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Makar-Limanov's conjecture on free subalgebras *

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Abstract

It is proved that over every countable field K there is a nil algebra R such that the algebra obtained from R by extending the field K contains noncommutative free subalgebras of arbitrarily high rank.

It is also shown that over every countable field K there is an algebra R without noncommutative free subalgebras of rank two such that the algebra obtained from R by extending the field K contains a noncommutative free subalgebra of rank two. This answers a question of Makar-Limanov [15].

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Key words: free subalgebras, extensions of algebras, nil rings

1 Introduction

In the last forty years free subobjects in groups and algebras have been extensively studied by many authors and enormous progress has been made

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[1, 4, 8, 11, 13, 14, 17, 18, 22]. In the influential paper of Makar-Limanov [12] several interesting open questions have been asked. In particular Makar-Limanov conjectured that *if R is a finitely generated infinite dimensional algebraic division algebra then R contains a free subalgebra in two generators.* Another question along this line was asked by Anick [1] in mid 1980's: *Let R be a finitely presented algebra with exponential growth. Does it follow that R contains a free subalgebra in two generators?* In the same paper he shown that finitely presented monomial algebras with exponential growth contain free subalgebras in two generators [1]. In [13] Makar-Limanov proved that the quotient algebra of the Weyl algebra contains a free subalgebra in two generators. He also conjectured that the following holds.

Conjecture 1.1 (Makar-Limanov, [15], [2]) *If R is an algebra without free subalgebras of rank two and S is an extensions of R obtained by extending the field K then S doesn't contain a free K- algebra of rank two.*

Makar-Limanov mentioned that the truth of this conjecture would imply that we need only to consider algebras over uncountable fields in his mentioned above conjecture on the division algebras [12]. Conjecture 1.1 in the case of skew-fields, as stated in [12], attracted a lot of attention and is known to be true in several important cases [3, 4, 5, 6, 10, 14, 17, 19]. In 1996 Reichstein showed that Conjecture 1.1 holds for algebras over uncountable fields [16]. The purpose of this paper is to show that the situation is completely different for algebras over countable fields, as shown in the next theorem.

Theorem 1.1 *Over every countable field K there is an algebra A without free noncommutative subalgebras of rank two such that the polynomial ring $A[x]$ in one indeterminate x over A contains a free noncommutative K-algebra of rank two.*

Note that if an algebra contains a noncommutative free algebra of rank two then it also contains a noncommutative free algebra of arbitrarily high rank. As an application the following result is obtained.

Theorem 1.2 *For every countable field K there is a field F with $K \subseteq F$ and a K-algebra A without noncommutative free subalgebras of rank two such that the algebra $A \otimes_K F$ contains a noncommutative free K-subalgebra of rank two.*

In the case of skew-fields Makar-Limanov conjecture is still open.

A ring R is nil if every element $r \in R$ is nilpotent, i.e. for every $r \in R$ there is n such that $r^n = 0$. Jacobson radical rings and nil rings are useful for investigating the general structure of rings. In addition nil rings have applications in group theory. For example the famous construction of Golod and Shafarevich, [7, 9], in the 1960s produced a finitely generated nil algebra that was not nilpotent. This was then used to construct a counterexample to the Burnside Conjecture, one of the biggest outstanding problems in group theory at that time. The Golod-Shafarevich construction gave also a counterexample to the Kurosh Problem: *let R be a finitely generated algebra over a field F such that R is algebraic over F, is R finite dimensional over F?* However, the Kurosh Problem is still open for the key special case of a division ring. There are connections with problems in nil rings. A nil element is obviously algebraic, and in the converse direction, it is possible to construct an associated graded algebra connected with an algebraic algebra in such a way that the positive part is a graded nil algebra [21].

It was shown by Amitsur in 1973 that if R is a nil algebra over an uncountable field then polynomial rings in many commuting variables over R are also nil [7, 9]. However in general polynomial rings over nil rings need not be nil [20, 21]. Our next result shows that polynomial rings over some nil rings contain noncommutative free algebras of rank two, and hence are very far from being nil.

Theorem 1.3 *Over every countable field K there is a nil algebra N such that the polynomial ring $N[X_1, \ldots, X_6]$ in six commuting indeterminates $X_1, \ldots, X_6$ over N contains a noncommutative free K-algebra of rank two.*

As an application the following result is induced.

Theorem 1.4 *Over every countable field K there is a field F, $K \subseteq F$ and a nil algebra R such that the algebra $R \otimes_K F$ contains a noncommutative free K-algebra of rank two.*

