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# METRIC LIE 3-ALGEBRAS IN BAGGER-LAMBERT THEORY

PAUL DE MEDEIROS, JOSÉ FIGUEROA-O'FARRILL AND ELENA MÉNDEZ-ESCOBAR

ABSTRACT. We recast physical properties of the Bagger–Lambert theory, such as shift-symmetry and decoupling of ghosts, the absence of scale and parity invariance, in Lie 3-algebraic terms, thus motivating the study of metric Lie 3-algebras and their Lie algebras of derivations. We prove a structure theorem for metric Lie 3-algebras in arbitrary signature showing that they can be constructed out of the simple and one-dimensional Lie 3-algebras iterating two constructions: orthogonal direct sum and a new construction called a double extension, by analogy with the similar construction for Lie algebras. We classify metric Lie 3-algebras of signature  $(2, p)$  and study their Lie algebras of derivations, including those which preserve the conformal class of the inner product. We revisit the 3-algebraic criteria spelt out at the start of the paper and select those algebras with signature  $(2, p)$  which satisfy them, as well as indicate the construction of more general metric Lie 3-algebras satisfying the ghost-decoupling criterion.

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## 1. INTRODUCTION

The foundational work of Bagger and Lambert [1] and Gustavsson [2], has led to the construction in [3] of a superconformal field theory in three-dimensional Minkowski spacetime as a model of multiple M2-branes in M-theory. This theory realises the maximal  $\mathfrak{osp}(8|4)$  superconformal symmetry of the near-horizon geometry of M2-branes in eleven-dimensional supergravity. The lagrangian and its equations of motion nicely encapsulate several other features expected [4,5] in the long sought-after low-energy description of multiple M2-branes. These encouraging properties have prompted a great deal of interest in the Bagger–Lambert theory [6–42].

The Bagger–Lambert theory has a novel local gauge symmetry which is not based on a Lie algebra, but rather on a Lie 3-algebra [43]. The analogue of the Lie bracket  $[-, -]$  here being the 3-bracket  $[-, -, -]$ , an alternating trilinear map on a vector space  $V$ , which satisfies a natural generalisation of the Jacobi identity (sometimes referred to as the fundamental identity). The dynamical fields in the Bagger–Lambert model take values in  $V$  and consist of eight real bosonic scalars and a fermionic spinor in three dimensions which transforms as a chiral spinor under the  $\mathfrak{so}(8)$  R-symmetry. There is also a non-dynamical gauge field which takes values in a Lie subalgebra of  $\mathfrak{gl}(V)$ . The on-shell closure of the supersymmetry transformations for these fields follows from the fundamental identity.

To obtain the correct equations of motion for the Bagger–Lambert theory from a lagrangian that is invariant under all the aforementioned symmetries seems to require the Lie 3-algebra to admit an invariant inner product. (Following remark 8 in [31], we will take this inner product to be non-degenerate.) The signature of the inner product here determines the relative signs of the kinetic terms for the scalar and fermion fields in the Bagger–Lambert lagrangian. It would therefore be desirable to choose this inner product to have positive-definite signature, to avoid the occurrence of states with negative-norm in the quantum theory. Unfortunately there are very few such euclidean metric Lie 3-algebras. Indeed, as shown in [44] (see also [17, 18]), they can always be written as the direct sum of abelian Lie 3-algebras plus multiple copies of the unique simple euclidean Lie 3-algebra  $S_{0,4}$  considered by Bagger and Lambert in their original construction. The moduli space for the Bagger–Lambert theory associated with  $S_{0,4}$  has been interpreted as describing two M2-branes on a certain M-theory orbifold in [12, 13].

