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Uniqueness of near-horizon geometries of rotating extremal AdS$_4$ black holes

Hari K. Kunduri^{a*} and James Lucietti^{b†}

^{a}*DAMTP, University of Cambridge*
Centre for Mathematical Sciences, Wilbeforce Road, Cambridge, CB3 0WA, UK

^{b}*Centre for Particle Theory, Department of Mathematical Sciences,*
University of Durham, South Road, Durham, DH1 3LE, UK

Abstract

We consider stationary extremal black hole solutions of the Einstein-Maxwell equations with a negative cosmological constant in four dimensions. We determine all non-static axisymmetric near-horizon geometries (with non-toroidal horizon topology) and all static near-horizon geometries for black holes of this kind. This allows us to deduce that the most general near-horizon geometry of an asymptotically globally AdS$_4$ rotating extremal black hole is the near-horizon limit of extremal Kerr-Newman-AdS$_4$. We also identify the subset of near-horizon geometries which are supersymmetric. Finally, we show which physical quantities of extremal black holes may be computed from the near-horizon limit alone, and point out a simple formula for the entropy of the known supersymmetric AdS$_4$ black hole. Analogous results are presented in the case of vanishing cosmological constant.

1   Introduction

Extremal black holes have been essential for uncovering the origin of the Bekenstein-Hawking entropy from a statistical counting of their microstates. It is well known that string theory first reproduced the entropy formula, for a set of asymptotically flat extremal black holes which are supersymmetric, by mapping the gravitational system to a strongly-coupled 2d conformal field theory [1]. Restricting attention to supersymmetric states is crucial as their degeneracy is not expected to change from weak to strong coupling. The analogous calculation for asymptotically AdS$_4$/AdS$_5$ supersymmetric black holes [2–6], where the gauge-theory/gravity correspondence is used to map the problem to one of enumerating appropriate operators in the dual CFT$_3$/CFT$_4$, remains an important open problem. The AdS$_5$ case is best understood and a certain amount of progress has been made [7–11].

A supersymmetric black hole is necessarily extremal, and there is growing evidence to suggest that it is this latter property which is behind the success of previous entropy calculations in flat space. There are now a number of examples of successful microstate counting for extremal, non-supersymmetric black holes [12–16], including some recent progress using asymptotic symmetries of near-horizon geometries [17–20]. More generally, for rotating extremal black holes an attractor mechanism has been demonstrated to exist under the assumption that their associated near-horizon geometry must possess an enhanced $SO(2,1)$ symmetry [21,22]. This assumption was subsequently proved in a generic class of theories in $D = 4, 5$ provided the black hole possesses $D - 3$ rotational

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*h.k.kunduri@damtp.cam.ac.uk

†james.lucietti@durham.ac.uk

symmetries [23][1]. In particular, in [23] it was proved that such a near-horizon geometry must take the form

$$ds^2 = \Gamma(\rho) \left[-C^2 r^2 dv^2 + 2 dv dr \right] + d\rho^2 + \gamma_{ij}(\rho) \left(dx^i + k^i r dv \right) \left(dx^j + k^j r dv \right) \qquad (1)$$

where the horizon is at $r = 0$, the Killing vector fields $\partial/\partial x^i$ generate the rotational symmetries ($i = 1, \ldots, D-3$), Γ and γ_{ij} are smooth functions of the horizon coordinate ρ and C, k^i are constants.

The standard rigidity theorem in 4d, which states that a stationary, rotating, non-extremal black hole must be axisymmetric, has recently been generalised to extremal black holes [25]. This implies the existence of one rotational symmetry and therefore *any* extremal rotating black hole in 4d must have a near-horizon limit given by (1) . For black holes in $D \geq 5$ only one rotational symmetry has been proved to exist (for non-extremal [26] and for partial results in the extremal case [25]) and thus the assumption of two in $D = 5$ may constitute a genuine restriction on the space of solutions (we note that all known examples belong to this class) [27].

It is remarkable that the classification problem for extremal black holes even in four dimensions remains unanswered. This is because the classic black hole uniqueness theorems assume that the horizon does not have degenerate components, i.e. the black hole is non-extremal. Moreover, these theorems are only valid for asymptotically flat spacetimes. Therefore, the classification of asymptotically AdS$_4$ black holes (extremal and non-extremal) also remains an open problem.

Extremality is a far weaker constraint than supersymmetry – for example, the vacuum Kerr black hole can be extremal but never supersymmetric. While there are systematic techniques for constructing supersymmetric solutions, it is also of interest to develop techniques for classifying generic extremal black holes. Indeed, every extremal black hole admits a near-horizon limit that yields a geometry that solves the same theory. The field equations reduce to a $D - 2$ dimensional problem of Riemannian geometry on a compact space. Classifying near-horizon geometries is thus a more tractable problem and one can deduce important information about the allowed extremal solutions in a given theory. The analysis reveals not only the allowed horizon topologies but also their explicit geometry. Hence, one may rule out the existence of black holes with certain horizon topologies. The only drawback of using this technique is that the existence of a near-horizon solution is not sufficient to imply the existence of an extremal black hole with that near-horizon geometry. However, the combination of a near-horizon classification along with global information of the black hole spacetime can provide a method to tackle the uniqueness/classification problem for extremal black holes. Indeed, such an approach was used to prove a uniqueness theorem for the supersymmetric BMPV solution in five dimensional minimal ungauged supergravity [27] and for the supersymmetric MP black holes of four dimensional minimal ungauged supergravity [28].

It has turned out to be a difficult task to classify all near-horizon geometries in a given theory with no extra assumptions. For supersymmetric solutions this has been possible in 4/5/6d ungauged supergravity [27–29]. For static near-horizon geometries it has also been possible in the vacuum (for any dimension and including the case of a cosmological constant) [30] and 4d electrovacuum [31]. However, in other cases it has been necessary to assume the existence of one or two rotational symmetries in 4d and 5d respectively. This assumption has allowed a classification of the near-horizon geometries of supersymmetric AdS$_5$ black holes in five-dimensional gauged supergravity [32, 33]. More recently, with the same assumptions, we performed such a classification for 4d (including a negative cosmological constant) and 5d rotating extremal vacuum black holes [34].

The purpose of this note is to perform a classification of all near-horizon geometries of stationary, extremal black hole solutions to four-dimensional Einstein-Maxwell theory including a non-positive cosmological constant. This theory is the bosonic sector of minimal (un)gauged supergravity, however we do not assume the solutions preserve any supersymmetry. Nevertheless, since these theories support supersymmetric black holes (which are necessarily extremal) our analysis will capture these as a subset to the space of all extremal solutions. Our main results may be summarized as follows.

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[1] For analogous statements in vacuum gravity in $D > 5$ see [24].

Result 1 Consider a four dimensional non-static, axisymmetric near-horizon geometry with a compact horizon section of non-toroidal topology, satisfying the Einstein-Maxwell equations with cosmological constant Λ. If $\Lambda = 0$ then it must be the near-horizon geometry of the rotating extremal Kerr-Newman black hole. If $\Lambda < 0$ it must be the near-horizon geometry of the rotating extremal Kerr-Newman-AdS$_4$.

Result 2 Consider a four dimensional static near-horizon geometry with a compact horizon section, satisfying the Einstein-Maxwell equations with cosmological constant Λ. For $\Lambda < 0$ it must be a direct product of AdS_2 with a metric of constant curvature on S^2, T^2, or (compact quotients of) H^2. For $\Lambda = 0$ it must be a direct product of AdS_2 and S^2.

Remarks

- These results do not employ any asymptotic information of the relevant black hole solutions. For $\Lambda = 0$ we are of course interested in asymptotically flat black holes. For $\Lambda < 0$ we are mostly interested in asymptotically *globally AdS* solutions. In these cases topological censorship [35] immediately implies that the horizon topology is S^2.

- In [25] it has been shown that an asymptotically flat or globally AdS extremal rotating black hole must be axisymmetric[2]. It follows that its near-horizon limit, which may be non-static or static, must be axisymmetric. If non-static our Result 1 implies that it must be given by the near-horizon geometry of the $J \neq 0$ extremal Kerr-Newman-(AdS$_4$). If static Result 2 and topological censorship (which tells us the horizon must be S^2) implies it is given by $AdS_2 \times S^2$ which is the near-horizon geometry of extremal Reissner-Nordstrom-(AdS$_4$) (which is the $J = 0$ extremal Kerr-Newman-(AdS$_4$)).