2 Notations

Let K be a countable field and let A be the free K- algebra generated by elements $x_1, x_2,\ x_3,\ y_1,\ y_2,\ y_3$. Let $G = \{x_1, x_2, x_3, y_1, y_2, y_3\}$. We say that an element $w \in R$ is a monomial, and write $w \in M$, if w is a product of elements from G. Given $e \in G, w \in M$ by $\deg_e(w)$ we will denote the number of occurrences of e in w. By M_i we denote the set of monomials of degree i. Let H_i be the K-linear space spanned by elements from M_i, i.e. $H_{m_i} = KM_i = span_K M_i$. Let D be the free K- algebra generated by elements $x,\ y$. Denote $x = z_1,\ y = z_2$. By $P \subseteq D$ we will denote the set of all monomials in x, y, and by P_i the set of monomials of degree i. Let $(i_1, \ldots, i_m), (j_1, \ldots, j_t)$ be integers. We say that $(i_1, \ldots, i_m) \prec (j_1, \ldots, j_t)$ if $(i_1, \ldots, i_m)$ is smaller than $(j_1, \ldots, j_t)$ in the lexicographical ordering, i.e. either $i_1 < j_1$ or $i_1 = j_1$ and $i_2 < j_2$, etc. Introduce a partial ordering on elements of P. Let $z, z' \in P$ and $z = \prod_{k=1}^{m} z_{i_k}\ z' = \prod_{i=1}^{m'} z_{j_k}$ where $i_k, j_k \in \{1, 2\}$ (recall that $z_1 = x, z_2 = y$). We will say that $z \prec z'$ if $m = m'$ and $(i_1, \ldots, i_m) \prec (j_1, \ldots, j_m)$. Let $\beta : M \to P$ be a semigroup homomorphism such that $\beta(x_1) = \beta(x_2) = \beta(x_3) = x$ and $\beta(y_1) = \beta(y_2) = \beta(y_3) = y$. Given $z \in P$, define $S(z) = span_K\{w \in M_{\deg z} : \beta(w) \prec z\}$, $Q(z) = span_K\{w \in M_{\deg z} : \beta(w) = z\}$. Similarly, given $z \in M$, define $S(z) = span_K\{w \in M_{\deg z} : \beta(w) \prec \beta(z)\}$, $Q(z) = span_K\{w \in M_{\deg z} : \beta(w) = \beta(z)\}$. Given integers $n_1, \ldots, n_6$ and a monomial $w \in P \cup M$, let $w(n_1, \ldots, n_6) = \sum\{v \in Q(w) : \deg_{x_1} v = n_1, \deg_{x_2} v = n_2, \deg_{x_3} v = n_3, \deg_{y_1} v = n_4, \deg_{y_2} v = n_5, \deg_{y_3} v = n_6\}$. We put $w(n_1, \ldots, n_6) = 0$ if either $\deg_x w \neq n_1 + n_2 + n_3$ or $\deg_y w \neq n_4 + n_5 + n_6$, because in this case the sum goes over the empty set.

Lemma 2.1 *For each $z \in P$ the set $U_z = \{z(n_1, \ldots, n_6) : 0 \leq n_1, \ldots, n_n,$*

$\deg_x z = n_1 + n_2 + n_3, \deg_y z = n_4 + n_5 + n_6\}$ is a free basis of a right module $U_z A$. Let $z_1, \ldots, z_n \in P_i$, for some i and assume that elements $z_1, \ldots, z_n$ are pairwise distinct. Then the set $T = T_{z_1} \bigcup T_{z_2} \bigcup \ldots \bigcup T_{z_n}$ is a free basis of a right module TA.

Proof. The proof follows from the fact that A is a free algebra and elements from U_z are linear combinations of pairwise distinct monomials of the same degree.

Lemma 2.2 *Let $0 < p, r$ be natural numbers and let $z = uv$ where $z \in P_{p+r}$, $u \in P_r$, $v \in P_p$. Then, for arbitrary integers $n_1, \ldots, n_t$, and $r < p+r$ we have $z(n_1, \ldots, n_6) = \sum\{u(r_1, \ldots, r_6)v(n_1 - r_1, \ldots, n_6 - r_6) : r_1 + \ldots + r_6 = r\}$.*

Proof. Observe first that if $p = 1$ then $z(n_1, \ldots, n_6) = \sum_{i=1}^{6} u_i v_i$ where $u_1 = u(n_1-1, n_2, n_3, n_4, n_5, n_6)$, $u_2 = u(n_1, n_2-1, n_3, n_4, n_5, n_6)$, $u_3 = u(n_1, n_2, n_3-1, n_4, n_5, n_6)$, $u_4 = u(n_1, n_2, n_3, n_4 - 1, n_5, n_6)$, $u_5 = u(n_1, n_2, n_3, n_4, n_5 - 1, n_6)$, $u_6 = u(n_1, n_2, n_3, n_4, n_5, n_6 - 1)$ and $v_1 = v(1, 0, 0, 0, 0, 0)$, $v_2 = v(0, 1, 0, 0, 0, 0)$, $v_3 = v(0, 0, 1, 0, 0, 0)$, $\ldots$, $v_6 = v(0, 0, 0, 0, 0, 1)$. Note that if $v = x$ then $v_4 = v_5 = v_6 = 0$. We will prove Lemma 2.2 by induction on n. For $n = 2$ the result holds because then $r = p = 1$. Suppose the result is true for some $n > 2$. We will show it is true for $n+1$. If $n = r+1$ and $p = 1$ then the result is true by the above observations. If $p > 1$ write $v = ww'$ for some $w \in P_{p-1}$, $w' \in P_1$.

Then by the case $p = 1$ we have $z(n_1, \ldots, n_6) = \sum_{i=1}^{6} (uw)_i w'_i$, where similarly as in the beginning of the proof $(uw)_1 = uw(n_1-1, n_2, n_3, n_4, n_5, n_6)$ and $w'_1 = w'(1, 0, 0, 0, 0, 0)$, $(uw)_2 = uw(n_1, n_2 - 1, n_3, n_4, n_5, n_6)$ and $w'_1 = w'(0, 1, 0, 0, 0, 0)$,etc.

By the inductive assumption, $uw(q_1, \ldots, q_6) = \sum\{u(r_1, \ldots, r_6)w(q_1 - r_1, \ldots, q_6 - r_6) : r_1 + \ldots + r_6 = r\}$. Now $(uw)_1 = \sum\{u(r_1, \ldots, r_6)w(n_1 - 1 - r_1, n_2 - r_2, \ldots, q_6 - r_6) : r_1 + \ldots + r_6 = r\}$.

