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THE DIXMIER-MOEGLIN EQUIVALENCE FOR TWISTED HOMOGENEOUS COORDINATE RINGS

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Abstract. Given a projective scheme X over a field k, an automorphism $\sigma : X \to X$, and a σ-ample invertible sheaf $\mathcal{L}$, one may form the twisted homogeneous coordinate ring $B = B(X, \mathcal{L}, \sigma)$, one of the most fundamental constructions in noncommutative projective algebraic geometry. We study the primitive spectrum of B, as well as that of other closely related algebras such as skew and skew-Laurent extensions of commutative algebras. Over an algebraically closed, uncountable field k of characteristic zero, we prove that that the primitive ideals of B are characterized by the usual Dixmier-Moeglin conditions whenever $\dim X \leq 2$.

1. Introduction

Let A be an algebra over a base field k. To better understand A, one of the basic properties in which one is interested is the structure of the simple right A-modules. Often an explicit understanding of the simple modules is difficult, and one settles for less exact information: one aims instead to understand the primitive ideals of A, that is, the possible annihilators of the simple modules. For this reason, questions about primitive ideals are clearly fundamental, and there is a long history in noncommutative ring theory of their study.

One of the seminal achievements in the subject is the work of Dixmier and Moeglin on the characterization of the primitive spectrum of the universal enveloping algebra $\mathfrak{U}(L)$ of a finite-dimensional Lie algebra L over $\mathbb{C}$. This work showed the that the primitive ideals can be recognized among all prime ideals in the prime spectrum $\operatorname{Spec}\mathfrak{U}(L)$ in two different ways. A prime ideal of an algebra A is called *locally closed* if $\{P\}$ is an open subset of its closure in the Zariski topology on $\operatorname{Spec} A$. Assuming A is also Goldie, the prime P is called *rational* if the center of the Goldie quotient ring $Q(A/P)$ is an algebraic extension of k. If L is as above and $A = \mathfrak{U}(L)$, then a prime P of A is primitive if and only if it is locally closed, if and only if it is rational. Thus any Goldie algebra A for which both of these characterizations hold is said to satisfy the *Dixmier-Moeglin (DM)-equivalence.*

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More recent research has extended the work on enveloping algebras in two directions. First, there are results which show that some of the implications among the properties of being primitive, rational, and locally closed (as well as some other related properties) hold under quite general hypotheses on an algebra A. The paper [RoSm] has a good review of what general implications are currently known; we also review in Section 2 below some of the ones we need. Second, many other interesting special classes of rings have been shown to satisfy the DM-equivalence. For example, the DM-equivalence holds for the most familiar quantized coordinate rings in the theory of quantum groups [BG, Corollary II.8.5]. Typically, in such classes of examples where the DM-equivalence has been proved, there is some connection of the algebras in question to geometry (in the quantized coordinate ring case, there is a torus action.)

In this paper, we consider an important class of graded rings with a strong connection to projective algebraic geometry, and study the DM-equivalence for them; in particular we prove that the DM-equivalence always holds in small-dimensional cases. The main examples of interest are twisted homogeneous coordinate rings, which are fundamental in the theory of noncommutative projective geometry. Given a projective k-variety X, an invertible sheaf $\mathcal{L}$ on X and an automorphism $\sigma : X \to X$, one defines from this data the *twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \sigma)$, as follows. Set $\mathcal{L}_0 = \mathcal{O}_X$ and for each $n \geq 1$, put $\mathcal{L}_n = \mathcal{L} \otimes \sigma^* \mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^* \mathcal{L}$. Then $B(X, \mathcal{L}, \sigma) = \bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{L}_n)$ is a naturally an $\mathbb{N}$-graded k-algebra under the multiplication $f \star g = f \cdot (\sigma^m)^* g$, where $f \in B_m$, $g \in B_n$. We only make this construction when $\mathcal{L}$ is *σ-ample*, an additional technical condition which we will define in the body of the paper. This condition ensures, among other things, that B is noetherian. See [AV] and [Ke1] for more background on these rings.

In fact, our methods for studying primitivity will apply not just to the rings $B = B(X, \mathcal{L}, \sigma)$ but also to several other graded algebras. One of the fundamental properties of the ring B, in case X is integral, is that its graded quotient ring has the form $Q_{\mathrm{gr}}(B) = k(X)[t, t^{-1}; \sigma]$, where $k(X)$ is the field of rational functions on X, and $\sigma : k(X) \to k(X)$ is the induced automorphism. We call any $\mathbb{Z}$-graded Ore domain A where $(Q_{\mathrm{gr}}(A))_0$ is a field *birationally commutative*, and in general one expects the properties of such rings to be closely connected to commutative geometry. In particular, this class also includes the skew and skew-Laurent extensions $R[t; \sigma]$ and $R[t, t^{-1}; \sigma]$, where R is a commutative domain, finitely generated over k, and $\sigma : R \to R$ is an automorphism. The primitivity of skew and skew-Laurent extensions has been studied by several authors, but our methods put these examples in an interesting new context.

We are now ready to state our main theorem.

Theorem 1.1. [Theorem 8.1] *Let k be an uncountable algebraically closed field of characteristic* 0.

(1) *Let X be a projective variety over k with automorphism $\sigma : X \to X$ and σ-ample invertible sheaf $\mathcal{L}$. If $\dim X \leq 2$, then $B(X, \mathcal{L}, \sigma)$ satisfies the Dixmier-Moeglin equivalence.*

(2) *Let S be a commutative finitely generated k-algebra, with automorphism $\sigma : S \to S$. Let A be either of the algebras $S[t; \sigma]$ or $S[t, t^{-1}; \sigma]$. If $\dim S \leq 2$ and $\mathrm{GK}\, A < \infty$, then A satisfies the Dixmier-Moeglin equivalence.*

We remark that part (2) of the theorem is false without the assumption that $\mathrm{GK}\, A < \infty$, as several well-known examples which fail the DM-equivalence are skew-Laurent rings of this type. Also, the assumption that $\mathcal{L}$ is σ-ample in part (1) forces $\mathrm{GK}\, B < \infty$, whereas twisted homogeneous coordinate rings over non-σ-ample sheaves typically have exponential growth. Thus the two parts of the theorem are more analogous than they at first appear. We also see no obstruction to this theorem being true in all dimensions, and we conjecture that this is so. Our current methods are highly dependent, however, on results specific to the surface case.

Now we begin to describe the structure of the paper, and the methods we employ, several of which are interesting in themselves. In Section 2, we review some well-known results and show that in the special case of a birationally commutative $\mathbb{Z}$-graded algebra A, to prove the Dixmier-Moeglin equivalence it often suffices to examine the *homogeneous* prime spectrum. Specifically, under mild extra assumptions (including the uncountability of k), the DM-equivalence will hold for A unless A has a prime graded factor algebra whose number of homogeneous height one primes is countably infinite. We then introduce in Section 3 the main examples A we will consider, and show that in each case there is a quasi-projective scheme X, with automorphism $\sigma : X \to X$, such that (X, σ) encodes all of the information about the homogeneous primes of the algebra. For $A = B(X, \mathcal{L}, \sigma)$, for example, the homogeneous primes of A are in one-to-one inclusion-reversing correspondence with *σ-irreducible* closed subsets Z of X—the closed σ-invariant subsets Z such that σ acts as a cycle on the irreducible components of Z. Then if X is integral, the homogeneous height one primes of A correspond to the *maximal σ-irreducible* closed subsets, in other words, the maximal elements among the set of all proper σ-irreducible subsets of X. A similar correspondence holds for the skew and skew-Laurent extensions as in Theorem 1.1(2), taking $X = \operatorname{Spec} S$.

In Sections 4 and 5, we study the possible structure of the maximal σ-irreducible subsets of a variety with automorphism σ, and prove the following very general geometric result, which is of independent interest.

Theorem 1.2. [Theorem 4.2, Theorem 5.7] *Let k be uncountable and algebraically closed, and let X be a quasi-projective integral k-scheme with automorphism $\sigma : X \to X$. Then the following are equivalent:*

(1) *X has uncountably many maximal σ-irreducible closed subsets.*

(2) *X has infinitely many maximal σ-irreducible closed subsets of codimension 1 in X.*

(3) *X has a nonconstant rational function $f \in k(X)$ such that $\sigma(f) = f$.*

We note that given the existence of a rational function f as in part (3) of the theorem, the fibers of the induced rational map $f : X \dashrightarrow \mathbb{P}^1$ give a cover of X by uncountably many σ-invariant codimension-1 closed subsets; it is surprising to find that this is forced by each of the seemingly much weaker conditions in parts (1) and (2).

The previous theorem allows us, for one, to reformulate the study of the cardinality of the set of maximal σ-invariant subsets in a more elegant way. Given $x \in X$, we write $\mathcal{O}_x = \{\sigma^n(x) | n \in \mathbb{Z}\}$. We say that (X, σ) is *ordinary* if for all σ-irreducible closed subsets $Z \subseteq X$, the set $\{z \in Z | \mathcal{O}_z \text{ is Zariski dense in Z}\}$ is open in Z. In the other words, ordinary automorphisms are those for which the property of lying on a dense orbit is an open condition, inside any closed subset Z to which the automorphism restricts. We show in Section 4, using the equivalence between (1) and (3) in Theorem 1.2, that any of our examples will satisfy the DM-equivalence as long as the corresponding geometric data (X, σ) is ordinary. In addition, the equivalence between (2) and (3) in Theorem 1.2 is very useful in the study of which pairs (X, σ) are ordinary, as it shows that any failure of ordinariness must be happening in codimension at least 2. The proof of this latter equivalence in Theorem 1.2 involves a reduction to the case of a countable base field, and so we also study more generally in Section 5 how the structure of the σ-invariant closed subsets is affected by changing the base field.

In Section 6, we restrict to the case that X is integral with $\dim X \leq 2$. We show how the growth of our birationally commutative algebra A is related to the geometric properties of its associated data (X, σ). In case $\mathrm{GK}\, A < \infty$, we say that (X, σ) has *finite growth type*, and we show in particular that this is equivalent to the condition $\mathrm{GK}\, k(X)[t, t^{-1}; \sigma] < \infty$. (If X is projective, finite growth type is also equivalent to σ being *quasi-unipotent* as in [Ke1].) These results extend some material from [Ro], which also relies on a dynamical classification of birational maps of surfaces by Diller and Favre [DF].

In Section 7, we prove that various special kinds of automorphisms of varieties are ordinary, and in particular show the following result:

Theorem 1.3. [Theorem 7.6] *Let k be uncountable, algebraically closed and of characteristic* 0. *Let X be a quasi-projective variety over k with* $\dim X = 2$, *and let $\sigma : X \to X$ be an automorphism of finite growth type. Then (X, σ) is ordinary.*

The proof of this theorem depends on many non-trivial results from the theory of projective surfaces. The main result given above, Theorem 1.1, essentially immediately follows; the proof is given in Section 8, where we also give a number of illustrative examples. Among other things, we review several examples of skew and skew-Laurent extensions that do not satisfy the DM-equivalence.

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2. Primitivity and the prime spectrum

Let A be a k-algebra. The following hypothesis is crucial for the results we are about to review.

Standing Hypothesis 2.1. Throughout this section, k is an uncountable base field.

Later sections of the paper will require varying assumptions on k, and so the appropriate standing hypothesis on k will be stated at the beginning of each section, as well as in the statements of major results.

We first recall some standard results relating the primitivity of a prime ideal P in A to the cardinality of the set of height one primes in A/P. We then specialize to the case of $\mathbb{Z}$-graded birationally commutative algebras A. Such algebras may have many non-homogeneous primes, but we show the somewhat surprising result that to detect the Dixmier-Moeglin equivalence, it typically suffices to examine only the homogeneous prime spectrum of A.

Recall from the introduction that an algebra A satisfies the Dixmier-Moeglin (DM)-equivalence if the sets of locally closed, rational, and primitive ideals of A, as defined there, all coincide. We say that a prime ideal P of a prime algebra A has *height one* if there does not exist a prime ideal Q with $(0) \subsetneq Q \subsetneq P$. Let $\operatorname{Spec} A$ be the set of all prime ideals of A, considered as a topological space using the Zariski topology, whose closed sets are the sets $V(I) = \{P \in \operatorname{Spec} A | P \supseteq I\}$ as I ranges over the ideals of A. For the definitions of other standard notions we use here, such as Jacobson ring, we refer the reader to [MR]. It is not obvious, in general, that a nonzero prime ideal of a noetherian algebra A must contain a height one prime. To ensure this we will usually assume that A has DCC on prime ideals, which will be easy to show in all of the examples in which we are interested. It is a well-known open question whether a noetherian ring must have DCC on prime ideals.

The next two results, Lemmas 2.2 and 2.5, are well-known, but for lack of a single reference we sketch the proofs for the convenience of the reader. The first result shows how the DM-equivalence for the prime (0) comes for free, given a cardinality assumption on the set of height one primes.

Lemma 2.2. *Let A be a prime noetherian, countably generated k-algebra with DCC on prime ideals.*

(1) *Suppose that A has finitely many height one primes. Then* (0) *is locally closed, primitive, and rational.*

(2) *Suppose that A has uncountably many height one primes. Then* (0) *is not locally closed, not primitive, and not rational.*

Proof. Since A is noetherian and $\dim_k A$ has smaller cardinality than $|k|$, it is a standard result that A is a Jacobson ring satisfying the Nullstellensatz over k [BG, Proposition II.7.12, Proposition II.7.16]. In this case, for any prime P we have P locally closed $\implies$ P primitive $\implies$ P rational [BG, Lemma II.7.15].

(1) Suppose that A has finitely many height one primes, say $\{P_1, P_2, \dots, P_n\}$. By the previous paragraph, we just need to prove that (0) is locally closed. Since every prime ideal contains a height one prime, we have $\{(0)\} = \operatorname{Spec} A \setminus V(I)$, where $I = P_1 \cap P_2 \cap \cdots \cap P_n \neq 0$. Thus $\{(0)\}$ is open in its closure, $\operatorname{Spec} A$.

