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Citation for published version:
de Medeiros, P, Figueroa-O'Farrill, J & Mendez-Escobar, E 2008, 'Lorentzian Lie 3-algebras and their Bagger-Lambert moduli space' Journal of High Energy Physics, vol. 2008, no. 7, 111, pp. -. DOI: 10.1088/1126-6708/2008/07/111

Digital Object Identifier (DOI):
10.1088/1126-6708/2008/07/111

Published In:
Journal of High Energy Physics

de Medeiros, P., Figueroa-O'Farrill, J. & Mendez-Escobar, E. 2008, 'Lorentzian Lie 3-algebras and their Bagger-Lambert Moduli Space', Journal of High Energy Physics. Vol. 2008, No. 7, pp. 111

LORENTZIAN LIE 3-ALGEBRAS AND THEIR BAGGER–LAMBERT MODULI SPACE

PAUL DE MEDEIROS, JOSÉ FIGUEROA-O'FARRILL AND ELENA MÉNDEZ-ESCOBAR

Abstract. We classify Lie 3-algebras possessing an invariant lorentzian inner product. The indecomposable objects are either one-dimensional, simple or in one-to-one correspondence with compact real forms of metric semisimple Lie algebras. We analyse the moduli space of classical vacua of the Bagger–Lambert theory corresponding to these Lie 3-algebras. We establish a one-to-one correspondence between one branch of the moduli space and compact riemannian symmetric spaces. We analyse the asymptotic behaviour of the moduli space and identify a large class of models with moduli branches exhibiting the desired $N^{3/2}$ behaviour.

Contents

Date: 19th June 2008.

1. Introduction and motivation

The existence of low-energy effective field theory descriptions of coincident D-branes in string theory in terms of nonabelian gauge theories has led to a deeper understanding of the dynamics of these nonperturbative objects. A similar approximation modelling the dynamics of multiple M2- and M5-branes in M-theory would also be greatly beneficial and could lead to a clearer understanding of the fundamental degrees of freedom here. However, the construction of interacting effective field theories which explicitly realise all the symmetries expected for M-theory branes has proved elusive.

This is perhaps not surprising given the analyses based on gravitational thermodynamics [1] and absorption cross-sections [2] that suggest the entropy of a large number of N coincident branes should obey a power law N^k, with $k = 2, \frac{3}{2}, 3$ for D-, M2-, M5-branes, respectively. Thus a supersymmetric Yang–Mills description of D-branes involving $N \times N$ matrices is natural while the appropriate gauge-theoretic description of multiple branes in M-theory is less clear. Based on the foundational work by Bagger and Lambert [3] and Gustavsson [4], a superconformal field theory in three-dimensional Minkowski spacetime was constructed in [5] as a model of multiple M2-branes in M-theory. The theory is maximally supersymmetric and scale-invariant with an explicit $\mathfrak{so}(8)$ R-symmetry. This is consistent with the full $\mathfrak{osp}(8|4)$ superconformal symmetry of the near-horizon geometry of M2-branes in eleven-dimensional supergravity. The Lagrangian and its equations of motion nicely encapsulate several other features expected [6, 7] in a low-energy description of multiple M2-branes. These encouraging properties have prompted a great deal of interest in the Bagger–Lambert theory [8–33].

A novel feature of the Bagger–Lambert theory is that it has a local gauge symmetry which is not based on a Lie algebra, but rather on a Lie 3-algebra. The analogue of the Lie bracket $[-,-]$ here being the 3-bracket $[-,-,-]$, an alternating trilinear map on a vector space V, which satisfies a natural generalisation of the Jacobi identity (sometimes referred to as the fundamental identity). The dynamical fields in the Bagger–Lambert model are taken to be valued in V and consist of eight real bosonic scalars and a fermionic spinor in three dimensions which transforms as a chiral spinor under the $\mathfrak{so}(8)$ R-symmetry. There is also a non-dynamical gauge field which takes values in a Lie subalgebra of $\mathfrak{gl}(V)$. The on-shell closure of the supersymmetry transformations for these fields follows from the fundamental identity.

To obtain the correct equations of motion from a Lagrangian that is invariant under all the aforementioned symmetries seems to require the Lie 3-algebra to admit an invariant inner product. In this paper we will consider only Lie 3-algebras with a nondegenerate inner product. The signature of this inner product determines the relative signs of the kinetic terms for the scalar fields in the Bagger–Lambert Lagrangian. As with ordinary gauge theory, taking this metric to be positive-definite would avoid potential issues concerning lack of unitarity in the quantum theory. The problem is that there are very few euclidean metric Lie 3-algebras. Indeed, as shown in [34] (see also [19, 20]), they can always be written as the direct sum of abelian Lie 3-algebras plus multiple copies of the unique simple euclidean Lie 3-algebra considered by Bagger and Lambert in their original construction. Therefore this assumption is too restrictive.

It should be noted that one could still construct the Lagrangian and gauge-invariant operators for the Bagger–Lambert theory for a Lie 3-algebra with a degenerate inner product. However, one could not obtain from such a Lagrangian the equations of motion for the fields that couple only to the degenerate components of this metric. Nonetheless, in certain cases it is possible to write down an auxiliary Lagrangian that is decoupled from the one involving the nondegenerate components of the inner product to account for the missing equations of motion (see e.g. [33] where the metric is degenerate in just one direction). The hope with this construction would be to evade the no-go theorem noted above and perhaps find Lie 3-algebras with invariant degenerate metrics whose nondegenerate components are positive-definite. A problem with such Lie 3-algebras is that they do not seem to allow any new interactions in the Bagger–Lambert Lagrangian. As shown in Remark 8, the canonical 4-form in (7) that appears in all the interaction terms in the Lagrangian can have no 'legs' in the degenerate directions. Hence the Lagrangian can always be written in terms of the quotient metric Lie 3-algebra corresponding to the nondegenerate directions. Since this quotient Lie 3-algebra is nondegenerate, the problem can be reduced to the nondegenerate case.

Of course, it is by no means guaranteed that the low-energy effective theory on multiple M2-branes should have a Lagrangian description and Gran et al [16] have considered, purely at the level of its classical equations of motion, the Bagger–Lambert theory for a class of Lie 3-algebras (defined by a Lie algebra of one dimension lower) that do not admit a metric. Following the novel Higgs mechanism technique introduced by Mukhi and Papageorgakis [9], they are able to reduce to the correct equations of motion for the supersymmetric Yang–Mills description of multiple D2-branes with arbitrary gauge algebra corresponding to the choice of Lie algebra defining the Lie 3-algebra.

An alternative approach considered recently [23–25] is to investigate the Bagger–Lambert theory for a class of Lie 3-algebras (defined by a euclidean Lie algebra in two dimensions lower) admitting an inner product of lorentzian signature. It is unclear at present whether the Bagger–Lambert theory associated with such 3-algebras is really unitary at the quantum level, but there are some encouraging signs noted in the aforementioned references, based on the specific structure of the interactions in the Bagger–Lambert model and the way the ghost-like fields that might give rise to negative-norm states seem to decouple from the physical Hilbert space. By giving a vacuum expectation value to one of the scalar fields in a null direction of the Lie 3-algebra and taking the Lie algebra to be $\mathfrak{su}(N)$, the correct reduction of the Bagger–Lambert to supersymmetric Yang–Mills effective Lagrangian for N D2-branes is obtained in [23, 25]. Moreover, in [24] it is suggested that the moduli space of this Bagger–Lambert theory has a branch that corresponds to the moduli space $(\mathbb{R}^8)^N/S_N$ of N M2-branes in Minkowski spacetime. This is to be contrasted with the moduli space of the Bagger–Lambert theory associated with the unique simple euclidean Lie 3-algebra used in the original construction of Bagger and Lambert. The semi-classical moduli space $(\mathbb{R}^8 \times \mathbb{R}^8)/D_{2k}$ for this theory was obtained in [14] for $k = 1, 2$ and in [15] for general k, where the integer k in the dihedral group D_{2k} of order $4k$ here corresponds to the value of the quantised level for the Chern-Simons term in the Bagger–Lambert Lagrangian. For $k > 2$, the M-theoretic interpretation of the moduli space is still not entirely clear, but is thought to describe two M2-branes on a so-called M-fold: essentially a $\mathbb{Z}_{2k}$ quotient which acts on both the background spacetime and the M2-branes.

Motivated by the recent results in [23–25], and by the tractability of the problem, we set ourselves the task of classifying the lorentzian metric Lie 3-algebras and studying in detail the corresponding moduli space of classical vacua. This paper contains our results and is organised as follows. In Section 2 we study the structure of metric Lie 3-algebras. The strong analogy with the case of metric Lie algebras turns out to be very fruitful and after introducing the necessary, yet standard, algebraic concepts, we are able to classify the lorentzian 3-algebras in Section 3. The main result in this section is Theorem 9, which says that the indecomposable ones are either one-dimensional, simple or belong to a class whose objects are in one-to-one correspondence with the compact real forms of metric semisimple Lie algebras. These latter ones are precisely the class of lorentzian metric Lie 3-algebras already discovered in [23–25]!

In Section 4 we investigate the structure of the moduli space of maximally supersymmetric vacua of the Bagger–Lambert theory for indecomposable lorentzian Lie 3-algebras. This adds to and complements the preliminary analysis in [24]. This moduli space can be described as the quotient by residual gauge symmetries of a certain variety in the space of constant scalar fields in the Bagger–Lambert theory. We perform a detailed calculation of the residual symmetries of the vacuum which correspond to automorphisms of the Lie 3-algebra. To some extent, we are able to factorise the moduli space and find it to have two distinct branches (which we label *degenerate* and *nondegenerate*), according to whether one of the scalar fields in a null direction of the Lie 3-algebra is zero or not (in agreement with [24], whose *abelian* branch corresponds to our nondegenerate branch).

The nondegenerate branch has the simpler structure and is of the form

$$\mathscr{M}_{\text{nondeg}} = \mathbb{R}^{16} \times \mathbb{R}^{8r}/\mathfrak{W} ,$$

for a compact semisimple Lie algebra $\mathfrak{g}$ of rank r whose Weyl group is $\mathfrak{W}$. The second factor can be identified with (the strong coupling limit of) the classical moduli space of $N{=}8$ super Yang-Mills theory with gauge algebra $\mathfrak{g}$ in three-dimensional Minkowski space (see e.g. [35]). The scalars in the two null directions of the 3-algebra, spanning $\mathbb{R}^{16}$, are related to the extra $\mathfrak{u}(1)$ degree of freedom and Yang-Mills coupling in the D2-brane reduction described in [23].

The degenerate branch has a much more intricate structure and is defined by subspaces $\mathfrak{p} \subset \mathfrak{g}$ satisfying $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}^\perp$, i.e., the Lie bracket on $\mathfrak{p}$ spans the orthogonal complement $\mathfrak{p}^\perp$ of $\mathfrak{p} \subset \mathfrak{g}$. We find a large class of such subspaces $\mathfrak{p}$ that are in one-to-one correspondence with compact riemannian symmetric spaces, and have maximal dimension $\frac{1}{2}(\dim \mathfrak{g} + \operatorname{rank} \mathfrak{g})$. After properly performing the quotient, it turns out the dimension of the degenerate branch can be very different for different choices of symmetric spaces. It is often of larger dimension than the nondegenerate branch. It would be very interesting to understand what the M-theoretic interpretation of this degenerate branch is.

We end by exploring numerically the asymptotic properties of these branches as one one takes the dimension of the Lie algebra $\mathfrak{g}$ to be large and we exhibit a large class of models with the infamous $N^{3/2}$ scaling at large N.

Acknowledgments

It is a pleasure to thank Costis Papageorgakis, Patricia Ritter, Joan Simón and especially Dmitri Alekseevsky for useful conversations and/or correspondence. PdM is supported by a Seggie-Brown Postdoctoral Fellowship of the School of Mathematics of the University of Edinburgh.

