Non-bipartite graphs of small Mahler measure

NON-BIPARTITE GRAPHS OF SMALL MAHLER MEASURE

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ABSTRACT. The problem of describing the set of all small Mahler measures of polynomials with integer coefficients is a difficult one. One approach is to look for possible candidates among polynomials attached to combinatorial objects. In this paper we study the Mahler measure of polynomials coming from non-bipartite graphs: we classify all such graphs that have Mahler measure below $\phi = \frac{1}{2}(1+\sqrt{5})$. The bound of ϕ is natural in that it is found to be the smallest limit point of the set of Mahler measures of non-bipartite graphs. (The bipartite case was covered in an earlier paper by the second and third authors.)

1. Introduction

For a monic polynomial $p(z) \in \mathbb{Z}[z]$, its *Mahler measure*, written $M(p)$, is defined by

$$M(p) = \prod_{p(\alpha)=0} \max(1, |\alpha|) \ , \tag{1}$$

where multiple roots contribute to the product according to their multiplicity. The description of all 'small' Mahler measures of polynomials in $\mathbb{Z}[z]$ is a notorious open problem: see [18] for a recent survey of results. For $p(z)$ irreducible and $\neq z$ or $z-1$, Breusch [2] showed that unless $p(z)$ is a reciprocal polynomial (meaning that $z^{\deg(p)}p(1/z) = p(z)$) one has $M(p) \geq 1.1796\ldots$; this constant was later improved ([17]) to the best-possible one $M(z^3 - z - 1) = 1.3247\ldots$. The smallest-known Mahler measure greater than 1 is $1.17628\ldots$, the larger real root of what is now called 'Lehmer's polynomial', the reciprocal polynomial $L(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1$ [9]. Is this the smallest Mahler measure greater than 1? This celebrated question remains unresolved, although several interesting special cases have been settled [18, 12]. In view of Breusch's result, the hunt for small values of $M(p) > 1$ can be restricted to reciprocal polynomials.

A fruitful way of studying certain algebraic objects is by associating them with combinatorial structures. Indeed Lehmer's polynomial itself was discovered in this way: $L(-z)$ is the Alexander polynomial of a pretzel knot [15]. There is a natural way to attach reciprocal polynomials to graphs, and it becomes an interesting question to ask about the spectrum of possible Mahler measures for reciprocal polynomials that arise in this way (not all reciprocal polynomials do, but $L(z)$ is an example of one that does).

Let G be a finite graph, with n vertices. (For definitions of graph-theoretical terms, see [1] or [7].) The notion of the *Mahler measure of a graph* was introduced in [10]. If $\chi_G(x)$ is the characteristic polynomial of G, then G has the *associated reciprocal polynomial* $R_G(z) = z^n \chi_G(z + 1/z)$. The Mahler measure of a graph G,

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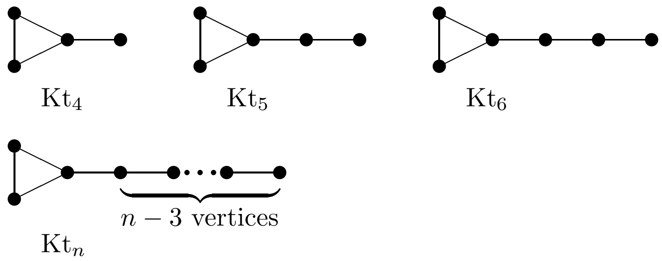


FIGURE 1. The kite graphs: Kt_n has n vertices.

written $M(G)$, is defined to be the Mahler of measure of its associated reciprocal polynomial.

It is convenient to translate (1), with $p = R_G$, into an equation involving the eigenvalues of G:

$$M(G) = \prod_{\chi_G(\lambda)=0,\ |\lambda|>2} \tfrac{1}{2}\left(|\lambda| + \sqrt{\lambda^2-4}\right) . \tag{2}$$

Again one treats multiple roots according to multiplicity. As a shorthand, we shall say simply that a graph G has *small* Mahler measure to mean that $M(G) < \phi := \frac{1}{2}(1+\sqrt{5})$.