Thus, in order to find new interacting Bagger–Lambert lagrangians and despite the possibility of negative-norm states, one is led to contemplate Lie 3-algebras with an invariant inner product of indefinite signature. This idea was pioneered in [20–22] for a class of Lie

3-algebras (defined by a euclidean semisimple Lie algebra in two dimensions lower) admitting an inner product of lorentzian signature. (It was subsequently proved in [31] that every indecomposable lorentzian Lie 3-algebra belongs to this class, unless it is the unique simple lorentzian Lie 3-algebra  $S_{1,3}$  or one-dimensional.) A feature of these lorentzian Lie 3-algebras is that their canonical 4-form (built from the 3-algebra structure constants and invariant inner product) has precisely one leg in only one of the two null directions of the 3-algebra (the remaining three legs are in the directions spanned by the euclidean Lie algebra). It is this 4-form which dictates the structure of the interactions appearing in the Bagger-Lambert lagrangian. Consequently the scalar and fermion fields in the other null direction of the 3-algebra appear only in the free kinetic terms of the Bagger-Lambert lagrangian here. This might suggest one could simply integrate out these components, thus setting the scalar and fermion components in the complementary null direction to obey their free field equations. However, the absence of interaction terms involving scalars and fermions in one null direction gives rise to an additional symmetry for their kinetic terms upon shifting them by constant values. The gauging of this extra shift symmetry in a superconformally-invariant manner has been analysed in some detail recently in [33, 36] where it is found that, after fixing the gauged symmetry, the resulting lagrangian can be simply expressed as the sum of the ungauged lagrangian plus a maximally supersymmetric lagrangian involving the Faddeev-Popov ghosts for the shift symmetry. The BRST transformations for this gauge-fixed theory mix the fields and ghosts in the two lagrangians as expected and the BRST cohomology is found to be free of negative-norm states.

It would be good to understand whether this remarkable absence of negative-norm states in the Bagger-Lambert theory for lorentzian Lie 3-algebras persists for inner products of more indefinite signature, or at least whether one can establish clear 3-algebraic criteria that would guarantee that the construction noted above could be employed for more general algebras. The resulting moduli spaces for such theories and their possible M-theoretic interpretation might also be of interest. In this paper we will begin to address some of these questions.

Given the central rôle played by the Lie 3-algebra in the Bagger-Lambert theory, we first recast some desirable physical properties, such as the possibility of decoupling of negative-norm states and the scale and parity invariances of the model, in 3-algebraic language. This then motivates the study of metric Lie 3-algebras in arbitrary signature subject to some 3-algebraic criteria which can be explicitly checked for a given class of Lie 3-algebras or else built *ab initio* into a general construction of such Lie 3-algebras.

Indeed, by analogy with the structure theorem [45, 46] of metric Lie algebras, we will prove that any metric Lie 3-algebra can be constructed from the one-dimensional and simple Lie 3-algebras by iterating the operations of orthogonal direct sum and ‘double extension’ (see Definition 10). Furthermore, following [31], we will classify Lie 3-algebras with inner products of  $(2, p)$  signature and find the non-simple indecomposable ones to fall into one of two distinct classes. We find that only one of these two classes describes metric Lie 3-algebras with a canonical 4-form having no legs in precisely half of the null directions of the 3-algebra (this is similar to what happened in the lorentzian case). For the Bagger-Lambert theory with Lie 3-algebra in this class, this implies that the scalars and

fermions in these two null directions appear only in the free kinetic terms of the lagrangian, suggesting that one might be able to remove the negative-norm states from the physical Hilbert space here also after appropriate fixing of the gauged shift symmetries as in [33,36]. We will consider the physical properties of such theories and their corresponding moduli spaces in a forthcoming publication.

We should emphasise that even if the existence of extra shift symmetries allows one to obtain a positive-definite Hilbert space, in general one would need to impose extra constraints to make contact with the effective description of M2-branes. For instance, one would also like the effective field theory to have no coupling constant. This would be guaranteed provided the Lie 3-algebra admits a suitable outer automorphism which absorbs the formal coupling dependence by rescaling the inner product in the Bagger–Lambert lagrangian. With this in mind, we compute the Lie algebra of derivations (i.e. the infinitesimal form of the automorphism group) of each class of metric Lie 3-algebra in  $(2, p)$  signature and highlight, when it exists, the appropriate outer automorphism that could be used to fix the coupling constant in the Bagger–Lambert theory. It would also be desirable for the theory to be parity-invariant. This condition is satisfied provided the Lie 3-algebra admits an isometric anti-automorphism (which effectively reverses the sign of the Lie 3-algebra structure constants in such a way that it compensates the parity inversion on the M2-branes' worldvolume). We will find four new classes of  $(2, p)$  signature metric Lie 3-algebras that satisfy all of the above conditions.