2. Metric Lie 3-algebras

In this section we study the structure of metric Lie 3-algebras. Lie n-algebras (for $n > 2$) were introduced by Filippov [36] and studied in a number of subsequent papers by Filippov and other authors. Metric Lie n-algebras (for $n > 2$) seem to have been considered for the first time in [37], albeit tangentially.

2.1. **Basic definitions.**

Let V be a finite-dimensional real vector space. Recall that a Lie algebra structure on V is a linear map $[,] : \Lambda^2 V \to V$ obeying the Jacobi identity. There are many equivalent ways to think of the Jacobi identity. One such way is to say that the endomorphisms ad_x of V for all $x \in V$, defined by $\operatorname{ad}_x(z) = [x, z]$, are a derivation over the bracket, or in other words,

$$[x, [y, z]] = [[x, y], z] + [y, [x, z]] \ , \tag{1}$$

for all $x, y, z \in V$. This formulation admits a straight-forward generalisation to n-ary brackets. In this note we will be interested in the case where $n = 3$. Thus we define a **Lie 3-algebra** (also known as a **Filippov 3-algebra**) structure on V to be a linear map $\Phi : \Lambda^3 V \to V$, often written simply as a 3-bracket $\Phi(x, y, y) = [x, y, z]$, such that for all $x, y \in V$, the endomorphism $\operatorname{ad}_{x,y} : V \to V$, defined by

$$\operatorname{ad}_{x,y}(z) = - \operatorname{ad}_{y,x}(z) = [x, y, z] \ , \tag{2}$$

is a derivation over Φ. In other words, we demand that for all $x, y, z_1, z_2, z_3 \in V$,

$$[x, y, [z_1, z_2, z_3]] = [[x, y, z_1], z_2, z_3] + [z_1, [x, y, z_2], z_3] + [z_1, z_2, [x, y, z_3]] \ , \tag{3}$$

which we call the **3-Jacobi identity**. An endomorphism $\delta : V \to V$ is called a **derivation** of the Lie 3-algebra if for all $x, y, z \in V$,

$$\delta[x, y, z] = [\delta x, y, z] + [x, \delta y, z] + [x, y, \delta z] \ . \tag{4}$$

The 3-Jacobi identity says that for all $x, y \in V$, the endomorphism $\operatorname{ad}_{x,y}$ is a derivation. Such derivations are said to be **inner**. Derivations form a Lie subalgebra of $\mathfrak{gl}(V)$ of which the inner derivations are an ideal.

Now recall that a Lie algebra structure on V is said to be **metric**, if there is an inner product (i.e., a nondegenerate symmetric bilinear form) $b \in S^2V^*$ on V, often written simply as $b(x,y) = \langle x, y\rangle$, which is invariant under the action of ad_x for all $x \in V$; that is,

$$\langle [x,y], z\rangle + \langle y, [x,z]\rangle = 0 \ , \tag{5}$$

for all $x, y, z \in V$. In the same spirit, we say that a Lie 3-algebra structure (V, Φ, b) is **metric** if the inner product b is invariant under the inner derivations $\mathrm{ad}_{x,y}$ for all $x, y \in V$; that is,

$$\langle [x,y,z], w\rangle + \langle z, [x,y,w]\rangle = 0 \ , \tag{6}$$

for all $x, y, z, w \in V$. In other words, the inner derivations $\mathrm{ad}_{x,y}$ lie in the Lie subalgebra $\mathfrak{so}(V) < \mathfrak{gl}(V)$ preserving b. Just like a metric Lie algebra possesses a canonical three-form $\Omega \in \Lambda^3V^*$, given by $\Omega(x,y,z) = \langle [x,y], z\rangle$, a metric Lie 3-algebra possesses a canonical 4-form $F \in \Lambda^4V^*$, defined by

$$F(x,y,z,w) = \langle [x,y,z], w\rangle \ . \tag{7}$$

Given two metric Lie 3-algebras (V_1, Φ_1, b_1) and (V_2, Φ_2, b_2), we may form their **orthogonal direct sum** $(V_1 \oplus V_2, \Phi_1 \oplus \Phi_2, b_1 \oplus b_2)$, by declaring that

$$[x_1, x_2, y] = 0 \qquad \text{and} \qquad \langle x_1, x_2\rangle = 0 \ ,$$

for all $x_i \in V_i$ and all $y \in V_1 \oplus V_2$. The resulting object is again a metric Lie 3-algebra. A metric Lie 3-algebra is said to be indecomposable if, roughly speaking, it cannot be written as an orthogonal direct sum of metric Lie 3-algebras $(V_1 \oplus V_2, \Phi_1 \oplus \Phi_2, b_1 \oplus b_2)$ with $\dim V_i > 0$.

In order to classify the metric Lie 3-algebras, it is clearly enough to classify the indecomposable ones. In order to do this we will find it convenient to introduce some basic 3-algebraic concepts by analogy with the theory of Lie algebras. Most of these concepts can be found in the foundational paper of Filippov [36].

2.2. **Some structure theory.** Let (V, Φ) be a Lie 3-algebra. A subspace $W \subset V$ is a **subalgebra**, written $W < V$, if $[W,W,W] \subset W$, which is a convenient shorthand notation for the following: for all $x, y, z \in W$, $[x,y,z] \in W$. We will this shorthand notation without further comment in what follows. A subalgebra $W < V$ (which could be all of V) is said to be **abelian** if $[W,W,W] = 0$.

If V, W are Lie 3-algebras, then a linear map $\phi : V \to W$ is a **homomorphism** if

$$\phi[x,y,z] = [\phi(x), \phi(y), \phi(z)] \ ,$$

for all $x, y, z \in V$. If ϕ is also a vector space isomorphism, we say that it is an **isomorphism** of Lie 3-algebras.

It is clear that the image of a homomorphism is a subalgebra and we expect that the kernel ought to be an ideal. Indeed, if $x \in \ker\phi$, then $\phi[x,y,z] = 0$ for all $y, z \in V$. This suggests the following definition: a subspace $I \subset V$ is an **ideal**, written $I \lhd V$, if $[I, V, V] \subset I$.

Lemma 1. *There is a one-to-one correspondence between ideals and kernels of homomorphisms.*

Proof. The kernel of a homomorphism is an ideal, by definition. (In fact, this motivated the definition.) Conversely, if $I \lhd V$, then V/I is a Lie 3-algebra with bracket

$$[x+I, y+I, z+I] = [x,y,z] + I \ ,$$

and the canonical projection $V \to V/I$ is a homomorphism with kernel I. ☐

Lemma 2. *If I, J are ideals of V then so is their intersection $I \cap J$ and their linear span $I + J$, defined as the smallest vector subspace containing their union $I \cup J$.*

Proof. Since $I \cap J \subset I$, $[I \cap J, V, V] \subset I$ and since $I \cap J \subset J$, $[I \cap J, V, V] \subset J$, hence $[I \cap J, V, V] \subset I \cap J$. Similarly, $[I + J, V, V] \subset [I, V, V] + [J, V, V] \subset I + J$. ☐

An ideal $I \lhd V$ is **minimal** if any other ideal $J \lhd V$ contained in I is either 0 or I. Dually, an ideal $I \lhd V$ is **maximal** if any other ideal $J \lhd V$ containing I is either I or V.

A Lie 3-algebra is **simple** if it is not one-dimensional and every ideal $I \lhd V$ is either 0 or V.

Lemma 3. *If $I \lhd V$ is a maximal ideal, then V/I is simple or one-dimensional.*

Proof. Let $\pi : V \to V/I$ denote the natural surjection, suppose that $J \subset V/I$ is an ideal and let let $\pi^{-1}J = \{x \in V | \pi(x) \in J\}$. Then $\pi^{-1}J$ is an ideal of V: $\pi[\pi^{-1}J, V, V] = [J, V/I, V/I] \subset J$, whence $[\pi^{-1}J, V, V] \subset \pi^{-1}J$. Since $I = \ker\pi$, I is contained in $\pi^{-1}J$, but since I is maximal $\pi^{-1}J = I$ or $\pi^{-1}J = V$. In the former case, $J = \pi\pi^{-1}J = \pi I = 0$ and in the latter $J = \pi\pi^{-1}J = \pi V = V/I$. Hence V/I has no proper ideals. ☐

Simple Lie n-algebras have been classified. In particular, for $n = 3$ we have the following

Theorem 4 ([38])**.** *A simple real Lie 3-algebra is isomorphic to one of the four-dimensional Lie 3-algebras defined, relative to a basis $\boldsymbol{e}_i$, by*

$$[\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k] = \sum_{\ell=1}^{4} \varepsilon_{ijk\ell}\lambda_\ell \boldsymbol{e}_\ell \ , \tag{8}$$

for some λ_ℓ, all nonzero.

It is plain to see that simple real Lie 3-algebras admit invariant metrics of any signature: euclidean, lorentzian or split. Indeed, the Lie 3-algebra in (8) leaves invariant the diagonal metric with entries $(1/\lambda_1, 1/\lambda_2, 1/\lambda_3, 1/\lambda_4)$. One can further change to a basis where the λ_i are signs. In particular this shows that up to homothety (i.e., a rescaling of the inner product) there are unique simple metric Lie 3-algebras with euclidean and lorentzian signatures, corresponding to choosing λ_i to be $(1,1,1,1)$ and $(-1,1,1,1)$, respectively. The euclidean case is the original Lie 3-algebra which was used in Appendix A of [5].

The image $[V, V, V] \subset V$ of $\Phi : \Lambda^3 V \to V$ is an ideal called the **derived ideal** of V. Another ideal is provided by the **centre** Z, defined by

$$Z = \{z \in V | [z, x, y] = 0, \ \forall x, y \in V\} \ .$$

In other words, $[Z, V, V] = 0$. More generally the **centraliser** $Z(W)$ of a subspace $W \subset V$ is defined by

$$Z(W) = \{z \in V | [z, w, y] = 0, \ \forall w \in W, y \in V\} \ ,$$

or equivalently $[Z(W), W, V] = 0$ (thus $Z(V) = Z$). It follows from the Jacobi identity (3) that $Z(W)$ is a subalgebra.

From now on let (V, Φ, b) be a metric Lie 3-algebra. If $W \subset V$ is any subspace, we define

$$W^\perp = \{v \in V | \langle v, w\rangle = 0 \ , \forall w \in W\} \ .$$

Notice that $(W^\perp)^\perp = W$. We say that W is **nondegenerate**, if $W \cap W^\perp = 0$, whence $V = W \oplus W^\perp$; **isotropic**, if $W \subset W^\perp$; and **coisotropic**, if $W \supset W^\perp$. Of course, in positive-definite signature, all subspaces are nondegenerate. A metric Lie 3-algebra is said to be **indecomposable** if it is not isomorphic to a direct sum of orthogonal ideals or, equivalently, if it does not possess any proper nondegenerate ideals: for if $I \lhd V$ is nondegenerate, $V = I \oplus I^\perp$ is an orthogonal direct sum of ideals.

The proof of the following lemma is routine.

