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# ON DERIVATION OF EULER-LAGRANGE EQUATIONS FOR AREA-PRESERVING ENERGY-MINIMIZERS

NIRMALENDU CHAUDHURI AND ARAM L. KARAKHANYAN

ABSTRACT. Derivation of the system of Euler-Lagrange equations for volume-preserving, energy-minimizing $W^{1,2}$-deformations and establishing the existence of an integrable pressure associated with the volume constraint is an open problem. In this article we consider this problem for the case $n = 2$. For an area-preserving, elastic energy-minimizing deformation $\mathbf{u}$ with $|\nabla \mathbf{u}|^2$ in the Hardy space $\mathcal{H}^1$, we establish an explicit representation of the associated pressure $p \in L^1_{\rm loc}$ via Calderón-Zygmund type singular integral operators. We then derive the system of Euler-Lagrange equations for $W^{1,r}_{\rm loc}(\Omega, \mathbb{R}^2), r \geq 3$ area-preserving local minimizers and prove partial regularity under smallness assumption on pressure.

## 1. Introduction

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a smooth, bounded and simply connected domain. The classical Stokes problem in hydrodynamics involves minimizing the potential energy

$$I[\mathbf{w}] := \int_\Omega \frac{1}{2}|\nabla \mathbf{w}|^2 + \langle \mathbf{f}, \mathbf{w} \rangle$$

for all divergence free velocity fields $\mathbf{w} \in W^{1,2}_0(\Omega, \mathbb{R}^n)$ for a given force field $\mathbf{f} \in L^2(\Omega, \mathbb{R}^n)$. It follows that the problem has a unique incompressible minimizer $\mathbf{u} \in W^{1,2}_0(\Omega, \mathbb{R}^n)$. The linear incompressible constraint $\operatorname{div} \mathbf{u} = 0$ ensures the existence of a hydrostatic pressure $p \in L^2_{\rm loc}(\Omega)$ and the pair $(\mathbf{u}, p)$ satisfies the following system of Euler-Lagrange equations

$$\begin{cases} \Delta \mathbf{u}(x) = \nabla p(x) - \mathbf{f}(x), & \text{in } \Omega \\ \operatorname{div} \mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{1.1}$$

in the weak sense, see for example [Ev 98, pp 472-474]. The regularity of $(\mathbf{u}, p)$ is well understood and detailed analysis can be found in [Ga 94, Chapter IV].

An analogue of this problem appears in nonlinear elasticity. In such context, $\mathbf{w}$ represents the displacement of an incompressible elastic body which has the rest configuration $\Omega \subset \mathbb{R}^n$. For incompressible neo-Hookean materials [Ba 77], [TO 81], [Og 84], such as vulcanized rubber, in the equilibrium state, one is interested in

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$$E[\mathbf{w}] := \int_\Omega L(\nabla \mathbf{w}(x))dx\,, \tag{1.2}$$

for incompressible $W^{1,2}$-deformations $\mathbf{w} : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$, subject to its own boundary condition and corresponding to a given bulk energy $L : \mathbb{M}^{n\times n} \to \mathbb{R}$. The simplest $L$ is the Dirichlet energy, given by $L(X) = \frac{1}{2}|X|^2 := \frac{1}{2}\mathrm{tr}\,(X^t X)$. Let us denote the admissible set of deformations

$$\mathcal{A} := \left\{\mathbf{w} \in W^{1,2}(\Omega, \mathbb{R}^n) : \mathrm{cof}\,\nabla\mathbf{w} \in L^2(\Omega, \mathbb{M}^{n\times n}),\, \det \nabla\mathbf{w} = \mathbf{1} \text{ a.e.}\right\}, \tag{1.3}$$

where $W^{k,p}$ denotes the usual *Sobolev spaces* [Ad 75] and cof $P$ is the *cofactor* matrix, whose $ij$-th entries is the determinant of $(n-1)\times(n-1)$ submatrix obtained by deleting the $i$-th row and the $j$-th column from the $n \times n$ matrix $P$ . We call $\mathbf{u} \in \mathcal{A}$ to be a *local minimizer* of $E[\cdot]$ if and only if

$$E[\mathbf{u}] \le E[\mathbf{w}] \quad \text{for all } \mathbf{w} \in \mathcal{A} \text{ and } \mathrm{supp}\,(\mathbf{w}-\mathbf{u}) \subset \Omega\,. \tag{1.4}$$

Under the hypothesis that the energy density $L$ is quasiconvex [Mo 52] and have quadratic growth, using direct methods in the calculus of variations together with weak continuity of determinant, Ball [Ba 77] proved the existence of local minimizers $\mathbf{u} \in \mathcal{A}$ of the energy $E[\cdot]$. However the derivation of the system of Euler-Lagrange equations for such minimizers and proving the existence of an integrable pressure associated with the volume constraint is a challenging open problem.

We will be concerned in this paper with the derivation of Euler-Lagrange equations for the area-preserving local minimizers and the existence of a locally integrable pressure in the planar case $n = 2$. Our main results are as follows.

**Theorem 1.1.** *Let $\Omega \subset \mathbb{R}^2$ be a smooth, simply connected and bounded domain. Assume that $\boldsymbol{u} \in W^{1,r}_{\mathrm{loc}}(\Omega, \mathbb{R}^2) \cap \mathcal{A} = \{\boldsymbol{w} \in W^{1,2}(\Omega, \mathbb{R}^2) : \det\nabla\boldsymbol{w}(x) = 1, \text{a. e. in } \Omega\}$, for some $r \ge 3$ is a local minimizer of $E[\cdot]$. Then there exists a scalar function $q \in L^{r/2}_{\mathrm{loc}}(\boldsymbol{u}(\Omega))$ such that the pair $(\boldsymbol{u}, p)$ satisfies*

$$\int_\Omega DL(\nabla\boldsymbol{u}(x)) : \nabla\phi(x)\,dx = \int_\Omega p(x)\,\mathrm{cof}\,(\nabla\boldsymbol{u}(x)) : \nabla\phi(x)\,dx\,, \tag{1.5}$$

*for all $\phi \in C^1_0(\Omega, \mathbb{R}^2)$, where $p := q \circ \boldsymbol{u} \in L^{r/2}_{\mathrm{loc}}(\Omega)$ and $A : B := \sum_{ij} a_{ij}b_{ij}$, for $A, B \in \mathbb{M}^{2\times 2}$. In other words, the pair $(\boldsymbol{u}, p)$ satisfies the system of Euler-Lagrange equations*

$$\mathrm{div}\,[DL(\nabla\boldsymbol{u}(x)) - p(x)\,\mathrm{cof}\,(\nabla\boldsymbol{u}(x))] = 0 \qquad \text{in } \Omega, \tag{1.6}$$

*in the sense of distribution, where the divergence is taken in each rows.*

Under the stronger hypothesis that the local minimizers of $E[\cdot]$ are classical, namely in Sobolev spaces $W^{2,r}, r > 2$, Tallec and Oden [TO 81] established the above system of equations. Whereas, our approach to establish the existence of a pressure $p \in L^{r/2}$ associated with the local minimizer $\mathbf{u}$, we only require $\mathbf{u} \in W^{1,r}_{\mathrm{loc}}$, $r > 2$ and to derive the system of equilibrium equations (1.6) for $(\mathbf{u}, p)$ in $\Omega$ we need $r \ge 3$.

Recall that for $f \in L^1(\mathbb{R}^n)$ the maximal function $Mf$ is defined by

$$(Mf)(x) := \sup_{\rho>0} \frac{1}{\mathrm{meas}\,B_\rho(x)} \int_{B_\rho(x)} |f(y)|\,dy\,.$$

From the classical results in singular integrals due to Stein [St 69, Theorem 1] or [St 70, pp 23], it follows that if $f \in L^1(\mathbb{R}^n)$ and is supported on a finite ball $B \subset \mathbb{R}^n$, then $Mf \in L^1(B)$ is and only if

$$f \in L\log L := \left\{ g : B \to \mathbb{R} \,:\, \int_B |g| \log^+ |g|\, dx < \infty \right\} \equiv \left\{ g : B \to \mathbb{R} \,:\, \int_B |g| \log(2 + |g|)\, dx < \infty \right\},$$

where $\log^+ |x| = 0$ for $0 < |x| \le 1$ and $\log^+ |x| = \log |x|$ for $|x| > 1$. A standard result states that a positive function $f$ is in the *Hardy space* $\mathcal{H}^1$ (the pre dual of BMO) if and only if $f \in L\log^+ L$. Notice that without any further higher integrability assumption on $\nabla \mathbf{u}$, we cannot ensure integrability of the maximal function $M|\nabla \mathbf{u}|^2$. However, under the additional assumption that $M|\nabla \mathbf{u}|^2$ is integrable, which is equivalent to $|\nabla \mathbf{u}|^2 \in \mathcal{H}^1$, we prove that the pressure $q$ on the deformed domain $\mathbf{u}(\Omega)$ is locally integrable and $(\mathbf{u}, q)$ satisfies the same system of differential equations a very weak sense. More precisely, we prove the following theorem.

