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Citation for published version:
Smyth, C & McKee, J 2013, 'Single polynomials that correspond to pairs of cyclotomic polynomials with interlacing zeros' Central European Journal of Mathematics, vol. 11, no. 5, pp. 882-899. DOI: 10.2478/s11533-013-0209-9

Digital Object Identifier (DOI):
10.2478/s11533-013-0209-9

Published In:
Central European Journal of Mathematics

PAIRS OF CYCLOTOMIC POLYNOMIALS WITH INTERLACING ROOTS

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ABSTRACT. We give a complete classification of all pairs of cyclotomic polynomials whose roots interlace on the unit circle, making explicit a result essentially contained in work of Beukers and Heckman. We show that each such pair corresponds to a single polynomial from a certain special class of integer polynomials, the 2-*reciprocal disc-bionic* polynomials. We also show that each such pair also corresponds (in four different ways) to a single Pisot polynomial from a certain restricted class, the *cyclogenic* Pisot polynomials. We investigate properties of this class of Pisot polynomials.

1. INTRODUCTION

As usual, let $\Phi_n(z)$ denote the cyclotomic polynomial whose roots are the primitive nth roots of unity. Suppose that $P(z)$ and $Q(z)$ are cyclotomic polynomials, which for us (following [Bo1]) means that each is a product of one or more of the polynomials $\Phi_n(z)$. We say that the pair $\{P, Q\}$ is an *interlacing (cyclotomic) pair* if $P(z)$ and $Q(z)$ are coprime, all their roots are simple, and these roots interlace on the unit circle (in the sense that between every pair of roots of $P(z)$ there is a root of $Q(z)$, and between every pair of roots of $Q(z)$ there is a root of $P(z)$). In particular, $P(z)$ and $Q(z)$ must have the same degree, and both 1 and -1 must appear amongst the roots of $P(z)Q(z)$. Thus one of P and Q, say $P(z)$, is a reciprocal polynomial, and the other ($Q(z)$) is $z-1$ times a reciprocal polynomial. As PQ has no repeated roots, it is a product of polynomials Φ_n for distinct values of n. Finally, we say that the interlacing pair $\{P(z), Q(z)\}$ is *imprimitive* if both are polynomials in z^ℓ for some $\ell > 1$. Otherwise the pair is *primitive*. It is known (see [MS1], or Corollary 4 below) that if $\{P(z), Q(z)\}$ is an interlacing pair then so is $\{P(z^k), Q(z^k)\}$ for any $k \in \mathbb{N}$. Conversely, if $\{P(z^k), Q(z^k)\}$ is an interlacing pair for some k, then so is $\{P(z), Q(z)\}$. We also need to remark that $\{P(z), Q(z)\}$ is an interlacing pair if and only if $\{P(-z), Q(-z)\}$ is an interlacing pair. Thus to describe all interlacing pairs it is clearly enough to describe all primitive interlacing pairs, and indeed only one of the two interlacing pairs $\{P(z), Q(z)\}$ and $\{P(-z), Q(-z)\}$.

The following result is essentially contained in the paper [BH] of Beukers and Heckman, although a little work is needed to extract it – see Section 8.

Date: 21 September 2011.

2010 *Mathematics Subject Classification.* 11R06, 11C08.

Key words and phrases. Cyclotomic polynomials, interlacing, Pisot polynomials, disc-bionic.

Theorem 1. *The primitive interlacing cyclotomic pairs are the pairs $\{P(z), Q(z)\}$ and $\{P(-z), Q(-z)\}$, where one of these pairs is given in Table 1 (two infinite families) or Table 2 (26 sporadic cases).*

The proof of Theorem 1 uses, among other results, the Shephard-Todd classification of finite irreducible complex reflection groups. Our aim in undertaking this work had originally been to try to find a more direct, and hopefully simpler, proof of Beukers and Heckman's result. Such a proof would, for instance, follow immediately from a derivation, independent of Theorem 1, of the classification (below, Corollary 3) of 2-reciprocal disc-bionic polynomials. We have not been able to do this. Instead, we have produced a number of bijections between the set of all interlacing cyclotomic pairs and certain sets of polynomials (including the set of 2-reciprocal disc-bionic polynomials). Then application of Theorem 1 enables us to describe these sets precisely. These bijections in turn imply the existence of other bijections between these sets of polynomials. We describe many of these bijections explicitly. We hope that our work will be a stepping-stone to producing a simpler proof of Theorem 1.

Family	1	2
$P(z)$	$\dfrac{z^{n+1}-1}{z-1}$	z^n+1
$Q(z)$	$\dfrac{(z^j-1)(z^{n+1-j}-1)}{z-1}$	$(z^j-1)(z^{n-j}+1)$
Range of n, j	$n \geq 1, \quad \gcd(j, n+1) = 1$ $1 \leq j \leq (n+1)/2$	$n \geq 3, \quad \gcd(j, 2n) = 1$ $1 \leq j < n$
disc-bionic polynomial	$\dfrac{2z^{n+1}-z^{n+1-j}-z^j}{z-1}$	$2z^n - z^{n-j} + z^j$
[BH]	# 1, Table 8.3	Theorem 5.8

Table 1. The two infinite families of primitive interlacing cyclotomic pairs, with the disc-bionic polynomials associated to them by Theorem 2.

In Section 2 we give a bijection between interlacing cyclotomic pairs $\{P(z), Q(z)\}$ and what we call 2-reciprocal disc-bionic polynomials. In Section 3 a we define the set $\mathcal{C}$ of cyclogenic Pisot polynomials, which we partition in two ways:

$$\mathcal{C} = \mathcal{C}_{\text{just}} \cup \mathcal{C}_{\text{strictly}} = \mathcal{C}_{\leq 2} \cup \mathcal{C}_{\geq 2}.$$

In Sections 4 and 5 we produce various explicit bijections, as indicated in Figure 1.

In Section 6 we find the cyclogenic Pisot polynomials corresponding to $\{P(-z), Q(-z)\}$ and $\{P(z^\ell), Q(z^\ell)\}$ in terms of the cyclogenic Pisot polynomial corresponding to $\{P(z), Q(z)\}$. In Section 7 we prove that the largest cyclogenic Pisot number is 3. In Section 8 we prove Theorem 1. Finally, in Section 10 we describe how all primitive interlacing cyclotomic pairs arise naturally from the study of rooted bipartite cyclotomic signed graphs.

Label	Degree	$P(z)$	$Q(z)$	$P_-(z)$	$Q_-(z)$	[BH]
A	4	12	1·2·3	12	1·2·6	37
B	6	3·12	1·2·8	6·12	1·2·8	45
C	6	3·12	1·2·5	6·12	1·2·10	46
D	6	9	1·2·4·6	18	1·2·3·4	47
E	6	9	1·2·8	18	1·2·8	48
F	6	9	1·2·5	18	1·2·10	49
G	7	2·18	1·3·12	2·6·12	1·9	58
H	7	2·18	1·3·5	2·6·10	1·9	59
I	7	2·18	1·7	2·14	1·9	60
J	7	2·14	1·3·12	2·6·12	1·7	61
K	7	2·14	1·3·5	2·6·10	1·7	62
L	8	30	1·2·18	15	1·2·9	63
M	8	30	1·2·3·12	15	1·2·6·12	64
N	8	30	1·2·3·8	15	1·2·6·8	65
O	8	30	1·2·4·5	15	1·2·4·10	66
P	8	30	1·2·3·5	15	1·2·6·10	67
Q	8	30	1·2·7	15	1·2·14	68
R	8	30	1·2·4·12	15	1·2·4·12	69
S	8	30	1·2·4·8	15	1·2·4·8	70
T	8	20	1·2·3·12	20	1·2·6·12	71
U	8	20	1·2·3·8	20	1·2·6·8	72
V	8	20	1·2·7	20	1·2·14	73
W	8	20	1·2·9	20	1·2·18	74
X	8	24	1·2·4·5	24	1·2·4·10	75
Y	8	24	1·2·7	24	1·2·14	76
Z	8	24	1·2·9	24	1·2·18	77

Table 2. The 26 sporadic cases of primitive interlacing cyclotomic pairs. The numbers in the $P(z), Q(z)$ columns refer to products of cyclotomic polynomials $n \leftrightarrow \Phi_n$. The final column refers to Table 8.3 of [BH]. Also $P_-(z), Q_-(z)$ are $(-1)^{\deg P}P(-z), (-1)^{\deg Q}Q(-z)$, if necessary interchanged so that $Q_-(1) = 0$.