Lemma 5. *Let $I \lhd V$ be a coisotropic ideal of a metric Lie 3-algebra. Then $I/I^\perp$ is a metric Lie 3-algebra.*

Lemma 6. *Let V be a metric Lie 3-algebra. Then the centre is the orthogonal subspace to the derived ideal; that is, $[V, V, V] = Z^\perp$.*

Proof. Let $z \in Z$, then for all $x, y, w \in V$, $0 = \langle [x, y, z], w\rangle = -\langle [x, y, w], z\rangle$, whence $z \in [V, V, V]^\perp$ and $Z \subset [V, V, V]^\perp$. Conversely, let $z \in [V, V, V]^\perp$. This means that for all $x, y, w \in V$,

$$0 = \langle z, [x, y, w]\rangle = -\langle [x, y, z], w\rangle \ ,$$

which implies that $[x, y, z] = 0$ for all $x, y \in V$ and hence $z \in Z$. In other words, $[V, V, V]^\perp \subset Z$. □

Proposition 7. *Let V be a metric Lie 3-algebra and $I \lhd V$ be an ideal. Then*

(1) *$I^\perp \lhd V$ is also an ideal;*
(2) *$I^\perp \lhd Z(I)$; and*
(3) *if I is minimal then $I^\perp$ is maximal.*

Proof.
(1) For all $x, y \in V$, $u \in I$ and $v \in I^\perp$, $\langle [v, x, y], u\rangle = -\langle [x, y, u], v\rangle = 0$, since $[x, y, u] \in I$. Therefore $[v, x, y] \in I^\perp$.
(2) For all $u \in I^\perp$, $v \in I$ and $x, y \in V$, consider $\langle [u, v, x], y\rangle = -\langle [x, y, v], u\rangle = 0$ since I is an ideal, which means that $[u, v, x] = 0$, whence $[I, I^\perp, V] = 0$.
(3) Let $J \supset I^\perp$ be an ideal. Taking perpendiculars, $J^\perp \subset I$. Since I is minimal, $J^\perp = 0$ or $J^\perp = I$, whence $J = V$ or $J = I^\perp$ and $I^\perp$ is maximal.

□

Remark 8. Although we have been assuming that the inner product is nondegenerate, let us make a remark concerning the possibility of a degenerate inner product. Let $V^\perp$ denote the radical of the inner product; that is $x \in V^\perp$ if $\langle x, y\rangle = 0$ for all $y \in V$. It follows immediately by invariance of the inner product that $V^\perp \lhd V$ is an ideal. This means that if F is the 4-form in (7), then $F(x, -, -, -) = 0$ for all $x \in V^\perp$. In other words, the 4-form is the pull-back of the 4-form on the quotient metric Lie 3-algebra $V/V^\perp$. This means that the degrees of freedom corresponding to $V^\perp$ seem to effectively decouple, yielding a Bagger–Lambert Lagrangian with Lie 3-algebra $V/V^\perp$. Of course, the symmetry transformations themselves (that do not involve the metric) will generally mix up the degrees of freedom on $V/V^\perp$ and $V^\perp$, though clearly the truncation to $V/V^\perp$ is consistent.

2.3. **Structure of metric Lie 3-algebras.** We now investigate the structure of metric Lie 3-algebras. As in the case of Lie algebras [39–41], there is a subtle interplay between ideals and the inner product.

If a Lie 3-algebra is not simple or one-dimensional, then it has a proper ideal and hence a minimal ideal. Let $I \lhd V$ be a minimal ideal of a metric Lie 3-algebra. Then $I \cap I^\perp$, being an ideal contained in I, is either 0 or I. In other words, minimal ideals are either nondegenerate or isotropic. If nondegenerate, $V = I \oplus I^\perp$ is decomposable. Therefore if V is indecomposable, I is isotropic. Moreover, by Proposition 7 (2), I is abelian and furthermore, because I is isotropic, $[I, I, V] = 0$.

It follows that if V is euclidean and indecomposable, it is either one-dimensional or simple, whence of the form (8) with all λ_i positive. As we will see below, one can choose an orthogonal (but not orthonormal) basis for V where the λ_i are equal to 1. This result, originally due to [34], was conjectured in [37], both of which also treat the case of Lie n-algebras for $n > 3$. This result has been rediscovered more recently for Lie 3-algebras in [19, 20] and for $n > 3$ in [21]. Here it is seen to follow structurally as a corollary of Theorem 4. The result for $n > 3$ also follows structurally from the $n > 3$ version of that theorem along the same lines.

Let V be an indecomposable metric Lie 3-algebra. Then V is either simple, one-dimensional, or possesses a proper minimal ideal I which is isotropic and obeys $[I, I, V] = 0$. The perpendicular ideal $I^\perp$ is maximal and hence by Lemma 3, $U := V/I^\perp$ is simple or one-dimensional, whereas by Lemma 5, $W := I^\perp/I$ is a metric Lie 3-algebra.

The inner product on V induces a nondegenerate pairing $g : U \otimes I \to \mathbb{R}$. Indeed, let $[u] = u + I^\perp \in U$ and $v \in I$. Then we define $g([u], v) = \langle u, v\rangle$, which is clearly independent of the coset representative for $[u]$. In particular, $I \cong U^*$ is either one- or four-dimensional. If the signature of the metric of W is (p, q), that of V is $(p + k, q + k)$ where $k = \dim I = \dim U$. So that if V is to have lorentzian signature, $k = 1$ and W must be euclidean; although not necessarily indecomposable.

In the next section we will classify indecomposable lorentzian Lie 3-algebras. The technique is analogous to the classification of indecomposable metric Lie algebras given in [39] and its refinement in [40, 41]. The lorentzian Lie algebras have been classified in [42] (see also [43, §2.3]). By the same techniques it is possible [44] to classify metric Lie 3-algebras with signature $(2, *)$ and to prove a structure theorem for the case of general signature. Similarly it is possible to classify lorentzian Lie n-algebras for $n > 3$ [45].

3. Lorentzian Lie 3-algebras

A lorentzian Lie 3-algebra decomposes into one lorentzian indecomposable factor and zero or more indecomposable euclidean factors. As discussed above, the indecomposable euclidean Lie 3-algebras are either one-dimensional or simple. On the other hand, an indecomposable lorentzian Lie 3-algebra is either one-dimensional, simple or else possesses a one-dimensional isotropic minimal ideal. It is this latter case which remains to be treated and we do so now.

The quotient Lie 3-algebra $U = V/I^\perp$ is also one-dimensional. Let $u \in V$ be such that $u \notin I^\perp$, whence its image in U generates it. Because $I \cong U^*$, there is $v \in I$ such that $\langle u, v\rangle = 1$. Complete it to a basis (v, x_a) for $I^\perp$. Then (u, v, x_a) is a basis for V, with (x_a) spanning a subspace isomorphic to $W = I^\perp/I$ and which, with a slight abuse of notation, we will also denote W. It is possible to choose u so that $\langle u, u\rangle = 0$ and such that $\langle u, x\rangle = 0$ for all $x \in W$. Indeed, given any u, the map $x \mapsto \langle u, x\rangle$ defines an element in the dual W^*. Since the restriction of the inner product to W is nondegenerate, there is some $z \in W$ such that $\langle u, x\rangle = \langle z, x\rangle$ for all $x \in W$. We let $u' = u - z$. This still obeys $\langle u', v\rangle = 1$ and now also $\langle u', x\rangle = 0$ for all $x \in W$. Finally let $u'' = u' - \frac{1}{2}\langle u', u'\rangle v$, which still satisfies $\langle u'', v\rangle = 1$, $\langle u'', x\rangle = 0$ for all $x \in W$, but now satisfies $\langle u'', u''\rangle = 0$ as well.

In this basis, there are four kinds of 3-brackets: $[u, v, x]$, $[u, x, y]$, $[v, x, y]$ and $[x, y, z]$ where $x, y, z \in W$. From Proposition 7 (2), it is immediate that $[u, v, x] = 0 = [v, x, y]$, whence v is central. In summary, the only nonzero 3-brackets are, using the summation convention,

$$[u, x_a, x_b] = f_{ab}{}^c x_c \qquad \text{and} \qquad [x_a, x_b, x_c] = -f_{abc} v + \phi_{abc}{}^d x_d \ , \tag{9}$$

where $f_{abc} = \langle [u, x_a, x_b], x_c\rangle$. (Notice that an additional $\omega_{ab} v$ term that might have occurred on the right hand side of the first 3-bracket must vanish by taking the inner product with u.) The 3-Jacobi identity is equivalent to the following two conditions:

(1) $[x_a, x_b] := f_{ab}{}^c x_c$ defines a Lie algebra structure on W, which leaves the inner product invariant due to the skewsymmetry of $f_{abc} = \langle [x_a, x_b], x_c\rangle$; and
(2) $[x_a, x_b, x_c]_W := \phi_{abc}{}^d x_d$ defines a euclidean Lie 3-algebra structure on W which is ad-invariant with respect to the Lie algebra structure.

We will show below that for V indecomposable, $\phi \equiv 0$ so that the Lie 3-algebra structure on W is abelian, but not before discussing a family of Lie algebras associated to every Lie 3-algebra.

3.1. **A family of metric Lie algebras.**

Let (V, Φ) be a Lie 3-algebra. It was already observed in [36] that every $z \in V$ defines a bracket $[-,-]_z : \Lambda^2 V \to V$ by

$$[x, y]_z := [x, y, z] \ , \tag{10}$$

which obeys the Jacobi identity as a consequence of the 3-Jacobi identity (3). Thus $[-,-]_z$ defines on V a Lie algebra structure for which z is a central element. In other words, a Lie 3-algebra V defines a family of Lie algebras on V parametrised linearly by V itself. Letting $\mathscr{L}_V \subset \Lambda^2 V^* \otimes V$ denote the space of Lie algebra structures on V, a Lie 3-algebra structure on V defines a linear embedding $V \hookrightarrow \Lambda^2 V^* \otimes V$ whose image lies in $\mathscr{L}_V$. Although it would be tempting to characterise Lie 3-algebras in this way, it is known [46, 47] however that this condition is strictly weaker than the 3-Jacobi identity. It is not known whether this is still the case for metric Lie 3-algebras.

If (V, Φ, b) is a metric Lie 3-algebra, then each of the Lie algebras $(V, [-,-]_z, b)$ is a metric Lie algebra. Let V be a simple euclidean Lie 3-algebra. It is possible to change to a basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_4)$ where the 3-bracket is

$$[\boldsymbol{e}_i, \boldsymbol{e}_j, \boldsymbol{e}_k] = \varepsilon_{ijk\ell}\boldsymbol{e}_\ell \ , \tag{11}$$

using the summation convention. Moreover, such a basis is orthogonal, but not necessarily orthonormal. Thus there is a one parameter family of such metric Lie 3-algebras, distinguished by the scale of the inner product. We will denote the simple Lie 3-algebra with the above 3-brackets by $\mathfrak{s}$. Fixing any nonzero $x \in \mathfrak{s}$, the Lie algebra $[-,-]_x$ is isomorphic to $\mathfrak{so}(3) \oplus \mathbb{R}$, where the $\mathfrak{so}(3)$ is the orthogonal Lie algebra in the perpendicular complement of the line containing x. Under the adjoint action of this Lie algebra, the vector space $\mathfrak{s}$ decomposes into $\mathfrak{s} = \mathbb{R}x \oplus x^\perp$.

3.2. **Indecomposable lorentzian Lie 3-algebras.** We are now ready to classify the indecomposable lorentzian Lie 3-algebras. We have previously shown that such an algebra is given in a basis (u, v, x_a) by the 3-bracket in (9). We will now show that if V is indecomposable, then ϕ necessarily vanishes.

The tensor ${\phi_{abc}}^d$ defines a euclidean Lie 3-algebra structure on W. The most general euclidean Lie 3-algebra is an orthogonal direct sum $W = \mathfrak{a} \oplus \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_m$, where $\mathfrak{a}$ is an n-dimensional abelian Lie 3-algebra and the $\mathfrak{s}_i$ are m copies of the simple Lie 3-algebra with 3-brackets given by (11). The inner product is such that the above direct sums are orthogonal, and the inner products on each of the factors is positive-definite.

