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DCPT-09/39

On the uniqueness of extremal vacuum black holes

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Abstract

We prove uniqueness theorems for asymptotically flat, stationary, extremal, vacuum black hole solutions, in four and five dimensions with one and two commuting rotational Killing fields respectively. As in the non-extremal case, these problems may be cast as boundary value problems on the two dimensional orbit space. We show that the orbit space for solutions with one extremal horizon is homeomorphic to an infinite strip, where the two boundaries correspond to the rotational axes, and the two asymptotic regions correspond to spatial infinity and the near-horizon geometry. In four dimensions this allows us to establish the uniqueness of extremal Kerr amongst asymptotically flat, stationary, rotating, vacuum black holes with a single extremal horizon. In five dimensions we show that there is at most one asymptotically flat, stationary, extremal vacuum black hole with a connected horizon, two commuting rotational symmetries and given interval structure and angular momenta. We also provide necessary and sufficient conditions for four and five dimensional asymptotically flat vacuum black holes with the above symmetries to be static (valid for extremal, non-extremal and even non-connected horizons).

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1   Introduction

The black hole uniqueness theorems are one of the most striking results of four dimensional General Relativity. In the absence of matter the precise statement may be summarised by: *the only asymptotically flat, stationary, black hole solution of the Einstein vacuum equations, with a connected non-degenerate horizon, is the non-extremal Kerr black hole solution* [1–7][1]. This remarkable uniqueness theorem has been a cornerstone to our understanding of semi-classical and quantum properties of equilibrium black holes. This is because it shows that equilibrium black holes are uniquely specified by their conserved charges (the mass M, the angular momentum J), thus providing a simple physical way of identifying such states within quantum gravity. In fact, so far our most successful quantum descriptions have been for *extremal* black holes (within string theory). This is because they are simpler objects within quantum gravity due to the fact they don't radiate. Curiously though, the classical uniqueness theorems always assumed the black hole is non-extremal. Thus, it is of interest to fill this gap in the uniqueness theorems.

The black hole uniqueness theorem has taken many years to prove and in fact follows from a number of more specific results each of which must be proved separately. It is worth reviewing how this theorem has been proved for non-extremal black holes. Firstly, one establishes that spatial sections of the horizon must have $\mathbf{S}^2$ topology (this was first argued by Hawking [9,10] and later strengthened using topological censorship [11, 12]). A stationary black hole, by definition, possesses a Killing field ξ timelike near infinity. Since ξ must leave the event horizon invariant it must be tangent to it. This leads to two cases: either ξ is normal to the horizon (*i.e.* tangent to the null geodesic generators) or ξ is spacelike on the horizon. These are referred to as non-rotating and rotating respectively. In the non-rotating case one can prove the solution must be also static [13]. One can then appeal to a theorem that states that the only static solution in this class is Schwarzschild [14,15] (even without assuming a connected horizon). The rotating case is more involved. First one establishes the rigidity theorem which allows one to deduce that a rotating black hole must be axisymmetric [9,10,16,17]. The way this is proved is to show that the event horizon is a Killing horizon of some Killing field χ which is not parallel to ξ (as its rotating). Then one shows that the Killing field $\chi - \xi$ has closed (space-like) orbits, *i.e.* there exists a $U(1)$ isometry. Having established this, the rotating case reduces to a study of stationary, axisymmetric vacuum spacetimes, which by definition are Weyl solutions. Einstein's equations for such solutions may be recast as an integrable 2d non-linear sigma model with target space given by the symmetric space $SU(1,1)/S(U(1) \times U(1))$ [18, 19]. Properties of this sigma model then allow one to show uniqueness of Kerr within the class of asymptotically flat, stationary, axisymmetric vacuum solutions. The main technical tool is the so called Mazur identity (see *e.g.*, [7]), which together with a careful study of boundary conditions on the horizon, the axis of symmetry and asymptotic infinity allows one to establish uniqueness.

The purpose of this paper is to extend the known uniqueness theorems to extremal black holes. Even in the 4d case discussed above this does not seem to have been discussed in the literature. We will focus on the rotating case. However, we should note that to the best of our knowledge the non-rotating case has not been fully ruled out (the existence of static extremal vacuum black holes has been ruled out [20], however it has not been shown that a non-rotating

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[1]We have omitted various technical assumptions, one of the main ones being the assumption of analyticity. See [8] for a self-contained rigorous proof which discusses the precise technical conditions and fills in some gaps in the literature.

extremal black hole must be static). In the rotating case the rigidity theorem has also been established in the extremal case [21]. Thus we know that such solutions must also be stationary and axisymmetric. We will show that the methods used for the uniqueness theorem for non-extremal black holes can be easily adapted to extremal black holes. The only difference is the 2d space which the sigma model is defined on. The 2d space in fact can be identified with the orbit space of the $\mathbb{R} \times U(1)$ isometry group[2]. This is an orientable simply connected manifold with boundaries and corners. For non-extremal black holes (with one horizon) this orbit space is homeomorphic to a semi-infinite strip, with the boundary corresponding to the two rotation axis and the horizon, and the one asymptotic region corresponding to spatial infinity [2][3]. For an extremal black hole (with one horizon) we will show that the orbit space is an infinite strip with one of the asymptotic regions corresponding to the horizon, the other corresponding to spatial infinity and the two (disconnected) boundaries the rotation axis. The asymptotic region corresponding to the horizon is fully determined by the near-horizon geometry of such solutions. Therefore the new ingredient required to prove uniqueness of extremal black holes (for given M, J) is a general understanding of their possible near-horizon geometries. This has in fact already been achieved [22] (furthermore it turns out such vacuum near-horizon geometries are unique [23–25]). This allows us to deduce our first result:

Theorem 1. *The only four dimensional, asymptotically flat, stationary and axisymmetric, rotating, black hole solution of the Einstein vacuum equations, with a connected degenerate horizon with non-toroidal sections, is the extremal Kerr solution.*

The assumption of a non-toroidal horizon is used to determine the structure of the orbit space[4], and in the context of black holes is of course justified by topological censorship. Similarly, axisymmetry is justified by the rigidity theorem of [21]. Together with these results, our Theorem 1 provides a complete classification of rotating vacuum black holes with a single degenerate horizon.[5]

The uniqueness/classification problem for higher dimensional black holes remains a much tougher challenge. This is shown by the fact that even in five dimensions explicit examples of asymptotically flat black holes are known with $\mathbf{S}^3$ [26] and $\mathbf{S}^2 \times \mathbf{S}^1$ [27–29] horizon topology, which demonstrates explicitly black hole non-uniqueness. Again we will focus on asymptotically flat, stationary vacuum black holes with a single connected component of the horizon. Some general results are known. Firstly, by generalising Hawking's original argument, the topology of the horizons sections have been shown to be of positive Yamabe type [30, 31]. Static non-extremal black hole solutions have been shown to be given by higher dimensional versions of Schwarzschild [32]. Furthermore, it has been shown that non-extremal non-rotating black holes of this kind must be static [33]. This leaves the rotating case. To this end, a generalisation of the rigidity theorem has been established: stationary, non-extremal, rotating black holes must admit of $U(1)$ isometry with spacelike orbits [34,35]. This guarantees the existence of two commuting Killing fields. However, the generalisation of Weyl's class to higher

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[2]Given a manifold M with a group action of a group G, the orbit space is the quotient M/G, which is defined by the equivalence relation "two points in M are equivalent if they are related by some element of G".

[3]It is worth noting that the orbit space in this case is also homeomorphic to the upper half complex plane with the real line corresponding to the boundary and the asymptotic regions spatial infinity.

[4]It is an assumption of the uniqueness theorem of the near-horizon geometry [25]. In fact it can be shown that the most general vacuum near-horizon geometry with toroidal horizon sections is simply the direct product $R^{1,1} \times T^2$ [25].

[5]We have not spelled out all technical assumptions, the main one being analyticity of the metric. As in the non-extremal case, it would be interesting to remove such assumptions.

dimensions requires $D-2$ commuting Killing fields [27, 36] (something the $D=4$ uniqueness theorem relies on crucially). For $D>4$ we see this is more than guaranteed by rigidity. Luckily for $D=5$ asymptotic flatness is consistent with the existence of $U(1)^2$ isometry with spacelike orbits (this is the maximal abelian subgroup of the rotation group).[6] Therefore all progress so far has been made by focusing on solutions with this symmetry as they belong to the Weyl class[7]. In fact an analogue of the 4d uniqueness theorem has already been proved: *asymptotically flat, stationary, non-extremal black hole vacuum solutions with $U(1)^2$ rotational symmetry are uniquely specified by the conserved charges J_1, J_2 and the interval structure* [38]. The interval structure [38, 39], is a set of data which consists of an ordered set of open intervals which correspond to boundary segments of the orbit space[8]. These segments either correspond to the horizon orbit space, or sets where certain combination of the rotational Killing fields vanish. Each of these intervals is specified by its length and, for the parts not corresponding to the horizon, a 2d vector which determines which linear combination of rotational Killing fields vanishes on that interval. Note that a consequence of the theorem is that all other data, such as the mass, is in fact determined for such solutions by the interval structure and angular momentum.

In this paper we will also extend this theorem to extremal black holes in this class. We will focus on the non-static case (the static case has been ruled out [20]), although we should note that, as in 4d, it has not been shown that non-rotating implies static for this class of solutions. Once again the same method as in the non-extremal case can be used although the structure of the orbit space changes. A general understanding of near-horizon geometries of such black holes [22] allows us to show that, as in 4d, the orbit space is an infinite strip with one asymptotic region fully determined by the near-horizon geometry. The interval structure in this case only corresponds to the axes of rotation, as the horizon orbit space is no longer treated as part of the boundary of the full orbit space and instead consists of an asymptotic region. Thus our second result is:

Theorem 2. *Consider a five dimensional, asymptotically flat, stationary black hole solution of the vacuum Einstein equations, with $U(1)^2$ isometry with spacelike orbits and a connected degenerate horizon (with non-toroidal sections). There exists at most one such solution with given angular momenta J_1, J_2 and a given interval structure.*

Again, the assumption of a non-toroidal horizon is used to determine the structure of the orbit space – the general form of a near-horizon geometry [22] is used for this. In fact near-horizon geometries for such black holes have been recently classified [25]. Naturally, the results are more complicated than in 4d and one can have three possible near-horizon geometries: two classes of $\mathbf{S}^3$ horizon geometries and one class of $\mathbf{S}^2 \times \mathbf{S}^1$ horizon geometry. Despite this lack of near-horizon uniqueness our Theorem shows that specifying the interval structure and angular momenta is sufficient to uniquely specify a full extremal black hole solution (and therefore also its near-horizon geometry!).

While the above uniqueness theorems mainly concern stationary but non-static black holes, it is also of interest to make general statements regarding static black holes. In [2] Carter proved that a stationary and axisymmetric $D=4$ electro-vacuum black hole solution with a non-degenerate connected horizon and zero angular momentum is necessarily static. This theorem does not apply to the case of a horizon with several connected components. In fact

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[6]In fact all known 5d black hole solutions have $\mathbb{R} \times U(1)^2$ symmetry.