We will also make some remarks on how the decoupling of negative-norm states might work for metric Lie 3-algebras of general indefinite signature. The abstract criterion for the existence of additional shift symmetries in half of the null directions will be that the Lie 3-algebra should admit a maximally isotropic centre. We show that the obstruction to having a maximally isotropic centre corresponds roughly speaking to the existence of a simple ideal among the ingredients of the metric Lie 3-algebra. This allows us to give a prescription for how to construct *in principle* all such metric Lie 3-algebras.

The paper is organised as follows. In Section 2 we will briefly review the main features of the Bagger–Lambert theory from the perspective of Lie 3-algebras. In other words, this section will translate desirable physical properties of the theory into 3-algebraic criteria. These will be revisited at the end of the paper in Section 6 in light of the structural results and classifications obtained in the paper. We interpret metricity and indecomposability in terms of the Bagger–Lambert theory and show why these properties are desirable. We then translate physical requirements such as ghost decoupling, absence of scale and parity invariance into 3-algebraic criteria, which can be easily checked given either explicit metric Lie 3-algebras as in Section 4 or a general construction scheme as in Section 3. After this physical motivation, the paper contains a number of technical algebraic sections. In Section 3 we prove a structure theorem for metric Lie 3-algebras. After the observation that every metric Lie 3-algebra is an orthogonal direct sum of indecomposables, we set out to prove Theorem 12, which says that every indecomposable metric Lie 3-algebra is either one-dimensional, simple or else it is obtained by double extending a (not necessarily indecomposable) metric Lie 3-algebra by either a one-dimensional or simple Lie 3-algebra. A simple induction argument then allows us to state Corollary 13 describing the class of

metric Lie 3-algebras as the class of Lie 3-algebras generated by the one-dimensional and simple Lie 3-algebras under the operations of double extension and orthogonal direct sum. The notion of double extension appears in Definition 10. Section 4 contains a classification of indecomposable metric Lie 3-algebras of signature  $(2, p)$ , culminating with Theorem 24. They fall into two main types, which we call *type Ia*, defined in (16), and *type IIIb*, defined in (19). They are distinguished by the fact that type IIIb possesses a maximally isotropic centre, whereas type Ia does not. In fact, type IIIb is a very general class of metric Lie 3-algebras and we deconstruct it into several non-isomorphic classes at the end of Section 4.3.2. Section 5 contains the analysis of the Lie algebra of derivations of the indecomposable metric Lie 3-algebras found in Section 4, as well as to the subalgebras of derivations which preserve (the conformal class of) the inner product. Finally in Section 6 we revisit the 3-algebraic criteria obtained in Section 2 in light of the explicit classification and structural results obtained in the previous sections. We focus first on the absence of ghosts, and using the structure results in Section 3.2 we indicate how metric Lie 3-algebras satisfying the ghost-decoupling criterion can be constructed. For the  $(2, p)$  metric Lie 3-algebras classified in Section 4, we determine the indecomposable ones which satisfy the ghost decoupling criterion as well as the absence of a coupling constant and the parity invariance of the theory. This results in four classes of indecomposable  $(2, p)$  metric Lie 3-algebras deserving of further study.

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## 2. BAGGER–LAMBERT THEORY IN LIE 3-ALGEBRAIC LANGUAGE

We will start by summarising how some of the physical properties of the Bagger–Lambert theory relate to the general structure of metric Lie 3-algebras discussed in [31] and later in this paper. In Section 6 we will revisit these properties in light of our classification of  $(2, p)$  signature metric Lie 3-algebras in Section 4 and our structure theorem of Section 3.