Now $uw_1w'_1 = \sum\{u(r_1, \ldots, r_6)w(n_1 - 1 - r_1, n_2 - r_2, \ldots, q_6 - r_6)w'_1 : r_1 + \ldots + r_6 = r\}$. Similarly, $uw_2w'_2 = \sum\{u(r_1, \ldots, r_6)w(n_1 - r_1, n_2 - r_2 - 1, n_3 - r_3, \ldots, n_6 - r_6)w'_2 : r_1 + \ldots + r_6 = r\}$, etc. Therefore, $z(n_1, \ldots, n_6) =$

$\sum\{u(r_1,\ldots,r_6)[w(n_1-r_1-1,n_2-r_2,\ldots,n_6-r_6)w'_1+w(n_1-r_1,n_2-r_2-1,\ldots,n_6-r_6)w'_2+\ldots+w(n_1-r_1,n_2-r_2,\ldots,n_6-r_6-1)w'_6]:r_1+\ldots+r_6=r\}$. Observe that $w(n_1-r_1-1,n_2-r_2,\ldots,n_6-r_6)w'_1+w(n_1-r_1,n_2-r_2-1,\ldots,n_6-r_6)w'_2+\ldots+w(n_1-r_1,n_2-r_2,\ldots,n_6-r_6-1)w'_6]=ww'(n_1-r_1,\ldots,n_6-r_6)$, as in the beginning of the proof. Therefore, $z(n_1,\ldots,n_6)=\sum\{u(r_1,\ldots,r_6)v(n_1-r_1,\ldots,n_6-r_6):r_1+\ldots+r_6=r\}$, as desired.

Lemma 2.3 *Let p,q be natural numbers. Let $f:H_p\to H_p$, $g:H_q\to H_q$, and $h:H_{p+q}\to H_{p+q}$ be K-linear mappings such that for all $w\in M_p$, $w'\in M_q$, $h(ww')=f(w)g(w')$. Let $z\in P_{p+q}$, $z=uv$, $u\in P_p$, $v\in P_q$. If $h(z(n_1,\ldots,n_6))\in h(S(z))$ for all $n_1+\ldots+n_6=p+q$ then either $f(u(p_1,\ldots,p_6))\in f(S(u))$ for all $p_1+\ldots+p_6=p$ or $g(v(q_1,\ldots,q_6))\in g(S(v))$ for all $q_1+\ldots+q_6=q$.*

Proof. Suppose that the result does not hold. Let $(p_1,\ldots,p_6)$ and $(q_1,\ldots,q_6)$ be minimal with respect to the ordering $\prec$ and such that $p_1+\ldots+p_6=p$, $q_1+\ldots+q_6=q$ and $f(u(p_1,\ldots,p_6))\notin f(S(u))$, $g(v(q_1,\ldots,q_6))\notin g(S(v))$. Let $D=H_p\cap f(S(u))$ and $B=H_q\cap g(S(v))$. By Lemma 2.2, $z(p_1+q_1,\ldots,p_6+q_6)=\sum_{r_1+\ldots+r_6=p}u(r_1,\ldots,r_6)v(p_1+q_1-r_1,\ldots,p_6+q_6-r_6)$. It follows that $h(z(p_1+q_1,\ldots,p_6+q_6))=\sum_{r_1+\ldots+r_6=p}f(u(r_1,\ldots,r_6))g(v(p_1+q_1-r_1,\ldots,p_6+q_6-r_6))$. Note that if $(p_1,\ldots,p_6)\prec(r_1,\ldots,r_6)$ with respect to the lexicographical ordering then $(p_1+q_1-r_1,\ldots,p_6+q_6-r_6)\prec(q_1,\ldots,q_6)$. By the assumptions about the minimality of $(p_1,\ldots,p_6)$ if $(r_1,\ldots,r_6)\prec(p_1,\ldots,p_6)$ then $f(u(r_1,\ldots,r_6))\in f(S(u))$. Similarly, if $(v_1,\ldots,v_6)\prec(q_1,\ldots,q_6)$ then $g(v(v_1,\ldots,v_6))\in g(S((v))$. Therefore $h(z(p_1+q_1,\ldots,p_6+q_6))\in h(z(p_1,\ldots,p_6))g(z(q_1,\ldots,q_6))+DH_q+H_pB$. By the assumptions of our theorem, $h(z(p_1+q_1,\ldots,p_6+q_6))\in h(S(z))$. Note that since A is generated in degree one $S(z)\subseteq H_pS(v)+S(u)H_q$ and so $h(S(z))\subseteq H_pg(S(v))+f(S(u))H_q=H_pD+BH_q$. It follows that $h(z(p_1+q_1,\ldots,p_6+q_6))\in DH_q+H_pB$. Therefore, $f(z(p_1,\ldots,p_6))g(z(q_1,\ldots,q_6))\in DH_q+H_pB$. Recall that $f(z(p_1,\ldots,p_6))\in H_p$ and $D\in H_p$. Therefore either $f(u(p_1,\ldots,p_6))\in D\subseteq f(S(u))$ or $g(v(q_1,\ldots,q_6))\in B\subseteq g(S(v))$ a contradiction.