2. The bionic polynomial bijection

For any polynomial $F(z)$ we denote by $F^*(z)$ its reciprocal polynomial $z^{\deg F}F(1/z)$. Note that $(z^r F(z))^* = F^*(z)$: multiplying F by a power of z does not affect its reciprocal polynomial. Note that we always have $F(z) = z^{\deg F}F^*(1/z)$ (though not always $F(z) = z^{\deg F^*}F^*(1/z)$). We say that a polynomial $F(z)$ is 2-*reciprocal* if $F(z) \equiv F^*(z) \pmod 2$, i.e., all the coefficients of $F(z) - F^*(z)$ are even. It is clear that if $F(z)$ is 2-reciprocal, then so is $F(z^\ell)$ for every $\ell \geq 1$. We say that $F(z)$ is *primitive* if it is not of the form $F_1(z^\ell)$ for any polynomial F_1 and integer $\ell > 1$.

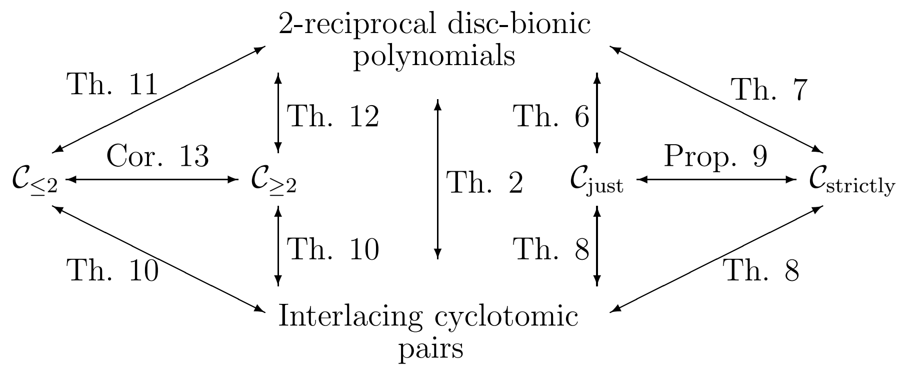

Figure 1. Various bijections

We call a nonconstant polynomial with integer coefficients *bionic* if its leading coefficient is 2. We call a bionic polynomial *disc-bionic* if all its roots lie in the open unit disc $\{z : |z| < 1\}$. It is easy to see that a disc-bionic polynomial must be either of the form $2z^n$ for some $n \geq 1$, or the product of an irreducible disc-bionic polynomial and a nonnegative power of z. Also, it is clear that a 2-reciprocal disc-bionic polynomial $H(z)$ must be divisible by a positive power of z. Furthermore, then $(-1)^{\deg H} H(-z)$ and $H(z^\ell) \quad (\ell \geq 1)$ are also 2-reciprocal disc-bionic.

A *Garsia polynomial* (see Garsia [Ga], Brunotte [Br], Hare [Ha]) is a nonconstant monic polynomial with integer coefficients, all of whose roots have modulus greater than 1, and with the product of the roots being ± 2. Disc-bionic polynomials and Garsia polynomials are closely related: if $G(z)$ is a Garsia polynomial then $(-1)^{\deg G} G^*(z)$ is a disc-bionic polynomial. The converse is almost true: if $H(z)$ is disc-bionic, then either $H^*(z)$ is the constant polynomial 2, or $(-1)^{\deg H} H^*(z)$ is a Garsia polynomial.

Theorem 2. *There is a bijection between the set of pairs $\{P, Q\}$ of interlacing cyclotomic polynomials and the set of 2-reciprocal disc-bionic polynomials H. It is given explicitly by*

$$H(z) = P(z) + Q(z). \tag{1}$$

In the other direction,

$$P(z) = \tfrac{1}{2} \{H(z) + H^*(z)\}; \tag{2}$$

$$Q(z) = \tfrac{1}{2} \{H(z) - H^*(z)\}. \tag{3}$$

Proof. Let $\{P, Q\}$ be a pair of interlacing cyclotomic polynomials of degree n, with say $Q(1) = 0$. Then, $P + Q$ has leading term 2 and, by [MS3, Proposition 9.3], all its roots inside the unit circle. Hence $H(z) := P(z) + Q(z)$ is a disc-bionic polynomial. Further,

$$H^*(z) = P(z) - Q(z), \tag{4}$$

so that $H(z) - H^*(z) = 2Q(z)$, showing that $H(z)$ is also 2-reciprocal.

Note, too, that then P and Q are given in terms of H and H^* by (2) and (3).

Conversely, let $H(z)$ be a 2-reciprocal disc-bionic polynomial of degree d say, with $2Q(z) := H(z) - H^*(z)$ having all coefficients even. Then $2P(z) := H(z) + H^*(z)$ also has all coefficients even. We claim that P and Q, so defined, are interlacing cyclotomic polynomials. We consider the rational function $R(z) := H(z)/H^*(z)$. Now $|R(z)| = 1$ for $|z| = 1$, and $R(1) = 1$, since $Q(1) = 0$. Also R has d zeros and no poles in $|z| < 1$. So as z winds once around the unit circle, anticlockwise, $R(z)$ performs d circuits of the unit circle. Hence $R(z)$ takes the value 1 at least d times, and similarly the value -1 at least d times. But for any λ with $\lambda = 1$ the polynomial $H(z) - \lambda H^*(z)$ has degree d, so in fact $R(z)$ takes the value λ exactly d times. Hence as z winds once around the unit circle, $R(z)$ winds monotonically around the circle (no doubling back). Thus $R(z)$ must take each of the values 1 and -1 *exactly* d times, and the values of z where $R(z) = 1$ must interlace on the unit circle with the values of z where $R(z) = -1$. Hence $\{P, Q\}$ forms a pair of interlacing cyclotomic polynomials. $\square$

Thus, corresponding to primitive interlacing cyclotomic pairs, there are primitive 2-reciprocal disc-bionic polynomials. Also, we make a canonical choice between such a polynomial $H(z)$ and $(-1)^{\deg H} H(-z)$, as follows. If $H(z) = (-1)^{\deg H} H(-z)$, then there is no choice to make. Otherwise H has a term of lowest even degree and a term of lowest odd degree. These two terms have the same signs in exactly one of $H(z)$ and $(-1)^{\deg H} H(-z)$. We choose H canonically to be the one where these two terms have the same sign.

We can list all canonical 2-reciprocal disc-bionic polynomials. Combining Theorem 2 with Theorem 1, we obtain the following.

Corollary 3. *The canonical* 2*-reciprocal disc-bionic polynomials consist of the following:*

- *the infinite family* $(2z^{n+m} - z^n - z^m)/(z-1)$ *for all* $m, n \in \mathbb{N}$ *with* $n \geq m$ *and* $\gcd(m, n) = 1$*;*
- *the infinite family* $2z^{n+m} + (-1)^{nm} z^n + z^m$ *for all* $m, n \in \mathbb{N}$ *with* $n > m$ *and* $\gcd(m, n) = 1$*;*
- *the* 26 *polynomials in Table 3.*

Since $\gcd(n, m) = 1$, n and m are not both even! In the first case, the coefficients are a (nonempty) string of 2's followed by a (possibly empty) string of 1's, then a (nonempty) string of 0's, the smallest-degree such example being $2z$.

Proof. The disc-bionic polynomials corresponding to the infinite families can be read off from Table 1. For the first infinite family we put $m = j$ and replace $n + 1 - j$ by n to obtain the first bullet-point of the corollary.

For the second infinite family we note that j is odd. If n is also odd then $n - j$ is even, so that, on replacing z by $-z$ and multiplying by -1 the disc-bionic polynomial of the second family becomes $2z^n + z^{n-j} + z^j$. Putting $m = j$ and replacing $n - j$ by n we have $(-1)^{nm} = 1$, so that this disc-bionic polynomial becomes $2z^{n+m} + z^n + z^m = 2z^{n+m} + (-1)^{nm} z^n + z^m$, which is canonical.

