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METRIC LIE N-ALGEBRAS AND DOUBLE EXTENSIONS

JOSÉ FIGUEROA-O'FARRILL

ABSTRACT. We prove a structure theorem for Lie n-algebras possessing an invariant inner product. We define the notion of a double extension of a metric Lie n-algebra by another Lie n-algebra and prove that all metric Lie n-algebras are obtained from the simple and one-dimensional ones by iterating the operations of orthogonal direct sum and double extension.

CONTENTS

1. INTRODUCTION

A (finite-dimensional, real) Lie n-algebra consists of a finite-dimensional real vector space V together with a linear map $\Phi : \Lambda^n V \to V$, denoted simply as an n-bracket, obeying a generalisation of the Jacobi identity. To define it, let us recall that an endomorphism $D \in \operatorname{End} V$ is said to be a **derivation** if

$$D[x_1 \ldots x_n] = [Dx_1 \ldots x_n] + \cdots + [x_1 \ldots Dx_n] \ ,$$

for all $x_i \in V$. Then (V, Φ) defines a **Lie n-algebra** if the endomorphisms $\operatorname{ad}_{x_1 \ldots x_{n-1}} \in \operatorname{End} V$, defined by $\operatorname{ad}_{x_1 \ldots x_{n-1}} y = [x_1 \ldots x_{n-1} y]$, are derivations. When $n = 2$ this clearly agrees with the Jacobi identity of a Lie algebra. For $n > 2$ we will call it the n-Jacobi identity. The vector space of derivations is a Lie subalgebra of $\mathfrak{gl}(V)$ denoted $\operatorname{Der} V$. The derivations $\operatorname{ad}_{x_1 \ldots x_{n-1}} \in \operatorname{Der} V$ span the ideal $\operatorname{ad} V \lhd \operatorname{Der} V$ consisting of **inner derivations**.

From now on, whenever we write Lie n-algebra, we will assume that $n > 2$ unless otherwise stated. In this paper we will only work with finite-dimensional real Lie n-algebras.

Lie n-algebras were introduced by Filippov [1] and have been studied further by a number of people. We mention here only two outstanding works beyond Filippov's original paper: the pioneering work of Kasymov [2] and the PhD thesis of Ling [3]. Kasymov studied the various notions of solvability and nilpotency for Lie n-algebras, introduced the notion of representation of a Lie n-algebra and proved an Engel-type theorem and a Cartan-like criterion for solvability. Ling classified simple Lie n-algebras and proved a very useful Levi-type decomposition. It is perhaps remarkable that most structural results in the theory of Lie n-algebras are actually consequences of similar results for the Lie algebra of derivations. In this sense it is to be expected that results for Lie algebras should have their analogue in the theory of Lie n-algebras; although it seems that Lie n-algebras become more and more rare as n increases, due perhaps to the fact that as n increases, the n-Jacobi identity imposes more and more conditions.

For example, over the complex numbers there is up to isomorphism a unique simple Lie n-algebra for every $n > 2$, of dimension $n + 1$ and whose n-bracket is given relative to a basis $(\boldsymbol{e}_i)$ by

$$[\boldsymbol{e}_1 \dots \widehat{\boldsymbol{e}_i} \dots \boldsymbol{e}_{n+1}] = (-1)^i \boldsymbol{e}_i \ ,$$

where a hat denotes omission. Over the reals, they are all given by attaching a sign ε_i to each $\boldsymbol{e}_i$ on the right-hand side of the bracket.

A class of Lie n-algebras which have appeared naturally in mathematical physics are those which possess a nondegenerate inner product which is invariant under the inner derivations. We call them **metric Lie n-algebras**. They seem to have arisen for the first time in work of Papadopoulos and the author [4] in the classification of maximally supersymmetric type IIB supergravity backgrounds [5], and more recently, for the case of $n = 3$, in the work of Bagger and Lambert [6, 7] and Gustavsson [8] on a superconformal field theory for multiple M2-branes. It is this latter work which has revived the interest of part of the mathematical physics community on metric Lie n-algebras.