Lemma 2.4 *Let p, r be integers such that $p > 10^8$, $r > 10p$, 40 divides $p+r$. Let $f : H_p \to H_p$, $g : H_{r+p} \to H_{r+p}$ be K-linear mappings such that for $w \in M_r$, $w' \in M_p$, $g(ww') = wf(w')$. Let $z = uv$, $z \in M_{p+r}, u \in M_r$, $v \in M_p$. Suppose that for all $n_1 + \ldots + n_6 = p + r$, we have*

$$g(z(n_1, \ldots, n_6)) \in \sum_{r_1,\ldots,r_6: r_1+\ldots r_6=r} u(r_1, \ldots, r_6)f(S(v)) + c + \sum_{i=1}^{10^{-4}(r+p)^2} Kh_i$$

for some $h_i \in H_{p+r}$, and some $c \in \sum_w wA$ where $w \in M_r$ are monomials which are linearly independent from the elements $z(r_1, \ldots, r_6)$ with $r_1 + \ldots + r_6 = r$. Then $f(v(p_1, \ldots, p_6)) \in f(S(v))$ for all $p_1 + \ldots + p_6 = p$.

Proof. We may assume that $\deg_x z \geq \frac{\deg z}{2} = \frac{p+r}{2}$. In the case when $\deg_y z \geq \frac{\deg z}{2}$ the proof is similar. Note that $f(z(p_1, \ldots, p_6)) = 0$ if $p_i < 0$ for some i, because then $z(p_1, \ldots, p_6) = 0$. Hence, it suffices to show that each $f(v(p_1, \ldots, p_6))$ is a linear combination of $f(v(q_1, \ldots, q_6))$ with $(q_1, \ldots, q_6) \prec (p_1, \ldots, p_6)$ and elements from $f(S(v))$. Let $q_1, \ldots, q_6$ be such that $v(q_1, \ldots, q_6) \neq 0$. Then $\deg_x v = q_1 + q_2 + q_3$ and $\deg_y v = q_4 + q_5 + q_6$ by the definition of $v(q_1, \ldots, q_6)$. We will show that $f(v(q_1, \ldots, q_6)) = 0$. Let $S = \{(n_1 \ldots, n_6) : \frac{1}{6}(p + r) < n_1 < (p + r)(\frac{1}{6} + \frac{1}{40}), \frac{1}{6}(p + r) < n_2 < (p+r)(\frac{1}{6} + \frac{1}{40}), n_1+n_2+n_3 = \deg_x z$ and moreover $n_4 = q_4+\deg_y u$, $n_5 = q_5$, $n_6 = q_6\}$.

First we shall prove that $card(S) \geq (p + r)^2 10^{-4}$. Observe that there are at least $(p + r)40^{-1} - 2$ natural numbers laying between $(p + r)\frac{1}{6}$ and $(p+r)(\frac{1}{6} + \frac{1}{40})$. We can choose $((p+r)(40)^{-1} - 2)^2$ distinct pairs (n_1, n_2) such that $\frac{1}{6}(p+r) < n_1 < (p+r)(\frac{1}{6}+\frac{1}{40})$ and $\frac{1}{6}(p+r) < n_2 < (p+r)(\frac{1}{6}+\frac{1}{40})$. For each such pair we can choose a natural number n_3 such that $n_1 + n_2 + n_3 = \deg_x z$ and $(\frac{1}{6} - \frac{1}{20})(p + r) \leq n_3$ because $\deg_x z \geq \frac{p+r}{2}$. Since $p + r > 10^8$, we get that $card(S) \geq ((p + r)(40)^{-1} - 2)^2 > 10^{-4}(p + r)^2$.

Hence the assumption of the theorem implies that

$$\sum_{(n_1,\ldots,n_6)\in S} l_{n_1,\ldots,n_6} g(z(n_1, \ldots, n_6)) \in \sum_{r_1,\ldots,r_6: r_1+\ldots r_6=r} u(r_1, \ldots, r_6)f(S(v)) + c,$$

for some $l_{n_1,\ldots,n_6} \in K$, not all of which are zeros (c is as in the thesis). Let $(j_1, \ldots, j_6)$ be the maximal element in S, with respect to $\prec$, such that

$l_{j_1,\ldots,j_6} \neq 0$. Then $g(z(j_1,\ldots,j_6)) = \sum k_{n_1,\ldots,n_6} g(z(n_1,\ldots,n_6)) + q$ where the sum runs over all $(n_1,\ldots,n_6) \in S$ with $z(n_1,\ldots,n_6) \prec (j_1,\ldots,j_6)$. Moreover, $q \in \sum_{r_1,\ldots,r_6: r_1+\ldots r_6=r} u(r_1,\ldots,r_6) f(S(v)) + c$ for some $k_{r_1,\ldots,r_6} \in K$.

Now $g(v(n_1,\ldots,n_6)) = \sum_{r_1+\ldots+r_6=r} u(r_1,\ldots,r_6) f(v(n_1-r_1,\ldots,n_6-r_6))$, by Lemma 2.2. Similarly, $g(z(j_1,\ldots,j_6)) = \sum_{r_1+\ldots+r_6=r} u(r_1,\ldots,r_6) f(v(j_1-r_1,\ldots,j_6-r_6))$.