Bipartite graphs having small Mahler measure were classified in [10, Theorem 10.2] (the word 'bipartite' was mistakenly omitted from the statement), and the remarks following it. If a graph is bipartite, then its roots are symmetric about the origin [4], and consequently having Mahler measure below ϕ implies that the spectral radius is below $\theta = \sqrt{2+\sqrt{5}}$ (with $\lambda = \sqrt{2+\sqrt{5}}$ one has $(\lambda + \sqrt{\lambda^2-4})/2 = \sqrt{\phi}$, and in the bipartite case both λ and $-\lambda$ contribute to (2)). The set of connected graphs with largest eigenvalue in the interval $(2, \theta]$ is described completely in the survey paper of Cvetković and Rowlinson [6, Theorem 2.4], drawing on work of Brouwer and Neumaier [3] and Cvetković, Doob, and Gutman [5]. The work of [10] identifies the intersection of this set of graphs with the set of those that have Mahler measure below ϕ, and hence deals with the bipartite case. But in the non-bipartite case, it is possible for the spectral radius to be larger, with the Mahler measure still below ϕ. The current paper completes the classification of all graphs that have small Mahler measure by dealing with the non-bipartite case.

Theorem 1. *Every connected non-bipartite graph that has Mahler measure below $\phi = \frac{1}{2}(1+\sqrt{5})$ is of one of the following types:*

- *an odd cycle;*
- *a 'kite' graph, shown in Figure 1;*
- *a 'balloon' graph, shown in Figure 2;*
- *one of eight sporadic examples,* Sp_a, …, Sp_h, *shown in Figure 3.*

Following [10], we shall call a graph *cyclotomic* if all its eigenvalues are in the interval $[-2, 2]$. Equivalently, G is cyclotomic if and only if $M(G) = 1$. Cyclotomic graphs were classified by Smith [16]. In particular, he showed that the only connected cyclotomic non-bipartite graphs are the odd cycles.

From Theorem 1 and [10, Theorem 10.2], it is easy to describe all (not necessarily connected) non-bipartite graphs of small Mahler measure. See also the remark in

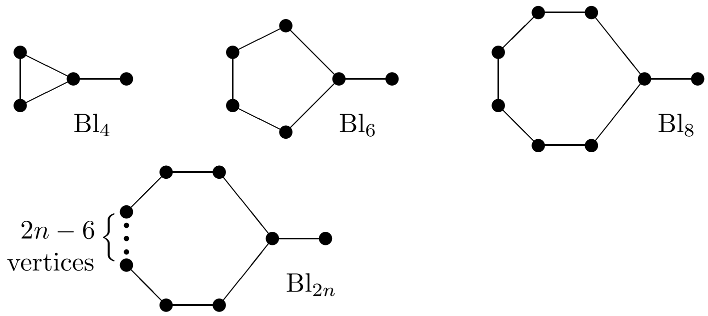


FIGURE 2. The balloon graphs: Bl_{2n} has $2n$ vertices. The smallest balloon is also a kite.

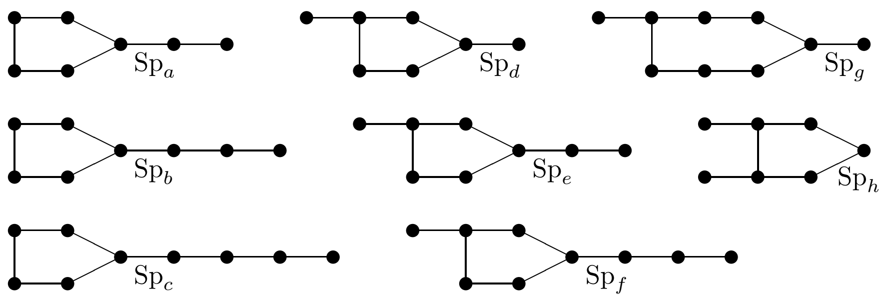
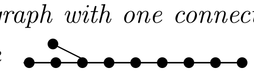


FIGURE 3. The sporadic graphs $\mathrm{Sp}_a, \ldots, \mathrm{Sp}_h$.

[10], following Theorem 10.2, concerning non-connected bipartite graphs of small Mahler measure.