Metric Lie algebras are not as well understood as the simple Lie algebras; although, shy of a classification, a number of structural results are known. It is a classic result that Lie algebras possessing a positive-definite invariant inner product are reductive, whence isomorphic to an orthogonal direct sum of simple and one-dimensional Lie algebras. In lorentzian signature (i.e., index 1) there is a classification due to Medina [9]. The indecomposable lorentzian Lie algebras are constructed out of the one-dimensional Lie algebra by iterating two constructions: orthogonal direct sum and *double extension*. This was later extended by Medina and Revoy [10] (see also work of Stanciu and the author [11]), who showed that indecomposable metric Lie algebras are constructed by again iterating the operations of direct sum and the (generalised) double extension, using again as ingredients the simple and one-dimensional Lie algebras. This was used in [9] to construct all possible indecomposable metric Lie algebras of index 2 (i.e., signature $(2, p)$). Contrary to the lorentzian case, there is a certain ambiguity in this construction, which prompted Kath

and Olbrich [12] to approach the classification problem for metric Lie algebras from a co-homological perspective. In particular they classified indecomposable metric Lie algebras with index 2, a result which had been announced in [13]. For more indefinite signatures, the classification problem is still largely open.

Much less is known about metric Lie n-algebras. There is a classification for euclidean [14] (see also [15]) and lorentzian [16] metric Lie n-algebras and also a classification of index-2 metric Lie 3-algebras [17]. In that paper there is also a structure theorem for metric Lie 3-algebras and a definition of double extension. In this note we will extend these results to $n > 3$. We prove a structure theorem for metric Lie n-algebras and in particular introduce the notion of a double extension of a metric Lie n-algebra by another Lie n-algebra.

Acknowledgments

2. Metric Lie n-algebras

We recall that a metric Lie n-algebra is a triple (V, Φ, b) consisting of a finite-dimensional real vector space V, a linear map $\Phi : \Lambda^n V \to V$, denoted simply by an n-bracket, and a nondegenerate symmetric bilinear form $b : S^2 V \to \mathbb{R}$, denoted simply by $\langle -, - \rangle$, subject to the n-Jacobi identity

$$[x_1 \ldots x_{n-1}[y_1 \ldots y_n]] = [[x_1 \ldots x_{n-1}y_1] \ldots y_n] + \cdots + [y_1 \ldots [x_1 \ldots x_{n-1}y_n]] , \qquad (1)$$

and the invariance condition of the inner product

$$\langle [x_1 \ldots x_{n-1}y_1], y_2 \rangle = - \langle [x_1 \ldots x_{n-1}y_2], y_1 \rangle , \qquad (2)$$

for all $x_i, y_i \in V$.

Given two metric Lie n-algebras (V_1, Φ_1, b_1) and (V_2, Φ_2, b_2), we may form their **orthogonal direct sum** $(V_1 \oplus V_2, \Phi_1 \oplus \Phi_2, b_1 \oplus b_2)$, by declaring that

$$[x_1 x_2 y_1 \ldots y_{n-2}] = 0 \qquad \text{and} \qquad \langle x_1, x_2 \rangle = 0 ,$$

for all $x_i \in V_i$ and all $y_i \in V_1 \oplus V_2$. The resulting object is again a metric Lie n-algebra. A metric Lie n-algebra is said to be **indecomposable** if it is not isomorphic to an orthogonal direct sum of metric Lie n-algebras $(V_1 \oplus V_2, \Phi_1 \oplus \Phi_2, b_1 \oplus b_2)$ with $\dim V_i > 0$. In order to classify the metric Lie n-algebras, it is clearly enough to classify the indecomposable ones. In Section 3 we will prove a structure theorem for indecomposable Lie n-algebras.