**2.1. Brief review of Bagger–Lambert theory.** Let us begin by reviewing some details of the Bagger–Lambert theory. Consider a metric Lie 3-algebra, with 3-bracket  $[-, -, -]$  and inner product  $\langle -, - \rangle$  of general indefinite signature  $(k, k + n)$  and define a null basis  $e_A = (u_i, v_i, e_a)$ , with  $i = 1, \dots, k$ ,  $a = 1, \dots, n$ , such that  $\langle u_i, v_j \rangle = \delta_{ij}$ ,  $\langle u_i, u_j \rangle = 0 = \langle v_i, v_j \rangle$  and  $\langle e_a, e_b \rangle = \delta_{ab}$ . With respect to this basis, the components of the canonical 4-form for the metric Lie 3-algebra are  $F_{ABCD} := \langle [e_A, e_B, e_C], e_D \rangle$  (indices will be raised and lowered using the invariant 3-algebra metric  $\langle e_A, e_B \rangle$ ). The fields in the Bagger–Lambert theory have components  $X_I^A, \Psi^A, (\tilde{A}_\mu)^A_B = F^A_{BCD} A_\mu^{CD}$ , corresponding respectively to the scalars ( $I = 1, \dots, 8$  in the vector of the  $\mathfrak{so}(8)$  R-symmetry), fermions and gauge field ( $\mu = 0, 1, 2$  on  $\mathbb{R}^{1,2}$  Minkowski space). Although the supersymmetry transformations and equations of motion can be expressed in terms of  $(\tilde{A}_\mu)^A_B$ , the lagrangian requires it to be expressed



as above in terms of  $A_\mu^{AB}$ . Indeed, recall that the Bagger–Lambert lagrangian [3] can be written

$$\begin{aligned} \mathcal{L} = & -\frac{1}{2} \langle D_\mu X_I, D^\mu X_I \rangle + \frac{i}{2} \langle \bar{\Psi}, \Gamma^\mu D_\mu \Psi \rangle - \frac{i}{4} F_{ABCD} (\bar{\Psi}^A \Gamma^{IJ} \Psi^B) X_I^C X_J^D \\ & - \frac{1}{12} \langle [X_I, X_J, X_K], [X_I, X_J, X_K] \rangle + \mathcal{L}_{CS} , \end{aligned} \quad (1)$$

where

$$\mathcal{L}_{CS} = \frac{1}{2} \left( A^{AB} \wedge d\tilde{A}_{AB} + \frac{2}{3} A^{AB} \wedge \tilde{A}_{AC} \wedge \tilde{A}^C{}_B \right) , \quad (2)$$

and  $D_\mu \phi^A = \partial_\mu \phi^A + (\tilde{A}_\mu)^A{}_B \phi^B$  defines the action on any field  $\phi$  valued in  $V$  of the derivative  $D$  that is gauge-covariant with respect to  $\tilde{A}^A{}_B$ . The fermion  $\Psi^A$  transforms as a Majorana spinor in  $\mathbb{R}^{1,10}$  subject to the projection  $\Gamma_{012} \Psi^A = -\Psi^A$ , where  $\Gamma_\mu$  and  $\Gamma_I$  denote respectively the  $\mathbb{R}^{1,2}$  and  $\mathbb{R}^8$  components of the Clifford algebra on  $\mathbb{R}^{1,10}$ .

The integral of the lagrangian (1) is invariant under the supersymmetry transformations

$$\begin{aligned} \delta X_I^A &= i \bar{\epsilon} \Gamma_I \Psi^A \\ \delta \Psi^A &= D_\mu X_I^A \Gamma^\mu \Gamma^I \epsilon + \frac{1}{6} F^A{}_{BCD} X_I^B X_J^C X_K^D \Gamma^{IJK} \epsilon \\ \delta (\tilde{A}_\mu)^A{}_B &= i F^A{}_{BCD} X_I^C \bar{\epsilon} \Gamma_\mu \Gamma^I \Psi^D , \end{aligned}$$

where the parameter  $\epsilon$  transforms as a Majorana spinor on  $\mathbb{R}^{1,10}$  subject to the projection  $\Gamma_{012} \epsilon = \epsilon$ . Using the equations of motion obtained from (1), these supersymmetry transformations are found to close, up to translations in  $\mathbb{R}^{1,2}$  and gauge transformations.