**Theorem 1.2.** *Let $\Omega \subset \mathbb{R}^2$ be a smooth, simply connected, bounded domain. Assume that $\boldsymbol{u} \in \mathcal{A}$ is a local minimizer of $E[\cdot]$ such that $|\nabla \boldsymbol{u}|^2 \in \mathcal{H}^1_{\text{loc}}(\Omega)$. Then there exists $q \in L^1_{loc}(\boldsymbol{u}(\Omega))$ such that the pair $(\boldsymbol{u}, q)$ satisfies the integral identity*

$$\int_\Omega DL(\nabla \boldsymbol{u}(x)) \,:\, \nabla(\boldsymbol{v} \circ \boldsymbol{u})\, dx = \int_{\boldsymbol{u}(\Omega)} q(z)\ \operatorname{div} \boldsymbol{v}(z)\, dz \tag{1.7}$$

*for all $\boldsymbol{v} \in C_0^\infty(\boldsymbol{u}(\Omega), \mathbb{R}^2)$.*

The proof of Theorem 1.2 is quite delicate. The main ideas in our proof are to localize the *mollified pressure* on the deformed domain $\mathbf{u}(\Omega)$, its explicit representation using Green's function of the unit disc in $\mathbb{R}^2$ and finding its uniform bound by using Calderón-Zygmund estimate [CZ 52]. Finally we show that the pressure on $\mathbf{u}(\Omega)$ is locally represented as the sum of certain singular integral operators of $|\nabla \mathbf{u}|^2$ involving Calderón-Zygmund type kernels (see equation (4.17) in Section 4) [CZ 52].

**Theorem 1.3.** [CZ 52, **Calderón-Zygmund Theorem**] *Let $f \in L\log^+ L$ and let $\Gamma$ be a $C^1$ function on $\mathbb{R}^n \setminus \{0\}$ homogeneous of degree 0 with mean value 0 over the unit sphere $\mathbb{S}^{n-1}$, that is*

$$\int_{\mathbb{S}^{n-1}} \Gamma(x)\, dS(x) = 0. \tag{1.8}$$

*Then the function defined as*

$$f^*(x) := \lim_{\delta \to 0} \int_{|x-y| \ge \delta} \frac{\Gamma(x-y)}{|x-y|^n} f(y)\, dy \tag{1.9}$$

*exists a.e. and integrable. Furthermore,*

$$\int_K |f^*|\, dy \le C \int_{\mathbb{R}^n} |f| \left(1 + \log^+ \left( (\operatorname{meas} K)^{\frac{n+1}{n}} |f| \right)\right) dy + C (\operatorname{meas} K)^{-\frac{1}{n}}, \tag{1.10}$$

*for all measurable subset $K$ of $\mathbb{R}^n$ with finite measure.*

For $n = 2$, through a series of papers, Bauman, Owen and Phillips [BOP 91], [BOP 91a], [BOP 92] proved that any $W^{2,r}$, $r > 2$ solutions of (1.6) are smooth solutions. In 1999, Evans and Gariepy [EG 99] proved that any *non-degenerate*, Lipschitz area-preserving local minimizers of $E[\cdot]$ are $C^{1,\alpha}(\Omega_0)$, for some $0 < \alpha < 1$ for a dense open subset $\Omega_0 \subset \Omega$. However, as a consequence of the Euler-Lagrange equations (1.6) together with the standard elliptic estimates [GM 79]we prove the following theorem.

**Theorem 1.4.** *Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded domain and $L : \mathbb{M}^{2\times 2} \to \mathbb{R}$ be a smooth and uniformly convex function having quadratic growth. Assume that $\boldsymbol{u} \in \mathcal{A} \cap W^{1,3}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ be a local minimizer of $E[\cdot]$ and $q(z) \in C^\alpha$ for some positive $\alpha$. Then $\boldsymbol{u}$ has Hölder continuous first derivatives in sumbdomain $\Omega_0$. Moreover*

$$|\Omega \setminus \Omega_0| = 0.$$

In a forthcoming paper [CHK 08] we will discuss the regularity of $W^{1,r}_{\mathrm{loc}}$, $r > 2$- area-preserving local minimizers and the derivation of system of Euler-Lagrange equations for the case $n \geq 3$.

## 2. The First Variation of Energy

In the study of regularity of finite energy deformations, Šverák [Sv 88] proved that for any $W^{1,n}$-deformation $\mathbf{w}$ with $\det \nabla \mathbf{w}(x) > 0$, a.e., there exists a continuous function $\omega$ on $\mathbb{R}$ with $\omega(0) = 0$ such that

$$|\mathbf{w}(x) - \mathbf{w}(y)| \leq \omega(|x - y|), \quad \text{for any } x, y \in \Omega \subset\subset \mathbb{R}^n.$$

In connection to the study of quasi-regular maps for $n = 2$, Iwaniec and Šverák [IS 93] proved that any $W^{1,2}$-deformation $\mathbf{w}$ with the *distortion* function $K(\cdot, \mathbf{w}) := |\nabla \mathbf{w}(\cdot)|^2 / \det \nabla \mathbf{w}(\cdot)$ being integrable, $\mathbf{w}$ is a homeomorphism. Thus in particular, area-preserving $W^{1,2}$-deformations in the plane are continuous and open maps. For $n \geq 3$, it is still unknown whether a map $\mathbf{u} \in \mathcal{A}$ is a homeomorphism.

**Theorem 2.1.** *Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a smooth bounded domain. Let $L : \mathbb{M}^{n\times n} \to \mathbb{R}$ be a smooth function and $\boldsymbol{u} \in \mathcal{A}$ be a local minimizer of $E[\cdot]$. For $n \geq 3$, we further assume that $\boldsymbol{u}$ is a continuous and an open map. Then $\boldsymbol{u}$ satisfies the following integral identity*

$$\int_\Omega DL(\nabla \boldsymbol{u}(x)) : \nabla(\boldsymbol{v} \circ \boldsymbol{u})(x)\, dx = 0\,, \tag{2.1}$$

*for all smooth, compactly supported and divergence free vector fields $\boldsymbol{v}$ on $\mathbf{u}(\Omega)$, where $A : B := \mathrm{tr}(\mathrm{A^t B}) = \sum_{\mathrm{i,j=1}}^{\mathrm{n}} \mathrm{a_{ij} b_{ij}}$ is the scalar product on $\mathbb{M}^{n\times n}$.*

**Proof:** Let $\mathbf{v} \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^n)$ be a vector field with $\operatorname{div} \mathbf{v} = 0$. For each $y \in \mathbf{u}(\Omega)$, consider the unique smooth flow $\phi(y, \cdot) : \mathbb{R} \to \mathbf{u}(\Omega)$ given by

$$\frac{d\phi}{dt}(y,t) = \mathbf{v}(\phi(y,t)) \quad \text{in} \quad \mathbb{R}, \quad \phi(y,0) = y. \tag{2.2}$$

Using the relations $\dfrac{\partial}{\partial P_{ij}} \det P = (\operatorname{cof} P)_{ij}$ and $P\,(\operatorname{cof} P)^t = I_n \det P$, by a direct calculations we observe that

$$\frac{d}{dt}\left(\det \nabla_y \phi(y,t)\right) = \det \nabla_y \phi(y,t)\ \operatorname{div} \mathbf{v} = 0. \tag{2.3}$$

Since $\det \nabla_y \phi(y, 0) = 1$, from (2.3) it follows that $\det \nabla_y \phi(y,t) = 1$ for all $t \in \mathbb{R}$ and $y \in \mathbf{u}(\Omega)$. Consider the map $\mathbf{w} : \Omega \times \mathbb{R} \to \mathbf{u}(\Omega)$ defined by

$$\mathbf{w}(x,t) := \phi(\cdot, t) \circ \mathbf{u}\,(x) = \phi(\mathbf{u}(x), t) \quad \text{for any } t \in \mathbb{R},\ x \in \Omega.$$

Let $V := \operatorname{supp}\mathbf{v} \subset \mathbf{u}(\Omega)$, then $\mathbf{v}(\mathbf{u}(x)) = 0$ for $\mathbf{u}(x) \notin V$. This in conjunction with the uniqueness of $\phi$ implies that $\phi(\mathbf{u}(x),t) = \mathbf{u}(x)$ for all points $x$ such that $\mathbf{u}(x) \notin V$. Since $\Omega$ is bounded, $\mathbf{u}$ is continuous and $V$ is compact, $\Omega' = \mathbf{u}^{-1}(V)$ is a compact subset of $\Omega$. Hence $\operatorname{supp}(\mathbf{w}(x,t)-\mathbf{u}(x)) \subset \Omega'$. Furthermore, $\det \nabla_x \mathbf{w}(x,t) = \det \nabla_y \phi(y,t) \det \nabla \mathbf{u}(x) = 1$. Therefore, $\mathbf{w}(\cdot,t) \in \mathcal{A}$ and $\operatorname{supp}(\mathbf{u} - \mathbf{w}(\cdot,t)) \subset \Omega$ for all $t \in \mathbb{R}$. Since $\mathbf{u}$ is a local minimizer of $E[\cdot]$,

$$E[\mathbf{u}] \le E[\mathbf{w}(\cdot,t)] \qquad \text{for all} \quad t \in \mathbb{R}.$$