If n is even then $n - j$ is odd, so that $2z^n - z^{n-j} + z^j$ is canonical. Again putting $m = j$ and replacing $n - j$ by n, this becomes $2z^{n+m} - z^n + z^m = 2z^{n+m} + (-1)^{nm} z^n + z^m$. So in each case we get the second bullet-point of the corollary.

□

The following result is an immediate consequence of Theorem 2 and the fact that if $H(z)$ is a 2-reciprocal disc-bionic polynomial then so is $H(z^\ell)$ for every $\ell \geq 1$.

Corollary 4. *If $\{P(z), Q(z)\}$ is an interlacing cyclotomic pair, then so is $\{P(z^\ell), Q(z^\ell)\}$ for every $\ell \geq 1$.*

3. Cyclogenic Pisot polynomials

A *Pisot number* is a real algebraic integer $\theta > 1$, all of whose Galois conjugates $\neq \theta$ have modulus strictly less than 1. As in [MS3], we define a *Pisot polynomial* to be a monic integer polynomial having one real root > 1, with all other roots in $|z| < 1$. It is thus the minimal polynomial of a Pisot number, possibly multiplied by a power of z. For a Pisot polynomial $A(z)$ of degree d we say that it is *cyclogenic* if $2A'(1) \geq (d-2)A(1)$. It is easy to check that a Pisot polynomial A is cyclogenic if and only if $z^2A(z) - A^*(z)$ is positive on the interval $(1, \infty)$. Also, we say that a Pisot number is cyclogenic if it is the root of some cyclogenic Pisot polynomial. It turns out that the largest cyclogenic Pisot number is 3 (see Proposition 16 below.) For our purposes we need to extend the cyclogenic Pisot polynomials a little: we define $\mathcal{C}$ to be the set of all cyclogenic Pisot polynomials, plus the polynomials z^r and $z^r(z-1)$ for $r \geq 0$. (So we include the constant polynomial 1.)

We take $\mathcal{C}_{\geq 2}$ to be the polynomials in $\mathcal{C}$ whose largest root is in $(2, 3]$, along with the polynomial $z(z-2) \in \mathcal{C}$, and $\mathcal{C}_{\leq 2}$ to be the polynomials in $\mathcal{C}$ having no root in $[2, 3]$, along with the polynomial $z - 2 \in \mathcal{C}$. After Proposition 16, we have $\mathcal{C} = \mathcal{C}_{\leq 2} \cup \mathcal{C}_{\geq 2}$.

For $A \in \mathcal{C}$ of degree d we say that A is *just cyclogenic* if $2A'(1) = (d-2)A(1)$, and *strictly cyclogenic* if $2A'(1) > (d-2)A(1)$. We define $\mathcal{C}_{\text{just}}$ and $\mathcal{C}_{\text{strictly}}$ to be respectively the set of just cyclogenic polynomials and strictly cyclogenic polynomials. We define the *Boyd number* of $A \in \mathcal{C}$ of degree d to be $d - 2 - 2A'(1)/A(1)$ (compare Boyd [Bo1, eq.(17)]). In particular $z^r(z-1)$ $(r \geq 0)$ is defined to have Boyd number $+\infty$. So the Boyd number is 0 for $A \in \mathcal{C}_{\text{just}}$, and positive for $A \in \mathcal{C}_{\text{strictly}}$, with the exception that z^r $(r \geq 0)$ has Boyd number $-(r+2)$.

The significance of the Boyd number lies in the following.

Proposition 5. *Suppose that α is a cyclogenic Pisot number with minimal polynomial $A_\alpha(z)$ having Boyd number b. Then $z^kA_\alpha(z)$ is a cyclogenic Pisot polynomial precisely for $k = 0, \dots, \lfloor b \rfloor$. Further, $z^kA_\alpha(z)$ is a just cyclogenic Pisot polynomial if and only if b is an integer and $k = b$.*

Proof. It is easily checked that if $A(z)$ has Boyd number b then $zA(z)$ has Boyd number $b - 1$, from which the results follow immediately. □

Note that this Proposition concerns all cyclogenic Pisot polynomials, apart from the exceptional ones z^r and $z^r(z-1)$ whose roots are not (cyclogenic) Pisot numbers.

Label	Canonical 2-reciprocal disc-bionic	Pisot polynomials in $\mathcal{C}_{\text{just}}$	root	Pisot polynomials in $\mathcal{C}_{\text{strictly}}$	root	Boyd #
A	$2z^4+z^3-z^2-z$	$(z^2-z-1)z^2$	1.6180	z^3-z^2-3z-2	2.5115	1/5
B	$2z^6+z^5-z^4-z^3+z^2+z$	$z^6-2z^5-2z^4+z^3+2z^2-z-2$	2.5519	$(z^3-z-1)z^2$	1.3247	3
−C	$2z^6-2z^5+z^3$	$z^6-3z^5+2z^4-z^2+z-1$	2.1431	$z^5-2z^4+z^2-1$	1.7846	1
−D	$2z^6+z^5+z^4-z^3-z^2-z$	$(z^3-z^2-1)z^3$	1.4656	$z^5-z^4-2z^3-4z^2-3z-2$	2.5878	1/11
E	$2z^6-z^4+z^3+z^2$	$z^6-2z^5-z^4+z^3-z-1$	2.2973	$z^5-z^4-z^3+z^2-1$	1.4433	3
−F	$2z^6-z^5-z^3+z$	$z^6-3z^5+z^4+z^2+z-2$	2.5402	$(z^3-z^2-1)z^2$	1.4656	1
G	$2z^7+z^6-z^5-2z^4+z^2+z$	$z^7-2z^6-2z^5+3z^3+z^2-z-2$	2.5923	$(z^3-z-1)z^3$	1.3247	2
H	$2z^7+2z^6+z^5-z^4-z^3-z^2$	$z^7-z^6-z^5-z^4+z^3-1$	1.7475	$z^6-2z^4-4z^3-4z^2-3z-1$	2.2201	2/13
−I	$2z^7-z^6+z^4-z^3+z$	$z^7-3z^6+z^5+z^4-2z^3+z^2+z-2$	2.5488	$(z^4-z^3-1)z^2$	1.3803	2
J	$2z^7-z^5-z^4+z^3+z^2$	$z^7-2z^6-z^5+2z^3-z-1$	2.2929	$z^6-z^5-z^4+z^2-1$	1.5016	2
K	$2z^7+z^6+z^5-z^2-z$	$(z^5-z^4-z^2-1)z^2$	1.5701	$z^6-z^5-2z^4-3z^3-3z^2-3z-2$	2.5334	2/13
L	$2z^8+z^7-z^6-2z^5-z^4+z^2+z$	$z^8-2z^7-2z^6+2z^4+2z^3+z^2-z-2$	2.6082	$(z^3-z-1)z^4$	1.3247	1
M	$2z^8+2z^7-z^6-3z^5-z^4+z^3+z^2$	$z^8-z^7-3z^6-z^5+3z^4+3z^3-2z-1$	2.1857	$z^7-2z^5-2z^4-z-1$	1.8042	1/5
N	$2z^8+2z^7-2z^5-z^4$	$z^8-z^7-2z^6-z^5+2z^4+2z^3-z-1$	1.9145	$z^7-2z^5-3z^4-2z^3-2z^2-2z-1$	2.0686	1/11
O	$2z^8+2z^7+z^6-z^4-2z^3-z^2$	$z^8-z^7-z^6+z^2-1$	1.5737	$z^7-2z^5-4z^4-5z^3-5z^2-3z-1$	2.2896	1/19
P	$2z^8+3z^7+2z^6-z^4-2z^3-2z^2-z$	$(z^3-z-1)z^5$	1.3247	$z^7-3z^5-6z^4-7z^3-7z^2-5z-2$	2.6143	1/29
Q	$2z^8+2z^7-z^5-z^4-z^3$	$z^8-z^7-2z^6+z^4+z^3+z^2-z-1$	1.8326	$z^7-2z^5-3z^4-3z^3-3z^2-2z-1$	2.1241	1/13
R	$2z^8+z^7-z^6-z^5-z^4-z^3+z^2+z$	$z^8-2z^7-2z^6+z^5+z^4+z^3+2z^2-z-2$	2.5354	$(z^5-z^3-z^2-z-1)z^2$	1.5342	1/3
−S	$2z^8-z^7+z^5-z^4+z^3-z$	$(z^6-2z^5+z^4-z^2+z-1)z^2$	1.5618	$z^7-2z^6-z^5-2z^3-z-2$	2.5345	1/7
−T	$2z^8-z^7-2z^6+2z^5+z^4-2z^3+z$	$z^8-3z^7+4z^5-2z^4-3z^3+3z^2+z-2$	2.5469	$(z^5-z^4-z^3+z^2-1)z^2$	1.4433	1
U	$2z^8+z^7-z^6-z^5+z^4+z^3-z^2-z$	$(z^6-z^5-z^4+z^2-1)z^2$	1.5016	$z^7-z^6-3z^5-2z^4-z^2-3z-2$	2.5460	1/11
V	$2z^8+z^7-z^6+z^4-z^2-z$	$(z^5-z^4-z^3+z^2-1)z^3$	1.4433	$z^7-z^6-3z^5-2z^4-z^3-2z^2-3z-2$	2.5904	1/13
−W	$2z^8-2z^6-z^5+z^4+z^3$	$z^8-2z^7-z^6+z^5+z^4-1$	2.1532	$z^7-z^6-2z^5+2z^3+z^2-z-1$	1.7583	1
X	$2z^8+z^7+z^6+z^5-z^4-z^3-z^2-z$	$(z^4-z^3-1)z^4$	1.3803	$z^7-z^6-2z^5-3z^4-5z^3-4z^2-3z-2$	2.6078	1/19
Y	$2z^8+z^7-z^4-z$	$(z^7-z^6-z^5+z^2-1)z$	1.5452	$z^7-z^6-2z^5-2z^4-3z^3-2z^2-2z-2$	2.4455	1/13
−Z	$2z^8-z^6-z^5-z^4+z^3+z^2$	$z^8-2z^7-z^6+z^4+2z^3-z-1$	2.2907	$z^7-z^6-z^5+z^2-1$	1.5452	1