**2.2. Degeneracy implies decoupling.** We have been assuming throughout that the Lie 3-algebra inner product is nondegenerate. Recall that the supersymmetry transformations and equations of motion for the Bagger–Lambert theory do not require the Lie 3-algebra to admit an inner product at all. It is the existence of a lagrangian which realises all the symmetries and gives rise to all the correct equations of motion appearing in the on-shell closure of supersymmetry algebra that seems to require that the Lie 3-algebra should admit an invariant inner product. As shown in remark 8 of [31], if we allowed this invariant inner product to be degenerate then the corresponding canonical 4-form  $F$  can have no legs along any of the degenerate directions. Since it is this 4-form which dictates the structure of interactions in the Bagger–Lambert lagrangian (1), it is clear that only the fields in the nondegenerate directions can appear in the interactions. Moreover, even the free scalar and fermion kinetic terms cannot appear along the degenerate directions of the inner product. Hence there is always a consistent truncation of the theory to the quotient Lie 3-algebra (with nondegenerate inner product) along the nondegenerate directions. Therefore, in terms of characterising which kinds of new interactions can occur in the Bagger–Lambert lagrangian, nothing is lost by assuming a nondegenerate inner product.

**2.3. Decomposability implies decoupling.** By definition, a decomposable metric Lie 3-algebra  $V = V_1 \oplus V_2$  can be written as the orthogonal direct sum of two metric Lie 3-algebras  $V_1$  and  $V_2$  (with  $[V_1, V_2, V] = 0$ ). This implies that neither the inner product  $\langle -, - \rangle$  nor the canonical 4-form  $F$  of a decomposable Lie 3-algebra can ever have ‘mixed legs’ in both  $V_1$  and  $V_2$ . Hence the Bagger–Lambert lagrangian and supersymmetry transformations for

$V = V_1 \oplus V_2$  completely decouple in terms of the individual factors  $V_1$  and  $V_2$ . Thus, as one would expect, the indecomposable metric Lie 3-algebras really are the basic building blocks for the Bagger–Lambert theory.

#### 2.4. Maximally isotropic centres, shift symmetries and decoupling of ghosts.

The free kinetic terms for the scalars and fermions

$$\sum_{i=1}^k \left( -\partial_\mu X_I^{u_i} \partial^\mu X_I^{v_i} + i \bar{\Psi}^{u_i} \Gamma^\mu \partial_\mu \Psi^{v_i} \right) - \frac{1}{2} \partial_\mu X_I^a \partial^\mu X_I^a + \frac{i}{2} \bar{\Psi}^a \Gamma^\mu \partial_\mu \Psi^a \quad (3)$$

are the only terms which do not couple to components  $F_{ABCD}$  of the canonical 4-form in (1). The  $(u_i, v_i)$  components correspond to the  $2k$  null directions in the Lie 3-algebra and are related to the  $k$  kinetic terms with the ‘wrong’ sign in the lagrangian that need to be exorcised.

From (1) we see that there are two ways in which the fields in the Bagger–Lambert theory couple to the components of  $F$  in the interaction terms in the lagrangian; either linearly via contraction with  $F_{ABCD}$  or quadratically via contraction with  $F_{ABCG} F_{DEF}^G$ . The first type gives rise to the  $X^2 \Psi^2$ ,  $AX^2$ ,  $A\Psi^2$  interactions and the free kinetic term for the gauge field. The second type gives rise to the  $X^6$ ,  $A^2 X^2$  and  $A^3$  interactions.

Clearly a sufficient condition for the existence of extra shift symmetries for, say, the null components  $X^{v_i}$  and  $\Psi^{v_i}$  in (3) would be that these fields do not appear in any of the interactions. If this condition is met for all  $k$  of these null components then a naive counting argument would suggest that gauge-fixing the theory obtained from gauging these extra shift symmetries, à la [33,36], should have a BRST cohomology free of negative-norm states.

The question of whether the criterion above is met can be posed at the level of the Lie 3-algebra as follows. Of the various terms in the Bagger–Lambert lagrangian, the most restrictive in terms of satisfying the condition above comes from the quartic  $X^2 \Psi^2$  scalar-fermion interaction. For this to not involve any of the null components  $X^{v_i}$  and  $\Psi^{v_i}$  requires  $F_{v_i ABC} = 0$ . This condition on the Lie 3-algebra is equivalent to saying that the  $v_i$  are central, whence they span a maximally isotropic subspace of the centre. Notice that  $F_{v_i ABC} = 0$  further implies  $F_{v_i ABG} F_{CDE}^G = 0$  and so, as required, the fields  $X^{v_i}$ ,  $\Psi^{v_i}$  in these  $k$  null directions do not appear in any of the interactions (furthermore  $A_\mu^{v_i A}$  does not appear at all in the lagrangian).

