system

$$\Delta u_i^\varepsilon = \frac{u_i^\varepsilon}{\varepsilon} \sum_{k \neq i} u_k^\varepsilon.$$

For this section we may replace Δu by a uniformly elliptic operator $Lu = D_i(a_{ij} D_j u)$ with bounded measurable coefficients a_{ij}.

We will assume that the u_i^ε are bounded ($0 \leq u_i^\varepsilon \leq M$), and that $\vec{u}^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, \ldots, u_m^\varepsilon)$ is in L^1 or L^2, since being subharmonic (or a-subharmonic i.e. $Lu_i^\varepsilon \geq 0$) the mean value theorem implies that the u_i^ε are bounded in B_{1-h} for any sufficiently small $h > 0$.

Of course u_i^ε are smooth, with bounds depending on ε. Our first theorem is

Theorem 1. *In $B_{1/2}$, for any ε, $\vec{u}^\varepsilon$ is C^α for some $\alpha > 0$ independent of ε, and*

$$\|\vec{u^\varepsilon}\|_{C^\alpha(B_{1/2})} \leq C(M) \ ,$$

with $C(M)$ also independent of ε.

Remark. For this first theorem we may replace the Laplacian by any other operator Lu, linear elliptic or parabolic with the following three properties:

Let w satisfy $Lw = f \geq 0$ and

$$\operatorname*{osc}_{B_2} w = \sup_{B_2} w - \inf_{B_2} w = 1.$$

Then for a positive constant $\mu(\gamma_0)$ depending on γ_0 one has

a) If $|\{f \geq \gamma_0 > 0\}| \geq \gamma_0 > 0$ then

$$\sup_{B_1} w \leq \sup_{B_2} w - \mu(\gamma_0).$$

b) If $|\{w \leq \sup w - \gamma_0\}| \geq \gamma_0$ then

$$\sup_{B_1} w \leq \sup_{B_2} w - \mu(\gamma_0).$$

c) If $|f| \ll \operatorname*{osc}_{B_2} w = 1$, then

$$\operatorname*{osc}_{B_1} w \leq \operatorname*{osc}_{B_2} w - \gamma_0.$$

This is true for uniformly elliptic or parabolic equations with bounded measurable coefficients from DeGiorgi-Nash-Moser (and the Littman-Stampacchia-Weinberger estimate [LSW]) for divergence equations, and the Alexandrov-Backelman-Pucci and Krylov-Safanov theory for non-divergence equations. In the parabolic case we must take consecutive parabolic cylinders.

In order to prove Theorem 1 we first state the following

Lemma 2. *Let* $m_i(R) = \min_{B_R} u_i^\varepsilon, M_i(R) = \max_{B_R} u_i^\varepsilon$, *and* $\mathcal{O}_i(R) = \operatorname*{osc}_{B_R} u_i^\varepsilon$. *Suppose that either of the following is satisfied for some positive constant* γ_0*:*

a) $|\{x \in B_{1/4}, u_i^\varepsilon(x) \leq M_i - \gamma_0 \mathcal{O}_i\}| \geq \gamma_0$ *where* $m_i = m_i(1), M_i = M_i(1), \mathcal{O}_i = \mathcal{O}_i(1)$,

b) $|\{x \in B_{1/4},\ \Delta u_i^\varepsilon(x) \geq \gamma_0 \mathcal{O}_i\}| \geq \gamma_0$,

c) $|\{x \in B_{1/4},\ \Delta u_i^\varepsilon \geq \gamma_0 u_i^\varepsilon\}| \geq \gamma_0$.

Then there exists a small positive constant $c_0 = c_0(\gamma_0)$ *such that the following decay estimate is valid*

$$M_i(\frac{1}{4}) \leq M_i - c_0 \mathcal{O}_i.$$

Proof. It is well-known that if $Lu \geq 0$ in $D \subset B_R$ where $u \geq K$ in D and $u = K$ on $\partial D \cap B_R$ then

$$\sup_{D \cap B_{R/4}} u \leq \frac{1}{1 + C\frac{|B_R \cap D^c|}{R^n}} \cdot \sup_D u.$$

This estimate is classical (see [La]) and as an application to our problem

$$\Delta u_i^\varepsilon = \frac{u_i^\varepsilon \sum_{j \neq i} u_j^\varepsilon}{\varepsilon}, \quad \text{in} \quad B_1$$

with $u = u_i^\varepsilon$ we have that

$$\begin{aligned} M_i\left(\frac{1}{4}\right) = \sup_{B_{1/4}} u_i^\varepsilon &\leq \max\left(\frac{M_i}{1 + C\gamma_0},\ M_i - \gamma_0(M_i - m_i)\right) \\ &= c_0 M_i \qquad c_0 < 1. \end{aligned}$$

In particular $\operatorname*{osc}_{B_R} u_i^\varepsilon = \mathcal{O}_i(R)$ decays

$$\begin{aligned} \mathcal{O}_i\left(\frac{1}{4}\right) = M_i\left(\frac{1}{4}\right) - m_i\left(\frac{1}{4}\right) &\leq c_0 M_i - m_i\left(\frac{1}{4}\right) \\ &\leq c_0(M_i - m_i) = c_0 \mathcal{O}_i(1) \end{aligned}$$

since $m_i(1/4) \geq m_i(1) \geq c_0 m_i$, $c_0 < 1$ so part a) follows.

To prove part b) we use Green's representation formula

$$u_i^\varepsilon(x) = v_i^\varepsilon(x) - \int_{B_1} G(x,y)\Delta u_i^\varepsilon(y)\,dy, \qquad G(x,y) \geq 0$$

where v_i^ε is the harmonic replacement of u_i^ε in B_1 and $G(x,y)$ is the Green's function of B_1.

Then we have

$$\begin{aligned} 0 \leq v_i^\varepsilon(x) - u_i^\varepsilon(x) = \int_{B_1} G(x,y)\Delta u_i^\varepsilon(y)\,dy &\geq \int_{B_{1/4}\cap\{\Delta u_i^\varepsilon \geq \gamma_0 \mathcal{O}_i\}} G(x,y)\Delta u_i^\varepsilon(y)\,dy \\ &\geq \gamma_0 \mathcal{O}_i \int_{B_{1/4}\cap\{\Delta u_i^\varepsilon \geq \gamma_0 \mathcal{O}_i\}} G(x,y) \\ &\geq C\gamma_0^2 \mathcal{O}_i \end{aligned}$$

that is $u_i^\varepsilon(x) \leq v_i^\varepsilon(x) - C\gamma_0\mathcal{O}_i(1)$ and $M_i(1/4) \leq c_0 M_i$ for $c_0 < 1$.

Now consider $A_i = \{x \in B_{1/4}, \Delta u_i^\varepsilon \geq \gamma_0 u_i^\varepsilon\}$ and $H_i = \left\{x \in A_i, u_i^\varepsilon(x) < \frac{M_i}{2}\right\}$. First let us assume that

$$|A_i \setminus H_i| \geq \frac{1}{2}|A_i|. \tag{1.1}$$

Then

$$\left\{x \in A_i,\ u_i^\varepsilon(x) \geq \frac{M_i}{2}\right\} \subset \left\{\Delta u_i^\varepsilon \geq \gamma_0 \frac{M_i}{2}\right\}.$$

Therefore

$$\begin{aligned} \left|\left\{\Delta u_i^\varepsilon \geq \frac{\gamma_0 M_i}{2}\right\}\right| \geq \left|\left\{x \in A_i,\ u_i^\varepsilon(x) \geq \frac{M_i}{2}\right\}\right| &= \\ = |A_i \setminus H_i| \geq \frac{1}{2}|A_i| \geq \frac{1}{2}\gamma_0 \quad &\text{by (1.1)}. \end{aligned}$$

So part b) applies and we have that

$$M_i\left(\frac{1}{4}\right) \leq c_0 M_i(1) \qquad c_0 < 1\ .$$

Now assume that

$$|A_i \setminus H_i| \leq \frac{1}{2}|A_i|. \tag{1.2}$$

Then

$$|H_i| = |A_i \setminus (A_i \setminus H)| = |A_i| - |A_i \setminus H_i| \geq \frac{1}{2}|A_i| \geq \frac{1}{2}\gamma_0 \qquad \text{by (1.2)}.$$

This implies

$$\left|\left\{x \in B_{1/4},\ u_i^\varepsilon(x) \geq \frac{M_i}{2}\right\}\right| \geq |H_i| \geq \frac{\gamma_0}{2}$$

and from part a) the result follows. □

We now return to the proof of the Hölder regularity. To simplify the notations we shall denote $\vec{u}^\varepsilon$ by u.

Proof. The proof is inductive, based on reducing the oscillation of the vector u in consecutive balls B_{λ^k} by a fixed constant $\mu < 1$, for some (fixed) $\lambda < 1$. Since we can always renormalize the system to the unit ball by

$$u^*(x) = \frac{1}{M} u(\lambda^k x) \ ,$$

with $M = \sup_{j,x} u_j$ where $x \in B_{\lambda^k}$ into the same system (with a different ε), it is enough to show that the largest of the individual oscillations decays from B_1 to B_λ, for a system u, with $\max_{j,x} u_j(x) = 1$ on B_1. Let $\mathcal{O}_i = \operatorname*{osc}_{B_1} u_j$ and without loss of generality we assume that $\operatorname{osc}(u_j) = \mathcal{O}_j$, $1 \geq \mathcal{O}_1 \geq \mathcal{O}_2, \ldots, \mathcal{O}_k$.

We start with several simple cases in which the oscillation of a given component decreases by a fixed proportion (see Lemma 2)

a) If $m_i \leq u_i \leq M_i$ and $|\{u_i \leq M_i - \gamma_0(M_i - m_i)\}| \geq \gamma_0$ then in $B_{1/2}$, M_i decays to $M_i - \mu(\gamma_0)(M_i - m_i)$ and $\mathcal{O}_i$ decays to $[1 - \mu(\gamma_0)]\mathcal{O}_i$.

b) If $|\{Lu_i \geq \gamma_0(M_i - m_i)\}| \geq \gamma_0 > 0$, then again M_i and $\mathcal{O}_i$ decay by amount proportional to $\mathcal{O}_i$.

c) If $|A| = |\{Lu_i \geq \gamma_0 u_i\}| \geq \gamma_0$ then, either $u_i \geq M_i/2$ half of the time in A (and M_i decays from b) or $u_i \leq M_i/2$ and M_i decays from a) (in both cases the amounts of decays in M_i are proportional to $\mathcal{O}_i$).

d) If $\sum_{j>1} \mathcal{O}_j \leq \delta\mathcal{O}_1, \delta > 0$, then we let w be the solution of $Lw = 0$, $w|_{\partial B_1} = u_1$.

Since $u_1 + \sum_{j>1}(M_j - u_j)$ is a super solution, we have

$$u_1 \leq w \leq u_1 + \sum_{j>1}(M_j - u_j) \leq u_1 + \delta\mathcal{O}_1 \ .$$

But

$$\operatorname*{osc}_{B_{1/2}} w \leq (1-\mu) \operatorname*{osc}_{B_1} w \leq \mu_1 \mathcal{O}_1$$

hence

$$\operatorname*{osc}_{B_{1/2}} u_1 \leq ((1-\mu) + \delta)\mathcal{O}_1 \leq \mathcal{O}_1$$

and $\operatorname{osc} u_1$ decreases proportionally to $\mathcal{O}_1$.

Therefore, to establish our basic iterative decay estimate for oscillations, it is sufficient to prove that either

a) Among those $\mathcal{O}_j$'s with $\mathcal{O}_j \geq \delta\mathcal{O}_1$ there is at least one that decays by a factor whenever the sizes of balls shrink by a half;

or

b) All $\mathcal{O}_j$'s with possible exception of $(\mathcal{O}_1)$, decay by a factor.

Indeed, applying a) or b) a finite number of times we will force all $\mathcal{O}_j$ bigger than $\delta\mathcal{O}_1$ to decrease.

Case 1: We first discuss the case $\varepsilon > 1$, i.e., for $\theta = 1/\varepsilon$:

$$\begin{cases} 0 \leq u_j \leq 1 \\ \Delta u_i = \theta u_i \sum_{j \neq i} u_j \quad \text{with} \quad \theta < 1 \\ \max u_{j_0} = 1 \quad \text{for some} \quad j_0 \ . \end{cases}$$

Consider two sub-cases: i) $\mathcal{O}_1 \sim 1$, i.e., $\mathcal{O}_1 \geq \delta_0$ and ii) $\mathcal{O}_1 \ll 1$.