[7]See [37] for some recent rigorous results.

[8]This data is also known as rod-structure in the earlier literature [27, 36].

for $D = 5$ stationary non-static multi black hole states with zero total angular momentum are certainly possible: the Black Saturn [50] provides the first explicit example of such a spacetime. Therefore it is of interest to generalise Carter's staticity theorem. We will derive necessary and sufficient staticity conditions for asymptotically flat $D = 4, 5$ stationary vacuum black hole solutions with $U(1)^{D-3}$ rotational symmetry and a possibly non-connected horizon (whose components may be either degenerate or non-degenerate). Indeed we find that in the case of non-connected horizons, the vanishing of the total angular momentum of the spacetime is not sufficient to guarantee staticity in accordance with the known examples. However, the vanishing of the Komar angular momentum or angular velocity for each component of the horizon is sufficient.

This paper is organised as follows. First, we introduce some preliminary concepts required for the proof and we use them to derive the necessary and sufficient staticity conditions. Then we analyse the orbit space for an extremal black hole, highlighting the differences to the non-extremal case. Then we prove the uniqueness theorem, which requires a detailed analysis of the behaviour of a certain function at asymptotic infinity, the axes and the near-horizon region. We end with a discussion of our results and provide two Appendices with the proofs of the staticity theorems and the Einstein equations written in a certain coordinate system.

Note added: While this paper was nearing completion, a paper [59] appeared which also proves the uniqueness of the four dimensional extremal Kerr black hole. After the first version of our paper appeared we were informed of an existing uniqueness proof for extremal Kerr [60] (that also assumes axisymmetry) which uses a different method.

2 Stationary vacuum black holes with rotational symmetries

2.1 Preliminaries

Consider stationary asymptotically flat black hole spacetimes $(\mathcal{M}, g_{\mu\nu})$ which are solutions to Einstein's vacuum equations $R_{\mu\nu} = 0$. We will restrict to $D = 4, 5$ and assume there is an $\mathbb{R} \times U(1)^{D-3}$ isometry group. For four-dimensional black holes, the rigidity theorem [9, 10], which guarantees the existence of a $U(1)$ isometry, has recently been extended to extremal black holes [21]. In five dimensions rigidity only guarantees the existence of a $U(1)$ isometry and has only been proved for non-extremal black holes (see [21] for partial results in the extremal case), and thus the assumption of a $U(1)^2$ isometry appears to be a genuine restriction. We assume that spatial sections of the event horizon are connected $(D - 2)$-dimensional compact (closed and orientable) manifolds which we denote by $\mathcal{H}$. Therefore $\mathcal{H}$ inherits an $U(1)^{D-3}$ isometry group which defines an (effective) $U(1)^{D-3}$ action on $\mathcal{H}$ which constrains its topology as follows. In $D = 4$ we must have $\mathbf{S}^2, \mathbf{T}^2$, whereas in $D = 5$ we must have $\mathbf{S}^3, \mathbf{S}^2 \times \mathbf{S}^1, L(p, q), \mathbf{T}^3$ (see *e.g.*, [40, 41]).[9] In four dimensions only $\mathbf{S}^2$ is allowed by topological censorship [12]. In five dimensions only $\mathbf{S}^3$ (and $L(p, q)$) and $\mathbf{S}^2 \times \mathbf{S}^1$ horizon sections are allowed [30, 31]. Thus in all cases we will assume non-toroidal horizon sections. It is worth noting that the orbit space $\mathcal{H}/U(1)^{D-3}$ is then always a compact interval with a certain combination of the Killing fields vanishing at the endpoints.

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[9]So far, all known asymptotically flat black hole solutions with topology $L(p, q)$ are singular [42].

We will denote the stationary Killing field by ξ and the rotational Killing fields by m_i where $i = 1, \ldots, D - 3$. Thus we have a vacuum solution with $D - 2$ commuting Killing fields (ξ, m_i) and hence belongs to the (generalised) Weyl class [27]. As is well known for such solutions one can always find coordinates (ξ^α, x^a) such that the spacetime metric is

$$ds^2 = G_{\alpha\beta}(x)d\xi^\alpha d\xi^\beta + g_{ab}(x)dx^a dx^b \tag{1}$$

where α, β run over $(0, i)$ and $\xi = \partial/\partial\xi^0$ and $m_i = \partial/\partial\xi^i$ and a, b run over the 2d base space $\mathcal{B}$. The vacuum Einstein equations are then equivalent to the following equations on $\mathcal{B}$:

$$D_a(\rho\, G^{-1} D^a G) = 0 \tag{2}$$

$$R_{ab} = D_a D_b \log \rho - \frac{1}{4}\text{Tr}\left(D_a G^{-1} D_b G\right) \tag{3}$$

where $\rho^2 \equiv -\det G \geq 0$, and D_a and R_{ab} are the metric connection and Ricci tensor associated to the base metric g_{ab}. Note that G is a matrix of scalar fields on $\mathcal{B}$ with components $G_{\alpha\beta}$ and we will suppress these matrix indices. We will find it useful to write the Killing part of the metric $G_{\alpha\beta}$ as

$$G_{\alpha\beta}d\xi^\alpha d\xi^\beta = -\frac{\rho^2}{\det\lambda}\,dt^2 + \lambda_{ij}(d\phi^i - \omega^i\, dt)(d\phi^j - \omega^j\, dt) \tag{4}$$

where we have introduced coordinates adapted to the Killing fields $\xi = \partial/\partial t$ and $m_i = \partial/\partial\phi^i$.

Define the twist one-forms

$$\Omega_i = \star\left(m_1 \wedge \cdots \wedge m_{D-3} \wedge dm_i\right), \tag{5}$$

where $\star$ is the Hodge dual in the full spacetime (we choose the orientation to be $\epsilon_{t\phi_1\ldots\phi_{D-3}ab} = +\rho\,\epsilon_{ab}^{(2)}$.) The Einstein vacuum equations imply that the Ω_i are closed, so locally we can find the corresponding twist potentials Y_i so that $\Omega_i = dY_i$. In fact, Einstein's equations also imply that $\xi \cdot \Omega_i$ is a constant function and thus assuming at least one m_i vanishes somewhere in space-time (as is the case for asymptotically flat spacetimes) it follows that $\xi \cdot \Omega_i = 0$. Therefore we see the Ω_i are also closed one-forms on spatial sections Σ_t of the spacetime. In fact topological censorship guarantees that for asymptotically flat black holes the exterior of the black hole (domain of outer communications) is simply connected – it follows that the functions Y_i exist globally (i.e. Ω_i are exact). Note that Y_i are only defined up to an additive constant: this is the only gauge freedom associated to them. Now we can state an important fact: at a fixed point of any of the rotational Killing fields one must have $\Omega_i = 0$ for all i, and thus Y_i are constant on these subspaces.

Using the explicit parametrisation introduced above one can check that

$$dY_i = \frac{\det\lambda}{\rho}\lambda_{ij} \star_2 d\omega^j\,, \tag{6}$$

which can be inverted:

$$d\omega^i = -\frac{\rho}{\det\lambda}\lambda^{ij} \star_2 dY_j\,, \tag{7}$$

where $\lambda^{ik}\lambda_{kj} = \delta^i_{\ j}$. In terms of the twist potentials equation (2) is equivalent to[10]

$$D^a(\rho\,\lambda^{ik}\, D_a\lambda_{kj}) = -\frac{\rho}{\det\lambda}\,\lambda^{ik}\,(D^a Y_k)(D_a Y_j) \tag{8a}$$

$$D^a\left(\frac{\rho}{\det\lambda}\,\lambda^{ij}\, D_a Y_j\right) = 0\,. \tag{8b}$$

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[10]From (4) it appears that (2) yields four sets of equations for λ_{ij} and Y_i. However, one can show that only two of them are independent and these are the ones that we display.

It is worth noting that for $D = 4$, in which case there is only one rotational Killing field m_1, we have two scalar functions $X \equiv \lambda_{11}$ and $Y \equiv Y_1$. The above equations can then be written as a single equation for the complex Ernst potential $E = X + \mathrm{i}\, Y$ [18].

We will now show that the event horizon must be a Killing horizon (see *e.g.*, [43]).[11] Consider the hypersurface $\mathcal{N} = \{\rho = 0, \det \lambda > 0\}$. Since $\det G = -\rho^2$ we immediately see that the induced metric on $\mathcal{N}$ is degenerate with signature $(0, +, \ldots, +, +)$. Therefore $\mathcal{N}$ is a null hypersurface. Consider the vector field $\xi + \omega^i\, m_i$ which is tangent and null on $\mathcal{N}$. It follows it is also normal to $\mathcal{N}$. From (7) we see that the ω^i's are constant on $\mathcal{N}$ and therefore the normal $\xi + \omega^i\, m_i$ is also Killing on $\mathcal{N}$. Hence we deduce that $\mathcal{N}$ is a Killing horizon of the Killing vector field $\chi \equiv \xi + \omega^i|_{\mathcal{N}}\, m_i$. The trace of the extrinsic curvature associated to the normal χ must vanish since it is a Killing vector, and thus $\mathcal{N}$ is the apparent horizon. From standard results (see *e.g.*, [10]) for asymptotically flat, stationary spacetimes the apparent horizon coincides with the event horizon and therefore we deduce that $\mathcal{N}$ is the event horizon. We will assume the spacetime does contain a black hole, *i.e.* the set $\mathcal{N}$ is not empty. Note that the angular velocities of the horizon are defined by the constants $\omega_H^i \equiv \omega^i|_{\mathcal{N}}$ and the black hole is referred to as non-rotating if $\omega_H^i = 0$.

So far we have treated extremal and non-extremal solutions on equal footing. We can now we distinguish them as follows. One can compute

$$\chi^2 = -\frac{\rho^2}{\det \lambda} + \lambda_{ij}\left(\omega^i - \omega^i|_{\mathcal{N}}\right)\left(\omega^j - \omega^j|_{\mathcal{N}}\right), \tag{9}$$

from which it follows that

$$d\chi^2\big|_{\mathcal{N}} = -\frac{d\rho^2}{\det \lambda}\bigg|_{\mathcal{N}} \tag{10}$$

The surface gravity κ is defined by $d\chi^2|_{\mathcal{N}} = -2\,\kappa\,\chi|_{\mathcal{N}}$ and thus we see that the horizon is extremal if and only if $d\rho^2|_{\mathcal{N}} = 0$. Unfortunately this equation does not seem to be a useful way of imposing extremality.[12]

Similarly we may also define axes of rotation (*i.e.*, the set of fixed points of the rotational Killing fields) as the set $\mathcal{A} = \{\rho = 0, \det \lambda = 0, 0 < \rho^2/\det \lambda < \infty\}$. Since $\lambda_{ij} = m_i \cdot m_j$ we see that on $\mathcal{A}$ the m_i are not all linearly independent and hence the twist 1-forms $\Omega_i = dY_i = 0$. Thus, since Y_i are continuous (in fact smooth) functions on the whole spacetime, they are constants on each connected component of a $\mathcal{A}$. For a black hole spacetime with a connected horizon the set $\mathcal{A}$ has two disconnected components $\mathcal{A}^\pm$, so $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-$ with $\mathcal{A}^\pm$ connected, each of which is connected to asymptotic infinity (see next section)[13] . Therefore it follows that the value of the constants $Y_i^\pm \equiv Y_i|_{\mathcal{A}^\pm}$, which thus determine Y_i on the whole of $\mathcal{A}$, can be fixed by comparing to their values in the asymptotically flat region of the spacetime (it turns out they are related to the angular momenta of the spacetime). This fact is important in the uniqueness proofs.