(2) Suppose that A has uncountably many height one primes. By the first paragraph of the proof, we need show only that (0) is not rational. Suppose that A has a countable separating set of ideals, in other words there exist nonzero ideals $\{I_1, I_2, I_3, \dots\}$ of A with the following property: for any nonzero ideal J of A, there is some $n \geq 1$ so that $J \supseteq I_n$. Each height one prime P contains I_n for some n, and since P/I_n is a minimal prime of the noetherian ring A/I_n, we see that at most finitely many height one primes contain a given I_n. This contradicts the assumption that A has uncountably many height 1 primes, so A does not

have a countable separating set of ideals. Since A is noetherian and countably generated, Irving's theorem [Ir, Theorem 2.2] now applies and shows that (0) is not rational. □

The preceding lemma shows that the existence of algebras with a countably infinite number of height one primes is the main obstruction to the DM-equivalence. This suggests the following definition.

Definition 2.3. Given a k-algebra A, we say that $\operatorname{Spec} A$ is *countable-avoiding* if every prime factor algebra of A has either finitely many or uncountably many height one primes.

The following is an immediate consequence of the preceding definition and Lemma 2.2.

Corollary 2.4. *Let A be a noetherian, countably generated k-algebra such that A has DCC on prime ideals and* $\operatorname{Spec} A$ *is countable-avoiding. Then A satisfies the DM-equivalence.*

The preceding corollary applies to many standard kinds of algebras, for example algebras satisfying a polynomial identity (PI algebras).

Lemma 2.5. *Let R be a noetherian PI algebra which is countably generated over k. Then R has DCC on prime ideals,* $\operatorname{Spec} R$ *is countable-avoiding, and R satisfies the DM-equivalence.*

Proof. The fact that R has DCC on prime ideals is a theorem of Small [Sm]. By Corollary 2.4, we need only show that every prime factor ring R/P has a finite or uncountable number of height one primes. Since the hypotheses pass to factor rings, we may assume that R is prime and prove that R has either finitely many or uncountably many height one primes.

Suppose first that R is primitive. By Kaplansky's Theorem, a primitive PI ring is a central simple algebra, so (0) is the only prime of R in this case.

If R is not primitive, let Z be the center of R. By Posner's Theorem, RS^{-1} is a finite-dimensional central simple $L = ZS^{-1}$-algebra, where S is the multiplicative system $Z \setminus \{0\}$. If L/k is algebraic, then RS^{-1} is an algebraic k-algebra and hence R itself is as well. But then every regular element of R is already a unit, and so $RS^{-1} = R$. In this case, R is simple and so certainly primitive, a contradiction. Thus L/k is transcendental. Now suppose that R has at most countably many height one primes $\{P_1, P_2, \dots\}$. Each such prime P_i contains a regular central element $z_i \in P_i$. If T is the multiplicative system generated by the elements $\{z_1, z_2, \dots\}$, then RT^{-1} is already simple (since every prime contains a height one prime) and so $RT^{-1} = RS^{-1}$. But then RS^{-1} is countably generated over k, in particular countable dimensional over k. This is a contradiction because k is uncountable and L/k is transcendental, which forces L and hence RS^{-1} to be uncountable dimensional over k. Thus R has uncountably many height one primes. □

In this paper, we are primarily interested in studying primitivity for certain classes of $\mathbb{Z}$-graded algebras. Let $A = \bigoplus_{n \in \mathbb{Z}} A_n$ be a prime $\mathbb{Z}$-graded noetherian k-algebra. Then the graded quotient ring $Q = Q_{\mathrm{gr}}(A)$ of A exists by [GS, Theorem 1], and by [NV, Theorem I.5.8], Q must have the form $M_n(S)$ (with some choice of grading) for some graded division ring S. Moreover, either $A = A_0$ is trivially graded and $S = S_0 = D$ is a division ring in degree 0, or else $S \cong D[t, t^{-1}; \sigma]$ for some division ring D in degree 0 and element t of positive degree [NV, Corollary I.4.3]. We call A *birationally PI* if D is PI. In practice, all of the specific examples we study later will actually satisfy the stronger condition that either $Q_{\rm gr}(A) = K$ or $Q_{\rm gr}(A) \cong K[t, t^{-1}; \sigma]$ for some commutative ring K and element t of positive degree, in which case we say that A is *birationally commutative*.

Birationally PI algebras are special in the following way.

Lemma 2.6. *Let A be a prime $\mathbb{Z}$-graded noetherian k-algebra which is birationally PI, and let $Q = Q_{\rm gr}(A)$.*

(1) *Q is either PI or a simple ring.*

(2) *If Q is simple, then every nonzero prime ideal of A contains a nonzero homogeneous prime ideal.*

Proof. (1) If $A = A_0$ is trivially graded, then by assumption $Q = D$ is a PI division ring, so that case is certainly fine. Thus assume that $Q \cong M_n(D[t, t^{-1}; \sigma])$ with some choice of grading, where D is PI. Suppose first that some positive power of σ is inner, say $\sigma^n(x) = a^{-1}xa$ for some $0 \neq a \in D$ and $n > 0$. Then $D[t, t^{-1}; \sigma]$ is finitely generated as a module over the subring $D[t^n, t^{-n}; \sigma^n] \cong D[z, z^{-1}]$, where $z = at^n$. So $D[z, z^{-1}]$ is PI [MR, Corollary 13.1.11], and thus $D[t, t^{-1}; \sigma]$ is PI [MR, Corollary 13.4.9]. Then $Q = M_n(D[t, t^{-1}; \sigma])$ is also PI [MR, Theorem 13.4.8]. Otherwise, no power of σ is inner. In this case $D[t, t^{-1}; \sigma]$ is simple by [MR, Theorem 1.8.5], so Q is also simple.

(2) Since the graded quotient ring Q of A is simple, every nonzero prime ideal P of A contains a homogeneous regular element $0 \neq a \in P$. Let $\widetilde{P}$ be the homogeneous ideal generated by all homogeneous elements in P. If I and J are homogeneous ideals such that $IJ \subseteq \widetilde{P}$, then $IJ \subseteq P$ and so $I \subseteq P$ or $J \subseteq P$. But since I and J are homogeneous, this means that $I \subseteq \widetilde{P}$ or $J \subseteq \widetilde{P}$ by the definition of $\widetilde{P}$; we conclude that $\widetilde{P}$ is a nonzero homogeneous prime ideal contained in P. □

In the next proposition, we justify the claim made at the beginning of this section, namely that to prove the Dixmier-Moeglin equivalence for birationally PI algebras, it often is enough to study only the homogeneous prime spectrum. For this purpose, we need the following homogeneous analog of Definition 2.3.

Definition 2.7. Let A be a $\mathbb{Z}$-graded algebra, and let $\operatorname{hSpec} A$ be the homogeneous prime spectrum with the Zariski topology. If A is prime, a homogeneous prime P of A has *h-height one* if there does not exist a homogeneous prime Q with $(0) \subsetneq Q \subsetneq P$. We say that $\operatorname{hSpec} A$ is *countable-avoiding* if every factor ring $B = A/P$ by a homogeneous prime P has either finitely many or uncountably many h-height one primes.

Proposition 2.8. *Let A be a noetherian, countably generated $\mathbb{Z}$-graded algebra such that for every homogeneous prime P of A, A/P is birationally PI; A has DCC on homogeneous primes; and* $\operatorname{hSpec} A$ *is countable-avoiding. Then A has DCC on primes,* $\operatorname{Spec} A$ *is countable-avoiding, and A satisfies the Dixmier-Moeglin equivalence.*

Proof. Since the minimal primes of A are homogeneous [NV, Corollary C.I.1.9], we may assume without loss of generality that A itself is prime.

We first prove that A has DCC on prime ideals. By noetherian induction, we can assume that all proper homogeneous prime factor rings of A have DCC on prime ideals. Applying Lemma 2.6 to the ring A, we see that either A is PI or else every nonzero prime P of A contains a nonzero homogeneous prime $\widetilde{P}$. In the former case, A has DCC on prime ideals by Lemma 2.5, so assume the latter case. Recall from the proof of Lemma 2.6 that $\widetilde{P}$ is the ideal generated by the homogeneous elements in P. Thus a descending chain of nonzero prime ideals of A, $P_1 \supseteq P_2 \supseteq P_3 \supseteq \dots$, leads to a descending chain of nonzero homogeneous primes $\widetilde{P}_1 \supseteq \widetilde{P}_2 \supseteq \dots$ which stabilizes by assumption, with $\widetilde{P}_n = \widetilde{P}_{n+1} = \dots$, say. Now $A/\widetilde{P}_n$ is a proper homogeneous prime factor ring of A and so has DCC on primes by the induction hypothesis. Thus the original chain $P_1 \supseteq P_2 \supseteq \dots$ of primes, all of which contain $\widetilde{P}_n$, must also stabilize.

Now we prove that $\operatorname{Spec} A$ is countable-avoiding. Let M be any prime ideal of A, and let $P \subseteq M$ be a homogeneous prime ideal maximal among the homogeneous primes contained in M. Applying Lemma 2.6 to the birationally PI ring A/P, we have two cases to consider. If $Q = Q_{\mathrm{gr}}(A/P)$ is PI, then A/P is PI, and hence A/M is PI. Then A/M has finitely many or uncountably many height one primes by Lemma 2.5. Otherwise, Q is simple and every nonzero prime of A/P contains a nonzero homogeneous prime. In particular, in this case $M = P$ by choice of P, so M is itself homogeneous. Moreover, since every nonzero prime of A/M contains a nonzero homogeneous prime, it follows that the h-height one primes of A/M are the same as the height one primes of A/M. Thus the assumption that $\operatorname{hSpec} A$ is countable-avoiding implies that A/M has finitely many or uncountably many height one primes. Since M was an arbitrary prime, we conclude that $\operatorname{Spec} A$ is countable-avoiding.

Finally, A satisfies the Dixmier-Moeglin equivalence by Corollary 2.4. □

3. Some examples of birationally commutative algebras

Beginning with this section, we will work with graded algebras whose homogeneous ideals can be described using algebraic geometry. For simplicity, we adopt the following blanket convention on the base field.

Standing Hypothesis 3.1. In this section, k stands for an uncountable, algebraically closed field.

We begin now to introduce some important classes of examples to which Proposition 2.8 applies. The examples in which we are especially interested are twisted homogeneous coordinate rings, because these rings are so important in the theory of noncommutative projective geometry. We will see that for such rings B, the condition that $\operatorname{hSpec} B$ is countable-avoiding, as defined in the previous section, corresponds to a subtle algebro-geometric condition. We also show that this same geometric condition can be used to study the more familiar examples of skew and skew-Laurent extensions of commutative algebras, which will shed new light on the primitive spectra of these examples.

Example 3.2. Given a projective k-scheme X, a k-automorphism σ of X, and an invertible sheaf $\mathcal{L}$ on X, we defined in the introduction the *twisted homogeneous coordinate ring* $B(X, \mathcal{L}, \sigma) = \bigoplus_{n=0}^{\infty} \mathrm{H}^0(X, \mathcal{L}_n)$, where here $\mathcal{L}_n = \mathcal{L} \otimes \sigma^*\mathcal{L} \otimes (\sigma^2)^*\mathcal{L} \otimes \cdots \otimes (\sigma^{n-1})^*\mathcal{L}$ for each n. We always assume when making this construction that $\mathcal{L}$ is σ-*ample*: this means that for every coherent sheaf $\mathcal{F}$ on X, $\mathrm{H}^i(X, \mathcal{F} \otimes \mathcal{L}_n) = 0$ for all $n \gg 0$ and $i > 0$.

The σ-ample hypothesis is necessary in order for the algebra $B(X, \mathcal{L}, \sigma)$ to have well-behaved properties. In particular, in this case B is noetherian, and its homogeneous prime spectrum is completely determined by the following lemma.

Lemma 3.3. *Let $B = B(X, \mathcal{L}, \sigma)$ be a twisted homogeneous coordinate ring as in Example 3.2. Suppose that $Y \subseteq X$ is a reduced closed subscheme such that $\sigma(Y) = Y$ and σ acts on the irreducible components of Y in a single cycle. If $\mathcal{J}$ is the ideal sheaf on X corresponding to Y, then $\bigoplus_{n=0}^{\infty} \mathrm{H}^0(X, \mathcal{J} \otimes \mathcal{L}_n)$ is a homogeneous prime ideal of B. Conversely, every homogeneous prime ideal of B is of this form.*

Proof. The proof of the first assertion is straightforward; we prove the second. Let J be a homogeneous prime ideal of B. By [AV, Theorem 3.12] and [AS1, Lemma 4.4], there is an ideal sheaf $\mathcal{J}$ on X such that $\sigma^*\mathcal{J} = \mathcal{J}$ and, for some N, we have $J_n = H^0(X, \mathcal{J} \otimes \mathcal{L}_n)$ for $n \geq N$. Clearly,

$$J \subseteq \bigoplus_{n \geq 0} H^0(X, \mathcal{J} \otimes \mathcal{L}_n) = J'.$$

If Y is the subscheme of X defined by $\mathcal{J}$, it is easy to see that Y is reduced and that σ acts on the components of Y in a single cycle.

Let $b \in J'_m$ for some $m \geq 1$. As $\mathcal{J}$ is σ-invariant, $(bB)^N \subseteq (J')_{\geq N} = J_{\geq N}$. Primeness of J implies that $b \in J$; thus $J = J'$. □

Given the previous result, the structure of the homogeneous prime spectrum of a twisted homogeneous coordinate ring (with a σ-ample sheaf) is a purely geometric concern, and the study of the geometry involved will occupy us for much of the later sections of the paper. Since we will use it several times later, we note here that it follows easily from the definition that σ-ampleness restricts to any σ-invariant subscheme. More specifically, if Z is a subscheme of X with $\sigma(Z) = Z$, then given a σ-ample sheaf $\mathcal{L}$ on X, $\mathcal{L}|_Z$ is $\sigma|_Z$-ample.

The next examples are likely to be quite familiar to the reader.

