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Asset-Liability Management Modelling with Risk Control by Stochastic Dominance

Xi Yang [*] Jacek Gondzio [†] Andreas Grothey [‡]

School of Mathematics and Maxwell Institute for Mathematical Sciences
University of Edinburgh
James Clerk Maxwell Building
King's Buildings
Mayfield Road
Edinburgh EH9 3JZ
U.K.

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[*]University of Edinburgh, UK, X.Yang@ed.ac.uk
[†]University of Edinburgh, UK
[‡]University of Edinburgh, UK

Abstract

An Asset-Liability Management model with a novel strategy for controlling risk of underfunding is presented in this paper. The basic model involves multiperiod decisions (portfolio rebalancing) and deals with the usual uncertainty of investment returns and future liabilities. Therefore it is well-suited to a stochastic programming approach. A stochastic dominance concept is applied to measure (and control) risk of underfunding. A small numerical example is provided to demonstrate advantages of this new model which includes stochastic dominance constraints over the basic model.

Adding stochastic dominance constraints comes with a price. It complicates the structure of the underlying stochastic program. Indeed, new constraints create a link between variables associated with different scenarios of the same time stage. This destroys the usual tree-structure of the constraint matrix in the stochastic program and prevents the application of standard stochastic programming approaches such as (nested) Benders decomposition. A structure-exploiting interior point method is applied to this problem. A specialized interior point solver OOPS can deal efficiently with such problems and outperforms the industrial strength commercial solver CPLEX. Computational results on medium scale problems with sizes reaching about one million of variables demonstrate the efficiency of the specialized solution technique. The solution time for these nontrivial asset liability models seems to grow sublinearly with the key parameters of the model such as the number of assets and the number of realizations of the benchmark portfolio, and this makes the method applicable to truly large scale problems.

1 Introduction

The Asset-Liability Management (ALM) problem has crucial importance to pension funds, insurance companies and banks where business involves large amount of liquidity. Indeed, the financial institutions apply ALM to guarantee their liabilities while pursuing profit. The liabilities may take different forms: pensions paid to the members of the scheme in a pension fund, savers' deposits paid back in a bank, or benefits paid to insurees in the insurance company. A common feature of these problems is the uncertainty of liabilities and the resulting risk of underfunding. The other major uncertainty originates from asset returns. Together they constitute a nontrivial difficulty in how to manage risk in the model applied by the financial institution. The need for multi-period planning additionally complicates the problem.

A paradigm of stochastic programming [1, 22] is well-suited to tackle these problems and has already been applied in this context as shown in [33] and in many references therein. One of the first industrially applied models of this type was the stochastic linear program with simple recourse developed by Kusy and Ziemba in [24]. This model captured certain characteristics of ALM problems: it maximized revenues for the bank in the objective under legal, policy, liquidity, cash flow and budget constraints to make sure that deposit liability is met as much as possible. Under computational limits at the time when it was developed, this model took the advantage of stochastic linear programming so as to be practical even for the large problems faced in banks. It was shown to be superior compared to a sequential decision theoretical model in terms of maximizing both the initial profit and the mean profit. However, the risk management was not considered in this work; only expected penalties of constraints violation were taken into account.

A major difficulty in such models consists in risk management. One may follow the Markowitz risk-averse paradigm [26] and optimize the multiple objectives: maximize the return and minimize the associated risk, e.g. [29]. A successful example of optimization-based ALM modelling which took risk management issues into account was the Russell-Yasuda Kasai model for a Japanese insurance company by Frank Russell consulting company, which used multi-stage stochastic programming [4, 5]. This dynamic stochastic model took into account multiple accounts, regulatory rules and liabilities to enable the managing of complex issues arising in the Yasuda Fire and Marine Insurance company. Expected shortfall, i.e. the expected amount by which the goals were not achieved, was applied to measure risk more accurately than the calculation of expected penalties and it was easy to handle in the solution process. Moreover, the model proved to be easy to understand by decision-makers. The implementation results showed the advantages of the Russell-Yasuda model over the mean-variance model in multi-period and multi-account problems.

There are various ways to measure risk such as variance and expected shortfall, to mention a few. Stochastic dominance is an alternative measure and it has recently gained substantial interest from the research community. It has several attractive features but two of them are particularly important: stochastic dominance is consistent with utility functions and it considers the whole probability distribution. We will discuss these issues in detail in Section 3. The stochastic dominance concept dates from the work of Karamata in 1932 (see [25] for a survey). Subsequently, stochastic dominance has been applied in statistics [2], economics [19, 20] and finance. Stochastic dominance involves comparison of (nonlinear) probability distribution functions and this makes its straightforward application difficult.

An application of the first-order stochastic dominance in the stochastic programming context leads to a non-convex mixed integer programming formulation. In contrast, the second-order stochastic dominance can be incorporated in a form of linearized constraints [8] which makes it a more attractive option. In a series of papers Dentcheva and Ruszczyński analyzed several aspects of the use of stochastic dominance such as its optimality and duality [8], applications to nonlinear dominance constraints [9] and an application to static portfolio selection [10]. An introduction of non-convex constraints by the use of first-order stochastic dominance introduces serious complications into the optimization models and makes their solution difficult. Relaxations of these problems were analyzed in [27]; stability and sensitivity of first-order stochastic dominance with respect to general perturbation of the underlying probability measures were studied in [7]. Noyan, et al. also introduced interval second-order stochastic dominance which is equivalent to first-order stochastic dominance and generated a mixed integer problem based on this dominance relation in [27]. Roman, et al. proposed a multi-objective portfolio selection model with second-order stochastic dominance constraints [30] and Fábián et al. [12] developed an efficient method to solve this model based on a cutting-plane scheme. The application of stochastic dominance in dispersed energy planning and decision problems has been illustrated in [13, 14, 15], including both first-order and second-order stochastic dominances. The use of multivariate stochastic dominance to measure multiple random varables jointly was discussed in [11].

To the best of our knowledge, stochastic dominance has not been applied in the ALM context yet and in this paper we demonstrate how this can be done. Further, we develop a chance constraint from relaxed interval second-order stochastic dominance and show that it is an intermediate dominance constraint between first-order and second-order in the problem with discrete probability distribution. By combining second-order stochastic dominance and relaxed interval second-order stochastic dominance, the model can help generate portfolio strategies with better management of risk and better control of underfunding. We illustrate this issue with a small example analysed in Section 5.1.