- It has been shown that asymptotically flat, non-extremal, non-rotating black holes must be static [36, 37]. If this result extends to extremal and AdS black holes then our Result 2 shows it must be given by the near-horizon limit of extremal Reissner Nordstrom-(AdS$_4$).

- The $\Lambda = 0$ case of Result 1 has been proved in the context of extremal isolated horizons [38]. The classification of extremal isolated horizons is in fact mathematically equivalent to that of near-horizon geometries. Also see [39] for some results on the AdS case.

- The near-horizon geometry of the supersymmetric AdS$_4$ black hole [2, 40] is non-static and hence given by a subset of the solutions in Result 1. In fact we will show this subset is the most general non-static, axisymmetric, supersymmetric near-horizon geometry (with non-toroidal horizon topology). We will also present a simple formula for their entropy.

- Many of our proofs are also valid for $\Lambda > 0$. Indeed our method determines the near-horizon geometry in this case too, although we have not analysed the resulting solutions in detail.

This note is organized as follows. First we set up the near-horizon equations which must be satisfied by a near-horizon geometry solution to the Einstein-Maxwell equations with a cosmological constant. Next, we determine all static near-horizon geometries. We then determine all non-static axisymmetric near-horizon geometries. We then identify which subset of the derived near-horizon geometries are supersymmetric. Finally, we present general formulas for the electric and magnetic charges and the angular momentum of extremal black hole solutions to this theory, written as integrals of the near-horizon data over the horizon.

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[2]Although this theorem is only stated explicitly for the asymptotically flat case, it also applies to asymptotically globally AdS black holes. In fact, the only parts of the proof which employ the asymptotic information, are where it is used to establish that the stationary Killing field does not vanish on the event horizon (so one can define a spacelike foliation of the event horizon) and that the exterior is simply connected (topological censorship). Both of these properties are still true for asymptotically globally AdS black holes.

2   Electro-vacuum near-horizon equations

We consider solutions of $D = 4$ Einstein-Maxwell theory with a cosmological constant. The field equations are

$$R_{\mu\nu} = T_{\mu\nu} \equiv 2\left(\mathcal{F}_{\mu}{}^{\delta}\mathcal{F}_{\nu\delta} - \frac{1}{4}g_{\mu\nu}\mathcal{F}^2\right) + \Lambda g_{\mu\nu} \tag{2}$$

$$d \star \mathcal{F} = 0, \qquad d\mathcal{F} = 0$$

where $\mathcal{F}$ is the Maxwell two form and we write $\mathcal{F} = d\mathcal{A}$. We will be mainly interested in the cases $\Lambda = 0$ and $\Lambda < 0$ which correspond to the bosonic sectors of minimal ungauged or gauged supergravity (with gauge coupling given by $\Lambda = -3g^2$) respectively, although many of our proofs also work for $\Lambda > 0$.

The event horizon of a four dimensional stationary extremal black hole (asymptotically flat or AdS) must be a Killing horizon of a Killing vector field V [25]. In a neighbourhood of such a Killing horizon we can always introduce Gaussian null coordinates (v, r, x^a) such that $V = \partial/\partial v$, the horizon is at $r = 0$ and x^a $(a = 1, 2)$ are coordinates on $\mathcal{H}$, a spatial section of the horizon. Note that $\mathcal{H}$ is a two-dimensional compact manifold without boundary. The black hole metric and Maxwell field in these coordinates are

$$ds^2 = r^2 F(r, x)dv^2 + 2dvdr + 2rh_a(r, x)dvdx^a + \gamma_{ab}(r, x)dx^a dx^b \tag{3}$$

$$\mathcal{F} = \mathcal{F}_{vr}dv \wedge dr + \mathcal{F}_{ra}dr \wedge dx^a + \mathcal{F}_{va}dv \wedge dx^a + \frac{1}{2}\mathcal{F}_{ab}dx^a \wedge dx^b \,. \tag{4}$$

The near-horizon limit [23, 27] is obtained by taking the limit $v \to v/\epsilon$, $r \to \epsilon r$ and $\epsilon \to 0$. The resulting metric is

$$ds^2 = r^2 F(x)dv^2 + 2dvdr + 2rh_a(x)dvdx^a + \gamma_{ab}(x)dx^a dx^b \tag{5}$$

where F, h_a, γ_{ab} are a function, a one-form, and a Riemannian metric respectively, defined on $\mathcal{H}$. In general the Maxwell field (4) does *not* admit a near-horizon limit due to the $\mathcal{F}_{va}$ component. However, we can use the field equations to show that it must for solutions. It is well known that for a Killing horizon $\mathcal{N}$ of ξ one must have $R_{\mu\nu}\xi^\mu\xi^\nu|_{\mathcal{N}} = 0$. Taking $\mathcal{N}$ to be the event horizon with $\xi = V$, and using (2) one finds:

$$R_{\mu\nu}\xi^\mu\xi^\nu|_{\mathcal{N}} = 2\gamma^{ab}\mathcal{F}_{va}\mathcal{F}_{vb}|_{r=0} \tag{6}$$

which implies $\mathcal{F}_{va} = 0$ at $r = 0$. It follows (assuming analycity) that $\mathcal{F}_{va} = r\hat{\mathcal{F}}_{va}$ for some regular functions $\hat{\mathcal{F}}_{va}$. This guarantees that the near-horizon limit of the Maxwell field always exists, and is given by:

$$\mathcal{F} = \mathcal{F}_{vr}(x)dv \wedge dr + r\hat{\mathcal{F}}_{va}(x)dv \wedge dx^a + \frac{1}{2}\mathcal{F}_{ab}(x)dx^a \wedge dx^b \,. \tag{7}$$

Note that the Bianchi identity $d\mathcal{F} = 0$ further constrains the Maxwell field and implies it can be written as

$$\mathcal{F} = d(\Delta(x)rdv) + \hat{\mathcal{F}} \tag{8}$$

where $\hat{\mathcal{F}} \equiv \frac{1}{2}\mathcal{F}_{ab}(x)dx^a \wedge dx^b$ is a closed two-form and $\Delta \equiv -\mathcal{F}_{vr}$ is a function, both defined on $\mathcal{H}$. Note that we can locally introduce a potential $\hat{A}$ on $\mathcal{H}$ such that $\hat{\mathcal{F}} = d\hat{A}$.

The purpose of this note is to determine all electrovacuum extremal black hole near-horizon geometries with a cosmological constant in four dimensions. This is equivalent to finding the most general metric and Maxwell field of the form (5) and (8) that satisfy (2). A lengthy calculation

reveals that the spacetime field equations are equivalent to the following set of equations on $\mathcal{H}$:

$$R_{ab} = \frac{1}{2}h_a h_b - \nabla_{(a}h_{b)} + \Lambda\gamma_{ab} + 2\hat{\mathcal{F}}_{ac}\hat{\mathcal{F}}_{bd}\gamma^{cd} + \Delta^2\gamma_{ab} - \frac{\gamma_{ab}}{2}\hat{\mathcal{F}}^2 \tag{9}$$

$$F = \frac{1}{2}h_a h^a - \frac{1}{2}\nabla_a h^a + \Lambda - \Delta^2 - \frac{\hat{\mathcal{F}}^2}{2} \tag{10}$$

$$d\star_2\hat{\mathcal{F}} = \star_2 i_h\hat{\mathcal{F}} + \star_2(d\Delta - \Delta h). \tag{11}$$

where R_{ab}, ∇ and $\star_2$ are the Ricci tensor, the covariant derivative and Hodge dual of the 2d metric γ_{ab}. In particular, (9) is the ab component of the Einstein equations, (10) is the vr component and (11) is the Maxwell equation, all written covariantly on $\mathcal{H}$. It can be shown that the rest of the Einstein equations are satisfied as a consequence of the above set of equations. It is useful to note that using equations (9), (10) and (11), it can be shown that the contracted Bianchi identity $\nabla^b(R_{ab} - \frac{1}{2}\gamma_{ab}R) = 0$ is equivalent to

$$\nabla_a F = Fh_a + 2h_b\nabla_{[a}h_{b]} - \nabla_b\nabla_{[a}h_{b]} - 2(\hat{\mathcal{F}}_{ab} + \Delta\gamma_{ab})(\nabla^b\Delta - \Delta h^b). \tag{12}$$