Example 3.4. Let S be a commutative finitely generated k-algebra, with k-algebra automorphism $\sigma : S \to S$. Let $U = S[t; \sigma]$ be the skew polynomial ring and let $T = S[t, t^{-1}; \sigma]$ be the skew-Laurent ring. Our convention is to write coefficients on the left; so either ring satisfies the relations $ts = \sigma(s)t$ for all $s \in S$. Let $X = \operatorname{Spec} S$; we use the same name $\sigma : X \to X$ for the scheme automorphism of X corresponding to the algebra automorphism σ.

Notice the similarity between the following characterization of homogeneous primes of such rings and Lemma 3.3.

Lemma 3.5. *Consider Example 3.4.*

(1) *Suppose that* $Z \subseteq X$ *is a reduced closed subscheme such that* $\sigma(Z) = Z$ *and* σ *acts as a single cycle on the irreducible components of* Z*. If* I *is the (radical) ideal of* S *corresponding to* Z*, then* $P = \bigoplus_{n \in \mathbb{Z}} It^n$ *is a homogenous prime ideal of* T*; moreover, all homogeneous prime ideals of* T *have this form.*

(2) *The homogeneous primes of* U *are exactly those of the form* $P \cap U = \bigoplus_{n \geq 0} It^n$ *where* P *is as in part (1), together with those of the form* $Q = J \oplus St \oplus St^2 \oplus \dots$ *where* J *is any prime ideal of* S*.*

Proof. This result is well-known. Cf. [Jo, Lemma 2.3, 2.4] for the claim about T. The claim for U follows since T is the localization of U at the multiplicative system $\{1, t, t^2, \dots\}$. □

Primitivity of prime rings of the form T is characterized in [Jo], and for prime rings of the form U in [LM]; we review the criteria in Section 8. However, it is not clear from these characterizations (which go by constructing faithful simple modules in the primitive case) when the Dixmier-Moeglin equivalence holds for these rings. Our methods will allow us to address this question for all of the examples we have given above, through a study of the geometry of (X, σ). The following definitions will be useful for the geometric point of view.

Definition 3.6. Let X be a scheme of finite type over k, with k-automorphism $\sigma : X \to X$. A closed subset Z of X is called *σ-invariant* if $\sigma(Z) = Z$ and *σ-periodic* if $\sigma^n(Z) = Z$ for some $n \geq 1$. A σ-invariant closed subset Z is called *σ-irreducible* if there does not exist a decomposition $Z = Z_1 \cup Z_2$ with $Z_1, Z_2 \subsetneq Z$ closed σ-invariant subsets. If X is itself σ-irreducible, a *maximal σ-irreducible* subset Z of X is a maximal element of the set of all *proper* σ-irreducible subsets of X under inclusion.

For use in Section 5, we note that the notions defined here make sense for an arbitrary ground field k.

Notice that a σ-invariant subset $Z \subseteq X$ is σ-irreducible if and only if σ permutes the finitely many irreducible components of Z in a single cycle.

Definition 3.7. Let X be a scheme of finite type over k, with k-automorphism $\sigma : X \to X$. We say that the pair (X, σ) is *countable-avoiding* if it satisfies the following property: for every σ-irreducible closed subset $Z \subseteq X$, $(Z, \sigma|_Z)$ has either finitely many or uncountably many maximal σ-irreducible closed subsets.

In the main result of this section, we now show that for all of the examples A above, the property of (X, σ) being countable-avoiding is the geometric equivalent of the property of $\operatorname{hSpec} A$ being countable-avoiding. We also verify that all of these examples satisfy the other hypotheses of Proposition 2.8, so that to prove the DM-equivalence for them it will be enough to study the geometry of (X, σ).

Proposition 3.8. *Let* A *be any one of the following* k*-algebras:*

(1) *A twisted homogeneous coordinate ring* $A = B = B(X, \mathcal{L}, \sigma)$ *as in Example 3.2;*

(2) *A skew-Laurent ring* $A = T = S[t, t^{-1}; \sigma]$ *as in Example 3.4 ; or*

(3) *A skew polynomial ring* $A = U = S[t; \sigma]$ *as in Example 3.4.*

In each case, we have the corresponding geometric data (X, σ)*; we remind the reader that in cases* (2) *and* (3)*,* $X = \operatorname{Spec} S$*. Then* (X, σ) *is countable-avoiding if and only if* $\operatorname{hSpec} A$ *is countable-avoiding, and in this case* $\operatorname{Spec} A$ *is countable-avoiding and* A *satisfies the Dixmier-Moeglin equivalence.*

Proof. In cases (1) and (2), Lemmas 3.3 and 3.5 show that there is a one-to-one inclusion-reversing correspondence between the homogeneous primes of A and σ-irreducible subschemes of X. Then it follows directly from definitions that (X, σ) is countable-avoiding if and only if $\operatorname{hSpec} A$ is countable-avoiding. Now consider case (3), where $A = U = S[t; \sigma] \subseteq T = S[t, t^{-1}; \sigma]$. Lemma 3.5 shows that the topological space $\operatorname{hSpec} U$ is the disjoint union of a subspace $W_1 \cong \operatorname{hSpec} T$ and a subspace $W_2 \cong \operatorname{Spec} S$. Moreover, for a prime $P \in W_2$, the primes of height one over P lie again in W_2; while for a prime P in W_1, the primes of height one over P are the primes of height one over P in W_1 together with finitely many primes from W_2. Since $\operatorname{Spec} S$ is automatically countable-avoiding by Lemma 2.5, it follows that $\operatorname{hSpec} U$ is countable-avoiding if and only if $\operatorname{hSpec} T$ is countable-avoiding, which is if and only if (X, σ) is countable-avoiding as we have already seen.

To finish the proof, we will apply Proposition 2.8. It thus remains only to verify the hypotheses of that proposition, which we do case by case.

(1) Let $A = B = B(X, \mathcal{L}, \sigma)$ as in Example 3.2. Since $\mathcal{L}$ is σ-ample, B is noetherian [Ke1, Theorem 1.2]. Since B is noetherian and $\mathbb{N}$-graded with $B_0 = k$ and $\dim_k B_n < \infty$ for all $n \geq 0$, it follows that the right B-module $k = B/B_{\geq 1}$ has a minimal free resolution by free modules of finite rank, say

$$\cdots \to B^{n_1} \to B^{n_0} \to B \to k \to 0,$$

where n_0 is the minimal number of generators of B as a graded k-algebra and n_1 is the minimum number of relations; see, for example, [ATV1, p. 42-43]. Thus B is finitely presented. It is clear that B has DCC on homogeneous prime ideals, using Lemma 3.3 and the fact that X has finite dimension.

Now consider a homogeneous prime factor ring B/P of B, where P corresponds via Lemma 3.3 to the σ-irreducible closed reduced subscheme $Z \subseteq X$ with ideal sheaf $\mathcal{I}$. Then there is a natural homomorphism $\phi : B = B(X, \mathcal{L}, \sigma) \to B' = B(Z, \mathcal{L}|_Z, \sigma|_Z)$ given by restriction of sections, with kernel P. It is easy to see that ϕ is surjective in large degree, by the definition of σ-ampleness of $\mathcal{L}$. Thus the graded quotient ring of B/P is isomorphic to the graded quotient ring of B'. But $Q_{\mathrm{gr}}(B') \cong R[t, t^{-1}; \sigma]$, where R is the product of the function fields of the irreducible components of Z, and σ is the induced automorphism of R (see, for example, the proof of [RZ, Proposition 3.5].) Thus, every homogeneous prime factor ring of B is birationally commutative, so certainly birationally PI.

(2), (3) Since S is noetherian and finitely generated, both T and U are noetherian, finitely generated k-algebras. It is immediate from Lemma 3.5 that T and U have DCC on homogeneous primes. The fact that all homogeneous prime factor rings of T and U are birationally commutative is also clear from Lemma 3.5. □

We remark that the same idea as in the previous theorem applies equally well to many other birationally commutative algebras, for example the naïve blowups studied in [RS1], or the geometric idealizer rings of [Si]; in each case, the given ring A is a subring of a twisted homogeneous coordinate ring B for which one can show that $\operatorname{hSpec} A$ and $\operatorname{hSpec} B$ are isomorphic. We have chosen to focus here only on a few representative examples.

4. Orbital characterization of ordinary automorphisms

Hypothesis 3.1 remains in force throughout this section. Our eventual goal is to apply Proposition 3.8 to show that many twisted homogeneous coordinate rings satisfy the DM-equivalence. In order to do so, we need first to understand better the property of (X, σ) being countable-avoiding. Clearly, a condition on the cardinality of maximal σ-irreducible subsets is not very natural from a geometric standpoint. In this section we show that the countable-avoiding condition can be restated in a much nicer way, in terms of the orbits of the σ-action on X. This new characterization will be much more readily amenable to further study using geometric methods.

We first record for reference a standard result.

Lemma 4.1. *Let X be an integral scheme of finite type over k. Then X cannot be written as a union of countably many proper closed subsets.*

Proof. This is a well-known consequence of the uncountability of k. A proof in case X is affine may be found in [Bell, Corollary 3.4], and the general case follows immediately from this. □

Suppose that X is an integral scheme for which (X, σ) is countable-avoiding. If X has finitely many maximal σ-irreducible subsets, then the picture is quite clear: the union of these finitely many sets is a σ-invariant closed set Z containing all proper σ-invariant closed sets of X, and for every point $x \notin Z$, x lies on a dense σ-orbit. On the other hand, if X has uncountably many maximal σ-irreducible subsets, it is not immediately clear how to picture σ. For example, can X also have some dense σ-orbits? In the following theorem, we show that in the second case the orbital picture is also clear: surprisingly, X must be completely covered by σ-invariant codimension-1 subschemes arising as the fibers of a σ-invariant rational function. In particular, X has no dense σ-orbits. This proves part of Theorem 1.2 from the introduction.

Theorem 4.2. *Assume k is uncountable and algebraically closed. Let X be an integral scheme of finite type over k, with $\sigma : X \to X$ a k-automorphism and $\sigma : k(X) \to k(X)$ the induced automorphism of the field of rational functions. Then the following are equivalent:*

1. *There exists a non-constant σ-invariant rational function $f \in k(X)$.*
2. *There exists a non-constant σ^m-invariant rational function $f \in k(X)$ for some $m \geq 1$.*
3. *X has no dense σ-orbits.*
4. *X has uncountably many maximal σ-irreducible closed subsets.*

(5) *X has uncountably many σ-irreducible closed subsets of codimension-1.*

Proof. (1) $\implies$ (2) is immediate.

(2) $\implies$ (3): Suppose that $\sigma^m(f) = f \circ \sigma^m = f$ for some $m \geq 1$, and let U be the largest open set of X on which the rational function f is defined. For each $a \in k$, $f^{-1}(a)$ is a σ^m-invariant closed subset of U, which is of codimension-1 because f is nonconstant. The set U is covered by such fibers of f; moreover, $X \setminus U$ is also a σ^m-invariant codimension-1 closed set. So X is covered by uncountably many σ-periodic codimension-1 closed sets, and certainly then there are no dense σ-orbits.

This also shows that (2) $\implies$ (5), and (5) $\implies$ (4) is obvious.

(3) $\implies$ (4): This is a consequence of Lemma 4.1.

(4) $\implies$ (1): Let Ω be an uncountable set of maximal σ-irreducible closed subsets of X. Pick any open affine subset $U = \operatorname{Spec} R$ of X, where R is a domain finitely generated over k. Let $K = k(X)$, the fraction field of R. For each $n \in \mathbb{Z}$, put $V_n = \sigma^n(U)$ and $Y_n = X \setminus V_n$. We claim that Ω contains an uncountable subset Ω' so that for any $Z \in \Omega'$, we have that $Z \not\subseteq \bigcup_{n\in\mathbb{Z}} Y_n$. To see this, note that any finite union of some of the Y_n can contain at most finitely many of the sets in Ω, as the closure of the union of any infinite collection of sets in Ω is σ-invariant and so equal to X.

Without loss of generality, we may replace Ω by Ω'. Then as $\bigcup_{n\in\mathbb{Z}} Y_n$ is σ-invariant, we see that any irreducible component of any $Z \in \Omega$ intersects $V = \bigcap_{n\in\mathbb{Z}} V_n$ nontrivially.

Fix once and for all a countable k-basis $r_1, r_2, r_3, \ldots$ for R. Let T be the union of the closed points in $Z \cap V$ as Z ranges over all sets in Ω. For each $x \in T$, let $H = H_x$ be the closure in X of $\mathcal{O}_x = \{\sigma^n(x) | n \in \mathbb{Z}\}$. Since there is some $Z \in \Omega$ so that $H_x \cap U \subseteq Z \cap U$ is a proper nonempty closed subset of U, there is some nonzero regular function $g \in R$ such that g vanishes along H_x. Write $g = \sum_{i=1}^m a_i r_i$. We may choose such a g with minimal possible m; then g is uniquely determined by x up to scalar and we put $g = g_x$ and $m = m_x$. For each $p \geq 1$, let $T_p = \{x \in T | m_x = p\}$. Note that T_p is σ-invariant. For any $Z \in \Omega$, note also that

$$Z = \Big(\bigcup_{p\geq 1}(Z \cap T_p)\Big) \bigcup \Big(\bigcup_{n\in\mathbb{Z}}(Z \cap Y_n)\Big).$$

Since σ permutes the components of Z in a cycle and each subset $Z \cap T_p$ is σ-invariant, we see by Lemma 4.1 that $Z \cap T_p$ is dense in Z for some $p \geq 1$. As Ω is uncountable, we can find some fixed $p \geq 1$ such that $Z \cap T_p$ is dense in Z for uncountably many $Z \in \Omega$. Letting $S = T_p$ for this p, we see that S has the following properties: S is uncountable; $S \subseteq V$; S is dense in X (since its closure contains infinitely many maximal σ-irreducibles); and for each $x \in S$, $m_x = p$.