Due to the uncertainties of asset returns and liabilities, the stochastic programming involves many scenarios corresponding to the simulation of realisations of those random factors. As a result, the problem size increases significantly, especially when the problem has multiple stages, and this leads to more difficulties in the solution process. Consigli and Dempster [6] proposed the Computational-aided Asset/Liability Management (CALM) model as a "here and now" problem. Out of the simplex method, the interior point method and nested Benders decomposition, the last one is shown to be the most efficient in the sense of both solution time and memory requirements.

Stochastic dominance constraints link variables which are associated with different nodes at the same stage in the event tree. Adding such constraints to the stochastic programming problem destroys the usual tree-structure of the problem and prevents the use of Benders decomposition. We discuss this issue briefly in Section 5.2. We convey the structure of our ALM model with stochastic dominance constraints to a specialized structure-exploiting parallel interior point solver OOPS which takes advantage of such information in the solution process. OOPS is an interior point solver which uses object-oriented programming techniques and treats each sub-structure of the problem as an object carrying its own dedicated linear algebra routines [18]. OOPS can easily deal with complicated ALM problems which contain stochastic dominance constraints. The analysis of computational results confirms that, by exploiting the structure, OOPS outperforms the commercial optimization solver CPLEX 10.0 on these problems.

The basic multi-stage stochastic programming model applied to asset/liability management is discussed in Section 2. The theoretical issues of stochastic dominance are discussed in Section 3 with emphasis on second-order stochastic dominance and relaxed interval second-order stochastic dominance. The practical aspects of the application of different stochastic dominance constraints in the ALM model (second-order and relaxed interval second-order stochastic dominance) are covered in Section 4. These are followed with an analysis of a small example of the model proposed and a discussion of computational results for a selection of realistic medium scale problems in Section 5. Section 6 concludes the paper.

2 Asset-Liability Management

ALM models assist financial institutions in decision making on asset allocations considering full use of fund and resources available. The model aims to maximize the overall revenue, sometimes as well as revenue at intermediate stages, with restrictions of risk. Risk in the ALM problems is present in two aspects: a possible loss of investment and missing the ability to meet liabilities. The returns of assets and the liabilities are both uncertain. It is essential in ALM modelling to deal with uncertainties as well as with risks. Stochastic programming approach is naturally applicable to problems which involve uncertainties; an approach to deal with risk management is discussed in next section.

2.1 Multi-Stage ALM Modelling

Suppose a financial institution plans to invest in assets from a set I, with x_i denoting the investment in asset i. The return r_i of asset i has probability denoted as $Prob_i$ and the total return of the portfolio is R. We make a strong assumption that the probability distribution can be deduced (approximated) from the historical data or simulation. In this work, we will choose the former one in the implementation. Then we can calculate the expected return of the portfolio:

$$E[R] = \sum_i E[x_i * r_i] = \sum_i x_i E[r_i]. \tag{1}$$

Considering a risk function $\phi(x)$ measuring the the risk incurred by decision $x \in R^m$, a general portfolio selection problem, without taking into account the liabilities, can be formulated in one of the following three ways:

$$\min_x -E[x] + \phi(x), \quad x \in X, \tag{2}$$

$$\min_x \phi(x), \quad E[x] \geq R, \quad x \in X, \tag{3}$$

$$\min_x -E[x], \quad \phi(x) \leq \beta, \quad x \in X. \tag{4}$$

Suppose that constraints $E[x] \geq R$, $\phi(x) \leq \beta$ have strictly feasible points. It is proved in [23] that these three problems are equivalent in the sense that they can generate the same efficient frontier, given convex set X and convex risk measure function $\phi(x)$. The best-known example of formulation (2) is the Markowitz mean-variance multi-objective model (1959), which considers both return and risk in the objective. In formulation (3) risk is minimized with acceptable returns, while in formulation (4), the return is maximized subject to risk being kept at an acceptable level. The constraint in Equation (4) defines the feasible set with feasible risk so that in the objective the decision-maker can focus on maximizing the return. In this paper we will use formulation (4).

Besides the return and risk control, ALM model has also the following features:

- Transaction cost; each transaction will be charged certain percentage of total transaction value;
- Cash balance; liabilities should be paid to clients, meanwhile there is an inflow in terms of deposits or premiums; the model should make sure the outflow and inflow match;
- Inventories of assets and cash, which are essential in dynamical system of 2- or even multiple-stage problem;

- Legal and policy constraints align with the financial sector's requirements.

This work considers the first three points.

It is important for decision makers to rebalance the portfolio during the investment period as they may wish to adjust the asset allocations according to updated information on the market. The strategy which is currently optimal may not be optimal any more as the situation changes. Thus it is important to reconsider the strategy and make the necessary change in order to remain in the optimal position. Taking this into account, the problem is multi-period and at the beginning of each period, new decisions are made. Such a multi-stage ALM model allows different decisions through the investing process.

To make it easier to model, we consider the problem stage by stage and with portfolio rebalancing done at the beginning of each stage. Also, the uncertainties of asset returns are implemented with discrete distributions, in which case an event tree is used to capture the uncertainties in multiple stages throughout the whole decision process, e.g. as shown in Figure 1. Each node is labelled with (i, j) denoting node j at stage i. The nodes at each stage represent possible future events. Asset returns, liabilities and cash deposits are subject to uncertain future evolution. Meanwhile, the asset rebalancing is done after knowing which value the asset returns and liabilities take at each node.

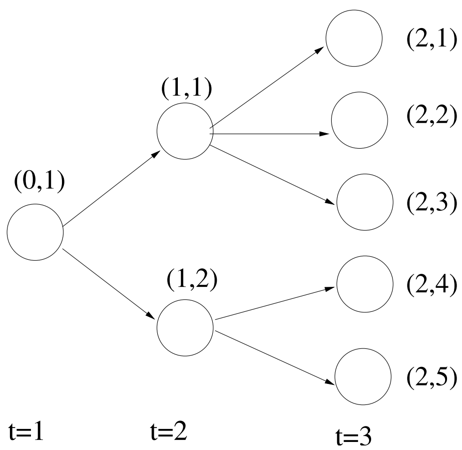

Figure 1: An example of event tree describing different return states of nature.