3 Static near-horizon geometries

A static near-horizon geometry is one for which the Killing field normal to the horizon is hypersurface orthogonal, i.e. $V \wedge dV = 0$ everywhere. Such solutions have been classified previously in the vacuum (including a negative cosmological constant) [30] and electrovacuum (with no cosmological constant) [31] and considered more generally in [23]. In [23] it was shown that staticity is equivalent to the following constraints on the metric: $dF = Fh$ and $dh = 0$. We first derive an analogous constraint for the Maxwell field. Defining the twist one-form $\omega = \frac{1}{2} \star (V \wedge dV)$, one can check $d\omega = - \star (V \wedge R(V))$ where $R(V)_\mu = R_{\mu\nu}V^\nu$. Therefore a static near-horizon must be Ricci-static, i.e. $V \wedge R(V) = 0$. From (2) it follows that

$$V \wedge R(V) = 2V \wedge \mathcal{F}_{\mu\rho}\mathcal{F}^{\rho\nu}V_\nu dx^\mu. \tag{13}$$

From this it is easy to check that a near-horizon geometry is Ricci static if and only if $d\Delta = h\Delta$. It follows that a static near-horizon geometry must have $d\Delta = h\Delta$.

We now turn to solving the staticity conditions. Since $dh = 0$, we see that *locally* we can always write $h = d\lambda$. If $\mathcal{H}$ is simply connected (as for S^2) then the function λ is actually globally defined. If $\mathcal{H}$ is not simply connected, consider an open cover of simply connected sets $\{U_i\}$; then in each set U_i one can write $h = d\lambda_i$. Defining $\psi = e^{-\lambda/2}$, we can integrate (in each U_i) the remaining staticity conditions to get $F = F_0^i\psi_i^{-2}$ and $\Delta = e_i\psi_i^{-2}$ for some constants F_0^i and e_i.

Now consider the near-horizon Maxwell equation (11). This simplifies considerably to

$$d\phi - \phi d\lambda = 0 \tag{14}$$

where $\phi = \star_2\hat{\mathcal{F}}$ is a (globally defined) function on $\mathcal{H}$. It follows that (in each U_i) $\phi = b_i\psi_i^{-2}$.

We may now deduce an important fact. First note that if $e_i = b_i = 0$ for all i then $\mathcal{F} = 0$ everywhere on $\mathcal{H}$ which is the vacuum case studied in [30]. Therefore, for a non-trivial Maxwell field we must have at least one of e_i or b_i non-zero, which we will assume. By comparing the expressions for Δ and ϕ on the overlaps $U_i \cap U_j$ we must have $e_i\psi_i^{-2} = e_j\psi_j^{-2}$ and $b_i\psi_i^{-2} = b_j\psi_j^{-2}$. We deduce that either all the e_i are non-zero (and have the same sign) or all the b_i are non-zero (and have the same sign). Using the fact that the ψ_i are only defined up to a multiplicative constant depending on i (since λ_i are defined up to an additive constant depending on i) we may set $e_i = e_j$ (if these are non-zero) or $b_i = b_j$ (if $e_i = 0$) to deduce $\psi_i = \psi_j$. It follows that $F_0^i = F_0^j$ and $b_i = b_j$. Thus we deduce that ψ is a globally defined function even in the non simply connected case and from now on

we drop all indices i and work on the whole of $\mathcal{H}$. Note we have determined: $F = F_0\psi^{-2}$, $\Delta = e\psi^{-2}$ and $\phi = b\psi^{-2}$.

We now turn to solving the rest of the equations on $\mathcal{H}$. Our analysis closely parallels that of [30]. First note that (12) is automatically satisfied as a consequence of staticity. Then (9) and (10) can be written as

$$\psi R_{ab} = 2\nabla_a \nabla_b \psi + \psi T_{ab}, \tag{15}$$

$$F_0 = \psi \nabla^2 \psi + |\nabla \psi|^2 + \psi^2 T_{vr} = \frac{1}{2}\nabla^2 \psi^2 + \psi^2 T_{vr} \tag{16}$$

where

$$T_{ab} = (e^2 + b^2)\psi^{-4}\gamma_{ab} + \Lambda\gamma_{ab}, \qquad T_{vr} = -(e^2 + b^2)\psi^{-4} + \Lambda . \tag{17}$$

Following the procedure given in [30], which uses the fact that $\mathcal{H}$ is two-dimensional so $R_{ab} = \frac{1}{2}g_{ab}R$, allows one to show that

$$\psi^3 R = 6(e^2 + b^2)\psi^{-1} + \frac{2\Lambda\psi^3}{3} + c \tag{18}$$

where c is a constant. Taking the trace of (15) gives $2\nabla^2 \psi = \psi(R - \gamma^{ab}T_{ab})$ and substituting (18) gives

$$2\nabla^2 \psi = c\psi^{-2} + 4(e^2 + b^2)\psi^{-3} - \frac{4\Lambda\psi}{3} \tag{19}$$

There are two cases to consider: either (i) $d\psi \neq 0$ in some open set in $\mathcal{H}$, or (ii) ψ is constant in $\mathcal{H}$.

Let us treat case (i) first. Since $d\psi \neq 0$ in some open set, we can use ψ as a coordinate there and introduce another coordinate φ such that

$$\gamma_{ab}dx^a dx^b = \frac{d\psi^2}{|\nabla\psi|^2} + H(\psi, \varphi)d\varphi^2 . \tag{20}$$

To calculate $|\nabla\psi|^2$ we proceed as in [30] and take the divergence of (15) leading to

$$|\nabla\psi|^2 + \frac{\psi^2 R}{2} - 2(e^2 + b^2)\psi^{-2} = \alpha \tag{21}$$

for some constant α. Using our expression for the Ricci scalar then gives

$$|\nabla\psi|^2 = \alpha - \frac{c}{2\psi} - \frac{e^2 + b^2}{\psi^2} - \frac{\Lambda\psi^2}{3} \equiv P(\psi) . \tag{22}$$

The Laplacian in (ψ, φ) coordinates is

$$\nabla^2 \psi = P'(\psi) + P\frac{\partial}{\partial\psi}\log\sqrt{\frac{H}{P}} \tag{23}$$

which implies that equation (19) becomes

$$\frac{\partial}{\partial\psi}\log\sqrt{\frac{H}{P}} = 0 \tag{24}$$

and thus $H(\psi, \phi) = g(\phi)P(\psi)$ for some function $g(\phi)$. Therefore, introducing a coordinate $d\tilde{\varphi}^2 = gd\varphi^2$ the metric on $\mathcal{H}$ is simply

$$\gamma_{ab}dx^a dx^b = \frac{d\psi^2}{P(\psi)} + P(\psi)d\tilde{\varphi}^2 \tag{25}$$

6

where $P(\psi)$ is given by (22). If one changes coordinates $r \to \psi^2 r$, then the full near-horizon geometry simplifies to

$$ds^2 = \psi^2(F_0 r^2 dv^2 + 2dvdr) + \frac{d\psi^2}{P(\psi)} + P(\psi)d\tilde{\varphi}^2 \qquad (26)$$

$$\mathcal{F} = edr \wedge dv + b\psi^{-2}d\psi \wedge d\varphi \qquad (27)$$

It is worth pointing out that this near-horizon solution can be obtained by an analytic continuation of the full non-extremal Reissner-Nordstrom-AdS black hole solution, and can be non-singular for non-compact $\mathcal{H} = R^2$. However, for compact $\mathcal{H}$ we can show that the horizon metric must be singular. To see this, write $P = \psi^{-2}Q$ where Q is a quartic in ψ. Compactness requires $d\psi$ to vanish at distinct maxima and minima of ψ and thus, noting $(d\psi)^2 = \psi^{-2}Q(\psi)$, we see that $Q(\psi)$ must have real roots at these points $\psi_1 < \psi_2$ with $\psi_1 \le \psi \le \psi_2$ and $Q(\psi) > 0$ inside the interval. The condition for the absence of conical singularities at these endpoints implies $Q(\psi)$ must be even (see [34] for an identical argument), i.e. $c = 0$ and thus $\psi_1 = -\psi_2 < 0$. However by definition $\psi > 0$ and therefore we have contradiction and hence one cannot simultaneously remove the conical singularities. We therefore rule out this case.