For any $n \in \mathbb{Z}$ we write $r_i^{\sigma^n}$ for the rational function $\sigma^n(r_i) = r_i \circ \sigma^n \in K$. For each $n \in \mathbb{Z}$ let $v_n = (r_1^{\sigma^n}, r_2^{\sigma^n}, \ldots, r_p^{\sigma^n}) \in K^p$, and let $W = \operatorname{span}_K\{v_n | n \in \mathbb{Z}\}$. Note that for any $x \in S$, we can evaluate v_n at x, obtaining $v_n(x) = (r_1(\sigma^n(x)), r_2(\sigma^n(x)), \ldots, r_p(\sigma^n(x))) \in k^p$. Let $W(x) = \operatorname{span}_k\{v_n(x) | n \in \mathbb{Z}\}$. Now by construction, $g_x = \sum_{i=1}^p a_i r_i$ is the unique (up to scalar) nonzero k-linear combination of $r_1, \ldots, r_p$ which vanishes at $\sigma^n(x)$ for all $n \in \mathbb{Z}$. Thus $(a_1, \ldots, a_p)$ spans the orthogonal complement to $W(x)$ in k^p, and so

$\dim_k W(x) = p-1$. In fact, then we can find a fixed sequence of integers $n_1, n_2, \ldots, n_{p-1}$ and a dense subset $S' \subseteq S$ such that $v_{n_1}(x), \ldots, v_{n_{p-1}}(x)$ is a k-basis for $W(x)$, for all $x \in S'$ (where here we use Lemma 4.1 again.)

We claim now that $\dim_K W = p-1$. First, suppose that $v_{n_1}, \ldots, v_{n_{p-1}}$ are K-linearly dependent, say $\sum_{j=1}^{p-1} b_j v_{n_j} = 0$ with $b_j \in K$ not all zero. Since S' is dense, we can choose $x \in S'$ such that b_j is defined at x for all $1 \leq j \leq p-1$ and $b_j(x) \neq 0$ for some j. Then evaluating at x we have $\sum_{j=1}^{p-1} b_j(x) v_{n_j}(x) = 0$ and it follows that $v_{n_1}(x), \ldots, v_{n_{p-1}}(x)$ are k-linearly dependent, a contradiction. Thus $v_{n_1}, \ldots, v_{n_{p-1}}$ are linearly independent over K. On the other hand, suppose that $\dim_K W = p$, so we may choose n_p such that $v_{n_1}, \ldots, v_{n_p}$ is a K-basis for W. If we consider the $p \times p$ matrix $M = (M_{ij}) = (r_j^{\sigma^{n_i}}) \in M_p(K)$, then $0 \neq \det M \in K$. However, we know from the previous paragraph that for any $x \in S$, $v_{n_1}(x), \ldots, v_{n_p}(x)$ are linearly dependent over k and so $(\det M)(x) = 0$. Since S is dense in X and $\det M$ is a nonzero rational function, this is a contradiction. So $\dim_K W = p-1$ as claimed.

Pick $F = (f_1, f_2, \ldots, f_p) \in K^p$ which spans the orthogonal complement to W in K^p. The vector $F^\sigma = (f_1^\sigma, f_2^\sigma, \ldots, f_p^\sigma)$ is in the orthogonal complement to W^σ, but by construction $W^\sigma = W$. Thus $F^\sigma = \lambda F$ for some $0 \neq \lambda \in K$ and we conclude that $(f_i/f_j)^\sigma = f_i/f_j$ for any i, j such that $f_j \neq 0$. Finally, we claim that if $f_j \neq 0$, then there is some i so that $b_i = f_i/f_j \notin k$. For if $b_i \in k$ for all i, then we would have a k-dependency $b_1 r_1 + b_2 r_2 + \cdots + b_p r_p = 0$, contradicting the initial choice of the r_i as a k-basis of R. So picking j such that $f_j \neq 0$ and i such that $h = f_i/f_j$ is not in k, we obtain a non-constant σ-invariant rational function h on X. □

With the preceding result in hand, our alternative characterization of the countable-avoiding property, which we call simply *ordinary*, is very easy to prove. Here are the relevant definitions.

Definition 4.3. Let X be a scheme and let $\sigma : X \to X$ be an automorphism. For any $x \in X$, let $\mathcal{O}_x = \{\sigma^n(x) | n \in \mathbb{Z}\}$ be the σ-orbit of X. We say that (X, σ) has *good dense orbits* if the set

$$U = \{x \in X | \mathcal{O}_x \text{ is Zariski dense in } X\}$$

is a (Zariski) open subset of X. We say that (X, σ) is *ordinary* if for all σ-irreducible closed subsets $Z \subseteq X$, $(Z, \sigma|_Z)$ has good dense orbits.

Corollary 4.4. *Let X be a scheme of finite type over k with k-automorphism $\sigma : X \to X$. Then (X, σ) is countable-avoiding if and only if (X, σ) is ordinary.*

Proof. Clearly it suffices to assume that X is reduced and σ-irreducible, and to prove that X has good dense orbits if and only if X has either finitely many or uncountably many maximal σ-irreducible subsets. Replacing σ by some power which fixes each component of X, we see that we may also reduce to the case that X is integral.

Now by Proposition 4.2, X has no dense σ-orbits if and only if X has uncountably many maximal σ-irreducible subsets. On the other hand, it is easy to see that the points lying on a dense σ-orbit form a nonempty open set U if and only if X has only finitely many maximal σ-irreducible subsets, say $Y_1, \dots, Y_n$, whose union is $Y = X \setminus U$. □

5. Base extension

In this section, we study how the maximal σ-irreducible subsets of a pair (X, σ) are related to those of (X', σ), where X' is some base field extension of X. We then offer two important applications. First, we will show that the property of an automorphism having good dense orbits, as in Definition 4.3, is invariant under extension of the base field. This will enable us in later sections, by the usual Lefschetz principle, to change the base field from an uncountable algebraically closed field k of characteristic 0 to the case $k = \mathbb{C}$, so that results from complex algebraic geometry can be applied. The second application is more subtle. Suppose that (X, σ) fails to have good dense orbits, for some integral scheme X. We already know from Theorem 4.2 that in this case, X must have a countably infinite number of maximal σ-irreducible closed subsets. In Theorem 5.7 below, we show the surprising result that only finitely many of those σ-irreducible closed subsets can be of codimension-1 in X. This will a key ingredient in our later analysis of the case where X is a surface. The idea of the proof of Theorem 5.7 is to reduce to the case of a base field F which is finitely generated over its prime subfield, and then use the fact that the divisor class group of a variety over such a field F is a finitely generated group. Thus, we will need to prove our general base field extension results for arbitrary fields.

Notation 5.1. In all of the results in this section, we assume the following setup and notation. Suppose that Y is an integral scheme of finite type over an arbitrary field F. Let $F \subseteq E$ be a field extension and set $X = Y \times_{\operatorname{Spec} F} \operatorname{Spec} E$. Let $\sigma : Y \to Y$ be an automorphism of Y as an F-scheme, and let $\sigma : X \to X$ also denote the induced E-automorphism of X. Let $\pi : X \to Y$ be the projection morphism.

We first single out a few simple observations about this setup.

Lemma 5.2. *Assume Notation 5.1. If F is algebraically closed, then X is also integral.*

Proof. This amounts to the ring theoretic statement that if R is a commutative F-algebra which is a domain, then $R \otimes_F E$ is also a domain. This fact is well-known; for instance see [Ber, Proposition 17.2] for a more general result. □

Lemma 5.3. *Assume Notation 5.1. For every irreducible closed subset $Z \subseteq Y$, every irreducible component of $\pi^{-1}(Z)$ has the same dimension as Z.*

Proof. This result is surely well-known, but for lack of an exact reference, we sketch the proof. One may reduce immediately to the affine case; thus one needs to prove the following fact about rings: given a finitely generated commutative F-algebra R which is a domain with $\dim R = d$, then every minimal prime P of $S = R \otimes_F E$ has $\dim(S/P) = d$. By Noether normalization, we may choose a polynomial ring $F[y_1, y_2, \ldots y_d] \subseteq R$ over which R is finite; then S is finite over $E[y_1, \ldots y_d]$ and so $\dim S = d$. Now if E is purely transcendental over F, then S is a domain and the result follows. Thus we can reduce to the case that E is algebraic over F. In this case, S is both flat (in fact free) over R and integral over R. Given any minimal prime P of S, $P \cap R = 0$ by the going-down theorem [Ma, Theorem 9.5]. Then $\dim S/P > \dim R$ by the going-up theorem [Ma, Theorem 9.4], so $\dim S/P = \dim R = d$. □

Now we prove our main lemma about base extension. The idea of the proof of part (1) is similar to some other results in the literature; for example, cf. [Ir, Theorem 2.1].

Lemma 5.4. *Assume Notation 5.1, and in addition assume that X is integral. Suppose that for every σ-invariant rational function $f \in E(X)$, f is algebraic over E. Then the following hold:*

(1) *If $Z \subsetneq X$ is a proper σ-invariant closed subset of X, then $\overline{\pi(Z)} \subsetneq Y$ is a proper σ-invariant closed subset of Y.*

(2) *The rule $Z \mapsto \overline{\pi(Z)}$ defines a surjective function*

$$\phi : \{maximal\ \sigma - irreducible\ subsets\ of\ X\} \to \{maximal\ \sigma - irreducible\ subsets\ of\ Y\}$$

*which is finite-to-*1 *and dimension-preserving.*

Proof. (1) Choose any open affine set $U' \subseteq Y$ and let $U = \pi^{-1}(U')$. If $U' = \operatorname{Spec} B$, then $U = \operatorname{Spec} A$ where $A = B \otimes_F E$. Let $\mathcal{I}$ be the ideal sheaf of the reduced subscheme Z. Let $I = \mathcal{I}(U)$, which is a nonzero ideal of A. The only nontrivial thing we need to show is that $\overline{\pi(Z)}$ is proper in Y, and for this it will suffice to show that $I \cap B \neq 0$.

Pick a nonzero element

$$x = \sum_{i=1}^{d} b_i \otimes \alpha_i \in I$$

with $b_i \in B$, $\alpha_i \in E$. We choose such an element with the minimal d for all possible choices of open affine subset U'. If $d = 1$, then $x(1 \otimes \alpha_1^{-1})$ is a nonzero element of $I \cap B$ and we are done, so we may assume that $d > 1$. We will deduce a contradiction. By minimality, the α_i are linearly independent over F. Since $\mathcal{I}$ is σ-invariant, we have

$$\sigma(x) = \sum_{i=1}^{d} \sigma(b_i) \otimes \alpha_i \in \mathcal{I}(\sigma^{-1}(U))$$

and thus working on the open affine set $U \cap \sigma^{-1}(U) = \pi^{-1}(U' \cap \sigma^{-1}(U'))$ we see that

$$b_d\sigma(x) - \sigma(b_d)x = \sum_{i=1}^{d-1} (b_d\sigma(b_i) - \sigma(b_d)b_i) \otimes \alpha_i \in \mathcal{I}(U \cap \sigma^{-1}(U)).$$

By minimality of d, this element must be zero. Hence, by linear independence of the α_i, we have

$$b_i/b_d = \sigma(b_i/b_d)$$

as elements of $F(Y) = Q(B)$. Thus $b_i/b_d \in E(X) = Q(A)$ is algebraic over E for $1 \leq i \leq d$ by hypothesis.

Since A is a domain, the localization $Q(B) \otimes_F E$ is also a domain, and it contains the elements b_i/b_d for all i. Since these elements are algebraic over E, $L = E[b_1/b_d, b_2/b_d, \ldots, b_{d-1}/b_d]$ is a finite-dimensional E-subspace of a domain, and hence a field. Clearly there is $m > 0$ such that $b_d^m \lambda \in B \otimes_F E = A$ for all $\lambda \in L$. Write $y = x(b_d \otimes 1)^{-1} = \sum_{i=1}^d b_i/b_d \otimes \alpha_i \in L$. Since $y \neq 0$ it has an inverse $y^{-1} = \sum_j \gamma_j \otimes \beta_j \in L$. Then

$$0 \neq x' = y^{-1}(b_d^m \otimes 1) = \sum_j (b_d^m \gamma_j) \otimes \beta_j \in A$$

and

$$0 \neq xx' = (b_d \otimes 1)yy^{-1}(b_d^m \otimes 1) = b_d^{m+1} \otimes 1 \in I,$$

contradicting the assumption that d is greater than 1. Thus the case $d > 1$ cannot occur.

(2) Given a σ-irreducible closed subset $W \subseteq Y$, $\pi^{-1}(W)$ is clearly σ-periodic, so is a finite union of σ-irreducibles of X. Moreover, all components of $\pi^{-1}(W)$ have the same dimension as W, by Lemma 5.3. Conversely, if $Z \subseteq X$ is a σ-irreducible closed subset, it is clear that $Z' = \overline{\pi(Z)}$ is σ-irreducible in Y.

Now if Z as above is a *maximal* σ-irreducible, then Z' is a proper subset of Y by part (1), so $\pi^{-1}(Z')$ is a proper subset of X and by maximality Z is a union of components of $\pi^{-1}(Z')$. In particular, in this case $\dim Z' = \dim Z$. Also, Z' is a maximal σ-irreducible of Y: otherwise $Z' \subsetneq W \subsetneq Y$ for some σ-irreducible W, and then $Z \subsetneq \pi^{-1}(W) \subsetneq X$ by dimension considerations, a contradiction. Thus there is a well-defined function ϕ as in the statement of the lemma, and it preserves dimensions. Conversely, if W is a maximal σ-irreducible of Y, then picking any maximal σ-irreducible Z of X such that Z contains a component of $\pi^{-1}(W)$, we have $W \subseteq Z' = \overline{\pi(Z)}$ and thus $Z' = W$, so ϕ is surjective. Moreover, we must have $\dim Z = \dim Z' = \dim W = \dim \pi^{-1}(W)$, so Z is a union of components of $\pi^{-1}(W)$. It follows that ϕ is finite-to-one. □

Next, we immediately apply the preceding lemma, to show that the property of having good dense orbits is invariant under extension of the base field.