2.1. **Basic facts about Lie n-algebras.** From now on let (V, Φ) be a Lie n-algebra. Given subspaces $W_i \subset V$, we will let

$$[W_1 \ldots W_n] = \{[w_1 \ldots w_n] | w_i \in W_i\} .$$

We will use freely the notions of subalgebra, ideal and homomorphisms as reviewed in [16]. In particular a **subalgebra** $W < V$ is a subspace $W \subset V$ such that $[W \ldots W] \subset W$, whereas an **ideal** $I \lhd V$ is a subspace $I \subset V$ such that $[IV \ldots V] \subset I$. A linear map

$\phi : V_1 \to V_2$ between Lie n-algebras is a **homomorphism** if $\phi[x_1 \ldots x_n] = [\phi(x_1) \ldots \phi(x_n)]$, for all $x_i \in V_1$. An **isomorphism** is a bijective homomorphism. There is a one-to-one correspondence between ideals and homomorphisms and all the standard theorems hold. In particular, intersection and sums of ideals are ideals. An ideal $I \lhd V$ is said to be **minimal** if any other ideal $J \lhd V$ contained in I is either 0 or I. Dually, an ideal $I \lhd V$ is said to be **maximal** if any other ideal $J \lhd V$ containing I is either V and I. If $I \lhd V$ is any ideal, we define the **centraliser** $Z(I)$ of I to be the subalgebra defined by $[Z(I)IV \ldots V] = 0$. Taking V as an ideal of itself, we define the **centre** $Z(V)$ by the condition $[Z(V)V \ldots V] = 0$. A Lie n-algebra is said to be **simple** if it has no proper ideals and $\dim V > 1$.

Lemma 1. *If $I \lhd V$ is a maximal ideal, then V/I is simple or one-dimensional.*

Simple Lie n-algebras have been classified.

Theorem 2 ([3, §3]). *A simple real Lie n-algebra is isomorphic to one of the $(n+1)$-dimensional Lie n-algebras defined, relative to a basis $\boldsymbol{e}_i$, by*

$$[\boldsymbol{e}_1 \ldots \widehat{\boldsymbol{e}_i} \ldots \boldsymbol{e}_{n+1}] = (-1)^i \varepsilon_i \boldsymbol{e}_i \ , \tag{3}$$

where a hat denotes omission and where the ε_i are signs.

It is plain to see that simple real Lie n-algebras admit invariant inner products of any signature. Indeed, the Lie n-algebra in (3) leaves invariant the diagonal inner product with entries $(\varepsilon_1, \ldots, \varepsilon_{n+1})$.

Complementary to the notion of semisimplicity is that of solvability. As shown by Kasymov [2], there is a whole spectrum of notions of solvability for Lie n-algebras. However we will use here the original notion introduced by Filippov [1]. Let $I \lhd V$ be an ideal. We define inductively a sequence of ideals

$$I^{(0)} = I \qquad \text{and} \qquad I^{(k+1)} = [I^{(k)} \ldots I^{(k)}] \subset I^{(k)} \ . \tag{4}$$

We say that I is **solvable** if $I^{(s)} = 0$ for some s, and we say that V is **solvable** if it is solvable as an ideal of itself. If $I, J \lhd V$ are solvable ideals, so is their sum $I + J$, leading to the notion of a maximal solvable ideal $\operatorname{Rad} V$, known as the **radical** of V. A Lie n-algebra V is said to be **semisimple** if $\operatorname{Rad} V = 0$. Ling [3] showed that a semisimple Lie n-algebra is isomorphic to the direct sum of its simple ideals. The following result is due to Filippov [1] and can be paraphrased as saying that the radical is a *characteristic* ideal.

Theorem 3 ([1, Theorem 1]). *Let V be a Lie n-algebra. Then $D \operatorname{Rad} V \subset \operatorname{Rad} V$ for every derivation $D \in \operatorname{Der} V$.*

We say that a subalgebra $L < V$ is a **Levi subalgebra** if $V = L \oplus \operatorname{Rad} V$ as vector spaces. Ling showed that, as in the theory of Lie algebras, Lie n-algebras admit a Levi decomposition.

Theorem 4 ([3, Theorem 4.1]). *Let V be a Lie n-algebra. Then V admits a Levi subalgebra.*

A further result of Ling's which we shall need is the following. Let us say that a Lie n-algebra is **reductive** if its radical coincides with its centre: $\operatorname{Rad} V = Z(V)$.

Theorem 5 ([3, Theorem 2.10]). *Let V be a Lie n-algebra. Then V is reductive if and only if the Lie algebra $\operatorname{ad} V$ of inner derivations is semisimple. If in addition $\operatorname{Der} V = \operatorname{ad} V$, V is semisimple.*

In turn this allows us to prove the following useful result.