We consider first the subcase that $\mathcal{O}_1 \sim 1$. If $\operatorname*{osc}_{B_{1/2}} u_1 = \mathcal{O}_1$ has not decreased, i.e., $\mathcal{O}_1 \geq (1-\gamma_0)\mathcal{O}_1$, then

$$\bar{m}_1 = \min_{B_{1/2}} u_1 \leq M_1 - \mathcal{O}_1 \leq M_1 - (1-\gamma_0)\mathcal{O}_1.$$

Note that there is a point x_0, in $B_{1/2}$, such that $u_1(x_0) = \bar{m}_1$.

Since the right hand side of the equation is < 1, $u_1(x) \leq \bar{m}_1 + \frac{1}{2}\delta_0$ in a neighborhood of size δ_0, i.e., for x in $B_{\delta_0}(x_0)$ one has

$$u_1(x) \leq M_1 - (1-\gamma_0)\mathcal{O}_1 + \frac{1}{2}\delta_0 \leq M_1(1-\delta_0).$$

From observation a), $\mathcal{O}_i(1/2)$ has decreased proportionally to $\mathcal{O}_i$. We thus obtain a contradiction.

The second subcase is then: ii) $\mathcal{O}_1 \ll 1$.

Again we divide it into two subcases. Assume first that all $\mathcal{O}_j \leq \frac{1}{2}M_j$ (i.e., $\frac{1}{2}M_j \leq u_j \leq M_j$ for all j). Then the right hand side of $\Delta u_j \sim \theta M_j$ for $j \neq j_0$, $(u_{j_0} \sim 1)$.

If $\theta M_j \geq \gamma_0\mathcal{O}_j$ for some γ_0 small, we have decay from observation b).

If $\theta M_j \leq \gamma_0\mathcal{O}_j$, we have decay from regularity (the right hand side is much smaller than the oscillation).

On the other hand if for some j $(j \neq j_0)$, $\mathcal{O}_j \geq \frac{1}{2}M_j$, we have

$$Lu_j \sim \theta u_j \leq 2\theta\mathcal{O}_j.$$

If $\theta \ll 1$ we have decay of $\mathcal{O}_j$ from regularity since the right hand side $\ll \mathcal{O}_j$.

If $\theta \sim 1$, we have it from observation c).

Case 2: We go now to the case $\varepsilon < 1$.

We consider two subcases: i) $M_1 \geq \varepsilon$ and ii) $M_1 < \varepsilon$.

i) $M_1 \geq \varepsilon$.

If $\mathcal{O}_1$ does not decay, we must have (from observation a) Lemma 2)

$$\left|\left\{u_1 \geq \frac{\varepsilon}{2}\right\}\right| \geq \frac{1}{2} \,.$$

Then, for $i \neq 1$

$$\left|\left\{Lu_i \geq \frac{u_i}{2}\right\}\right| \geq \frac{1}{2} \,.$$

Thus all M_i decay for $i \neq 1$.

ii) M_1 and hence all $\mathcal{O}_j$ are smaller than ε (since $\mathcal{O}_j \leq \mathcal{O}_1 \leq M_1$).

Since $M_{j_0} = 1$, $u_{j_0} \geq 1 - \varepsilon$, so

$$Lu_1 \geq \frac{1}{\varepsilon} u_1,$$

and b) applies.

The proof is complete. □

Corollary 3. *Given a family of solutions $(\vec{u}\,)^{\varepsilon_k}$, with ε_k going to zero, there is a subsequence that converges uniformly to a C^α function $\vec{u}$.*

2. General properties of the limit u

We now restrict ourselves to the Laplace operator.

Lemma 4. *Let $\vec{u}(x) = (u_1(x), u_2(x), \ldots, u_m(x))$ be the limit function from Corollary 3. Then we have*

i) *Δu_i is a positive measure and*

$$\Delta u_i \leq \sum_{j \neq i} \Delta u_j,$$

ii) *$\Delta u_i = 0$ whenever $u_i > 0$.*

Proof. i) follows from the fact that all u_i^ε are subharmonic and for each ε, and a nonnegative function $\varphi \in C_0^\infty(\Omega)$

$$\int (\Delta\varphi) u_i^\varepsilon = \int \varphi \Delta u_i^\varepsilon \leq \int \varphi \Big(\sum_{j \neq i} \Delta u_j^\varepsilon \Big) = \int (\Delta\varphi) \sum_{j \neq i} u_j^\varepsilon \,.$$

To prove ii) we will use the formula

$$\fint_{\partial B_r(x)} [u(y) - u(x_0)] = \int_0^r \left(\int_{B_\rho} \Delta u \right) \frac{d\rho}{\rho^{n-1}} \geq r^2 \fint_{B_r} \Delta u.$$

By Hölder continuity, if $u_1^\varepsilon(x_0) = \alpha_0 > 0$, then $|u_1^\varepsilon(y) - u_1^\varepsilon(x_0)| \leq \alpha_0/2$ in a neighborhood $B_h(x_0)$. Then

$$\fint_{B_h} u_1^\varepsilon \Big(\frac{1}{\varepsilon} \sum_{j \neq 1} u_j^\varepsilon \Big) = \fint_{B_h} \Delta u_1^\varepsilon \leq \frac{\alpha_0}{2h^2}$$

for ε small, from the uniform convergence. Since in B_h $u_1^\varepsilon \geq \alpha_0/2$ we get

$$\fint_{B_h} \frac{1}{\varepsilon} \Big(\sum_{j \neq 1} u_j^\varepsilon \Big) \leq \frac{1}{h^2}.$$

As $\varepsilon \to 0$, $\sum_{j \neq 1} u_j^\varepsilon$ goes to zero. □

Corollary 5. *The Hölder continuous functions u_j have disjoint supports and are harmonic when positive.*

In order to show the linear decay of u_i away from the boundary of its support, we recall the monotonicity formula introduced in [ACF] (see [CSa] for details).

Corollary 6. *Let v_1 and v_2 be defined as*

$$v_1 = \sum_1^{j_0} u_j \ , \quad v_2 = \sum_{j_0+1}^{k} u_j$$

and x_0 is a point on the boundary of $\operatorname{supp} u_{j_0}$. *Then we have that $J(R) = D(v_1, R) D(v_2, R) \nearrow$ as $R \nearrow$, where $D(v, R)$ denotes the Dirichlet average $D(v,R) = \frac{1}{R^2} \int_{B_R(x_0)} \frac{(\nabla v)^2}{|x - x_0|^{n-2}}\, dx$, furthermore*

$$\left(\frac{1}{R} \fint_{\partial B_R} v_i \right)^2 \leq C D(v_i, R) \leq \frac{C}{R^2} \fint_{B_{2R} \setminus B_R} v_i^2, \ i = 1, 2. \tag{2.1}$$

Proof. The second inequality in (2.1) follows from (12.16) [CSa]. For the proof of the first inequality we refer to [ACF]. □

Lemma 7 (Linear decay of u at the boundary of its support)**.** *Let $x_0 \in B_{1/2} \cap \partial \operatorname{supp}(u_1)$. Then*

a) $\displaystyle \frac{1}{R} \fint_{B_R(x_0)} u_1 \leq C J_R \Big(u_1, \sum_{j \neq 1} u_j \Big) \leq \|u\|_{L^2(B_{1/2})}$,

b) $\displaystyle \sup_{B_R} u_1 \leq CR.$

Proof. $u_1 - \sum_{j \neq 1} u_j$ is super harmonic. Since $u_j(x_0) = 0$ for all j

$$\fint_{\partial B_R} u_1 \leq \fint_{\partial B_R} \sum_{j \neq 1} u_j.$$

Thus

$$\theta_R = \frac{1}{R} \fint_{\partial B_R} u_1 \leq C[D(u_1, R)]^{1/2}$$

and also

$$\theta_R \leq \frac{1}{R} \fint_{\partial B_R} \sum_{j \neq 1} u_j \leq C[D(\sum_{j \neq 1} u_j, R)]^{1/2}.$$

Hence

$$\theta_R^4 \leq J_R(u_1, \sum_{j \neq 1} u_j) \leq J_{1/2} \leq C\|u\|^2_{L^2(B_{1/2})} \| \sum_{j \neq 1} u_j \|^2_{L^2(B_{1/2})}.$$

Now part b) follows from subharmonicity. For y in $B_R(x_0)$,

$$u_1(y) \leq \fint_{B_R(y)} u_1 \leq C \fint_{B_{2R}(x_0)} u_1 \leq 2R\|u\|_{L^2(B_{1/2}(x_0))}.$$

□

Corollary 8. *u_1 is Lipschitz in $B_{1/4}(x_0)$ and*

$$\|u_1\|_{\mathrm{Lip}(B_{1/4})} \leq C\|u\|_{L^2(B_{1/2})}.$$

Proof. Let $y \in B_{1/4} \cap \sup u_1$ and $d(y, \partial \sup u_1) = h$ ($h < 1/4$). Then, in $B_h(y)$, u_1 is positive, harmonic and $\sup(u_j) \leq Ch$. Therefore $|\nabla u_1(y)| \leq \frac{C}{h} h = C$. □

3. Geometric description of the interphase

In this section we start to analyze the geometric properties of the free boundary. First, a simple

Lemma 9. *If in $B_\rho(x_0)$, $\sum_{j>2} u_j \equiv 0$, then $u_1 - u_2$ is harmonic.*

Proof. $u_1 - \sum_{j \geq 2} u_j$ and $u_2 - \sum_{j \neq 2} u_j$ are superharmonic. □

This is not a very interesting result, since it is not clear when this hypothesis will hold.

To reach a reasonable description of the interphase, we will complement it with two lemmas:

a) A "clean-up" lemma that asserts that if in B_ρ the "density" of the components "of u_j" is very small, for $j \neq 1, 2$, then in $B_{\rho/2}$, $\sum_{j \neq 1,2} u_j \equiv 0$

and

b) "Almgren" monotonicity formula, that says that in the complementary situation $\vec{u}$ has a tangent "cone" of homogeneity strictly bigger than one.

We start with the "clean-up" lemma. The clean-up lemma consists of two parts.

The first part, a consequence of the monotonicity formula, says that if one of the components, u_1, goes to zero at a point x_0 in a "non-degenerate" fashion, i.e.,

$$\frac{1}{r} \fint_{B_r(x_0)} u_1 \geq \theta > 0 \quad \text{as} \quad r \quad \text{goes to zero,}$$

the whole configuration is a "small perturbation" of a linear function.

Lemma 10. (See [CSa].) *Assume that at* x_0

$$D(u_1, u_2, 0) = \lim_{R \to 0} D(u_1, u_2, R) = \alpha_0 > 0.$$

Then

a) *any convergent sequence of dilations*

$$\frac{1}{\lambda_k} u(\lambda_k x) \ , \quad \textit{for} \quad \lambda_k \to 0$$

converges to

$$\bar{u}_1 = \alpha_1 x_1^+ \ , \quad \bar{u}_2 = \alpha_2 x_1^- \ , \quad \bar{u}_j \equiv 0 \quad \textit{for} \quad j > 2.$$

b) *Further* $(\bar{u}_1 - \bar{u}_2)$ *must be harmonic. So*

$$\alpha_1 = \alpha_2 = (\alpha_0)^{1/4} \ .$$

Proof. Property a) is proven in [CSa], note that $\frac{1}{\lambda_k} u_j(\lambda_k x)$ is Lipschitz and supported in narrower and narrower domains, so $\bar{u}_j \equiv 0$.

b) follows from the fact that $u_i - \sum_{j \neq i} u_j$ is superharmonic. □

In this circumstance, the "clean-up" lemma says that the components u_j, for $j \neq 1, 2$ disappear before reaching x_0.

Theorem 11. *Assume hypotheses of the previous lemma. Then, in a neighborhood of* x_0, $\sum_{j>2} u_j \equiv 0$.

Before going into the proof, we need some preliminaries: after a large dilation, we can start with a configuration satisfying the following hypothesis.