Lemma 5.5. *Assume Notation 5.1, and suppose in addition that $F = k$ and $E = \ell$ are algebraically closed uncountable fields. Then (X, σ) has good dense orbits if and only if (Y, σ) has good dense orbits.*

Proof. Since k is algebraically closed, and Y is integral, the extended scheme $X = Y \times_{\operatorname{Spec} k} \operatorname{Spec} \ell$ is also integral by Lemma 5.2. Suppose first that (X, σ) has a dense orbit. Then by Theorem 4.2, we see that X has no non-constant σ-invariant rational functions, and so the hypotheses of Lemma 5.4 hold. Thus by Lemma 5.4(2), the number of maximal σ-irreducible closed subsets of Y is either finite or uncountable if and only if the same holds for X. Then by the proof of Corollary 4.4, (X, σ) has good dense orbits if and only if (Y, σ) does.

If, on the other hand, (X, σ) has no dense orbits, then it will suffice to prove that (Y, σ) also has no dense orbits. Note that given a closed point $y \in Y$, then $x = \pi^{-1}(y)$ is also a closed point of X, since k

is algebraically closed. We claim that if y lies on a dense σ-orbit, then so does x. In fact, a more general statement holds: given a set of closed points $S \subseteq Y$, if S is dense in Y then $\pi^{-1}(S)$ is dense in X. To see this, one may reduce to the affine case, say $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$ with $A = B \otimes_k \ell$. If for $y \in S$ we let I_y be the corresponding maximal ideal of B, then by assumption $\bigcap I_y = 0$; thus $\bigcap (I_y \otimes \ell) = 0$ in A and the claim follows. □

We now move toward our second application of the base field extension lemma. We first recall in the next lemma some needed background results, which are presumably well-known. We work in the rest of this section with Weil divisors on a normal variety X; see [Ha, Section II.6] for more background. Recall that the divisor class group of X, $\operatorname{Cl} X$, is the group of Weil divisors modulo linear equivalence (which might be different from the Picard group $\operatorname{Pic} X$ if X is not locally factorial.)

Lemma 5.6. *Let X be a normal quasi-projective variety over a field F.*

(1) *If F is a finitely generated extension of its prime subfield, then $\operatorname{Cl} X$ is a finitely generated abelian group.*
(2) *If $G = \Gamma(X, \mathcal{O}_X)^*$ is the group of units of the global regular functions on X, then $G/(\overline{F}^* \cap G)$ is a finitely generated abelian group.*

Proof. (1) By definition, X can be embedded as an open subset of a projective variety Y; replacing Y by its normalization if necessary (which does not affect the open set X), we may assume that Y is normal and projective. Now $\operatorname{Cl} X$ is a surjective image of $\operatorname{Cl} Y$ by [Ha, Proposition II.6.5], so we need only show that $\operatorname{Cl} Y$ is finitely generated. This is immediate from the version of the Mordell-Weil-Néron-Severi theorem in [La2, Corollary 6.6.2].

(2) As in part (1), we may embed X as an open subset of a normal projective Y. Then $Y \setminus X$ has finitely many irreducible components, some of which are of codimension 1, say $Z_1, \dots, Z_n$. We thus have a homomorphism from G to the additive group $\mathbb{Z}^n$, given by $f \mapsto (v_{Z_1}(f), \dots, v_{Z_n}(f))$, where v_{Z_i} is the discrete valuation associated to the divisor Z_i. The kernel of this homomorphism is $\Gamma(Y, \mathcal{O}_Y)^*$, which, since Y is projective, is contained in $\overline{F}^*$. The result follows. □

We now give the result promised at the beginning of this section, which shows in particular that if (X, σ) fails to have good dense orbits, all but a finite number of the countably many maximal σ-irreducible subsets must be of codimension-2 or smaller. This completes the proof of Theorem 1.2 from the introduction.

Theorem 5.7. *Let k be uncountable and algebraically closed, and let X be a quasi-projective integral scheme over k with k-automorphism $\sigma : X \to X$. If X has infinitely many σ-irreducible codimension-1 closed subsets, then there exists a non-constant σ-invariant rational function $f \in k(X)$. Consequently, σ does not have a dense orbit, and there are uncountably many σ-irreducible codimension-1 closed subsets.*

Proof. Let X have infinitely many σ-irreducible codimension-1 closed subsets. Since the hypotheses pass to the normalization $\widetilde{X}$ of X, we may assume that X is normal. It is standard that we can pick a finitely generated extension F of the prime subfield of k so that X and σ are defined over F. In other words, we can find a quasi-projective F-variety Y such that $X = Y \times_{\operatorname{Spec} F} \operatorname{Spec} k$ and an automorphism $\sigma : Y \to Y$ which induces the given $\sigma : X \to X$. Thus we are in the situation of Notation 5.1, with $E = k$. As there, let $\pi : X \to Y$ be the projection. Note that Y is also normal.

We are certainly done if there exists a nonconstant $f \in k(X)$ with $\sigma(f) = f$, so we may assume that this is not the case, and then the hypotheses of Lemma 5.4 hold. Note that as a special case of the correspondence in part (2) of that lemma, the rule $Z \to \overline{\pi(Z)}$ defines a finite-to-one mapping from the set of σ-irreducible codimension-1 closed subsets of X to the set of σ-irreducible codimension-1 closed subsets of Y. In particular, we conclude that Y also has infinitely many σ-irreducible codimension-1 closed subsets.

Now by Lemma 5.6, $\operatorname{Cl} Y$ and $H = \Gamma(Y, \mathcal{O}_Y)^*/(\Gamma(X, \mathcal{O}_Y)^* \cap \overline{F}^*)$ are finitely generated abelian groups. Moreover, it easy to see that H is torsionfree; let d be its torsionfree rank. We may find a sequence of distinct irreducible codimension-one closed subsets $Y_1, Y_2, \dots$ which are σ-periodic. Let $[Y_i]$ be the divisor class in $\operatorname{Cl} Y$ determined by Y_i. Then there is some n such that in $\operatorname{Cl} Y$,

$$[Y_m] \in \sum_{i=1}^{n} \mathbb{Z}[Y_i]$$

for all m. For each $m > n$, there is some rational function f_m on Y with corresponding principal divisor $(f_m) = [Y_m] - a_{1,m}[Y_1] - \cdots - a_{n,m}[Y_n]$ for some integers $a_{i,j}$. By assumption, there is some q (depending on m) such that $\sigma^{qa}(f_m)$ has the same divisor as f_m for all integers a. Thus we may pick some number p such that for $m \in [n+1, n+d+1]$ we have

$$g_m := \sigma^p(f_m)/f_m$$

is in $\Gamma(Y, \mathcal{O}_Y)^*$. It follows that the images of $g_{n+1}, g_{n+2}, \dots, g_{n+d+1}$ in H have some nontrivial relation. That is, for some integers b_j (not all zero), we have

$$\alpha = \prod_{j=1}^{d+1} g_{n+j}^{b_j} \in \overline{F}^*.$$

Then

$$f = \prod_{j=1}^{d+1} f_{n+j}^{b_j}$$

is a rational function such that $\sigma^p(f) = \alpha f$. Moreover, if j is the largest index such that $b_j \neq 0$, then f has a zero or pole in the dvr corresponding to Y_{n+j}, so it is not in $\overline{F}$.

Notice that we used only finitely many of the Y_i above. Throwing those away, we can repeat the process above and construct another non-constant rational function f' with $\sigma^q(f') = \alpha' f'$, and such that the codimension-1 subsets appearing as zeroes or poles in (f') are completely different from those appearing in (f). Repeating this process, we can construct rational functions $h_0, h_1, \dots, h_s$, where $s = \dim Y$ is the transcendence degree of $F(Y)$ over F, satisfying $\sigma^p(h_i) = \alpha_i h_i$ for some fixed p, such that $(h_i) \neq 0$ and no codimension-1 irreducible subset of Y appears in more than one divisor (h_i). The elements $h_0, \dots, h_s$ are algebraically dependent over F, so we may pick a nonzero polynomial $P(x_0, \dots, x_s) \in F[x_0, \dots, x_s]$ such that $P(h_0, \dots, h_s) = 0$. Choose such a P with a minimal number of monomial terms with nonzero coefficient. By construction, $\sigma^p(P(h_0, \dots, h_s)) = P(\alpha_0 h_0, \dots, \alpha_s h_s) = 0$. If $x_0^{i_0} \cdots x_s^{i_s}$ and $x_0^{j_0} \cdots x_s^{j_s}$ are distinct monomials with nonzero coefficient in P, then we must have

$$\prod_\ell \alpha_\ell^{i_\ell} = \prod_\ell \alpha_\ell^{j_\ell},$$

since otherwise we would contradict the minimality of P. Thus

$$h \ = \ \prod_\ell h_\ell^{i_\ell - j_\ell}$$

is σ^p-invariant, and since the (h_ℓ) all involve different codimension-1 subsets, there is no cancelation and so $(h) \neq 0$. Thus $h \notin \overline{F}$, so the element $(h \otimes 1)$ of $F(Y) \otimes_F k \subseteq k(X)$ is a rational function on X which is σ^p-invariant and not in k.

All statements in the theorem now follow immediately from Theorem 4.2. □

6. Growth type

For the results of this section only, the cardinality of k is not relevant, and so we assume the following

Standing Hypothesis 6.1. Throughout this section, k is an algebraically closed field.

Suppose that A is one of the examples introduced in Section 3, with corresponding geometric data (X, σ) where X is a quasi-projective scheme. Theorem 1.3 suggests that there is some connection between the growth of A and the geometry of (X, σ), and in this section we begin to explain this connection.

In fact, we will work with birationally commutative algebras more generally. Suppose that A is a finitely generated connected $\mathbb{N}$-graded k-algebra, which is a noetherian domain with $Q_{\mathrm{gr}}(A) = K[t, t^{-1}; \sigma]$ where K is a field. In [RZ, Theorem 1.4], it was shown that whether A has finite GK-dimension or not depends only on (K, σ). This idea was explored further in [Ro], in the special case where K/k is a finitely generated extension with $\operatorname{tr.deg} K/k = 2$; specifically, it was shown how to exactly characterize which (K, σ) correspond to algebras A of finite GK-dimension. In this section, we review the results of [Ro] and prove some generalizations which are useful to know in themselves, as well as being applicable to our study of ordinary automorphisms in the next section. In particular, we show that the case where $\operatorname{GK} A < \infty$ above can in fact be characterized by the condition $\operatorname{GK} K[t, t^{-1}; \sigma] < \infty$; this extends [RZ, Theorem 1.6]. This will also allow us to extend the theory of [Ro] to $\mathbb{Z}$-graded algebras such as skew and skew-Laurent extensions.

We now recall some needed definitions from [Ro].

Definition 6.2. Let $\sigma : K \to K$ be an automorphism of a finitely generated field extension of k with $\operatorname{tr.deg} K/k = 2$. We define associated *growth data* (ρ, j) with real number $\rho \geq 1$ and integer $j \geq 0$, as follows. Pick any nonsingular projective surface X with $k(X) = K$, and consider the birational map $\sigma : X \dashrightarrow X$ induced by $\sigma : K \to K$. Let $N^1(X)$ be the group of Cartier divisors on X modulo numerical equivalence. Each power σ^n induces a homomorphism $(\sigma^n)^* : N^1(X) \to N^1(X)$, defined essentially by pullback of divisors (see [Ro, Definition 2.1] and the discussion following.) The group $N^1(X)$ is free abelian of finite rank and so each $(\sigma^n)^*$ corresponds to an integer matrix. Finally, choosing any matrix norm, the sequence $f(n) = ||(\sigma^n)^*||$ is equivalent to the sequence $g(n) = n^j \rho^n$ for a unique (ρ, j) as above (independent of X), in the sense that $\lim_{n\to\infty} f(n)/g(n) < \infty$ [Ro, Lemma 2.12]. Finally, we say that (K, σ) (or (X, σ)) has *finite growth type* if the associated growth data has $\rho = 1$, and *infinite growth type* if $\rho > 1$.

The main results of [Ro] depend on a beautiful classification of the possible values of (ρ, j) that can occur in Definition 6.2, which is due (in case $k = \mathbb{C}$) to Diller and Favre [DF]. We recall in the next theorem the part of the classification corresponding to finite growth type, for which we need a few more definitions. Given a birational self-map of a projective variety $\sigma : X \dashrightarrow X$, we say that it is *conjugate* to a birational map $\tau : Y \dashrightarrow Y$ if there is a birational map $\theta : X \dashrightarrow Y$ such that $\tau\theta = \theta\sigma$ (as birational maps). A surjective morphism $f : X \to C$ from a integral nonsingular surface X to a nonsingular curve C is called a *fibration*, and it is called *σ-invariant* if there is an automorphism $\sigma' : C \to C$ such that $\sigma' f = f\sigma$. The fibration is *rational* if the generic fiber is a rational curve and *elliptic* if the generic fiber is elliptic.

Theorem 6.3. [DF, Theorem 0.2] *Let $\sigma : X \dashrightarrow X$ be a birational map of an integral projective surface X over $\mathbb{C}$. Define the growth data (ρ, j) of $(k(X), \sigma)$ as in Definition 6.2. If $\rho = 1$, then possibly after replacing $\sigma : X \dashrightarrow X$ with a conjugate map $\tau : Y \dashrightarrow Y$, where Y is a nonsingular projective surface, we can assume that exactly one of the following cases occurs:*

(1) *$j = 0$, $\tau : Y \to Y$ is an automorphism, and $(\tau^n)^* : N^1(Y) \to N^1(Y)$ is the identity map for some $n \geq 1$.*
(2) *$j = 1$, and there is a τ-invariant rational fibration $f : Y \to C$.*
(3) *$j = 2$, $\tau : Y \to Y$ is an automorphism, and there is a τ-invariant elliptic fibration $f : Y \to C$.*

Theorem 6.3 will be used in a crucial way in the next section, where we will use the detailed information the classification provides to help us prove that if (X, σ) has finite growth type, then it is ordinary. In the remainder of this section, we prove an alternative characterization of the property of finite growth type which includes the case of $\mathbb{Z}$-graded algebras. Consider the $\mathbb{Z}$-graded ring $Q = K[t, t^{-1}; \sigma]$ where K/k is a finitely generated field extension and t has degree 1. We say that a $\mathbb{Z}$-graded k-subalgebra $A \subseteq Q$ is *big* in Q if $A \neq A_0$ and there exist finitely many homogeneous elements $a_i, b_i \in A$, where a_i, b_i have the same degree d_i for each i, such that the elements $\{a_i b_i^{-1}\}$ generate $Q_0 = K$ as a field extension of k. If A is a graded Ore domain, it is clearly big in Q if and only if $Q_{\mathrm{gr}}(A)$ is equal to some Veronese ring of Q; the definition of big allows us to avoid worrying about the Ore condition.