It is worth remarking that the criterion above is satisfied for all indecomposable lorentzian Lie 3-algebras except the unique simple one  $S_{1,3}$ . This follows from Theorem 9 in [31] where, in the Witt basis  $(u, v, e_a)$ , the canonical 4-form has only the legs  $F_{uabc} = f_{abc}$  (where  $f_{ab}^c$  are the structure constants of a compact semisimple Lie algebra). On the other hand  $S_{1,3}$  has a non-vanishing  $F_{uvab}$  component, thus violating the condition above. We will notice a similar structure in Theorem 24 below.

**2.5. Conformal automorphisms and the coupling constant.** The formal coupling dependence of the interactions in the Bagger–Lambert theory can be brought to an overall factor  $\frac{1}{\kappa^2}$  multiplying the lagrangian by rescaling the canonical 4-form  $F_{ABCD} \rightarrow \kappa^2 F_{ABCD}$



followed by the field redefinitions  $X_I^A \rightarrow \frac{1}{\kappa} X_I^A$ ,  $\Psi^A \rightarrow \frac{1}{\kappa} \Psi^A$  and  $A_\mu^{AB} \rightarrow \frac{1}{\kappa^2} A_\mu^{AB}$ . Since the scalars and fermions are valued in  $V$ , inspection of all the terms in the lagrangian not involving the gauge field shows that the overall  $\frac{1}{\kappa^2}$  factor can be absorbed for these terms if the Lie 3-algebra admits a conformal automorphism  $\varphi : V \rightarrow V$  with  $\langle \varphi(x), \varphi(y) \rangle = \kappa^2 \langle x, y \rangle$  for all  $x, y \in V$ . This would then follow by redefining  $X_I \rightarrow \varphi(X_I)$  and  $\Psi \rightarrow \varphi(\Psi)$  in the lagrangian. This induces the transformation  $\tilde{A}_\mu \rightarrow \varphi \tilde{A}_\mu \varphi^{-1}$  on the gauge field  $\tilde{A}_\mu$  that appears in the covariant derivatives. The Chern-Simons term then requires  $A_\mu \rightarrow \varphi A_\mu \varphi^t$ , which indeed follows from the definition  $(\tilde{A}_\mu)^A_B = F^A_{BCD} A_\mu^{CD}$ , in order to get the correct scaling for all the terms involving the gauge field in the Bagger–Lambert lagrangian.

Let us recall how this works for the case of indecomposable lorentzian Lie 3-algebras, as noted in [21, 22], using the language of Proposition 11 in [31]. Relative to the Witt basis  $(u, v, e_a)$  noted above, the appropriate conformal automorphism maps  $\varphi(u) = \beta u$ ,  $\varphi(v) = \beta^{-3} v$  and  $\varphi(e_a) = \beta^{-1} e_a$ , which rescales the inner product by a factor of  $\beta^{-2}$  and so absorbs the coupling if we take  $\beta = \kappa^{-1}$ . To be completely explicit,  $X_I \rightarrow \varphi(X_I)$  transforms the components of the scalars as  $X_I^u \rightarrow \kappa^{-1} X_I^u$ ,  $X_I^v \rightarrow \kappa^3 X_I^v$ ,  $X_I^a \rightarrow \kappa X_I^a$  (and likewise for the fermions), whereas  $A_\mu \rightarrow \varphi A_\mu \varphi^t$  transforms the components of the gauge field as  $A_\mu^{uv} \rightarrow \kappa^2 A_\mu^{uv}$ ,  $A_\mu^{ua} \rightarrow A_\mu^{ua}$ ,  $A_\mu^{va} \rightarrow \kappa^4 A_\mu^{va}$ ,  $A_\mu^{ab} \rightarrow \kappa^2 A_\mu^{ab}$ . (Recall though that the central components  $A_\mu^{vA}$  do not appear in the lagrangian.) The components  $B_a := \frac{1}{2} f_{abc} A^{bc}$  and  $A^a := A^{ua}$  of the gauge field appear in the Chern-Simons term of (1) as a so-called BF term (with  $F$  being the field strength of gauge field  $A$  here). This automorphism then matches equation (2.30) in [21] if we identify their  $+$ ,  $-$  and  $g$  with our  $u, v$  and  $\kappa$ .