Table 3. The 26 sporadic cases: canonical 2-reciprocal disc-bionic polynomials, Cyclogenic Pisot polynomials in $\mathcal{C}_{\text{just}}$ and in $\mathcal{C}_{\text{strictly}}$, with their Pisot number roots, and Boyd numbers. (Those in $\mathcal{C}_{\text{just}}$ have Boyd number 0.) The negative labels in column 1 refer to the fact that the canonical polynomial in column 2 is associated with the interlacing cyclotomic pair $\{P_-, Q_-\}$ rather than $\{P, Q\}$ in the relevant row of Table 2.

Family	Cyclogenic Pisot	Boyd number
1 Just	$\dfrac{z^{n+1}-2z^n-z^{n+1-j}+z^{n-j}-z^j+z^{j-1}+1}{z-1}$	0
1 Strictly	$\dfrac{z^{n+1}-2z^n+z^{n-j}+z^{j-1}-1}{(z-1)^2}$	$\begin{cases} \infty & \text{if } j=1; \\ \dfrac{n+1}{(j-1)n-(j^2-j+1)} & \text{if } j\geq 2. \end{cases}$
2 Just	$z^n-2z^{n-1}-z^{n-j}+z^{n-1-j}+z^j-z^{j-1}-1$	0
2 Strictly	$\dfrac{z^n-2z^{n-1}+z^{n-1-j}-z^{j-1}+1}{z-1}$	$\begin{cases} \infty & \text{if } j=1; \\ \dfrac{1}{j-1} & \text{if } j\geq 2. \end{cases}$

Table 4. The cyclogenic Pisot polynomials associated via Theorem 8 to the two infinite families of interlacing cyclotomic pairs given in Table 1.

4. Cyclogenic Pisot polynomial bijections

Our first result in this section is the following.

Theorem 6. *There is a bijection between the set of* 2*-reciprocal disc-bionic polynomials* $H(z)$ *and the set* $\mathcal{C}_{\text{just}}$ *is given explicitly by*

$$A(z) = \tfrac{1}{2}\bigl\{(1-2/z)H(z) - H^*(z)\bigr\}. \tag{5}$$

In the other direction it is given by

$$H(z) = \frac{z}{(z-1)^2}\bigl\{(2z-1)A(z) - A^*(z)\bigr\}. \tag{6}$$

Proof. Given a 2-reciprocal disc-bionic polynomial H, define A by (5). Now $|(1-2/z)H(z)| > |H^*(z)|$ for $|z|=1$, $z\neq 1$. Since $A(1) = -H(1) \neq 0$, $A(z)$ has by Rouché's Theorem the same number of roots inside the unit circle as $(1-2/z)H(z)$, namely (using the fact that H is disc-bionic) $\deg A - 1$. Since H does not change sign on $[1,\infty)$, we have $A(1) < 0$. As H is 2-reciprocal $(1-2/z)H(z)-H^*(z)$ has all coefficients even. Hence $A \in \mathbb{Z}[z]$ and is monic. Thus it is a Pisot polynomial. Direct calculation shows that $2A'(1) - (\deg A - 2)A(1) = 0$, so that A is just cyclogenic.

Conversely, given a polynomial $A \in \mathcal{C}_{\text{just}}$, define H by (6). By applying Rouché's Theorem to $\lambda(2z-1)A(z) - A^*(z)$ for $\lambda > 1$ and then letting $\lambda \to 1$ we see that H has at least $\deg A$ roots in $|z|<1$, with the remaining roots on the unit circle, which must be at $z=1$. Since $A \in \mathcal{C}_{\text{just}}$, both $(2z-1)A(z) - A^*(z)$ and its derivative are zero at $z=1$, and we see that H, as defined by (6), is a polynomial with all its roots inside the unit circle. Moreover, H has leading coefficient 2.

We now show that H is 2-reciprocal. Let $a = \deg A$. Then $\deg H^* < \deg H = a$. Now

$$z^a H(1/z) = \frac{1/z}{(1/z-1)^2}\bigl\{(2/z-1)z^a A(1/z) - z^a A^*(1/z)\bigr\},$$

giving

$$H^*(z) = -\frac{1}{(z-1)^2}\big\{zA(z) + (z-2)A^*(z)\big\}. \tag{7}$$

Hence $H(z) - H^*(z) = \frac{2}{(z-1)^2}\big\{z^2A(z) - A^*(z)\big\}$. Thus H is 2-reciprocal.

An easy check shows that (5) and (6) are mutually inverse maps. □

Theorem 7. *The bijection between the set of 2-reciprocal disc-bionic polynomials $H(z)$ and the set of polynomials $B \in \mathcal{C}_{\text{strictly}}$ is given explicitly by*

$$B(z) = \frac{1}{2(z-1)}\big\{(1-2/z)H(z) + H^*(z)\big\}. \tag{8}$$

Furthermore, the Boyd number of B is

$$\frac{2H(1)}{2H'(1) - (\deg(H)+2)H(1)}. \tag{9}$$

In the other direction it is given by

$$H(z) = \frac{z}{(z-1)}\big\{(2z-1)B(z) - B^*(z)\big\}. \tag{10}$$

The proof of this theorem is similar to that of Theorem 6. The computation of the Boyd number is straightforward, using (8) and the fact that $d = \deg H = \deg B + 1 = d_B + 1$, say. We note the following formulae:

$$\begin{aligned} 2B(1) &= (d+2)H(1) - 2H'(1);\\ 4B'(1) &= (d^2-d-4)H(1) - 2(d-3)H'(1);\\ H(1) &= 2B'(1) - (d-3)B(1) = 2B'(1) - (d_B-2)B(1);\\ H'(1) &= (d_B+3)B'(1) - \tfrac{1}{2}(d_B^2 + d_B - 4)B(1). \end{aligned} \tag{11}$$

Theorem 8. *Suppose that $\{P, Q\}$ is an interlacing pair of cyclotomic polynomials, with say $(z-1) \mid Q$. Then*

(i) *there is a bijection between such pairs $\{P, Q\}$ and polynomials A in $\mathcal{C}_{\text{just}}$. This bijection is given explicitly in one direction for A by*

$$P(z) = \frac{zA(z) - A^*(z)}{z-1}, \quad Q(z) = \frac{z^2A(z) - A^*(z)}{(z-1)^2}. \tag{12}$$

In the other direction it is given by

$$zA(z) = (z-1)Q(z) - P(z). \tag{13}$$

(ii) *There is a bijection between such pairs $\{P, Q\}$ and polynomials B in $\mathcal{C}_{\text{strictly}}$. This bijection is given explicitly in one direction by*

$$P(z) = \frac{z^2B(z) - B^*(z)}{z-1}, \quad Q(z) = zB(z) - B^*(z). \tag{14}$$

In the other direction it is given by

$$zB(z) = P(z) - \frac{Q(z)}{(z-1)}. \tag{15}$$

(iii) *For A and B as in (i) and (ii), one of them is in $\mathcal{C}_{\leq 2}$ and the other is in $\mathcal{C}_{\geq 2}$.*

Proof. The proof is by composing the bijections in Theorems 2 and 6.