Thus in particular,

$$\begin{aligned}
0 &= \frac{d}{dt}\int_\Omega L(\nabla \mathbf{w}(x,t))\,dx\Big|_{t=0} \\
&= \sum_{i,j=1}^{2}\int_\Omega L_{ij}(\nabla \mathbf{w}(x,t))\,\frac{d}{dt}\left(\frac{\partial w^i}{\partial x_j}(x,t)\right)dx\Big|_{t=0} \\
&= \sum_{i,j=1}^{2}\int_\Omega L_{ij}(\nabla \mathbf{w}(x,t))\,\frac{\partial}{\partial x_j}\left(\frac{d\phi^i}{dt}(\mathbf{u}(x),t)\right)dx\Big|_{t=0} \\
&= \sum_{i,j=1}^{2}\int_\Omega L_{ij}(\nabla \mathbf{w}(x))\,\frac{\partial}{\partial x_j}\left(v^i(\phi(u(x),t)\right)dx\Big|_{t=0} \\
&= \sum_{i,j=1}^{2}\int_\Omega L_{ij}(\nabla \mathbf{u}(x))\,\frac{\partial}{\partial x_j}\left(v^i(\mathbf{u}(x))\right)dx \\
&= \int_\Omega DL(\nabla \mathbf{u}(x)) \,:\, \nabla(\mathbf{v}\circ\mathbf{u})(x)\,dx\,,
\end{aligned}$$

for all smooth, compactly supported and divergence free vector fields on $\mathbf{u}(\Omega)$, where $L_{ij}(P) := \dfrac{\partial L}{\partial P_{ij}}(P)$ . This proves the Theorem. □

## 3. Derivation of Euler-Lagrange Equations for $n = 2$

Let $\Omega \subset \mathbb{R}^2$ be a smooth, bounded and simply connected domain. Assume that the bulk energy $L : \mathbb{M}^{2\times 2} \to \mathbb{R}$ is smooth such that, $|L(P)| \le C(1+|P|^2)$, $|DL(P)| \le C(1+|P|)$ and $|D^2L(P)| \le C$ for all $P \in \mathbb{M}^{2\times 2}$, for some $C > 0$. Since $|\operatorname{cof} P| = |P|$ for $P \in \mathbb{M}^{2\times 2}$, the area-preserving maps in the plane $\mathcal{A}$ defined in (1.3) is equivalent to the family $\{\mathbf{w} \in W^{1,2}(\Omega,\mathbb{R}^2) \,:\, \det\nabla\mathbf{w}(x) = 1, \text{a.e. in } \Omega\}$. Let $\mathbf{u} \in \mathcal{A}$ be a local minimizer of $E[\cdot]$. Then $\mathbf{u} : \Omega \to \mathbf{u}(\Omega)$ is an open map and a local homeomorphism [Sv 88], [IS 93]. Throughout this section we denote $V \subset\subset \mathbf{u}(\Omega)$, a smooth and simply connected sub-domain, $C$ is a generic absolute constant depending only on $\Omega$, $V$, and $L$. Its value can vary from line to line, but each line is valid with $C$ being a pure positive number.

Let $\mathbf{v} = (v^1, v^2) \in C_0^\infty(V,\mathbb{R}^2)$ such that $\operatorname{div}\mathbf{v} = 0$. Let $\rho$ be the usual mollification kernel. For $0 < \varepsilon < \operatorname{dist}(V, \partial\,\mathbf{u}(\Omega))$, let $\mathbf{v}_\varepsilon := (v_\varepsilon^1, v_\varepsilon^2)$ be the mollification of $\mathbf{v}$, where

$$v_\varepsilon^i(y) := (v^i * \rho_\varepsilon)(y) = \int_{\mathbb{R}^2}\rho_\varepsilon(y-z)v^i(z)dz = \int_V \rho_\varepsilon(y-z)v^i(z)dz, \quad y \in \mathbf{u}(\Omega).$$

Thus $\mathbf{v}_\varepsilon \in C_0^\infty(\mathbf{u}(\Omega), \mathbb{R}^2)$ and $\operatorname{div} \mathbf{v}_\varepsilon = 0$. Hence by testing the identity (2.1) with $\mathbf{v} = \mathbf{v}_\varepsilon$, we obtain

$$\sum_{i,j=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \, \frac{\partial}{\partial x_j} (v_\varepsilon^i \circ \mathbf{u})(x) \, dx = 0,$$

or in more explicitly

$$\sum_{i,j,k=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \, \frac{\partial v_\varepsilon^i}{\partial y_k}(\mathbf{u}(x)) \, \frac{\partial u^k}{\partial x_j}(x) \, dx = 0. \tag{3.1}$$

From the definition of mollification, by taking $y = \mathbf{u}(x)$, for $x \in \Omega$, we obtain

$$\frac{\partial v_\varepsilon^i}{\partial y_k}(\mathbf{u}(x)) = \int_V \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \, v^i(z) dz. \tag{3.2}$$

Therefore by pluging (3.2) into (3.1) and Fubini's Theorem yields

$$\begin{aligned}
0 &= \sum_{i,j,k=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \frac{\partial u^k}{\partial x_j}(x) \left( \int_V \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \, v^i(z) dz \right) dx \\
&= \sum_{i,j,k=1}^{2} \int_V \left( \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \, \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \, \frac{\partial u^k}{\partial x_j}(x) \, dx \right) v^i(z) \, dz \\
&= \sum_{i,j=1}^{2} \int_V \left( \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \sum_{k=1}^{2} \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \, \frac{\partial u^k}{\partial x_j}(x) \, dx \right) v^i(z) \, dz \\
&= \sum_{i=1}^{2} \int_V \left[ \sum_{j=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \, \frac{\partial}{\partial x_j} \Big( \rho_\varepsilon(\mathbf{u}(x) - z) \Big) \, dx \right] \, v^i(z) dz \, .
\end{aligned} \tag{3.3}$$

Let us define the smooth function $g_\varepsilon^i : V \to \mathbb{R}$, for $i = 1, 2$ by

$$g_\varepsilon^i(z) := \sum_{j=1}^{2} \int_\Omega L_{ij}(\nabla \mathbf{u}(x)) \, \frac{\partial}{\partial x_j} \Big( \rho_\varepsilon(\mathbf{u}(x) - z) \Big) dx. \tag{3.4}$$

Then $\mathbf{g}_\varepsilon = (g_\varepsilon^1, g_\varepsilon^2) \in C^\infty(V, \mathbb{R}^2)$ and

$$\begin{aligned}
|\mathbf{g}_\varepsilon(z)| &\le \sum_{ij} \int_\Omega \left| L_{ij}(\nabla \mathbf{u}(x)) \, \frac{\partial \rho_\varepsilon}{\partial y_k}(\mathbf{u}(x) - z) \, \frac{\partial u^k}{\partial x_j}(x) \right| dx \\
&\le \frac{C}{\varepsilon^3} \Big( (\operatorname{meas} \Omega)^{1/2} + \|\nabla \mathbf{u}\|_{L^2(\Omega)} \Big) \, \|\nabla \mathbf{u}\|_{L^2(\Omega)}.
\end{aligned}$$

Thus combing (3.3) and (3.4) we get

$$\int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z) \rangle \, dz = 0 \quad \text{for } \mathbf{v} \in C_0^\infty(V, \mathbb{R}^2) \text{ such that } \operatorname{div} \mathbf{v} = 0 \text{ in } V \, , \tag{3.5}$$

where $\langle \cdot, \cdot \rangle$ the usual scalar product in $\mathbb{R}^2$. Let $\phi \in C_0^\infty(V)$ and define $\mathbf{v}(z) := J \nabla \phi(z)$ for $z \in V$, where $J$ be the $90^0$ planar rotation given by

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then it follows that $\operatorname{div} \mathbf{v} = 0$ and hence by testing (3.5) with this particular choice of $\mathbf{v}$ and integrating by parts we obtain,

$$\begin{aligned} 0 &= \int_V \langle \mathbf{g}_\varepsilon(z), J\nabla\phi(z)\rangle \, dz \\ &= \int_V \left\langle J^t\mathbf{g}_\varepsilon(z), \nabla\phi(z)\right\rangle \, dz \\ &= -\int_V \phi(z) \operatorname{div}(J^t\mathbf{g}_\varepsilon(z))\, dz \qquad \text{for all } \phi \in C_0^\infty(V). \end{aligned}$$

Hence $\operatorname{curl} \mathbf{g}_\varepsilon := \dfrac{\partial g_\varepsilon^1}{\partial z_2} - \dfrac{\partial g_\varepsilon^2}{\partial z_1} = \operatorname{div}(J^t\mathbf{g}_\varepsilon) = 0$ in $V$. Since $V$ is simply connected, there exists $q_\varepsilon \in C^\infty(V)$, such that

$$\mathbf{g}_\varepsilon(z) = -\nabla q_\varepsilon(z), \quad \text{for all } z \in V, \tag{3.6}$$

modulo translation of a constant.

**Lemma 3.1.** *Consider the family* $\boldsymbol{g}_\varepsilon$ *be given by (3.4). Then* $\boldsymbol{g}_\varepsilon \rightharpoonup \boldsymbol{g}$ *weakly in the dual space* $(C_0^1(V,\mathbb{R}^2))^*$.