Proposition 6. *Let $0 \to A \to B \to \overline{C} \to 0$ be an exact sequence of Lie n-algebras. If A and $\overline{C}$ are semisimple, then so is B.*

Proof. Since A is semisimple, Theorem 5 says that $\operatorname{ad} A$ is semisimple. B is a representation of $\operatorname{ad} A$, hence fully reducible. Since A is an $\operatorname{ad} A$-submodule of B, we have $B = A \oplus C$, where C is a complementary $\operatorname{ad} A$-submodule. Since $A \lhd B$ is an ideal (being the kernel of a homomorphism), $\operatorname{ad} A(C) = 0$, whence $[A \ldots AC] = 0$.

The subspace C is actually a subalgebra, since the component $[C \ldots C]_A$ of $[C \ldots C]$ along A is $\operatorname{ad} A$-invariant by the n-Jacobi identity and the fact that C is $\operatorname{ad} A$-invariant. This means that $[C \ldots C]_A$ is central in A, but A is semisimple, whence it must vanish. Hence, $[C \ldots C] \subset C$. Since the projection $B \to \overline{C}$ maps C isomorphically to $\overline{C}$, we see that this isomorphism is one of Lie n-algebras, hence $C < B$ is semisimple and indeed $[C \ldots C] = C$.

Next we show that $[AC \ldots C] = 0$. Indeed, for $c_1, \ldots, c_{n-1} \in C$, the map $a \mapsto [c_1 \ldots c_{n-1}a]$ is a derivation of A. Since A is semisimple, it is an inner derivation. However since $\operatorname{ad} A$ acts trivially on C, this derivation is $\operatorname{ad} A$-invariant, which means that it is central. Since $\operatorname{ad} A$ has trivial centre, we see that it must be zero. This shows that $B = A \oplus C$ is also a direct sum of $\operatorname{ad} C$-modules, with A being a trivial $\operatorname{ad} C$-module.

Now consider $W_k := [\underbrace{A \ldots A}_{n-k}\underbrace{C \ldots C}_{k}]$. We have seen that $W_0 = A$, $W_1 = 0 = W_{n-1}$ and $W_n = C$. We claim that $W_{1<k<n-1} = 0$ as well. Indeed, W_k is the image of $\Lambda^{n-k}A \otimes \Lambda^k C \to A$ (since A is an ideal) under the bracket. Since the bracket is $\operatorname{ad} V$-equivariant, it is in particular $\operatorname{ad} C$-equivariant, but now A is a trivial $\operatorname{ad} C$-module and C, being semisimple, decomposes into nontrivial irreducible $\operatorname{ad} C$-modules. Therefore the only $\operatorname{ad} C$-equivariant map $\Lambda^{n-k}A \otimes \Lambda^k C \to A$, for $k \geq 1$, is the zero map.

In other words, $[ACB \ldots B] = 0$, whence $B = A \oplus C$ is the direct sum of the two commuting ideals A and C. Since A and C are themselves direct sum of simple ideals, so is B. $\qquad\square$

A useful notion that we will need is that of a representation of a Lie n-algebra. A **representation** of Lie n-algebra V on a vector space W is a Lie n-algebra structure on the direct sum $V \oplus W$ satisfying the following three properties:

(1) the natural embedding $V \to V \oplus W$ sending $v \mapsto (v, 0)$ is a Lie n-algebra homomorphism, so that $[V \ldots V] \subset V$ is the original n-bracket on V;
(2) $[V \ldots VW] \subset W$; and
(3) $[V \ldots VWW] = 0$.

We will often say that W is a V**-module**, although this is slightly misleading in the absence of a notion of a "universal enveloping algebra" for a Lie n-algebra. The second of the above

conditions says that if W is a representation of V, we have a map $\operatorname{ad} V \to \operatorname{End} W$ from inner derivations of V to linear transformations on W. The n-Jacobi identity for $V \oplus W$ says that this map is a representation of the Lie algebra $\operatorname{ad} V$. Viceversa, any representation $\operatorname{ad} V \to \operatorname{End} W$ defines a Lie n-algebra structure on $V \oplus W$ extending the Lie n-algebra structure of V and demanding that $[V, \ldots, V, W, W] = 0$. Taking $W = V$ gives rise to the **adjoint representation**, whereas taking $W = V^*$ gives rise to the **coadjoint representation**, where if $\alpha \in V^*$ then

$$[v_1, \ldots, v_{n-1}, \alpha] = \beta \in V^* \qquad \text{where} \qquad \beta(v) = -\alpha\left([v_1, \ldots, v_{n-1}, v]\right) . \tag{5}$$

2.2. **Basic notions about metric Lie n-algebras.** Let us now introduce an inner product, so that (V, Φ, b) is a metric Lie n-algebra.