Corollary 2. *Every non-bipartite graph of small Mahler measure is of one of the following types:*

- *A (not necessarily connected) bipartite graph of small Mahler measure, with one or more additional connected components consisting of odd cycles;*
- *A graph with one connected component as given in Theorem 1, with any other components cyclotomic;*
- *A graph with one connected component* Bl_8*, one connected component the tree*  *, with any other components cyclotomic.*

As an immediate consequence of Theorem 1, and the computations involved in its proof, we find the following lower bound on Mahler measures greater than 1 for non-bipartite graphs.

Corollary 3. *Let G be a non-bipartite graph. Then either $M(G) = 1$ or $M(G) \geq M(\mathrm{Bl}_8) = 1.35098\ldots$, the larger real root of $z^{10} - z^9 - z^6 + z^5 - z^4 - z + 1$.*

We note that if H is an induced subgraph of G, then by interlacing [7, Theorem 9.1] one has $M(H) \leq M(G)$.

All computations were performed using either PARI [14] or Maple [13].

2. Proof of Theorem 1

The plan of the proof is as follows. After Smith's result [16] we are reduced to considering non-cyclotomic graphs. We prove that all kites (§2.1) and balloons (§2.2) have small Mahler measure. We record the results of some computations (§2.3) that deal with all small examples. We list some special graphs whose Mahler measure is not small (§2.4): by interlacing these examples cannot appear as induced subgraphs of graphs that have small Mahler measure. We then prove that any connected graph that has small Mahler measure and contains a triangle must be a kite (Lemma 9). To complete the proof, we show that all remaining cases of connected, non-bipartite graphs that have small Mahler measure are in fact balloons (Lemma 11). The paper ends with the proof of Corollary 2, and some open problems.

2.1. All kites have small Mahler measure.

The spectrum of a kite is no doubt well-known and in any event is not difficult to derive. For completeness we give a short argument that the Mahler measure of a kite is small.

The graph Kt_n is a line graph [7, §1.7], so has all eigenvalues in the interval $[-2,\infty)$ [1, Proposition 3.7]. Deleting one of the vertices in the triangle leaves a cyclotomic graph, as is seen from Smith's classification [16]. But Kt_n itself is not one of Smith's graphs, so does not have all eigenvalues in $[-2,2]$, and so by interlacing [7, Theorem 9.1.1] Kt_n has a unique eigenvalue larger than 2, and this is the only eigenvalue that contributes to the Mahler measure via (2). (In the language of [10], Kt_n is a *Salem graph*.) Let λ_n be the largest eigenvalue of Kt_n.

As n increases, so does λ_n, and indeed it strictly increases [7, Theorem 8.8.1(b)]. Write $\lambda_n = z_n + 1/z_n$, with $z_n > 1$; then z_n also strictly increases with n, and equals the Mahler measure of Kt_n. By [10, Lemma 4.3], using the explicit formula in the proof of [10, Lemma 4.1], z_n converges to a root of $z^2 - z - 1 = 0$, and it must be the positive root ϕ. Hence $z_n = M(\mathrm{Kt}_n) < \phi$ for all $n \geq 4$, and we see that ϕ is a limit point of the set of Mahler measures of non-bipartite graphs.

2.2. All balloons have small Mahler measure.

Balloons cause more trouble than kites, as (apart from small cases) they have two eigenvalues outside the interval $[-2,2]$. Computing the characteristic polynomial by expanding along the row corresponding to the leaf, one readily computes that the reciprocal polynomial of Bl_{2n} is

$$\frac{z^{2n-1}-1}{z-1}\cdot\frac{(z^4-z^2-1)z^{2n-1}-(z^4+z^2-1)}{z+1}\,.$$

Removing cyclotomic factors from this reciprocal polynomial, and multiplying by $z+1$, gives $(z^4-z^2-1)z^{2n-1}-(z^4+z^2-1) = P_n(z)$, say. To show that Bl_{2n} has small Mahler measure, we must show that $M(P_n) < \phi$. For $n < 5$, we check this by direct computation. It remains to deal with $n \geq 5$.