**Proof:** Since $\rho_\varepsilon$ is radially symmetric

$$\frac{\partial\rho_\varepsilon}{\partial y_k}(|y-z|) = {\rho_\varepsilon}'(|y-z|)\frac{y_k - z_k}{|y-z|} = -\frac{\partial\rho_\varepsilon}{\partial z_k}(|y-z|). \tag{3.7}$$

Therefore from the definition of $g_\varepsilon^i$ in (3.4), we have

$$\begin{aligned} g_\varepsilon^i(z) &= -\sum_{j,k=1}^{2}\int_\Omega L_{ij}(\nabla\mathbf{u}(x))\,\frac{\partial u^k}{\partial x_j}(x)\frac{\partial\rho_\varepsilon}{\partial z_k}(\mathbf{u}(x)-z)dx \\ &= -\sum_{k=1}^{2}\int_\Omega \sigma_{ik}(x)\frac{\partial}{\partial z_k}\Big(\rho_\varepsilon(\mathbf{u}(x)-z)\Big)dx\,, \end{aligned} \tag{3.8}$$

where

$$\sigma_{ik}(x) := \sum_{j=1}^{2} L_{ij}(\nabla\mathbf{u}(x))\,\frac{\partial u^k}{\partial x_j}(x) \quad \text{for } x \in \Omega\,. \tag{3.9}$$

Since $\mathbf{u}$ is a $W^{1,2}$ area-preserving homeomorphism, $\nabla\mathbf{u}^{-1}(\mathbf{u}(x)) = (\operatorname{cof}\nabla\mathbf{u}(x))^t$. Thus it follows that $\mathbf{u}^{-1} \in W^{1,2}(\mathbf{u}(\Omega),\Omega)$. Using the structural assumptions on $L$ in (3.9), we get

$$\int_{\mathbf{u}(\Omega)} |(\sigma_{ik}\circ\mathbf{u}^{-1})(z))|\,dz = \int_\Omega |\sigma_{ik}(x)|dx \le C\int_\Omega |\nabla\mathbf{u}(x)|^2dx < \infty\,,$$

and hence $\tilde\sigma_{ik} := \sigma_{ik}\circ\mathbf{u}^{-1} \in L^1(\mathbf{u}(\Omega))$, for $i,k = 1,2$. Now observe that for any test function $\mathbf{v} \in C_0^\infty(V,\mathbb{R}^2)$, using Fubini, integration by parts and change of variable

$\xi = \mathbf{u}(x)$ we obtain

$$\begin{aligned}
(3.10)\qquad \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z)\rangle\, dz &= -\sum_{i,j=1}^{2} \int_\Omega \sigma_{ij}(x) \left( \int_V \frac{\partial}{\partial z_j} (\rho_\varepsilon(\mathbf{u}(x)-z))\, v^i(z)\, dz \right) dx \\
&= \sum_{i,j=1}^{2} \int_\Omega \sigma_{ij}(x) \left( \int_V \rho_\varepsilon(\mathbf{u}(x)-z)\, \frac{\partial v^i}{\partial z_j}(z)\, dz \right) dx \\
&= \sum_{i,j=1}^{2} \int_V \frac{\partial v^i}{\partial z_j}(z) \left( \int_\Omega \sigma_{ij}(x)\, \rho_\varepsilon(\mathbf{u}(x)-z)\ dx \right) dz \\
&= \sum_{i,j=1}^{2} \int_V \frac{\partial v^i}{\partial z_j}(z) \left( \int_{\mathbf{u}(\Omega)} \sigma_{ij}(\mathbf{u}^{-1}(\xi)\, \rho_\varepsilon(\xi - z)\ d\xi \right) dz \\
&= \sum_{i,j=1}^{2} \int_V \frac{\partial v^i}{\partial z_j}(z)\ (\tilde\sigma_{ij})_\varepsilon(z)\, dz\,,
\end{aligned}$$

where

$$(3.11)\qquad (\tilde\sigma_{ij})_\varepsilon(z) := ((\sigma_{ij}\circ \mathbf{u}^{-1}) * \rho_\varepsilon)(z) = \int_{u(\Omega)} \sigma_{ij}(\mathbf{u}^{-1}(\xi))\, \rho_\varepsilon(\xi - z)\, d\xi,$$

is the usual mollification of $\sigma_{ij}\circ\mathbf{u}^{-1}$. Since $(\sigma_{ij}\circ\mathbf{u}^{-1}) * \rho_\varepsilon \to \sigma_{ij}\circ\mathbf{u}^{-1}$ in $L^1(\mathbf{u}(\Omega))$ as $\varepsilon\to 0$, by passing through the limit as $\varepsilon \to 0$ in (3.10) we conclude that

$$(3.12)\qquad \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z)\rangle\, dz \to \sum_{i,j=1}^{2} \int_V \sigma_{ij}(\mathbf{u}^{-1}(z))\, \frac{\partial v^i}{\partial z_j}(z)\,, dz \quad \text{as } \varepsilon \to 0$$

for all $\mathbf{v} \in C_0^\infty(V, \mathbb{R}^2)$. Now let us define the functional $\mathbf{g} : C_0^1(V,\mathbb{R}^2) \to \mathbb{R}$ as

$$(3.13)\qquad \langle \mathbf{g}, \mathbf{v}\rangle := \lim_{\varepsilon\to 0} \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}(z)\rangle\, dz = \int_V \sigma(\mathbf{u}^{-1}(z)) : \nabla\mathbf{v}(z)\, dz\,,$$

for $\mathbf{v} \in C_0^1(V,\mathbb{R}^2)$, where $\sigma(x) := (\sigma_{ij}(x)) \in \mathbb{M}^{2\times 2}$. Then from (3.13) it follows that

$$(3.14)\qquad |\langle \mathbf{g}, \mathbf{v}\rangle| \le C\|\sigma\|_{L^1(\Omega)}\, \|\nabla \mathbf{v}\|_{L^\infty(\mathbf{u}(\Omega))} \le C\|\nabla\mathbf{u}\|^2_{L^2(\Omega)}\, \|\nabla\mathbf{v}\|_{L^\infty(\mathbf{u}(\Omega))},$$

for any $\mathbf{v} \in C_0^1(\mathbf{u}(\Omega, \mathbb{R}^2)$. Hence $\mathbf{g}$ is a continuous linear functional on $C_0^1(\mathbf{u}(\Omega), \mathbb{R}^2)$. Therefore, from the definition of $\mathbf{g}_\varepsilon$ in (3.4), it follows that $\mathbf{g}_\varepsilon \rightharpoonup \mathbf{g}$ weakly in the dual space $(C_0^1(V,\mathbb{R}^2))^*$. This proves the lemma. □

**Lemma 3.2.** *Assume that $\boldsymbol{u} \in W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)\cap\mathcal{A}$ for some $r > 2$. Then the family $q_\varepsilon$ defined by $-\nabla q_\varepsilon = \boldsymbol{g}_\varepsilon$ in (3.6) is uniformly bounded in $L^{r/2}_{\mathrm{loc}}(\boldsymbol{u}(\Omega))$.*

**Proof** Since $\mathbf{u} \in W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$ for some $r > 2$, from the definition of $\sigma_{ij}$ in (3.9) and the growth condition on $L$, it follows that for any $V \subset\subset \mathbf{u}(\Omega)$

$$(3.15)\qquad \int_V |(\sigma_{ij}\circ\mathbf{u}^{-1})(z))|^{r/2} dz = \int_{\mathbf{u}^{-1}(V)} |\sigma_{ij}(x)|^{r/2} dx \le C \int_{\mathbf{u}^{-1}(V)} |\nabla\mathbf{u}(x)|^r dx,$$

and hence $\tilde\sigma_{ij} := \sigma_{ij}\circ\mathbf{u}^{-1} \in L^{r/2}(V)$, for $i, j = 1, 2$. Let $f_\varepsilon : V \to \mathbb{R}$ be defined as $f_\varepsilon(z) := q_\varepsilon(z)|q_\varepsilon(z)|^{\frac{r}{2}-2}$, $z \in V$, so that for any $1 < s < \infty$,

$$\int_V |f_\varepsilon(z)|^s\, dz = \int_V |q_\varepsilon(z)|^{s\left(\frac{r}{2}-1\right)} dz = \left\| |q_\varepsilon|^{\frac{r}{2}-1}\right\|^s_{L^s(V)}\,.$$

Translating $f_\varepsilon$ to $f_\varepsilon - \frac{1}{\operatorname{meas} V}\int_V f_\varepsilon(z)\,dz$, if necessary, so that $\int_V f_\varepsilon(z)\,dz = 0$. In view of this normalization, there exists a smooth vector field $\mathbf{w}_\varepsilon : V \mapsto \mathbb{R}^2$, such that

$$\begin{cases} \operatorname{div} \mathbf{w}_\varepsilon = f_\varepsilon & \text{in } V \\ \mathbf{w}_\varepsilon = 0 & \text{on } \partial V\,. \end{cases} \tag{3.16}$$

Furthermore we have the estimate

$$\|\mathbf{w}_\varepsilon\|_{W^{1,s}(V)} \le C\|f_\varepsilon\|_{L^s(V)} = C\left\||q_\varepsilon|^{\frac{r}{2}-1}\right\|_{L^s(V)}\,, \tag{3.17}$$

for $C > 0$ independent of $\varepsilon$, see Dacorogna-Moser [DM 90]. Then for sufficiently small $\varepsilon > 0$

$$\begin{aligned}
\int_V |q_\varepsilon(z)|^{r/2} dz &= \int_V q_\varepsilon(z)|q_\varepsilon(z)|^{r/2\,-\,2} q_\varepsilon(z)\,dz \\
&= \int_V q_\varepsilon(z)\operatorname{div}\mathbf{w}_\varepsilon(z) dz && \text{by (3.16)} \\
&= -\int_V \langle \nabla q_\varepsilon(z), \mathbf{w}_\varepsilon(z)\rangle\, dz \\
&= \int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{w}_\varepsilon(z)\rangle\, dz && \text{by (3.6)} \\
&= \sum_{i,j=1}^{2} \int_V ((\sigma_{ij}\circ \mathbf{u}^{-1}) * \rho_\varepsilon)(z)\, \frac{\partial w_\varepsilon^i}{\partial z_k}(z) dz && \text{by (3.10)} \\
&\le C \sum_{i,j=1}^{2} \left(\int_V |\sigma_{ij}(\mathbf{u}^{-1}(z))|^{r/2} dz\right)^{2/r} \left(\int_V \left|\frac{\partial w_\varepsilon^i}{\partial z_k}(z)\right|^{r/(r-2)} dz\right)^{(r-2)/r} \\
&\le C\left\||q_\varepsilon|^{\frac{r}{2}-1}\right\|_{L^{r/(r-2)}(V)} \sum_{i,j=1}^{2} \|\sigma_{ij}\circ \mathbf{u}^{-1}\|_{L^{r/2}(V)} && \text{by (3.17)} \\
&= C\left(\int_V |q_\varepsilon(z)|^{r/2} dz\right)^{1-2/r} \|\sigma\|_{L^{r/2}(\mathbf{u}(\Omega),\mathbb{M}^{2\times 2})} \\
&\le C\left(\int_V |q_\varepsilon(z)|^{r/2} dz\right)^{1-2/r} \|\nabla \mathbf{u}\|^2_{L^r(\Omega)}\,. && \text{by (3.15)}
\end{aligned}$$