If $W \subset V$ is any subspace, we define

$$W^\perp = \{v \in V | \langle v, w \rangle = 0 , \forall w \in W\} .$$

Notice that $(W^\perp)^\perp = W$. We say that W is **nondegenerate**, if $W \cap W^\perp = 0$, whence $V = W \oplus W^\perp$; **isotropic**, if $W \subset W^\perp$; and **coisotropic**, if $W \supset W^\perp$. Of course, in positive-definite signature, all subspaces are nondegenerate.

An equivalent criterion for decomposability is the existence of a proper nondegenerate ideal: for if $I \lhd V$ is nondegenerate, $V = I \oplus I^\perp$ is an orthogonal direct sum of ideals. The proofs of the following results can be read off *mutatis mutandis* from the similar results for metric Lie 3-algebras in [18, §2.2].

Lemma 7. *Let $I \lhd V$ be a coisotropic ideal of a metric Lie n-algebra. Then $I/I^\perp$ is a metric Lie n-algebra.*

Lemma 8. *Let V be a metric Lie n-algebra. Then the centre is the orthogonal subspace to the derived ideal; that is, $[V, \ldots, V] = Z^\perp$.*

Proposition 9. *Let V be a metric Lie n-algebra and $I \lhd V$ be an ideal. Then*

(1) *$I^\perp \lhd V$ is also an ideal;*
(2) *$I^\perp \lhd Z(I)$; and*
(3) *if I is minimal then $I^\perp$ is maximal.*

3. STRUCTURE OF METRIC LIE n-ALGEBRAS

We now investigate the structure of metric Lie n-algebras. If a Lie n-algebra V is not simple or one-dimensional, then it has a proper ideal and hence a minimal ideal. Let $I \lhd V$ be a minimal ideal of a metric Lie n-algebra. Then $I \cap I^\perp$, being an ideal contained in I, is either 0 or I. In other words, minimal ideals are either nondegenerate or isotropic. If nondegenerate, $V = I \oplus I^\perp$ is decomposable. Therefore if V is indecomposable, I is isotropic. Moreover, by Proposition 9 (2), I is abelian and furthermore, because I is isotropic, $[IIV \ldots V] = 0$.

It follows that if V is euclidean and indecomposable, it is either one-dimensional or simple, whence of the form (3) with all $\varepsilon_i = 1$. This result, originally due to Nagy [14] (see

also [15]), was conjectured in [4], albeit in the guise of a conjectural generalisation of the classical Plücker identity.

Let V be an indecomposable metric Lie n-algebra. Then V is either simple, one-dimensional (provided the index of the inner product is < 2) or possesses an isotropic proper minimal ideal I which obeys $[IIV\ldots V] = 0$. The perpendicular ideal $I^\perp$ is maximal and hence by Lemma 1, $\overline{U} := V/I^\perp$ is simple or one-dimensional, whereas by Lemma 7, $\overline{W} := I^\perp/I$ is a metric Lie n-algebra. The inner product on V induces a nondegenerate pairing $g : \overline{U} \otimes I \to \mathbb{R}$. Indeed, let $[u] = u + I^\perp \in \overline{U}$ and $v \in I$. Then we define $g([u], v) = \langle u, v \rangle$, which is clearly independent of the coset representative for $[u]$. In particular, $I \cong \overline{U}^*$ is either one- or $(n+1)$-dimensional. If the signature of the metric of $\overline{W}$ is (p, q), that of V is $(p + r, q + r)$ where $r = \dim I = \dim \overline{U}$.

There are two possibilities for $\overline{U}$: either it is one-dimensional or else it is simple. We will treat both cases separately.