Now consider case (ii), i.e. the case when ψ is a constant. It follows that λ is constant, so $h = 0$. Then we see that F, Δ and ϕ are constants and $F = -\Delta^2 - \phi^2 + \Lambda$. Observe that the full near-horizon geometry is then

$$ds^2 = (-\Delta^2 - \phi^2 + \Lambda)r^2 dv^2 + 2dvdr + \gamma_{ab}dx^a dx^b \qquad (28)$$

$$\mathcal{F} = \Delta dr \wedge dv + \phi \star_2 1$$

and since $F = -\Delta^2 - \phi^2 + \Lambda < 0$ (for $\Lambda \le 0$ and $\mathcal{F} \neq 0$) it is simply the direct product $AdS_2 \times \mathcal{H}$. The remaining equation is

$$R_{ab} = (\Delta^2 + \phi^2 + \Lambda)\gamma_{ab} \qquad (29)$$

which implies the metric γ_{ab} is locally isometric to one of the maximally symmetric metrics on S^2, T^2, H^2 depending on the sign of $\Delta^2 + \phi^2 + \Lambda$.

Topological censorship implies that in the case of asymptotically flat or globally AdS spacetimes, then $\mathcal{H} = S^2$. We have shown that the only *regular* static near-horizon geometry in this case is a direct product of $AdS_2 \times S^2$, given by (28), parameterised by constants Δ, ϕ which satisfy $\Delta^2 + \phi^2 + \Lambda > 0$. This solution is simply the near-horizon limit of extremal Reissner-Nordstrom-AdS$_4$. Furthermore, in the asymptotically locally AdS case, where higher genus horizons are allowed, we have shown that the only static near-horizon geometries are given again by (28) with $\Delta^2 + \phi^2 + \Lambda = 0$ for $\mathcal{H} = T^2$, or $\Delta^2 + \phi^2 + \Lambda < 0$ for $\mathcal{H} = \Sigma_g$.

4 Non-static near-horizon geometries

It has been shown that a 4d stationary rotating extremal black hole must be axisymmetric (for both asymptotically flat and globally AdS) [25]. Thus, we assume the existence of a rotational Killing vector field m for the full black hole spacetime, with closed spacelike orbits. The near-horizon limit of such a black hole inherits this symmetry and therefore we need only solve for the most general axisymmetric near-horizon geometry. It follows that m leaves the near-horizon data $(F, h_a, \gamma_{ab}, \Delta, \hat{\mathcal{F}})$ invariant. Therefore m is also a Killing vector field of the horizon metric. It should be noted that a compact 2d manifold with a global $U(1)$ action must be topologically S^2 or T^2 and therefore $\mathcal{H}$ can only have these topologies [42]. As in the vacuum case [34], some of our global arguments do not apply to toroidal horizons and therefore we will assume S^2 horizon topology.[3] Note that for S^2 topology, m must vanish at two distinct points (the poles).

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[3]It should be noted that for $\Lambda = 0$ it is easy to show that the only T^2 case (even without assuming axisymmetry) is the flat solution $R^{1,1} \times T^2$ with vanishing Maxwell field. To see this note that using (9), the vanishing of the Euler

We introduce coordinates (ρ, x) on $\mathcal{H}$ adapted to this symmetry, so $m = \partial/\partial x$, and

$$\gamma_{ab}dx^a dx^b = d\rho^2 + \gamma(\rho)dx^2, \qquad h = \Gamma^{-1}[\gamma(\rho)k(\rho)dx - \Gamma' d\rho] \tag{30}$$

$$\hat{\mathcal{F}} = B(\rho)d\rho \wedge dx \tag{31}$$

which define the functions (γ, k, Γ, B) with $\Gamma > 0$ and $f'(\rho) = df/d\rho$. These coordinates are then valid everywhere on $\mathcal{H}$ except at the points where the rotational Killing field m vanishes, where a separate analysis of regularity will be performed.

It is useful to first consider the near-horizon Maxwell equation (11). It is equivalent to two equations, corresponding to the x and ρ components of (11), which are:

$$kB = (\Delta\Gamma)', \qquad \left(\frac{B\Gamma}{\sqrt{\gamma}}\right)' + k\Delta\sqrt{\gamma} = 0 . \tag{32}$$

respectively.

For completeness we will now show how the symmetry enhancement result we proved in [23] emerges in the present formalism. Firstly, note that $T_{\rho x} = 0$, so equation (9) gives $R_{\rho x} = -\frac{1}{2}\Gamma^{-1}\gamma k'$ and since $R_{\rho x} = 0$ automatically for a metric of the form (30), we see that the function k must be constant. Now, the ρ component of (12) is

$$F' = -\frac{\Gamma'}{\Gamma}F + \Gamma^{-1}k(\Gamma^{-1}\gamma k)' + 2\Delta\Gamma^{-1}(kB - (\Delta\Gamma)') \tag{33}$$

which using the x component of the Maxwell equation is identical to the vacuum case. Defining $A = \Gamma^2 F - \gamma k^2$ then implies $A' + \frac{\Gamma'}{\Gamma}A + k'k = 0$ and thus using the fact that $k' = 0$ we learn that $A = A_0\Gamma$ for constant A_0. This is sufficient to prove the near-horizon geometry symmetry enhancement result of [23]. This can be seen by changing coordinates $r \to \Gamma(\rho)r$, in which case the metric and Maxwell field simplify to

$$ds^2 = \Gamma(\rho)[A_0 r^2 dv^2 + 2dvdr] + d\rho^2 + \gamma(\rho)(dx + krdv)^2 \tag{34}$$

$$\mathcal{F} = Edr \wedge dv + Bd\rho \wedge (dx + rdv) \tag{35}$$

where we have defined $E \equiv \Delta\Gamma$ and used $kB = E'$ to rewrite the Maxwell field. Note that $k = 0$ implies the near-horizon geometry is static and therefore henceforth we assume $k \neq 0$. By rescaling the coordinate x we will set $k = 1$.

Let us now return to the Maxwell equations (32). The x component allows one to solve for $B = E'$. This can then be substituted into the ρ component to give

$$\frac{\Gamma}{\sqrt{\gamma}}\left(\frac{\Gamma E'}{\sqrt{\gamma}}\right)' + E = 0 . \tag{36}$$

In the analysis of vacuum near-horizon geometries [34], it proved essential to introduce the coordinate σ defined by $\sigma' = \sqrt{\gamma}$. Note that σ is a globally defined function [34] and can be used as a coordinate everywhere except the points where m vanishes. In terms of this coordinate the remaining part of the Maxwell equation simplifies to

$$\Gamma\frac{d}{d\sigma}(\Gamma\dot{E}) + E = 0 \tag{37}$$

where for functions of σ we write $\dot{f} = df/d\sigma$. This equation can be integrated by noting that it is a total derivative if one multiplies it by $\dot{E}$. The result is

$$(\Gamma\dot{E})^2 + E^2 = e^2 \tag{38}$$

---

number gives $0 = \int_{\mathcal{H}} R = \int_{\mathcal{H}} \left(\frac{1}{2}h^2 + \hat{\mathcal{F}}^2 + 2\Delta^2\right)$, which implies $h \equiv 0$, $\hat{\mathcal{F}} \equiv 0$ and $\Delta \equiv 0$. This in turn implies that $F \equiv 0$ and the metric γ_{ab} is flat. This argument fails when $\Lambda < 0$ and it is an interesting problem to classify solutions in this case (even assuming axisymmetry).

where e is an integration constant and we assume $e > 0$ (otherwise $E = 0$ and hence $\mathcal{F} = 0$ which corresponds to the vacuum case).