Then the multi-stage ALM problem concerning the investment strategy can be represented as:

$$\max \sum_{i \in I, j \in N_T} \pi_j^T((1-\gamma) w_i x h_{i,j}^T + Cash_j^T - \lambda b_j^T) \tag{5a}$$

$$s.t. \quad (1+\gamma) \sum_{i \in I} w_i x h_{i,0}^0 + Cash_0 = Budget - l_0 + c_0 \tag{5b}$$

$$(1-\gamma)\sum_{i\in I} w_i xs^t_{i,j} + Cash^t_j = (1+\gamma)\sum_{i\in I} w_i xb^t_{i,j} + (1+r^t_{c,j})Cash^{t-1}_{a(j)} - l^t_j + c^t_j, \quad j, \quad t \tag{5c}$$

$$(1+r^t_{i,j})xh^{t-1}_{i,a(j)} + xb^t_{i,j} - xs^t_{i,j} = xh^t_{i,j}, \quad i, \quad j, \quad t \tag{5d}$$

$$\sum_{i\in I}(1-\gamma)w_i xh^T_{i,j} + Cash^T_j + b^T_j \geq l^T_j, \tag{5e}$$

$$\phi(x) \leq \beta, \tag{5f}$$

$$xh^t_{i,j} \geq 0, \quad xs^t_{i,j} \geq 0, \quad xb^t_j \geq 0, \quad b^T_j \geq 0$$

$$xh^t_j, \quad xs^t_j, \quad xb^t_j \in R^m$$

$$i \in I = \{1,\ldots,m\}, \quad j \in N_t = \{1,\ldots,n_t\}, \quad t = 1,\ldots,T,$$

where, $xh^t_{i,j}$, $xs^t_{i,j}$ and $xb^t_{i,j}$ are units of asset i held, sold and bought in node j at stage t, similarly for $Cash^t_j$; w_i is the price of asset i and $r^t_{i,j}$ is the return of asset i in node j at stage t while $r^t_{c,j}$ is the interest rate in node j at stage t; b^T_j is the amount of underfunding at terminal stage that cannot be satisfied and λ is the penalty coefficient of underfunding; γ is the transaction fee; $Budget$ is the fund available to manage; l^t_j and c^t_j denote the outflow and inflow of resources, e.g. liabilities to pay, new deposits received; $\phi(x)$ is the risk measure function; β is the upper bound on the risk measure; I is the asset set, N_t is the set of nodes belonging to stage t with π as the probability measure of N_t, $a_{(j)}$ is the ancestor of node j and this is a $T-stage$ problem. In this stochastic programming model, scenarios are possible outcomes of random variables, i.e. the asset returns, liabilities and cash deposits here.

The decision maker does not seek the strategy to strictly satisfy the liability at the end, but penalises the underfunding. The objective (5a) aims to maximize the final wealth of the fund taking into account the penalties of underfunding. Equation (5b) balances the initial wealth at the first stage while Equations (5c) do the same for the following stages, both taking into account transaction cost, proportional to the total trade volume. The inventories of each asset at each stage are captured in Equation (5d). Equation (5e) defines the underfunding level b_j at the terminal stage. Risk control is expressed in Equation (5f) with the risk measure function $\phi(x)$ and the maximum acceptable level of risk β. This constraint will be discussed in more detail in the following section. If the risk constraint is linear, the model (5) is a linear program.

The risk control in ALM problem involves many aspects. Two of the most important are the overall performance and the underfunding. The overall performance is analyzed considering all possible outcomes of the portfolio, e.g. variance. We will use stochastic dominance to control the risk of overall performance and discuss the modelling issues involved in Section 3. Underfunding concerns the possibility of unsatisfied liability only. To avoid underfunding completely is expensive to implement and in many situations impossible. We will control underfunding through stochastic dominance constraints discussed in Section 4.

3 Stochastic Dominance

Stochastic dominance, as a coherent risk measure [31], has been considered to be a reference to other risk measures by Ogryczak and Ruszczyński in [28]. Below we demonstrate how it can be incorporated into our ALM model. First we briefly recall the definitions of stochastic dominance following closely the exposition in [28]. The reader familiar with these definitions may skip Section 3.1.

3.1 Definition and Properties of Stochastic Dominance

Given a random variable ω, we consider the first performance function, which is actually the probability distribution function, as:

$$F^1_\omega(\eta) = P(\omega \leq \eta). \tag{6}$$

Then we say that random variable Y dominates L by first-order stochastic dominance (FSD) if:

$$F^1_Y(\eta) \leq F^1_L(\eta), \quad \forall \eta \in R, \tag{7}$$

denoted as

$$Y \succeq_1 L. \tag{8}$$

Next, we define the second performance function as:

$$F^2_\omega(\eta) = \int_{-\infty}^{\eta} F^1_\omega(\zeta) d\zeta, \quad \forall \eta \in R. \tag{9}$$

Then we say that random variable Y dominates L by second-order stochastic dominance (SSD) if:

$$F^2_Y(\eta) \leq F^2_L(\eta), \quad \forall \eta \in R, \tag{10}$$

denoted as

$$Y \succeq_2 L. \tag{11}$$

Hence, if y and l are returns of two portfolio strategies satisfying (7) (or (10)), then Y dominates L and Y is preferable. Iteratively, we can define higher order stochastic dominance. And it has also been proved that the lower order dominance relations can guarantee the dominance of higher orders. See [28, 32].

Stochastic dominance has been widely used today in decision theory and economics. The most important reason for that is its consistency with utility theory. Utility measures a degree of satisfaction. The value of portfolio depends only on itself and is equal for every investor; the utility, however, is dependent on the particular circumstances of the person making the estimate. Investors seek to maximize their utilities. In general, utility functions are nondecreasing, which means most people prefer more fortune to less. It is known that $X \succeq_1 Y$ if and only if $E[U(X)] \geq E[U(Y)]$ for any nondecreasing utility function U for which these expected values are finite. And, $X \succeq_2 Y$ if and only if $E[U(X)] \geq E[U(Y)]$ for any nondecreasing and concave utility function U for which these expected values are finite. A nondecreasing and concave utility function reflects that the investor prefers more fortune but the speed of increase in satisfaction decreases. Details of stochastic dominance and utility theory can be found in [25]. Generally, a reasonable investor has nondecreasing and concave utility function. Hence, we will incorporate SSD in ALM models also because of its computational advantage we show later, while FSD leads to a mixed integer formulation which can be found in [15, 27].

3.2 Linear Formulation of SSD

In ALM modelling, a benchmark can be set as the market index or competitors' performance. SSD constraints will make sure that the resulting portfolio strategy performs no worse than this benchmark. However, the integration of probability distribution function in SSD definition can lead to difficulty in computation. Hence, we will consider a relaxed form in discrete probability distribution case in the following part.