Deleting the vertex of valency 3 leaves a (disconnected) cyclotomic graph, so by interlacing P_n has at most two roots outside the unit disc. Note that $P_n(-\sqrt{\phi}) < 0$, $P_n(-1) = 0$, $P_n'(-1) = 9 - 2n < 0$ for $n \geq 5$, $P_n(\sqrt{\phi}) < 0$, $P_n(\infty) = +\infty$, so that for $n \geq 5$, P_n has a root z_n^- in $(-\sqrt{\phi}, -1)$ and a root z_n^+ in $(\sqrt{\phi}, \infty)$, and these account for all possible roots outside the unit disc.

From $P_n(z)=0$, we get

$$\frac{\log\left|\dfrac{z^4+z^2-1}{z^4-z^2-1}\right|}{\log|z^2|}=\frac{2n-1}{2}\,. \tag{3}$$

Putting $z^2=(1+x)\phi$ in (3), the left-hand side becomes

$$g(x):=\frac{\log|1/x|+C+\log|1+xR(x)|}{\log\phi+\log|1+x|}\,, \tag{4}$$

where $C=\log\dfrac{2\phi}{\phi+2}\approx -0.11157$, and $R(x)=\dfrac{(4\phi+3)x+7\phi+5}{(4\phi+2)x+6\phi+2}$.

The two roots z_n^- and z_n^+ correspond to real roots of the equation $g(x)=\frac{2n-1}{2}$: call these $-u_n'$ and u_n, say, where $z_n^-=-\sqrt{(1-u_n')\phi}$ and $z_n^+=\sqrt{(1+u_n)\phi}$. We easily see that $g(x)$ is decreasing for $x>0$: rearranging the numerator in (4) as $\log|1+\frac{2(1+x)}{x(x\phi+\phi+1/\phi)}|$ one checks that it is decreasing, and the denominator is increasing. Since $g(0.1)<9/2$, we see from (3) that $u_n\in(0,0.1)$, for all $n\geq 5$. We have

$$M(P_n)=\phi\sqrt{(1+u_n)(1-u_n')}=\phi\sqrt{1+u_n-u_n'-u_nu_n'}\,,$$

which is less than ϕ if $u_n'>u_n$. We now show that this is indeed the case.

Knowing that $g(x)$ is decreasing in $(0,\infty)$, and $u_n\in(0,0.1)$, it will be enough to show that $g(-x)>g(x)$ for $x\in(0,0.1)$. One readily checks that $g(-x)-g(x)\sim\dfrac{2x}{(\log\phi)^2}\log|1/x|$ as $x\to 0+$, and simple estimates involving approximations to the logarithm function show that this positive main term dominates in the interval $(0,0.1)$, as desired.

2.3. **Details of some computations.** A consequence of interlacing is that any connected graph that has small Mahler measure can be 'grown' from smaller connected examples by adding vertices. For non-bipartite graphs this process proceeds as follows. The complete list of connected, non-bipartite graphs that have three vertices (and Mahler measure below ϕ) is very short: just the triangle. Consider all possible ways of adding a new vertex to produce a connected non-bipartite graph with 4 vertices; keep only those that have Mahler measure below ϕ, and keep only one representative of each isomorphism class. Grow similarly to get a list of 5-vertex graphs, now adding to this list the 5-cycle (which cannot be grown from a triangle). One can proceed in this way for larger and larger graphs, until computational limitations prevent further growing. In particular, growing up to 8 vertices is a trivial matter, and it establishes the following Lemma.

Lemma 4. *Let G be a connected, non-bipartite graph with $1<M(G)<\phi$ and with at most 8 vertices. Then G is either a kite (Figure 1), a balloon (Figure 2), or one of Sp_a, Sp_b, Sp_d, Sp_e, Sp_h (Figure 3).*

The growing process can also be used to investigate connected, non-bipartite graphs that have small Mahler measure and contain a particular induced subgraph H. One starts with the singleton graph H, and applies the growing process. For certain subgraphs H this process terminates, revealing only a finite number of possible larger graphs. In particular, we record the results of growing from a pentagon and from a heptagon.

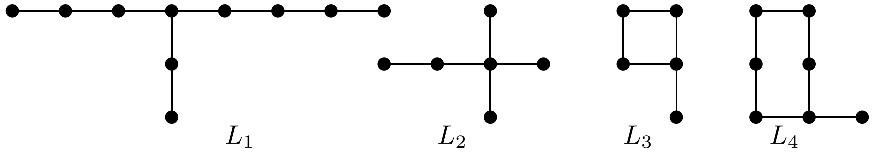


FIGURE 4. The graphs $L_1, \ldots, L_4$.