Let $\bar{u}_i, i = 1, 2$ be the λ-dilatation of u_i at the origin, i.e. $\bar{u}_i(x) = u_i(\lambda x)/\lambda$ and let us write $\bar{u}_1 - \bar{u}_2$ as

$$\begin{aligned}\bar{u}_1 - \bar{u}_2 &= v_0 + \int G(x,y)\Delta(\bar{u}_1 - \bar{u}_2) \\ &= v_0 + v_1\end{aligned}$$

with v_0 harmonic, $v_0|_{\partial B_1} = \bar{u}_1 - \bar{u}_2$, v_1 is the part that comes from the presence of u_j, $j \neq 1, 2$ and is supposed to be small.

From the previous lemma, we may renormalize $\alpha_0 = 1$, and assume that

$$|(\bar{u}_1 - \bar{u}_2) - x_1| \leq h, \tag{3.1}$$

in particular $\operatorname{supp}_{j \neq 1,2} \bar{u}_j \subset |\{|x_1| \leq h\}|$, and each $\bar{u}_j$ has Lipschitz norm less than ch.

We also recall a decay property of harmonic functions in narrow domains.

Lemma 12. *Let w be continuous in B_1, supported in Ω and harmonic in its support. Assume that Ω is "narrow" in the sense that any ball of radius h, $B_h(y)$, contained in B_1, intersects the complement of Ω, $\mathcal{C}\Omega$, say, half of the time, i.e.,*

$$\frac{|B_h \cap \mathcal{C}\Omega|}{|B_h|} > \frac{1}{2}\ .$$

Then

$$w(x) \leq \sup_{\partial B_1} w \cdot e^{-C\frac{(1-|x|)}{h}}\ .$$

Proof. We prove that in the ball B_{1-kh}, $k = 1, 2, \ldots, N$, where $N \sim h^{-1}$

$$w(x) \leq \frac{1}{2} \sup_{B_{1-(k-1)h}} w\ .$$

Indeed, by the mean value theorem

$$w(x) \leq \fint_{B_h(x)} w$$

But $w \equiv 0$ "half of the time in such a ball." Hence the estimate follows. □

Before going back to the proof of the theorem, we slightly transform (3.1) into a convenient inductive hypothesis. Mainly, in a) we change the x_1 to the harmonic replacement v_0 of $u_1 - u_2$ in B_1, i.e., v_0 is harmonic and

$$v_0|_{\partial B_1} = u_1 - u_2\ .$$

Since $v_0 - x_1$ is harmonic in B_1 and $(v_0 - x_1)|_{\partial B_1} = h$, we have that $|v_0 - (u_1 - u_2)| \leq 2h$ and in $B_{1/2}$, $|\nabla(v_0 - x_1)| = |(\nabla v_0) - e_1| \leq ch$.

Therefore, for a small number h, to be chosen, we have the starting hypothesis:

Decompose $(u_1 - u_2) = v_0 + v_1$, with v_0 the harmonic replacement in B_1, then

a) $|v_0 - (u_1 - u_2)| \leq h$

b) $|\nabla v_0 - e| \leq h$

c) support of (u_j) for $j \geq 2$ is contained in the N_h neighborhood of the Lipschitz level surface $v_0 = 0$.

Note also that $\sum_{j \neq 1,2}(\sup_{B_1} u_j - u_j) = \beta$ provides a barrier for v_1 since $\beta \geq 0$ and $\Delta\beta \leq \Delta(u_1 - u_2)$. Hence $-\beta \leq v_1 \leq \beta \leq Ch$.

Let us see now what kind of improvement we can gain by going from B_1 to B_{1-s}. We note that u_j has decreased from $h = h_0$, to $h_1 = he^{-Cs/h} \leq h_0^2$ whenever $s \sim h^{1/2}/2$.

In particular if we decompose $u_1 - u_2 = \tilde{v}_0 + \tilde{v}_1$, now in B_{1-s}, $\tilde{v}_1 \leq h_0^2$. Therefore

$$|u_1 - u_2 - \tilde{v}_0| \leq h_0^2$$

while $|v_0 - \tilde{v}_0| \leq |v_0 - u_1 - u_2| \leq h_0$.

To see how v_1 decays, we first estimate the total mass of measure $\Delta u_j, j \neq 1, 2$ in B_{1-s}. If $B_{2\rho} \subset B_1$ then

$$⨍_{B_\rho} \Delta u_j \leq \frac{C}{\rho^2} ⨍_{B_{2\rho}} u_j, j \neq 1, 2$$

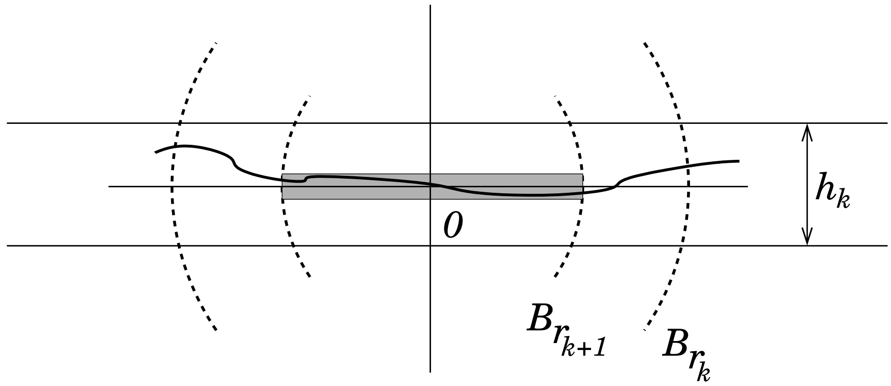
implying

$$\mu_{B_\rho}(\Delta u_j) \leq \frac{C}{\rho^2} \operatorname*{osc}_{B_{2\rho}} u_j |B_\rho|.$$

Choosing a family of balls $B_k = B_{\rho_k}, \rho_k \leq h^{1/2}$ which covers $\operatorname{supp} u_j$ and using exponential decay we conclude that $\mu_{B_{1-s}} \leq \sum_k \rho_k^{-2} \operatorname*{osc}_{B_{2\rho_k}} u_j |B_{\rho_k}| \leq h^{2n-1}|B_{1-s}|$, provided we take $\operatorname*{osc}_{B_{2\rho}} u_j \leq h^{2n}$. Now for $x \in B_1, \operatorname{dist}(x, \operatorname{supp} u_j) \geq h^{\frac{1}{2n}}$ we see from Green's representation formula that

$$\tilde{v}_1(x) \leq h^{2n-1}.$$

Thus we have on B_{1-2s} the estimate $|\nabla \tilde{v}_0 - \nabla v_0| \leq h_0^{1/2}$.



This suggests the following iterative scheme: Start with h_0 small. Consider the inductive sequence $h_k = (h_{k-1})^2$ (that converges to zero very fast) and the sequence

$$r_k, \quad \text{with} \quad r_1 = 1, \quad r_{k+1} = r_k - h_k^{1/2}$$

that converges to $1 - \mu$ with $\mu \leq 1/2$ if h_0 is small. Then:

Lemma 13. *In B_{r_k} there is a harmonic function v_k such that*

a) $|v_k - (u_1 - u_2)| \leq h_k$

b) $|\nabla(v_k - v_{k-1})| \leq h_{k-1}^2$

c) $|\nabla v_k - e_1| \leq \sum_0^{k-1} (h_\ell)^{1/2} \leq \frac{1}{4}$

d) *The level surface $v_k = 0$ is Lipschitz with the Lipschitz constant less than one for every k.*

The proof is exactly the discussion above.

Note that we take as v_k the harmonic replacement of $u_1 - u_2$ half way between r_k and r_{k+1}, so it does not coincide with $u_1 - u_2$ on $\partial B_{r_{k+1}}$, but still satisfies a) and this allows us to establish the estimate b).

4. Almgren monotonicity formula and control of the singular set

We will now prove, at the points of the interphases, a monotonicity formula due to Almgren that shows that at each such point $\vec{u}$ is asymptotically homogeneous and bounds this homogeneity from below.

First we note that

Lemma 14. $(\nabla u)^2$ *is a continuous function across the interphase.*

Proof. If $J_0(x_0) = \lim_{R\to 0^+} J_R(x_0) \neq 0$, then according to the clean-up lemma, $(u_1 - u_2)$ is harmonic.

If $J_0(x_0) = 0$ for every pair, then $|\nabla u(x)|^2$ goes to zero as x goes to x_0.

Indeed from semicontinuity, given $\varepsilon > 0$, there exist a δ and τ such that

$$J_\delta(x) \leq \varepsilon \quad \text{for} \quad x \in B_\tau(x_0).$$

If $y \in B_{\tau/2}(x_0) \cap \{u_1 > 0\}$ and $B_s(y)$ is the largest ball around y contained in $\{u_1 > 0\}$, $(s < \tau/2)$, then $B_s(y)$ has a point $x_1 \in \partial B_s(y) \cap \{u_1 = 0\}$.

From earlier discussions, we have

$$\frac{1}{2s} \fint_{B_{2s}(x_1)} (u_1)^2 \leq \varepsilon^{1/2}$$

and $|\nabla u_1(y)| \leq \varepsilon^{1/2}$. □

We can prove now the Almgren's monotonicity theorem [A] adapted to our setting.

Theorem 15. *For x_0 in the interphase let us define*

$$F(u, R, x_0) = \frac{R \displaystyle\int_{B_R(x_0)} |\nabla u|^2}{\displaystyle\int_{\partial B_R(x_0)} u^2}.$$

Then, $F'(R) \geq 0$.

Proof. By scale invariance it suffices to show that $(\log F)' \geq 0$ for $R = 1$

$$(\log F)'(1) = 1 + \frac{\displaystyle\int_{\partial B_1} |\nabla u|^2}{\displaystyle\int_{B_1} |\nabla u|^2} - \frac{(n-1)\displaystyle\int_{\partial B_1} u^2 + 2\displaystyle\int_{\partial B_1} u\, u_r}{\displaystyle\int_{\partial B_1} u^2}.$$

Assume for a moment that $\Delta \frac{u^2}{2} = (\nabla u)^2$ as measures. Then

$$\int_{B_1} |\nabla u|^2 = \int_{B_1} \Delta \frac{u^2}{2} = \int_{\partial B_1} u\, u_\nu \ .$$

Since u^2 is subharmonic, Δu^2 is a positive measure, and the identity is correct except on the interface. At a regular point of the interphase, where $\nabla u \neq 0$, this is also true. So we need to prove that Δu^2 is absolutely continuous with respect to the Lebesgue measure and that it vanishes in the Lebesgue sense at every point where $|\nabla u|^2 = 0$ and $u = 0$.

At those points x_0 where $|\nabla u|$ goes to zero, $u^2(x) \leq o(|x - x_0|^2)$ we have:

(4.1) $$\fint \Delta u = o(1).$$

We go on with the formal computation

$$(\log F)' = -(n-2) + \frac{\displaystyle\int_{\partial B_1} |\nabla u|^2}{\displaystyle\int_{\partial B_1} u\, u_r} - \frac{2\displaystyle\int_{\partial B_1} u\, u_r}{\displaystyle\int_{\partial B_1} u^2}.$$

We need to transform $\int_{\partial B_1} (\nabla u)^2$ into integrals involving u and u_r.

We use the following Rellich identity (see [GL])

$$\text{div }(x|\nabla u|^2) = n|\nabla u|^2 + 2x_i u_j u_{ij}$$

and

$$\text{div }\langle x, \nabla u\rangle \nabla u = |\nabla u|^2 + \langle x, \nabla u\rangle \Delta u + x_i u_j u_{ij}$$

or

$$\text{div}(x|\nabla u|^2 - 2\langle x, \nabla u\rangle \nabla u) = (n-2)|\nabla u|^2 - 2\langle x, \nabla u\rangle \Delta u\ .$$

We now integrate (assuming that $2\langle x, \nabla u\rangle \Delta u = 0$)

$$\int_{B_1} (n-2)|\nabla u|^2 = \int_{B_1} \text{div }(x|\nabla u|^2 - 2\langle x, \nabla u\rangle \nabla u) = \int_{\partial B_1} |\nabla u|^2 - 2(u_\nu)^2$$

or

$$\int_{\partial B_1} |\nabla u|^2 = 2\int (u_\nu)^2 + \int_{B_1} (n-2)|\nabla u|^2.$$

Substituting in

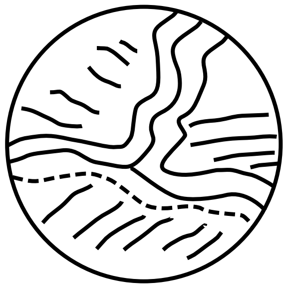
$$(\log F)' = 2\left[\frac{\displaystyle\int_{\partial B_1} (u)^2}{\displaystyle\int_{\partial B_1} u\, u_r} - \frac{\displaystyle\int_{\partial B_1} u\, u_r}{\displaystyle\int_{\partial B_1} u^2}\right] \geq 0.$$

To complete the proof we have to make sense of

$$\int_{B_1} \langle x, \nabla u\rangle \Delta u = 0.$$

We start by separating B_1 into two parts: first one is S_ε, an epsilon neighborhood of $S = \{x : \vec{u}(x) = 0\ \}$, and second one is $G_\varepsilon = B_1 \setminus S_\varepsilon$.