Changing the order of integration in Equation (9), we have

$$F_\omega^2(\eta) = E[(\eta - \omega)_+]. \tag{12}$$

With SSD as the risk measure, Equation (5f) is replaced with

$$E[(\eta - xh^t)_+] \leq E[(\eta - Benchmark^t)_+], \quad \eta \in R. \tag{13}$$

To make the problem easier for modelling and computations, consider a relaxed formulation of this constraint valid in interval $[a,b]$:

$$E[(\eta - xh^t)_+] \le E[(\eta - Benchmark^t)_+], \quad \eta \in [a,b]. \tag{14}$$

Denote the shortfall as $v_j^t : [a,b] \times \Omega \to R$, and observe that (14) is equivalent to:

$$xh_j^t + v_j^t \ge \eta \tag{15a}$$

$$E[v^t] \le E[(\eta - Benchmark^t)_+], \quad \eta \in [a,b] \tag{15b}$$

$$v \ge 0. \tag{15c}$$

If the $Benchmark$ has discrete probability distribution with realizations $Benchmark_l$, $l = 1,\dots,n$, $a \le Benchmark_l \le b$, then Equations (15) can be rewritten as

$$E[(Benchmark_l - xh^t)_+] \le E[(Benchmark_l - Benchmark^t)_+], \tag{16}$$

or evaluating E over the considered scenarios:

$$xh_j^t + v_{l,j}^t \ge Benchmark_l^t \tag{17a}$$

$$\sum_j \pi_j v_{l,j}^t \le \widehat{v}_l^t, \tag{17b}$$

$$v \ge 0, \tag{17c}$$

where $\widehat{v}_l^t = E[(Benchmark_l^t - Benchmark^t)_+]$. It is easy to see that Equations (17) are linear.

3.3 Interval Second-order Stochastic Dominance and Chance Constraints

The interval SSD was firstly introduced by Noyan et al in [27] and proved to be a sufficient as well as a necessary condition of FSD. Here, we will consider a relaxed interval SSD in the discrete case as a intermediate risk measure between FSD and SSD, i.e. a weaker condition than FSD, but stronger than SSD.

We say that random variable Y dominates L by interval second-order stochastic dominance (ISSD) if:

$$E[(\eta_2 - Y))_+] - E[(\eta_1 - Y)_+] \leq E[(\eta_2 - L)_+] - E[(\eta_1 - L)_+], \tag{18}$$

for any η_1, $\eta_2 \in R$ and $\eta_1 \leq \eta_2$.

The Proposition below establishes a relation between FSD and ISSD. It was first proved in [27] in a case of discrete probability distribution. We shall prove it in a general form.

Proposition 1. *$Y \succeq_1 L$ if and only if Y dominates L by ISSD.*

Proof. The proof of necessity is simple. If $Y \succeq_1 L$, then for any given $\eta_1 \leq \eta_2$ and t, $\eta_1 \leq t \leq \eta_2$,

$$0 \leq F_Y^1(t) \leq F_L^1(t).$$

Hence, the integration

$$\int_{\eta_1}^{\eta_2} F_Y^1(t)dt \leq \int_{\eta_1}^{\eta_2} F_L^1(t)dt. \tag{19}$$

Similarly to Equation (12), Equation (19) is equivalent to the definition of ISSD, i.e. Equation (18).

We prove the sufficiency by contradiction. Suppose there exists t such that

$$F_Y^1(t) > F_L^1(t).$$

Let (a^*, b^*) be an interval such that $t \in (a^*, b^*)$ and

$$\begin{aligned} a^* &= max\{a, F_Y(a) > F_L(a), a < t\} \\ b^* &= min\{b, F_Y(b) > F_L(b), b > t\} \end{aligned}$$

Then, we have

$$\int_{a^*}^{b^*} F_Y^1(\alpha)d\alpha > \int_{a^*}^{b^*} F_L^1(\alpha)d\alpha$$

which violates the definition of ISSD. The sufficiency is proved.

□

If Y and L both have discrete probability distribution with realizations $y_1 < y_2 < \cdots < y_M$, and $l_1 < l_2 < \cdots < l_K$, the ISSD in this case can be written as:

$$E[(l_k - Y)_+] - E[(y_m - Y))_+] \leq E[(l_k - L)_+] - E[(y_m - L))_+], \tag{20}$$

for all $m \in \{1, \ldots, M\}$ and $k \in \{1, \ldots, K\}$ such that $l_k \geq y_m$ and

$$\{l_1, \cdots, l_K, y_1, \cdots, y_M\} \cap (y_m, l_k) = \emptyset, \tag{21}$$

where (y_m, l_k) is the open interval with endpoints y_m and l_k [27].

Incorporating constraints (20) into ALM model (5) leads to a mixed integer formulation. The integer variables are induced by the dependence of y_m on decision variables in the model. Hence, we consider a relaxed form of ISSD in the situation with discrete distribution:

$$E[(l_k - Y)_+] - E[(l_{k-1} - Y)_+] \leq E[(l_k - L)_+] - E[(l_{k-1} - L)_+], \quad k \in 1, \cdots, K \tag{22}$$

where l_k, $k = 1, \cdots, K$ are the realizations of L and l_0 is any real number such that $l_0 < l_1$. And denote above relation of Y and L as

$$Y \succeq_{1\frac{1}{2}} L. \tag{23}$$

It is easy to prove that this relaxed ISSD is weaker than FSD but stronger than SSD, i.e.

$$FSD \Rightarrow Relaxed \quad ISSD \Rightarrow SSD. \tag{24}$$

The first implication was proved in [27]. We will give a full picture of these three dominance relations in the following Proposition.

Proposition 2. *If Y dominates L by FSD, then $Y \succeq_{1\frac{1}{2}} L$; If $Y \succeq_{1\frac{1}{2}} L$, then Y dominates L by SSD.*

Proof. By Proposition 1, if FSD is true, ISSD is satisfied, which is sufficient for relaxed ISSD.