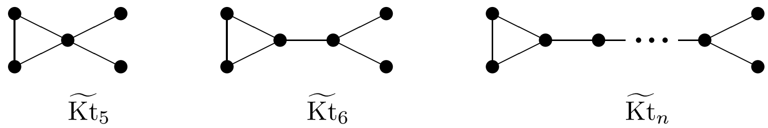


FIGURE 5. The tailed kites $\widetilde{\mathrm{Kt}}_n$, n vertices, $n \geq 5$.

Lemma 5. *The only connected, non-bipartite graphs of Mahler measure in the interval* $(1, \phi)$ *that contain either a* 5*-cycle or a* 7*-cycle are* Bl_6, Bl_8, *and the eight sporadic graphs of Figure 3.*

2.4. **Some graphs that do not have small Mahler measure.** We present here some graphs that have Mahler measure greater than ϕ. There are, of course, many others—we merely list those which will play a rôle in our later proofs. First we list some bipartite examples, for which we can appeal to [10, Theorem 10.2].

Lemma 6. *The four graphs* L_1, L_2, L_3, L_4 *in Figure 4 all have Mahler measure greater than* ϕ.

The graphs L_3 and L_4 are the first two members of an infinite family of balloons containing an even cycle (by contrast to the balloons of Figure 2). We leave it as an exercise to check that the corresponding sequence of Mahler measures forms a decreasing sequence, converging to ϕ.

Lemma 7. *The 'tailed kites'* $\widetilde{\mathrm{Kt}}_n$ *of Figure 5 (*n *vertices,* $n \geq 5$*) all have Mahler measure greater than* ϕ.

Proof. Note that $\widetilde{\mathrm{Kt}}_n$ is not one of Smith's graphs [16], but that deleting one of the degree-2 vertices in the triangle of $\widetilde{\mathrm{Kt}}_n$ leaves a subgraph of one of those graphs. By interlacing, $\widetilde{\mathrm{Kt}}_n$ has at most one eigenvalue greater than 2, and indeed exactly one, since the spectral radius of a graph always equals an eigenvalue [7, Lemma 8.7.3]. On the other hand, $\widetilde{\mathrm{Kt}}_n$ is a generalised line graph [1, 3h]. Hence $\widetilde{\mathrm{Kt}}_n$ has all eigenvalues at least -2. Thus $\widetilde{\mathrm{Kt}}_n$ has a unique eigenvalue outside the interval $[-2, 2]$, and this is > 2. From [8, Proposition 2.4], the Mahler measure of $\widetilde{\mathrm{Kt}}_n$ strictly decreases as n increases. In the limit, using [10, §4], this sequence of Mahler measures converges to ϕ. Hence $M(\widetilde{\mathrm{Kt}}_n) > \phi$ for all $n \geq 5$. $\square$

Lemma 8. *Let* $\widetilde{Q}(d, e)$ *be the graph shown in Figure 6, where* d, $e \geq 1$ *and* $d+e > 2$. *Then with the exceptions of* Sp_d, Sp_g, Sp_h $\big($*corresponding to* $(d, e) = (2, 3)$, $(3, 4)$, $(1, 4)\big)$ *one has* $M(\widetilde{Q}(d, e)) > \phi$.

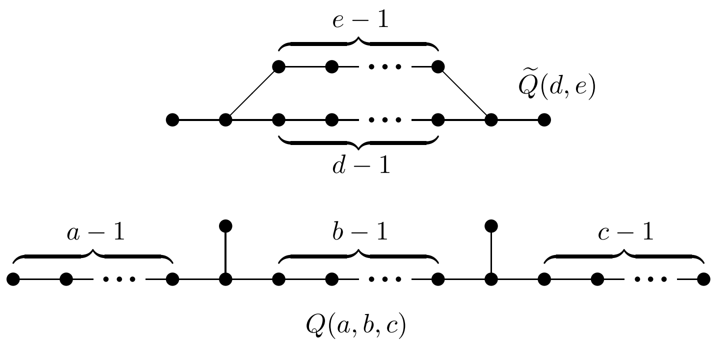


FIGURE 6. The graphs $\widetilde{Q}(d,e)$ and $Q(a,b,c)$.