Hence there exists a constant $C > 0$, independent of $\varepsilon$ such that

$$\|q_\varepsilon\|_{L^{r/2}(V)} \le C\|\nabla \mathbf{u}\|^2_{L^r(\Omega)}. \tag{3.18}$$

Since $r > 2$, there exists a function $q \in L^{r/2}(V)$, such that $q_\varepsilon \rightharpoonup q$ weakly in $L^{r/2}(V)$. This proves the lemma. □

**Proof of Theorem 1.1** Using the change of variables, recalling the definitions of $\mathbf{g}$ in (3.13), and $\sigma_{ij}$ in (3.9), we obtain

$$
\begin{aligned}
\langle \mathbf{g}, \mathbf{v}\rangle &= \sum_{i,j=1}^{2} \int_V \sigma_{ij}(\mathbf{u}^{-1}(z)) \frac{\partial v^i}{\partial z_j}(z)\, dz \\
&= \sum_{i,j=1}^{2} \int_{\mathbf{u}^{-1}(V)} \sigma_{ij}(x) \frac{\partial v^i}{\partial z_j}(\mathbf{u}(x))\, dx \\
&= \sum_{i,k=1}^{2} \int_{\mathbf{u}^{-1}(V)} L_{ik}(\nabla \mathbf{u}(x)) \left( \sum_{j=1}^{2} \frac{\partial v^i}{\partial z_j}(\mathbf{u}(x)) \frac{\partial u^j}{\partial x_k}(x) \right) dx \\
&= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}\circ \mathbf{u})(x) dx \quad \text{for } \mathbf{v} \in C_0^1(V, \mathbb{R}^2)\,.
\end{aligned} \tag{3.19}
$$

Since $\mathbf{u}^{-1} \in W^{1,r}(V, \mathbf{u}^{-1}(V))$, for any $\phi \in C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^2)$, the composition $\phi\circ\mathbf{u}^{-1} \in W_0^{1,r}(V, \mathbb{R}^2)$. Hence there exists $\mathbf{v}_\delta \in C_0^1(V, \mathbb{R}^2)$ such that $\mathbf{v}_\delta \to \psi := \phi\circ\mathbf{u}^{-1}$ strongly in $W^{1,r}(V, \mathbb{R}^2)$ as $\delta \to 0$. Then Hölder inequality yields

$$
\begin{aligned}
&\int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \Big( \nabla(\mathbf{v}_\delta \circ \mathbf{u})(x) \; - \; \nabla(\psi \circ \mathbf{u})(x) \Big) dx \\
&\qquad = \int_{\mathbf{u}^{-1}(V)} (\nabla \mathbf{u}(x))^t DL(\nabla \mathbf{u}(x)) : \Big( \nabla_z \mathbf{v}_\delta(\mathbf{u}(x)) - \nabla_z \psi(\mathbf{u}(x)) \Big) dx \\
&\qquad \le C \|\nabla \mathbf{u}\|_{L^{2r'}(\mathbf{u}^{-1}(V))} \, \|\nabla(\mathbf{v}_\delta - \psi)\|_{L^r(V)},
\end{aligned}
$$

where $r' = r/(r-1)$. Notice that $r \ge 3$ yields $2r' \le r$ and hence $\nabla \mathbf{u} \in L^r_{\mathrm{loc}}(\Omega) \subseteq L^{2r'}_{\mathrm{loc}}(\Omega)$. Therefore, from (3.19) we obtain

$$
\begin{aligned}
\langle \mathbf{g}, \mathbf{v}_\delta \rangle &= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\mathbf{v}_\delta \circ \mathbf{u})(x)\, dx \\
&\to \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla(\phi \, \circ \, \mathbf{u}^{-1} \circ \mathbf{u})(x)\, dx \quad \text{as } \delta \to 0 \\
&= \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla\phi(x)\, dx\,.
\end{aligned} \tag{3.20}
$$

Now define the linear functional $\mathbf{g}\circ\mathbf{u} : C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^2) \to \mathbb{R}$ by

$$
\langle \mathbf{g}\circ\mathbf{u}, \phi\rangle := \langle \mathbf{g}, \phi \, \circ \, \mathbf{u}^{-1}\rangle = \lim_{\delta\to 0} \langle \mathbf{g}, \mathbf{v}_\delta\rangle = \int_{\mathbf{u}^{-1}(V)} DL(\nabla \mathbf{u}(x)) : \nabla\phi(x)\, dx, \tag{3.21}
$$

for any $\phi \in C_0^1(\mathbf{u}^{-1}(V), \mathbb{R}^2)$. Hence $\mathbf{g}\circ\mathbf{u}$ defines a continuous linear functional on $W_0^{1,2}(\mathbf{u}^{-1}(V), \mathbb{R}^2)$. On the other hand, since $q_\varepsilon \rightharpoonup q$ weakly in $L^{r/2}(V)$, using the definition of $\mathbf{g}$, the representation of $\mathbf{g}_\varepsilon = -\nabla q_\varepsilon$ and integration by parts we conclude that

$$\begin{aligned}(3.22)\qquad \langle \mathbf{g}, \mathbf{v}_\delta\rangle &= \lim_{\varepsilon\to 0}\int_V \langle \mathbf{g}_\varepsilon(z), \mathbf{v}_\delta(z)\rangle\; dz\\ &= -\lim_{\varepsilon\to 0}\int_V \langle \nabla q_\varepsilon(z), \mathbf{v}_\delta(z)\rangle\; dz\\ &= \lim_{\varepsilon\to 0}\int_V q_\varepsilon(z)\,\mathrm{div}\,\mathbf{v}_\delta(z)\,dz\\ &= \int_V q(z)\,\mathrm{div}\,\mathbf{v}_\delta(z)\,dz\\ &= \int_V q(z)\;\mathrm{tr}\,(\nabla_z\mathbf{v}_\delta(z))\;dz\,.\end{aligned}$$

The area constraint $\det\nabla\mathbf{u}(x) = 1$ a.e., and $\nabla(\mathbf{v}\circ\mathbf{u})(x) = \nabla_z\mathbf{v}(\mathbf{u}(x))\;\nabla\mathbf{u}(x)$, yields $\nabla_z\mathbf{v}(\mathbf{u}(x)) = \nabla(\mathbf{v}\circ\mathbf{u})(x)\,(\mathrm{cof}\,\nabla\mathbf{u}(x))^t$. Using $\mathbf{u}\in W^{1,r}_{\mathrm{loc}}(\Omega,\mathbb{R}^2)$ together with the fact that $|\mathrm{cof}\,P| = |P|$ for any $P\in\mathbb{M}^{2\times 2}$, we conclude that $\mathrm{cof}\,\nabla\mathbf{u}\in L^r_{\mathrm{loc}}(\Omega,\mathbb{M}^{2\times 2})$. Since $q\in L^{r/2}(V)$ and $L^{r/2}_{\mathrm{loc}}\subseteq L^{r/(r-1)}_{\mathrm{loc}}$ for $r\geq 3$, applying change of variables in (3.22), we obtain

$$\begin{aligned}(3.23)\qquad \langle \mathbf{g}, \mathbf{v}_\delta\rangle &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x))\;\mathrm{tr}\Big(\nabla_z\mathbf{v}_\delta(\mathbf{u}(x))\Big)\,dx\\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x))\;\mathrm{tr}\Big(\nabla(\mathbf{v}_\delta\circ\mathbf{u})(x)\;(\mathrm{cof}\,\nabla\mathbf{u}(x))^t\Big)\,dx\\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x))\;\mathrm{cof}\,(\nabla\mathbf{u}(x)) : \nabla(\mathbf{v}_\delta\circ\mathbf{u})(x)\,dx,\\ &\to \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x))\;\mathrm{cof}\,(\nabla\mathbf{u}(x)) : \nabla(\phi\,\circ\,\mathbf{u}^{-1}\circ\mathbf{u})(x)\,dx\quad\text{as }\delta\to 0\\ &= \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x))\;\mathrm{cof}\,(\nabla\mathbf{u}(x)) : \nabla\phi(x)\,dx\,.\end{aligned}$$