If relaxed ISSD is satisfied, we have

$$\int_{l_{k-1}}^{l_k} F_Y^1(t)dt \leq \int_{l_{k-1}}^{l_k} F_L^1(t)dt,$$

for $k = 1, \ldots, K$. Since $F_L^1(t) = 0$, for any t that $l_0 < t < l_1$,

$$\int_{l_0}^{l_1} F_Y^1(t)dt \leq \int_{l_0}^{l_1} F_L^1(t)dt = 0.$$

We have $F_Y(t) = 0$, a.e., for $t < l_1$. Hence, for any real number $\eta < l_1$,

$$\int_{-\infty}^{\eta} F_Y^1(t)dt \leq \int_{-\infty}^{\eta} F_L^1(t)dt = 0. \tag{25}$$

Also,

$$\begin{aligned}
\int_{-\infty}^{l_k} F_Y^1(t)dt &= \int_{-\infty}^{l_1} F_Y^1(t)dt + \sum_{j=1,\ldots,k-1} \int_{l_j}^{l_{j+1}} F_Y^1(t)dt \\
&\leq 0 + \sum_{j=1,\ldots,k-1} \int_{l_j}^{l_{j+1}} F_L^1(t)dt \\
&= \int_{-\infty}^{l_k} F_L^1(t)dt,
\end{aligned}$$

$k = 1, \ldots, K$. Suppose there exists $\eta \in [l_k, l_{k+1}]$ such that

$$\int_{-\infty}^{\eta} F_Y^1(t)dt > \int_{-\infty}^{\eta} F_L^1(t)dt.$$

Since

$$\int_{-\infty}^{l_k} F_Y^1(t)dt \leq \int_{-\infty}^{l_k} F_L^1(t)dt,$$

we have

$$\int_{l_k}^{\eta} F_Y^1(t)dt > \int_{l_k}^{\eta} F_L^1(t)dt. \tag{26}$$

In addition, for $t \in [l_k, l_{k+1})$, $F_L^1(t) = F_L^1(l_k)$. From Equation (26), using monotonicity of F_Y^1,

$$F_Y^1(\eta) > F_L^1(l_k). \tag{27}$$

As a result,

$$\int_{\eta}^{l_{k+1}} F_Y^1(t)dt > \int_{\eta}^{l_{k+1}} F_L^1(t)dt. \tag{28}$$

Equations (26) and (28) together imply

$$\int_{l_k}^{l_{k+1}} F_Y^1(t)dt > \int_{l_k}^{l_{k+1}} F_L^1(t)dt, \tag{29}$$

which contradicts the relaxed ISSD condition. Therefore, for all $\eta \in [l_1, l_K]$, SSD is satisfied.

For $\eta > l_K$,

$$\begin{aligned}\int_{-\infty}^{\eta} F_L^1(t)dt &= \int_{-\infty}^{l_K} F_L^1(t)dt + \int_{l_K}^{\eta} F_L^1(t)dt \\ &= \int_{-\infty}^{l_K} F_L^1(t)dt + \int_{l_K}^{\eta} 1dt \\ &\geq \int_{-\infty}^{l_K} F_Y^1(t)dt + \int_{l_K}^{\eta} F_Y^1(t)dt.\end{aligned}$$

The sufficiency of SSD is proved. □

An interesting question arises whether any reverse implication to (24) holds. Two examples are given below to illustrate that the other direction of the relation is not true. The first demonstrates that the relaxed ISSD does not imply FSD and the second shows that SSD does not imply the relaxed ISSD.

Example 1: Consider two assets L and Y with the following probability distributions of returns: $P(L = 100) = \frac{1}{3}$, $P(L = 200) = \frac{1}{3}$, $P(L = 300) = \frac{1}{3}$; $P(Y = 150) = \frac{1}{2}$, $P(Y = 300) = \frac{1}{2}$. For these distributions, we find:

$$E[(\eta - L)_+] = \begin{cases} 0, & \eta \leq 100 \\ \frac{1}{3}(\eta - 100), & 100 < \eta \leq 200 \\ \frac{2}{3}(\eta - 200) + \frac{1}{3}(200 - 100), & 200 < \eta \leq 300 \\ (\eta - 300) + \frac{2}{3}(300 - 200) + \frac{1}{3}(200 - 100), & 300 < \eta \end{cases}$$

$$E[(\eta - Y)_+] = \begin{cases} 0, & \eta \leq 150 \\ \frac{1}{2}(\eta - 150), & 150 < \eta \leq 300 \\ (\eta - 300) + \frac{1}{2}(300 - 150), & 300 < \eta \end{cases}$$

and collect the values of $E[(l_k - X)_+] - E[(l_{k-1} - X)_+]$ for both variables L and Y for all intervales $(l_{k-1}, l_k]$ in the table below:

$E[l_k - X]_+ - E[l_{k-1} - X]_+$	$[0, 100]$	$(100, 200]$	$(200, 300]$
X=L	0	33.3	66.7
X=Y	0	25	50

Table 1: The relaxed ISSD values of assets L and Y.

Obviously, inequality (22) is always satisfied hence the relaxed ISSD is satisfied, i.e. $Y \succeq_{1\frac{1}{2}} L$. However, $P(L \leq 150) < P(Y \leq 150)$, which means FSD is violated.

Example 2: Consider two assets L and Y, where L is the same as in Example 1. Asset Y

has three possible returns: $P(Y = 150) = \frac{1}{2}$, $P(Y = 200) = \frac{1}{4}$ and $P(Y = 300) = \frac{1}{4}$. Y dominates L by SSD but Y does not dominate L by relaxed ISSD, because

$$F^2_\omega = E[(\eta - \omega)_+] = \int_{-\infty}^{\eta} F_\omega(\xi) d\xi,$$

$$F^2_L(\eta) = E[(\eta - L)_+] = \begin{cases} 0, & \eta \leq 100 \\ \frac{1}{3}(\eta - 100), & 100 < \eta \leq 200 \\ \frac{2}{3}(\eta - 200) + \frac{1}{3}(200 - 100), & 200 < \eta \leq 300 \\ (\eta - 300) + \frac{2}{3}(300 - 200) + \frac{1}{3}(200 - 100), & 300 < \eta \end{cases}$$

$$F^2_Y(\eta) = E[(\eta - Y)_+] = \begin{cases} 0, & \eta \leq 150 \\ \frac{1}{2}(\eta - 150), & 150 < \eta \leq 200 \\ \frac{3}{4}(\eta - 200) + \frac{1}{2}(200 - 150), & 200 < \eta \leq 300 \\ (\eta - 300) + \frac{3}{4}(300 - 200) + \frac{1}{2}(200 - 150), & 300 < \eta \end{cases}$$

illustrating that $E[(\eta - L)_+] \geq E[(\eta - Y)_+]$, while

$$E[(300 - L)_+] - E[(200 - L)_+] = \frac{200}{3} \leq E[(300 - Y)_+] - E[(200 - Y)_+] = 75.$$

Below we prove one more technical result regarding relaxed ISSD which has important consequences for a practical way of modelling relaxed ISSD constraints as stated in the two remarks at the end this section.