3.1. $\overline{U}$ is one-dimensional.

If the quotient Lie n-algebra $\overline{U} = V/I^\perp$ is one-dimensional, so is the minimal ideal I. Let $u \in V$ be such that $u \notin I^\perp$, whence its image in $\overline{U}$ generates it. Because $I \cong \overline{U}^*$ is induced by the inner product, there is $v \in I$ such that $\langle u, v \rangle = 1$. The subspace spanned by u and v is therefore nondegenerate, and hence as a vector space we have an orthogonal decomposition $V = \mathbb{R}(u, v) \oplus W$, where W is the perpendicular complement of $\mathbb{R}(u, v)$. It is clear that $W \subset I^\perp$, and that $I^\perp = I \oplus W$ as a vector space. Indeed, the projection $I^\perp \to \overline{W}$ maps W isomorphically onto $\overline{W}$.

From Proposition 9 (2), it is immediate that $[u, v, x_1 \ldots, x_{n-2}] = 0 = [v, x_1 \ldots, x_{n-1}]$, for all $x_i \in W$, whence v is central. Metricity then implies that the only nonzero n-brackets take the form

$$\begin{aligned}
[ux_1 \ldots x_{n-1}] &= [x_1 \ldots x_{n-1}] \\
[x_1 \ldots x_n] &= (-1)^n \langle [x_1 \ldots x_{n-1}], x_n \rangle v + [x_1 \ldots x_n]_W\ ,
\end{aligned} \tag{6}$$

which defines $[x_1 \ldots x_{n-1}]$ and $[x_1 \ldots x_n]_W$ and where $x_i \in W$. The n-Jacobi identity is equivalent to the following two conditions:

(1) $[x_1 \ldots x_{n-1}]$ defines a Lie $(n-1)$-algebra structure on W, which leaves the inner product invariant due to the skewsymmetry of $\langle [x_1 \ldots x_{n-1}], x_n \rangle$; and
(2) $[x_1 \ldots x_n]_W$ defines a metric Lie n-algebra structure on W which is invariant under the $(n-1)$-algebra structure.

As we will see below, this makes V into the *double extension* of the metric Lie n-algebra W by the one-dimensional Lie n-algebra $\overline{U}$.

3.2. $\overline{U}$ is simple.

Consider $I^\perp$ as a Lie n-algebra in its own right and let $R = \operatorname{Rad} I^\perp$ denote its radical. By Theorem 4, $I^\perp$ admits a Levi subalgebra $L < I^\perp$. Since $I^\perp \lhd V$ and $R \lhd I^\perp$ is a characteristic ideal, $R \lhd V$. Indeed, for all $x_i \in V$, $\operatorname{ad}_{x_1\ldots x_{n-1}}$ is a derivation of $I^\perp$ (since $I^\perp \lhd V$) and by Theorem 3, it preserves R. Let $M = V/R$. Notice that

$$\overline{U} = V/I^\perp \cong (V/R)/(I^\perp/R) = M/L\ ,$$

by the standard homomorphism theorems. Since L and $\overline{U}$ are semisimple, Proposition 6 says that so is M and moreover that $M \cong L \oplus \overline{U}$. This means that R is also the radical of V, whence M is a Levi factor of V. This discussion is summarised by the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
R & == & R \\
\downarrow & & \downarrow \\
0 \longrightarrow I^{\perp} \longrightarrow & V \longrightarrow & \overline{U} \longrightarrow 0 \\
\downarrow & \downarrow & \| \\
0 \longrightarrow L \longrightarrow & M \longrightarrow & \overline{U} \longrightarrow 0 \\
\downarrow & \downarrow & \\
0 & 0 &
\end{array}
$$

The map $M \to \overline{U}$ admits a section, so that M has a subalgebra $\widetilde{U}$ isomorphic to $\overline{U}$ and such that $M = \widetilde{U} \oplus L$. Then the vertical map $V \to M$ also admits a section, whence there is a subalgebra $U < V$ isomorphic to $\overline{U}$ such that $V = I^{\perp} \oplus U$ (as vector space). Furthermore, the inner product on V pairs I and U nondegenerately, whence $I \oplus U$ is a nondegenerate subspace. Let W denote its perpendicular complement, whence $V = W \oplus I \oplus U$. Clearly $I^{\perp} = W \oplus I$, whence the canonical projection $I^{\perp} \to \overline{W}$ maps W isomorphically onto $\overline{W}$.