Now substitute these expressions in the equation

$$g(z(j_1,\ldots,j_6)) = \sum k_{n_1,\ldots,n_6} g(z(n_1,\ldots,n_6)) + q.$$

We get $\sum_{r_1+\ldots+r_6=r} u(r_1,\ldots,r_6)[f(v(j_1-r_1,\ldots,j_6-r_6)) - \sum_{n_1,\ldots,n_6 \in S} f(v(n_1-r_1,\ldots,n_6-r_6)] \in \sum_{r_1+\ldots+r_6=r} u(r_1,\ldots,r_6) S(v) + c$ where the sum runs over all $(n_1,\ldots,n_6) \in S$ with $z(n_1,\ldots,n_6) \prec (j_1,\ldots,j_6)$.

Now, compare the elements starting with nonzero $u(r_1,\ldots,r_6)$ (they are linearly independent by Lemma 2.1). We get the following equations

$f(z(j_1-r_1,\ldots,j_6-r_6)) \in \sum k_{n_1,\ldots,n_6} f(z(n_1-r_1,\ldots,n_6-r_6)) + f(S(v))$ where the sum runs over all $(n_1,\ldots,n_6) \in S$ with $(n_1,\ldots,n_6) \prec (j_1,\ldots,j_6)$ (provided that $u(r_1,\ldots,r_6) \neq 0$). Consider now elements $r_1 = j_1 - q_1$, $r_2 = j_2 - q_2$, $r_3 = j_3 - q_3$ and $r_4 = \deg_y u$, $r_5 = r_6 = 0$. We will show that $u(r_1,\ldots,r_6) \neq 0$. Observe first that all $r_i \geq 0$. It follows because, the definition of S and the assumption $r > 10p$ imply that $j_i > p$ for $i = 1,2,3$. By the assumptions $q_1+q_2+q_3 = \deg_x v \leq \deg v = p$. Hence for the integers $r_1 = j_1 - q_1$, $r_2 = j_2 - q_2$, $r_3 = j_3 - q_3$ are positive and $r_1 + r_2 + r_3 = (j_1 + j_2 + j_3) - (q_1 + q_2 + q_3) = \deg_x z - \deg_x v = \deg_x u$. Observe also that $r_4 + r_5 + r_6 = \deg_y u$ as required. Hence, $u(r_1,\ldots,u_6) \neq 0$. Therefore, $f(z(q_1,\ldots,q_6)) = f(z(j_1-r_1,\ldots,j_6-r_6)) \in \sum_{n_1,\ldots,n_6 \prec (j_1,\ldots,j_6)} k_{n_1,\ldots,n_6} f(z(n_1-r_1,\ldots,n_6-r_6)) + f(S(v))$. Clearly, $(n_1-r_1,\ldots,n_6-r_6) \prec (j_1-r_1, j_2-r_2, j_6-r_6)$, so the result holds.

3 Some results from other papers

In this section we quote some results from [20]. These results will be used in the last section to get the main result.

Let A be a K- algebra generated by elements $x_1, x_2,\ x_3,\ y_1,\ y_2,\ y_3$ with gradation one. Write $A = H_1 + H_2 + \ldots$. Recall that $H_i = KM_i$. We will write $M_0 = \{1\} \subseteq K,\ H_0 = K$. Given a number n and a set $F \subseteq A$ by $B_n(F)$ we will denote the right ideal in A generated by the set $\bigcup_{k=0}^{\infty} M_{nk}F$, i.e., $B_n(F) = \sum_{k=0}^{\infty} H_{nk}FA$.

Theorem 3.1 *Let f_i, $i = 1, 2, \ldots$ be polynomials in A with degrees t_i, and let m_i, $i = 1, 2, \ldots$ be an increasing sequence of natural numbers such that $m_i > 6^{6t_i}$ and $m_1 > 10^8$. There exists subsets $F_i \subseteq H_{m_i}$ with card$(F_i) < 10^{-4}m_i^2$ such that the ideal I of A generated by $f_i^{10m_{i+1}}$, $i = 1, 2, \ldots$ is contained in the right ideal $\sum_{i=0}^{\infty} B_{m_{i+1}}(F_i)$. Moreover, for every k, $I \cap H_{m_{k+1}} \subseteq \sum_{i=0}^{k} B_{m_{i+1}}(F_i)$.*

Proof. Let I_i be the smallest homogeneous ideal in A containing $f_i^{10m_{i+1}}$, for $i = 1, 2, \ldots$. By considering algebras generated by 6 elements instead of 3 elements and using the same proof as the proof of Theorem 2 in [20] for $k = m_i$, $w = m_{i+1}$, $f = f_i$ and changing constants from 3 to 6, we get the following result. There exists a set $F_i \subseteq H_{m_i}$, such that card$F_i < m_i 6^{6t_i} t_i^2$ such that the (two sided) ideal of A generated by $f_i^{10m_{i+1}}$ is contained in $B_{m_{i+1}}(F_i)$. Note that card$F_i < 10^{-4}m_i^2$ since $m_i > 6^{6t_i}$ and $m_i > m_1 > 10^8$ by the assumptions. Observe now that $I \subseteq \sum_{i=1}^{\infty} I_i$. Note that I_{k+1} is generated by elements with degrees larger than m_{k+1}. Recall that ideals I_i are homogeneous. Therefore, $I \cap H_{m_{k+1}} \subseteq \sum_{i=1}^{k} I_i$. Hence, $I \cap H_{m_{k+1}} \subseteq \sum_{i=1}^{k} B_{m_{i+1}}(F_i)$ as required. This finishes the proof.