Proof. We may assume that $d \le e$. For $e < 9$, we check the result by direct computation. For $e \ge 9$, delete suitable vertices from the middle of the longer path between the two degree-3 vertices to leave a subgraph $Q(a,d,c)$ (see Figure 6; here $(a-1)+(c-1) \le 7 < e-1$) in the following list: $Q(3,1,3)$, $Q(3,2,3)$, $Q(3,3,3)$, $Q(3,4,4)$, $Q(3,5,5)$, $Q(3,6,5)$, $Q(3,7,6)$, $Q(4,8,5)$, or $Q(4,d,4)$ if $d \ge 9$. From the computations in the proof of [10, Theorem 10.2], this (bipartite) subgraph has Mahler measure greater than ϕ, and hence by interlacing so does $\widetilde{Q}(d,e)$. □

2.5. **All large enough connected, non-cyclotomic, non-bipartite graphs of small Mahler measure are either kites or balloons.**

Lemma 9. *Let G be a connected graph, with Mahler measure in the interval $(1,\phi)$. If G contains a triangle, then G is a kite.*

Proof. We use induction on $n \ge 1$. For $n \le 8$, the direct computations in §2.3 establish the result.

Suppose that $n > 8$ and that the result is known for relevant graphs with fewer vertices. Let T be a triangle in G, and for any vertex v define the distance from v to T to be the minimal number of edges in a path from v to one of the vertices in T. Take v a vertex of maximal distance from T. Let G' be the subgraph obtained by deleting v and all incident edges. Maximality of the distance from v to T ensures that G' is connected. By interlacing, the Mahler measure of G' is at most that of G, so either equals 1 or is in the interval $(1,\phi)$. The former is excluded by inspection of Smith's graphs [16], so by our inductive hypothesis $G' = \mathrm{Kt}_{n-1}$. Let x be the leaf in G', with y its neighbour. By maximality of the distance of v from T, the only possible neighbours of v in G are x and y.

First consider the possibility that v is adjacent to both x and y. Using $n-1 \ge 8$, we could then delete vertices from the middle of the path from y to T to leave two disjoint copies of Kt_4. By interlacing, we would have $M(G) \ge M(\mathrm{Kt}_4)^2 > 1.50613^2 > \phi$, contradicting $M(G) < \phi$. We deduce that v is adjacent to exactly one of x and y.

Next consider the possibility that v is adjacent to y only. Then G is a tailed kite (Figure 5), and Lemma 7 gives a contradiction.

We are forced to the conclusion that v is adjacent to x only, and therefore that $G = \mathrm{Kt}_n$. □

Lemma 10. *Let G be a connected, non-bipartite graph, with Mahler measure below ϕ. Let C be an odd cycle in G, of shortest length. If v is a vertex not in C, then v is adjacent to at most one vertex of C.*

Proof. If G contains a triangle, then the result follows from Lemma 9. We may therefore suppose that G contains no triangles.

Suppose that v is a vertex not in C that is adjacent to two vertices x and y on C (and perhaps adjacent to others). The cycle C provides us with two paths from x to y, and since C has odd length one of these paths P contains an even number of edges. If P had more than two edges, then following the odd-length path from x to y, then going from y to v and from v to x would give an odd cycle shorter than C. Hence P has exactly two edges; let z be the vertex on P between x and y, and let u be the other neighbour of y on C. Since G has no triangles, and u cannot be a neighbour of x (else we could shorten C by replacing the path $xzyu$ by the path xu) the subgraph induced by x, y, z, u, v is L_3 in Figure 4. Lemma 6 records that $M(L_3) > \phi$, hence by interlacing we have $M(G) > \phi$, which is a contradiction. We conclude that no such vertex v exists, which is the claim of the current Lemma. □

We complete the proof of Theorem 1 by showing that any connected, non-bipartite graph with Mahler measure in the interval $(1, \phi)$ and with no odd cycle of length below 9 is a balloon.