The rest of the analysis closely follows that of the vacuum case in [34]. We will work directly in (σ, x) coordinates and as in the vacuum case it is convenient to define the function $Q(\sigma) \equiv \Gamma \sigma'^2 = \Gamma \gamma$ so the metric on $\mathcal{H}$ is

$$\gamma_{ab} dx^a dx^b = \frac{\Gamma(\sigma)}{Q(\sigma)} d\sigma^2 + \frac{Q(\sigma)}{\Gamma(\sigma)} dx^2 \ . \tag{39}$$

The method involves both local and global arguments and therefore at this stage it is worth noting the following. Since σ is a globally defined function, compactness of $\mathcal{H}$ implies that $d\sigma$ must vanish at distinct maxima and minima and thus since $(d\sigma)^2 = Q/\Gamma$, we learn that Q must vanish at these two distinct points. Further since $(d\sigma)^2 = m \cdot m$ we see that Q can only vanish at the fixed points of m and thus Q must only have two zeros on $\mathcal{H}$. Therefore we see that $\sigma_1 \leq \sigma \leq \sigma_2$ with $Q > 0$ inside this interval and vanishing at the end points.

Equation (10) gives

$$A_0 + \frac{Q}{2\Gamma^2} - \frac{1}{2}\nabla^2\Gamma = \Gamma\left(-\frac{E^2}{\Gamma^2} - \dot{E}^2 + \Lambda\right) \tag{40}$$

and integrating over $\mathcal{H}$ shows that $A_0 < 0$ and so we define $C > 0$ such that $A_0 = -C^2$. Noting that $\nabla^2 f = \frac{d}{d\sigma}\left(\frac{Q\dot{f}}{\Gamma}\right)$, this equation then reads

$$\frac{d}{d\sigma}\left(\frac{Q\dot{\Gamma}}{\Gamma}\right) = -2C^2 + \frac{Q}{\Gamma^2} + 2\left(\frac{E^2}{\Gamma} + \dot{E}^2\Gamma - \Lambda\Gamma\right) \ . \tag{41}$$

Now, consider the xx component of (9). Using the fact that $R_{xx} = -\frac{1}{2}\nabla^2\gamma + \frac{1}{2}\dot{\gamma}^2$, it gives

$$\ddot{Q} - \frac{d}{d\sigma}\left(\frac{Q\dot{\Gamma}}{\Gamma}\right) + 2\left(\frac{E^2}{\Gamma} + \dot{E}^2\Gamma + \Lambda\Gamma\right) + \frac{Q}{\Gamma^2} = 0 \ . \tag{42}$$

Combining equations (41) and (42) in such a way to eliminate the $\nabla^2\Gamma$ term implies

$$\ddot{Q} + 2C^2 + 4\Lambda\Gamma = 0 \tag{43}$$

which is the same as in the vacuum case. Using (38), equation (41) can be written as

$$Q = \Gamma^2 \frac{d}{d\sigma}\left(\frac{Q\dot{\Gamma}}{\Gamma}\right) + 2\Gamma\left(C^2\Gamma - e^2 + \Lambda\Gamma^2\right) \tag{44}$$

and differentiating this expression with respect to σ implies

$$\dot{Q} = Q\Gamma\frac{d^3\Gamma}{d\sigma^3} + \ddot{\Gamma}(2\dot{Q}\Gamma - Q\dot{\Gamma}) + 2\dot{\Gamma}(C^2\Gamma + \Lambda\Gamma^2 - e^2) \tag{45}$$

where we have used (43) to eliminate the $\dddot{Q}$ generated by the differentiation. Subtracting equation (44) (times $\dot{\Gamma}$) from equation (45) (times Γ) leads to:

$$Q\frac{d^3\Gamma}{d\sigma^3} + \left(\dot{Q} - \frac{\dot{\Gamma}Q}{\Gamma}\right)\left(2\ddot{\Gamma} - \frac{\dot{\Gamma}^2}{\Gamma} - \frac{1}{\Gamma}\right) = 0 \ . \tag{46}$$

This equation is identical to the one encountered in the vacuum case in [34] and thus we may solve it using the same technique.

This involves noticing that if we define $\mathcal{P} = 2\ddot{\Gamma} - \frac{\dot{\Gamma}^2}{\Gamma} - \frac{1}{\Gamma}$ then (46) can be written as

$$\dot{\mathcal{P}} = \left(\frac{\dot{\Gamma}}{\Gamma} - \frac{2\dot{Q}}{Q}\right)\mathcal{P} \tag{47}$$

and thus can be integrated to give $Q^2\mathcal{P} = k\Gamma$ where k is the integration constant. In [34] it was shown that $Q^2\mathcal{P}$ is a globally defined function which vanishes where Q does and thus evaluating at these points we learn that the constant $k = 0$ and hence $\mathcal{P} = 0$. The differential equation $\mathcal{P} = 0$ is easily integrated to give

$$\dot{\Gamma}^2 + 1 = \beta\Gamma \tag{48}$$

where β is an integration constant which must be positive. There are two solutions to this equation. One is $\Gamma = \beta^{-1}$ which implies Q is a constant and is thus incompatible with $\mathcal{H}$ being compact. The other solution is

$$\Gamma = \beta^{-1} + \frac{\beta(\sigma - \sigma_0)^2}{4} \tag{49}$$

where σ_0 is an integration constant which we will set to zero using the freedom that σ has only been defined up to an additive constant. We can now integrate (43) for Q to get

$$Q = -\frac{\beta\Lambda}{12}\sigma^4 - (C^2 + 2\Lambda\beta^{-1})\sigma^2 + c_1\sigma + c_2 \tag{50}$$

where c_i are two integration constants. Substituting back into (44) gives

$$c_2 = 4\beta^{-3}(C^2\beta + \Lambda - e^2\beta^2)\ . \tag{51}$$

Finally we may integrate (38) (which determines the Maxwell field) resulting in

$$E = \frac{\sigma e \cos\alpha - \left(\beta^{-1} - \frac{\beta\sigma^2}{4}\right)e\sin\alpha}{\Gamma} \tag{52}$$

where without loss of generality we have chosen a sign for E and α is a constant. The rest of the near-horizon equations are now satisfied identically.

We now complete the global analysis of the horizon metric. The procedure follows the case with vanishing Maxwell field [34] closely. In general the horizon metric we have derived has conical singularities at $\sigma = \sigma_1, \sigma_2$, the zeros of Q. Although the precise details of the argument depend on whether Λ vanishes or not [34], in both cases the condition for simultaneous removal of these conical singularities is equivalent to $Q(\sigma)$ being even, i.e. $c_1 = 0$, so $\sigma_2 = -\sigma_1 > 0$. In this case, the horizon metric is regular with $\partial/\partial x$ vanishing at the endpoints $\sigma = \pm\sigma_2$. This is of course consistent with $\mathcal{H}$ having S^2 topology.

Summary of non-static near-horizon geometries

We have shown that there is a unique axisymmetric non-static near-horizon geometry with a compact horizon section of S^2 topology. It is given by:

$$ds^2 = \Gamma[-C^2r^2dv^2 + 2dvdr] + \frac{\Gamma d\sigma^2}{Q} + \frac{Q}{\Gamma}(dx + rdv)^2 \tag{53}$$

$$\mathcal{F} = d[E(rdv + dx)] \tag{54}$$

where

$$\Gamma = \beta^{-1} + \frac{\beta\sigma^2}{4}, \qquad Q = -\frac{\beta\Lambda}{12}\sigma^4 - (C^2 + 2\Lambda\beta^{-1})\sigma^2 + 4\beta^{-3}(C^2\beta + \Lambda - e^2\beta^2) \tag{55}$$

and E is given by (52), where $C, \beta, e > 0$ and α are constants. This near-horizon solution is invariant under the scaling

$$C^2 \to KC^2, \qquad \beta \to K^{-1}\beta, \qquad e \to Ke, \quad \sigma \to K\sigma, \qquad (v, x) \to K^{-1}(v, x) \qquad (56)$$

where $K > 0$ is a constant. This allows one to fix one, or a combination of, the parameters to any desired value. Therefore, it is a 3-parameter family. The horizon is at $r = 0$ and the coordinates on spatial sections of this ($v = \text{const}$) are (σ, x). The polynomial $Q(\sigma)$ must have a roots at $\pm\sigma_2$ and the coordinate ranges are $-\sigma_2 \le \sigma \le \sigma_2$ (with $Q > 0$ inside this interval) and x is periodically identified in such a way to remove the conical singularities at $\sigma = \pm\sigma_2$. Therefore $\partial/\partial x$ has fixed points at $\sigma = \pm\sigma_2$. We will now show that this near-horizon geometry is in fact identical to the near-horizon limit of extremal Kerr-Newman for $\Lambda = 0$ and extremal Kerr-Newman-AdS for $\Lambda < 0$.