Let us now write the possible n-brackets for $V = W \oplus I \oplus U$. First of all, by Proposition 9 (2), $[V, \ldots, V, I^{\perp}, I] = 0$. Since $U < V$, $[U, \ldots, U] \subset U$ and since I is an ideal, $[U, \ldots, U, I] \subset I$. Similarly, since $W \subset I^{\perp}$ and $I^{\perp} \lhd V$ is an ideal, $[W, \ldots, W] \subset W \oplus I$. We write this as

$$[w_1 \ldots w_n] := [w_1 \ldots w_n]_W + \varphi(w_1 \ldots w_n) \,,$$

where $[w_1 \ldots w_n]_W$ defines an n-bracket on W, which is isomorphic to the Lie n-bracket of $\overline{W} = I^{\perp}/I$, and $\varphi : \Lambda^n W \to I$ is to be understood as an abelian extension. At the other extreme we have the bracket $[U \ldots UW] \subset W$ which makes W into an $\mathrm{ad}\,U$-module. Metricity forbids a nonzero I-component to the above bracket:

$$\langle [U \ldots UW], U \rangle = \langle [U \ldots U], W \rangle = 0 \,,$$

since U is a subalgebra. Finally we have a sequence of brackets

$$V_k := [\underbrace{U \ldots U}_{k} \underbrace{W \ldots W}_{n-k}] \subset W \oplus I \,,$$

for $0 < k < n - 1$. We notice that

$$\langle V_k, U \rangle = \left\langle [\underbrace{U \ldots U}_{k} \underbrace{W \ldots W}_{n-k}], U \right\rangle = \left\langle [\underbrace{U \ldots U}_{k+1} \underbrace{W \ldots W}_{n-k-1}], W \right\rangle = \langle V_{k+1}, W \rangle \,,$$

whence the component of V_k along I agrees up to a sign with the component of V_{k+1} along W. In principle all such brackets occur and the only conditions apart from the metricity come from the Jacobi identity of V.

Similarly to the case when $\overline{U}$ is one-dimensional, we will interpret V as the *double extension* of the metric Lie n-algebra $\overline{W}$ by the simple Lie n-algebra U.

3.3. **Double extensions and the structure theorem.** More generally we have the following definition.

Definition 10. Let W be a metric Lie n-algebra and let U be a Lie n-algebra. Then by the **double extension of W by U** we mean the metric Lie n-algebra on the vector space $W \oplus U \oplus U^*$ with the following nonzero n-brackets subject to the Jacobi identity for V:

- $[U \dots U] = [U \dots U]_U$, making U into a subalgebra;
- $[U \dots U U^*] \subset U^*$, making U^* into the coadjoint representation of U;
- $[U \dots U W] \subset W$, making W into an $\mathrm{ad}\, U$-module;
- $[w_1 \dots w_n] = [w_1 \dots w_n]_W + \varphi(w_1, \dots, w_n)$ for $w_i \in W$, where $[\dots]_W$ is the bracket of the Lie n-algebra W and $\varphi : \Lambda^n W \to U^*$ is an $\mathrm{ad}\, U$-equivariant map; and
- $\mathrm{ad}\, U$-equivariant brackets $[\underbrace{U \dots U}_{k} \underbrace{W \dots W}_{n-k}] \subset W \oplus U^*$ for $0 < k < n-1$, where
  metricity identifies (perhaps up to a sign) the W component of $[\underbrace{U \dots U}_{k} \underbrace{W \dots W}_{n-k}]$
  with the U^* component of $[\underbrace{U \dots U}_{k-1} \underbrace{W \dots W}_{n-k+1}]$.

The resulting Lie n-algebra is metric, with inner product which extends the one on W by the dual pairing between U and U^*. One can also add any invariant symmetric bilinear form on U, even if degenerate.

For $n = 2$ this construction is due to Medina and Revoy [10], whereas for $n = 3$ it is due to the authors of [17].