Next we truncate each one of the u_j by taking $u_j^\delta = (u_j - \delta)^+$. Each of the u_j^δ has now separated support



and we apply the previous calculation in each domain D_j^δ which is the interior of $\operatorname{supp} u_j^\delta$. Then we are left with the extra boundary term

$$\int \langle x \cdot \nu \rangle |\nabla u|^2 dA_\delta - 2\langle x, \nabla u\rangle \langle \nabla u, \nu\rangle dA$$

along the analytic surfaces $u_j = \delta$. Since these are level surfaces of u_i, we have that $\langle u_i, \nu\rangle = -|\nabla u_i|$, and, also, $|\nabla u_i| dA = d\mu_{i\delta}$ where $\mu_{i\delta}$ is the primitive measure Δu_i^δ. The integrals above are then equal to

$$\int \langle x, \nu\rangle |\nabla u_i| d\mu_{i\delta} - 2\langle x, \nabla u\rangle d\mu_{i\delta} = -\int \langle x, \nabla u_i\rangle d\mu_{i\delta}.$$

For ε fixed we now let δ go to zero.

Outside of S_ε, we have a sequence of smooth level surfaces and the integrals cancel in the limit. Inside S_ε, $|\nabla u_i| = o(1)$ therefore the integrals inside S_ε are all bounded by

$$(\text{Total mass } \mu_j) \cdot o(1).$$

We then let ε go to zero and the formula is complete. □

5. The singular set

At this point, we have verified all the hypotheses necessary to develop the interphase regularity theory, as in [CL2]. Therefore, we obtain the same final theorem: (Theorem 4.7).

Theorem 16. *The set of interphases $S = \{x : \vec{u}(x) = 0\ \}$ consists of two parts:*

a) *A singular set, $\Sigma = \{|\nabla u_j|^2 = 0\}$ of Hausdorff dimension $n-2$ and*

b) *A family of analytic surfaces, level surfaces of harmonic functions.*

(Note that in our case, the proof of the part b) immediately follows from the "clean up" lemma.)

6. A final remark on the regularity of the ε-system

From the Lipschitz continuity of the limiting solutions we can deduce the following regularity theorem.

Theorem 17. *Let $\vec{u}^\varepsilon(x) = (u_1^\varepsilon(x), \ldots, u_k^\varepsilon(x))$ be a solution of the ε-problem in B_1 of $\mathbb{R}^n$, such that*

$$\|u\| \le 1 \ .$$

Then, for any $\alpha < 1$, and any $1 < p < \infty$, u is in $C^\alpha(B_{1/2})$ and $W^{1,p}(B_{1/2})$ with

$$\|u\|_{C^\alpha} \le C(\alpha) \ ,$$

$$\|u\|_{W^{1,p}} \le C(p)$$

independently of ε.

Proof. The proof follows from the techniques described in [CP], using the following approximation lemma.

Lemma 18. *Given δ, $\exists\ \varepsilon_0 > 0$ so that if $\varepsilon \le \varepsilon_0$, and u^ε is a solution as in Theorem 17 above, there exists a solution u of the limiting problem that satisfies*

$$\|u_\varepsilon - u\|_{L^\infty(B_{3/4})} \le \delta$$

$$\|\nabla(u_\varepsilon - u)\|_{L^2(B_{3/4})} \le \delta \ .$$

Proof. The first bound follows from equicontinuity and compactness. For the L^2 norm estimate we first point out that the total mass

$$\sum_i \int_{B_{3/4}} \Delta u_i^\varepsilon$$

and

$$\int_{B_{3/4}} (\nabla u^\varepsilon)^2$$

are uniformly bounded since

$$\int_{B_{3/4}} \Delta u_i^\varepsilon \le C \int_{B_1} u_i^\varepsilon$$

and the gradient bound follows from Caccioppoli's inequality.

Next, notice that

$$\Delta u^2 = 2(u\Delta u + (\nabla u)^2 \ ).$$

Then for a cut-off function φ we write

$$\int_{B_1} \varphi|\nabla(u^\varepsilon - u)|^2 = -\int \varphi(u-u^\varepsilon)\Delta(u-u^\varepsilon) + \int \varphi\Delta\frac{(u-u^\varepsilon)^2}{2} \ .$$

The first integral on the right-hand side goes to zero since $(u-u^\varepsilon)$ goes to zero uniformly. The second integral, after integration by parts, takes the form

$$\int \Delta\varphi\frac{(u-u^\varepsilon)^2}{2}$$

which goes to zero. □

We will now extend our results to the evolution system

$$(6.1)\qquad \begin{cases} \Delta u_i^\varepsilon - (u_i^\varepsilon)_t = \dfrac{u_i^\varepsilon}{\varepsilon}\displaystyle\sum_{j\neq i} u_j^\varepsilon & \text{in } \Omega\times(-T,0) \\ u_i^\varepsilon(x,0) = f_i(x) & x\in\Omega \\ u_i^\varepsilon(x,t) = h_i(x,t) & \text{on } \partial\Omega\times(-T,0) \end{cases}$$

with $T > 0$. It models a problem from population dynamics: the configuration of competing species which cannot coexist on the same region (competition rate is ∞). We assume that $\partial\Omega$, the initial a boundary data are sufficiently smooth so that for every $\varepsilon > 0$ we have a smooth solution.

More generally one can consider the Fisher's equation: logistic growth equation supplemented by an extra diffusion term Δ

$$\Delta u_i^\varepsilon(x,t) - (u_i^\varepsilon(x,t))_t = \frac{1}{\varepsilon} u_i^\varepsilon(x,t) \sum_{j\neq i} u_j^\varepsilon(x,t) + g_i(x,t,u_\varepsilon^i)$$

where Δu_i^ε is the spatial diffusion, $(u_i^\varepsilon(t,x))_t$ is the instantaneous rate of change of the i-th population's density, $\frac{1}{\varepsilon}u_i^\varepsilon(t,x)\sum_{j\neq i} u_j^\varepsilon(t,x)$ describes the interaction between different species with competition rate $\frac{1}{\varepsilon}$, and $g_i(t,x,u_\varepsilon^i)$ is the growth rate.

As the competition rate $\frac{1}{\varepsilon}$ becomes larger and larger the populations undergo a segregation and this process leads to a final configuration where the populations are separated.

As we pointed out above, the Hölder regularity theory for the elliptic ε-system extends to the parabolic case.

Lemma 19. *If $\|u\|_{L^\infty(B_1)} \le 1$, then $u|_{B_{1/2}} \in C^\alpha$ and $\|u\|_{C^\alpha(B_{1/2})} \le C$ with α and C independent of ε.*

As before we will consider limits u of a convergent sequence of solutions u^ε as ε goes to zero. We start with the Lipschitz regularity of the limit function u.

Lipschitz regularity. Since we have uniform Hölder estimate for u^ε the limit function u is also Hölder continuous. Following the elliptic theory we start by proving the following:

Lemma 20. *Let $u = (u_1, \ldots, u_m)$ be the limit function as $\varepsilon \to 0$, then u_i and u_j have disjoint supports $(i \neq j)$ and $H(u_i) \equiv \Delta u_i - D_t u_i = 0$ on the interior of the support of u_i.*

Proof. Note that

$$H(u_i^\varepsilon) = \Delta u_i^\varepsilon - D_t u_i^\varepsilon = \frac{1}{\varepsilon} u_i^\varepsilon \sum_{k \neq i} u_k^\varepsilon$$

then

$$H(u_1^\varepsilon) \leq \sum_{k \neq 1} H(u_k^\varepsilon) \ .$$

Indeed

$$\sum_{k \neq 1} H(u_k^\varepsilon) = \frac{u_2^\varepsilon}{\varepsilon} \sum_{i \neq 2} u_i^\varepsilon + \frac{u_3^\varepsilon}{\varepsilon} \sum_{i \neq 3} u_i^\varepsilon + \cdots + \frac{u_m^\varepsilon}{\varepsilon} \sum_{i \neq m} u_i^\varepsilon =$$

$$= \frac{u_1^\varepsilon}{\varepsilon} \sum_{k \neq 1} u_k^\varepsilon + \text{ positive terms} \geq$$

$$\geq H(u_1^\varepsilon) \ .$$

Now let us assume that $(x_0, t_0) \in Q = \Omega \times (-T, 0)$ and $u_1^\varepsilon(x_0, t_0) = \alpha_0 > 0$. From Hölder continuity of u_1^ε we conclude that

$$u_1^\varepsilon(x, t) \geq \frac{\alpha_0}{2} \quad \text{in} \quad Q_h(x_0, t_0)$$

with $Q_h(x_0, t_0) = B_h(x_0) \times (t_0 - \frac{h^2}{2}, t_0 + \frac{h^2}{2})$ for some small $h > 0$. Let $\varphi(x)$ be the standard cut-off function of $B_{2h}(x_0)$, $\varphi \equiv 1$ in $B_h(x_0)$. Then in $Q_h = Q_h(x_0, t_0)$

$$\iint_{Q_h} H(u_1^\varepsilon) \leq \iint_{Q_{2h}} \varphi(x) H(u_1^\varepsilon) =$$

$$= \iint_{Q_{2h}} \Delta\varphi(x) u_1^\varepsilon(x, t) + \int_{B_{2h}(x_0)} \varphi(x) \left[u_1^\varepsilon(x, t_0 + 2h^2) - u_1^\varepsilon(x, t_0 - 2h^2) \right]$$

$$\leq C(h).$$

On the other hand, $H(u_1^\varepsilon) = \frac{u_1^\varepsilon}{\varepsilon} \sum_{k \neq 1} u_k^\varepsilon$, and then we have

$$\iint_{Q_h} H(u_1^\varepsilon)\, dx\, dt \leq C(h) \qquad \text{and} \qquad \iint_{Q_h} \frac{u_1^\varepsilon}{\varepsilon} \sum_{k \neq 1} u_k^\varepsilon \geq \frac{\alpha_0}{2\varepsilon} \iint_{Q_h} \sum_{k \neq 1} u_k^\varepsilon.$$

So we conclude that

$$\iint_{Q_h(x_0, t_0)} \sum_{k \neq 1} u_k^\varepsilon \, dx\, dt \leq \frac{2}{\alpha_0} C(h) \varepsilon \ .$$

Since u_k^ε's are subsolutions, this implies that $\sum_{k \neq 1} u_k^\varepsilon \to 0$ uniformly in $Q_{h/2}$. □

To prove that u_i is caloric in the interior of its support we use our observation

$$H(u_1^\varepsilon) \le \sum_{k\neq 1} H(u_k^\varepsilon) \ .$$

Therefore

$$\begin{aligned}\iint_{Q_{h/2}(x_0,t_0)} H(u_1^\varepsilon) &\le \iint_{Q_{h/2}(x_0,t_0)} \sum_{k\neq 1} H(u_k^\varepsilon) \le \\ &\le \sum_{k\neq 1} \left(\iint_{Q_h(x_0,t_0)} \Delta\eta u_k^\varepsilon(x,t) + \int_{B_h(x_0)} \eta(x) \left[u\Big(x, t_0 + \frac{h^2}{2}\Big) - u\Big(x, t_0 - \frac{h^2}{2}\Big)\right] \right. \\ &\le C(h)\varepsilon \ .\end{aligned}$$

Here $\eta(x)$ is the standard cut-off function for $B_h(x_0)$. □

Now we are ready to prove the Lipschitz regularity. We use a parabolic version of the monotonicity formula [CSa].