The *Lie algebra* structure on W is such that its adjoint representation preserves both the 3-brackets and the inner product, whence $\operatorname{ad} W$ is contained in $\mathfrak{so}(\mathfrak{a}) \oplus \mathfrak{so}(\mathfrak{s}_1) \oplus \cdots \oplus \mathfrak{so}(\mathfrak{s}_m)$. Indeed, for any $x \in W$, ad_x preserves the Lie 3-bracket, whence also the "volume" forms on each of the simple factors. In turn this means that ad_x preserves the subspaces $\mathfrak{s}$ themselves. To see this, let $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4)$ be a basis for one of the simple factors, say $\mathfrak{s}_1$, and let $\boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_4$ be the corresponding volume form. Invariance under ad_x means

$$[x, \boldsymbol{e}_1] \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_4 + \boldsymbol{e}_1 \wedge [x, \boldsymbol{e}_2] \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_4 + \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge [x, \boldsymbol{e}_3] \wedge \boldsymbol{e}_4 + \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge [x, \boldsymbol{e}_4] = 0 \ .$$

Now by invariance of the inner product, $[x, \boldsymbol{e}_i] \perp \boldsymbol{e}_i$, whence we may write it as $[x, \boldsymbol{e}_i] = y_i + z_i$, where $y_i \in \mathfrak{s}_1 \cap \boldsymbol{e}_i^\perp$ and $z_i \in \mathfrak{s}_1^\perp$. Back into the above equation,

$$z_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_4 + \boldsymbol{e}_1 \wedge z_2 \wedge \boldsymbol{e}_3 \wedge \boldsymbol{e}_4 + \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge z_3 \wedge \boldsymbol{e}_4 + \boldsymbol{e}_1 \wedge \boldsymbol{e}_2 \wedge \boldsymbol{e}_3 \wedge z_4 = 0 \ .$$

Each of the above four terms is linearly independent, whence $z_i = 0$ and ad_x indeed preserves $\mathfrak{s}_1$. This means that each simple factor is a submodule of the adjoint representation and, hence that so is their direct sum. Finally, by invariance of the inner product, so is its perpendicular complement $\mathfrak{a}$. In other words, the adjoint representation is contained in $\mathfrak{so}(\mathfrak{a}) \oplus \mathfrak{so}(\mathfrak{s}_1) \oplus \cdots \oplus \mathfrak{so}(\mathfrak{s}_m)$.

This decomposition of the adjoint representation now implies a decomposition of the Lie algebra itself as $W = \mathfrak{g} \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_m$, where $\mathfrak{g}$ is an n-dimensional euclidean Lie algebra (i.e., with $\operatorname{ad} \mathfrak{g} < \mathfrak{so}(\mathfrak{a})$) and each $\mathfrak{h}_i$ is a four-dimensional euclidean Lie algebra (i.e., $\operatorname{ad} \mathfrak{h}_i < \mathfrak{so}(\mathfrak{s})$). Indeed, if x and y belong to different orthogonal summands of the vector space W, then $[x, y]$ belongs to the same summand as y when understood as $\operatorname{ad}_x(y)$

and to the same summand as x when understood as $\mathrm{ad}_y(x)$. Since these summands are orthogonal, $[x,y]=0$.

Now euclidean Lie algebras are reductive; that is, a direct sum of a compact semisimple Lie algebra and an abelian Lie algebra. By inspection there are precisely two isomorphism classes of four-dimensional euclidean Lie algebras: the abelian 4-dimensional Lie algebra $\mathbb{R}^4$ and $\mathfrak{so}(3)\oplus\mathbb{R}$. Hence $\mathfrak{so}(\mathfrak{s})$ has to be isomorphic to one of those.

We will now show that every $\mathfrak{s}$ summand in W factorises in V, contradicting the assumption that V is indecomposable.

Consider one such $\mathfrak{s}$ summand, say $\mathfrak{s}_1$. The corresponding Lie algebra $\mathfrak{h}_1$ is either abelian or isomorphic to $\mathfrak{so}(3)\oplus\mathbb{R}$. If $\mathfrak{h}_1$ is abelian, so that the structure constants vanish, then for any $x\in\mathfrak{s}_1$, $[u,x,V]=0$ and $[x,y,V]=0$ for any $y\in W$ perpendicular to $\mathfrak{s}_1$. Hence $\mathfrak{s}_1\lhd V$ is a nondegenerate ideal, contradicting the indecomposability of V.

If $\mathfrak{h}_1\cong\mathfrak{so}(3)\oplus\mathbb{R}$, its adjoint algebra $\mathrm{ad}\,\mathfrak{h}_1$ is an $\mathfrak{so}(3)$ subalgebra of $\mathfrak{so}(\mathfrak{s}_1)\cong\mathfrak{so}(4)$, which therefore leaves a line $\ell\subset\mathfrak{s}_1$ invariant. The Lie algebra structure on $\mathfrak{h}_1$ thus coincides with that given by the Lie bracket $[-,-]_x=[x,-,-]_W$, for some $x\in\ell$, induced from the Lie 3-algebra structure on $\mathfrak{s}_1$. In other words, $[u,y,z]=[y,z]_x=[x,y,z]_W$ for all $y,z\in W$. This allows us to "twist" $\mathfrak{s}_1$ into a nondegenerate ideal of V. Indeed, define now

$$u'=u-x-\tfrac{1}{2}|x|^2v \qquad\text{and}\qquad y'=y+\langle y,x\rangle\,v\ , \tag{12}$$

for all $y\in\mathfrak{s}_1$. Then $[u',y',z']=0$ for all $y,z\in\mathfrak{s}_1$, and, using that v is central,

$$\begin{aligned}[y',z',w']=[y,z,w]&=-\langle [y,z],w\rangle\,v+[y,z,w]_W\\ &=-\langle [x,y,z]_W,w\rangle\,v+[y,z,w]_W\\ &=\langle [y,z,w]_W,x\rangle\,v+[y,z,w]_W\\ &=[y,z,w]_W'\ .\end{aligned}$$

Moreover, for every $y\in\mathfrak{s}_1$,

$$\langle u',y'\rangle=\left\langle u-x-\tfrac{1}{2}|x|^2v,y+\langle x,y\rangle\,v\right\rangle=\langle x,y\rangle\,\langle u,v\rangle-\langle x,y\rangle=0\ ,$$

and finally

$$\langle u',u'\rangle=\left\langle u-x-\tfrac{1}{2}|x|^2v,u-x-\tfrac{1}{2}|x|^2v\right\rangle=-|x|^2\,\langle u,v\rangle+\langle x,x\rangle=0\ .$$

In other words, the subspace of V spanned by the y' for $y\in\mathfrak{s}_1$ is a nondegenerate ideal of V, contradicting again the fact that V is indecomposable.

Consequently there can be no $\mathfrak{s}$'s in W, whence as a Lie 3-algebra, W is abelian. As a Lie algebra it is euclidean, whence reductive. However the abelian summand commutes with u, hence it is central in V, again contradicting the fact that it is indecomposable. Therefore as a Lie algebra W is compact semisimple.

In summary, we have proved the following

Theorem 9. *Let (V,Φ,b) be an indecomposable lorentzian Lie 3-algebra. Then it is either one-dimensional, simple, or else there is a Witt basis (u,v,x_a), with u,v complementary null directions, such that the nonzero 3-brackets take the form*

$$[u,x_a,x_b]=f_{ab}{}^c x_c \qquad\textit{and}\qquad [x_a,x_b,x_c]=-f_{abc}v\ ,$$

where $[x_a, x_b] = f_{ab}{}^c x_c$ *makes the span of the* (x_a) *into a compact semisimple Lie algebra and* $f_{abc} = \langle [x_a, x_b], x_c\rangle$.

These latter Lie 3-algebras have been discovered independently in [23–25], albeit in some cases in a slightly different form. It should be remarked that they provide explicit counterexamples to the lorentzian conjecture of [37], simply by taking the semisimple Lie algebra to be anything but a direct product of $\mathfrak{so}(3)$'s. Since the main focus in [37] was on middle-dimensional forms in low dimension, such examples did not arise.

Paraphrasing the theorem, the class of indecomposable lorentzian Lie 3-algebras are in one-to-one correspondence with the class of euclidean metric semisimple Lie algebras, by which we mean a compact semisimple Lie algebra *and* a choice of invariant inner product. This choice involves a choice of scale for each simple factor.

A final remark is that the classification of indecomposable lorentzian Lie 3-algebras is analogous to the classification of indecomposable lorentzian Lie algebras, which as shown in [42] (see also [43, §2.3]) are either one-dimensional, simple, or obtained as a double extension [39–41] of an abelian euclidean Lie algebra $\mathfrak{g}$ by a one-dimensional Lie algebra acting on $\mathfrak{g}$ via a skew-symmetric endomorphism. In the Lie 3-algebra case, we have an analogous result, with the action of the endomorphism being replaced by a semisimple Lie algebra.

4. The Bagger–Lambert moduli space

4.1. **Basic definitions.** The space of (maximally supersymmetric) **classical vacua** of the Bagger–Lambert theory associated to a Lie 3-algebra V is defined as follows:

$$\mathscr{V} = \left\{\phi \in \operatorname{Hom}(\mathbb{R}^8, V) \middle| [\phi(\boldsymbol{x}), \phi(\boldsymbol{y}), \phi(\boldsymbol{z})] = 0 \ \forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^8\right\} .$$

In other words, a linear map $\phi : \mathbb{R}^8 \to V$ belongs to $\mathscr{V}$ if and only if its image lies in an abelian subalgebra of V. This assumption guarantees that all the Bagger–Lambert supersymmetry transformations vanish if one sets the gauge field and fermions to zero and the scalars equal to the constants ϕ. If $A < V$ is an abelian subalgebra, then let us define $\mathscr{V}_A := \operatorname{Hom}(\mathbb{R}^8, A)$, whence

$$\mathscr{V} = \bigcup_{\substack{A<V\\ \text{abelian}}} \mathscr{V}_A .$$

If A, B are abelian subalgebras of V with $A < B$, then $\mathscr{V}_A \subset \mathscr{V}_B$, whence we can write $\mathscr{V}$ as a union

$$\mathscr{V} = \bigcup_{\substack{A<V\\ \text{maximal abelian}}} \mathscr{V}_A$$

of maximal subspaces, in the sense that no two subspaces appearing in the above sum are contained in one another. We see that $\mathscr{V}$ is therefore given by the set union of linear subspaces in $\operatorname{Hom}(\mathbb{R}^8, V)$, parametrised by the set of maximal abelian subalgebras of V. In other words, a maximal abelian subalgebra of V determines a "branch" of the classical space of vacua. Some of these branches will be gauge-related, hence the need to quotient by gauge transformations. Since we have chosen a gauge in which the gauge field vanishes, we are only allowed to quotient by gauge transformations which preserve this choice of gauge. In particular they are constant, whence they define an (invariant) subgroup of the automorphisms of the Lie 3-algebra.

We define the **automorphism group** $\operatorname{Aut} V$ of the Lie 3-algebra V to be the subgroup of $\mathrm{GL}(V)$ which preserves the 3-bracket; that is,

$$\operatorname{Aut} V = \{g \in \mathrm{GL}(V) | g[x,y,z] = [gx, gy, gz],\ \forall x,y,z \in V\} \ .$$

Its Lie algebra $\operatorname{Der} V$ is the Lie subalgebra of $\mathfrak{gl}(V)$ consisting of derivations of the 3-bracket. Let $\operatorname{ad} V = \{\mathrm{ad}_{x,y} | x,y \in V\}$ denote the Lie subalgebra of $\operatorname{Der} V$ consisting of inner derivations. It is the Lie algebra of a normal subgroup $\operatorname{Ad} V \lhd \operatorname{Aut} V$, which we will call the group of **inner automorphisms**; although we should keep in mind that the nomenclature is somewhat misleading in the absence of a notion of Lie 3-group. The group $\operatorname{Ad} V$ is the closest thing one has to a gauge group in the Bagger–Lambert theory.