Before moving on to the uniqueness theorems, we will derive simple necessary and sufficient conditions for such black holes to be static, *i.e.* for the stationary Killing field ξ to be hypersurface orthogonal. First note that staticity is equivalent to the functions $\omega^i \equiv 0$.

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[11]Since we are assuming the existence of rotational symmetries this is easy to prove and is sometimes referred to as the "weak rigidity theorem". For $D > 4$, the strong rigidity theorem (*i.e.* without assuming the existence of rotational symmetries) has only been proved for non-extremal black holes.

[12] Although note that this implies that ρ^2 is not a good coordinate on a degenerate horizon, whereas it is in the non-degenerate case [2].

[13]Note this does not assume that the same linear combination of m_i vanishes everywhere on each of $\mathcal{A}^\pm$.

Therefore for this class of black hole solutions it is clear that static implies non-rotating (*i.e.*
$\omega_H^i = 0$). In fact the converse statement (*i.e.* non-rotating implies static) is also true in this
case and can be deduced from the staticity theorem we prove below. Furthermore, Carter
proved that for $D = 4$ non-extremal black holes in the above class, zero angular momentum
implies the spacetime is static [2]: our staticity theorem also generalises this result. We are
now ready to state our result (see Appendix A for proof):

Staticity theorem. Consider an asymptotically flat, stationary, $D = 4, 5$ black hole solution
of the vacuum Einstein equations, with a $U(1)^{D-3}$ isometry with spacelike orbits, a connected
non-degenerate or degenerate horizon with compact sections of non-toroidal topology. Such
spacetimes are static if and only if $\omega_H^i J_i = 0$ (where ω_H^i are the angular velocities of the
horizon and J_i the angular momenta).

Remark 1. We deduce that sufficient conditions for staticity are either $\omega_H^i = 0$ (non-rotating
horizon) or $J_i = 0$ (zero angular momentum). Note that in 4d one of these conditions is also
necessary. This allows a complete classification of the class of black holes specified in the
above theorem. In the non-extremal case it has been shown that staticity implies the solution
is Schwarzschild [14,15,32]. In the extremal case, static vacuum near-horizon geometries must
be direct products of $\mathbb{R}^{1,1}$ and a Ricci flat $D-2$ compact manifold [20] – this is not compatible
with the horizon topologies and symmetries we are considering and hence there can be no
extremal static black holes of this kind.

Remark 2. It is interesting to note how the above theorem changes if one drops the as-
sumption of a connected horizon. In particular vanishing of the total angular momentum of
the spacetime is no longer sufficient. Indeed multi black hole spacetimes in this class, which
are non-static but have zero angular momentum, are known explicitly (*e.g.*, the Black Sat-
urn [50]). The reason the above result fails in this case is that the axes set $\mathcal{A}$ has more than
two disconnected components one of which is necessarily *not* connected to asymptotic infinity
– this means one cannot fix the Y_i everywhere on $\mathcal{A}$ in terms of the angular momenta. In fact,
if in the above staticity theorem we drop the assumption of a connected horizon and instead
assume we have a horizon with $I = 1, \ldots N$ components (which may be either degenerate or
non-degenerate in any combination), it is possible to prove that the spacetime is static if and
only if

$$\sum_{I=1}^{N} \omega_I^i J_i^I = 0 \,, \tag{11}$$

where ω_I^i are the angular velocities and J_i^I are the Komar angular momenta of the Ith horizon.
The proof is given in Appendix A. Notice that the condition (11) may be satisfied if $J_i^I = 0$
or $\omega_I^i = 0$ for all $I = 1, \ldots N$. However, it follows that the vanishing of the *total* angular
momenta of the spacetime $\sum_{I=1}^{N} J_i^I$ is not sufficient to guarantee staticity of the spacetime.

2.2 Orbit space

Due to the symmetries of the spacetime, the natural space to work on is the orbit space
$\mathcal{M}/[\mathbb{R} \times U(1)^{D-3}]$ (*i.e.* the space of orbits of the isometry group, also known as the factor
space).[14] In the present context, the results of the previous section show that this can be

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[14]More precisely one considers the orbit space of the domain of outer communication of the black hole. In
general one expects this to be an orbifold, however due to topological censorship it is in fact a manifold [39].

identified with the 2d base space $\mathcal{B}$ which is defined by the integrable subspace orthogonal to the Killing fields. Thus $\mathcal{B} \cong \mathcal{M}/[\mathbb{R} \times U(1)^{D-3}]$ and we deduce that the orbit space has a metric g_{ab}. So far we have not introduced explicit coordinates on $\mathcal{B}$. It is well known that ρ is harmonic on $\mathcal{B}$, *i.e.* $D^a D_a \rho = 0$. This allows one to show that in the region $\rho > 0$ one can use ρ as a coordinate on $\mathcal{B}$ (*i.e.* it has no critical points [2, 8]). Furthermore one can use the harmonic conjugate function z, defined by $dz = - \star_2 d\rho$, as the other coordinate so

$$g_{ab}(x)dx^a dx^b = e^{2\nu(\rho,z)}(d\rho^2 + dz^2) \tag{12}$$

for some function $\nu(\rho, z)$. These coordinates are always valid outside the horizon and away from the axes, $\rho > 0$, irrespective of whether the horizon is extremal or not. It is important to understand the global structure of the orbit space.

2.2.1 Non-extremal case

In the non-extremal case it has been shown that $\mathcal{B}$ is a 2d orientable simply connected manifold with a connected boundary with corners [2, 38, 39] . The boundary consists of a union of segments which correspond to the fixed point sets of the rotational Killing fields or the horizon orbit space, joined by the corners. In 4d there is one rotational Killing field and thus the boundary segments are specified simply by their length. In 5d the boundary segments which do not correspond to the horizon are specified by a length and a 2d vector which determines which linear combination of the rotational Killing fields m_i vanishes – this is referred to as the *interval structure* (also known as rod data) [38, 39] (see also [36]). The 2d vectors of neighbouring boundary segments are also required to satisfy a compatibility condition. It has been shown that given the interval structure the spacetime manifold $\mathcal{M}$ may be reconstructed from the orbit space up to diffeomorphism. Furthermore, (ρ, z) are global coordinates on the orbit space[15] showing that $\mathcal{B}$ is conformal to the upper half of the complex plane $\zeta = z + i\rho$, with the boundary and corners mapped on to a set of intervals on the real axis $\rho = 0$. Assuming that spatial sections of the horizon $\mathcal{H}$ are connected and non-toroidal (which we will from now on), the part of the boundary representing the horizon is simply an open interval on the z-axis, say $(-\mu, \mu)$. One can in fact show that the length of this interval is $2\mu = \kappa A_H/(2\pi)^{D-3}$ where A_H and κ are the area and surface gravity of the horizon [2, 38]. As is well known [2] one can conformally map this to the semi-infinite strip $r \geq \mu$ and $|x| \leq 1$ in the $w = r + ix$ plane using $\rho^2 = (r^2 - \mu^2)(1 - x^2)$ and $z = rx$, with $r = \mu$ and $x = \pm 1$ corresponding to the horizon and axes respectively. More precisely the subsets of the orbit space corresponding to $\mathcal{H}$ and $\mathcal{A}^\pm$ are $\{r = \mu,\ |x| \leq 1\}$ and $\{r > \mu,\ x = \pm 1\}$ respectively. In these coordinates the base metric reads

$$g_{ab}dx^a dx^b = e^{2\nu}(r^2 - \mu^2 x^2)\left(\frac{dr^2}{r^2 - \mu^2} + \frac{dx^2}{1 - x^2}\right)\ . \tag{13}$$

The structure of $\mathcal{B}$ in both the complex ζ-plane and in the complex w-plane is depicted in Fig. 1.

2.2.2 Extremal case

In the extremal case the structure of $\mathcal{B}$ changes. Any point outside the horizon is at an infinite proper distance away from the horizon. Therefore the horizon actually corresponds

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[15]Although note that ρ is *not* a good coordinate on the horizon of the full spacetime metric, whereas in fact ρ^2 is [2].
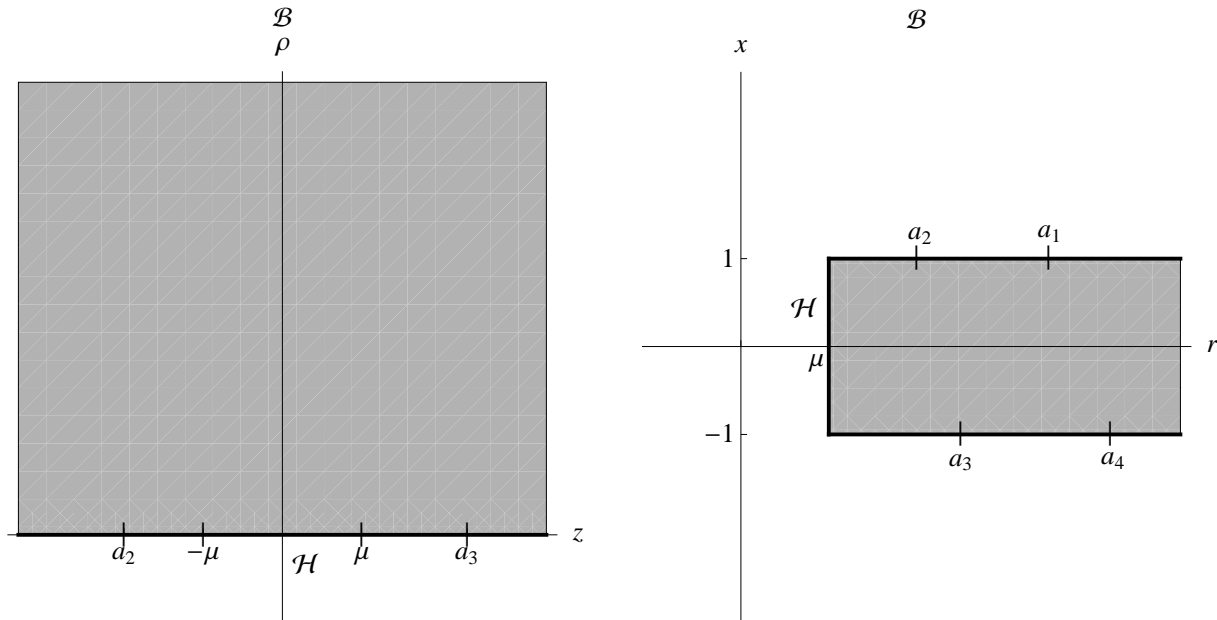
Figure 1: Orbit space $\mathcal{B}$ for non-extremal black holes. *Left:* $\mathcal{B}$ can be mapped to the upper half of the complex plane $\zeta = z + i\rho$. The boundaries and corners correspond to intervals on the real axis. *Right:* $\mathcal{B}$ can also be represented as a semi-infinite strip in the $w = r + ix$ plane.

to another asymptotic region rather than a part of the boundary. We will show that the orbit space of the exterior of an extremal black hole with a connected horizon is homeomorphic to an infinite strip, such that the two disconnected boundaries correspond to the axes and the two asymptotic regions correspond to spatial infinity and the horizon. As in the non-extremal case the boundaries are split into segments characterised by their interval structure (their length and in 5d a 2d vector which determines which combination of rotational Killing fields vanish). In terms of the standard (ρ, z) coordinates, we will argue below that the horizon is a point and in fact $\mathcal{B}$ is conformal to the upper half of the ζ-plane with the point on the real axis corresponding to the horizon removed (which has the same topology as an infinite strip).