Proof of equivalence to Kerr-Newman-AdS

First consider $\Lambda < 0$, so we set $\Lambda = -3g^2$. In this case the function Q (55) is a quartic with a positive σ^4 coefficient. As argued above, regularity requires that Q be even with at least two distinct roots at $\sigma = \pm\sigma_2$ and be positive for $-\sigma_2 < \sigma < \sigma_2$. Hence Q must have two additional real roots at $\sigma = \pm\sigma_3$ with $0 < \sigma_2 < \sigma_3$. Observe that our solution depends on three parameters which may be taken to be: $(\sigma_2, \sigma_3, e, \alpha)$ subject to the scaling symmetry. Note that

$$\sigma_2^2 + \sigma_3^2 = \frac{4C^2}{\beta g^2} - \frac{24}{\beta^2} \qquad (57)$$

$$\sigma_2^2 \sigma_3^2 = \frac{16}{\beta^4 g^2} \left(C^2\beta - 3g^2 - e^2\beta^2 \right) , \qquad (58)$$

which are equivalent to

$$(\beta\sigma_2)^2(\beta\sigma_3)^2 - 4(\beta\sigma_2)^2 - 4(\beta\sigma_3)^2 = 48 - \frac{16e^2\beta^2}{g^2}, \qquad (59)$$

$$(\beta\sigma_2)^2(\beta\sigma_3)^2 - 2(\beta\sigma_2)^2 - 2(\beta\sigma_3)^2 = \frac{8C^2\beta}{g^2} - \frac{16e^2\beta^2}{g^2} . \qquad (60)$$

Introducing positive parameters (a, r_+, q) (which are invariant under the scaling (56))

$$a \equiv \frac{\sigma_2}{g\sigma_3}, \qquad r_+ \equiv \frac{2}{g\beta\sigma_3}, \qquad q \equiv e\beta r_+^2 \qquad (61)$$

(note that $ag < 1$) and eliminating σ_2, σ_3 from (59), we find $g^2 r_+^2 < 1$ and

$$a^2 = \frac{r_+^2(1 + 3g^2 r_+^2) - q^2}{1 - g^2 r_+^2}. \qquad (62)$$

One may eliminate σ_2, σ_3 from (60), and using (62) it follows

$$\beta C^2 = \frac{1 + 6g^2 r_+^2 + a^2 g^2}{r_+^2} . \qquad (63)$$

Then using the scaling freedom (56) to set

$$C^2 = \frac{1 + 6g^2 r_+^2 + a^2 g^2}{\Xi(r_+^2 + a^2)}, \qquad (64)$$

where $\Xi \equiv 1 - a^2 g^2$, we find from (63) $\beta = \Xi(r_+^2 + a^2)/r_+^2$. Substituting into the definitions of (r_+, a) gives

$$\sigma_3 = \frac{2r_+}{g\Xi(r_+^2 + a^2)}, \qquad \sigma_2 = \frac{2r_+ a}{\Xi(r_+^2 + a^2)}. \tag{65}$$

Next, perform the coordinate change

$$\phi = \frac{2ar_+ x}{(r_+^2 + a^2)^2}, \qquad \cos\theta = \frac{\sigma}{\sigma_2}, \tag{66}$$

so $0 \le \theta \le \pi$ uniquely parameterises the interval, which implies

$$Q = \frac{4r_+^2 a^2 \sin^2\theta \Delta_\theta}{\Xi^3(r_+^2 + a^2)^3}, \qquad \Gamma = \frac{\rho_+^2}{\Xi(r_+^2 + a^2)}, \qquad k^\phi = \frac{2ar_+}{(r_+^2 + a^2)^2}, \tag{67}$$

where $\Delta_\theta = 1 - a^2 g^2 \cos^2\theta$ and $\rho_+^2 = r_+^2 + a^2 \cos^2\theta$. It is straightforward to verify

$$\gamma_{ab}dx^a dx^b = \frac{\Gamma d\sigma^2}{Q} + \frac{Q}{\Gamma}dx^2 = \frac{\rho_+^2 d\theta^2}{\Delta_\theta} + \frac{\sin^2\theta \Delta_\theta(r_+^2 + a^2)^2}{\rho_+^2 \Xi^2}d\phi^2 \tag{68}$$

and it is easy to see that absence of conical singularities implies $\phi \sim \phi + 2\pi$. Next, define

$$q_e \equiv -q\sin\alpha \qquad q_m \equiv -q\cos\alpha. \tag{69}$$

We find

$$E = \frac{1}{\rho_+^2 \Xi(r_+^2 + a^2)}\left[q_e(r_+^2 - a^2\cos^2\theta) - 2q_m r_+ a \cos\theta\right] \tag{70}$$

and hence the space-time gauge potential is

$$\mathcal{A} = \frac{1}{\rho_+^2 \Xi}\left(q_e\left[\frac{(r_+^2 - a^2\cos^2\theta)rdv}{r_+^2 + a^2} + r_+ a \sin^2\theta d\phi\right] - q_m\left[\frac{2r_+ a \cos\theta rdv}{r_+^2 + a^2} + (r_+^2 + a^2)\cos\theta d\phi\right]\right) \tag{71}$$

where we have performed a gauge transformation to ensure a good $a \to 0$ limit. We have thus rewritten our near-horizon solution (53), (55), (52) in terms of the coordinates (θ, ϕ) and parameters (a, r_+, q_e, q_m) satisfying the constraint (62). It is easy to check that this is exactly the same as the near-horizon limit of the extremal rotating Kerr-Newman-AdS$_4$ [41][4].

Finally consider $\Lambda = 0$. In this case the solution we have derived is simpler. We have $c_2 = 4(C^2 - e^2\beta)/\beta^2$ and since $Q = -C^2\sigma^2 + c_2$ we must have $c_2 > 0$ and $\sigma_2 = \sqrt{c_2}/C$. Changing variables to:

$$\cos\theta = \frac{\sigma}{\sigma_2}, \qquad \phi = \frac{2(C^2 - e^2\beta)^{1/2}Cx}{2 - e^2\beta^2 C^{-2}} \tag{72}$$

so $0 \le \theta \le \pi$ uniquely parameterises the interval, and introducing the positive parameters (which are invariant under the scaling (56))

$$r_+^2 \equiv \frac{1}{\beta C^2}, \qquad q \equiv \frac{e}{C^2}, \qquad a \equiv \sqrt{\frac{1}{\beta C^2}\left(1 - \frac{e^2\beta}{C^2}\right)}, \tag{73}$$

so $r_+^2 = a^2 + q^2$, shows that the near-horizon solution specified by $(\Gamma, k^\phi, \gamma_{ab}, E)$ is given by equations (67), (68) and (70) with $g = 0$. This confirms that our $\Lambda = 0$ solution is identical to the near-horizon limit of extremal Kerr-Newman.

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[4]In fact, the $a \to 0$ limit is simply the static near-horizon geometry of extremal Reissner-Nordstrom-AdS$_4$.

5 Supersymmetric near-horizon geometries

For $\Lambda = -3g^2$, the theory we have been considering is the bosonic sector of minimal gauged supergravity with gauge coupling $g > 0$, or minimal ungauged supergravity for $g = 0$. Since any supersymmetric black hole must be extremal[5], our classification of near-horizon geometries of extremal black holes includes a classification of near-horizon geometries of supersymmetric black holes. Therefore, we will now identify the subset of (axisymmetric) near-horizon geometries in these theories, which are supersymmetric. The strategy is to derive the integrability conditions arising from the existence of a Killing spinor in this supergravity and figure out the constraints this imposes on the near-horizon geometries we have derived. We will take $g > 0$, as for $g = 0$ a classification has been previously given in [28], where the result is simply the near-horizon geometry of extremal Reissner-Nordstrom $AdS_2 \times S^2$.