Theorem 21. *Let* $u = (u_1, \ldots, u_m)$ *be a solution in* Q_1*. Then*

$$\|u\|_{\mathrm{Lip}(Q_{1/2})} \le \|u\|_{L^2(Q_1)}$$

Proof. Recall the monotonicity formula for a pair of disjoint nonnegative subcaloric functions [CSa]: let u_1, u_2 verify

a) $\Delta u_i - D_t u_i \ge 0 \qquad i = 1, 2,$

b) $u_1 u_2 \equiv 0,$

c) $u_1(0,0) = u_2(0,0) = 0.$

Let $\varphi(x)$ be a cut-off function in x, such that $\varphi \equiv 0$ outside $B_{2/3}$ and $\varphi \equiv 1$ in $B_{1/2}$. Define

$$J(t) = J(w_1, w_2, t) = \frac{1}{t^2} \left(\int_{\mathbb{R}^n} \int_{-t}^{0} |\nabla w_1|^2 G(x,-s)\, dx\, ds \times \int_{\mathbb{R}^n} \int_{-t}^{0} |\nabla w_2|^2 G(x,-s)\, dx\, ds \right)$$

where $G(x,t) = \frac{1}{t^{n/2}} e^{-|x|^2/4t}$, $w_i = u_i \varphi$. Then

$$J(0^+) - J(t) \le A e^{-c/t} \|u_1\|^2_{L^2(Q_1)} \|u_2\|_{L^2(Q_1)} \ .$$

We divide the proof of the Lipschitz continuity into several steps. We start by observing that in all the estimates below there are underlying Lipschitz homogeneities.

In the first step we show that $J(t)$ controls the (weighted) product of the L^2 norm of w_i in some strip.

Next we show that, due to the inequality $Hu_1 \le \sum_{j\neq 1} Hu_j$ the w_2 factor controls the w_1 factor implying its boundedness at every scale. Finally we show that this implies spatial Lipschitz continuity for u_1.

Step 1. L^2 bound on w_i.

Let $w(x,t) = u(x,t)\varphi(x)$ then by direct computation

$$H(w^2) = \Delta(w^2) - D_t w^2 = 2w\Delta w + 2|\nabla w|^2 - 2ww_t \ .$$

Hence

$$\begin{aligned} |\nabla w|^2 &= \frac{1}{2}H(w^2) - w\Delta w + ww_t = \\ &= \frac{1}{2}H(w^2) - w[\Delta u\varphi + 2\nabla u\nabla\varphi + u\Delta\varphi] + wu_t\varphi. \end{aligned}$$

Integrating this identity with respect to the measure $d\mu = G(x,-s)\,dx\,ds$ we get

$$\begin{aligned} \int_{-t}^{0}\int_{\mathbb{R}^n} |\nabla w|^2 d\mu \qquad &= \int_{-t}^{0}\int_{\mathbb{R}^n} \frac{1}{2}H(w^2)d\mu - \int_{-t}^{0}\int_{\mathbb{R}^n} w[\Delta u\varphi + 2\nabla u\nabla\varphi + u\Delta\varphi]d\mu \\ &+ \int_{-t}^{0}\int_{\mathbb{R}^n} wu_s\varphi d\mu \ . \end{aligned}$$

Note that

$$\begin{aligned} &\int_{-t}^{0}\int_{\mathbb{R}^n} \frac{1}{2}H(w^2)G(x,-s)\,dx\,ds = \\ &= \frac{1}{2}\int_{-t}^{0}\int_{\mathbb{R}^n} [w^2\Delta G(x,-s) - D_s w^2 G(x,-s)]\,dx\,ds = \\ &= \frac{1}{2}\int_{-t}^{0}\int_{\mathbb{R}^n} w^2[\Delta G(x,-s) + D_s G(x,-s)]\,dx\,ds + \frac{1}{2}\int_{\mathbb{R}^n} w^2(x,-t)G(x,t)\,dx = \\ &= \frac{1}{2}w^2(0,0) + \frac{1}{2}\int_{\mathbb{R}^n} w^2(x,-t)G(x,t)\,dx \end{aligned}$$

since $\Delta G(x,-s) + D_s G(x,-s) = \delta_{0,0}$.

Therefore we conclude that

$$\begin{aligned} \int_{-t}^{0}\int_{\mathbb{R}^n} |\nabla w|^2 G(x,-s)\,dx\,ds = \frac{1}{2}w^2(0,0) + \frac{1}{2}\int_{\mathbb{R}^n} w^2(x,-t)G(x,t)\,dx \\ - \int_{-t}^{0}\int_{\mathbb{R}^n} w\varphi H(u)G(x,-s)\,dx\,ds \\ - \int_{-t}^{0}\int_{\mathbb{R}^n} w[2\nabla u\nabla\varphi + u\Delta\varphi]G(x,-s)\,dx\,ds \ . \end{aligned}$$

Now if $u = u_1$, then $w_1 = u_1 \cdot \varphi$ and

$$\int_{-t}^{0}\int_{\mathbb{R}^n} w\varphi H(u_1)G(x,-s)\,dx\,ds = \int_{-t}^{0}\int_{\mathbb{R}^n} \varphi^2 u_1 H(u_1)G(x,-s)\,dx\,ds = 0$$

since $u_1 H(u_1) = 0$ and $w_1(0,0) = 0$ so

$$I_1(t) = \int_{-t}^{0} \int_{\mathbb{R}^n} |\nabla w_1|^2 G(x,-s)\, dx\, ds =$$

$$= \frac{1}{2} \int_{\mathbb{R}^n} w_1^2(x,-t) G(x,t)\, dx - \int_{-t}^{0} \int_{\mathbb{R}^n} w_1 [2\nabla u_1 \nabla \varphi + u_1 \Delta \varphi] G(x,-s)\, dx\, ds \ .$$

Observe that the last term on the right admits an estimate

$$\int_{-t}^{0} \int_{\mathbb{R}^n} w_1 [2\nabla u_1 \nabla \varphi + u_1 \Delta \varphi] G(x,-s)\, dx\, ds \le$$

$$\le C \int_{-t}^{0} \int_{B_{2/3} \setminus B_{1/2}} |w| (|\nabla u_1| + u_1) G(x,-s)\, dx\, ds \le$$

$$\le C\ e^{-c/t}\ ,$$

where C depends on the L^2 norm of u_1. Now we consider $w_2 = \tilde{u}\varphi$, where $\tilde{u} = \sum_{k\neq 1} u_k$. Note that u_1 and $\sum_{k\neq 1} u_k$ satisfy the assumption of the monotonicity formula.

Next, for w_2 we have

$$I_2(t) = \int_{-t}^{0} \int_{\mathbb{R}^n} |\nabla w_2|^2 G(x,-s)\, dx\, ds =$$

$$= \frac{1}{2} \int_{\mathbb{R}^n} w_2^2 G(x,t)\, dx - \int_{-t}^{0} \int_{\mathbb{R}^n} w_2 \varphi H(\tilde{u}) G(x,-s)\, dx\, ds$$

$$- \int_{-t}^{0} \int_{\mathbb{R}^n} w_2 [w \nabla \tilde{u} \nabla \varphi + \tilde{u} \Delta \varphi] G(x,-s)\, dx\, ds\ .$$

If at (x,t) we have that $u_2(x,t) > 0$, then $Hu_2(x,t) = 0$, and since u_k's have disjoint supports

$$\tilde{u} H(\tilde{u}) = 0\ .$$

If (x,t) is a free boundary point, then $\tilde{u}(x,t) = 0$. Hence

$$\int_{-t}^{0} \int_{\mathbb{R}^n} \varphi^2 \tilde{u} H(\tilde{u}) G(x,-s)\, dx\, ds = 0$$

and as in the case of w_1,

$$\int_{-t}^{0} \int_{\mathbb{R}^n} w_2 [2\nabla \tilde{u} \nabla \varphi + \tilde{u} \Delta \varphi] G(x,-s)\, dx\, ds \le C\ e^{-c/t}\ .$$

Combining these estimates for I_1 and I_2 we have

$$J(t) = \frac{1}{t^2} I_1(t) I_2(t) \ge \frac{1}{t^2} \left(\frac{1}{2} \int_{\mathbb{R}^n} w_1^2(x,-t) G(x,t)\, dx + O(e^{-c/t}) \right) \times$$

$$\times \left(\frac{1}{2} \int_{\mathbb{R}^n} w_2^2(x,-t) G(x,t)\, dx + O(e^{-c/t}) \right).$$

This means that

$$\frac{1}{4t^2}\int_{\mathbb{R}^n} w_1^2(x,-t)G(x,t)\,dx \int_{\mathbb{R}^n} w_2^2(x,-t)G(x,t)\,dx \le J(t) + O(e^{-c/t}).$$

Step 2. Next we want to show that the w_1-term is controlled by the w_2-term. Recall that $H(u_1) \le H(\tilde{u})$ so

$$\begin{aligned}
0 &\le \int_{-t}^{0}\int_{\mathbb{R}^n} H(\tilde{u}-u_1)\varphi G(x,-s)\,dx\,ds \\
&= \int_{-t}^{0}\int_{\mathbb{R}^n} (\tilde{u}-u_1)\Delta[\varphi G(x,-s)]\,dx\,ds - \int_{-t}^{0}\int_{\mathbb{R}^n} (\tilde{u}-u_1)_s \varphi G(x_1-s)\,dx\,ds = \\
&= \int_{-t}^{0}\int_{\mathbb{R}^n} (\tilde{u}-u_1)\left[\Delta\big(\varphi G(x_1-s)\big) + D_s\big(\varphi G(x_1-s)\big)\right]\,dx\,ds \\
&\quad + \int_{\mathbb{R}^n} (\tilde{u}-u_1)\varphi(x,-t)G(x,t)\,dx \\
&= \int_{-t}^{0}\int_{\mathbb{R}^n} (\tilde{u}-u_1)\left[\Delta\varphi G(x,-s) + 2\nabla\varphi\nabla G(x,-s) + \varphi\Delta G(x,-s) + \varphi(x)D_sG(x_1-s)\right]\,dx\,ds \\
&\quad + \int_{\mathbb{R}^n} (w_2(x,-t)-w_1(x,-t))G(x,t)\,dx\,ds \\
&= \int_{-t}^{0}\int_{\mathbb{R}^n} (\tilde{u}-u_1)\left[\Delta\varphi + \nabla\varphi\cdot\frac{x}{t}\right]G(x,-s)\,dx\,ds + \int_{\mathbb{R}^n} (w_2(x,-t)-w_1(x,-t))G(x,t)\,dx \\
&= O(e^{-c/t}) + \sqrt{t}(\theta_2(t)-\theta_1(t))
\end{aligned}$$

where

$$\theta_i(t) = \frac{1}{\sqrt{t}}\int_{\mathbb{R}^n} w_i(x,-t)G(x,t)\,dx\ , \qquad i=1,2\ .$$

Therefore

$$\theta_1(t) \le \theta_2(t) + O(e^{-c/t}).$$

then we have that after applying the Cauchy Schwartz inequality

$$\theta_1(t) \le \left(\frac{1}{t}\int_{\mathbb{R}^n} w_1^2(x,-t)G(x,t)\,dx\right)^{1/2},$$

$$\theta_1(t) \le \theta_2(t) \le \left(\frac{1}{t}\int_{\mathbb{R}^n} w_2^2(x,-t)G(x,t)\,dx\right)^{1/2}.$$

Multiplying both inequalities we get

$$\theta_1^4(t) \le \frac{1}{t^2}\int_{\mathbb{R}^n} w_1^2(x,-t)G(x,t)\,dx \int_{\mathbb{R}^n} w_2^2(x,-t)G(x,t)\,dx \le 4(J(t) + O(e^{-c/t}))\ .$$

Therefore the monotonicity formula theorem implies that $\theta_1(t)$ is bounded for any t small.

Step 3. Since the heat equation is translation invariant, we can extend the previous estimate to any free boundary point $(x_0,-t_0) \in Q$ with $t_0 > 0$. For $\rho_0 > 0$ we let $B_{\rho_0}(x_0) \times (-t_0-\rho_0, -t_0) \subset Q$. ρ_0

depends only on the distance of $(x_0, -t_0)$ from the parabolic boundary of Q. Then we let $\eta = x - x_0, \tau = t + t_0$ and consider $v_i(\eta, \tau) = u_i(x_0 + \eta, \tau - t_0)$ is also a solution. Taking $t = r^2$ in the definition of $\theta_i(t), t > 0$ and using a change of variables $x = ry$ we have that

$$\frac{1}{r}\int_{\mathbb{R}^n} u_1(x_0 + yr, -t_0 - r^2)\varphi(x_0 + yr)G(y, 1) \leq C_0 \tag{6.2}$$

for any point $(x_0, -t_0)$ such that $\text{dist}((x_0, -t_0), \partial_p Q) \geq \rho_0$ and C_0 depends on ρ_0.