Proposition 6.4. *Let K/k be a finitely generated field extension with* $\operatorname{tr.deg} K/k = 2$*, where k is algebraically closed, and let $\sigma \in \operatorname{Aut}_k K$. Put $Q = K[t, t^{-1}; \sigma]$. Then the following are equivalent:*

(1) *(K, σ) has finite growth type;*

(2) $\operatorname{GK} Q < \infty$*;*

(3) *There exists a $\mathbb{Z}$-graded big subalgebra $A \subseteq Q$ such that* $\operatorname{GK} A < \infty$.

Proof. (1) $\implies$ (2): If there exists a nonsingular projective surface X with $K = k(X)$ such that the corresponding induced map on X is an *automorphism* $\sigma : X \to X$, the result $\operatorname{GK} Q < \infty$ has already been shown in [RZ, Theorem 1.6]. We now show that a similar argument works in general, using also the techniques from [Ro]. (Thus this argument is needed only if case (2) of Theorem 6.3 occurs, but it does not seem to help to restrict to this case.)

First, given any nonsingular projective surface X with $K = k(X)$, we have an induced birational map $\sigma : X \dashrightarrow X$. The birational map σ induces a homomorphism $\sigma^* : \operatorname{Pic} X \to \operatorname{Pic} X$ which is essentially pullback of divisors [Ro, Definition 2.1]. Now for an invertible sheaf $\mathcal{M} \subseteq \mathcal{K}$, where $\mathcal{K}$ is the constant sheaf of rational functions on X, we write $\mathcal{M}^\sigma$ for $\sigma^*\mathcal{M} \subseteq \mathcal{K}$. We then set $\mathcal{M}_n = \mathcal{M} \otimes \mathcal{M}^\sigma \otimes \cdots \otimes \mathcal{M}^{\sigma^{n-1}} \subseteq \mathcal{K}$ for each $n \geq 1$. For a k-subspace $V \subseteq K$, we also write $V^\sigma = \sigma^{-1}(V)$. Now by [Ro, Theorem 2.10], we can choose X initially so that σ is *stable* in the sense of [Ro, Definition 2.7]; the important consequence of this is that if $V = \mathrm{H}^0(X, \mathcal{M}) \subseteq K$, then $V V^\sigma \dots V^{\sigma^{n-1}} \subseteq \mathrm{H}^0(X, \mathcal{M}_n)$ [Ro, Lemma 4.1]. If $\mathcal{M}$ is a very ample invertible sheaf on X, possibly after replacing $\mathcal{M}$ by a large tensor power we also have the following good vanishing properties: $\mathrm{H}^i(X, \mathcal{M}_n^{\otimes m}) = 0$ for all $i \geq 1$, $n \geq 1$, $m \geq 1$ [Ro, Lemma 4.4].

We now follow the proof of [RZ, Theorem 1.6]. The argument there shows that the growth of any finitely generated subalgebra of Q is bounded above by the growth of the sequence $(2n+1)\dim_k \left(W W^\sigma \dots W^{\sigma^{2n}}\right)^n$ for some subspace $W \subseteq K$ with $1 \in W$. We may choose a very ample sheaf $\mathcal{M} \subseteq \mathcal{K}$ with good vanishing properties as above, and such that $W \subseteq \mathrm{H}^0(X, \mathcal{M}) \subseteq K$ [RZ, Lemma 5.2]. Then

$$\left(W W^\sigma \dots W^{\sigma^{2n}}\right)^n \subseteq \mathrm{H}^0(X, \mathcal{M}_{2n+1}^{\otimes n}) \subseteq K.$$

Since all of the sheaves $\mathcal{M}_{2n+1}^{\otimes n}$ have vanishing higher cohomology, the Riemann Roch formula gives that $\dim_k \mathrm{H}^0(X, \mathcal{M}_{2n+1}^{\otimes n})$ grows like the intersection numbers $(\mathcal{M}_{2n+1}^{\otimes n}.\mathcal{M}_{2n+1}^{\otimes n} \otimes \omega^{-1})$, where ω is the canonical sheaf on X. Finally, these intersection numbers have polynomial growth in n, by [Ro, Lemma 5.4, 5.6]; in fact, they grow no faster than n^{j+4}, where $(1, j)$ is the growth data of σ as in Definition 6.2. So one has the even stronger result that $\operatorname{GK} Q \leq j + 5$.

(2) $\implies$ (3): This is trivial.

(3) $\implies$ (1): We show the contrapositive, so suppose that (K, σ) has infinite growth type. The idea is to reduce to the $\mathbb{N}$-graded case, where the result has already been shown in [Ro]. Suppose that $A \subseteq Q$ is a $\mathbb{Z}$-graded big subalgebra, so $A \neq A_0$. Choose homogeneous elements $a_i, b_i \in A$, where $a_i, b_i \in A_{d_i}$ for each i, such that the elements $\{a_i b_i^{-1}\}$ generate K as a field extension of k. Suppose that A contains a

nonzero homogeneous element of positive degree, say $s \in A_d$ for some $d > 0$. Then for $n \gg 0$, the elements $a'_i = a_i s^n, b'_i = b_i s^n$ all have positive degree, and $A' = k\langle a'_1, \dots, a'_n, b'_1, \dots, b'_n \rangle$ is now a connected finitely generated $\mathbb{N}$-graded big subalgebra of Q with $A' \subseteq A$. By [Ro, Theorem 1.1, Theorem 7.1], since (K, σ) has infinite growth type, A' has exponential growth, so the same is true of A. If A contains instead an element of negative degree, apply the same argument to the graded ring $\widetilde{A} = \bigoplus_{n \in \mathbb{Z}} A_{-n}$. $\square$

To conclude this section, we show that the property of finite growth type is invariant under base extension.

Lemma 6.5. *Let $k \subseteq \ell$ with ℓ also algebraically closed. Given an extension $k \subseteq K$ where $\operatorname{tr.deg}_k K = 2$, let L be the field of fractions of $K \otimes_k \ell$. Given an automorphism $\sigma \in \operatorname{Aut}_k K$, consider the induced automorphism $\sigma' \in \operatorname{Aut}_\ell L$. Then (K, σ) has finite growth type if and only if (L, σ') does.*

Proof. Note that $K \otimes_k \ell$ is a domain by Lemma 5.2, so we can indeed form the field of fractions L. Let $Q = K[t, t^{-1}; \sigma]$, $Q' = L[t, t^{-1}; \sigma']$, and $R = Q \otimes_k \ell \cong (K \otimes_k \ell)[t, t^{-1}; \sigma']$. Then $\operatorname{GK}_k Q = \operatorname{GK}_\ell R$, and R is a big ℓ-subalgebra of Q'. The result now follows immediately from Proposition 6.4. $\square$

7. Ordinary automorphisms

Suppose that X is a scheme of finite type over k with automorphism σ. The goal of this section is to study when σ is ordinary using the methods of algebraic geometry. Thus we assume Hypothesis 3.1 throughout this section.

The most general method we present for proving that an automorphism σ is ordinary is the following: if σ is an element of an algebraic group of automorphisms, then σ is ordinary (see Proposition 7.4 below.) This result applies to some important special cases of automorphisms of varieties of arbitrary dimension, for example any automorphism of a complete toric variety. We then examine the case where X is a surface in greater detail. Here, we will need to apply much more detailed information about the automorphism groups of surfaces to prove Theorem 1.3 from the introduction, namely that automorphisms of finite growth type are ordinary.

In this section, all divisors on X will be Cartier divisors, and we use $\sim$ for linear equivalence of divisors. We start by recording some of the more elementary observations about ordinary automorphisms. We will use the following easy lemma, whose proof we leave to the reader, without comment below. In particular, the lemma shows that in studying ordinary automorphisms we can reduce to the case of an integral scheme.

Lemma 7.1. *Let X be a finite type k-scheme with k-automorphism σ.*

(1) *(X, σ) is ordinary if and only if (X_{red}, σ) is.*
(2) *For any $n \geq 1$, (X, σ) is ordinary if and only if (X, σ^n) is ordinary.*
(3) *Let X be a reduced scheme with irreducible components $X_1, \dots, X_d$. Then choosing any $n \geq 1$ so that $\sigma^n(X_i) = X_i$ for all i, (X, σ) is ordinary if and only if $(X_i, \sigma^n|_{X_i})$ is ordinary for all i.*

Next, we see that the case of a curve is easy. We remark that part (1) of the next lemma also follows from Theorem 5.7, but we prefer to give a more direct proof.

Lemma 7.2. *Let $\sigma : X \to X$ be an automorphism of a finite-type k-scheme X.*

(1) *If $\dim X \leq 1$, then (X, σ) is ordinary.*

(2) *If X is σ-irreducible and $\dim X = 2$, then (X, σ) is ordinary if and only if (X, σ) has good dense orbits.*

Proof. (1) By Lemma 7.1, we may assume X is integral. The case $\dim X = 0$ is trivial, so assume $\dim X = 1$. There is a birational map $f : Y \dashrightarrow X$ where Y is the unique nonsingular projective curve with $k(Y) = k(X)$, and an induced automorphism $\tau : Y \to Y$ such that $f\tau = \sigma f$; it clearly suffices to show that (Y, τ) is ordinary. As is well-known, any automorphism of Y is either of finite order or else its set of periodic points consists of $0, 1$, or 2 fixed points (for example, see [AS1, p.14]). The result easily follows.

(2) Again we may reduce to the integral case, and the result is an immediate consequence of part (1). □

The property of having good dense orbits passes between schemes related by a dominant rational map.

Lemma 7.3. *Let X and Y be integral finite-type k-schemes with $\dim X = \dim Y$, and let $f : X \dashrightarrow Y$ be a dominant rational map. Suppose that $\sigma : X \to X$ and $\tau : Y \to Y$ are automorphisms such that $f\sigma = \tau f$ (as rational maps). Then (X, σ) has good dense orbits if and only if (Y, τ) has good dense orbits.*

Proof. Let $U \subseteq X$ be the maximal open set on which f is defined; then U is σ-invariant. Now (X, σ) clearly has good dense orbits if and only if $(U, \sigma|_U)$ does, so we may replace X by its open subscheme U and suppose from now on that f is a dominant morphism.

Let $V \subseteq X$ be the set of points lying on a dense σ-orbit, and let $W \subseteq Y$ be the set of points which lie on a dense τ-orbit. It follows easily from the facts that $\dim X = \dim Y$ and f is dominant that $f^{-1}(W) = V$. Thus if (Y, τ) has good dense orbits then so does (X, σ).

Suppose conversely that (X, σ) has good dense orbits. Since f is dominant, $f(X)$ contains an open dense subset of Y; letting $T = \overline{Y \setminus f(X)}$, it is clear that T is a proper closed τ-invariant subset of Y. Thus if X has no dense σ-orbit, then Y has no dense τ-orbit. Assume now that X does have a dense orbit, so V is an open dense subset. Putting $Z = X \setminus V$, we see that $\overline{f(Z)}$ is a proper closed τ-invariant subset in Y. Thus $W = Y \setminus (T \cup \overline{f(Z)})$ is open and (Y, τ) has good dense orbits in this case as well. □

Next, we turn to our most general methods for proving an automorphism has good dense orbits. We emphasize that for us an algebraic group is a reduced group scheme of finite type over the ground field.

Proposition 7.4. *Let X be an integral scheme of finite type over k with automorphism σ.*

(1) *If an algebraic group G acts on X with some $g \in G$ acting by σ, then (X, σ) is ordinary.*

(2) *If D is an ample Cartier divisor on X with $(\sigma^n)^* D \sim D$ for some $n \geq 1$, then (X, σ) is ordinary.*

(3) *If D is a big Cartier divisor on X with $(\sigma^n)^*D \sim D$ for some $n \geq 1$, then (X, σ) has good dense orbits.*

Proof. (1) By Corollary 4.4, we need to show, given any σ-irreducible closed subset $Z \subseteq X$, that $(Z, \sigma|_Z)$ has good dense orbits. Let $H = \overline{\{g^n | n \in \mathbb{Z}\}} \subseteq G$. Then H acts on X, and the H-invariant subschemes of X are precisely the σ-invariant subschemes of X. Since H thus also acts on any σ-invariant Z, it is enough to show that (X, σ) has good dense orbits. If X has no dense σ-orbits, this is true by definition. If $x \in X$ has a dense σ-orbit, let $U = Hx$. Any orbit of an algebraic group action is locally closed [Sp, Lemma 2.3.3], so U is open inside $\overline{U} = X$. As H is abelian, U is precisely the set of those $y \in X$ with a dense σ-orbit.

(2) We may replace σ by σ^n without loss of generality. Then some multiple mD of D is very ample and $\sigma^*(mD) \sim mD$. Taking the projective embedding $\phi : X \to \mathbb{P}^d$ corresponding to the complete linear series $|mD|$, we see that σ lifts to an automorphism $\widetilde{\sigma} : \mathbb{P}^d \to \mathbb{P}^d$. Since the automorphism group $\mathrm{PGL}(d+1, k)$ of $\mathbb{P}^d$ is an algebraic group, the result follows from part (1).