$\operatorname{Aut} V$, and hence $\operatorname{Ad} V$, act on $\operatorname{Hom}(\mathbb{R}^8, V)$ by ignoring the $\mathbb{R}^8$ and acting on V via the defining representation. In other words, if $g \in \operatorname{Aut} V$ and $\phi : \mathbb{R}^8 \to V$, then $(g \cdot \phi)(\boldsymbol{x}) = g\phi(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathbb{R}^8$. It is clear that if $\phi \in \mathscr{V}$, then $g \cdot \phi \in \mathscr{V}$, whence we may define the (classical) **moduli space** of the Bagger–Lambert theory associated with V as the gauge-equivalence classes of vacuum configurations in $\mathscr{V}$, where two vacuum configurations are gauge-equivalent if they related by the action of $\operatorname{Ad} V$.

4.2. **Automorphisms.** We will now study the automorphisms $\operatorname{Aut} V$ of the indecomposable lorentzian Lie 3-algebra V built out of a euclidean semisimple Lie algebra $\mathfrak{g}$ as described in Theorem 9, paying close attention to the inner automorphisms $\operatorname{Ad} V$.

As seen above, $V = \mathbb{R}u \oplus \mathbb{R}v \oplus \mathfrak{g}$ as a vector space, with inner product given by extending the inner product on $\mathfrak{g}$ to V in such a way that u, v are orthogonal to $\mathfrak{g}$ and obey $\langle u,u\rangle = 0 = \langle v,v\rangle$ and $\langle u,v\rangle = 1$, and where the nonzero 3-brackets are given by

$$[u,x,y] = [x,y] \qquad \text{and} \qquad [x,y,z] = -\langle [x,y], z\rangle\, v \ ,$$

for all $x,y,z \in \mathfrak{g}$.

Let $\varphi \in \operatorname{Aut} V$. It follows that φ preserves the centre $Z \lhd V$. Indeed, if $z \in Z$, then for all $x,y \in V$, $[\varphi(z), \varphi(x), \varphi(y)] = \varphi[z,x,y] = 0$. This means that $[\varphi(z), \varphi(V), \varphi(V)] = 0$, but $\varphi(V) = V$ since φ is vector space isomorphism, whence $\varphi(z) \in Z$. Similarly, φ preserves the derived ideal $[V,V,V]$. Indeed, $\varphi[V,V,V] = [\varphi(V), \varphi(V), \varphi(V)] = [V,V,V]$. This means that φ must take the following form:

$$\begin{aligned}
\varphi(v) &= \alpha v \\
\varphi(u) &= \beta u + \gamma v + t \\
\varphi(x) &= f(x) + \langle w, x\rangle\, v \ ,
\end{aligned}$$

for all $x \in \mathfrak{g}$ and where $\alpha, \beta, \gamma \in \mathbb{R}$, $w, t \in \mathfrak{g}$ and $f : \mathfrak{g} \to \mathfrak{g}$. Invertibility of φ forces α, β to be nonzero and f to be invertible.

We will now determine the most general φ preserving the 3-brackets. For all $x,y,z \in \mathfrak{g}$,

$$\varphi[x,y,z] = -\langle [x,y], z\rangle\, \varphi(v) = -\alpha \langle [x,y], z\rangle\, v \ ,$$

but also

$$\varphi[x,y,z] = [\varphi(x),\varphi(y),\varphi(z)] = [f(x),f(y),f(z)] = -\langle [f(x),f(y)],f(z)\rangle\, v\ ,$$

whence we arrive at

$$\langle [f(x),f(y)],f(z)\rangle = \alpha\,\langle [x,y],z\rangle\ . \tag{13}$$

For all $x,y \in \mathfrak{g}$,

$$\varphi[u,x,y] = \varphi[x,y] = f[x,y] + \langle w,[x,y]\rangle\, v\ ,$$

but also

$$\varphi[u,x,y] = [\varphi(u),\varphi(x),\varphi(y)] = [\beta u + t, f(x), f(y)] = \beta[f(x),f(y)] - \langle t,[f(x),f(y)]\rangle\, v\ ,$$

which yields two equations

$$\beta[f(x),f(y)] = f[x,y]\ , \tag{14}$$

and

$$\langle w,[x,y]\rangle = -\langle t,[f(x),f(y)]\rangle\ . \tag{15}$$

Multiplying equation (13) by β (which is nonzero) and using equation (14), we arrive at

$$\langle f[x,y],f(z)\rangle = \alpha\beta\,\langle [x,y],z\rangle\ ,$$

since $\mathfrak{g}$ is semisimple, $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, whence this is equivalent to

$$\langle f(x),f(y)\rangle = \alpha\beta\,\langle x,y\rangle\ , \tag{16}$$

for all $x,y \in \mathfrak{g}$. Now we multiply equation (15) by β and again use equation (14) to obtain

$$\beta\,\langle w,[x,y]\rangle = -\langle t, f[x,y]\rangle = -\langle f^*t,[x,y]\rangle\ ,$$

and again using that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$, we see that $w = -\beta^{-1}f^*t$, where f^* is the adjoint of f relative to the inner product.

Let us define

$$\operatorname{Aut}_\lambda \mathfrak{g} = \{f : \mathfrak{g} \to \mathfrak{g} \text{ invertible } | \lambda[f(x),f(y)] = f[x,y]\}\ .$$

For $\lambda = 1$ we have the automorphism group of $\mathfrak{g}$. The proof of the following lemma is routine.

Lemma 10. *If $f \in \operatorname{Aut}_\lambda \mathfrak{g}$ then $f^{-1} \in \operatorname{Aut}_{1/\lambda} \mathfrak{g}$. If in addition $g \in \operatorname{Aut}_\mu \mathfrak{g}$ then $f \circ g \in \operatorname{Aut}_{\lambda\mu} \mathfrak{g}$.*

Equation (14) says that $f \in \operatorname{Aut}_\beta \mathfrak{g}$. Now if $f_1, f_2 \in \operatorname{Aut}_\beta \mathfrak{g}$, the lemma says that $f_1^{-1} \circ f_2 \in \operatorname{Aut} \mathfrak{g}$, whence any two elements of $\operatorname{Aut}_\beta \mathfrak{g}$ are related by composition with an automorphism. Now the map $x \mapsto \beta^{-1}x$ is invertible and belongs to $\operatorname{Aut}_\beta \mathfrak{g}$. Hence the most general solution to equation (14) is given by $f(x) = \beta^{-1}a(x)$ for some Lie algebra automorphism $a \in \operatorname{Aut} \mathfrak{g}$. Substituting this into equation (16), we find

$$\langle a(x),a(y)\rangle = \alpha\beta^3\,\langle x,y\rangle\ . \tag{17}$$

Since the inner product is positive definite, this means that $\alpha\beta^3 > 0$.

Now decompose $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_N$ into simple factors. On each of the factors, the inner product is a (negative) multiple of the Killing form, which is preserved by automorphisms. Therefore if $a \in \operatorname{Aut} \mathfrak{g}$ preserves each of the factors (this is the case, e.g., if no two factors are isomorphic) then it preserves the inner product and we see that $\alpha\beta^3 = 1$. If a does not preserve the factors, it permutes them as well as acting by automorphisms of each of the factors. However, since N is finite, some power of a will again preserve the factors and hence some power of $\alpha\beta^3$ must be equal to 1, but since $\alpha\beta^3 > 0$ we again conclude that $\alpha\beta^3 = 1$, whence equation (17) says that a is an isometry. Let us denote by $\operatorname{Aut}^0 \mathfrak{g}$ the subgroup of automorphisms which are also isometries. In summary, we have proved the following

Proposition 11. *Every 3-algebra automorphism $\varphi \in \operatorname{Aut} V$ is given by*

$$\begin{aligned}\varphi(v) &= \beta^{-3} v\\ \varphi(u) &= \beta u + \gamma v + t\\ \varphi(x) &= \beta^{-1} a(x) - \beta^{-2}\langle t, a(x)\rangle\, v\ ,\end{aligned}$$

for all $x \in \mathfrak{g}$ and where $\beta \in \mathbb{R}^\times$, $\gamma \in \mathbb{R}$, $t \in \mathfrak{g}$ and $a \in \operatorname{Aut}^0 \mathfrak{g}$.

Let $\operatorname{Aut}^0 V$ denote the subgroup of 3-algebra automorphisms of V which also preserve the inner product. It is easy to determine such automorphisms.

Proposition 12. *Every 3-algebra automorphism $\varphi \in \operatorname{Aut}^0 V$ preserving the inner product is given by*

$$\begin{aligned}\varphi(v) &= v\\ \varphi(u) &= u - \tfrac{1}{2}|t|^2 v + t\\ \varphi(x) &= a(x) - \langle t, a(x)\rangle\, v\ ,\end{aligned}$$

for all $x \in \mathfrak{g}$ and where $t \in \mathfrak{g}$ and $a \in \operatorname{Aut}^0 \mathfrak{g}$.

Proof. The condition $\langle \varphi(u), \varphi(v)\rangle = 1$ fixes $\beta = 1$. The condition $\langle \varphi(u), \varphi(u)\rangle = 0$ fixes $\gamma = -\frac{1}{2}|t|^2$. The rest of the conditions are satisfied identically. □

The automorphism generated by β (that does not preserve the inner product) can be identified with the transformation used in [24, 25] to fix the value of the coupling constant in the Bagger–Lambert theory.

As we will now show, the connected component of $\operatorname{Aut}^0 V$ consists of the inner automorphisms obtained by exponentiating the inner derivations of the Lie 3-algebra V. Indeed, the inner derivations are thus given by $\operatorname{ad}_{u,x} = \operatorname{ad}_x$, $\operatorname{ad}_{x,y} = [x,y] \otimes v^\flat - v \otimes [x,y]^\flat$ for all $x, y \in \mathfrak{g}$, with $\flat : V \to V^*$ denoting the musical isomorphism induced by the inner product; that is, $x^\flat(y) = \langle x, y\rangle$. Since $\operatorname{ad}_{x,y}$ only depends on x, y via their Lie bracket, and since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, we see that the image of the $\operatorname{ad}_{x,y}$ is the abelian subalgebra of $\mathfrak{gl}(V)$ given by

$$\mathfrak{g}_{\mathrm{ab}} := \left\{ t \otimes v^\flat - v \otimes t^\flat \middle| t \in \mathfrak{g} \right\}\ .$$

Similarly, the image of the $\operatorname{ad}_{u,x}$ is the adjoint Lie algebra $\operatorname{ad}\mathfrak{g}$ of $\mathfrak{g}$, and it is clear that $\operatorname{ad}\mathfrak{g}$ acts on $\mathfrak{g}_{\mathrm{ab}}$ by restricting the defining representation of $\mathfrak{gl}(\mathfrak{g})$. In other words, the inner derivations of V span a Lie algebra

$$\operatorname{ad} V \cong \mathfrak{g}_{\mathrm{ab}} \rtimes \operatorname{ad}\mathfrak{g}\ .$$

As a subalgebra of $\mathfrak{so}(V)$, this is contained in the stabiliser of the null vector v, with $\mathfrak{g}_{\mathrm{ab}}$ acting as null rotations and $\operatorname{ad}\mathfrak{g}$ as transverse rotations. Exponentiating $\operatorname{ad} V$, we obtain the group $\operatorname{Ad} V = \mathfrak{g}_{\mathrm{ab}} \rtimes \operatorname{Ad} G$, which is an affinisation of the adjoint group. Indeed, exponentiating $\operatorname{ad}\mathfrak{g}$ we obtain $\operatorname{Ad} G$, whereas exponentiating an element of the form $t \otimes v^\flat - v \otimes t^\flat$, we obtain

$$1 + t \otimes v^\flat - v \otimes t^\flat - \tfrac{1}{2}|t|^2 v \otimes v^\flat \ .$$

In summary, elements $\psi \in \operatorname{Ad} V$ are parametrised by $a \in \operatorname{Ad} G$ and $t \in \mathfrak{g}$ and act by

$$\begin{aligned} \psi(v) &= v \\ \psi(u) &= u + t - \tfrac{1}{2}|t|^2 v \\ \psi(x) &= a(x) - \langle t, a(x)\rangle\, v \ , \end{aligned} \tag{18}$$

whence $\operatorname{Ad} V$ is precisely the connected component of the identity of $\operatorname{Aut}^0 V$, as claimed.