To determine the topology of the orbit space for a vacuum extremal black hole we can exploit some general results which have been derived for their corresponding near-horizon geometries. In [22] it was shown that the near-horizon limit of any stationary extremal black hole with $U(1)^{D-3}$ rotational symmetries (and non-toroidal horizon sections), in a large class of theories which include vacuum gravity, can be written as

$$ds^2 = \Gamma(\sigma)[-C^2 r^2 dv^2 + 2dvdr] + \frac{d\sigma^2}{\det \gamma(\sigma)} + \gamma_{ij}(\sigma)(d\phi^i + k^i rdv)(d\phi^j + k^j rdv), \qquad (14)$$

where Γ is a strictly positive function and C, k^i are constants. The constant C is actually a trivial parameter (due to a scaling symmetry) and may be set to any convenient value – this is a freedom we will exploit shortly. Note that the horizon is at $r = 0$ and spatial sections are $(D-2)$-dimensional compact manifolds with (σ, ϕ^i) coordinates. The Killing field $\chi = \partial/\partial v$ is tangent to the null geodesic generators of the horizon. The Killing fields $m_i = \partial/\partial \phi^i$ generate the $U(1)^{D-3}$ isometry and σ takes values in a closed interval corresponding to the orbit space of $\mathcal{H}$ (see [25]). Since the near-horizon geometry is a vacuum solution with $D - 2$ commuting

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Killing fields (χ, m_i) one can always find coordinates (ξ^α, x^a) to write the metric in the Weyl form (1) with $\chi = \partial/\partial\xi^0$ and $m_i = \partial/\partial\xi^i$. Explicitly, for $r > 0$ define coordinates

$$t = v + \frac{1}{C^2 r}, \qquad \bar{\phi}^i = \phi^i + k^i C^{-2} \log r \qquad (15)$$

in which the near-horizon geometry becomes

$$ds^2 = \Gamma(\sigma)\left[-C^2 r^2 dt^2 + \frac{dr^2}{C^2 r^2}\right] + \frac{d\sigma^2}{\det\gamma(\sigma)} + \gamma_{ij}(\sigma)(d\bar{\phi}^i + k^i r dt)(d\bar{\phi}^j + k^j r dt) \qquad (16)$$

which allows us to read off the base metric

$$g_{ab}(x)dx^a dx^b = \Gamma\left(\frac{dr^2}{C^2 r^2} + \frac{d\sigma^2}{Q}\right) \qquad (17)$$

where $Q \equiv \Gamma\gamma$, and the Killing part of the metric

$$G_{\alpha\beta}d\xi^\alpha d\xi^\beta = -\Gamma(\sigma)C^2 r^2 dt^2 + \gamma_{ij}(\sigma)(d\bar{\phi}^i + k^i r dt)(d\bar{\phi}^j + k^j r dt) . \qquad (18)$$

Recall that the function ρ is defined by $\rho^2 \equiv -\det G$. In this case we therefore have $\rho^2 = C^2 r^2 \Gamma\gamma = C^2 r^2 Q$. The vacuum equations imply that ρ is harmonic on the base space and this allows one to introduce the harmonic conjugate function z via $dz = -\star_2 d\rho$. One can check that $\star_2 d\rho = -rC^2 d\sigma + \frac{1}{2}\dot{Q}dr$ where we use an orientation defined by $\epsilon_{r\sigma} > 0$ and "dots" refer to σ derivatives. It follows that ρ is harmonic (i.e. $d \star_2 \rho = 0$) if and only if $\ddot{Q} + 2C^2 = 0$. Note that this agrees exactly with what was found in the classification of vacuum near-horizon geometries in [25]. Then, the equation for z can be integrated to give $z = -\frac{1}{2}r\dot{Q}$, where we have set the integration constant to zero. Since $Q \geq 0$ with equality only at two isolated points which define the boundary of the horizon orbit space [25], we integrate to get $Q = C^2(-\sigma^2 + \sigma_0^2)$ where σ_0 is a positive constant so $-\sigma_0 \leq \sigma \leq \sigma_0$ (wlog the linear term has been set to zero by translating the σ coordinate). Now define a coordinate $0 \leq \theta \leq \pi$ by $\cos\theta = x = \sigma/\sigma_0$ so $Q = C^2\sigma_0^2(1 - x^2) = C^2\sigma_0^2 \sin^2\theta$ and the 2d base metric is simply

$$g_{ab}(x)dx^a dx^b = \frac{\Gamma}{C^2 r^2}\left(dr^2 + r^2 d\theta^2\right) , \qquad (19)$$

with $\rho = C^2\sigma_0 r \sin\theta$ and $z = C^2\sigma_0 r \cos\theta$. We will now exploit the scaling freedom associated to the constant C to set $C^2\sigma_0 = 1$. Thus $d\rho^2 + dz^2 = dr^2 + r^2 d\theta^2$ and hence

$$e^{2\nu} = \frac{\Gamma}{C^2 r^2} . \qquad (20)$$

Thus we have showed that for any vacuum near-horizon geometry the horizon $r = 0$ is located at the origin of the upper half plane $\rho = z = 0$, and the coordinate r (which originates from the Gaussian null coordinates) is simply the polar coordinate $r = \sqrt{\rho^2 + z^2}$. Furthermore, the coordinate $x \equiv \cos\theta$, so $|x| \leq 1$, which parametrises the horizon orbit space corresponds to the polar angle $\tan\theta = \rho/z$ of the (ρ, z) plane. However, this description is potentially misleading as the proper length of the horizon segment is in fact finite and the proper distance from any point to the horizon is infinite (since the conformal factor is singular at the origin of the upper half plane). A better description of the base space in this region is achieved by defining a coordinate $y = \log r$ so $-\infty < y < \infty$ and

$$g_{ab}(x)dx^a dx^b = \frac{\Gamma(x)}{C^2}\left(dy^2 + \frac{dx^2}{1 - x^2}\right) . \qquad (21)$$

This shows that the base space in the near-horizon limit is conformal to the strip $|y| < \infty$ and $|x| \leq 1$. This space has two disconnected boundaries given by $x = \pm 1$, which correspond to the axes, and two asymptotic ends, namely the horizon, which is at $y \to -\infty$, and spatial infinity, which is at $y \to \infty$. Note that in these coordinates it is easy to see that the proper length of the horizon orbit space is $l_H = \int_0^\pi d\theta \sqrt{\Gamma} C^{-1} > 0$. In terms of the original Weyl coordinates (ρ, z), the orbit space is homeomorphic to the upper half of the complex plane with a point which corresponds to the location of the horizon (say the origin) removed.

We may use this result to deduce the structure of the orbit space for the full extremal black hole (*i.e.* not just its near-horizon geometry). Cutting the spacetime off outside the horizon, we see that the region connected to asymptotic infinity is identical to that for a non-extremal black hole. The region connected to the horizon is well approximated by the near-horizon geometry provided we cut sufficiently near the horizon. Thus gluing together these shows that the orbit space for the full extremal black hole must also be homeomorphic to an infinite strip with the horizon and asymptotic infinity located at the two asymptotic ends and the axes are the two boundaries. The structure of the orbit space for extremal black holes is depicted in Fig. 2. We note that for $\rho > 0$ the functions (r, θ) defined by $\rho = r \sin \theta$ and $z = r \cos \theta$, are good coordinates (since (ρ, z) are). As shown above $r \to 0$ corresponds to the asymptotic region near the horizon ($r = 0$ is not part of the orbit space) as it must coincide with the near-horizon limit. For the full black hole $r \to \infty$ corresponds to asymptotic infinity. Therefore (r, θ) are global coordinates on the orbit space of the exterior of the full extremal black hole whose metric is then[16]

$$g_{ab}dx^a dx^b = e^{2\nu(r,\theta)}(dr^2 + r^2 d\theta^2) \tag{22}$$

with $r > 0$ and $0 \leq \theta \leq \pi$, and the boundary consists of $\{r > 0, \ \theta = 0\}$ and $\{r > 0, \ \theta = \pi\}$ which correspond to the two disconnected parts of the axes $\mathcal{A}^\pm$.

3   Uniqueness proof

Both the classic uniqueness theorem in $D = 4$, as well as its more recent extension to $D = 5$, rely on a number of remarkable properties of the Einstein vacuum equations for Weyl solutions. In particular, one can recast the equation of motion (8) as

$$D_a \left(\rho \, \Phi^{-1} D^a \Phi \right) = 0 \tag{23}$$

where

$$\Phi = \begin{pmatrix} \dfrac{1}{\det \lambda} & -\dfrac{1}{\det \lambda} Y_i \\[2ex] -\dfrac{1}{\det \lambda} Y_j & \lambda_{ij} + \dfrac{1}{\det \lambda} Y_i Y_j \end{pmatrix}, \tag{24}$$

is a real, $(D-2) \times (D-2)$, *positive-definite*, symmetric matrix with unit determinant. It is useful to define the current $J = \Phi^{-1} d\Phi$ which satisfies the conservation law $D^a(\rho J_a) = 0$, and the "square-root" matrix S such that $\Phi = S^T S$ and $\det S = 1$.