Let mappings $R_i : H_{m_i} \to H_{m_i}$ and $c_{R_i(F_i)}$ be defined as in section 2 in [20] with $F_i = \{f_{i,1}, \ldots, f_{i,r_i}\} \subseteq H_{m_i}$ be as in Theorem 3.1. Recall that $c_{R_i(F_i)} : H_{m_i} \to H_{m_i}$ is a K-linear mapping with ker$c_{R_i(F_i)} = \{R_i(f_{i,1}), \ldots, R_i(f_{i,r_i})\}$. Given $w = x_1 \ldots x_{m_{i+1}} \in M_{m_{i+1}}$, $R_{i+1} : H_{m_{i+1}} \to H_{m_{i+1}}$ is a K-linear mapping such that

$$R_{i+1}(w) = c_{R_i(F_i)}(R_i(x_1 \ldots x_{m_i})) \prod_{j=2}^{m_{i+1}m_i^{-1}} R_i(x_{(j-1)m_i+1} \ldots x_{jm_i}).$$

Moreover, $R_1 = Id$. The fact that the algebra A is generated by 6 elements instead of 3 elements doesn't change the proof of Theorem 4 in [20].

Theorem 3.2 (Theorem 4, [20]) *Suppose that* $w \in H_{m_{l+1}} \cap \sum_{i=0}^{l} B_{m_{i+1}}(F_i)$. *Then* $R_{l+1}(w) = 0$.

4 Linear mappings

In this section we will prove some technical results about the mappings R_i. The algebra $A = H_1 + H_2 + \dots$ is as in the previous sections. We will use the following notations. $M_0 = \{1\}$ and $H_0 = K$. In this section we will assume that $R_i : H_{m_i} \to H_{m_i}$ are as in section 3 and moreover $40m_i$ divides m_{i+1} and $m_{i+1} > 2^{i+101}m_i$, $m_1 > 10^8$ for $i = 1, 2, \dots$.

Lemma 4.1 *Let* k *be a natural number. Then there are non-negative integers* e_i, d_i *with* $\sum_i e_i > 50 \sum_i d_i$ *and* $\sum_i e_i + d_i = m_k$ *such that if* $w \in M_{m_i}$ *and* $w = \prod_i u_i v_i$ *with* $u_i \in M_{e_i}$, $v_i \in M_{d_i}$ *then* $R_k(w) = \prod_i u_i g_{i,k}(v_i)$ *for some* K*-linear mappings* $g_{i,k} : H_{d_i} \to H_{d_i}$.

Let σ *be a permutation on a set of* m_k *elements, such that* $(\prod_{i=1} u_i v_i)^{\sigma} = \prod_i u_i \prod_i v_i$. *Denote* $u = \prod_i u_i$, $v = \prod_i v_i$. *Let* $T_k(uv) = R_k((uv)^{\sigma^{-1}})^{\sigma}$. *Then* $T_k(uv) = ufk(v)$, *where* $f_k : H_{\deg v} \to H_{\deg v}$ *is a* K*-linear mapping defined as follows* $f_k(v) = f_k(\prod_i v_i) = \prod_i g_{i,k}(v_i)$.

Proof. The proof of the first part of Lemma 4.1 is the same as the proof of Theorem 6 in [20]. Note that $e_1 = 0$ and $u_1 = 1 \in K$. To prove the second part of Lemma 4.1, observe that $T_k(uv) = R_k(w)^{\sigma} = R_k(\prod_i u_i v_i)^{\sigma} = (\prod_i u_i g_{i,k}(v_i))^{\sigma} = \prod_i u_i \prod_i g_{i,k}(v_i) = u f_k(v)$, as required.

Lemma 4.2 *Let* $w = \prod_i u_i v_i$, $u = \prod_i u_i$, $v = \prod_i v_i$, e_i, d_i, T_k *be as in Lemma 4.1. Let* k *be a natural number. Then*

$$(R_k(S(w)))^{\sigma} \subseteq \sum_{c \in M_{\deg u} : c \notin Q(u)} cA + \sum_{c \in M : c \in Q(u)} c f_k(S(v)).$$

Moreover

$$R(w(n_1, \dots, n_6)) = (\sum_{p_1 + \dots + p_6 = \deg u} u(p_1 \dots p_6) f_k(v(n_1 - p_1, \dots, n_6 - p_6)))^{\sigma^{-1}},$$

for all $n_1, \ldots, n_6$.

Proof. Observe first that $S(w)$ is a linear combination of some elements $t = \prod_i q_i r_i$ with $q_i \in M_{e_i}$, $r_i \in M_{d_i}$. If $\prod_i q_i \in Q(u)$ then $q_i \in Q(u_i)$ for each i. In this case, since $\prod_i q_i r_i \in S(w)$ we have $\prod_i r_i \in S(v)$.

By the definition of the mapping R_k we have $R_k(t) = \prod_i q_i g_{i,k}(r_i)$. Now $(R_k(t))^\sigma = \prod_i q_i \prod_i g_{i,k}(r_i) = \prod_i q_i f_k(\prod_i r_i)$. Recall that, if $\prod_i q_i \in Q(u)$ then $\prod_i r_i \in S(v)$. Consequently, $f_k(\prod_i r_i) \in f_k(S(v))$, and so $(R_k(S(w)))^\sigma \subseteq \sum_{c \in M_{\deg u} : c \notin Q(u)} cA + \sum_{c \in M : c \in Q(u)} c f_k(S(v))$.