(i) We obtain (12) by substituting (6) and (7) into (2). Then elimination of A^* from (12) gives (13).

(ii) From (8) we obtain via $z \mapsto 1/z$ and $\deg B = \deg H - 1$ that

$$H^*(z) = \frac{1}{(z-1)}\{zB(z) + (z-2)B^*(z)\}. \tag{16}$$

Then substituting (8) and (16) into (2) we obtain (14). Elimination of B^* from (14) gives (15).

Then proceed as in (i).

(iii) We see that $A(2) = Q(2) - P(2)$, while $B(2) = P(2) - Q(2)$. So if each of A and B are Pisot polynomials with a root in $[1, 2) \cup (2, 3]$, or of the form z^a with $a \geq 0$ then the result follows immediately. If A has 2 as a root, it must be $A(z) = z(z-2) \in \mathcal{C}_{\geq 2}$, which corresponds using (6) to $H(z) = 2z - 1$. Then, using (8), we see that H corresponds to $B(z) = z - 2 \in \mathcal{C}_{\leq 2}$.

□

Proposition 9. *For the two cyclogenic Pisot polynomials $A \in \mathcal{C}_{\text{just}}$ and $B \in \mathcal{C}_{\text{strictly}}$ corresponding to the same pair of interlacing cyclotomic polynomials we have*

$$B = \frac{1}{(z-1)^3}\left\{(z^2 - 3z + 1)A(z) - (z-2)A^*(z)\right\}; \tag{17}$$

$$A = \frac{1}{(z-1)}\left\{(z^2 - 3z + 1)B(z) - (z-2)B^*(z)\right\}.$$

Proof. We apply equations (6) and (10), and eliminate H to obtain

$$(2z-1)A(z) - A^*(z) = (z-1)\big((2z-1)B(z) - B^*(z)\big). \tag{18}$$

Noting that $\deg B = \deg H - 1 = \deg A - 1$, we replace z by $1/z$ in (18) and multiply by $z^{\deg A}$ to obtain

$$zA(z) + (z-2)A^*(z) = -(z-1)\big(zB(z) + (z-2)B^*(z)\big). \tag{19}$$

Then we obtain (17) by successively eliminating one of A^* and B^* from (18) and (19). □

5. Bijections involving $\mathcal{C}_{\leq 2}$ and $\mathcal{C}_{\geq 2}$

There are similar bijections between pairs of interlacing cyclotomic polynomials and polynomials in $\mathcal{C}_{\leq 2}$, and also polynomials in $\mathcal{C}_{\geq 2}$. These follow readily from our earlier results.

Theorem 10. *Suppose that $\{P, Q\}$ is an interlacing pair of cyclotomic polynomials, with say $(z-1) \mid Q$. Then*

(i) *there is a bijection between such pairs $\{P, Q\}$ and polynomials A in $\mathcal{C}_{\le 2}$. This bijection is given explicitly in one direction for A strictly cyclogenic by*

$$P(z) = \frac{z^2 A(z) - A^*(z)}{z-1}, \quad Q(z) = zA(z) - A^*(z),$$

and by

$$P(z) = \frac{zA(z) - A^*(z)}{z-1}, \quad Q(z) = \frac{z^2 A(z) - A^*(z)}{(z-1)^2}$$

if A is just cyclogenic. In the other direction it is given by

$$zA(z) = \begin{cases} P(z) - \dfrac{Q(z)}{(z-1)} & \text{if } P(2) \ge Q(2) \\ (z-1)Q(z) - P(z) & \text{if } P(2) < Q(2); \end{cases}$$

(ii) *there is a bijection between such pairs $\{P, Q\}$ and polynomials B in $\mathcal{C}_{\ge 2}$. This bijection is given explicitly in one direction for B strictly cyclogenic by*

$$P(z) = \frac{zB(z) - B^*(z)}{z-1}, \quad Q(z) = \frac{z^2 B(z) - B^*(z)}{(z-1)^2},$$

and by

$$P(z) = \frac{z^2 B(z) - B^*(z)}{z-1}, \quad Q(z) = zB(z) - B^*(z)$$

if B is just cyclogenic. In the the other direction it is given by

$$zB(z) = \begin{cases} (z-1)Q(z) - P(z) & \text{if } P(2) \ge Q(2) \\ P(z) - \dfrac{Q(z)}{(z-1)} & \text{if } P(2) < Q(2). \end{cases}$$

Theorem 11. *The bijection between the set of 2-reciprocal disc-bionic polynomials $H(z)$ and the set of polynomials $C(z) \in \mathcal{C}_{\le 2}$ is given explicitly by*

$$C(z) = \begin{cases} A(z) & \text{given by (5) if } H(\tfrac{1}{2}) > 0; \\ B(z) & \text{given by (8) if } H(\tfrac{1}{2}) \le 0. \end{cases}$$

In the other direction it is given by

$$H(z) = \begin{cases} A(z) & \text{given by (6) if } C \text{ is just cyclogenic;} \\ B(z) & \text{given by (10) if } C \text{ is strictly cyclogenic.} \end{cases}$$

Here C replaces A in (6) or B in (10).

The significance of $H(\frac{1}{2})$ comes from equations (5) and (8), from which we see that it determines the sign of $A(2)$ and $B(2)$, and thus whether the associated Pisot number is less than, equal to, or greater than 2.

Note that, since $H(z)$ has no roots in the interval $(0, \frac{1}{2})$, $H(\frac{1}{2})$ is positive if and only if the coefficient of the lowest-degree monomial in H is $+1$. So this coefficient is -1 if and only if $H(\frac{1}{2}) \le 0$. Thus, for instance, as a check, we see that in Table 3 the just cyclogenic Pisot numbers greater (less) than 2 correspond to the polynomials H with smallest-degree coefficient positive (negative).

Theorem 12. *The bijection between the set of 2-reciprocal disc-bionic polynomials $H(z)$ and the set of polynomials $C(z) \in \mathcal{C}_{\ge 2}$ is given explicitly by*

$$C(z) = \begin{cases} B(z) & \text{given by (8) if } H(\frac{1}{2}) > 0; \\ A(z) & \text{given by (5) if } H(\frac{1}{2}) \le 0. \end{cases}$$

In the other direction it is given by

$$H(z) = \begin{cases} B(z) & \text{given by (10) if } C \text{ is just cyclogenic;} \\ A(z) & \text{given by (6) if } C \text{ is strictly cyclogenic.} \end{cases}$$

Here C replaces B in (10) or A in (6).

The following is a consequence of Proposition 9.

Corollary 13. *For the two cyclogenic Pisot polynomials $A \in \mathcal{C}_{\le 2}$ and $B \in \mathcal{C}_{\ge 2}$ corresponding to the same pair of interlacing cyclotomic polynomials we have*

$$B = \frac{1}{(z-1)^{e_A}} \left\{ (z^2 - 3z + 1)A(z) - (z-2)A^*(z) \right\}; \tag{20}$$
$$A = \frac{1}{(z-1)^{e_B}} \left\{ (z^2 - 3z + 1)B(z) - (z-2)B^*(z) \right\},$$

where the exponent e_A (respectively e_B) is 3 or 1 depending on whether A (respectively B) is just cyclogenic or strictly cyclogenic.

Proof. This follows easily from Proposition 9. From (17) we see that for A and B as in Proposition 9 we have $A(2) = -B(2)$. Also, $A(z) = z(z-2)$ iff $B(z) = z - 2$. Since we have specified that $z - 2 \in \mathcal{C}_{\le 2}$ and $z(z-2) \in \mathcal{C}_{\ge 2}$, we see that one of A, B in (17) is in $\mathcal{C}_{\le 2}$ and the other is in $\mathcal{C}_{\ge 2}$. Thus (20) gives the required bijection. □

6. The relationship between cyclogenic Pisot polynomials from $\{P(z), Q(z)\}$ and those from $\{P(-z), Q(-z)\}$ and $\{P(z^\ell), Q(z^\ell)\}$.