Hence from (3.21) and (3.23) we obtain

$$\int_{\mathbf{u}^{-1}(V)} DL(\nabla\mathbf{u}(x)) : \nabla\phi(x)\,dx = \int_{\mathbf{u}^{-1}(V)} q(\mathbf{u}(x))\;\mathrm{cof}\,(\nabla\mathbf{u}(x)) : \nabla\phi(x)\,dx\,,$$

for any $\phi\in C^1_0(\mathbf{u}^{-1}(V),\mathbb{R}^2)$. Finally choose a sequence of smooth, simply connected sets $V_k\subset\subset V_{k+1}\subset\subset\mathbf{u}(\Omega)$ sub-domains such that $\mathbf{u}(\Omega) = \cup_{k=1}^\infty V_k$. Utilizing the foregoing arguments and lemmas 3.1-3.2, there exists $q_k\in L^{r/2}(V_k)$, $k\geq 1$ such that

$$(3.24)\qquad \int_{\mathbf{u}^{-1}(V_k)} DL(\nabla\mathbf{u}(x)) : \nabla\phi(x) = \int_{\mathbf{u}^{-1}(V_k)} q_k(\mathbf{u}(x))\;\mathrm{cof}\,(\nabla\mathbf{u}(x)) : \nabla\phi(x)\,,$$

for $\phi\in C^1_0(\mathbf{u}^{-1}(V_k),\mathbb{R}^2)$. Since $\mathbf{u}$ is locally area-preserving homeomorphism, $\Omega = \cup_{k=1}^\infty\mathbf{u}^{-1}(V_k)$ is an open covering of $\Omega$ and $\mathbf{u}^{-1}(V_k)\subset\subset\mathbf{u}^{-1}(V_{k+1})$. Using the identity $\mathrm{div}\;\mathrm{cof}\;\nabla\mathbf{u}(x) = 0$ and invertibility of $\nabla\mathbf{u}(x)$, from (3.24) it follows that $q_k$ is unique up to a translation of a constant. Thus adding constant terms as necessary to each $q_k$, we deduce from (3.24) that for each fixed $k\geq 1$

$$q_i(z) = q_k(z)\quad\text{for } z\in V_i,\quad 1\leq i\leq k.$$

We finally define $q : \mathbf{u}(\Omega) \to \mathbb{R}$ as $q(z) := q_k(z)$, for $z \in V_k$, so that $q \in L^{r/2}_{\mathrm{loc}}(\mathbf{u}(\Omega))$. This proves that for any $\phi \in C^1_0(\Omega, \mathbb{R}^2)$, the pair $(\mathbf{u}, q)$ satisfies

$$\int_\Omega DL(\nabla \mathbf{u}(x)) : \nabla\phi(x)\, dx = \int_\Omega q(\mathbf{u}(x))\ \mathrm{cof}\,(\nabla \mathbf{u}(x)) : \nabla\phi(x)\, dx\,.$$

Now let us define the pressure $p$ on $\Omega$ by

$$p(x) := q(\mathbf{u}(x)) \quad \text{for } x \in \Omega.$$

Then for any $k \geq 1$,

$$\int_{\mathbf{u}^{-1}(V_k)} |p(x)|^{r/2} = \int_{\mathbf{u}^{-1}(V_k)} |q(\mathbf{u}(x))|^{r/2} dx = \int_{V_k} |q(z)|^{r/2} dz < \infty,$$

and hence $p \in L^{r/2}_{\mathrm{loc}}(\Omega)$ and the pair $(\mathbf{u}, p)$ satisfies

$$\int_\Omega DL(\nabla \mathbf{u}(x)) : \nabla\phi(x)\, dx = \int_\Omega p(x)\ \mathrm{cof}\,(\nabla \mathbf{u}(x)) : \nabla\phi(x)\, dx\,, \tag{3.25}$$

for any $\phi \in C^1_0(\Omega, \mathbb{R}^2)$. In other words, $(\mathbf{u}, p)$ satisfies the system of Euler-Lagrange equations

$$\mathrm{div}\,[DL(\nabla \mathbf{u}(x)) - p(x)\,\mathrm{cof}\,(\nabla \mathbf{u}(x))] = 0, \quad \text{in } \ \Omega.$$

in the sense of (3.25). This completes the proof. □

## 4. Local $L^1$-Estimate and the Integral Representation of the Pressure

In this section we establish an explicit representation of the pressure on the deformed domain $\mathbf{u}(\Omega)$ in terms of Calderón-Zygmund type singular integral operator of the energy $|\nabla \mathbf{u}|^2$. Our main ideas in the proof are to localize the mollified pressure on the deformed domain $\mathbf{u}(\Omega)$, finding its explicit representation using Green's function of the unit disc in $\mathbb{R}^2$ and finding an uniform estimate by using Calderón-Zygmund Theorem [CZ 52] for $L\log^+ L$ functions.

**Proof of Theorem 1.2**. Let us assume that $\mathbf{u} \in \mathcal{A}$ minimizes the energy $E[\cdot]$ and $|\nabla \mathbf{u}|^2 \in L\log^+ L$. Let $V \subset\subset \mathbf{u}(\Omega)$ be a smooth and simply connected sub-domain of $\mathbf{u}(\Omega)$. Without loss of generality let us assume that $0 \in V$ and $V = B_1 := \{z \in \mathbb{R}^2 : |z| < 1\}$ be the unit disc. Recall the family $(\mathbf{g}_\varepsilon)_{\varepsilon>0}$ defined by (3.4) and the family $(q_\varepsilon)_{\varepsilon>0}$ in (3.6) represented by

$$-\nabla q_\varepsilon = \mathbf{g}_\varepsilon, \tag{4.1}$$

modulo an additive constant. Applying the divergence operator to the both sides of the above equation, we obtain

$$-\Delta q_\varepsilon = \mathrm{div}\,\mathbf{g}_\varepsilon. \tag{4.2}$$

Now our idea is to localize the equation (4.2) and find appropriate uniform estimates for the localized $q_\varepsilon$. Let $\eta \in C^\infty_0(B_1)$, $0 \leq \eta \leq 1$ be a cut-off function such that $\eta \equiv 1$ in $B_{2/3}$. Let $\bar{q}_\varepsilon = \eta q_\varepsilon$ be the localized pressure. Then $\bar{q}_\varepsilon$ is the solution to the Dirichlet problem

$$\begin{cases} -\Delta \bar{q}_\varepsilon = f_\varepsilon & \text{in } B_1 \\ \bar{q}_\varepsilon = 0 & \text{on } \partial B_1, \end{cases} \tag{4.3}$$

where $f_\varepsilon := \eta\Delta q_\varepsilon + 2\langle\nabla q_\varepsilon, \nabla\eta\rangle + q_\varepsilon\Delta\eta$. Therefore $\bar{q}_\varepsilon$ is the Green's potential of $f_\varepsilon$ in $B_1$. In other words,

$$\bar{q}_\varepsilon(y) = \int_{B_1} G(z-y) f_\varepsilon(z)\, dz\,, \tag{4.4}$$

where $G(z,y)$ Green's function of the unit disc $B_1 \subset \mathbb{R}^2$ given by

$$G(z,y) := -\frac{1}{2\pi}\log|z-y| + \frac{1}{2\pi}\log(|y||z-\hat{y}|),\ \ \hat{y} := \frac{y}{|y|^2}\,. \tag{4.5}$$

Using (4.1), (4.2) and (4.5) in (4.4), we obtain

$$\begin{aligned} \bar{q}_\varepsilon(y) \;&=\; -\frac{1}{2\pi}\int_{B_1}\Big(\eta\,\Delta q_\varepsilon + 2\,\langle\nabla q_\varepsilon,\nabla\eta\rangle + q_\varepsilon\,\Delta\eta\Big)\log|z-y|\,dz \\ &\quad+\frac{1}{2\pi}\int_{B_1} f_\varepsilon(z)\log(|y||z-\hat{y}|)\,dz \\ &= \frac{1}{2\pi}\int_{B_1}\Big(\eta\,\mathrm{div}\,\mathbf{g}_\varepsilon + 2\,\langle\mathbf{g}_\varepsilon(z),\nabla\eta(z)\rangle - q_\varepsilon\Delta\eta\Big)\,\log|z-y|\;dz \\ &\quad+\frac{1}{2\pi}\int_{B_1} f_\varepsilon(z)\log(|y||z-\hat{y}|)\,dz \\ &= \frac{1}{2\pi}\,I^1_\varepsilon(y) + \frac{1}{\pi}\,I^2_\varepsilon(y) + \frac{1}{2\pi}\,I^3_\varepsilon(y) + \frac{1}{2\pi}\,I^4_\varepsilon(y) \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} I^1_\varepsilon(y) \;&:=\; \int_{B_1}\eta(z)\,\log|z-y|\;\mathrm{div}\,\mathbf{g}_\varepsilon(z)\,dz \\ I^2_\varepsilon(y) \;&:=\; \int_{B_1}\langle\mathbf{g}_\varepsilon(z),\nabla\eta(z)\rangle\;\log|z-y|\,dz \\ I^3_\varepsilon(y) \;&:=\; -\int_{B_1} q_\varepsilon(z)\Delta\eta(z)\log|z-y|\,dz \\ I^4_\varepsilon(y) \;&:=\; \int_{B_1} f_\varepsilon(z)\;\log|y|(|z-\hat{y}|)\;dz\,. \end{aligned}$$

We now establish an uniform local $L^1$-estimate for $q_\varepsilon$ through the following steps.