Recall we showed that for $g > 0$, a static near-horizon geometry must be the direct product of AdS_2 with S^2, T^2 or H^2. The supersymmetry conditions for these solutions have been already considered in [46] where it was shown that only the H^2 case with $\Delta^2 + \phi^2 = g^2$ is supersymmetric and preserves half the supersymmetries. We will now analyse the non-static near-horizon geometry.

In fact, rather than calculating the integrability conditions explicitly for our geometry, we will employ the following trick. We will show that the non-static near-horizon geometry we derived is in fact related by an analytic continuation to the Reissner-Nordstrom-Taub-NUT-AdS$_4$ solution. The conditions for this solution to be supersymmetric are given in [46], which we will exploit.

To see this, note the non-static near-horizon geometry we have derived has an $SO(2,1)$ isometry with 3d orbits which are circle bundles over AdS_2. Consider analytic continuations of this geometry, of the kind introduced in [23], in such a way to map $AdS_2 \to S^2$. The result will be an $SO(3)$ invariant metric with 3d orbits, hence the 4d NUT charge.

More explicitly, first it is convenient to introduce global coordinates on the AdS$_2$ in which case our near-horizon solution reads

$$
ds^2 = \Gamma\left(-(1 + C^2 Y^2)dT^2 + \frac{dY^2}{1 + C^2 Y^2}\right) + \frac{\Gamma}{Q}d\sigma^2 + \frac{Q}{\Gamma}(d\chi + Y dT)^2 \tag{74}
$$

$$
\mathcal{F} = d[E(d\chi + Y dT)] \tag{75}
$$

where (Y, T, χ) are defined in terms of (r, v, x) as in [23]. For the sake of generality we will leave the constant c_1 appearing in the polynomial Q (50) arbitrary. Now, define the parameters:

$$
N^2 \equiv -\frac{1}{C^2\beta}, \qquad M \equiv \frac{c_1}{4NC^4}, \qquad z \equiv \frac{ie}{C^2}, \qquad p \equiv z\sin\alpha, \qquad q \equiv -z\cos\alpha \tag{76}
$$

and coordinates

$$
r \equiv \frac{\sigma}{2NC^2}, \qquad \cos\theta \equiv -iCY, \qquad \phi \equiv CT, \qquad \tau \equiv -2iC^2 N\chi. \tag{77}
$$

It is then easy to check that

$$
ds^2 = R^2(d\theta^2 + \sin^2\theta d\phi^2) + \frac{R^2 dr^2}{\lambda} - \frac{\lambda}{R^2}(d\tau + 2N\cos\theta d\phi)^2, \tag{78}
$$

$$
\mathcal{A}_\tau = \frac{qr - Np}{R^2} + \frac{p}{2N}, \qquad \mathcal{A}_\phi = \frac{\cos\theta}{R^2}\left(p(r^2 - N^2) + 2Nqr\right) \tag{79}
$$

---

[5]This can be seen as follows. Supersymmetry implies the existence of a globally defined non-spacelike Killing vector field V (this is constructed as a bilinear of the Killing spinor). It follows that V must be null and tangent on the event horizon $\mathcal{N}$ of a supersymmetric black hole. Hence V is normal to $\mathcal{N}$, i.e. $\mathcal{N}$ is a Killing horizon of V. Since $V^2 \leq 0$ it follows that $d(V^2) = 0$ on $\mathcal{N}$ (i.e. the function V^2 is at an extremum on $\mathcal{N}$). Therefore the surface gravity vanishes, i.e. the black hole is extremal.

where $R^2 = r^2 + N^2$ and $\lambda = g^2 R^4 + (1 + 4g^2 N^2)(r^2 - N^2) - 2Mr + z^2$. This is the Reissner-Nordstrom-Taub-NUT-AdS solution exactly as given in [46].

As we showed earlier regularity of the horizon requires $c_1 = 0$ and thus we need only consider the $M = 0$ Reissner-Nordstrom-Taub-NUT-AdS solution above. In this case the integrability conditions simplify a little and are [46]:

$$gqN(1 + 4g^2 N^2) = 0 \tag{80}$$

$$(1 \pm 2gp + 4g^2 N^2)(N^2(1 + 4g^2 N^2) - p^2 - q^2) = 0 \tag{81}$$

where one may take either sign in (81). From these we may deduce the integrability conditions for our non-static near-horizon geometry by using the continuation above to convert to our parameters. Recall the parameter ranges for our near-horizon geometry are $\beta > 0, C > 0, e > 0$. Note that $1 + 4g^2 N^2 = 1 - \frac{4g^2}{C^2 \beta}$ cannot vanish as if $\beta = 4g^2/C^2$ then our polynomial $Q(\sigma)$ has a single stationary point at $\sigma = 0$ and thus is not of the required form for a compact horizon (see previous section). Since $N \neq 0$ we see that (80) implies $q = 0$ and thus $\cos \alpha = 0$ in terms of our near-horizon geometry parameters. Now consider the second condition (81). The factor $1 \pm 2gp + 4g^2 N^2 = 1 \pm 2gieC^{-2}\sin\alpha - 4g^2 C^{-2}\beta^{-1}$ can only vanish if the real and imaginary parts vanish separately. The imaginary part gives $\sin \alpha = 0$ which is not allowed since we also have $\cos \alpha = 0$, and therefore we see that this factor can never vanish. It follows that $p^2 = N^2(1 + 4g^2 N^2)$. In terms of the near-horizon parameters this is $e^2 = \frac{C^2}{\beta} - \frac{4g^2}{\beta^2}$.

Therefore, to summarise the integrability conditions for the existence of a Killing spinor for our non-static near-horizon geometry are

$$\cos \alpha = 0, \qquad e^2 = \frac{C^2}{\beta} - \frac{4g^2}{\beta^2} \ . \tag{82}$$

Furthermore, we may deduce the number of supersymmetries preserved by these near-horizon geometries. From [46], the $M = q = 0$ and $p = N^2(1 + 4g^2 N^2)$ RN-Taub-NUT-AdS solution actually preserves $1/2$ of the supersymmetries. From the analytic continuation, it immediately follows that the supersymmetric limit of our non-static near-horizon geometry preserves the same number. In fact, as we show in the next section this supersymmetric non-static near-horizon geometry is identical to that of the known supersymmetric AdS$_4$ black hole [2, 41].

6 Conserved charges

We will now discuss what physical quantities of an extremal charged black hole can be computed from the near-horizon data alone, i.e. $(F, h_a, \gamma_{ab}, \Delta, \hat{\mathcal{F}})$ which is data defined purely on $\mathcal{H}$. In particular we consider solutions to (2) and we will assume they are asymptotically flat for $\Lambda = 0$ and asymptotically globally AdS for $\Lambda < 0$, and in both cases $\mathcal{F} \to 0$ (at a suitable rate) at infinity. We assume the black hole solution is axisymmetric, with m being the associated rotational Killing field (which also leaves $\mathcal{F}$ invariant) which we normalised to have period 2π.

First consider the electric and magnetic charges:

$$Q_e = \frac{1}{4\pi G} \int_{S^2_\infty} \star\mathcal{F}, \qquad Q_m = \frac{1}{4\pi G} \int_{S^2_\infty} \mathcal{F} \ . \tag{83}$$

By using Stokes' theorem $\int_\Sigma d\alpha = \int_{S^2_\infty} \alpha - \int_\mathcal{H} \alpha$ for a two form α and space-like hypersurface Σ, and noting that $(\star\mathcal{F})|_\mathcal{H} = \Delta \star_2 1$, we find

$$Q_e = \frac{1}{4\pi G} \int_\mathcal{H} \Delta \sqrt{\gamma}, \qquad Q_m = \frac{1}{4\pi G} \int_\mathcal{H} \hat{\mathcal{F}} \tag{84}$$

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where we have used the Maxwell equation and Bianchi identity to show the integrals over Σ vanish. It is thus clear that these physical quantities can be computed from the near-horizon data alone.