Suppose that $A(z)$ is the just cyclogenic polynomial related to the interlacing cyclotomic pair $\{P(z), Q(z)\}$, as described in Theorem 8. Then what is the just cyclogenic polynomial $A_-(z)$ related to the interlacing cyclotomic pair $\{P(-z), Q(-z)\}$? More accurately, $A_-(z)$ is related to the pair $\{P(-z), Q(-z)\}$ only if P and Q have even degree. If their degree is odd, then we must take the pair $\{-Q(-z), -P(-z)\}$ so that the polynomials are monic, and the second one is divisible by $z - 1$. (The factors $z - 1$ and $z + 1$ always occur in PQ, and are the only ones of odd degree. So if P and Q have even degree then Q is divisible by z^2-1, while if they have odd degree then P is divisible by $z+1$ and Q is divisible by $z-1$.)

The following result describes A_- in terms of A, and B_- in terms of B.

Theorem 14. (i) *Given a just cyclogenic Pisot polynomial $A(z)$, the polynomial*

$$A_-(z) = \begin{cases} \dfrac{1}{(z+1)^2}\left\{(z^2-2z-1)A(-z)-2A^*(-z)\right\} & \text{if } A \text{ has even degree;} \\ \dfrac{1}{(z+1)^2}\left\{-(z^2-z-1)A(-z)-zA^*(-z)\right\} & \text{if } A \text{ has odd degree.} \end{cases} \tag{21}$$

is also a just cyclogenic Pisot polynomial. Moreover, if A is related to the interlacing cyclotomic pair $P(z), Q(z)$, as described in Theorem 8, then $A_-(z)$ is related to the pair $\{P(-z), Q(-z)\}$ if P and Q have even degree, and to the pair $\{-Q(-z), -P(-z)\}$ if P and Q have odd degree.

(ii) *Given a strictly cyclogenic Pisot polynomial $B(z)$, the polynomial*

$$B_-(z) = \begin{cases} \dfrac{1}{z^2-1}\left\{(z^2-z-1)B(-z)+zB^*(-z)\right\} & \text{if } B \text{ has even degree;} \\ \dfrac{1}{z^2-1}\left\{-(z^2-2z-1)B(-z)+2B^*(-z)\right\} & \text{if } B \text{ has odd degree.} \end{cases} \tag{22}$$

is also a strictly cyclogenic Pisot polynomial. Its Boyd number is

$$\begin{cases} \dfrac{-4B(-1)}{2B'(-1)+(d_B+2)B(-1)} & \text{if } B \text{ has even degree;} \\ \dfrac{-B(-1)}{2B'(-1)+d_BB(-1)} & \text{if } B \text{ has odd degree.} \end{cases} \tag{23}$$

(Here $d_B = \deg(B)$.) Moreover, if A is related to the interlacing cyclotomic pair $\{P(z), Q(z)\}$, as described in Theorem 8, then $A_-(z)$ is related to the pair $\{P(-z), Q(-z)\}$ if P and Q have even degree, and to the pair $\{-Q(-z), -P(-z)\}$ if P and Q have odd degree.

Proof. For (i): From (13) we see that

$$zA_-(z) = \begin{cases} (z-1)Q(-z)-P(-z) & \text{if } A \text{ has even degree;} \\ Q(-z)-(z-1)P(-z) & \text{if } A \text{ has odd degree.} \end{cases} \tag{24}$$

We then replace z by $-z$ in the formulae (12) for P and Q, and substitute into (24).

The proof for (22) of (ii) is similar. The formula for the Boyd number is a routine calculation using (22). □

Theorem 15. (i) *Suppose that A is the just cyclogenic Pisot polynomial corresponding to the interlacing cyclotomic pair $\{P(z), Q(z)\}$. Then for any $\ell \in \mathbb{N}$ the just cyclogenic Pisot polynomial A_ℓ corresponding to the interlacing cyclotomic pair $\{P(z^\ell), Q(z^\ell)\}$ is given by*

$$A_\ell(z) = \frac{1}{(z^\ell-1)^2}\left\{z^{\ell-1}(z^{\ell+1}-2z^\ell+1)A(z^\ell)+(z^{\ell-1}-1)A^*(z^\ell)\right\}. \tag{25}$$

(ii) *Suppose that B is the strictly cyclogenic Pisot polynomial corresponding to the interlacing cyclotomic pair $\{P(z), Q(z)\}$. Then for any $\ell \in \mathbb{N}$ the strictly cyclogenic Pisot polynomial B_ℓ corresponding to the interlacing cyclotomic pair $\{P(z^\ell), Q(z^\ell)\}$ is given by*

$$B_\ell(z) = \frac{1}{(z-1)(z^\ell - 1)}\left\{z^{\ell-1}(z^{\ell+1} - 2z^\ell + 1)B(z^\ell) + (z^{\ell-1} - 1)B^*(z^\ell)\right\}. \qquad (26)$$

Furthermore, writing b for the Boyd number of $B(z)$, the Boyd number of $B_\ell(z)$ is

$$\frac{1}{\ell\left(1 + \frac{1}{b}\right) - 1},$$

(So, in particular, it is $1/(\ell - 1)$ if $b = \infty$.)

(iii) *As $\ell \to \infty$ the Pisot number roots of both $A_\ell(z)$ and $B_\ell(z)$ tend to 2.*

Proof. (i) From (5) we have, first on replacing z by z^ℓ and then on replacing $H(z)$ by $H(z^\ell)$, that

$$A(z^\ell) = \tfrac{1}{2}\left\{(1 - 2/z^\ell)H(z^\ell) - H^*(z^\ell)\right\};$$
$$A_\ell(z) = \tfrac{1}{2}\left\{(1 - 2/z)H(z^\ell) - H^*(z^\ell)\right\}.$$

Taking the reciprocal of the first of these equations, and using the fact that A and H have the same degree, we obtain

$$A^*(z^\ell) = \tfrac{1}{2}\left\{(1 - 2z^\ell)H^*(z^\ell) - H(z^\ell)\right\}.$$

We now eliminate $H(z^\ell)$ and $H^*(z^\ell)$ from these three equations to obtain the result.

(ii) Similarly, we obtain from (8) that

$$B(z^\ell) = \frac{1}{2(z^\ell - 1)}\left\{(1 - 2/z^\ell)H(z^\ell) + H^*(z^\ell)\right\}; \qquad (27)$$
$$B_\ell(z) = \frac{1}{2(z-1)}\left\{(1 - 2/z)H(z^\ell) + H^*(z^\ell)\right\};$$
$$B^*(z^\ell) = \frac{1}{2(z^\ell - 1)}\left\{(2z^\ell - 1)H^*(z^\ell) - H(z^\ell)\right\}.$$

For the third equation we have used the fact that $\deg B = \deg H - 1$. Again, elimination of $H(z^\ell)$ and $H^*(z^\ell)$ from these three equations gives the result.

The calculation of the Boyd number of B_ℓ is routine, if tedious, using its definition and (26). Alternatively, one can use (27) along with the formulae from (11).

(iii) Suppose that $A(z) = z^d + \cdots + az^k$, where $a \neq 0$ and $k \geq 0$, so that $A^*(z) = az^{d-k} + \cdots + 1$. Take $\delta \neq 0$ small and put $z = 2 + \delta$. Now for fixed real $z > 1$ we have $\ell \to \infty$ we have $A(z^\ell) = z^{d\ell} + O(z^{d-1}\ell)$ and $A^*(z^\ell) = az^{(d-k)\ell} + O(z^{(d-k-1)\ell})$.