We will now prove the second part of the theorem. Let $z = uv$, by Lemma 2.2, we have $\sum_{p_1 + \ldots + p_6 = \deg u} u(p_1 \ldots p_6) f_k(v(n_1 - p_1, \ldots, n_6 - p_6)) = T_k(z(n_1, \ldots, n_6))$. Note that $z^{\sigma^{-1}} = w$. Therefore, $T_k(z(n_1, \ldots, n_6)) = R_k(z(n_1, \ldots, n_6)^{\sigma^{-1}})^\sigma = R_k(w(n_1, \ldots, n_6))^\sigma$. The result follows.

Lemma 4.3 *Let* $w = \prod_i u_i v_i$, $u = \prod_i u_i$, $v = \prod_i v_i$, e_i, d_i, T_k, f_k *be as in Lemma* 4.2. *Let* k *be a natural number. Suppose that* $f_k(v(n_1, \ldots, n_6) \in f_k(S(v))$ *for all* $n_1 + \ldots + n_6 = \deg v$. *Then* $R_k(w(n_1, \ldots, n_6)) \subseteq R_k(S(w))$ *for all* $n_1 + \ldots + n_6 = m_i$.

Proof. By the assumption that $f_k(v(n_i, \ldots, n_6) \in f_k(S(v))$. Let $z = uv$. Hence, by Lemma 2.2, $z(n_1, \ldots, n_6) \in Q(u)S(v)$ for all $n_1, \ldots, n_6$. Consequently, $T_k(z(n_1, \ldots, n_6)) \in Q(u) f_k(S(v))$ for all $n_1, \ldots, n_6$. Now, by Lemma 4.1 we have $R_k(w(n_1, \ldots, n_6)) \in [Q(u)S(v)]^{\sigma^{-1}}$. An element in $S(v)$ is a linear combination of some elements $\prod_i r_i \in S(v)$, with $r_i \in M_{d_i}$. An element $p \in Q(u)$ is a linear combination of products $\prod_i q_i$, with $q_i \in Q(u_i)$. Therefore elements from the set $Q(u)S(v)$ are linear combinations of products $\prod_i q_i \prod_i r_i$. It follows that elements from the set $[Q(u) f_k(S(v))]^{\sigma^{-1}}$ are linear combinations of products $[\prod_i q_i \prod_i g_{i,k}(r_i)]^{\sigma^{-1}} = \prod_i q_i g_{i,k}(r_i) = R_k(\prod_i q_i r_i)$. It follows that $\prod_i q_i r_i \in S(w)$ since $\prod_i q_i \in Q(u)$ and $\prod_i r_i \in S(v)$, as required.

Theorem 4.1 *Let* T_k, $u = \prod_i u_i$, $v = \prod_i v_i$, $w = \prod_i u_i v_i$, *be as in Lemma* 4.2. *If* $R_k(w(n_1, \ldots, n_6)) \subseteq R_k(S(w)) + \sum_{i=1}^{m_k^2 10^{-4}} K g_i$ *for some* $g_i \in A$ *then* $R_k(w(n_1, \ldots, n_6)) \subseteq R_k(S(w))$ *for all* $n_1, \ldots, n_6$.

Proof. By Lemma 4.2 we have $T_k(z(n_1,\ldots,n_6)) = R_k(\bar{z}(n_1,\ldots,n_6)^{\sigma^{-1}})^\sigma = R_k(w(n_1,\ldots,n_6))^\sigma$ for all $n_1,\ldots,n_6$. By assumption $R_k(w(n_1,\ldots,n_6))^\sigma \subseteq R_k(S(w))^\sigma + \sum_{i=1}^{m_k^2 10^{-4}} Kg_i^\sigma$. Denote $g_i^\sigma = h_i$. By Lemma 4.2 $(R_k(S(w)))^\sigma \subseteq \sum_{c\in M_{\deg u}:c\notin Q(u)} cA+\sum_{c\in M:c\in Q(u)} cf_k(S(v))$. It follows that $T_k(z(n_1,\ldots,n_6)) \subseteq \sum_{c\in M_{\deg u}:c\notin Q(u)} cA + \sum_{c\in M:c\in Q(u)} cf_k(S(v)) + \sum_{i=1}^{m_i^2 10^{-4}} Kf_i$. Therefore T_i satisfies the assumptions of Lemma 2.4. Consequently, $f_k(v(n_1,\ldots,n_6)) \in f_k(S(v))$ for all $n_1,\ldots,n_6$. By Lemma 4.3 we get that $R_k(w(n_1,\ldots,n_6)) \subseteq R_k(S(w))$ for all $n_1,\ldots,n_6$, as required.

Theorem 4.2 *Let $i > 0$, $F_i = \{f_{i,1},\ldots,f_{i,r_i}\} \subseteq H_{m_i}$, with $r_i < 10^{-4}m_i^2$. For every monomial $w \in P$ of degree m_i for some i, there are $n_1,\ldots,n_6$ such $n_1+\ldots+n_6 = m_i$ such that $R_i(w(n_1,\ldots,n_6)) \notin R_i(S(w))$.*