Now consider the angular momentum

$$J = \frac{1}{16\pi G} \int_{S^2_\infty} \star dm \ . \tag{85}$$

Using Stokes' theorem to convert this to an integral over $\mathcal{H}$, we find a non-trivial volume integral:

$$J = \frac{1}{16\pi G} \int_{\mathcal{H}} \star dm - \frac{1}{8\pi G} \int_{\Sigma} \star R(m) \tag{86}$$

where we have used $\star d \star m = -2R(m)$ and $R(m)_\mu \equiv R_{\mu\nu} m^\nu$. In [24] it was shown that

$$J_H \equiv \frac{1}{16\pi G} \int_{\mathcal{H}} \star dm = \frac{1}{16\pi G} \int_{\mathcal{H}} h \cdot m \sqrt{\gamma} \ . \tag{87}$$

The new ingredient here is the volume integral over Σ, which can be evaluated using Einstein's equations and the Maxwell equations. The derivation is somewhat involved and very similar to the proof of the Smarr relation (see e.g [43]) and thus we omit details. However, we do point out that it is convenient to work in a gauge where $\mathcal{L}_m \mathcal{A} = 0$ (of course we have $\mathcal{L}_m \mathcal{F} = 0$ by assumption). In this gauge we find[6]

$$J_\Sigma \equiv -\frac{1}{8\pi G} \int_{\Sigma} \star R(m) = \frac{1}{4\pi G} \int_{\mathcal{H}} (m \cdot \mathcal{A}) \star \mathcal{F} \ . \tag{88}$$

Collecting these results we can therefore write[7]

$$J = \frac{1}{16\pi G} \int_{\mathcal{H}} \left(h \cdot m + 4(m \cdot \hat{A})\Delta \right) \sqrt{\gamma} \tag{89}$$

which is valid in the gauge $\mathcal{L}_m \hat{A} = 0$. We deduce that the angular momentum can also be computed purely from the near-horizon data.

It is also natural to enquire whether the mass M of an extremal black hole can be computed from the near-horizon data alone. In general this is not the case[8], as one requires knowledge of how the stationary Killing field is related to V and m which is not contained in the near-horizon data. Similarly the angular velocity Ω_H and the potential Φ_H cannot be deduced without asymptotic information which one loses in the near-horizon limit. Of course one can compute the area $A = \int_{\mathcal{H}} \sqrt{\gamma}$ from the horizon metric γ_{ab}.

For the non-static near-horizon geometry (written in Kerr-Newman-AdS coordinates and parameters) we have derived one can check that the above integrals give

$$Q_e = \frac{q_e}{G\Xi}, \qquad Q_m = \frac{q_m}{G\Xi}, \qquad J = \frac{r_+(1 + 2g^2 r_+^2 + a^2 g^2)a}{G\Xi^2}, \qquad A = \frac{4\pi(r_+^2 + a^2)}{\Xi} \tag{90}$$

in agreement with the corresponding quantities of the extremal Kerr-Newman-AdS black hole (see e.g. [41] and note in our conventions $a > 0$ so $J > 0$). For completeness we also give the mass of the extremal Kerr-Newman-AdS, which cannot be computed from the near-horizon geometry:

$$M = \frac{r_+(1 + 2g^2 r_+^2 + a^2 g^2)}{G\Xi^2} \ . \tag{91}$$

---

[6]This expression is (as it should be) invariant under residual gauge transformations, i.e. $\mathcal{A} \to \mathcal{A} + d\lambda$ such that $\mathcal{L}_m d\lambda = 0$. To see this, note that the constraint on λ is equivalent to $m \cdot d\lambda$ being constant, and further this constant must be zero since m has fixed points on $\mathcal{H} = S^2$.

[7]It is worth noting that this expression is also valid for Einstein-Maxwell-Λ theory for $D > 4$. Also note that the same expression has been derived for rotating isolated horizons [45].

[8]In the special case of BPS black holes one can determine the mass (see below).

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The theory we have been considering is the bosonic sector of gauged minimal supergravity with gauge coupling g, whose solutions must satisfy the BPS inequality $M \geq Z + gJ$ where $Z = \sqrt{Q_e^2 + Q_m^2}$ [2]. Such solutions admit Killing spinors (i.e. are supersymmetric) only when the BPS condition is saturated i.e. $M = Z + gJ$. For the extremal Kerr-Newman solution, it can be checked this is satisfied if and only if $r_+^2 = a/g$ [2]. However, in [40] it was shown that this is not a sufficient condition for the existence of a Killing spinor – in addition, for the Kerr-Newman solution one needs $q_m = 0$. In terms of our derived parameters this supersymmetric locus is $e^2 = C^2 \beta^{-1} - 4g^2 \beta^{-2}$ and $\cos \alpha = 0$. Observe that this agrees exactly with the conditions we derived (82) for the non-static near-horizon geometry to be supersymmetric[9]. However, it is more interesting to write this in terms of the physical quantities:

$$J|Q_e| = g(G^2 Q_e^4 - J^2), \qquad Q_m = 0 . \tag{92}$$

The supersymmetric black hole is thus a one-parameter family of solutions. We note that for this solution, the area as a function of the conserved charges takes on a particularly simple form[10]:

$$A = \frac{4\pi J}{g|Q_e|} = \frac{4\pi(G^2 Q_e^4 - J^2)}{Q_e^2} . \tag{93}$$

This leads to a very simple expression for the Hawking-Bekenstein entropy of the supersymmetric AdS_4 black hole. This is analogous to the expression found for supersymmetric AdS_5 black holes [44]. Finally it is worth noting that this supersymmetric black hole preserves $1/4$ of the supersymmetries [40]. From the previous section we deduce that its near-horizon geometry in fact preserves $1/2$ of the supersymmetries. We thus find that supersymmetry is enhanced in the near-horizon limit, as has been observed in five-dimensional gauged supergravities [32, 33, 47].

7 Discussion

In this note we have shown that the near-horizon geometry of any rotating, (globally) AdS_4 extremal black hole in Einstein-Maxwell theory must be given by that of the known extremal Kerr-Newman-AdS black hole. We exploited the result of [25] which show that such a solution must be axisymmetric. Therefore we assumed axisymmetry (and non-toroidal horizon topology), which is enough to allow a complete classification of near-horizon geometries. This is a first step towards proving a full uniqueness theorem for the extremal Kerr-Newman-AdS_4 black hole. We have also shown (with no assumptions) that a static near-horizon geometry must be a direct product of AdS_2 and S^2, T^2, H^2. Since we are mainly interested in asymptotically globally AdS_4 black holes, topological censorship then only allows the $AdS_2 \times S^2$ solution. This latter solution is simply the near-horizon geometry of extremal Reissner-Nordstrom-AdS_4 which is the sub-case of the extremal Kerr-Newman-AdS_4 black hole with $J = 0$.

One might be tempted to conclude that our combined results show that the near-horizon geometry of any extremal black hole rotating or otherwise must be given by that of the extremal Kerr-Newman-AdS black hole (for some J, possibly vanishing). However this is not quite correct, as it has not been shown that a non-rotating AdS black hole must be either static or axisymmetric (the analogous statement for asymptotically flat non-extremal black holes has been shown [36, 37]). There is thus a potential gap, corresponding to a non-axisymmetric, non-rotating, non-static extremal black hole with a non-static near-horizon limit (which has no axisymmetry). While it is tempting to dismiss such a possibility, we point out that in an analysis of gravitational perturbations of higher dimensional

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[9]This shows that the most general supersymmetric non-static and axisymmetric near-horizon geometry (with non-toroidal horizon topology) is in fact identical to that of the known supersymmetric black hole.

[10]Note that as $g \to 0$, from the constraint (92) $J \to 0$ (one cannot have $Q_e \to 0$ as then from the BPS relation $M \to 0$), giving the correct answer for extremal Reissner Nordstrom.

vacuum rotating AdS black holes [48], it is suggested that the endpoint of the superradiant instability (which occurs when the rotation is sufficiently fast as is the case for the extremal limit) is precisely a black hole with such properties.

The results and techniques used in this note turned out to be a straightforward generalisation of those for the analogous 4d vacuum problem [34], to include the effect of a Maxwell field. One may contemplate further generalisations such as coupling to uncharged scalar fields as occurs in (un)gauged supergravity coupled to extra multiplets. It turns out that the resulting coupled ODEs do not seem to admit a simple method of solution. Of perhaps more interest is to generalise the 5d vacuum classification of [34] to include a Maxwell field, as would be required for classifying non-supersymmetric near-horizon geometries in 5d minimal (un)gauged supergravity. We are currently examining this problem [49], which does not appear to be straightforward generalisation of the vacuum case.

Acknowledgements

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