(3) Again we replace σ by σ^n. In this case, for some multiple mD, the rational map $\phi : X \dashrightarrow \mathbb{P}^d$ associated to the complete linear series $|mD|$ has the property $\dim Y = \dim X$, where $Y = \overline{\phi(X)}$. As in part (2), there is a compatible automorphism $\widetilde{\sigma} : \mathbb{P}^d \to \mathbb{P}^d$, so $(Y, \widetilde{\sigma}|_Y)$ is ordinary by part (1). Then (X, σ) has good dense orbits by Lemma 7.3. □

We remark that an automorphism of a variety of general type, which satisfies part (3) of the preceding lemma for the canonical divisor, is ordinary for a more trivial reason: such a variety has a finite automorphism group.

In general, it is not obvious how to check if an automorphism σ belongs to an algebraic group of automorphisms, so that part (1) of the preceding proposition applies. The following general remarks help to clarify the issue. Let X be a projective integral k-scheme. Then $\mathrm{Aut}(X)$ has the natural structure of a group scheme [Han, Proposition 2.3], which is constructed as an open subscheme of the Hilbert scheme $\mathrm{Hilb}(X \times X)$. This Hilbert scheme is a countable disjoint union of projective schemes, so $\mathrm{Aut}(X)$ is a countable disjoint union of finite type schemes. Then $\mathrm{Aut}^0(X)$, the irreducible component of $\mathrm{Aut}(X)$ containing the identity, is an algebraic group. Clearly a sub-group scheme of $\mathrm{Aut}(X)$ is an algebraic group if and only if it is contained in finitely many components.

The techniques in the preceding proposition are enough to show that a number of special kinds of automorphisms of varieties of arbitrary dimension are ordinary. We will not attempt to exhaust all of the possibilities here, but we give in the next result some important examples. Note that we say that an automorphism of X is *numerically trivial* if its induced action on $N^1(X)$ is the identity map.

Corollary 7.5. *Let X be an integral scheme of finite type over k with k-automorphism σ. Then (X, σ) is ordinary in any of the following cases:*

(1) *X is a complete simplicial toric variety.*

(2) *X is projective,* $\mathrm{Pic}(X)$ *is discrete and σ^n is numerically trivial for some $n \geq 1$.*

(3) *X is an abelian variety and σ^n is numerically trivial for some $n \geq 1$.*

(4) *X is a nonsingular projective Fano variety.*

Proof. (1) By [Cox, Corollary 4.7], the automorphism group of a complete simplicial toric variety is an algebraic group. Now apply Theorem 7.4(1).

(2) Let D be an any ample divisor on X. We have that $(\sigma^n)^*D \equiv D$, where $\equiv$ means numerical equivalence; this implies that $(\sigma^n)^*(mD)$ is algebraically equivalent to mD for some $m \geq 1$ [SGA6, XIII, Theorem 4.6]. Since $\mathrm{Pic}(X)$ is discrete, the Picard variety $\mathrm{Pic}^0(X) \subseteq \mathrm{Pic}(X)$ of divisor classes algebraically equivalent to 0 is a point, so $(\sigma^n)^*(mD) \sim mD$, and by Proposition 7.4(3) we obtain that (X, σ^n) is ordinary. Thus (X, σ) is also.

(3) We may assume that σ is itself numerically trivial. We verify that the same proof as in [Do, Proposition 1] shows that if σ is a numerically trivial automorphism on an abelian variety X, then σ is contained in an algebraic group of automorphisms. For $x \in X$, let $T_x : X \to X$ be the translation automorphism $y \mapsto y + x$. Let L be a line bundle on X, and let $\phi_L : X \to \mathrm{Pic}(X)$ be the morphism defined by $\phi_L(x) = T_x^*(L) \otimes L^{-1}$. As is shown in [M, II.8], the image of any ϕ_L is contained in $\mathrm{Pic}^0(X)$.

Now let L be a very ample line bundle on X. Since σ is numerically trivial, we have $\sigma^*L \otimes L^{-1} \in \mathrm{Pic}^0(X)$, because numerical equivalence and algebraic equivalence are the same on an Abelian variety [La1, Corollary, p.135]. In addition, [M, II.8, Theorem 1] shows that for an ample line bundle L, the map $\phi_L : X \to \mathrm{Pic}^0(X)$ is surjective. Thus there is some x such that

$$\sigma^*L \cong T_x^*L.$$

That is, $T_{-x}^*\sigma^*$ leaves the isomorphism class of L invariant. But the group of automorphisms of X leaving L invariant is an algebraic group $G \subseteq \mathrm{Aut}(X)$, by the proof of Theorem 7.4(2); thus σ lies in the right coset GT_x. The group of translation automorphisms of X is isomorphic to the irreducible algebraic group X, so it must be contained in $\mathrm{Aut}^0(X)$. Since G is an algebraic group, it is contained in some finite set of irreducible components of $\mathrm{Aut}(X)$; the union of these components must be an algebraic group containing σ. Now apply Theorem 7.4(1).

(4) In this case, by definition the anti-canonical divisor $-K_X$ is ample, so the result follows from Theorem 7.4(2). □

Now we turn from techniques that work for varieties of any dimension and give our main result on the case of surfaces. We note that the proof of the next theorem depends in an essential way on a result about the automorphism groups of surfaces, [Do, Proposition 1], which is proved case-by-case by appealing to the classification of projective surfaces. We also use the classification of the possible growth data of an automorphism with finite growth type from [DF], as we reviewed in Theorem 6.3. Thus there is a lot of highly non-trivial geometry specific to surfaces supporting this result.

Theorem 7.6. *Let k be an uncountable algebraically closed field of characteristic 0. Let $\sigma : X \to X$ be an automorphism of an integral quasi-projective k-scheme X with $\dim X = 2$. If (X, σ) has finite growth type, then (X, σ) is ordinary.*

Proof. By Lemma 7.2(2), we need only prove that (X, σ) has good dense orbits. Moreover, we can reduce to the case $k = \mathbb{C}$, by the following standard application of the Lefschetz principle. There is some subfield $E \subseteq k$ which is a finitely generated field extension of $\mathbb{Q}$ and such that (X, σ) is defined over E. We can find an algebraically closed uncountable field F with $E \subseteq F \subseteq k$ such that F is also isomorphic to a subfield of $\mathbb{C}$. Then (X, σ) is also defined over F, in other words there is an F-scheme Y with $\sigma' : Y \to Y$ such that $X = Y \times_{\operatorname{Spec} F} \operatorname{Spec} k$ and σ' induces σ. We also have an extended scheme $\widetilde{X} = Y \times_{\operatorname{Spec} F} \operatorname{Spec} \mathbb{C}$ with induced automorphism $\widetilde{\sigma}$. Since F is algebraically closed, $\widetilde{X}$ is still integral by Lemma 5.2. Now by Lemma 5.5, (X, σ) has good dense orbits if and only if $(\widetilde{X}, \widetilde{\sigma})$ does, and by Lemma 6.5, $(\widetilde{X}, \widetilde{\sigma})$ still has finite growth type. Thus, replacing X by $\widetilde{X}$, from now on we can assume that $k = \mathbb{C}$.

By assumption, in the growth data (ρ, j) of $\sigma : k(X) \to k(X)$ we have $\rho = 1$. We use the classification in Theorem 6.3, which shows in particular that we only need to consider the cases $j = 0, 1, 2$. Using Lemma 7.3, if convenient we can replace $\sigma : X \to X$ with a conjugate automorphism $\tau : Y \to Y$ and it is enough to prove that (Y, τ) has good dense orbits.

In case $j = 0$, using Theorem 6.3 we change to a conjugate automorphism $\tau : Y \to Y$ where Y is nonsingular projective and some power of τ is numerically trivial. Without loss of generality, we may assume that τ is numerically trivial.

Let $\mathrm{O}(\mathrm{N}^1(Y))$ be the group of linear transformations on the lattice $\mathrm{N}^1(Y)$ that preserve the intersection form. Considering the action of automorphisms on $\mathrm{N}^1(Y)$ by pushforward, we have group a homomorphism

$$r : \operatorname{Aut}(Y) \to \mathrm{O}(\mathrm{N}^1(Y))$$

which induces (since $\operatorname{Aut}^0(Y)$ is irreducible and $\mathrm{O}(\mathrm{N}^1(Y))$ is discrete) a homomorphism

$$\overline{r} : \operatorname{Aut}(Y)/\operatorname{Aut}^0(Y) \to \mathrm{O}(\mathrm{N}^1(Y)).$$

In the case at hand, $\tau \in \operatorname{Ker}(r)$. By [Do, Proposition 1], $\operatorname{Ker}(\overline{r})$ is finite; therefore, τ is an element of a finite extension of the algebraic group $\operatorname{Aut}^0(Y)$, namely $\operatorname{Ker}(r)$, which is an algebraic group. Then (Y, τ) has good dense orbits by Proposition 7.4(1).

The arguments for the remaining cases $j = 1$, $j = 2$ are similar to each other. Using Theorem 6.3, if $j = 1$, then there is a birational map $g : X \dashrightarrow Z$ for some nonsingular projective surface Z such that passing to the conjugate $\tau : Z \dashrightarrow Z$ of σ, there is a τ-invariant rational fibration $f : Z \to C$ for some curve C. Consider the rational map $h = fg : X \dashrightarrow C$. The locus where h is defined is a σ-invariant open subset $Y \subseteq X$, and so we can consider the morphism $h' : Y \to C$; then it is enough to prove that $(Y, \sigma|_Y)$ has good dense orbits. In case $j = 2$, Theorem 6.3 immediately gives us a map $\tau : Y \to Y$ conjugate to $\sigma : X \to X$, where Y is nonsingular projective, and such that there is a τ-invariant elliptic fibration $f : Y \to C$ for some curve C.

Thus to finish both cases, it is enough to consider an integral quasi-projective surface Y with automorphism $\tau : Y \to Y$, together with a τ-invariant fibration $f : Y \to C$ for some curve C, and prove that (Y, τ) has good dense orbits. By assumption, there is a compatible automorphism $\theta : C \to C$ with $f\tau = \theta f$. If (Y, τ) has no dense orbit, we are done, so assume this is not the case. Then θ must have infinite order, so θ has finitely many periodic points on C by Lemma 7.2. The union of the fibers in Y over those points is a proper τ-invariant closed subset of Y which must contain all of the τ-periodic points of Y. Again since we assume (Y, τ) has a dense orbit, by Theorem 5.7 there are finitely many τ-periodic curves in Y. Letting W be the union of these curves, we also know that W contains all of the τ-periodic points of Y, so we conclude that $U = Y \setminus W$ is the set of points lying on dense τ-orbits, and U is open as required. □

Remark 7.7. The restriction $\operatorname{char} k = 0$ in Theorem 7.6 is probably unnecessary, but removing it would likely take significant work. To do so, one would need to verify that [Do, Proposition 1] and Theorem 6.3 both work over algebraically closed fields of positive characteristic. (In [Ro, Theorem 3.2], most of Theorem 6.3 is shown to hold over any algebraically closed field, but not the conclusion that there is a conjugate surface with a τ-invariant elliptic fibration in case $j = 2$.)

We conjecture that the converse to the preceding theorem also holds: namely, that on a quasi-projective integral surface X, an automorphism σ with infinite growth type never has good dense orbits. It is not too hard to show that such an automorphism cannot have a σ-invariant rational function, and thus must have finitely many σ-periodic curves by Theorem 5.7. Thus the failure of good dense orbits is equivalent in this case with having a countably infinite number of periodic points which do not lie on periodic curves. The result [DF, Theorem 0.6] does show that if X is projective, σ has infinite growth type *and no curves of periodic points*, then σ has a countably infinite number of periodic points. We see no obvious argument that works in general, though, and so we do not pursue this question further here. For the special case of $X = \mathbb{A}^2$, however, see Theorem 8.8 below.

8. Summary theorems and examples

We assume Hypothesis 3.1 throughout this section. We are now ready to prove Theorem 1.1 from the Introduction, which shows that the Dixmier-Moeglin equivalence often does hold for the examples in Section 3. At this point, the proof is simply a matter of piecing together results we have already proved. We then give a number of illustrative examples of skew and skew-Laurent rings which do not satisfy the DM-equivalence. These examples are well-known, but it is interesting to see how they fit into our geometric framework.

Theorem 8.1. *Let k be an algebraically closed uncountable field, and let A be one of the following k-algebras:*

(1) *A twisted homogeneous coordinate ring $A = B = B(X, \mathcal{L}, \sigma)$ as in Example 3.2, so X is projective with automorphism $\sigma : X \to X$ and $\mathcal{L}$ is σ-ample; or*

(2) *either a skew polynomial ring* $A = U = S[t;\sigma]$*, or skew-Laurent ring* $A = T = S[t,t^{-1};\sigma]$ *as in Example 3.4, so* S *is a finitely generated commutative* k*-algebra. In this case, we set* $X = \operatorname{Spec} S$ *and write* $\sigma : X \to X$ *for the induced scheme automorphism.*

In all cases, if (X,σ) *is ordinary then* A *satisfies the DM-equivalence. In particular, this always holds when* $\operatorname{char} k = 0$*,* $\dim X \leq 2$*, and* $\operatorname{GK} A < \infty$ *(the last condition being automatic in case (1)).*

Proof. If (X,σ) is ordinary, then (X,σ) is countable-avoiding by Corollary 4.4. Then A satisfies the DM-equivalence by Proposition 3.8.