In summary we have proved the following

Theorem 11. *Every indecomposable metric Lie n-algebra is either one-dimensional, simple or else it is the double extension of a metric Lie n-algebra of smaller dimension by a one-dimensional or a simple Lie n-algebra.*

An easy induction argument and the fact that metric Lie n-algebras are orthogonal direct sums of their indecomposable components yields the following

Corollary 12. *The class of metric Lie n-algebras is generated by the simple and one-dimensional Lie n-algebras under the operations of orthogonal direct sum and double extension.*

It is clear that the subclass of euclidean metric Lie n-algebras is generated by the simple and one-dimensional euclidean Lie 3-algebras under orthogonal direct sum, since double extension always incurs in indefinite signature. Therefore an indecomposable euclidean

metric Lie n-algebra is either one-dimensional or simple [4, 14, 15]. The lorentzian inde-composables admit at most one double extension by a one-dimensional Lie n-algebra and are easy to classify [18, 16].

REFERENCES

[1] V. Filippov, "n-Lie algebras," *Sibirsk. Mat. Zh.* **26** (1985), no. 6, 126–140, 191.

[2] S. M. Kasymov, "On a theory of n-Lie algebras," *Algebra i Logika* **26** (1987), no. 3, 277–297, 398.

[3] W. X. Ling, *On the structure of n-Lie algebras*. PhD thesis, Siegen, 1993.

[4] J. M. Figueroa-O'Farrill and G. Papadopoulos, "Plücker-type relations for orthogonal planes," *J. Geom. Phys.* **49** (2004) 294–331, `math.AG/0211170`.

[5] J. M. Figueroa-O'Farrill and G. Papadopoulos, "Maximal supersymmetric solutions of ten- and eleven-dimensional supergravity," *J. High Energy Phys.* **03** (2003) 048, `hep-th/0211089`.

[6] J. Bagger and N. Lambert, "Modeling multiple M2's," *Phys. Rev.* **D75** (2007) 045020, `arXiv:hep-th/0611108`.

[7] J. Bagger and N. Lambert, "Gauge symmetry and supersymmetry of multiple M2-branes," *Phys. Rev.* **D77** (2008) 065008, `arXiv:0711.0955 [hep-th]`.

[8] A. Gustavsson, "Algebraic structures on parallel M2-branes," `arXiv:0709.1260 [hep-th]`.

[9] A. Medina, "Groupes de Lie munis de métriques bi-invariantes," *Tōhoku Math. J. (2)* **37** (1985), no. 4, 405–421.

[10] A. Medina and P. Revoy, "Algèbres de Lie et produit scalaire invariant," *Ann. scient. Éc. Norm. Sup.* **18** (1985) 553.

[11] J. M. Figueroa-O'Farrill and S. Stanciu, "On the structure of symmetric selfdual Lie algebras," *J. Math. Phys.* **37** (1996) 4121–4134, `hep-th/9506152`.

[12] I. Kath and M. Olbrich, "Metric Lie algebras with maximal isotropic centre," *Math. Z.* **246** (2004), no. 1-2, 23–53, `math/0209366`.

[13] H. Baum and I. Kath, "Doubly extended lie groups - curvature, holonomy and parallel spinors," *Differential Geometry and its Applications* **19** (2003) 253, `math/0203189`.

[14] P.-A. Nagy, "Prolongations of Lie algebras and applications," `arXiv:0712.1398 [math.DG]`.

[15] G. Papadopoulos, "On the structure of k-Lie algebras," `arXiv:0804.3567 [hep-th]`.

[16] J. Figueroa-O'Farrill, "Lorentzian Lie n-algebras," `arXiv:0805.4760 [math.RT]`.

[17] P. de Medeiros, J. Figueroa-O'Farrill, and E. Méndez-Escobar, "Metric Lie 3-algebras in Bagger–Lambert theory," `arXiv:0806.3242 [hep-th]`.

[18] P. de Medeiros, J. Figueroa-O'Farrill, and E. Méndez-Escobar, "Lorentzian Lie 3-algebras and their Bagger–Lambert moduli space," `arXiv:0805.4363 [hep-th]`.

MAXWELL INSTITUTE AND SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, UK
E-mail address: `J.M.Figueroa@ed.ac.uk`