We then see that

$$\begin{aligned} z^{\ell-1}(z^{\ell+1}-2z^{\ell}+1)A(z^{\ell})+(z^{\ell-1}-1)A^*(z^{\ell}) &= z^{2\ell-1}\delta(z^{d\ell}+O(z^{d-1}\ell)) \\ &\quad +(1+a)z^{(d+1)\ell-1}+O(z^{d\ell-1}) \\ &= z^{(d+2)\ell-1}\delta+O(z^{(d+1)\ell-1}). \end{aligned}$$

Hence for ℓ sufficiently large $A_\ell(z)$, and, similarly, $B_\ell(z)$, has a root in the interval $(2-|\delta|, 2+|\delta|)$, which of course must be the Pisot number root of $A_\ell(z)$ or $B_\ell(z)$. □

7. The largest cyclogenic Pisot number is 3

Proposition 16. *The largest cyclogenic Pisot number is* 3*. All other cyclogenic Pisot numbers have norm at most* 2*, the largest of norm* 2 *being* $1+\sqrt{3}=2.7320\ldots$*, and the largest of norm* 1 *being* $1+\sqrt{2}=2.4142\ldots$*.*

For the proof, we need the following simple lemma.

Lemma 17. *Let* $u_1, u_2, \ldots, u_n$ *be a sequence of* $n \geq 1$ *nonnegative numbers. Then*

$$\prod_i \frac{1-u_i}{1+u_i} \geq \frac{1-\sum_i u_i}{1+\sum_i u_i}.$$

This lemma is the instance $f(x) = \log((1-x)/(1+x))$ of the general inequality $\sum_i f(u_i) \geq f(\sum_i u_i)$, valid for all sets of u_i in a subinterval of $(0,\infty)$ and all f where $f(x)/x$ is decreasing on that interval– see [HLP, §103 p. 83].

Proof of Proposition 16. Suppose that the cyclogenic polynomial A has a root $\alpha > 3$, and that its other roots are the α_i. From the identity

$$\frac{1}{1-z}+\frac{1}{1-\bar{z}} = 1+\frac{1-|z|^2}{|1-z|^2}$$

we have that

$$\begin{aligned} \frac{d-2A'(1)}{A(1)} &= d+\frac{2}{\alpha-1}-\sum_{\alpha_i\neq\alpha}\frac{2}{1-\alpha_i} \\ &= 2+\frac{3-\alpha}{\alpha-1}-\sum_{\alpha_i\in\mathbb{R}}\frac{1+\alpha_i}{1-\alpha_i}-\sum_{\alpha_i\notin\mathbb{R}}\frac{1-|\alpha_i|^2}{|1-\alpha_i|^2} \qquad (28) \\ &\leq 2+\frac{3-\alpha}{\alpha-1}, \end{aligned}$$

which is less than 2 for $\alpha > 3$. Hence, as $A(1) < 0$, $2A'(1) < (d-2)A(1)$ for $\alpha > 3$, showing that A is not cyclogenic. On the other hand, it is easily checked that $z-3$ is cyclogenic.

We shall also need from (28) and the cyclogenic condition the fact that

$$\begin{aligned}\frac{3-\alpha}{\alpha-1} &\geq \sum_{\alpha_i\in\mathbb{R}} \frac{1+\alpha_i}{1-\alpha_i} + \sum_{\alpha_i\notin\mathbb{R}} \frac{1-|\alpha_i|^2}{|1-\alpha_i|^2} \qquad (29)\\ &\geq \sum_{\alpha_i\in\mathbb{R}} \frac{1-|\alpha_i|}{1+|\alpha_i|} + \sum_{\alpha_i\notin\mathbb{R}} \frac{1-|\alpha_i|^2}{(1+|\alpha_i|)^2}\\ &= \sum_{\alpha_i\neq\alpha} \frac{1-|\alpha_i|}{1+|\alpha_i|}.\end{aligned}$$

Now for $\alpha_i \neq \alpha$ put $r_i := |\alpha_i|$, $u_i := \frac{1-r_i}{1+r_i}$, so that also $r_i := \frac{1-u_i}{1+u_i}$, and thus (29) gives

$$\frac{3-\alpha}{\alpha-1} \geq \sum_i u_i.$$

Suppose that $\alpha < 3$. Then $\mathrm{Norm}(\alpha) = \alpha \prod_i r_i \leq 2$. So if $N := \mathrm{Norm}(\alpha) = 1$ or 2 then

$$\frac{N}{\alpha} = \prod_i r_i = \prod_i \frac{1-u_i}{1+u_i} \geq \frac{1-\sum_i u_i}{1+\sum_i u_i} \geq \alpha - 2, \qquad (30)$$

using Lemma 17 and (28). Hence $\alpha(\alpha-2) - N \leq 0$, giving the rest of the result. $\square$

8. Proof of Theorem 1

Proof. For the proof, we first recall some definitions from [BH]. Let $\mathbf{a} = (a_1, \ldots, a_n)$, $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{C}^{*n}$ be complex parameters, and $p_{\mathbf{a}}(t) = (t-a_1)\ldots(t-a_n)$, $p_{\mathbf{b}}(t) = (t-b_1)\ldots(t-b_n)$, with $A, B \in \mathbb{C}^{n\times n}$ being the companion matrices of $p_{\mathbf{a}}$ and $p_{\mathbf{b}}$ respectively. Then a hypergeometric group $H(\mathbf{a};\mathbf{b})$ is any group conjugate inside $\mathrm{GL}_n(\mathbb{C})$ to the group $\langle A, B\rangle$ generated by A and B (see [BH, 3.1, 3.5]). A matrix $C \in \mathrm{GL}_n(\mathbb{C})$ is a *complex reflection* if $C - I$ has rank 1. Now $A^{-1}B - I = A^{-1}(B-A)$ has rank 1, provided that $A \neq B$, so that $A^{-1}B$ is a complex reflection. The complex reflection subgroup $H_{\mathrm{r}}(\mathbf{a};\mathbf{b})$ of $H = H(\mathbf{a};\mathbf{b})$ is the subgroup generated by all complex reflections $A^{k-1}BA^{-k}$ for $k \in \mathbb{Z}$ ([BH, 3.5, 5.2]). We are interested in the case where $p_{\mathbf{a}}$ and $p_{\mathbf{b}}$ are both cyclotomic polynomials, and with the additional property that their roots interlace on the unit circle.

A subgroup G of $\mathrm{GL}_n(\mathbb{C})$ *acts reducibly* on $\mathbb{C}^n$ if there is a nonzero proper subspace of $\mathbb{C}^n$ that is invariant under the action of G. Otherwise G is called *irreducible*. Further, G is called *imprimitive* if there is a direct sum decomposition of $\mathbb{C}^n$ as $V_1 \oplus V_2 \oplus \cdots \oplus V_d$ into more than one nonzero subspace V_i such that the action of G permutes the V_i. Otherwise G is called *primitive*.

By [BH, Theorem 4.8], we know that we are looking for H finite, this being a necessary and sufficient condition for $p_{\mathbf{a}}$ and $p_{\mathbf{b}}$ to have interlacing roots. We separate three cases:

(i) *The case when H_{r} acts reducibly on $\mathbb{C}^n$.*

By [BH, Theorem 5.3], H_{r} is imprimitive, and further (for the moment viewing $\mathbf{a}$ and $\mathbf{b}$ as sets) $\zeta\mathbf{a} = \mathbf{a}$, $\zeta\mathbf{b} = \mathbf{b}$ for some primitive ℓth root of unity ζ with $\ell > 1$, where l is taken to be maximal, giving the interlacing pair $P_*(z^\ell), Q_*(z^\ell)$, where P_* and Q_* are a primitive cyclotomic interlacing pair.

(ii) *The case H primitive.*

By [BH, Theorem 7.1], (ii) holds precisely for the first infinite family (see [BH, Table 8.3, # 1]), and the 26 sporadic examples—see final column of Table 2 above.

(iii) *The case H_{r} irreducible and H imprimitive.*

By [BH, Theorem 5.8], we then know that the interlacing cyclotomic polynomials $P(z)$, $Q(z)$ must be of the form (up to interchanging) $P(z) = z^n - b^j c^{n-j}$, $Q(z) = (z^j - b^j)(z^{n-j} - c^{n-j})$, where $\gcd(j, n) = 1$. In particular j and $n - j$ are not both even, so, by interchanging j and $n - j$ if necessary, we can assume that j is odd. As P and Q are cyclotomic, $b^j = \pm 1$ and $c^{n-j} = \pm 1$. If $b^j = 1$, then c^{n-j} must be -1 in order that $(z-1)^2 \nmid Q(z)$, giving $P(z) = z^n + 1$, $Q(z) = (z^j - 1)(z^{n-j} + 1)$. If $b^j = -1$, then c^{n-j} must be $(-1)^n$ in order that $(z + 1)^2 \nmid Q(z)$, giving $P(z) = z^n + (-1)^n$, $Q(z) = (z^j + 1)(z^{n-j} - (-1)^n)$. Now replacing z by $-z$ and multiplying the polynomials by $(-1)^n$ gives $P(z) = z^n + 1$, $Q(z) = (z^j - 1)(z^{n-j} + 1)$ again. So in either case P and Q are in the second infinite family in Table 2.