**Step 1: Limits of $I^3_\varepsilon$ and $I^4_\varepsilon$** Let us fix $|y| < 1/2$. Since $\Delta\eta = 0$ for $|z| < 2/3$, both the integrals $I^3_\varepsilon(y)$ and $I^4_\varepsilon(y)$ are well defined for $|y| < 1/2$. Since $q_\varepsilon$ is determined up to a constant, we can add a constant to $z \mapsto \Delta\eta(z)\log z - y|$, if nessecary, to ensure that it has vanishing integral. For each fixed $|y| < 1/2$, let $\mathbf{v}_y : B_1 \to \mathbb{R}^2$ be the solution of the Dirichlet problem

$$\begin{cases} \mathrm{div}\,\mathbf{v}_y(z) = \Delta\eta(z)\log|z-y| & \text{for } z \in B_1 \\ \mathbf{v}_y = 0 & \text{on } \ \partial B_1\,. \end{cases} \tag{4.7}$$

Then using (4.7) and (3.13) we obtain

$$
\begin{aligned}
I^3_\varepsilon(y) &= -\int_{B_1} q_\varepsilon(z)\Delta\eta(z)\log|z-y|\,dz \\
&= -\int_{B_1} q_\varepsilon(z)\operatorname{div}\mathbf{v}_y(z)\,dz \\
&= \int_{B_1} \langle \nabla q_\varepsilon(z), \mathbf{v}_y(z)\rangle\,dz \\
&= -\int_{B_1} \langle \mathbf{g}_\varepsilon(z), \mathbf{v}_y(z)\rangle\,dz \\
&\to -\int_{B_1} \sigma(\mathbf{u}^{-1}(z)) : \nabla\mathbf{v}_y(z)\,dz \quad \text{as } \varepsilon\to 0 \\
&:= I^3_0(y)\,.
\end{aligned} \tag{4.8}
$$

Since $f_\varepsilon = \Delta(q_\varepsilon\eta)$ and for each fixed $|y|<1/2$ the function $z\mapsto \Delta\log(|y|(z-\hat{y}))$ is smooth on $B_1$. By taking $\mathbf{w}_y : B_1\to\mathbb{R}^2$ to be the solution of the Dirichlet problem

$$
\begin{cases}
\operatorname{div}\mathbf{w}_y(z) = \eta(z)\Delta\log(|y||z-\hat{y}| & \text{for } z\in B_1\\
\mathbf{w}_y = 0 & \text{on } \partial B_1,
\end{cases} \tag{4.9}
$$

and applying the above arguments we obtain

$$
\begin{aligned}
I^4_\varepsilon(y) &= \int_{B_1} \Delta\Big(q_\varepsilon(z)\eta(z)\Big)\log|y|(|z-\hat{y}|)\;dz \\
&= \int_{B_1} q_\varepsilon(z)\left(\eta(z)\Delta\log(|y||z-\hat{y}|)\right)dz \\
&\to \int_{B_1} \sigma(\mathbf{u}^{-1}(z)) : \nabla\mathbf{w}_y(z)\,dz \quad \text{as } \varepsilon\to 0 \\
&:= I^4_0(y)\,.
\end{aligned} \tag{4.10}
$$

**Step 2: Limit of $I^2_\varepsilon$** Since $\nabla\eta(z)=0$ for $|z|<2/3$, the integral $I^3_\varepsilon(y)$ is well-defined for $|y|<1/2$. Recall that from (3.8) and (3.9)

$$
\begin{aligned}
-g^i_\varepsilon(z) &= \sum_{j=1}^{2}\frac{\partial}{\partial z_j}\int_\Omega \sigma_{ij}(x)\,\rho_\varepsilon(\mathbf{u}(x)-z)\,dx \\
&= \sum_{j=1}^{2}\frac{\partial}{\partial z_j}\int_{\mathbf{u}(\Omega)} \sigma_{ij}(\mathbf{u}^{-1}(y))\,\rho_\varepsilon(\mathbf{y}-z)\,dy \\
&= \sum_{j=1}^{2}\frac{\partial}{\partial z_j}\Big((\sigma_{ij}\circ\mathbf{u}^{-1})*\rho_\varepsilon\Big)(z)\,.
\end{aligned}
$$

In other words,

$$
\mathbf{g}_\varepsilon = -\operatorname{div}\tilde{\sigma}_\varepsilon\,, \tag{4.11}
$$

where the divergence is taken in each rows of matrix $\tilde{\sigma}_\varepsilon := \Big((\tilde{\sigma}_{ij})_\varepsilon\Big)\in\mathbb{M}^{2\times 2}$, $(\tilde{\sigma}_{ij})_\varepsilon := (\sigma_{ij}\circ\mathbf{u}^{-1})*\rho_\varepsilon$. Notice that $(\tilde{\sigma}_{ij})_\varepsilon\to\tilde{\sigma}_{ij} := \sigma_{ij}\circ\mathbf{u}^{-1}$ in $L^1$ as $\varepsilon\to 0$ for each $i,j=1,2$.

Using the above representation of $\mathbf{g}_\varepsilon$ observe that

$$
\begin{aligned}
(4.12)\quad I_\varepsilon^2(y) &= -\int_{B_1} \Big\langle \operatorname{div}\tilde\sigma_\varepsilon(z), \log|z-y|\,\nabla\eta(z) \Big\rangle\, dz \\
&= \int_{B_1} \tilde\sigma_\varepsilon(z) : \nabla\Big( \log|z-y|\,\nabla\eta(z)\Big) dz \\
&= \int_{B_1\setminus B_{2/3}} \tilde\sigma_\varepsilon(z) : \Big( \log|z-y|\,\nabla^2\eta(z) + \frac{\nabla\eta\otimes(z-y)}{|y-z|^2}\Big) dz \\
&\to \int_{B_1\setminus B_{2/3}} \tilde\sigma(z) : \Big( \log|z-y|\,\nabla^2\eta(z) + \frac{\nabla\eta\otimes(z-y)}{|y-z|^2}\Big) dz \quad \text{as } \varepsilon\to 0 \\
&:= I_0^2(y)\,.
\end{aligned}
$$

**Step 3: Limit of $I_\varepsilon^1(y)$** Since we assumed $|\nabla\mathbf{u}|^2 \in \mathcal{H}^1_{\rm loc}(\Omega)$, from the definition of $\tilde\sigma_{ij}$ it follows that $\tilde\sigma_{ij} \in L\log^+ L$. Thus the mollification $(\tilde\sigma_{ij})_\varepsilon$ converges strongly to $\tilde\sigma_{ij}$ in $L\log^+ L$ as $\varepsilon \to 0$. Integrating by parts twice and using (4.11)

$$
\begin{aligned}
I_\varepsilon^1(y) &= \int_{B_1} \operatorname{div}\mathbf{g}_\varepsilon(z)\ \eta(z)\log|z-y|\,dz \\
&= -\int_{B_1} \tilde\sigma_\varepsilon(z) : \nabla^2(\eta(z)\log|z-y|)\,dz \\
&= -\int_{B_1\setminus B_{2/3}} \tilde\sigma_\varepsilon(z) : \Big( \log|z-y|)\,\nabla^2\eta(z) + 2\frac{\nabla\eta(z)\otimes(z-y)}{|z-y|^2}\Big)\ dz \\
&\quad -\int_{B_1} \tilde\sigma_\varepsilon(z) : \Big( Id - 2\frac{(z-y)\otimes(z-y)}{|z-y|^2}\Big)\ \frac{\eta(z)}{|z-y|^2}\,dz \\
&:= I_\varepsilon^{11}(y) + I_\varepsilon^{12}(y),
\end{aligned}
$$

where $Id$ is the $2\times 2$ identity matrix and

$$
\begin{aligned}
(4.13)\quad I_\varepsilon^{11}(y) &:= -\int_{B_1\setminus B_{2/3}} \tilde\sigma_\varepsilon(z) : \Big( \log|z-y|\,\nabla^2\eta + 2\frac{\nabla\eta\otimes(z-y)}{|z-y|^2}\Big)\ dz \\
&\to -\int_{B_1} \tilde\sigma(z) : \Big( \log|z-y|\,\nabla^2\eta + 2\frac{\nabla\eta\otimes(z-y)}{|z-y|^2}\Big)\ dz \quad \text{as } \varepsilon\to 0 \\
&:= I_0^{11}(y)\,,
\end{aligned}
$$

and

$$
(4.14)\qquad I_\varepsilon^{12}(y) := -\int_{B_1} \tilde\sigma_\varepsilon(z) : \Big( Id - 2\frac{(z-y)\otimes(z-y)}{|z-y|^2}\Big)\ \frac{\eta(z)}{|z-y|^2}\,dz
$$

is the sum of Calderón-Zygmund [CZ 52] type singular integrals with the homogeneous kernel

$$
(4.15)\qquad G_{ij}(z) := \delta_{ij} - 2\frac{z_i z_j}{|z|^2}, \quad z\in\mathbb{R}^2\setminus\{0\}, \quad i,j = 1,2\,.
$$

Observe that each $G_{ij}$ satisfies all the conditions of the Calderón-Zygmund Theorem 1.3 [CZ 52]. Since $\sigma_{ij} \in L\log^+ L$, the following sum of singular integrals

$$
(4.16)\qquad I_0^{12}(y) := -\int_{B_1} \tilde\sigma(z) : \Big( Id - 2\frac{(z-y)\otimes(z-y)}{|z-y|^2}\Big)\ \frac{\eta(z)}{|z-y|^2}\,dz
$$

exists for almost every $|y| < 1/2$ and is integrable.