Proposition 3. *Let Y and L be random variables, whose probability distributions are discrete with realizations $y_1, \cdots, y_M$ and $l_1, \cdots, l_K$, respectively. Let Y dominate L by relaxed ISSD. If there exists $k \in \{1, \ldots, K-1\}$, such that*

$$\{y_1, \cdots, y_M\} \cap (l_k, l_{k+1}) = \emptyset, \tag{30}$$

then $F^1_Y(t) \leq F^1_L(t)$ for all $t \in [l_k, l_{k+1}]$

Proof. For any k such that

$$\{y_1, \cdots, y_M\} \cap (l_k, l_{k+1}) = \emptyset, \tag{31}$$

$F^1_Y(t) = F^1_Y(l_k)$, $t \in [l_k, l_{k+1})$. Then by relaxed ISSD relation,

$$\begin{aligned}\int_{l_k}^{l_{k+1}} F^1_Y(t)dt = F^1_Y(l_k)(l_{k+1} - l_k) \quad &\leq \quad \int_{l_k}^{l_{k+1}} F^1_L(t)dt = F^1_L(l_k)(l_{k+1} - l_k) \\ &\Rightarrow \quad F^1_Y(l_k) \leq F^1_L(l_k).\end{aligned}$$

□

Remark 4. *By comparing relaxed ISSD and ISSD which is equivalent to FSD, we can see the relaxation is at the points of y_m. Assume relaxed ISSD is true. From above Proposition, the FSD is satisfied in any interval $[l_k, l_{k+1})$ which does not contain any y_m. Actually, even if y_m appears in this interval, FSD still holds if $F^1_Y(y_m) \leq F^1_L(l_k)$. FSD is violated only in the interval in which the probability of Y jumps over the probability of benchmark L. And this violation will not transfer to the next interval because of relaxed ISSD.*

Remark 5. *Proposition 3 illustrates a way to construct a chance constraint from relaxed ISSD constraints, which will be shown in the following section.*

4 Multi-Stage ALM Model with SSD and Relaxed ISSD Constraints

Now, we will apply SSD and relaxed ISSD in the multi-stage ALM model to control the risk. Either SSD or relaxed ISSD can be incorporated in the model independently. It would have been possible to have a single SD constraint. Both SSD and relaxed ISSD constraints are set at each stage to make sure that the portfolio is efficient and overperforms the benchmark through the whole investment period. The model is as follows:

$$\max \sum_{i \in I, j \in N_T} \pi^T_j ((1-\gamma) w_i xh^T_{i,j} + Cash^T_j - \lambda b^T_j) \tag{32a}$$

$$s.t. \quad (1+\gamma) \sum_{i \in I} w_i xh^0_{i,0} + Cash_0 = Budget - l_0 + c_0 \tag{32b}$$

$$(1-\gamma) \sum_{i \in I} w_i xs^t_{i,j} + Cash^t_j = (1+\gamma) \sum_{i \in I} w_i xb^t_{i,j} + (1+r^t_{c,a(j)}) Cash^{t-1}_{a(j)} - l^t_j + c^t_j, \quad j, \quad t \tag{32c}$$

$$(1 + r^t_{i,j}) xh^{t-1}_{i,a(j)} + xb^t_{i,j} - xs^t_{i,j} = xh^t_{i,j}, \quad i, \quad j, \quad t \tag{32d}$$

$$\sum_{i \in I} (1-\gamma) w_i xh^T_{i,j} + Cash^T_j + b^T_j \geq l^T_j, \quad j \tag{32e}$$

$$\sum_{i\in I}(1+r_{i,j}^t)w_i xh_{i,a(j)}^{t-1} + (1+r_{c,j}^t)Cash_{a(j)}^{t-1} - \psi l_{j,t} + s_{j,t}^{l_1} \geq \tau_{l_1}, \quad j, \quad l_1, \quad t \tag{32f}$$

$$\sum_{j\in N_t} \pi_j^t s_{j,t}^{l_1} \leq \hat{\tau}_{l_1}, \quad l_1 = 1,\ldots,K_1, \quad t \tag{32g}$$

$$\sum_{i\in I}(1+r_{i,j}^t)w_i xh_{i,a(j)}^{t-1} + (1+r_{c,j}^t)Cash_{a(j)}^{t-1} - \psi l_{j,t} + v_{j,t}^{l_2} \geq \mu_{l_2}, \quad j, \quad l_2, \quad t \tag{32h}$$

$$\sum_{j\in N_t} \pi_j^t v_{j,t}^{l_2} - \sum_{j\in N_t} \pi_j^t v_{j,t}^{l_2-1} \leq \hat{\mu}_{l_2} - \hat{\mu}_{l_2-1}, \quad l_2 = 2,\ldots,K_2, \quad t \tag{32i}$$

$$\sum_{j\in N_t} \pi_j^t v_{j,t}^1 \leq \hat{\mu}_1, \quad t \tag{32j}$$

$$xh_{i,j}^t \geq 0, \quad xs_{i,j}^t \geq 0, \quad xb_j^t \geq 0, \quad b_j^T \geq 0$$

$$xh_j^t, \quad xs_j^t, \quad xb_j^t \in R^m$$

$$i \in I = \{1,\ldots,m\}, \quad j \in N_t = \{1,\ldots,n_t\}, \quad t = 1,\cdots,T,$$

$$l_1 \in \Xi_1 = \{1,\ldots,K_1\}, \quad l_2 \in \Xi_2 = \{1,\ldots,K_2\},$$

where $l_1 \in \Xi_1 = \{1,\ldots,K_1\}$ and $l_2 \in \Xi_2 = \{1,\ldots,K_2\}$ are two benchmark value sets for SSD and relaxed ISSD respectively, and τ_{l_1}, μ_{l_2} are the benchmark realizations, $\hat{\tau}_{l_1} = E[(\tau_{l_1} - \tau)_+]$ $\hat{\mu}_{l_2} = E[(\mu_{l_2} - \mu)_+]$. Equations (32f) and (32g) are SSD constraints, while Equations (32h), (32i) and (32j) are relaxed ISSD constraints.