□

9. Small cyclogenic Pisot numbers

As is well-known, Siegel [Si] showed that the smallest element of the set S of Pisot numbers is $1.3247\ldots$, with minimal polynomial $z^3 - z - 1$. All elements of S in an interval not containing a limit point of S can be found, using the Dufresnoy-Pisot-Boyd algorithm [Bo2]. This algorithm can easily be tweaked to compute also the Boyd number of all Pisot numbers found. Thus we find that the smallest ten Pisot numbers are cyclogenic. The eleventh smallest, $1.5911843\ldots$, with minimal polynomial $z^9 - z^8 - z^7 + z^2 - 1$, is noncyclogenic.

As another example, Boyd [Bo2, p.1252] determined using this algorithm that the interval $(1.755, 1.839)$ contains 165 Pisot numbers. The largest of their degrees is 31. Of these Pisot numbers, 38 of their minimal polynomials are cyclogenic (i.e., have nonnegative Boyd number), with 19 being just cyclogenic (i.e., having Boyd number 0), 13 have Boyd number in the interval $(0, 1)$, while five have Boyd number 1 (so, by Proposition 5, $z\times$ their minimal polynomial is just cyclogenic). The polynomial $z^6 - z^4 - 2z^3 - 2z^2 - 2z - 1$, with Boyd number $8/7$, is the only one having Boyd number greater than 1.

The smallest element of S' is the cyclogenic Pisot number $1.618033989\ldots$, the golden ratio, with minimal polynomial $z^2 - z - 1$. It is a limit point both of the cyclogenic Pisot numbers having minimal polynomials $(z^n(z^2 - z - 1) + 1)/(z - 1)$ and the noncyclogenic Pisot numbers having minimal polynomials $z^n(z^2 - z - 1) + z^2 - 1$. This shows that the set of noncyclogenic Pisot numbers is not closed.

The smallest noncyclogenic element of S' is 1.90516616775 with minimal polynomial $z^4 - z^3 - 2z^2 + 1$. It is a limit point only of noncyclogenic Pisot numbers, for instance of the sequence with minimal polynomials $z^n(z^4 - z^3 - 2z^2 + 1) + z^3 + z^2 - z - 1$. This shows that the set of noncyclogenic Pisot numbers contains some but not all of its limit points.

The noncyclogenic Pisot number 1.933184981899, with minimal polynomial z^5-2z^4+z-1, is the limit point of the sequence of (just) cyclogenic Pisot numbers having minimal polynomials $z^n(z^5-2z^4+z-1)-z^4+z^3-1$, showing that the set of cyclogenic Pisot numbers is not closed, either. It is also the limit point of the sequence of noncyclogenic Pisot numbers having minimal polynomials $z^n(z^5-2z^4+z-1)-z^4+z-1$, showing that the set of Pisot numbers that are limit points both of the set of cyclogenic Pisot numbers and of the set of noncyclogenic Pisot numbers contains both cyclogenic Pisot numbers and noncyclogenic Pisot numbers.

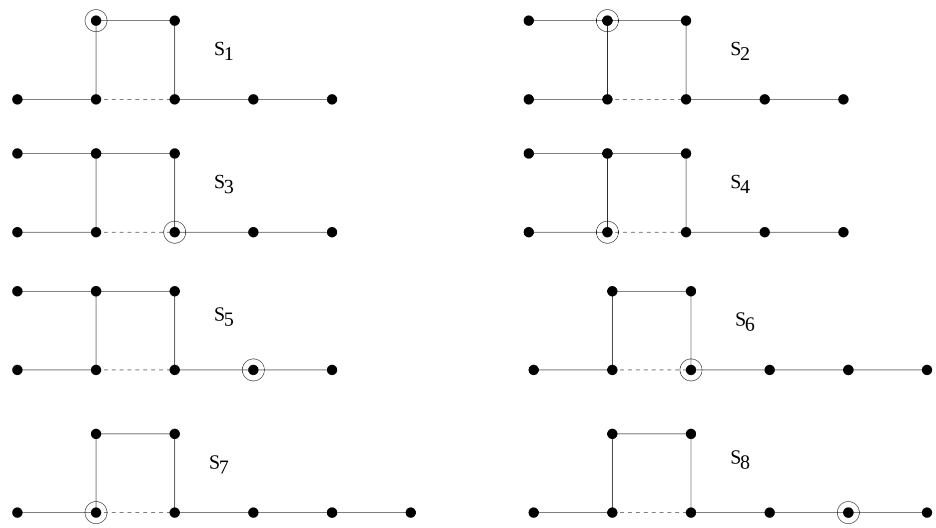

FIGURE 2. The distinguished-vertex signed graphs $S_1,\ldots,S_8$ used in Table 5. The solid edges correspond to an entry 1 in the adjacency matrix, while the broken edges correspond to an entry -1. The distinguished vertices are circled.

10. Interlacing cyclotomic polynomials from graphs and signed graphs

In this section we remark that all primitive interlacing cyclotomic pairs can also be produced from graphs or signed graphs. The method is as follows. Suppose that A is an $n\times n$ integer symmetric matrix with characteristic polynomial $\chi_A(x)$. For some i delete the ith row and column of A, to obtain A', with characteristic polynomial $\chi_{A'}(x)$. Then by Cauchy's Interlacing Theorem (see Fisk [Fi] for a slick proof) the eigenvalues $\lambda_1\le\lambda_2\le\cdots\le\lambda_n$ of A and the eigenvalues $\mu_1\le\mu_2\le\cdots\le\mu_{n-1}$ of B interlace, so that they satisfy

$$\lambda_1\le\mu_1\le\lambda_2\le\mu_2\le\cdots\le\mu_{n-1}\le\lambda_n\,.$$

We now assume that $\chi_A(x)$ is an even or odd function of x. This occurs, for instance, when A is the adjacency matrix of a bipartite graph or signed graph. Then also A has all entries 0 or 1 (respectively 0, 1 or -1). We assume further that A has all its eigenvalues in $[-2,2]$. Then it is readily checked (see [MS1]) that the pair of polynomials $z^{n/2}\chi_A(\sqrt{z}+$

Family	[MS2]
1	$A_n(j, n+1-j)$
2	$D_{n+1}(j, n+1-j)$
Label	[MS2]
A	$\tilde{E}_6(1)$
B	$E_6(1)$
C	$E_6(2)$
−D	$E_7(3)$
−E	$E_7(5)$
−F	$E_7(1)$
G	$E_7(6)$
H	$E_7(4)$
I	$E_7(7)$
J	S_1
K	$\tilde{E}_8(1)$
L	$E_8(7)$
M	$E_8(6)$
N	$E_8(5)$
O	$E_8(4)$
P	$E_8(3)$
Q	$E_8(2)$
R	$E_8(1)$
S	$E_8(8)$
T	S_2
U	S_3
V	S_4
W	S_5
X	S_6
Y	S_7
Z	S_8

TABLE 5. Graphs associated to the two infinite families of Table 1 and the 26 sporadic cases of interlacing cyclotomic pairs of Table 2. The second column refers to the graphs of [MS2, Figures 2,3,5] and the signed graphs S_i of Figure 2.

$1/\sqrt{z})$ and $(z-1)z^{(n-1)/2}\chi_{A'}(\sqrt{z}+1/\sqrt{z})$, when deprived of any common factor, is a pair of interlacing cyclotomic polynomials.

Table 5, with Figure 2, gives a graph or signed graph with a distinguished vertex for every primitive cyclotomic pair $\{P(z), Q(z)\}$. Take A to be the adjaceny matrix of the (signed) graph, and A' to be the adjacency matrix of the (signed) graph with the distinguished vertex removed. Then the construction above gives the pair $\{P(z), Q(z)\}$ of interlacing cyclotomic polynomials.

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