Lemma 11. *Let G be a connected, non-bipartite graph, with Mahler measure in the interval $(1, \phi)$. Suppose that G has n vertices and that the shortest odd cycle in G has length $2m - 1$. If $m \geq 5$ then $G = \mathrm{Bl}_{2m}$.*

Proof. We use induction on n. For $n \leq 9$ the result is vacuous.

We suppose that $n > 9$, and that the result is known for all relevant smaller graphs. Let C be a shortest odd cycle in G. We may assume that C has at least 9 edges, or there is nothing to prove. Since $M(C) = 1$, there must be other vertices in G. Let v be a vertex in G that is as far distant from C as possible. Deleting v leaves a connected graph H, containing C as a shortest odd cycle. If $M(H) = 1$, then $H = C$ (Case 1). Otherwise, by our inductive hypothesis, $n - 1$ is even and $H = \mathrm{Bl}_{n-1}$ (Case 2): we shall in fact show that this case cannot arise.

Case 1: $H = C$. Then $n - 1$ is odd, so n is even. And by Lemma 10, $G = \mathrm{Bl}_n$.

Case 2: $H = \mathrm{Bl}_{n-1}$. Let x be the leaf of H, and let y be its neighbour on C. We split into three subcases: (a) v is adjacent to x only; (b) v is adjacent to x and to a vertex z on C (exactly one such neighbour on C, after Lemma 10); (c) v is adjacent to a vertex z on C (again unique, after Lemma 10), but not to x.

Case 2(a). Noting that L_1 of Figure 4 is an induced subgraph, we see that this case is ruled out by Lemma 6.

Case 2(b). Consider the path P on C that connects y and z via an *odd* number of edges. By minimality of the length of C, the only possible lengths for P are 1 and 3 (else we could find a shorter odd closed walk by replacing the path P within C by the path $zvxy$). If P has length 1, then G contains L_3 of Figure 4 as an induced subgraph; if P has length 3, then it contains L_4. In either case we see that Lemma 6 gives a contradiction.

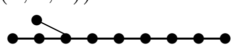
Case 2(c). We have two further subcases. If $z = y$, then we have L_2 of Figure 4 as an induced subgraph of G. If $z \neq y$, then we appeal to Lemma 8, noting that $m \geq 5$ excludes the sporadic cases.

Each subcase of Case 2 produces a contradiction, so we must be in Case 1: $H = C$ and $G = \mathrm{Bl}_n$. $\square$

3. Proof of Corollary 2

The proof of the Corollary 2 follows readily from Theorem 1, using the facts that

- A graph is non-bipartite if and only if at least one connected component is non-bipartite;
- The Mahler measure of a graph is the product of the Mahler measures of its connected components.

Let G be a non-cyclotomic graph of small Mahler measure. If all the non-cyclotomic components are bipartite, then at least one cyclotomic component must be non-bipartite, and so an odd cycle. This gives the first case of the Corollary. Otherwise, G has a non-bipartite non-cyclotomic component, as described by the Theorem. As all of these have Mahler measure at least $M(\mathrm{Bl}_8) = 1.350980338 > \sqrt{\phi}$, there can be only one of these components. If all other components are cyclotomic, we have the second case. Otherwise, some component is non-cyclotomic and bipartite, in which case, by [10, Theorem 10.2], it has Mahler measure at least $M(T(1,2,6)) = 1.176280818$. Here $T(1,2,6)$, defined in [10, Figure 15], is the tree  . But then the non-bipartite component of G can have Mahler measure at most $\phi/1.176280818 = 1.375550773$. But Bl_8 is the only such non-bipartite non-cyclotomic connected graph, all others having Mahler measure at least $M(\mathrm{Bl}_6) = 1.401268368$, and the only connected bipartite non-cyclotomic graph that has Mahler measure below $\phi/M(\mathrm{Bl}_8)$ is $T(1,2,6)$. This gives the third case.

4. Open problems

It would be nice to push knowledge of graphs of small Mahler measure beyond the ϕ boundary, in either the bipartite or non-bipartite case. In another direction, one might ask about signed graphs, or more generally the Mahler measure of integer symmetric matrices, as defined in [12]. The best result known in this setting is a classification of all indecomposable integer symmetric matrices that have Mahler measure below 1.3 ([12, Theorem 4], along with [11, §4] for a description of the cyclotomic cases).

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