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[16]Notice that if one takes the zero horizon interval length limit of the non-extremal case $\mu \to 0$, the coordinates in the semi-infinite strip description of the orbit space of a non-extremal black hole (13) coincide exactly with the polar coordinates of the ρ, z plane.
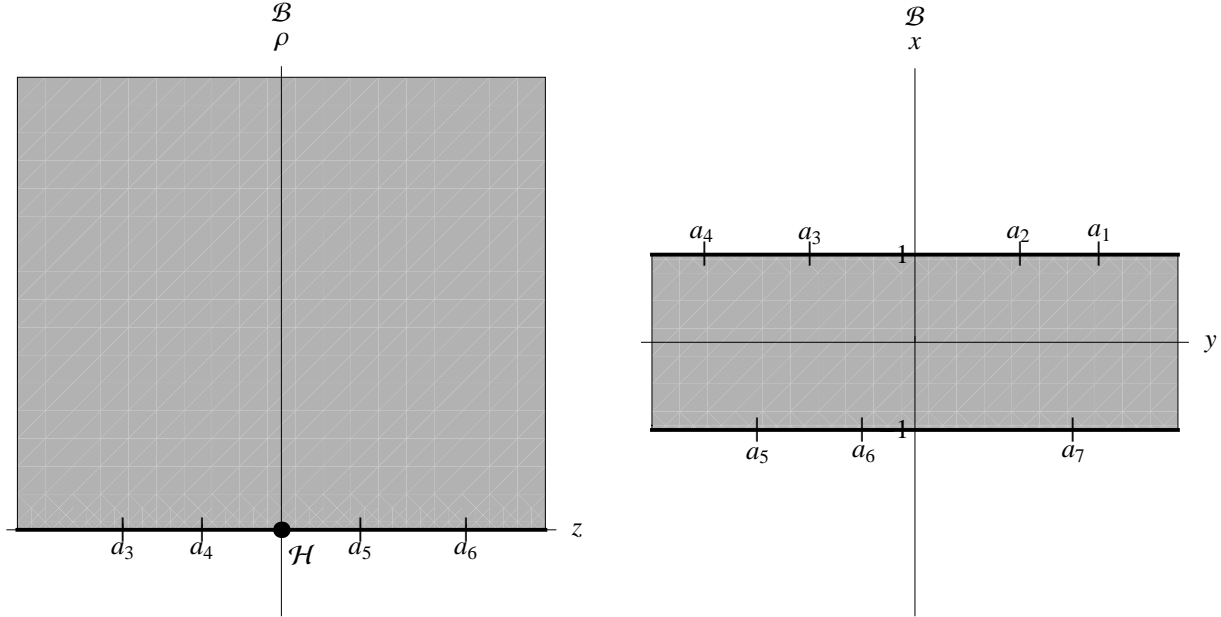
Figure 2: Orbit space $\mathcal{B}$ for extremal black holes. *Left:* $\mathcal{B}$ can be regarded as the upper half of the complex plane $\zeta = z + i\,\rho$, with the point on the boundary corresponding to the the horizon (the origin) removed. *Right:* $\mathcal{B}$ is better viewed as an infinite strip in the $y + i\,x$ plane with $r = \log y$. The horizon (not shown) is at $y \to -\infty$ and spatial infinity is at $y \to +\infty$. The two disconnected boundaries ($x = \pm 1$) correspond to the axes.

Now, consider two asymptotically flat vacuum black hole solutions $(\mathcal{M}_{[0]}, g_{[0]})$ and $(\mathcal{M}_{[1]}, g_{[1]})$ with a connected degenerate horizon and denote all data with the corresponding subscripts. Assume the two solutions have identical interval structure and angular momenta. We showed that the orbit space for any such black hole is homeomorphic to an infinite strip or equivalently the upper half $z + i\rho$ plane minus the origin. Therefore we may identify the orbit spaces and hence can assume $\rho_{[0]} = \rho_{[1]}$ and $z_{[0]} = z_{[1]}$. As in the non-extremal case, the orbit space together with the interval structure allows one to reconstruct the spacetime manifold together with the $\mathbb{R} \times U(1)^{D-3}$-action up to diffeomorphism and thus we may set $\mathcal{M}_{[0]} = \mathcal{M}_{[1]}$ and $\xi_{[0]} = \xi_{[1]}$, $m_{[0]\,i} = m_{[1]\,i}$. It now remains to show uniqueness of the spacetime metric.

First, consider the two corresponding solutions of (23), $\Phi_{[0]}$ and $\Phi_{[1]}$ and define the deviation matrix $\Psi \equiv \Phi_{[1]}\Phi_{[0]}^{-1} - \mathbf{1}$, where $\mathbf{1}$ denotes the identity matrix. Using (23) one can verify the Mazur identity

$$D_a(\rho \, D^a \mathrm{Tr}\, \Psi) = \rho \, \mathrm{Tr}\left(N_a^T N^a\right) \tag{25}$$

where $N \equiv S_{[1]}(J_{[0]} - J_{[1]})S_{[0]}^{-1}$. Explicitly

$$\mathrm{Tr}\,\Psi = -1 + \frac{\det \lambda_{[0]}}{\det \lambda_{[1]}} + \lambda_{[0]}^{ij}\left(\lambda_{[1]ij} - \lambda_{[0]ij}\right) + \frac{1}{\det \lambda_{[1]}}\lambda_{[0]}^{ij}\left(Y_{[1]i} - Y_{[0]i}\right)\left(Y_{[1]j} - Y_{[0]j}\right). \tag{26}$$

The Mazur identity is the key ingredient to proving uniqueness. We will follow the argument of [39, 44] closely. It can be shown that $\mathrm{Tr}\,\Psi \geq 0$ [39].[17] If we now define an auxiliary $\mathbb{R}^3 - \{0\}$ by $(x, y, z) = (\rho\cos\phi, \rho\sin\phi, z)$ where (ρ, z) are the coordinates on the orbit space

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[17]This is easy to see in $D = 4$ as $\mathrm{Tr}\,\Psi = \frac{1}{X_{[0]}X_{[1]}}[(X_{[1]} - X_{[0]})^2 + (Y_{[1]} - Y_{[0]})^2]$.

(recall $\rho = z = 0$ is not part of this space for an extremal black hole) the Mazur identity can be written as

$$\nabla^2 \operatorname{Tr} \Psi \geq 0 \tag{27}$$

on $\mathbb{R}^3 - \{z\text{-axis}\}$ where ∇ is the vector derivative and $\operatorname{Tr} \Psi$ is an axisymmetric function on $\mathbb{R}^3 - \{0\}$.

The uniqueness proof works as follows. Suppose $\operatorname{Tr} \Psi$ is bounded on $\mathbb{R}^3 - \{0\}$ and vanishes somewhere. Then by the maximum principle[18] $\operatorname{Tr} \Psi \equiv 0$ everywhere in $\mathbb{R}^3 - \{0\}$ and hence in $\mathcal{B}$. Since $\operatorname{Tr} \Psi = \operatorname{Tr}(F^T F) - (D - 2)$, where $F = S_{[1]} S_{[0]}^{-1}$ and $\det F = 1$, it follows that $F \equiv \mathbf{1}$ and hence $\Phi_{[0]} \equiv \Phi_{[1]}$, *i.e.* we must have $\lambda_{[0]ij} = \lambda_{[1]ij}$ and $Y_{[0]i} = Y_{[1]i}$ everywhere which establishes uniqueness of the solution (λ_{ij}, Y_i). Now, work in (r, θ) coordinates on $\mathcal{B}$. Since the conformal factor $e^{2\nu}$ cancels from equation (7), it determines ω^i (up to a constant which may be fixed from the fact that $\omega^i \to 0$ at spatial infinity). Finally, the metric on the base space $\mathcal{B}$ is then also determined by (3) as it reduces to a first order PDE for ν (see Appendix B) which is integrable and has a unique solution (up to some constant which may be fixed from the spatial asymptotics). Thus the whole problem of uniqueness boils down to showing that $\operatorname{Tr} \Psi$ is bounded everywhere on the orbit space and vanishes somewhere. We will do this next by working in (r, x) coordinates, where $x = \cos\theta$, throughout.[19]

3.1   Asymptotic infinity

Let us consider asymptotic spatial infinity. In both 4 and 5 dimensions we demand strict asymptotic flatness. Asymptotic spatial infinity in polar coordinates then corresponds to $r \to \infty$. In 4d asymptotic flatness implies

$$\lambda_{11} = r^2(1 - x^2)[1 + O(r^{-1})], \qquad Y_1 = 2\,J\,x(3 - x^2) + O(r^{-1}) \tag{28}$$

as $r \to \infty$, where J is the ADM angular momentum. Note that we have chosen a "symmetric" gauge for Y, *i.e.* $Y_{x=1} = -Y_{x=-1}$. It is then easy to see that two solutions with the same J must satisfy $\operatorname{Tr} \Psi = O(r^{-2})$ as $r \to \infty$.

In 5d asymptotic flatness implies that as $r \to \infty$

$$\lambda_{11} = r(1 + x)\left[1 + O(r^{-1})\right], \quad \lambda_{22} = r(1 - x)\left[1 + O(r^{-1})\right],$$
$$\lambda_{12} = (1 - x^2)\,O(r^{-1}),$$
$$Y_1 = y_1 - \frac{J_1\,x(2 - x)}{\pi} + O(r^{-1}) \qquad Y_2 = y_2 - \frac{J_2\,x(2 + x)}{\pi} + O(r^{-1}) \tag{29}$$

where J_i are the ADM angular momenta and y_i are integration constants (*i.e.* we will not fix the gauge for Y_i yet). Using (29) in (26), it follows that for two solutions with the same J_i (and in the same gauge y_i), we must have $\operatorname{Tr} \Psi = O(r^{-1})$ as $r \to \infty$.

Thus both in 4d and 5d we see that $\operatorname{Tr} \Psi \to 0$ as $r \to \infty$ thus proving that not only it is bounded near infinity but also vanishes there. Recall the fact that $\operatorname{Tr} \Psi$ vanishes somewhere is a necessary part of the uniqueness theorem.

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[18]In fact for the domain $\mathbb{R}^3 - \{z\text{-axis}\}$, one needs a modification of the standard maximum principle on $\mathbb{R}^3$. The necessary result is proved in [45] Proposition C.4. We thank Piotr Chrusciel for pointing this out.

[19]We should note that the (r, x) coordinates are also valid for non-extremal black holes in the region $r > 0$ and thus our analysis of asymptotic infinity and the axes is also valid in this case.

3.2 Axes

We now turn to the boundary conditions near the axes of rotation $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-$, which in
polar coordinates are given by $\mathcal{A}^\pm = \{r > 0,\ x = \pm 1\}$. These boundary conditions are chosen
to guarantee regularity of the spacetime metric near the fixed points of the rotational Killing
fields m_i.

In 4d there is only one rotational Killing field m_1 and regularity requires

$$\lambda_{11} = a(r)(1 - x^2) + O(1 - x^2)^2, \qquad G_{t1} = O(1 - x^2) \tag{30}$$

where $a(r) > 0$ and the axis is approached as $x \to \pm 1$. These equations may be expressed in
the more elegant equivalent form

$$(1 - x^2)\partial_x \log \lambda_{11} = \mp 2 + O(1 - x^2), \qquad Y_1 = \pm 4J + O(1 - x^2)^2 \tag{31}$$

as $x \to \pm 1$. Note that since all components of the axes of rotation set $\mathcal{A}$ are connected
to infinity (recall this follows from our assumption of a connected horizon) we can fix Y_1
everywhere on $\mathcal{A}$ by using the asymptotics for Y_1 near infinity. This shows that for two
solutions with the same J we have $\mathrm{Tr}\,\Psi = O(1)$ near the axis.