Next we want to show that u grows away from the free boundary linearly. Assume that $(x_1, -t_1) \in Q, t_1 > 0, u_1(x_1, -t_1) > 0$ and let ρ be the distance of $(x_1, -t_1)$ from the free boundary. Hence u_1 is caloric in $Q_1 = B_{\frac{\rho}{2}}(x_1) \times (-t_1 - \frac{\rho^2}{4}, -t_1)$. Suppose that for some x_2 we have that

$$u_1(x_2, -t_1) \geq MR$$

with $R = \frac{\rho}{2}$ and $M \gg 1$.

By Harnack inequality

$$\begin{aligned} \inf_{B_R(x_1)\times(-t_1-\frac{3R^2}{4}, -t_0-\frac{R^2}{2})} u_1 &\geq C_1 \sup_{B_R(x_1)\times(-t_1-\frac{R^2}{4}, -t_0)} u_1 \\ &\geq C_1 RM. \end{aligned}$$

Thus taking $r = 4R = 2\rho$ in (6.2) we obtain for every s

$$C_0 \geq \int_{\mathbb{R}^n} \frac{u_1(x_0 + 4Ry, -t_1 - \frac{R^2}{2} - (4R)^2)}{4R} G(y, 1)dy \geq c(n)MC_1 \tag{6.3}$$

which is a contradiction if $M > \frac{C_0}{c(n)C_1}$.

□

Theorem 22. *$u(x, t)$ is locally Lipschitz in the parabolic distance.*

Proof. It is a standard argument to show that the Lipschitz continuity in space implies $\frac{1}{2}$-Hölder continuity in time. □

7. The Clean-Up Lemma

We start by pointing out that in a "clean" neighborhood of a free boundary point, $u_1 - u_2$ is caloric.

Lemma 23. *If $\sum_{j>2} u_j \equiv 0$ in some cylinder $Q_\rho(x_0, t_0)$ then $u_1 - u_2$ is caloric in $Q_\rho(x_0, t_0)$.*

Proof. Since

$$H(u_1) \leq H\Big(\sum_{k\neq 1} u_k\Big) = H(u_2) + H\Big(\sum_{k>2} u_k\Big)$$

and

$$H(u_2) \leq H\Big(\sum_{k\neq 2} u_k\Big) = H(u_1) + H\Big(\sum_{k>2} u_k\Big)$$

it follows that $u_1 - u_2$ is caloric in $Q_\rho(x_0, t_0)$. □

Next we have the parabolic "clean-up" lemma, which plays a crucial role in the classification of singular points of the free boundary. It basically says that if at some free boundary point (x_0, t_0), $J(0^+) > 0$. That is $|\nabla u(x_0, t_0)| \neq 0$. Then at some neighborhood of (x_0, t_0) we have exactly two *phases.*

Clean-Up Lemma. *Assume that at* (x_0, t_0)

$$J(0^+) = \lim_{t\to 0^+} J(t) = \lambda > 0.$$

Then in a neighborhood of (x_0, t_0) $\sum_{j>2} u_j \equiv 0$.

First recall the following result [CSa].

Lemma 24. (See [CSa].) *Assume that at* (x_0, t_0)

$$J(u_1, u_2, 0) = \lim_{t\to 0} J(u_1, u_2, t) = \alpha_0 > 0.$$

Then

a) *any convergent sequence of dilations*

$$\frac{1}{\lambda_k} u(\lambda_k x, \lambda_k t) \ , \quad \textit{for} \ \ \lambda_k \to 0$$

converges to

$$\bar{u}_1 = \alpha_1 x_1^+ \ , \quad \bar{u}_2 = \alpha_2 x_1^- \ , \quad \bar{u}_j \equiv 0 \ \ \textit{for} \ \ j > 2.$$

b) *Further* $(\bar{u}_1 - \bar{u}_2)$ *must verify the heat equation. So*

$$\alpha_1 = \alpha_2 = (\alpha_0)^{1/4} \ .$$

In this circumstance, the "clean-up" lemma says that the components u_j, for $j \neq 1, 2$ decay faster than u_1, u_2 and vanish before reaching (x_0, t_0).

Theorem 25. *Let* u_1, u_2 *be as in lemma above. Then, in a neighborhood of* (x_0, t_0), $\sum_{j>2} u_j \equiv 0$.

Before going into the proof, we need some preliminaries. After a large dilation, we can start with a configuration satisfying the following hypothesis.

Let $\bar{u}_i, i = 1, 2$ be the λ-dilatation of u_i at the origin, i.e. $\bar{u}_i(x,t) = u_i(\lambda x, \lambda t)/\lambda$ and let us write $\bar{u}_1 - \bar{u}_2$ as

$$\bar{u}_1 - \bar{u}_2 = v_0 + v_1$$

with v_0 caloric, $v_0|_{\partial B_1} = \bar{u}_1 - \bar{u}_2$, and v_1 is the part that comes from the presence of u_j, $j \neq 1, 2$ and it is supposed to be small.

From the previous lemma, we may renormalize $\alpha_0 = 1$, and assume that

$$|(\bar{u}_1 - \bar{u}_2) - x_1| \leq h, \tag{7.1}$$

in particular $\text{supp}_{j \neq 1,2}\, \bar{u}_j \subset |\{|x_1| \leq h\}|$, and $\bar{u}_j$ being Lipschitz $\bar{u}_j \leq h$.

We also recall a decay property of harmonic functions in narrow domains.

Lemma 26. *Let w be continuous in $\mathcal{C}_1 = B_1 \times [-1, 1]$, supported in $\Omega \subset \mathcal{C}_1$ and harmonic in its support. Assume that Ω is "narrow" in the sense that any cylinder $Q_h = B(x_0) \times (t_0 - h^2, t_0)$, contained in $\mathcal{C}_1$, intersects $\mathcal{C}\Omega$, say, half of the time, i.e.,*

$$\frac{|Q_h \cap \mathcal{C}\Omega|}{|Q_h|} > \frac{1}{2} \ .$$

Then

$$w(x) \leq \sup_{\partial_p \mathcal{C}_1} w \cdot e^{-C\frac{(1-\sqrt{|x|^2+t)}}{h}} \ .$$

Proof. We prove that in $Q_{i,k} = Q_h(x_i, -1 + 2kh),\ k = 1, 2, \ldots, N, x_i \in h\mathbb{Z}^2 \cap \mathcal{C}_1$, where $N \sim h^{-2}$, we have that

$$w(x) \leq \frac{1}{C} \sup_{Q_{ik}} w$$

for some $C > 1$. Indeed, by a density estimate we have that

$$\sup_{Q_{i,k}} w(x,t) \leq \frac{1}{1 + c_0 \frac{|\mathcal{C}\Omega \cap Q_{i,k-1}|}{|Q_{i,k-1}|}} \sup_{Q_{i,k-1}} w.$$

But $w \equiv 0$ "half of the time " hence $\frac{|\mathcal{C}\Omega \cap Q_{i,k-1}|}{|Q_{i,k-1}|} \geq 1/2$. Repeating this for all i, k and combining the estimates the result follows. □

Now we start the proof of the parabolic clean-up lemma.

Proof. From the proof of the monotonicity formula [CSa], we have that the blow-up functions are a pair of linear functions, and from the $H(u)$ inequalities they have the same slope. This means that near (x_0, t_0) we have uniform flatness at every scale.

As in the elliptic case we want to start with a suitable inductive hypotheses.

In fact the iterative scheme is the same as in the elliptic case. Start with h_1 small. Consider the inductive sequence $h_k = (h_{k-1})^2$ (that converges to zero very fast) and the sequence

$$r_k, \quad \text{with} \quad r_1 = 1, \quad r_{k+1} = r_k - h_k^{1/2}$$

that converges to $1 - \mu$ with $\mu \leq 1/2$ if h_1 is small.

More precisely we can state

Lemma 27. *In* $\mathcal{C}_{r_k} = B_{r_k} \times (-1 + h_k^{1/2}, 1 - h_k^{1/2})$ *there is a caloric function* v_k *such that*

a) $|v_k - (u_1 - u_2)| \leq h_k$

b) $|\nabla(v_k - v_{k-1})| \leq h_k^{1/2}$

c) $|\nabla v_k - e_1| \leq \sum_1^k (h_\ell)^{1/2} \leq \frac{1}{4}$

d) *The level surface* $v_k = 0$ *is Lipschitz with Lipschitz constant less than one for every* k.

To prove this we proceed as follows. First from the exponential decay we can estimate $\tilde{v}_0 - (u_1 - u_2)$ in the cylinder of size $\mathcal{C}_{1-s}$. Next using the covering argument and computation from the previous section one can estimate the size of $\Delta u - u_t$ in C_{1-s} and then from Green's representation theorem we get that v_1 decays as h^{2n-1} away from $h^{\frac{1}{2n}}$ neighborhood of $\operatorname{supp} u_j, j > 2$. Finally, using gradient estimates we conclude that $|\nabla \tilde{v}_0 - \nabla v_0| \leq h^{1/2}$.

As in the elliptic theory, we have now a discontinuity. At the neighborhood of a clean point the free boundary is a transversal level surface of a caloric function. At a singular point the gradient of u goes to zero, and we want to classify such points.

8. Almgren's formula

Lemma 28. $(\nabla u)^2$ *is a continuous function across the interphase.*

Proof. If $J_0(x_0, t_0) = \lim_{t+0} J_t(x_0, t_0) \neq 0$, from the clean up lemma $(u_1 - u_2)$ is harmonic.

If $J_0(x_0)$ is zero for every pair, then $|\nabla u(x,t)|^2$ goes to zero as x goes to x_0, which follows from the estimates of $\theta_i(t), i = 1, 2$. □

We consider now the backward heat equation

$$\Delta u + u_t = 0 \text{ in } \mathbb{R}^{n+1}_+$$

For $t_0 > 0$, we define

$$H(t) = \int_{\mathbb{R}^n} |u(x,t)|^2 G(x,t)\, dx$$

where

$$G(x,t) = \frac{1}{(t+t_0)^{n/2}} e^{-\frac{|x|^2}{4(t+t_0)}} \text{ and } u = (u_1, u_2, \ldots, u_m).$$

Also

$$D(t) = \int_{\mathbb{R}^n} |\nabla u(x,t)|^2 G(x,t)\, dx.$$

Theorem 29 (Parabolic Almgren monotonicity formula)**.**

$$N(t) = \frac{(t+t_0)D(t)}{H(t)} \quad \textit{is monotone decreasing.}$$

Proof. A version of this theorem is due to [EFV] for the caloric functions. For completeness we give a proof in the Section 11 with the modification for our particular case. □

We are now in the following situation. Our solutions are only local and it is well-known that solutions of the heat equation in $B_1 \times (0, \infty)$ with suitable non-homogeneous time dependent boundary data prescribed on the lateral boundary $\partial B_1 \times (0, \infty)$ solution may become identically zero for $t \geq T$. We would like to prove the following: given a free boundary point, unless our solution is identically zero in a cylinder backwards in time (i.e. had already became identically zero all the way to the boundary) it is forced to have a polynomial decay at the point, so that we can "blow it up" to a nontrivial solution integrable at infinity against the Gaussian kernel.

We can ensure this by a modification to our setting of a theorem of L. Escauriaza, F.J. Fernandez, S. Vessella.

Theorem 30. *Let $(u_1, u_2, \dots, u_m)$ be a solution. Then there exists a constant C such that*

$$\int_{Q_{2r}} u^2 \le C \int_{Q_r} u^2.$$

This estimate is proved in [EFV] for a class of constant coefficient parabolic equations. The main ingredient of the proof is based upon a localization of Almgren's formula by multiplying u with a cut-off function. Since in our case $N(t)$ is a monotone function and all computations for derivatives of $D(t)$ and $H(t)$ remain valid, the doubling property of the solution now immediately follows from the proof of Theorem 29 and [EFV].