The Lie algebra $\operatorname{Der} V$ of $\operatorname{Aut} V$ consists of derivations of V. It is isomorphic to the real Lie algebra with generators D, S, L_x and T_x for $x \in \mathfrak{g}$, subject to the following nonzero Lie brackets:

$$[D, S] = -4S \ , \qquad [D, T_x] = -2T_x \ , \qquad [L_x, L_y] = L_{[x,y]} \qquad \text{and} \qquad [L_x, T_y] = T_{[x,y]} \ .$$

If we let $\mathfrak{a}$ denote the two-dimensional solvable Lie subalgebra spanned by D and S, then we find that $\operatorname{Der} V$ has the following structure

$$\operatorname{Der} V \cong \mathfrak{a} \ltimes \operatorname{ad} V \ .$$

4.3. **Maximal abelian subalgebras.** We now determine the maximal abelian subalgebras of V. Every maximal abelian subalgebra contains the centre $Z = \mathbb{R}v$. Let $A < V$ be a maximal abelian subalgebra. Then the restriction of the inner product to A is either degenerate or nondegenerate. If degenerate, it means that there can be no element of A of the form $u + \cdots$, whereas if it is nondegenerate, there is such an element, which can be taken to have the form $u + z$, for some $z \in \mathfrak{g}$.

In the nondegenerate case, $A = \mathbb{R}(u + z) \oplus \mathbb{R}v \oplus B$, where $B \subset \mathfrak{g}$ is a subspace obeying $[x, y] = 0$ for all $x, y \in B$. In other words, B is an abelian Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{g}$ is compact, maximal abelian Lie subalgebras coincide with the Cartan subalgebras. Hence nondegenerate maximal abelian subalgebras of V are of the form $A = \mathbb{R}(u + z) \oplus \mathbb{R}v \oplus \mathfrak{h}$, for some $z \in \mathfrak{g}$ and some Cartan subalgebra $\mathfrak{h} < \mathfrak{g}$.

The degenerate case is slightly more involved. Here $A = \mathbb{R}v \oplus \mathfrak{p}$, where $\mathfrak{p} \subset \mathfrak{g}$ is a subspace of $\mathfrak{g}$ on which the three-form $\Omega(x, y, z) = \langle [x, y], z\rangle$ vanishes identically. We call such subspaces **Ω-isotropic**. An equivalent condition for a subspace $\mathfrak{p}$ to be Ω-isotropic is that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p}^\perp$, whence the Ω-isotropic Lie subalgebras are necessarily abelian. The maximal Ω-isotropic subalgebras are therefore the Cartan subalgebras. However there is no need for $\mathfrak{p}$ to be a Lie subalgebra: it is A which has to be an abelian (3-)subalgebra of V and this only requires $\mathfrak{p}$ to be a subspace. We say that an Ω-isotropic subspace is **maximal**, if it is not properly contained in any Ω-isotropic subspace. The following is a useful characterisation of maximality.

Lemma 13. *An Ω-isotropic subspace $\mathfrak{p} \subset \mathfrak{g}$ is maximal if and only if $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}^\perp$.*

Proof. Let $\mathfrak{p} \subset \mathfrak{g}$ be an Ω-isotropic subspace properly contained in another Ω-isotropic subspace. Then there is some $x \in \mathfrak{p}^\perp$ such that $\hat{\mathfrak{p}} = \mathfrak{p} \oplus \mathbb{R}x$ is Ω-isotropic. This condition is equivalent to $x \in [\mathfrak{p},\mathfrak{p}]^\perp$ or dually that $[\mathfrak{p},\mathfrak{p}] \subset x^\perp$. In other words, $[\mathfrak{p},\mathfrak{p}] \subsetneq \mathfrak{p}^\perp$. ☐

A large class of maximally Ω-isotropic subspaces are in one-to-one correspondence with the compact riemannian symmetric spaces. Indeed, let $\mathfrak{k} \subset \mathfrak{g}$ be a Lie subalgebra and consider $\mathfrak{p} = \mathfrak{k}^\perp$. Since $\mathfrak{k}$ preserves the inner product, $\mathfrak{p}$ is stable under the adjoint action of $\mathfrak{k}$; that is, $[\mathfrak{k},\mathfrak{p}] \subset \mathfrak{p}$, whence the split $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is reductive. The split will be symmetric, so that $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$ precisely when $\mathfrak{p}$ is Ω-isotropic. Indeed, Ω is essentially the torsion of the canonical connection on G/K and precisely when G/K is a symmetric space, this connection agrees with the Levi-Cività connection, which is torsionless.

Type	$\mathfrak{g}$	$\mathfrak{k}$	$\dim \mathfrak{g}$	$\dim \mathfrak{p}$	rank
AI	$\mathfrak{su}(n)$	$\mathfrak{so}(n)$	n^2-1	$\frac{1}{2}(n-1)(n+2)$	$n-1$
AII	$\mathfrak{su}(2n)$	$\mathfrak{sp}(n)$	$4n^2-1$	$(n-1)(2n+1)$	$n-1$
AIII	$\mathfrak{su}(p+q)$	$\mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathbb{R}$	$(p+q)^2-1$	$2pq$	$\min(p,q)$
BDI	$\mathfrak{so}(p+q)$	$\mathfrak{so}(p) \oplus \mathfrak{so}(q)$	$\frac{1}{2}(p+q)(p+q-1)$	pq	$\min(p,q)$
DIII	$\mathfrak{so}(2n)$	$\mathfrak{u}(n)$	$n(2n-1)$	$n(n-1)$	$\lfloor n/2 \rfloor$
CI	$\mathfrak{sp}(n)$	$\mathfrak{u}(n)$	$n(2n+1)$	$n(n+1)$	n
CII	$\mathfrak{sp}(p+q)$	$\mathfrak{sp}(p) \oplus \mathfrak{sp}(q)$	$(p+q)(2p+2q+1)$	$4pq$	$\min(p,q)$
EI	$\mathfrak{e}_6$	$\mathfrak{sp}(4)$	78	42	6
EII	$\mathfrak{e}_6$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	78	40	4
EIII	$\mathfrak{e}_6$	$\mathfrak{so}(10) \oplus \mathfrak{so}(2)$	78	32	2
EIV	$\mathfrak{e}_6$	$\mathfrak{f}_4$	78	26	2
EV	$\mathfrak{e}_7$	$\mathfrak{su}(8)$	133	70	7
EVI	$\mathfrak{e}_7$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	133	64	4
EVII	$\mathfrak{e}_7$	$\mathfrak{e}_6 \oplus \mathfrak{so}(2)$	133	54	3
EVIII	$\mathfrak{e}_8$	$\mathfrak{so}(16)$	248	128	8
EIX	$\mathfrak{e}_8$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$	248	112	4
FI	$\mathfrak{f}_4$	$\mathfrak{sp}(3) \oplus \mathfrak{su}(2)$	52	28	4
FII	$\mathfrak{f}_4$	$\mathfrak{so}(9)$	52	16	1
G	$\mathfrak{g}_2$	$\mathfrak{so}(4)$	14	8	2

TABLE 1. Symmetric splits $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of compact simple Lie algebras

There are two types of compact irreducible riemannian symmetric spaces:

- **Type I**: G/K where G is a compact simple Lie group and K a subgroup with Lie algebra $\mathfrak{k} < \mathfrak{g}$ such that orthogonal decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a symmetric split; and
- **Type II**: compact simple Lie groups H relative to a bi-invariant metric. This can be written in terms of a symmetric split, with $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$, with the ad-invariant inner product on both copies of $\mathfrak{h}$ being the same, and $\mathfrak{k} = \{(x,x) | x \in \mathfrak{h}\}$ the diagonal

Lie subalgebra. Its perpendicular complement is $\mathfrak{p} = \{(x, -x)|x \in \mathfrak{h}\}$. It is easy to see that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, whence by Lemma 13, $\mathfrak{p}$ is maximal.

Table 1 lists the type I irreducible riemannian symmetric spaces in terms of their symmetric splits $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ using the Cartan nomenclature as described in [48]. There are some repetitions in the table, which can be eliminated by taking $p \geq q$ in AIII, BDI and CII; $n \geq 2$ in AI, AII and CI; $n \geq 5$ in DIII; $p \geq 2$ in AIII; and taking $p + q \geq 7$ in BDI in addition to $p = q = 1$.

Proposition 14. *The Ω-isotropic subspaces $\mathfrak{p} = \mathfrak{k}^\perp$ in Table 1 are maximal.*

Proof. We observe that the Jacobi identity says that $[\mathfrak{p}, \mathfrak{p}]$ is an ideal of $\mathfrak{k}$. Now many of the $\mathfrak{k}$ in Table 1 are simple, whence $[\mathfrak{p}, \mathfrak{p}]$, being nonzero, must be all of $\mathfrak{k}$ and by Lemma 13, $\mathfrak{p}$ is maximal. For the remaining entries but one, $\mathfrak{k} = \mathfrak{k}_1 \oplus \mathfrak{k}_2$, with $\mathfrak{k}_i$ a simple or one-dimensional ideal. Then $\mathfrak{p}$, if not maximal, must satisfy $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}_i$ for some $i = 1, 2$. Let's assume, without loss of generality, that $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}_1$. Then $\langle [\mathfrak{p}, \mathfrak{p}], \mathfrak{k}_2 \rangle = 0$, but this means that $\langle [\mathfrak{k}_2, \mathfrak{p}], \mathfrak{p} \rangle = 0$, whence $[\mathfrak{k}_2, \mathfrak{p}] = 0$, since it belongs to $\mathfrak{p}$. Since $[\mathfrak{k}_1, \mathfrak{k}_2] = 0$, $\mathfrak{k}_2$ would be an ideal of $\mathfrak{g}$, contradicting the fact that $\mathfrak{g}$ is simple. Therefore $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$ and again by Lemma 13 it is maximal. Finally, in case AIII, $\mathfrak{k} = \mathfrak{su}(p) \oplus \mathfrak{su}(q) \oplus \mathbb{R}$. In the same as when $\mathfrak{k}$ is the sum of two ideals, one shows that $[\mathfrak{p}, \mathfrak{p}]^\perp \cap \mathfrak{k}$ is an ideal of $\mathfrak{g}$. Since it cannot be all of $\mathfrak{g}$, simplicity of $\mathfrak{g}$ says that it must be zero. ☐

We have not been able to construct any maximal Ω-isotropic subspace $\mathfrak{p} \subset \mathfrak{g}$ which does not come from a symmetric split, but neither have we been able to prove that they all arise in this way; although it would be tempting to conjecture that this is the case.

4.4. **The moduli spaces.** We now quotient by inner automorphisms to arrive at the moduli spaces.

The classical space of vacua has two main branches, corresponding to degenerate and nondegenerate maximal abelian subalgebras $A < V$. The degenerate branch splits further into sub-branches labelled by the different types of maximal Ω-isotropic subspaces, many of which are given by (not necessarily irreducible) compact symmetric spaces.