Proof. Suppose on the contrary. Let i be the minimal number such that there is a monomial $w \in P_{m_i}$ with $R_i(w(n_1,\ldots,n_6)) \in R_i(S(w))$ for all $n_1,\ldots,n_6$. Clearly $i > 1$, since $R_1 = Id$, $m_1 > 10^8$ and A is a free algebra. Write $w = w_1w_2\ldots w_{\frac{m_i}{m_{i-1}}}$ where all $w_i \in H_{m_{i-1}}$. By the definition of R_i and by Lemma 2.3 we get that either for some $j > 1$ we have $R_{i-1}(w_j(p_1,\ldots,p_6)) \in S(w(j))$ for all $p_1+\ldots+p_6 = m_{i-1}$ or we have $c_{R_{i-1}(F_{i-1})}(R_{i-1}(w_1(p_1,\ldots,p_6)) \in c_{R_{i-1}(F_{i-1})}(S(w_1))$ for all $p_1+\ldots+p_6 = m_{i-1}$. Note that i was minimal, and hence the former is impossible. Thus suppose the later holds. Then, by the definition of the mapping $c_{R_{i-1}(F_{i-1})}$ we have $R_{i-1}(w_1(n_1,\ldots,n_6) - q(n_1,\ldots,n_6)) \in +\sum_{j=1}^{r_{i-1}} KR_{i-1}(f_{i-1,j})$, for some $q(n_1,\ldots,n_6) \in S(w_1)$. Therefore, $R_{i-1}(w_1(n_1,\ldots,n_6)) \in R_{i-1}(S(w_1)) + \sum_{j=1}^{r_{i-1}} KR_{i-1}(f_{i-1,j})$. By assumption $r_{i-1} < 10^{-4}m_{i-1}^2$. Theorem 4.1 applied for $k = i-1$ yields, $R_{i-1}(w_1(n_1,\ldots,n_6)) \in R_{i-1}(S(w_1))$. It is a contradiction, because i was minimal.

5 The main results

In this section we will prove Theorems 1.1−1.4. The general idea of the proof of Theorem 1.3 is a little similar to the proof that polynomial rings over nil rings need not be nil, in [20]. Theorems 1.1, 1.2 and 1.4 are consequences of Theorem 1.3.

Proof of Theorem 1.3**.** Let K be a countable field and let A be the free noncommutative associative K algebra in generators $x_1, x_2, x_3, y_1, y_2, y_3$. The field K is countable so elements of A can be enumerated, say $f_1, f_2, \ldots$ where degree of f_i is t_i. Let I be an ideal in A generated by the homogeneous components of elements $f_i^{10m_{i+1}}$, $i = 1, 2, \ldots$ where m_i, $i = 1, 2, \ldots$ is an increasing sequence of natural numbers such that Let $40m_i$ divide m_{i+1} and $m_{i+1} > 2^{i+101}m_i$, $m_1 > 10^8$ for $i = 1, 2, \ldots$. Denote $N = A/I$. Observe that N is nil. Let B be the subalgebra of $N[X_1, \ldots, X_6]$ generated by elements $X = x_1X_1 + x_2X_2 + x_3X_3 + I[X_1, \ldots, X_6]$ and element $Y = y_1X_4 + y_2X_5 + y_3X_6 + I[X_1, \ldots, X_6]$. Let Q be the subgroup of N generated by elements X, Y and let P be the free subgroup generated by elements x, y as in section 2 and let $\xi : P \to Q$ be a subgroup homomorphism such that $\xi(x) = X$, $\xi(y) = Y$. We will show that B is a free algebra. Note that the ideal I is homogeneous, hence we only need to show that linear combinations of non-zero elements of the same degree are non-zero (or else all coefficients are zero). Suppose on the contrary. Then there is $v \in P_{m_k}$ for some k such that $\xi(w) \in \sum_{v \prec w} K\xi(v)$. By rewriting this and comparing elements with a pre-fix $x_1^{n_1}x_2^{n_2}x_3^{n_3}y_1^{n_4}y_2^{n_5}y_3^{n_6}$ we get that $w(n_1, \ldots, n_6) + I \subseteq S(w) + I$, for all $n_1, \ldots, n_6$. Therefore, $w(n_1, \ldots, n_6) \subseteq S(w) + I$. Note that $w(n_1, \ldots, n_6) \in H_{\deg w} = H_{m_k}$. By Theorem 3.1 there exists subsets $F_i \subseteq H_{m_i} \subseteq A$, with $card(F_i) < 10^{-4}m_i^2$ such that $I \cap H_{m_k} \subseteq \sum_{i=1}^{k-1} B_{m_{i+1}}(F_i)$. It follows that, $w(n_1, \ldots, n_6) \subseteq S(w) + \sum_{i=1}^{k-1} B_{m_{i+1}}(F_i) \cap H_{m_k}$. By Theorem 3.2 $R_k(\sum_{i=1}^{k-1} B_{m_{i+1}}(F_i) \cap H_{m_k}) = 0$. Hence, $R_k(w(n_1, \ldots, n_6)) \subseteq R_i(S(w))$, for all $n_1, \ldots, n_6$. By Theorem 4.2 it is impossible.

Proof of Theorem 1.3**.** It follows from Theorem 1.3 when we take $F = K\{X_1, \ldots, X_6\}$, the field of rational functions in 6 commuting indeterminates over A where A is as in Theorem 1.3.

Proof of Theorem 1.1**.** Let A be as in Theorem 1.3. Consider rings $R_0 = A$, $R_1 = A[X_1], R_2 = A[X_1, X_2], \ldots, R_6 = A[X_1, \ldots, X_6]$. Note that

R_0 doesn't contain free algebras of rank two and R_6 contains a free algebra of rank 2. Then there is $0 \leq i < 6$, such that R_i doesn't contain free algebras of rank two and R_{i+1} contains a free algebra of rank 6. Then R_i satisfies the thesis of Theorem 1.1.

Proof of Theorem 1.2**.** It follows from Theorem 1.1 when we take $F = K\{X_1, \ldots, X_6\}$, the field of rational functions in 6 commuting indeterminates over A where A is as in Theorem 1.1.

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