9. Classification of the global solutions

If $N(t) = \lambda$ for all t and $\lambda > 0$, then from the proof of monotonicity formula we get that

$$u_t + \frac{x - x_0}{2(t + t_0)} \nabla u = c(t) u(x,t),$$

for some unknown function $c(t)$. We want to show that $c(t)$ is the homogeneity degree of u. Without loss of generality we may assume that $x_0 = 0$, $t_0 = 0$, then we have

$$u_t(x,t) + \frac{x}{2t} \nabla u = c(t) u(x,t).$$

For $\theta > 0$ we consider $u_\theta(x,t) = u(x\theta, t\theta^2)$, then

$$\frac{d}{d\theta} u_\theta = u_t(x\theta_1 + \theta^2) 2\theta t + \nabla(x\theta, t\theta^2) x.$$

u_θ satisfies to a differential equation on the path $(x\theta, t\theta^2)$ for fixed (x,t). Indeed

$$u_t(x\theta, t\theta^2) + \frac{x\theta}{2t\theta^2} \nabla u(x\theta, t\theta^2) = c(t\theta^2) u(x\theta, t\theta^2)$$
$$2t\theta\ u_t(x\theta, t\theta^2) + x \cdot \nabla u(x\theta, t\theta^2) = 2t\theta c(t\theta^2) u_\theta(x, \theta)$$

or

$$\frac{d}{d\theta} u_\theta = \frac{H(t\theta^2)}{\theta} u_\theta(x,t)$$

where $H(t\theta^2) = 2c(t\theta^2)t\theta^2$. Hence

$$\log u_\theta \big|_1^\theta = \int_1^\theta \frac{H(t\sigma^2)}{\sigma}\, d\sigma$$

and

$$u_\theta(x,t) = e^{\int_1^\theta \frac{H(t\sigma^2)}{\sigma} d\sigma} u(x,t).$$

Since $u_\theta(x,t)$ satisfies to the backward heat equation we get that

$$0 = H(u_\theta) = H(u) \cdot e^{\int_1^\theta \frac{H(t\sigma^2)}{\sigma} d\sigma} + u(x,t)\frac{d}{dt}e^{\int_1^\theta \frac{H(t\sigma^2)}{\sigma} d\sigma} .$$

Therefore $\int_1^\theta \frac{H(t\sigma^2)}{\sigma} d\sigma = c(\theta)$ does not depend on t. Differentiating this equality with respect to t we get

$$0 = \int_1^\theta H'(t\sigma^2)\sigma \, d\sigma = \frac{H(t\theta^2) - H(t)}{2t}$$

so H is a constant. Recall that $H(s) = 2c(s)s$ implying that

$$c(s) = \frac{\alpha}{2s} ,$$

where α is a constant therefore u satisfies

$$u_t + \frac{x}{2t}\nabla u \cdot x = \frac{\alpha}{2t} u.$$

Thus we conclude that

$$u(x\theta, t\theta^2) = \theta^\alpha u(x,t) , \qquad u = (u_1, u_2, \ldots, u_k)$$

that is u is homogeneous of degree α on the paths $(x\theta, t\theta^2)$.

Since u is homogeneous we can seek the solution u in the following form $t^{\alpha/2} f(x/\sqrt{t})$. In particular it can be a traveling wave $u(x,t) = (x^2 + ct)^{\alpha/2}$.

Consider

$$u(x,t) = t^{\alpha/2} f\left(\frac{x}{\sqrt{t}}\right).$$

Then

$$\begin{aligned} u_t &= \frac{\alpha}{2} t^{\frac{\alpha}{2}-1} f\left(\frac{x}{\sqrt{t}}\right) + t^{\alpha/2} \nabla f\left(\frac{x}{\sqrt{t}}\right)\left(-\frac{x}{2t^{3/2}}\right), \\ u_x &= t^{\alpha/2-1/2} f_x\left(\frac{x}{\sqrt{t}}\right), \\ u_{xx} &= t^{\frac{\alpha}{2}-1} f_{xx}\left(\frac{x}{\sqrt{t}}\right). \end{aligned}$$

Hence plugging these into the backward heat equation $\Delta u + u_t = 0$ we obtain

$$-\Delta f(z) + \frac{1}{2}\nabla f(z) \cdot z = \frac{\alpha}{2} f(z)$$

where $z = x/\sqrt{t}$. Therefore $\alpha/2$ is the eigenvalue of the operator $-\Delta + \frac{1}{2}\nabla \cdot z$ and u is the corresponding eigenfunction.

For the one dimensional case f satisfies an ODE

$$2f_{zz} - f_z z + \alpha f = 0.$$

Setting $w(z) = f(2z)$ one can easily verify that w solves $w_{zz} - 2w_z z + 2\alpha w = 0$. But the latter is the differential equation for the Hermite polynomials which can be explicitly given by

$$w(z) = \alpha! \sum_{k=0}^{[\frac{\alpha}{2}]} \frac{(-1)^k (2z)^{\alpha-2k}}{k!(n-2k)!}.$$

Hence returning to f we obtain

$$\begin{aligned} u(x,t) &= \alpha! t^{\alpha/2} f(\frac{x}{\sqrt{t}}) = \alpha! t^{\alpha/2} \sum_{k=0}^{[\frac{\alpha}{2}]} \frac{(-1)^k}{k!(n-2k)!} (\frac{x}{\sqrt{t}})^{\alpha-2k} \\ &= \alpha! \sum_{k=0}^{[\frac{\alpha}{2}]} \frac{(-1)^k}{k!(n-2k)!} x^{\alpha-2k} t^k \end{aligned} \tag{9.1}$$

which is the α-caloric polynomial for the backwards heat equation. Now if one has the heat equation (i.e. after replacing t with $-t$) then

$$h_m(x,s) = m! \sum_{k=0}^{[\frac{m}{2}]} \frac{1}{k!(n-2k)!} x^{m-2k} t^k$$

is the solution for our problem in the one dimensional case.

In n dimensions, $h_{m_1}(x_1, s) h_{m_2}(x_2, s) \cdots h_{m_n}(x_n, s)$, $\sum_{j=1}^n m_j = m, m_j \geq 0$ is the homogeneous solution of degree m of our problem. By the classical theory of Hermite polynomials they have only simple real zeros. Hence the polynomial $h_m(x, -1)$ has m simple zeros. Furthermore, $h_m(x, s)$ is even or odd in the variable x when m is an even or odd integer, respectively. Therefore we can describe the nature of the nodal sets of h_m in spacetime

$$\Sigma(h_m) = \{(x,s), h_m(x,s) = 0\}.$$

First notice that $h_0(x,s) \equiv 1$ so $\Sigma(h_0) = \emptyset$, $h_1(x,s) = x$ and $\Sigma(h_1)$ is the t-axis. Hence h_m has a degenerate zero if and only if $m \geq 2$.

10. Structure of the singular set

In this section we establish an estimate for the parabolic Hausdorff dimension of the set $\Sigma = \{(x,t), u(x,t) = 0, |\nabla u(x,t)| = 0\}$.

Theorem 31. *Let $\mathcal{P}$ be the parabolic Hausdorff measure. Then the parabolic Hausdorff dimension* $dim_{\mathcal{P}}\Sigma[u] \leq n$.

For the definition of $\mathcal{P}$ see [L2]. The proof is based on Federer's dimension reduction argument. We sketch it here. Let $\mathcal{F}$ be the set of all solutions and take $u \in \mathcal{F}$ and let $\mathcal{S} : \mathcal{F} \mapsto \mathcal{C}$, where $\mathcal{S}$ is the singular map, $\mathcal{S}(u) = \Sigma$, and $\mathcal{C}$ is the collection of all closed sets in $\mathbb{R}^n \times \mathbb{R}$. First notice that the following hypotheses are satisfied (see [L2] page 51)

- H1 $\mathcal{F}$ is closed under translation and scaling
- H2 Existence of homogeneous degree zero tangent functions
- H3 Singular set hypothesis i.e. the existence of mapping Σ.

If H1-H2 are satisfied then the pair $(\mathcal{F}, \mathcal{S})$ is locally asymptotically self-similar.

It is easy to see that H1 is satisfied. Next notice that from Almgren's theorem and nondegeneracy (polynomial growth from below) the scaled function $\lambda^{-N}u(\lambda x, \lambda^2 t)$ converges to a caloric polynomial by our classification of the global profiles. Here N is an positive integer. Finally H3 is satisfied in view of the local regularity of u. Hence the dimension reduction theorem applies (see [Ch] theorem 2.3) and we conclude that the parabolic Hausdorff dimension of Σ is smaller than n. Furthermore it also implies that

$$\dim_{\mathcal{H}}\{x \in \Omega, |\nabla u(x,t)| = 0\} \leq n - 2.$$

11. Proof of Almgren's formula

Here we present the proof of Almgren's monotonicity formula which works in our setting.

Recall that $G_t = \Delta G$ and

$$\nabla G = -\frac{x}{2(t+t_0)} G \ .$$

Compute

$$
\begin{aligned}
\frac{d}{dt}H(t) &= \int_{\mathbb{R}^n} \frac{d}{dt}\left[\sum_{k=1}^{m} u_k^2(x,t)G(x,t)\,dx\right] = \\
&= \int_{\mathbb{R}^n} 2uu_tG(x,t) + u^2G_t\,dx = \\
&= \int_{\mathbb{R}^n} 2uu_tG(x,t) + u^2\Delta G\,dx = \\
&= \int_{\mathbb{R}^n} 2uu_tG - 2u\nabla u\cdot\nabla G\,dx = \\
&= \int_{\mathbb{R}^n} 2u\left[u_t + \frac{\nabla u}{2(t+t_0)}\right]G(x,t)\,dx\ .
\end{aligned}
\tag{11.1}
$$

Next we transform $D(t)$,

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\nabla u_i|^2G\,dx = \int_{\mathbb{R}^n} \nabla u_i\nabla u_iG\,dx &= \\
= -\int_{\mathbb{R}^n} [u_i\Delta u_iG + \nabla u_i\nabla G]\,dx &= \\
= \int_{\mathbb{R}^n} u_i\left[u_t + \frac{x\cdot\nabla u_i}{2(t+t_0)}\right]G\,dx\ .&
\end{aligned}
\tag{11.2}
$$

Summing up with respect to all $i = 1, 2, \ldots, m$ we get

$$
D(t) = \int_{\mathbb{R}^n} u\left[u_t + \frac{\nabla u\cdot x}{2(t+t_0)}\right]G\,dx
$$

where

$$
u\nabla u\cdot x = \sum_{i,j} u_iD_ju_ix_j\ .
$$

Finally to compute $\frac{d}{dt}(D(t))$ we use the *Rellich-Nečas identity*

$$
\operatorname{div}(\nabla G(\nabla u_i)^2) - 2\operatorname{div}((\nabla u_i\cdot\nabla G)\cdot\nabla u_i) = \Delta G|\nabla u_i|^2 - 2(\nabla^2G\nabla u_i)\cdot\nabla u_i - 2\nabla u_i\cdot\nabla G\Delta u_i\ .
$$