**Claim:** $I_\varepsilon^{12} \to I_0^{12}$ *strongly in* $L^1(B_{1/2})$.

**Proof.** Let $\rho > 1/2$ and extend $\tilde\sigma_{ij}$ by 0 outside the unit ball $B_1$. From the singular integrals (4.14) and (4.16), we have

$$I_\varepsilon^{12}(y) - I_0^{12}(y) = -\sum_{i,j=1}^{2} \int_{\mathbb{R}^2} \eta \left( (\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij} \right) \left( \delta_{ij} - 2\frac{(z_i - y_i)(z_j - y_j)}{|z-y|^2} \right) \frac{dz}{|z-y|^2}\,.$$

Extend $I_\varepsilon^{12}$ and $I_0^{12}$ by 0 outside the ball $B_{1/2}$. Then by using Calderón-Zygmund estimate in Theorem 1.3 and strong convergence of $(\tilde\sigma_{ij})_\varepsilon$ in $L\log^+ L$, for any $\rho > 1/2$ we obtain

$$\begin{aligned}
\int_{B_{1/2}} |I_\varepsilon^{12}(y) - I_0^{12}(y)|dy \;&=\; \int_{B_\rho} |I_\varepsilon^{12}(y) - I_0^{12}(y)|dy \\
&\le\; C \sum_{i,j=1}^{2} \int_{\mathbb{R}^2} \eta|(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}|\,dz + C(\operatorname{meas} B_\rho)^{-\frac12} \\
&\quad + C \sum_{i,j=1}^{2} \int_{\mathbb{R}^2} \eta|(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}|\,\log^+ \left( (\operatorname{meas} B_\rho)^{\frac32} \eta|(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}| \right) dz \\
&\le\; C(1 + \log^+ \rho) \sum_{i,j=1}^{2} \int_{B_1} \eta|(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}|\,dz + \frac{C}{\rho} \\
&\quad + C \sum_{i,j=1}^{2} \int_{B_1} |(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}|\,\log \left( 2 + |(\tilde\sigma_{ij})_\varepsilon - \tilde\sigma_{ij}| \right) dz \\
&\to\; \frac{C}{\rho} \quad \text{as } \varepsilon \to 0 \\
&\to\; 0 \quad \text{as } \rho \to \infty\,.
\end{aligned}$$

Hence $I_\varepsilon^{12} \to I_0^{12}$ strongly in $L^1(B_{1/2})$. This proves the claim. □

**Step 4: An explicit representation of the pressure** To complete the proof, let us define $q : B_{1/2} \to \mathbb{R}$ by

$$q(y) := \frac{1}{2\pi}\left(I_0^{11}(y) + I_0^{12}(y)\right) + \frac{1}{\pi} I_0^2(y) + \frac{1}{2\pi}\left(I_0^3(y) + I_0^4(y)\right)\,.$$

Then from (4.9), (4.11), (4.12), (4.13) and (4.16), we conclude that $q_\varepsilon \to q$ strongly in $L^1(B_{1/2})$ and $q$ is represented as

$$\begin{aligned}
(4.17)\quad q(y) \;=\; &\frac{1}{2\pi} \int_{B_1} \sigma(\mathbf{u}^{-1}(z)) : \left( \nabla_z(\mathbf{w}_y(z) - \mathbf{v}_y(z)) - 2\Delta\eta \log|z-y| \right) dz \\
&- \frac{1}{2\pi} \int_{B_1} \sigma(\mathbf{u}^{-1}(z)) : \left( Id - 2\frac{(z-y)\otimes(z-y)}{|z-y|^2} \right) \frac{\eta(z)}{|z-y|^2}\,dz,
\end{aligned}$$

where $\sigma(x) := (\sigma_{ij}(x)) \in \mathbb{M}^{2\times 2}$ given by the equation (3.9). Since $q$ is the strong limit of the family $q_\varepsilon$ in ball $B_{1/2}$, it is independent of the choice of the cut-off function $\eta$. Following the same arguments as in Section 3, we can extend $q$ to all of $\mathbf{u}(\Omega)$ such that $q \in L^1_{\mathrm{loc}}(\mathbf{u}(\Omega))$ and the pair $(\mathbf{u}, q)$ satisfies the identity (1.7). This completes the proof of Theorem 1.2. □

## 5. Partial Regularity

Let us denote $L(x, \nabla \mathbf{u}) = \nabla \mathbf{u} - p(x)\nabla \mathbf{u}^{-t}$, then the equation is $\operatorname{div} L(x, \nabla \mathbf{u}) = 0$. First let us examine the ellipticity condition $L_{ij}(x, \xi)\xi_{ij} \geq \lambda|\xi|^2$ for some $\lambda > 0$. Since the deformation is incompressible we obtain

$$\nabla \mathbf{u}^{-t} = \begin{pmatrix} u_2^2 & -u_1^2 \\ -u_2^1 & u_1^1 \end{pmatrix}. \tag{5.1}$$

Introduce $I = L_{ij}(x, \xi)\xi_{ij} = |\xi|^2 - 2p(x)\det \xi$, where $\xi$ is any $2 \times 2$ matrix. Then completing squares we get

$$\begin{aligned} I &= \xi_{11}^2 + \xi_{12}^2 + \xi_{21}^2 + \xi_{22}^2 - 2p(\xi_{11}\xi_{22} - \xi_{12}\xi_{21}) \\ &= (\xi_{11} - p\xi_{22})^2 + (\xi_{12} - p\xi_{21})^2 + (1 - p^2)(\xi_{22}^2 + \xi_{21}^2) \\ &= (\xi_{22} - p\xi_{11})^2 + (\xi_{21} - p\xi_{12})^2 + (1 - p^2)(\xi_{11}^2 + \xi_{12}^2). \end{aligned} \tag{5.2}$$

Adding both identities and dividing by 2 we arrive at

$$\begin{aligned} I &= \frac{1}{2}((\xi_{11} - p\xi_{22})^2 + (\xi_{12} - p\xi_{21})^2 + (\xi_{22} - p\xi_{11})^2 + (\xi_{21} - p\xi_{12})^2 + (1 - p^2)|\xi|^2 \\ &\geq \frac{1 - p^2}{2}|\xi|^2. \end{aligned}$$

This computation shows that ellipticity condition

$$L_{ij}(x, \xi)\xi_{ij} \geq \lambda|\xi|^2, \lambda > 0$$

is equivalent to assume that

$$p^2 \leq 1 - 2\lambda. \tag{5.3}$$

Note that $p$ is defined up to addition of arbitrary constant, thus (5.3) is satisfied in subdomain $D \subset \Omega$ if

$$\operatorname{osc}_D p^2 < 1.$$

Next we examine the strong ellipticity condition, i.e.

$$L_{ij,kl}(x, \eta)\xi_{ij}\xi_{kl} \geq \lambda|\xi|^2, \tag{5.4}$$

where $\eta$ stands as dummy variable for $\nabla \mathbf{u}$. Recall that

$$L_{ij}(x, \eta) = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} - p(x) \begin{pmatrix} \eta_{22} & -\eta_{21} \\ -\eta_{12} & \eta_{11} \end{pmatrix}. \tag{5.5}$$

For instance $L_{11,kl} = \delta_{11,kl} - p\delta_{22,kl}$, and it is easy to check that

$$L_{ij,kl}(x, \eta)\xi_{ij}\xi_{kl} = |\xi|^2 - 2p(x)\det \xi,$$

that is the ellipticity implies strong ellipticity.

In what follows we make the following two assumptions

**1** $\mathbf{u}$ is $W^{1,3}(\Omega)$

**2** $q(z)$ is $\alpha$-Hölder continuous with respect to $z$.

**Proposition 5.1.** *Under the assumptions* **1-2** *we have that*

(i)
$$|L_{ij}(x,\nabla \boldsymbol{u})| \leq L(1+|\nabla \boldsymbol{u}|)$$

(ii) *for any* $x_1, x_2 \in \overline{\Omega}, \eta \in M^{2\times 2}$
$$\frac{|L_{ij}(x_1,\eta)-L_{ij}(x_2,\eta)|}{1+|\eta|} \leq C|x_1-x_2|^\alpha$$

(iii) $L_{ij}$ *is differentiable with respect to* $\eta$ *with bounded and continuous derivatives*
$$|L_{ij,kl}(x,\eta)| \leq L$$

(iv) $L_{ij}$ *satisfies to strong ellipticity condition*
$$L_{ij,kl}(x,\eta)\eta_{ij}\eta_{kl} \geq \lambda|\eta|^2$$

**Proof:** Since $\mathbf{u} \in W^{1,3}$, Sobolev imbedding theorem implies that $\mathbf{u} \in C^{1/3}$ then $p(x) = q(\mathbf{u}(x))$ is Hölder continuous and (i)-(ii) follow. (iii)-(iv) follow from (5.3). □

*Remark* 5.2. Assumptions $(i)-(iv)$ are stated in [GM 79], if fact they consider more general systems of elliptic equations. Using their theorem 1 we can obtain the following partial regularity result.

**Theorem 5.3.** *Assume that assumptions* **1-2** *are satisfied, i.e.* $\boldsymbol{u} \in W^{1,3}(\Omega), q \in C^\alpha$. *Then the first derivatives of* $\boldsymbol{u}$ *are Hölder continuous on an open set* $\Omega_0$. *Moreover*
$$|\Omega \setminus \Omega_0| = 0.$$

**Proof:** It follows from proposition then the requirements of theorem 1 in [GM 79] are satisfied and the result follows. □

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School of Mathematics and Applied Statistics University of Wollongong, Wollongong, NSW 2522, Australia
*E-mail address*: chaudhur@uow.edu.au

Department of Mathematics, University of Texas at Austin,Austin, Texas 78712, USA
*E-mail address*: aram@math.utexas.edu