4.4.1. *Moduli of nondegenerate maximal abelian subalgebras.* Let us first consider the branch of the moduli space corresponding to nondegenerate maximal abelian subalgebras of V of the form $A(\mathfrak{h}, z) = \mathbb{R}(u + z) \oplus \mathbb{R}v \oplus \mathfrak{h}$ for some $z \in \mathfrak{g}$ and some Cartan subalgebra $\mathfrak{h} < \mathfrak{g}$. Such maximal abelian subalgebras are parametrised by $\mathrm{Cartan}(\mathfrak{g}) \times \mathfrak{g}$, where $\mathrm{Cartan}(\mathfrak{g})$ is the space of Cartan subalgebras of $\mathfrak{g}$, which we can think of as a submanifold of the grassmannian of rank $\mathfrak{g}$-planes in $\mathfrak{g}$, isometric to G/H, where H is the maximal torus of a fixed Cartan subalgebra.

The subgroup $\mathrm{Ad}\, V < \mathrm{GL}(V)$ acts on V and hence on the $A(\mathfrak{h}, z)$. Let $\psi = \psi(t, a) \in \mathrm{Ad}\, V$ with $t \in \mathfrak{g}$ and $a \in \mathrm{Ad}\, G$. Then from (18) we see that, for all $x \in \mathfrak{g}$,

$$\begin{aligned} \psi(v) &= v \\ \psi(u + z) &= u + az + t - \left(\langle t, az \rangle + \tfrac{1}{2}|t|^2\right) v \\ \psi(x) &= ax - \langle t, ax \rangle\, v\ . \end{aligned}$$

In other words, ψ maps the subspace $A(\mathfrak{h}, z)$ to $A(a\mathfrak{h}, az+t)$. Now fix a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}$ and let $A_0 = A(\mathfrak{h}_0, 0)$. Any other Cartan subalgebra of $\mathfrak{g}$ can be obtained from $\mathfrak{h}_0$ by acting with some $a \in \operatorname{Ad} G$. The translational component of $\operatorname{Ad} V$ allows us to shift z in $(\mathfrak{h}, z)$ to 0. In other words, given any $A(\mathfrak{h}, z)$, there is some $\psi \in \operatorname{Ad} V$ such that $A(\mathfrak{h}, z) = \psi A_0$.

The subset of classical vacua corresponding to such maximal subalgebras is given by

$$\begin{aligned} \mathscr{V}_{\text{nondeg}} &= \bigcup_{(\mathfrak{h},z)\in \operatorname{Cartan}(\mathfrak{g})\times\mathfrak{g}} \operatorname{Hom}(\mathbb{R}^8, A(\mathfrak{h}, z)) \\ &= \bigcup_{\psi\in \operatorname{Ad} V} \operatorname{Hom}(\mathbb{R}^8, \psi A_0) \\ &= \bigcup_{\psi\in \operatorname{Ad} V} \psi \operatorname{Hom}(\mathbb{R}^8, A_0)\ . \end{aligned}$$

In other words, it is the orbit of $\operatorname{Hom}(\mathbb{R}^8, A_0)$ under $\operatorname{Ad} V$. Quotienting by $\operatorname{Ad} V$ yields

$$\mathscr{M}_{\text{nondeg}} = \operatorname{Hom}(\mathbb{R}^8, A_0)/G_0$$

where G_0 is the stabiliser of A_0 (and hence of $\operatorname{Hom}(\mathbb{R}^8, A_0)$) in $\operatorname{Ad} V$. This quotient is not trivial because there are elements of $\operatorname{Ad} V$ which preserve A_0 as a subspace, but not A_0 pointwise. Indeed, the stabiliser of A_0 in $\operatorname{Ad} V$ is the same as the stabilizer of $(\mathfrak{h}_0, 0) \in \operatorname{Cartan}(\mathfrak{g}) \times \mathfrak{g}$. The translations move the origin, hence G_0 is a subgroup of $\operatorname{Ad} G$. In fact, it is $\operatorname{Ad} N(H_0)$, where $N(H_0)$ is the normaliser (in G) of the maximal torus H_0 corresponding to $\mathfrak{h}_0$. Clearly $H_0 < N(H_0)$ fixes every point of $\mathfrak{h}_0$, whence only the Weyl group $\mathfrak{W}_0 := N(H_0)/H_0$ of $\mathfrak{h}_0$ acts effectively.

In summary, the branch of the moduli space of classical vacua corresponding to nondegenerate maximal abelian subalgebras of V is given by

$$\mathscr{M}_{\text{nondeg}} = \operatorname{Hom}(\mathbb{R}^8, \mathbb{R}v \oplus \mathbb{R}u \oplus \mathfrak{h}_0)/\mathfrak{W}_0\ ,$$

where $\mathfrak{W}_0$ is the Weyl group of $\mathfrak{h}_0$. We see that $\dim \mathscr{M}_{\text{nondeg}} = 8(2 + \operatorname{rank} \mathfrak{g})$. In the next section we will study the asymptotics of this branch for large rank as $\mathfrak{g}$ varies among the compact semisimple Lie algebras.

4.4.2. *Moduli of degenerate maximal abelian subalgebras.* We now consider the branch of the moduli space corresponding to degenerate maximal abelian subalgebras of V of the form $A(\mathfrak{p}) = \mathbb{R}v \oplus \mathfrak{p}$, where $\mathfrak{p} \subset \mathfrak{g}$ is a subspace obeying $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{p}^\perp$. In other words, such subalgebras are parametrised by the space $\operatorname{Iso}_\Omega(\mathfrak{g})$ of maximal Ω-isotropic subspaces of $\mathfrak{g}$. The adjoint group $\operatorname{Ad} G$ preserves $\operatorname{Iso}_\Omega(\mathfrak{g})$ and decomposes it into orbits. Some of these orbits are in one-to-one correspondence with (isometry classes of) compact riemannian symmetric spaces. The group $\operatorname{Ad} V$ acts on the $A(\mathfrak{p})$ as follows. Let $\psi = \psi(a, t) \in \operatorname{Ad} V$ with $a \in \operatorname{Ad} G$ and $t \in \mathfrak{g}$. Then from (18) we see that $\psi(v) = v$ and, for $x \in \mathfrak{g}$, and $\psi(x) = ax - \langle t, ax\rangle\, v$, whence $\psi A(\mathfrak{p}) = A(a\mathfrak{p})$.

The subset of classical vacua corresponding to the $A(\mathfrak{p})$ is given by

$$\begin{aligned}\mathscr{V}_{\text{deg}} &= \bigcup_{\mathfrak{p}\in\operatorname{Iso}_\Omega(\mathfrak{g})} \operatorname{Hom}(\mathbb{R}^8, A(\mathfrak{p}))\\ &= \bigcup_{[\mathfrak{p}_0]\in\operatorname{Iso}_\Omega(\mathfrak{g})/\operatorname{Ad}G} \quad \bigcup_{\psi\in\operatorname{Ad}V} \operatorname{Hom}(\mathbb{R}^8, \psi A(\mathfrak{p}_0)) \ ,\end{aligned}$$

where $\mathfrak{p}_0$ stands for a representative subspace in the orbit $[\mathfrak{p}_0]$ of $\operatorname{Ad}G$ on $\operatorname{Iso}_\Omega(\mathfrak{g})$. A subset of orbits consists of isometry classes of compact riemannian symmetric spaces of the form G/K with $\mathfrak{p}_0 = \mathfrak{k}^\perp$.

Each orbit $[\mathfrak{p}_0]$ gives rise to a branch of the moduli space obtained by quotienting the $\operatorname{Ad}V$ orbit of $\operatorname{Hom}(\mathbb{R}^8, A(\mathfrak{p}_0)$ by $\operatorname{Ad}V$. As before, the result is

$$\mathscr{M}_{[\mathfrak{p}_0]} = \operatorname{Hom}(\mathbb{R}^8, A(\mathfrak{p}_0)/(\mathfrak{g}_{\text{ab}} \rtimes \operatorname{Ad}K_0) \ ,$$

where $\operatorname{Ad}K_0 < \operatorname{Ad}G$ is the stabiliser of $\mathfrak{p}_0$.

Every (isometry class of) compact riemannian symmetric space gives rise to one such $\mathscr{M}_{[\mathfrak{p}_0]}$ with $\dim\mathfrak{p}_0$ being the dimension of the symmetric space. Those symmetric spaces which are products of irreducibles of type AI, BDI (with $p = q$), CI, EI, EV, EVIII, FI and G have maximal dimension for a given $\mathfrak{g}$: their dimension being $\frac{1}{2}(\dim\mathfrak{g} + \operatorname{rank}\mathfrak{g})$. They are characterised by the property that $\mathfrak{p}$ contains a Cartan subalgebra of $\mathfrak{g}$, or equivalently, that they are of maximal rank. At the other extreme, symmetric spaces which are products of spheres, complex and quaternionic projective spaces (i.e., types AIII, BDI and CII, all with $q = 1$) have smallest possible dimension compared with the dimension of $\mathfrak{g}$. Whereas in the former class the dimension of the moduli space of vacua grows like the square of the rank of $\mathfrak{g}$, in the latter the dimension of the moduli space grows linearly.

It would be very interesting to find a natural interpretation of these maximal Ω-isotropic subspaces in M-theory. Whereas the non-degenerate branch seems to capture the expected vacua of the super Yang-Mills description on a stack of D2-branes (perhaps with an orientifold plane) lifted to M-theory, the configuration space of M-branes that might mimic the degenerate branch is less clear to us.

4.5. **Asymptotic behaviour.** It is expected [1, 2] that for a theory of N coincident M2-branes, the number of physical degrees of freedom should grow as $N^{3/2}$ for large N. In the present context, the number N is the dimension of the moduli space of vacua, whereas the number of degrees of freedom is the dimension of the Lie 3-algebra V. For the non-degenerate branch we have seen that whereas $\dim V = \dim\mathfrak{g} + 2$, the dimension of the moduli space of vacua is $8(\operatorname{rank}\mathfrak{g} + 2)$. It is therefore natural to ask how $(\dim\mathfrak{g}, \operatorname{rank}\mathfrak{g})$ are distributed for semisimple Lie algebras.

Table 2 lists the ranks and dimensions of the simple Lie algebras, as well as information on the Weyl group which may help to interpret the classical vacua in terms of configurations of M2 branes. This data allows us to write the following partition function for compact semisimple Lie algebras:

Lie algebra	Rank	Dimension	Weyl group $\mathfrak{W}$	order of $\mathfrak{W}$
A_n	n	$n(n+2)$	S_{n+1}	$(n+1)!$
B_n	n	$n(2n+1)$	$(\mathbb{Z}_2)^n \rtimes S_n$	$2^n n!$
C_n	n	$n(2n+1)$	$(\mathbb{Z}_2)^n \rtimes S_n$	$2^n n!$
D_n	n	$n(2n-1)$	$(\mathbb{Z}_2)^{n-1} \rtimes S_n$	$2^{n-1} n!$
E_6	6	78		$2^7 3^4 5$
E_7	7	133		$2^{10} 3^4 5\ 7$
E_8	8	248		$2^{14} 3^5 5^2 7$
F_4	4	52	$S_3 \ltimes (S_4 \ltimes (\mathbb{Z}_2)^3)$	$2^7 3^2$
G_2	2	14	D_6	$2^2 3$

TABLE 2. The simple Lie algebras

$$Z_{\text{SSLA}}(t,\omega) = \prod_{n\geq 1} \frac{1}{1-\omega^n t^{n(n+2)}} \prod_{n\geq 2} \frac{1}{1-\omega^n t^{n(2n+1)}} \prod_{n\geq 3} \frac{1}{1-\omega^n t^{n(2n+1)}} \prod_{n\geq 4} \frac{1}{1-\omega^n t^{n(2n-1)}} \times \frac{1}{1-\omega^6 t^{78}} \frac{1}{1-\omega^7 t^{133}} \frac{1}{1-\omega^8 t^{248}} \frac{1}{1-\omega^4 t^{52}} \frac{1}{1-\omega^2 t^{14}} .$$

Expanding as a power series in t and ω, we have

$$Z_{\text{SSLA}}(t,\omega) = 1 + \sum_{d,r\geq 1} N_{d,r} t^d \omega^r ,$$

where $N_{d,r}$ is the number of d-dimensional compact semisimple Lie algebras of rank r. A little computer experimentation shows that the ranks are normally distributed around the mean rank, which grows linearly with dimension. Figure 1 shows a plot of the average rank as a function of dimension for the 23,058,218,050,191,608 compact semisimple Lie algebras of dimension ≤ 1000, depicted by the blue line. The red line is the graph of $r = d^{2/3}$.