In 5d we have two Killing fields m_i which can vanish, although by regularity they can only
do so simultaneously at isolated points (which correspond to the corners in the orbit space).
Consider the part of the axes set $\mathcal{A}^-$ defined by $x = -1$ and suppose we are near infinity
where (wlog) $m_1 = 0$. Regularity then requires that as $x \to -1$

$$\lambda_{11} = a(r)(1 + x) + O(1 + x)^2, \qquad \lambda_{12} = O(1 + x), \qquad \lambda_{22} = b(r) + O(1 + x) \tag{32}$$

where $a(r), b(r) > 0$ and $G_{t1} = O(1 + x)$ and $G_{t2} = O(1)$ (the latter two equations are
equivalent to $\omega^i = O(1)$). Analogous expressions are valid for the behaviour near other parts
of $\mathcal{A}^-$ (where potentionally a different combination of the Killing fields vanishes) and also on
$\mathcal{A}^+$ where we take $m_2 = 0$ near infinity. The conditions on λ_{ij} may be more elegantly written
in the form

$$(1 \pm x)^2 \mathrm{Tr}\left[(\lambda^{-1}\partial_x \lambda)^2\right] = 1 + O(1 \pm x), \qquad (1 \pm x)\partial_x \log \det \lambda = \mp 1 + O(1 \pm x) \tag{33}$$

as $x \to \mp 1$. In this form one does not need to specify which combination of Killing fields
vanishes and thus these conditions provide a more convenient statement of regularity near all
parts of $\mathcal{A}^\pm$ (not just near infinity). The conditions on G_{ti} can be translated to conditions
on Y_i. These can be written in a symmetric form by fixing the constant term in Y_i to match
with the asymptotics at infinity and picking the gauge $y_1 = -J_1/\pi$ and $y_2 = J_2/\pi$:

$$Y_i = \mp \frac{2J_i}{\pi} + O(1 \pm x) \tag{34}$$

as $x \to \mp 1$.[20] As argued earlier, the twist potentials Y_i are constant everywhere on the
axes $x = \pm 1$, and therefore equation (34) fixes these constants not only on the parts of the
axes that are near asymptotic infinity but everywhere on $\mathcal{A}^\pm$ and thus $\mathcal{A}$ (recall since we
are assuming a connected horizon all connected components of $\mathcal{A}$ are connected to infinity).
Using these conditions one may check that for two solutions with the same J_i, near the axes
set $\mathcal{A}$ we have $\mathrm{Tr}\,\Psi = O(1)$.

Thus, we have shown that both in 4d and 5d, regularity of the spacetime metric implies
that $\mathrm{Tr}\,\Psi = O(1)$ near $x = \pm 1$ thus showing this function is bounded near the parts of the
z-axis away from the origin as required.

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[20]Note that if we are on a part of an axis where some combination of m_i vanishes, then the corresponding
combination of Y_i has an error term which vanishes as $O((1 \pm x)^2)$.

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3.3 Near the horizon

It remains to consider the behaviour of Ψ near the horizon. This is the only part of the uniqueness proof where one really needs to distinguish between extremal and non-extremal black holes. We will assume the existence of a single degenerate horizon. As argued earlier, in this case one can use the polar coordinates (r, θ) defined by $\rho = r \sin \theta$ and $z = r \cos \theta$ with the horizon at $r \to 0$. In the Appendix B we give expressions for all the equations of motion in the (r, θ) coordinates and again for convenience we will use $x = \cos \theta$.

Regularity of the horizon requires that the matrix components λ_{ij} and the potentials Y_i have the following expansion near $r = 0$:

$$\lambda_{ij}(r, x) = \lambda_{ij}(x) + O(r), \qquad Y_i(r, x) = Y_i(x) + O(r) \tag{35}$$

for some smooth functions $\lambda_{ij}(x)$ and $Y_i(x)$[21]. In fact this is sufficient to establish that $\mathrm{Tr}\,\Psi = O(1)$ as $r \to 0$ and $|x| \leq 1$ (*i.e.* even on the axes). This is easy to see in 4d. Regularity near the axis means $\lambda_{11}(r, x) = (1 - x^2) f(r, x)$ and $Y_{[0]} - Y_{[1]} = (1 - x^2) g(r, x)$ for smooth functions f, g such that $f > 0$ everywhere. It then follows that $\mathrm{Tr}\,\Psi = f_{[0]}^{-1} f_{[1]}^{-1} [(f_{[1]} - f_{[0]})^2 + (g_{[1]} - g_{[0]})^2]$ which is indeed bounded as $r \to 0$ for any x. A similar argument can be made in 5d although since the formulas are more cumbersome we do not give details. This completes the proof of Theorem 1 and Theorem 2. $\square$

For completeness we will now show how the Einstein equation near $r = 0$ imply that the functions $\lambda_{ij}(x), Y_i(x)$ obey a set of coupled ODEs which are equivalent to the classification problem of all possible near-horizon geometries in this class – this has already been solved [25]. At the lowest order, the Einstein equations are (see Appendix B)

$$\left[(1 - x^2) \lambda^{ik} \lambda'_{kj} \right]' = -\frac{\lambda^{ik}}{\det \lambda} Y'_k Y'_j, \tag{36a}$$

$$\frac{d}{dx} \left[\frac{(1 - x^2) \lambda^{ij} Y'_j}{\det \lambda} \right] = 0, \tag{36b}$$

$$\partial_r \nu = -\frac{1}{8\, r} (1 - x^2)^2 \left[(\ln \det \lambda)'^2 + \mathrm{Tr}\left((\lambda^{-1} \lambda')^2 \right) + \frac{2}{\det \lambda} \lambda^{ij} Y'_i Y'_j \right] + O(1), \tag{36c}$$

$$\partial_x \nu = -\frac{1}{2} (\ln \det \lambda)' - \frac{1}{8} x(1 - x^2) \left[(\ln \det \lambda)'^2 + \mathrm{Tr}\left((\lambda^{-1} \lambda')^2 \right) + \frac{2}{\det \lambda} \lambda^{ij} Y'_i Y'_j \right] + O(r), \tag{36d}$$

where the prime $'$ denotes the derivative with respect to x.

The second equation in (36) gives us a first integral for the system of equations for (λ_{ij}, Y_i)

$$\frac{(1 - x^2) \lambda^{ij} Y'_j}{\det \lambda} = -k^i \tag{37}$$

where k^i are constants. The significance of these constants is revealed by looking at the equations for ω^i which as $r \to 0$ are

$$\partial_r \omega^i = \frac{(1 - x^2) \lambda^{ij} Y'_j}{\det \lambda} + O(r), \qquad \partial_x \omega^i = O(r^2). \tag{38}$$

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[21]Since $r = 0$ is not strictly part of the orbit space we do not need to demand regularity at the origin of the auxiliary $\mathbb{R}^3$. This is why the first order terms are $O(r)$ and not $O(r^2)$.

Solving these gives

$$\omega^i = -k^i r + O(r^2) \tag{39}$$

and we have set the integration constant to zero which corresponds to working in co-rotating coordinates.

We now turn to solving the equations for ν. First note that using the first two equations in (36), a straightforward calculation shows that

$$(1 - x^2)^2 \left[(\ln \det \lambda)'^2 + \text{Tr}\left((\lambda^{-1}\lambda')^2\right) + \frac{2}{\det \lambda} \lambda^{ij} Y_i' Y_j' \right] = 2\beta, \tag{40}$$

where β is a positive constant. We can now integrate the last two equations in (36) to obtain

$$e^{2\nu} = \frac{K}{\det \lambda} \left(\frac{1-x^2}{r^2}\right)^{\beta/4} [1 + O(r)], \tag{41}$$

for some (positive) constant K. We now show that the constant β is fixed by regularity. Using (37), we can rewrite (40) as

$$(1 - x^2)^2 \left[(\ln \det \lambda)'^2 + \text{Tr}\left((\lambda^{-1}\lambda')^2\right) \right] + 2 \det \lambda \, \lambda_{ij} k^i k^j = 2\beta. \tag{42}$$

We can evaluate this expression on any axis, $x = \pm 1$. From the boundary conditions (33) as $x \to \pm 1$, the third term on the LHS of this expression vanishes and the contributions from the two other terms gives $\beta = 4$.

Now, collecting these results, we have shown that as $r \to 0$ the space-time metric is

$$\begin{aligned}
ds^2 &= \frac{(1-x^2)}{\det \lambda} \left[-r^2 dt^2 + \frac{(K + O(r))dr^2}{r^2} \right] + \frac{(K + O(r))dx^2}{\det \lambda} \\
&+ \lambda_{ij}[d\phi^i + (k^i r + O(r^2))dt][d\phi^j + (k^j r + O(r^2))dt].
\end{aligned} \tag{43}$$

To examine regularity of the metric at $r = 0$ change coordinates to $v = t - \sqrt{K}/r$ and $\varphi^i = \phi^i - k^i \sqrt{K} \log r$ which gives

$$\begin{aligned}
ds^2 &= \frac{(1-x^2)}{\det \lambda} \left[-r^2 dv^2 + 2\sqrt{K} dv dr + O(r^{-1})dr^2 \right] + \frac{(K + O(r))dx^2}{\det \lambda} \\
&+ \lambda_{ij}[d\varphi^i + (k^i r + O(r^2))dv][d\varphi^j + (k^j r + O(r^2))dv].
\end{aligned} \tag{44}$$

Note that the only singular term is $g_{rr} = O(r^{-1})$; by shifting $v \to v + O(\log r)$ one should be able to prove that $r = 0$ is a regular degenerate Killing horizon of the Killing field $\partial/\partial v$. This requires a higher order calculation which we will not pursue and we will simply assume that we have a regular horizon. In any case, one can define the near-horizon limit $r \to \epsilon r$ and $v \to v/\epsilon$ taking $\epsilon \to 0$. This gives

$$ds^2 = \Gamma(x)[-r^2 dv^2 + 2\sqrt{K} dv dr] + \frac{K\Gamma}{(1-x^2)} dx^2 + \lambda_{ij}(x)(d\varphi^i + k^i r dv)(d\varphi^i + k^i r dv) \tag{45}$$

where we have defined $\Gamma = (1 - x^2)/\det \lambda$. Rescaling $v \to v/\sqrt{K}$, redefining $k^i \to \sqrt{K} k^i$ and setting $K = C^{-2}$ puts it in exactly the form as the near-horizon geometries of [22] (discussed

in §2.2).[22] In order to determine the near-horizon geometry fully it remains to solve for $\lambda_{ij}(x)$. Our equation for λ_{ij} now reads

$$[(1-x^2)\lambda^{ik}\lambda'_{kj}]' = -\frac{k^i k_j}{\Gamma} \tag{46}$$

where we have defined $k_i \equiv \lambda_{ij}k^j$. It is easy to check this equation is identical to the ij component of the near-horizon equation given in [25] (written in the σ coordinate of [25]). The solutions to this equation where completely determined in [25] under the assumption that spatial sections of the horizon are compact and non-toroidal. We may therefore simply appeal to the results of [25].