Proposition 3 opens a way to express chance constraints in LP form by imposing relaxed ISSD constraints, i.e. have an interval $[\mu_l, \mu_{l+1}]$, μ_l and μ_{l+1} are benchmark values, such that the portfolio will not have any realization in this interval. If such interval exists, the probability, that the portfolio value is less than or equal to any number in this interval, will be less than or equal to the probability of the benchmark. Hence, the probability can be constrained for those values. There is an issue of how to guarantee the existence of such intervals. We address this problem below.

The risk control in ALM modelling reflects concerns about the underfunding which is the amount of unsatisfied liability. Bogentoft, et al [3] applied CVaR to control the return of the pension fund with certain percentage to cover the liability . While it is difficult and costly to avoid any underfunding at all, it seems highly desirable to limit the probability that any underfunding happens. We will show how to express such probability constraints in LP form. Suppose the portfolio is expected to satisfy the following probability distribution constraint:

$$P(finalwealth - liability < 0) \leq \alpha, \tag{33}$$

where α is a given threshold. An interval $[\theta_1, \theta_2]$ is assumed to exist such that the following two equations

$$finalwealth - liability < \theta_1 \tag{34}$$

$$finalwealth - liability < \theta_2 \tag{35}$$

are equivalent to

$$finalwealth - liability < 0. \tag{36}$$

For example, it is the same to the fund manager in practice to have either no underfunding or an underfunding of £1. Then this interval can be $[-1, 0]$. We assume that such interval always exists. Then, (33) can be modelled via relaxed ISSD constraints. Suppose the return of the portfolio is modelled by N scenarios. A benchmark can be constructed satisfying the following conditions:

- The benchmark value has K realizations and $K > N + 1$;
- Among K realizations, $N + 1$ are allocated in the interval $[\theta_1, \theta_2]$;
- The last but most important, $P(benchmark < 0) \leq \alpha$.

If a portfolio overperforms such benchmark by relaxed ISSD, there must be an interval $[\mu_l, \mu_{l+1}) \subset [\theta_1, \theta_2]$, where the portfolio value has no realization. Then by Proposition 3, this portfolio has return below μ_{l+1} with probability less than α. While there is no difference to the fund manager to have an underfunding of μ_{l+1} or 0, the chance constraints of the funding is successfully satisfied. For multiple chance constraints, separate relaxed ISSD constraints can be applied and the derivation is the same as in the single case.

Now we can see that, by incorporating SSD in ALM model, the risk involving overall performance is set to be lower than the banchmark; by incorporating relaxed ISSD, the risk of underfunding is controlled in terms of chance constraints.

5 Implementations

The models discussed in this paper are applicable in practice. We first demonstrate the advantages of taking SD constraints into account using a small example. Next we show how real-world problems can be solved. We apply structure-exploiting interior point solver OOPS to these problems and compare its performance against the general-purpose optimizer CPLEX 10.0 on a number of medium scale test examples.

5.1 A Model Example

Consider a small investment project with 2 stages and 4 stocks to be chosen from. There are 4 branches at the 1st stage and 2 branches from each node of the 1st stage. Both asset return and liabilities are random. The returns in percentage of these 4 stocks are in the table below and other parameters are presented in Table 3:

1st Stg	A	B	C	D	2nd Stg	A	B	C	D
1	0.0145	-0.1020	-0.0305	0.2299	1	0.1145	-0.2020	-0.0305	0.0299
					2	-0.1060	0.2450	0.0341	0.0167
2	0.0256	0.2050	0.1041	0.0036	1	0.1145	-0.2020	-0.0305	0.0299
					2	-0.1060	0.2450	0.0341	0.0167
3	-0.0013	0.0007	-0.0287	0.1858	1	0.1145	-0.2020	-0.0305	0.0299
					2	-0.1060	0.2450	0.0341	0.0167
4	0.1573	-0.0086	0.0645	-0.0743	1	0.1145	-0.2020	-0.0305	0.0299
					2	-0.1060	0.2450	0.0341	0.0167

Table 2: Returns of the assets in percentage.

	Parameter	Value
# of assets	m	4
# of leave nodes	n_T	8
# of SSD benchmarks	K_1	1
# of rISSD benchmarks	K_2	1
length of investment horizon	T	2
penalty coefficient for underfunding at horizon	λ	2
lower bound of funding ratio	ϕ	1.01
transaction fee ratio	γ	0.03

Table 3: Typical parameter values.

We generated the investment strategy using 3 models. In the first one (i), the underfunding is penalized in the objective without any SD constraint. In the second one (ii), an SSD constraint is set based on the first model (i) to restrict the portfolio to outperform a benchmark at the first stage. Then we apply the model (32) to this problem as the third model (iii), where the probability of underfunding is controlled at the second stage, i.e. the final stage, to be less than 5% by relaxed ISSD constraints, with other features the same as the second model (ii). Model (i) suggested to investing only in assets A and D, while both models (ii) and (iii) proposed also asset B with slight differences in the units of each asset respectively. Assets A and D have better performance in terms of expected return compared to the other two. However, the inclusion of asset B can lead to a better diversification. From the results presented in Table 4, we can see that taking SSD constraints into account can half the risk of underfunding while the expected return is reduced by 30%. Relaxed ISSD together with SSD can effectively reduce the probability of underfunding merely to 2% while the expected return is still worth anticipation.

Model		Portfolio	Return	Prob(underfunding)
(i)	No SD	A+D	9.8%	16%
(ii)	SSD	A+B+D	6.8%	8%
(iii)	SSD + rISSD	A+B+D	5.8%	2 %

Table 4: Portfolio properties generated from 3 models: portfolio composition, expected return and the probability of underfunding.

5.2 Numerical Results

The ALM stochastic programming model (32) proposed in previous section has the structure shown in Figure 2. Each diagonal block composed of small A and B matrices corresponds to a branch in the event tree. It contains the inventory, cash balance and underfunding definition at the last stage. The right column are the coefficients of first stage variables and the bottom diagonal block is the initial budget constraint. The bottom border line corresponds to the stochastic dominance constraints linking all the nodes of a given stage together. By exploring this special structure, using structure-exploiting interior-point solver OOPS we will solve the problem and save both time and storage.

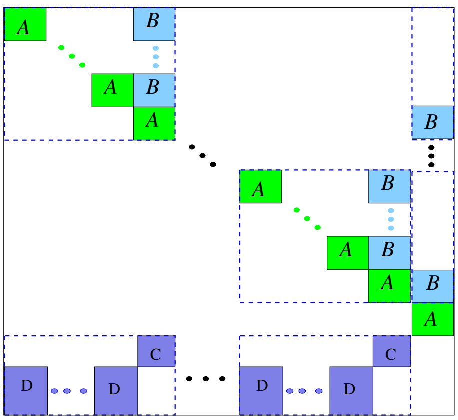

Figure 2: The structure of the two-stage ALM stochastic programming model with SSD constraints.