Despite not being the generic behaviour, it is not difficult to come up with series of semisimple Lie algebras whose rank and dimension obey $d = r^{3/2}$. Indeed, the classical simple Lie algebras of rank n have dimension which goes like n^2 for large n. Hence taking n such algebras yields a sequence of semisimple Lie algebras where $d \sim n^3$ and $r \sim n^2$ for large n. Indeed, let $\mathfrak{g}_n = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_n$ where the $\mathfrak{s}_i$ are any one of A_n, B_n, C_n or D_n and let V_n denote the indecomposable lorentzian Lie 3-algebra constructed out of $\mathfrak{g}_n$. The nondegenerate branch of the moduli space of the Bagger–Lambert models associated to V_n exhibit the desired behaviour between the number of M2-branes ($\frac{1}{8}\dim\mathscr{M}$) and the number of degrees of freedom ($\dim V$).

For the degenerate branch of the moduli space, the relevant question is how $(\dim\mathfrak{g}, \dim\mathfrak{p})$ are distributed. Using Table 1 for the Type I irreducible riemannian symmetric spaces and again Table 2 for the ones with Type II, and being careful to avoid repetitions due to low-dimensional isomorphisms, it is possible to write down the following generating functional

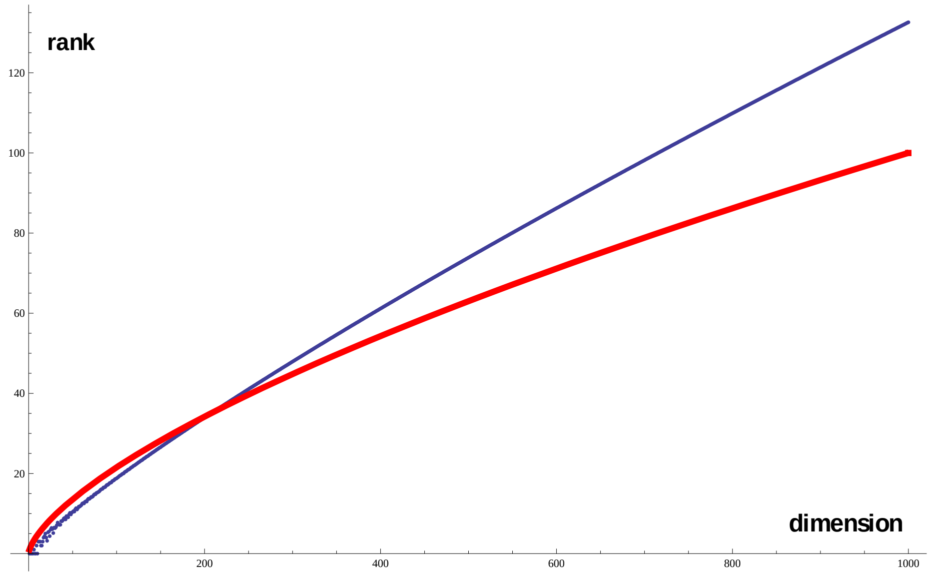

FIGURE 1. Average rank as a function of dimension for compact semisimple Lie algebras

for compact riemannian symmetric spaces

$$Z_{\mathrm{CRSS}}(t,\omega) = Z_{\text{type I}}(t,\omega) Z_{\text{type II}}(t,\omega) \ ,$$

where

$$\begin{aligned} Z_{\text{type I}}(t,\omega) = &\prod_{n\geq 2} \frac{1}{1-\omega^{(n-1)(n+2)}t^{n^2-1}} \prod_{n\geq 2} \frac{1}{1-\omega^{(n-1)(2n+1)}t^{4n^2-1}} \prod_{\substack{p\geq q\geq 1\\ p\geq 2}} \frac{1}{1-\omega^{2pq}t^{(p+q)^2-1}} \\ &\times \prod_{n\geq 2} \frac{1}{1-\omega^{(n-1)(n+2)}t^{n^2-1}} \prod_{n\geq 2} \frac{1}{1-\omega^{n(n-1)}t^{n(2n+1)}} \prod_{n\geq 5} \frac{1}{1-\omega^{n(n-1)}t^{n(2n-1)}} \\ &\times \prod_{\substack{p\geq q\geq 1\\ p+q\geq 7}} \frac{1}{1-\omega^{pq}t^{(p+q)(p+q-1)/2}} \prod_{p\geq q\geq 1} \frac{1}{1-\omega^{4pq}t^{(p+q)(2p+2q+1)}} \\ &\times \frac{1}{1-\omega t} \frac{1}{1-\omega^{42}t^{78}} \frac{1}{1-\omega^{40}t^{78}} \frac{1}{1-\omega^{32}t^{78}} \frac{1}{1-\omega^{26}t^{78}} \frac{1}{1-\omega^{70}t^{133}} \frac{1}{1-\omega^{64}t^{133}} \\ &\times \frac{1}{1-\omega^{54}t^{133}} \frac{1}{1-\omega^{128}t^{248}} \frac{1}{1-\omega^{112}t^{248}} \frac{1}{1-\omega^{28}t^{52}} \frac{1}{1-\omega^{16}t^{52}} \frac{1}{1-\omega^{8}t^{14}} \end{aligned}$$

is the partition function corresponding to Type I riemannian symmetric spaces and

$$Z_{\text{type II}}(t,\omega) = \prod_{n\geq 1} \frac{1}{1-(\omega t^2)^{n(n+2)}} \prod_{n\geq 2} \frac{1}{1-(\omega t^2)^{n(2n+1)}} \prod_{n\geq 3} \frac{1}{1-(\omega t^2)^{n(2n+1)}}$$

$$\times \prod_{n\geq 4} \frac{1}{1-(\omega t^2)^{n(2n-1)}} \frac{1}{1-(\omega t^2)^{78}} \frac{1}{1-(\omega t^2)^{133}} \frac{1}{1-(\omega t^2)^{248}} \frac{1}{1-(\omega t^2)^{52}} \frac{1}{1-(\omega t^2)^{14}}$$

is the corresponding to Type II riemannian symmetric spaces. Expanding as a power series in t and ω, we have

$$Z_{\mathrm{CRSS}}(t,\omega) = 1 + \sum_{d,s\geq 1} N_{d,s} t^s \omega^d \ ,$$

where $N_{d,s}$ is now the number of d-dimensional compact riemannian symmetric spaces with an s-dimensional group of isometries. Figure 2 shows a plot of the average dimension of a compact riemannian symmetric space ($\dim \mathfrak{p}$) as a function of the dimension of its isometry group ($\dim \mathfrak{g}$) for the $378,683,913,003,348,073,310,000,493,022$ compact riemannian symmetric spaces whose isometry group have dimension ≤ 1000, depicted by the blue line. The red line is the graph of $\dim \mathfrak{p} = (\dim \mathfrak{g})^{2/3}$.

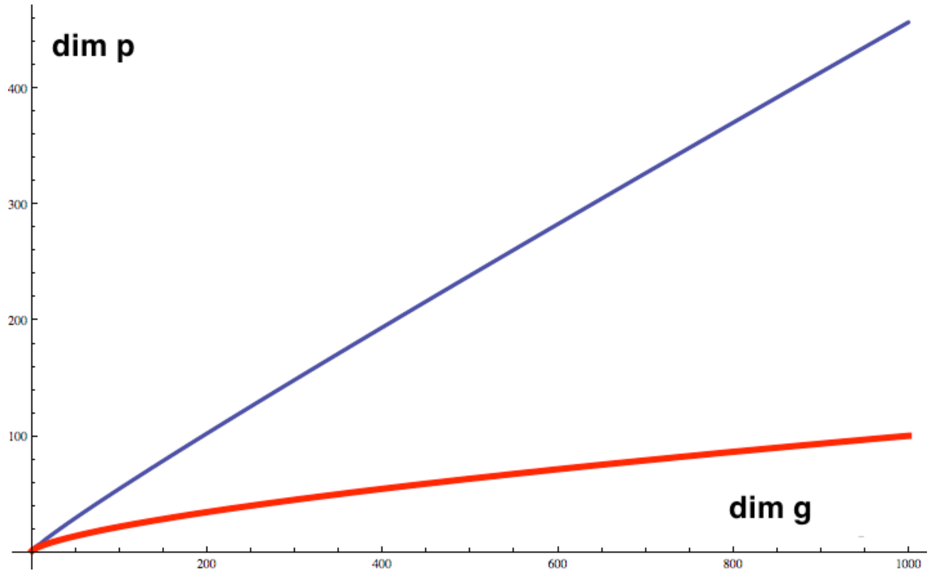

FIGURE 2. Average $\dim \mathfrak{p}$ as a function of $\dim \mathfrak{g}$, for $(\mathfrak{g}, \mathfrak{p}^\perp)$ a symmetric split

Despite the fact that the average behaviour is linear, just as in the case of the nondegenerate branch of the moduli space, it is easy to cook up models where the behaviour is of the desired form. Consider the type I symmetric spaces BDI, AIII and CII of rank 1; that is, those with $q = 1$. The dimension of the isometry group clearly goes like the square of the dimension of the symmetric space: $\dim \mathfrak{g} = n(n+1)/2$ and $\dim \mathfrak{p} = n$ for BDI; $\dim \mathfrak{g} = n(n+1)$ and $\dim \mathfrak{p} = 2n$ for AIII; and $\dim \mathfrak{g} = (n+1)(2n+3)$ and $\dim \mathfrak{p} = 4n$ for CII. Taking a product of n such symmetric spaces, yields $\dim \mathfrak{p} \sim n^2$ whereas $\dim \mathfrak{g} \sim n^3$. In fact, one is not restricted to rank 1 symmetric spaces. Taking p to infinity while keeping q fixed in the above cases also yields the right asymptotic behaviour.

A concrete series of models which realises this behaviour is to take the $(n(n^2-1)+2)$-dimensional lorentzian Lie 3-algebra V_n in Theorem 9 corresponding to the semisimple Lie algebra

$$\mathfrak{g}_n = \underbrace{\mathfrak{su}(n) \oplus \cdots \oplus \mathfrak{su}(n)}_{n \text{ times}} ,$$

and a choice of scale for the inner product on each simple factor. The classical moduli space of the Bagger–Lambert model associated to V_n has (at least) the following branches:

- a *nondegenerate* branch, where the moduli space becomes
$$\mathbb{R}^{16} \times \mathbb{R}^{8n(n-1)}/(S_n)^n ,$$
which has dimension $8(2+n(n-1))$;
- a number of *degenerate* branches, associated to the compact riemannian symmetric spaces with isometry $\mathfrak{g}_n$; that is, products of the following irreducible factors:
  - the type II symmetric space $(\mathfrak{su}(n) \oplus \mathfrak{su}(n), \mathfrak{su}(n))$, of dimension n^2-1;
  - the type I symmetric space AI, of dimension $\frac{1}{2}(n-1)(n+2)$;
  - if n is even, the type I symmetric space AII, of dimension n^2-1; and
  - the type I symmetric spaces AIII, of dimension $2q(n-q)$ for $1 \leq q \leq n-1$.

In particular, both the nondegenerate branch and the degenerate branch consisting of n type I symmetric spaces AIII with $q=1$ have the right asymptotic behaviour.

In summary, we believe that the question now is not whether there exist plausible Bagger–Lambert models with the $N^{3/2}$ asymptotic behaviour, but how to choose among the plethora of such models.

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