Suppose now that $\operatorname{GK} A < \infty$. This is always true in case (1), since $\mathcal{L}$ is σ-ample, by [Ke1, Theorem 1.2]. Choose an $n \geq 1$ such that σ^n restricts to an automorphism of each irreducible component X_α of X, and use the same name σ^n for the restriction $\sigma^n|_{X_\alpha}$. In case (1), $\mathcal{L}_n$ is σ^n ample on X, and so $\mathcal{L}_n|_{X_\alpha}$ is σ^n-ample on X_α. Thus the ring $A' = B(X_\alpha, \mathcal{L}_n|_{X_\alpha}, \sigma^n)$ also has $\operatorname{GK} A' < \infty$. In case (2), putting $X_\alpha = \operatorname{Spec} S_\alpha$, then $A' = S_\alpha[t,t^{-1};\sigma^n]$ is a factor algebra of the nth Veronese ring $A^{(n)} = S[t,t^{-1};\sigma^n]$ of A, so $\operatorname{GK} A' < \infty$ in this case also. Since in either case A' is a noetherian domain of finite GK-dimension, with graded quotient ring $k(X_\alpha)[t,t^{-1};\sigma^n]$, by Proposition 6.4 $(k(X_\alpha),\sigma^n)$ has finite growth type for each α. Now if we know in addition that $\operatorname{char} k = 0$ and $\dim X \leq 2$, then (X_α,σ^n) is ordinary for each α by Lemma 7.2 (in case $\dim X_\alpha \leq 1$) or Theorem 7.6 (in case $\dim X_\alpha = 2$). Then (X,σ) is ordinary by Lemma 7.1. □

Remark 8.2. Suppose that A is a prime twisted homogeneous coordinate ring as in case (1) of Theorem 8.1, for which (X,σ) is indeed ordinary. Suppose that A is primitive. Then it is also clear how to construct a faithful simple module for A. Since $\operatorname{Spec} A$ is countable-avoiding by Proposition 3.8, it must be that A has finitely many height one primes, say $P_1, P_2, \ldots P_n$, and these are necessarily homogeneous by the proof of Proposition 2.8. Choose any $0 \neq x \in \bigcap P_i$ which is homogeneous of positive degree, and let M be any maximal right ideal of A containing $x-1$ (note that $x-1$ is not a unit, as A is $\mathbb{N}$-graded.) Then A/M is a faithful simple right A-module.

We note next that our techniques allow us to give some partial results on the question of the DM-equivalence for twisted homogenous coordinate rings of higher-dimensional varieties.

Proposition 8.3. *Assume that* $\operatorname{char} k = 0$*, and let* X *be an integral projective three-fold with automorphism* σ *and* σ*-ample line bundle* $\mathcal{L}$*. If* (X,σ) *has good dense orbits, then* (X,σ) *is ordinary and* $B = B(X,\mathcal{L},\sigma)$ *satisfies the Dixmier-Moeglin equivalence.*

Proof. It is enough to show that for any proper σ-irreducible subset Z of X, $(Z,\sigma|_Z)$ has good dense orbits, and then apply Corollary 4.4 and Proposition 3.8. Since $\mathcal{L}|_Z$ is $\sigma|_Z$-ample and $\dim Z \leq 2$, the proof of Theorem 8.1 shows that $(Z,\sigma|_Z)$ has good dense orbits as required. □

If $\sigma : X \to X$ is an automorphism of a projective scheme X of arbitrary dimension, then σ is called *quasi-unipotent* if the induced action $\sigma^* : N^1(X) \to N^1(X)$ has eigenvalues which are roots of unity; this is the correct analog of σ having finite growth type, as defined in Section 6 for a surface, if X is projective of higher dimension. Keeler proved that an automorphism σ is quasi-unipotent if and only if X has a σ-ample invertible sheaf $\mathcal{L}$ [Ke1, Theorem 1.2]. If this holds, then given any σ-invariant subscheme $Z \subseteq X$, since $\mathcal{L}|_Z$ is $\sigma|_Z$-ample, it follows that $\sigma|_Z$ is also quasi-unipotent.

Proposition 8.4. *(X, σ) is ordinary for all projective k-schemes X and quasi-unipotent automorphisms σ, unless there exists such an (X, σ) with the following additional properties: X is integral, σ has a dense orbit, and X has precisely a countably infinite number of maximal σ-irreducible closed subsets, all but finitely many of which have codimension ≥ 2 in X.*

Proof. Suppose that (X, σ) is not ordinary, where $\sigma : X \to X$ is quasi-unipotent. Since quasi-unipotence restricts to σ-invariant subschemes as we observed above, by noetherian induction we may assume that X is σ-irreducible and that (X, σ) does not have good dense orbits. Replacing σ by a power and passing to an irreducible component, we may assume that X is integral. Then the remaining claimed properties of σ are implied immediately by Theorem 4.2 and Theorem 5.7. □

Since if (X, σ) is ordinary, then $B(X, \mathcal{L}, \sigma)$ satisfies the DM-equivalence for any σ-ample $\mathcal{L}$ (Theorem 8.1), the previous proposition shows that any counterexamples to the DM-equivalence for twisted homogeneous coordinate rings must come from automorphisms of a very restrictive and probably nonexistent kind. Thus we venture the following.

Conjecture 8.5. Given a projective k-scheme X of any dimension and automorphism $\sigma : X \to X$, with σ-ample sheaf $\mathcal{L}$, we conjecture that $B(X, \mathcal{L}, \sigma)$ satisfies the DM-equivalence.

In the remainder of the paper, we discuss the case of skew and skew-Laurent extensions further. Of course, in case (2) of Theorem 8.1, $\operatorname{GK} A < \infty$ does not always hold, and in this case one recovers some the standard examples which do not satisfy the DM-equivalence. In fact, we suspect that the skew and skew-Laurent extensions T and U never satisfy the DM-equivalence in the case they have infinite GK-dimension; see Theorem 8.8 below for some evidence for this contention. We now review some interesting examples.

Example 8.6. Let $T = S[t, t^{-1}; \sigma]$ be as in Theorem 8.1(2), where we assume for simplicity that S is a domain. Then the primitivity of the ring T is characterized by Jordan in [Jo, Theorem 5.10, Proposition 2.9]. Namely, T is primitive if and only if (X, σ) has a dense orbit. In fact, Jordan's results allow non-finitely-generated commutative algebras S; in the special case at hand, we can easily see why this characterization of primitivity holds using our previous results, as follows. If $p \in X$ lies on a dense orbit, then letting $\mathfrak{m}$ be the maximal ideal of S corresponding to the point p, it is easy to check that $\bigoplus_{n \in \mathbb{Z}} (S/\mathfrak{m}) t^n$ is a faithful simple right T-module. On the other hand, if (X, σ) has no dense orbit, then Proposition 4.2 implies that X has a non-constant σ-invariant rational function f, so f is in the center of $Q_{\mathrm{gr}}(T) = k(X)[t, t^{-1}; \sigma]$ and (0) is not a rational ideal. Then T is not primitive, as we saw in the proof of Lemma 2.2.

Example 8.7. Consider the situation of Theorem 8.1(2) with $S = k[u, v, u^{-1}, v^{-1}]$, so $X \cong (\mathbb{A}^1 - \{0\})^2$, and where $\sigma : S \to S$ is defined by $\sigma(u) = u^2 v, \sigma(v) = uv$. This was the original counterexample given by Lorenz in [Lo, 4.3] which showed that $T = S[t, t^{-1}; \sigma]$ fails the Dixmier-Moeglin equivalence, since the prime ideal (0) is primitive but not locally closed. It is easy to check directly that T has exponential growth. Theorem 8.1 shows that the lack of finite GK-dimension for T is a crucial property of this example.

In the case $X = \mathbb{A}^2$, we can characterize completely which automorphisms are ordinary, extending the analysis of [Jo, Proposition 7.8], which showed which automorphisms have a dense orbit. This gives a large number of examples that do not satisfy the DM-equivalence.

Theorem 8.8. *Let k be uncountable and algebraically closed with* $\operatorname{char} k = 0$*, and consider the situation of Theorem 8.1(2) with $S = k[u, v]$ and $X = \operatorname{Spec} S = \mathbb{A}^2$, so $T = S[t, t^{-1}; \sigma]$. Then the following are equivalent:*

(1) $\operatorname{GK} T < \infty$*;*
(2) $(\mathbb{A}^2, \sigma)$ *is ordinary;*
(3) T *satisfies the DM-equivalence.*

Proof. (1) $\implies$ (2) $\implies$ (3) is part of Theorem 8.1, so we need only prove (3) $\implies$ (1), for which we prove the contrapositive. Suppose that $\operatorname{GK} T = \infty$; then $(k(u, v), \sigma)$ has infinite growth type, by Proposition 6.4. We will show that T is primitive but that the ideal (0) is not locally closed. (By the first paragraph of Lemma 2.2, (0) will, however, be rational.)

We claim that $(\mathbb{A}^2, \sigma)$ is not ordinary, or equivalently by Lemma 7.2 that $(\mathbb{A}^2, \sigma)$ does not have good dense orbits. By the same Lefschetz principle argument as in the proof of Theorem 7.6, using Lemma 5.5 we may reduce to the case $k = \mathbb{C}$. But the dynamics of polynomial automorphisms of $\mathbb{C}^2$ are well-understood. Every such automorphism is conjugate in the group of polynomial automorphisms either to an *elementary automorphism* of the form $\tau(z, w) = (\alpha z + p(w), \beta w + \gamma)$ for some polynomial $p(w)$ and constants α, β; or else to a *Henon map*, which is a composition of Henon automorphisms of the form $\tau(z, w) = (p(z) - aw, z)$ with $\deg p(z) \geq 2$ [FM, Theorem 2.6]. Passing to a conjugate clearly does not affect the property of having good dense orbits. Now if τ is an elementary automorphism, then the degrees of the polynomials in the coordinates in the formula for τ^n stay bounded for all n, and it easily follows that such an automorphism has growth data $\rho = 1, j = 1$ in Definition 6.2. In particular, such automorphisms have finite growth type. Thus we may assume without loss of generality that σ is a Henon map. Now the first paragraph of the proof of [Jo, Proposition 7.8] shows that $(\mathbb{A}^2, \sigma)$ has a countably infinite set of periodic points, but no periodic curves, so clearly $(\mathbb{A}^2, \sigma)$ does not have good dense orbits, and the claim is proved.

Working again over our original field k, by Theorem 4.2 σ must have a dense orbit, and the number of its maximal σ-irreducible subsets is countably infinite. Thus T is primitive by Example 8.6, and T must have a countably infinite number of h-height one primes by Lemma 3.5; in fact, these are height one primes by the proof of Lemma 2.6, and thus (0) is not locally closed. □

All of the examples above concerned the skew-Laurent extension $T = S[t, t^{-1}; \sigma]$. The skew extension case $U = S[t; \sigma]$ behaves differently in certain ways. We close by discussing an example of Jordan in the context of our results.

Example 8.9. Consider Theorem 8.1(2), with S a domain for simplicity. The primitivity of the ring $U = S[t; \sigma]$ is characterized in [LM], as follows. The automorphism $\sigma : X \to X$ is called *special* if there exists a proper closed subset $Y \subsetneq X$ such that every proper irreducible σ-periodic subset Z of X satisfies $Z \subseteq \sigma^n(Y)$ for some $n \in \mathbb{Z}$. By [LM, Theorem 3.10], U is primitive if and only if σ is special.

Suppose in Example 8.9 that (X, σ) has good dense orbits. If σ has no dense orbit, then the automorphism $\sigma : X \to X$ is non-special (using Lemma 4.1). If σ has a dense orbit, then clearly σ is special (take Y to be the union of the finitely many maximal σ-irreducibles.) Thus in this case, $U = S[t; \sigma]$ is primitive if and only if σ has a dense orbit, just as in the case of the skew-Laurent ring T. However, in the cases where (X, σ) does not have good dense orbits (for example, the Henon maps of $X = \mathbb{A}^2$ as in Theorem 8.8), it seems to be very difficult to verify whether or not σ is special. Due to the work of Jordan, we do have one interesting example.

Example 8.10. Let $k = \mathbb{C}$ and $S = \mathbb{C}[u, v, u^{-1}, v^{-1}]$, with $\sigma : S \to S$ defined by $\sigma(u) = v, \sigma(v) = uv^{-1}$. Then [Jo, Propositions 7.11-13] shows that X has no σ-periodic curves, a countably infinite set of σ-periodic points, and that σ is *not* special. Thus, as is also pointed out in [LM, Example 3.11], for this automorphism we have that $U = S[t; \sigma]$ is not primitive, whereas $T = S[t, t^{-1}; \sigma]$ is primitive. Clearly $(\operatorname{Spec} S, \sigma)$ does not have good dense orbits, and so necessarily σ has infinite growth type (Theorem 7.6) and T and U have exponential growth by Proposition 6.4.

We note that the Dixmier-Moeglin equivalence fails for both rings T and U, but in different ways. Since the two rings have the same Goldie quotient ring, and we saw that (0) is a rational ideal of T in Theorem 8.8, (0) is a rational ideal of U. Also, a similar argument as in the last sentence of Theorem 8.8 shows that since σ has a countably infinite number of maximal σ-irreducible subsets, then (0) is not locally closed in $\operatorname{Spec} U$ either. To summarize, in T, (0) is primitive, rational, but not locally closed; whereas in U, (0) is not primitive and not locally closed, but it is rational.

We wonder if every automorphism with infinite growth type on an affine surface over k must fail to be special. Even in the case of $\mathbb{A}^2$, nothing seems to be known about this question beyond Jordan's example. The theory of the dynamics of maps of the affine plane has developed much in recent years; perhaps there are applications of that rich theory to the question of speciality.

We close with an intriguing question.

Question 8.11. Theorem 8.1 is quite suggestive that there may be a deeper relationship between the growth of an algebra and the Dixmier-Moeglin equivalence. Is there some more general ring-theoretic argument that shows that algebras of finite GK-dimension (perhaps with certain extra hypotheses) have an countable-avoiding prime spectrum?

This seems to be a deep question; a positive answer would imply, for example, that a finitely generated noetherian k-algebra A over an uncountable field k, which is a domain of GK-dimension 2, is either PI or primitive. (This is the well-known Dichotomy Conjecture of Small.) For, if A has finitely many height one primes, it is primitive by Lemma 2.2. On the other hand, if A has uncountably height one primes, then there are infinitely many such, say $P_1, P_2, \ldots$, such that the PI-degrees of the GK-1 factor rings A/P_i are all equal to a fixed integer d. Then as $\bigcap P_i = 0$, A embeds in $\prod A/P_i$ and thus is PI of degree d as well.

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