Hence after integration

$$
\begin{aligned}
\int_{\mathbb{R}^n} \Delta G|\nabla u_i|^2 &= 2\int_{\mathbb{R}^n} (\nabla^2G\nabla u_i)\nabla u_i + 2\int_{\mathbb{R}^n} \nabla u_i\nabla G\Delta u_i \\
&= 2\int_{\mathbb{R}^n} \left(\left[-\frac{\mathrm{Id}}{2(t+t_0)} + \frac{x\otimes x}{4(t+t_0)^2}\right]\nabla u_i\right)\cdot\nabla u_iG + 2\int_{\mathbb{R}^n} \nabla u_i\nabla G\Delta u_i \\
&= 2\int_{\mathbb{R}^n} \left(\frac{\nabla u_i\cdot x}{2(t+t_0)}\right)^2 G - \frac{1}{(t+t_0)}\int_{\mathbb{R}^n} |\nabla u_i|^2\cdot G - \\
&\quad - 2\int_{\mathbb{R}^n} \frac{\nabla u_i\cdot x}{2(t+t_0)}\Delta u_iG\,dx \qquad i = 1, 2, \ldots, m\ .
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\frac{d}{dt}D(t) &= \int_{\mathbb{R}^n} \frac{d}{dt}\left[\left(\sum D_j u_i\right)^2 G\right] dx \\
&= \int_{\mathbb{R}^n} \frac{d}{dt}\left(\sum_{i=1}^m |\nabla u_i|^2 G\right) dx = \\
&= \sum_{i=1}^m \int_{\mathbb{R}^n} \frac{d}{dt}\left(|\nabla u_i|^2 G\right) dx = \\
&= \sum_{i=1}^m \int_{\mathbb{R}^n} \left[2\nabla u_i \nabla (u_i)_t G + |\nabla u_i|^2 G_t\right] dx \\
&= \sum_{i=1}^m \int_{\mathbb{R}^n} 2\left[-(u_i)_t(\Delta u_i \cdot G + \nabla u_i \nabla G)\right] + |\nabla u_i|^2 G_t \, dx \\
&= \sum_{i=1}^m \int_{\mathbb{R}^n} \left(2\sum\left[((u_i)_t)^2 + (u_i)_t \frac{\nabla u_i \cdot x}{2(t+t_0)}\right] G + |\nabla u_i|^2 \Delta G\right) dx \\
&= \sum_{i=1}^m \left(\int_{\mathbb{R}^n} 2\left[(u_i)_t^2 + (u_i)_t \frac{\nabla u_i \cdot x}{2(t+t_0)}\right] G + \right. \\
&\qquad + 2\int_{\mathbb{R}^n}\left(\frac{\nabla u_i \cdot x}{2(t+t_0)}\right)^2 G - \frac{1}{(t+t_0)}\int_{\mathbb{R}^n} |\nabla u_i|^2 G \\
&\qquad \left. + 2\int_{\mathbb{R}^n} \frac{\nabla u_i \cdot x}{2(t+t_0)} (u_i)_t G \, dx\right) \\
&= 2\sum_{i=1}^m \int_{\mathbb{R}^n}\left[(u_i)_t + \frac{\nabla u_i \cdot x}{2(t+t_0)}\right]^2 G - \frac{1}{(t+t_0)} D(t) \ .
\end{aligned}
$$

Combining all these computations we have

$$
\begin{aligned}
\frac{d}{dt}N(t) &= \frac{D(t)}{H(t)} + \frac{(t+t_0)\frac{d}{dt}D(t)}{H(t)} - \frac{(t+t_0)D(t)\frac{d}{dt}H(t)}{H^2(t)} \\
&= \frac{t+t_0}{H^2(t)}\left[\frac{D(t)H(t)}{t+t_0} + H(t)\frac{d}{dt}D(t) - D(t)\frac{d}{dt}H(t)\right] \\
&= \frac{t+t_0}{H^2(t)}\left[\left(2\sum_{i=1}^m \int_{\mathbb{R}^n}\left[(u_i)_t + \frac{\nabla u_i \cdot x}{2(t+t_0)}\right]^2 - \frac{d}{dt}D(t)\right) H(t)\right. \\
&\qquad \left. + H(t)\frac{d}{dt}D(t) - D(t)\cdot 2\int_{\mathbb{R}^n} u\left[u_t + \frac{Du \cdot x}{2(t+t_0)}\right] G \, dx\right] \\
&= 2\frac{(t+t_0)}{H^2(t)}\left[\int_{\mathbb{R}^n} u^2 G \sum_{i=1}^m \int_{\mathbb{R}^n}\left[(u_i)_t + \frac{Du_i \cdot x}{2(t+t_0)}\right]^2 - \left(\int_{\mathbb{R}^n} u\left[u_t + \frac{\nabla u \cdot x}{2(t+t_0)}\right] G\right)^2\right]
\end{aligned}
$$

where the last line follows from a simple observation that

$$D(t) = \sum_{i=1}^{m} \int_{\mathbb{R}^n} |\nabla u_i|^2 G = \int_{\mathbb{R}^n} u \cdot \left[u_t + \frac{\nabla u \cdot x}{2(t+t_0)} \right] G \, dx \ .$$

Then from the Cauchy-Schwarz inequality we have that $N'(t) \geq 0$. □

Remark. It is important to point out that if $N(t) = const.$ then $u_t + \frac{\nabla u \cdot x}{2(t+t_0)} = c(t) u$. As we showed earlier, that $c(t)$ is, in fact, the degree of homogeneity.

Theorem 32. *Assume that* $u = (u_1, \dots, u_m)$ *is the solution to our free boundary problem. Then* $N(t)$ *is nondecreasing.*

Proof. Let's look back to those parts of the previous computations which contain integration by parts. Let S be the zero set of ∇u, and S_ε its ε-neighborhood. Furthermore let $\delta > 0$ and

$$u_i^\delta = (u_i - \delta)^+.$$

In equations (11.1) and (11.2) after integration by parts we have to deal with the following term

$$\int_{\mathbb{R}^n} u_i^\delta \left(\Delta u_i^\delta + (u_i^\delta)_t \right) G dx + \underbrace{\int u_i^\delta \nabla u_i^\delta \cdot G \, dA_{i,\delta}}_{\textit{boundary term}}$$

where $dA_{i,\delta}$ is the area measure on the δ-level surface of u_i. Both terms are well-defined and go to zero as $\delta \to 0$.

The next term that we have to deal with comes from the Rellich-Nečas *identity.* More precisely it consists of two parts:

$$I_1 = \int_{\mathbb{R}^n} \nabla u_i^\delta \cdot x \left[\Delta u_i^\delta + (u_i^\delta)_t \right] G$$

$$I_2 = \int \nabla G \cdot |\nabla u_i^\delta|^2 \cdot \vec{n} \, dA_{i,\delta} - 2 \int (\nabla u_i \cdot \nabla G) \cdot \nabla u_i \cdot \vec{n} \ dA_{i,\delta}$$

where $\vec{n}$ is the unit exterior normal to δ-level surface of u_i. Finally, we need to deal with the following term:

$$\int_{\mathbb{R}^n} \nabla u_i^\delta \nabla (u_i^\delta)_t G = \int (u_i^\delta)_t \nabla u_i^\delta G \, dA_{i,\delta} - \int_{\mathbb{R}^n} (u_i^\delta)_t (\Delta u_i^\delta G + \nabla u_i^\delta \nabla G).$$

We thus need to estimate

$$I_3 = \int (u_i^\delta)_t \nabla u_i^\delta G \, \vec{n} dA_{i,\delta} - \int_{\mathbb{R}^n} (u_i^\delta)_t (\Delta u_i^\delta + (u_i^\delta)_t) \cdot G.$$

□

These are the all "bad" terms that we are left with. First let us observe that on the boundary of $\Omega_t^\delta = \{u_i(x,t) > \delta\}$, $\nabla u_i^\delta = |\nabla u_i^\delta| \cdot \vec{n}$. Here we have

$$I_1 = \int_{\mathbb{R}^n} \nabla u_i^\delta \cdot x \left[\Delta u_i^\delta + (u_i^\delta)_t\right] G dx$$

$$I_2 = \int \nabla u_i^\delta \cdot \nabla G |\nabla u_i^\delta| dA_{i,\delta} - 2\int \nabla u_i^\delta \cdot \nabla G |\nabla u_i^\delta| dA_{i,\delta} = -\int \nabla u_i^\delta \cdot \nabla G |\nabla u_i^\delta| dA_{i,\delta}$$
$$= \int \frac{\nabla u_i^\delta \cdot x}{2(t+t_0)} |\nabla u_i^\delta| G dA_{i,\delta}$$

$$I_3 = \int (u_i^\delta)_t \nabla u_i^\delta G\, dA_{i,\delta} - \int_{\mathbb{R}^n} (u_i^\delta)_t (\Delta u_i^\delta + (u_i^\delta)_t) G\, dx.$$

Fix ε and let $\delta \to 0$, then the terms with $dA_{i,\delta}$ go to 0 (outside of S_ε). Since for x near S_ε, $|\nabla u| = o(1)$, we thus obtain

$$I_1 = o(1) \int_{\mathbb{R}^n} (\Delta u_i + (u_i)_t) G\, dx \xrightarrow[\varepsilon \to 0]{} 0$$

and similarly $I_3 \xrightarrow[\varepsilon \to 0]{} 0$. □

References

[A] Almgren, F. J., Jr. Q valued functions minimizing Dirichlets integral and the regularity of area minimizing rectifiable currents up to codimension two. Bull. Amer. Math. Soc. (N.S.) 8 (1983), no. 2, 327-328.

[ACF] H.W. Alt, L.A. Caffarelli, and A. Friedman, *Variational problems with two phases and their free boundaries*, Trans. Amer. Math. Soc. **282**, no.2 (1984), 431–461.

[BS] Burke, S. P.; Schuman, T. E. W. Diffusion flames.Industr. Eng. Chem. 20 (1928), no. 10, 998-1004.

[CL] L.A. Caffarelli and F-H. Lin, *Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries.* preprint

[CL1] Caffarelli, L. A.; Lin, F. An optimal partition problem for eigenvalues. J. Sci. Comput. 31 (2007), no. 1, 5-18. Available online at: http://www.springerlink.com/content/v671661832u5/?p=d2c1296d08c347ee80e9718c28995186?=4

[CL2] Caffarelli, L. A.; Lin, F. In preparation.

[CP] L.A. Caffarelli and I. Peral, *On $W^{1,p}$ estimates for elliptic equations in divergence form.* Comm. Pure Appl. Math. 51 (1998), no. 1, 1–21

[CSa] L.A. Caffarelli and S. Salsa, *A geometric approach to the free boundary problems* Graduate Studies in Mathematics, 68. American Mathematical Society, Providence, RI, 2005

[Ch] Chen, Xu-Yan A strong unique continuation theorem for parabolic equations. Math. Ann. 311 (1998), no. 4, 603–630.

[CLLL] Chang, S.-M.; Lin, C.-S.; Lin, T.-C.; Lin, W.-W. Segregated nodal domains of two-dimensional multispecies Bose-Einstein condensates. Phys. D 196 (2004), no. 3-4, 341-361.

[CR] Caffarelli, L. A.; Roquejoffre, J.-M. Uniform Hölder estimates in a class of elliptic systems and applications to singular limits in models for diffusion flames. Arch. Ration. Mech. Anal. 183 (2007), no. 3, 457-487.

[CTV] Conti, M.; Terracini, S.; Verzini, G. Neharis problem and competing species systems. Ann. Inst. H. Poincar Anal. Non Linaire 19 (2002), no. 6, 871-888.

[CTV2] Conti, M.; Terracini, S.; Verzini, G. An optimal partition problem related to nonlinear eigenvalues. J. Funct. Anal. 198 (2003), no. 1, 160-196.

[CTV3] Conti, M.; Terracini, S.; Verzini, G. Asymptotic estimates for the spatial segregation of competitive systems. Adv. Math. 195 (2005), no. 2, 524-560.

[CTV4] Conti, M.; Terracini, S.; Verzini, G. A variational problem for the spatial segregation of reaction- diffusion systems. Indiana Univ. Math. J. 54 (2005), no. 3, 779-815.

[EFV] L. Escauriaza, F.J. Fernandez, S. Vessella, *Doubling properties of caloric functions.* Appl. Anal. 85 (2006), no. 1-3, 205–223.

[GL] Garofalo, N.; Lin, F.-H. Monotonicity properties of variational integrals, A p weights and unique continuation. Indiana Univ. Math. J. 35 (1986), no. 2, 245-268.

[La] E. Landis, *Second order equations of elliptic and parabolic type.* Translations of Mathematical Monographs, 171. American Mathematical Society, Providence, RI, 1998.

[L] Lin, F.-H. Static and moving vortices in Ginzburg-Landau theories. Nonlinear partial differential equations in geometry and physics (Knoxville, TN, 1995), 71-111. Progr. Nonlinear Differential Equations Appl., 29. Birkhuser, Basel, 1997.

[L2] Lin, F.; Yang, X. Geometric measure theory. An introduction. International Press, Boston, 2002.

[L3] Lin, F.-H. Nodal sets of solutions of elliptic and parabolic equations. Comm. Pure Appl. Math. 44 (1991), no. 3, 287-308.

[LSW] W. Littman, G. Stampacchia, and H.F. Weinberger, *Regular points for elliptic equations with discontinuous coefficients.* Ann. Scuola Norm. Sup. Pisa (3) 17 1963 43–77.

L.A. Caffarelli, Department of Mathematics, University of Texas at Austin, 1 University Station – C1200, Austin, TX 78712-0257

E-mail address: `caffarel@math.utexas.edu`

Aram L. Karakhanyan, Department of Mathematics, University of Texas at Austin, 1 University Station – C1200, Austin, TX 78712-0257

E-mail address: `aram@math.utexas.edu`

Fang-Hua Lin, Department of Mathematics, Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012

E-mail address: `linf@cims.nyu.edu`