In four dimensions it was shown [23–25] that there is a unique near-horizon geometry which corresponds to:

$$\lambda_{11} = \frac{4a^2(1-x^2)}{1+x^2}, \\ Y_1 = \frac{8a^2\,x}{1+x^2}, \tag{47}$$

where $a > 0$ is some integration constant (and we have chosen the symmetric gauge for Y again). This is the near-horizon geometry of the extremal Kerr solution [46, 47]. We may fix the constant a as follows. Near the axis the behaviour of Y_1 is given by equation (31), which upon comparison to (47) implies $a^2 = J$. Therefore it follows that two asymptotically flat extremal black hole solutions with the same J must have isometric near-horizon geometries (this of course also follows from our Theorem 1!). This implies that we must have $\lambda_{[0]11}(x) - \lambda_{[1]11}(x) = O(r)$ and $Y_{[0]1}(x) - Y_{[1]1}(x) = O(r)$, and therefore $\mathrm{Tr}\Psi = O(r)$ (for all $|x| \le 1$). Note that this is in fact stronger than what we needed to establish Theorem 1; in other words knowledge of the near-horizon uniqueness theorem is sufficient but apparently not necessary to prove Theorem 1.

In five dimensions the situation is more complicated. As shown in [25] there are three different types of near-horizon geometry. There is a two parameter family with $\mathbf{S}^3$ horizon section topology which is in fact isometric to the near-horizon geometry of the extremal Myers-Perry black hole. By comparing Y_i near the axes these two parameters maybe be related to J_1, J_2 thus fully fixing the near-horizon geometry in terms of the conserved charges. The other two types of near-horizon geometry are both three parameter families, one with $\mathbf{S}^3$ horizon topology and one with $\mathbf{S}^2 \times \mathbf{S}^1$ topology. Again, comparing the Y_i near the axis allows one to fix two combinations of these parameters and thus in both cases there is one extra parameter. Despite this lack of near-horizon uniqueness, in view of our Theorem 2 we know that given an interval structure and angular momenta one can have at most one asymptotically flat extremal black hole. As a result its near-horizon geometry is also determined uniquely and therefore this extra parameter must be determined by this data.

4   Discussion

In this paper we have proved uniqueness theorems for asymptotically flat, stationary, rotating, extremal, vacuum black hole solutions in $D = 4, 5$ dimensions with $D-3$ commuting rotational symmetries and a single connected component of the horizon. We have employed the same

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[22]Recall such near-horizon geometries necessarily have an $SO(2,1)$ isometry group.

methods used in the non-extremal case, both for the classic 4d uniqueness theorem and the more recent 5d theorems [21, 38]. As in the non-extremal case, we have shown that such extremal solutions are uniquely specified by the angular momenta and the interval structure. The interval structure consists of a set of ordered intervals which correspond to the boundary segments of the orbit space where the rotational Killing fields have fixed points. These intervals possess a length and also a $D - 3$ vector that specifies what linear combination of the rotational Killing fields vanishes there. In $D = 4$ the interval structure (for a single black hole) is trivial[23]. Hence in 4d we have shown that such black hole solutions are uniquely specified by their angular momentum and therefore are exhausted by the extremal Kerr family. This fills in an important gap in the classic 4d uniqueness theorems.

A new feature arising in the extremal case is the existence of a well defined near-horizon geometry. Recently the general structure of such near-horizon geometries has been understood [22]. We used this in our proof to determine the structure of the orbit space near the horizon. In fact, in 4d it had been shown that such vacuum near-horizon geometries are unique [23–25] (and specified by the angular momentum). Therefore our Theorem 1 may be thought of as a generalisation of this result (although note that the near-horizon geometry uniqueness proof does not assume asymptotic flatness).

In $D = 5$ the situation is more complicated for a number of reasons. Firstly, as in the non-extremal case the interval structure is non-trivial, and given an interval structure currently there is no general way to know whether a corresponding regular black hole solution exists (other than constructing it explicitly!). This is a remaining crucial issue that must be settled in the non-extremal case in order to achieve results as strong as in 4d (at least with the assumed symmetries). Secondly, in the extremal case a new complication arises as the near-horizon geometries of such black holes are not unique. These have been classified (for non-toroidal horizons) and found to fall into three classes: two types of $\mathbf{S}^3$ horizon geometry, say type A and type B, and one type of $\mathbf{S}^2 \times \mathbf{S}^1$ geometry [25]. Currently, it is not known what conditions are required in order for a near-horizon geometry to arise as a near-horizon limit of a (asymptotically flat) black hole solution. These kinds of problems concern existence of a regular black hole solution given a set of data (the near-horizon data, asymptotic flatness, the interval structure and the angular momenta). New ideas will be required in order to answer such questions. It is worth emphasising that, for these reasons, Theorem 2 is not a generalisation of the explicit classification of possible near-horizon geometries. It tells us that the near-horizon geometry of an (asymptotically flat) extremal black hole is uniquely determined by its angular momentum and interval structure – this statement clearly has less content than the explicit classification of all possible near-horizon geometries.

In fact there are only two known explicit examples of 5d asymptotically flat, stationary, extremal black holes: the extremal Myers-Perry black hole [26] and the extremal black ring [29] both two parameter families uniquely specified by their (non-vanishing) angular momenta J_1, J_2. The near-horizon geometry of the extremal Myers-Perry is isometric to the type A $\mathbf{S}^3$ horizon geometry discussed above. The near-horizon geometry of the black ring corresponds to a special case of the $\mathbf{S}^2 \times \mathbf{S}^1$ class. The type B $\mathbf{S}^3$ horizon geometry and the most general ring topology case are not known to arise as near-horizon limits of asymptotically flat black holes. An interesting question which our work does not answer is whether there are corresponding asymptotically flat black holes of this kind (there are asymptotically KK ones). An argument against the spherical topology class has been given in [25]. As for the ring topology case

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[23]In the non-extremal case there is also an interval corresponding to the horizon and its length (which vanishes in the extremal limit) can be related to the mass and angular momentum.

one can give a physical argument against the existence of asymptotically flat black ring
solutions with more general near-horizon geometry. This is because in the classification of
near-horizon geometries it was shown that the most general ring topology horizon geometry
corresponds (essentially) to the boosted extremal Kerr-string. This object is tensionless for
a particular value of the boost, and its near-horizon geometry is in fact isometric to that of
the known extremal black ring. Therefore, if there are more general solutions with $\mathbf{S}^2 \times \mathbf{S}^1$
horizon geometries, they would correspond to the near-horizon geometries of black strings
with tension. This seems unlikely as heuristically one can imagine constructing a black ring
from a black string by bending a string into a loop and this can only be done (without
inputting external forces) for a tensionless string [48].

The uniqueness theorem that we have proved could also be extended without difficulty
We have only considered asymptotically flat black holes. However, in 5d vacuum spacetime
may also be asymptotically Kaluza-Klein. The most general such case is to have the KK $\mathbf{S}^1$
fibered over 4d Minkowski space (*i.e.* the KK monopole). In the non-extremal case such
a uniqueness theorem has been proved in the case where the $\mathbf{S}^1$ at infinity is not fibered
(*i.e.* the asymptotics are the direct product of 4d Minkowski and $\mathbf{S}^1$) [39]. It should be
straightforward to extend our results to these cases too, with the only change involving
the boundary conditions at spatial infinity. There are in fact three known extremal KK
vacuum black holes.[24] There are two $\mathbf{S}^3$ topology KK black hole which asymptote to the
KK monopole, one termed the slowly rotating solution and the other the fast rotating one.
The other solution is the boosted extremal Kerr string which is asymptotically the direct
product of 4d Minkowski and $\mathbf{S}^1$. Interestingly, in contrast to the asymptotically flat case, all
the possible vacuum near-horizon geometries classified in [25] actually correspond to those of
known extremal KK vacuum black holes.[25]

The uniqueness theorem that we have proved could also be extended without difficulty
to stationary black hole spacetimes with disconnected horizons, with at least one degenerate
component. In 4d, regular spacetimes of this kind are not expected to exist (indeed even
regular stationary non-extremal multi black holes are not expected). In 5d though, this type
of spacetime should exist, although we are not aware of any explicit example. For instance,
consider a five dimensional solution consisting of two doubly spinning black holes, one of
which is a black ring (*e.g.*, a doubly spinning version of the black saturn [50] or the various
known multi-rings [51]). One expects such configurations to admit extremal limits in which
either one or both the horizons become degenerate. This is because vacuum black rings (either
non-extremal or extremal) can be made arbitrarily thin and therefore the gravitational field
at the centre of the ring can be made arbitrarily weak. Therefore, one could consider putting
a sufficiently small (in comparison with the radius of the ring) doubly spinning black hole
at the centre and the gravitational attraction between the two objects would be arbitrarily
small. Furthermore, it should be possible to tune the angular momenta of the central object
until it becomes extremal, while equilibrium is maintained. Clearly this argument can also be
applied to spacetimes containing multiple black rings. For such spacetimes, the orbit space
would contain the asymptotic end corresponding to spatial infinity together with as many
asymptotic ends as degenerate horizons (each corresponding to a near-horizon geometry).
Therefore, it would also contain multiple disconnected boundaries, each of which in turn

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[24]One also expects an extremal black ring which asymptotes to the KK monopole to exist. In the non-
extremal a special case has been constructed [49].

[25] The two parameter family type A $\mathbf{S}^3$ case is also isometric to the slow extremal limit of the KK black
hole, the three parameter type B $\mathbf{S}^3$ case includes the fast extremal limit of the KK black hole and the three
parameter $\mathbf{S}^2 \times \mathbf{S}^1$ case includes the boosted extremal Kerr string.

would be divided into intervals. Once the multiple asymptotic ends and boundaries are accounted for, the proof of the theorem should go through unchanged.

It would also be interesting to extend these theorems to theories containing charged black holes. This is realistic in four-dimensional Einstein-Maxwell or five-dimensional minimal supergravity as they are known to have non-linear sigma model descriptions [4,52,53]. Therefore, the Mazur argument can be applied and an analogous uniqueness theorem follows once the boundary conditions for the gauge fields are properly taken into account. Indeed [4] proved the uniqueness of the non-extremal Kerr-Newman solution, and recently an analogous result has been shown in the context of 5d minimal supergravity [54] (see also [55] for a result in 5d pure Einstein-Maxwell system). In the extremal case one would again require an understanding of the form of the near-horizon geometry. In 4d these have been classified and found to be given by the near-horizon limit of the extremal Kerr-Newman black hole [24, 56] and thus a uniqueness theorem for this case will be straightforward. In 5d such near-horizon geometries have not been classified (they have for the special case of supersymmetric black holes [57]) and in fact this appears to be a difficult problem in itself [58]. However, we do have a general understanding of the structure of such near-horizon geometries [22] which should be sufficient to prove a uniqueness theorem analogous to our Theorem 2 (as this does not require one to specify the explicit near-horizon geometry).