The computational tests were performed using the FTSE100 and FTSE250 daily data from 01/01/2003 to 01/10/2008 to construct the scenarios of portfolio return. Table 5 summarises the statistics of ALM problems tested. All the problems are modeled following the equations (32) and are linear programs. “Stages” and “Total Nodes” refer to the geometry of the event tree for these problems. “Blocks” is the number of second stage nodes. All problems use asymmetric event trees, i.e. the number of branches are different from stage to stage. There are more branches at second stage than in the following stages, e.g. 80 branches at second stage and 2 branches for all later stages. “Assets” is the number of assets that can be invested in, which are the FTSE stocks. “Bnmk” is the number of realizations of the benchmark portfolio.

Problem	Stages	Blocks	Assets	Bnmk	Total Nodes	Constraints	Variables
	T	B	I	L	$\lvert N\rvert = \sum_{t=0}^{T-1} N_t$	(I+L+2)|N|	(3I+L+2)|N|
ALM1a	2	80	64	20	81	6966	17334
ALM1b	2	40	128	20	41	6150	16646
ALM1c	2	80	128	20	81	12150	32886
ALM1d	2	160	128	20	161	24150	65366
ALM2a	2	80	64	40	81	8586	18954
ALM2b	2	40	128	40	41	6970	17466
ALM2c	2	80	128	40	81	13770	34506
ALM2d	2	160	128	40	161	27370	68586
ALM3a	2	80	64	80	81	11826	22194
ALM3b	2	40	128	80	41	8610	19106
ALM3c	2	80	128	80	81	17010	37746
ALM3d	2	160	128	80	161	33810	75026
ALM4a	3	40	128	10	201	28140	79596
ALM4b	3	80	128	10	241	33740	95436
ALM5a	4	40	128	10	1641	229740	649836
ALM5b	4	40	128	10	2921	408940	1156716
ALM5c	4	80	128	10	1681	235340	665676
ALM5d	4	80	128	10	3281	459340	1299276

Table 5: Problems scales for comparison of OOPS with CPLEX.

The size of ALM problems grows exponentially with the number of stages. There are two sets of SSD constraints (32f), (32g) and three sets of ISSD constraints (32h), (32i), (32j) for each benchmark at each stage. Suppose there are T stages, N nodes, A_1 and A_2 benchmarks in total for SSD and relaxed ISSD respectively, and each benchmark a_1 (or a_2) has K_{a_1} (or K_{a_2}) realizations, $a_1 = 1, \ldots, A_1$ and $a_2 = 1, \ldots, A_2$. SSD requirements are captured by $(N+T)\sum_{a_1} K_{a_1}$ linear constraints and relaxed ISSD requirements are taken into account by means of $(N+T)\sum_{a_2} K_{a_2}$ linear constraints. The presence of these SD constraints makes the problem very difficult for standard optimization approaches. For example, it makes impossible the application of Benders decomposition (which is otherwise a powerful method for stochastic programming [1]).

All computations were done on the Intel Core2 Duo machine. This machine features 2 2.66GHz processors and a total of 2016MB of memory.

Problem	CPLEX 10.0			OOPS		
	Time(s)	Itr	MEM(Mb)	time(s)	Itr	MEM(Mb)
ALM1a	53.47	14	100.3	19.93	24	38.9
ALM1b	26.73	20	55.3	21.05	27	38.9
ALM1c	133.91	19	184.3	41.467	26	75.8
ALM1d	9.72	42	106.5	104.132	33	147.5
ALM2a	95.07	18	114.7	37.59	28	61.4
ALM2b	63.29	18	92.2	51.16	25	59.4
ALM2c	447.85	20	335.9	111.695	27	114.7
ALM2d	5021.74	35	1265.7	316.92	39	223.3
ALM3a	124.23	19	147.5	61.49	25	102.4
ALM3b	138.89	25	143.4	92.99	29	98.3
ALM3c	1072.28	30	421.9	180.91	28	190.5
ALM3d	7709.53	28	1316.9	593.562	47	376.8
ALM4a	96.89	15	196.6	72.179	28	133.1
ALM4b	588.11	15	536.6	160.20	30	262.1
ALM5a	1291.18	29	1357.8	890.44	41	1075.2
ALM5b	–	–	–	1557.15	41	1843.2
ALM5c	1542.12	20	1597.4	589.65	26	1118.2
ALM5d	–	–	–	1140.16	25	1822.7

Table 6: Comparing solution time in seconds of CPLEX with OOPS.

The numerical results are collected in Table 6. We report the solution time, number of iterations and memory requirements for CPLEX 10.0 barrier [21] and OOPS [16, 17, 18] for each problem. Most of the problems can be solved within reasonable time and IPM iterations. Both solvers did very well for small problems. However, CPLEX ran out of memory for problems ALM5b and ALM5d, while OOPS could solve them within half an hour. For most of the problems, OOPS was faster than CPLEX, CPLEX generally took less iterations though. The solution time of OOPS increases steadily with the scaling of problems. When the number of assets is doubled, the solution time of OOPS increases by a factor smaller than two, which can be seen from the comparison of solution statistics of ALM1a and ALM1c, ALM2a and ALM2c, ALM3a and ALM3c. By comparing solution statistics of problems ALM1a, ALM2a and ALM3a, we can observe the influence of the number of benchmark realizations on the efficiency of both solvers compared. The solution statistics of ALM1b-d, ALM2b-d and ALM3b-d demonstrate that the solution time of OOPS increases linearly with the number of blocks. In comparison, CPLEX solution time is badly affected by the increase of the number of benchmark realizations. The memory requirements of OOPS are generally smaller than those of CPLEX.

6 Conclusions

In addition to the operation constraints, i.e. inventory and cash balance, ALM models require sophisticated risk control to ensure that liabilities are met. As a consequence, underfunding which measures the amount of liability dissatisfaction is expected to be zero. Stochastic dominance as a standard of efficient risk measures can manage the risk in ALM problems effectively by the consistency with utility theory. Furthermore, the concept of relaxed interval second-order stochastic dominance is developed and is used to model chance constraints in linear form, which can manage underfunding in line with other stochastic dominance constraints. Object-oriented parallel solver OOPS [16, 18] can handle such problems efficiently in terms of both storage requirements and solution time.

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