Acknowledgements

A   Staticity theorem

In this appendix we provide the proofs of the Staticity Theorem and Remark 2 following it, which can be found in section 2.1.

Proof of staticity theorem: First note the following identity on $\mathcal{B}$:

$$D^a \left(\frac{\rho \lambda^{ij} Y_i D_a Y_j}{\det \lambda} \right) = \frac{\rho}{\det \lambda} \lambda^{ij} D^a Y_i D_a Y_j - Y_i D^a \left(\frac{\rho}{\det \lambda} \lambda^{ij} D_a Y_j \right) . \qquad (48)$$

Using the Einstein equations (8) and integrating over $\mathcal{B}$ gives

$$\int_{\mathcal{B}} \frac{\rho}{\det \lambda} \lambda^{ij} D^a Y_i D_a Y_j = \int_{\partial \mathcal{B}} \frac{\rho}{\det \lambda} \lambda^{ij} Y_i \star_2 dY_j \qquad (49)$$

where $\partial \mathcal{B}$ denotes the boundary of $\mathcal{B}$ (including any asymptotic regions). As discussed in Section 2.2, $\partial \mathcal{B}$ consists of $\rho = 0$ (which corresponds to the horizon and the axes) and asymptotic infinity $\rho \to \infty$. In order to evaluate the boundary integral on the RHS of (49) it is helpful to note that, using (7), it can be written as $- \int_{\partial \mathcal{B}} Y_i d\omega^i$. For asymptotically flat spacetimes $\omega^i \to 0$ and Y_i are bounded as $\rho \to \infty$ and hence the contribution to the boundary integral from this region must vanish. Further, as argued in section (2.1), on the horizon $d\omega^i = 0$ and the Y_i are well behaved. Therefore the contribution to the boundary integral from the horizon must also vanish (this argument is valid in both the non-extremal and

extremal cases). This leaves to evaluate $-\int_{\mathcal{A}} Y_i d\omega^i$. For a connected horizon $\mathcal{A} = \mathcal{A}^+ \cup \mathcal{A}^-$, where $\mathcal{A}^\pm$ are the two axes which extend to infinity, and thus

$$-\int_{\mathcal{A}} Y_i d\omega^i = -Y_i^+ \int_{\mathcal{A}^+} d\omega^i - Y_i^- \int_{\mathcal{A}^-} d\omega^i = \omega_H^i(Y_i^+ - Y_i^-) = \frac{8}{(2\pi)^{D-4}} \omega_H^i J_i \qquad (50)$$

where ω_H^i is the angular velocity of the horizon, and $Y_i^\pm = \pm 4 J_i/(2\pi)^{D-4}$ (see section 3.2). Therefore, the RHS of (49) is equal to $8\omega_H^i J_i/(2\pi)^{D-4}$. Since the integrand on the LHS of (49) is positive definite, we deduce that $\omega_H^i J_i = 0$ if and only if $D_a Y_i = 0$ everywhere in $\mathcal{B}$. But, $Y_i = \text{const}$ is equivalent to $\omega^i = \text{const}$ which is equivalent to $\xi = \partial/\partial t$ being hypersurface orthogonal (since the asymptotic condition $\omega^i \to 0$ fixes $\omega^i \equiv 0$). This completes the proof. $\qquad \square$

Proof of Remark 2. To prove the last statement in this remark we proceed as in the proof above, we argue that the contribution to the RHS of (49) coming from the boundary at infinity and the N components of the horizon vanishes. Next we consider the contribution from the axes set. However, now one has to bear in mind that since for a non-connected horizon $\mathcal{A}$ has more than two disconnected components, at least one of them is necessarily not connected to asymptotic infinity. Therefore, one cannot fix the value of Y_i in (50) to be proportional to J_i on these parts of the axes set. However, the integral $-\int_{\mathcal{A}} Y_i d\omega^i$ can be evaluated in general in this case too. Suppose that there are N disconnected components of the horizon. This implies that $\mathcal{A}$ is a disjoint union of $N+1$ disconnected sets $\mathcal{A}^I$, with $I = 0, 1 \ldots N$, and define the constants $Y_i^I \equiv Y_i|_{\mathcal{A}^I}$. Therefore

$$-\int_{\mathcal{A}} Y_i d\omega^i = -\sum_{I=0}^{N} Y_i^I \int_{\mathcal{A}^I} d\omega^i = \sum_{I=0}^{N} Y_i^I(\omega_{I+1}^i - \omega_I^i) = \sum_{I=1}^{N}(Y_i^{I-1} - Y_i^I)\omega_I^i \qquad (51)$$

where ω_I^i for $I = 1, \ldots, N$ are the angular velocities of the Ith component of the horizon and $\omega_0^i = \omega_{N+1}^i = 0$. Now in general[26]

$$\begin{aligned} Y_i^{I-1} - Y_i^I &= \int_{\mathcal{H}_I/U(1)^{D-3}} \Omega_i \\ &= \frac{1}{(2\pi)^{D-3}} \int_{\mathcal{H}_I} \star dm_i = \frac{16\pi}{(2\pi)^{D-3}} J_i^I, \end{aligned} \qquad (52)$$

where J_i^I are the Komar angular momenta of the Ith component of the horizon. In going from the first to the second line we have used the identity $(\star dm_i)|_{\mathcal{H}} = (\Omega_i \wedge d\phi^1 \wedge \cdots \wedge d\phi^{D-3})|_{\mathcal{H}}$ and the fact $L_{m_i}\Omega_j = 0$ (*i.e.* Ω_i are 1-forms on the orbit space) to convert the integral over the Ith horizon orbit space to an (spacetime) integral over the Ith horizon. We have thus derived the general identity

$$\int_{\mathcal{B}} \frac{\rho}{\det \lambda} \lambda^{ij} D^a Y_i D_a Y_j = \frac{8}{(2\pi)^{D-4}} \sum_{I=1}^{N} \omega_I^i J_i^I . \qquad (53)$$

Therefore, the spacetime is static if and only if $\sum_{I=1}^{N} \omega_I^i J_i^I = 0$. This completes the proof. $\quad \square$

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[26]This expression is a coordinate-independent generalisation of the relation derived in [38] valid for both non-degenerate and degenerate horizons.

B   Weyl solutions in spherical polar coordinates

Assume the event horizon has a single connected component with zero surface gravity. As shown in the text we can use spherical polar coordinates

$$\rho = r \sin\theta, \qquad z = r \cos\theta, \tag{54}$$

where $0 \le \theta \le \pi$ (since $\rho \ge 0$), so that the horizon now is located at the origin $r \to 0$. These coordinates are valid everywhere for $r > 0$, even away from the near-horizon limit. The results in §2.2 show that they coincide near the horizon (strictly in the near-horizon limit) with the coordinates originating from Gaussian null coordinates. In this Appendix we have written Einstein equations (2) and (3) in these spherical polar coordinates:

$$\left[\partial_r(r^2\,\partial_r) + \frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta)\right]\lambda_{ij} = \lambda^{mn}\left[r^2\,(\partial_r\lambda_{im})(\partial_r\lambda_{jn}) + (\partial_\theta\lambda_{im})(\partial_\theta\lambda_{jn})\right]$$
$$- \frac{1}{\det\lambda}\left[r^2\,(\partial_r Y_i)(\partial_r Y_j) + (\partial_\theta Y_i)(\partial_\theta Y_j)\right], \tag{55a}$$

$$\left[\partial_r(r^2\,\partial_r) + \frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta)\right]Y_i = \lambda^{mn}\left[r^2\,(\partial_r\lambda_{im})(\partial_r Y_n) + (\partial_\theta\lambda_{im})(\partial_\theta Y_n)\right]$$
$$+ r^2\,(\partial_r \ln\det\lambda)(\partial_r Y_i) + (\partial_\theta \ln\det\lambda)(\partial_\theta Y_i), \tag{55b}$$

$$\partial_r\nu = -\tfrac{1}{2}\,\partial_r\ln\det\lambda + \tfrac{1}{8}\left\{r\,\sin^2\theta\left[(\partial_r\ln\det\lambda)^2 - \tfrac{1}{r^2}(\partial_\theta\ln\det\lambda)^2\right.\right.$$
$$+\mathrm{Tr}\big((\lambda^{-1}\partial_r\lambda)^2\big) - \tfrac{1}{r^2}\mathrm{Tr}\big((\lambda^{-1}\partial_\theta\lambda)^2\big)$$
$$\left.+\tfrac{2}{\det\lambda}\lambda^{ij}\big((\partial_r Y_i)(\partial_r Y_j) - \tfrac{1}{r^2}(\partial_\theta Y_i)(\partial_\theta Y_j)\big)\right]$$
$$+2\,\sin\theta\,\cos\theta\left[(\partial_r\ln\det\lambda)(\partial_\theta\ln\det\lambda) + \mathrm{Tr}\big((\lambda^{-1}\partial_r\lambda)(\lambda^{-1}\partial_\theta\lambda)\big)\right.$$
$$\left.\left.+\tfrac{2}{\det\lambda}\lambda^{ij}(\partial_r Y_i)(\partial_\theta Y_j)\right]\right\}, \tag{56}$$

$$\partial_\theta\nu = -\tfrac{1}{2}\,\partial_\theta\ln\det\lambda - \tfrac{1}{8}\left\{r^2\,\sin\theta\,\cos\theta\left[(\partial_r\ln\det\lambda)^2 - \tfrac{1}{r^2}(\partial_\theta\ln\det\lambda)^2\right.\right.$$
$$+\mathrm{Tr}\big((\lambda^{-1}\partial_r\lambda)^2\big) - \tfrac{1}{r^2}\mathrm{Tr}\big((\lambda^{-1}\partial_\theta\lambda)^2\big)$$
$$\left.+\tfrac{2}{\det\lambda}\lambda^{ij}\big((\partial_r Y_i)(\partial_r Y_j) - \tfrac{1}{r^2}(\partial_\theta Y_i)(\partial_\theta Y_j)\big)\right]$$
$$-2\,r\,\sin^2\theta\left[(\partial_r\ln\det\lambda)(\partial_\theta\ln\det\lambda) + \mathrm{Tr}\big((\lambda^{-1}\partial_r\lambda)(\lambda^{-1}\partial_\theta\lambda)\big)\right.$$
$$\left.\left.+\tfrac{2}{\det\lambda}\lambda^{ij}(\partial_r Y_i)(\partial_\theta Y_j)\right]\right\} \tag{57}$$

Also note

$$\partial_r\omega_i = -\frac{\sin\theta}{\det\lambda}\lambda^{ij}\partial_\theta Y_j, \qquad \partial_\theta\omega_i = \frac{r^2\sin\theta}{\det\lambda}\lambda^{ij}\partial_r Y_j. \tag{58}$$

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