In Section 6 we will present a similar story for several of the classes of metric Lie 3-algebras in  $(2, p)$  signature in Section 4.

**2.6. Isometric anti-automorphisms and parity invariance.** Recall [4] that the effective field theory on the worldvolume of M2-branes is expected to be invariant under the  $\mathbb{Z}_2$  symmetry generated by a parity inversion. Despite the existence of a Chern-Simons term, this condition is satisfied for the Bagger–Lambert theory based on the euclidean simple Lie 3-algebra  $S_{0,4}$  (see [8, 10]) and for the class of indecomposable lorentzian Lie 3-algebras in [21, 22].

Following [3, 8], let us define a parity inversion to map the  $\mathbb{R}^{1,2}$  coordinates  $(x^0, x^1, x^2) \rightarrow (x^0, x^1, -x^2)$  (thus reversing the orientation on  $\mathbb{R}^{1,2}$ ) and mapping spinors  $\Psi \rightarrow \Gamma_2 \Psi$ . This implies that the spinor bilinear terms  $\tilde{\Psi}^A \Gamma^\mu \partial_\mu \Psi^B$  and  $\tilde{\Psi}^A \Gamma^{IJ} \Psi^B$  in the Bagger–Lambert Lagrangian (1) are respectively even and odd under this parity transformation. The structure of covariant derivatives in (1) further implies that  $\tilde{A}_0, \tilde{A}_1$  should be parity-even whilst  $\tilde{A}_2$  is parity-odd. Given these parity assignments, inspection of the terms in the Bagger–Lambert Lagrangian (1) leads one to deduce that a sufficient condition for invariance is that the Lie 3-algebra admits an isometric anti-automorphism, i.e. a linear map  $\gamma : V \rightarrow V$  obeying  $\langle \gamma x, \gamma y \rangle = \langle x, y \rangle$  and  $[\gamma x, \gamma y, \gamma z] = -\gamma[x, y, z]$  for all  $x, y, z$  in  $V$ . (This results in effectively reversing the sign of all the structure constants in the Lie 3-algebra, a condition already noted in [3] for parity-invariance.) The transformation  $\gamma$  maps  $X_I \rightarrow \gamma X_I$

and  $\tilde{A}_\mu \rightarrow \gamma \tilde{A}_\mu \gamma^{-1}$ , which implies  $A_\mu \rightarrow -\gamma A_\mu \gamma^t$ . Combining this transformation with the action of parity on the fields thus leaves (1) invariant. Hence the criterion of parity-invariance can be reduced to the existence of an isometric anti-automorphism of the Lie 3-algebra. Notice that the composition of any two isometric anti-automorphisms is an isometric automorphism, i.e. a symmetry of (1) on its own. Thus the generator of the isometric anti-automorphism needed for parity-invariance is essentially unique modulo a symmetry of the Lagrangian.

For the euclidean case  $S_{0,4}$ , defined with respect to an orthonormal basis  $(e_1, e_2, e_3, e_4)$ , the appropriate isometric anti-automorphism can be taken to be the map  $\gamma(e_1, e_2, e_3, e_4) = (-e_1, -e_2, -e_3, e_4)$ , as was clarified in [10]. For the lorentzian case discussed in [21,22], relative to the Witt basis  $(u, v, e_a)$  we have defined above, the appropriate anti-automorphism maps  $\gamma(u, v, e_a) = (u, v, -e_a)$ . Reading off the corresponding transformations of the components of the fields above, one finds that  $X_I^a$ ,  $\Psi^a$ ,  $A_\mu^{uv}$  and  $A_\mu^{ab}$  are odd under the action of  $\gamma$  whilst all the other components are even (in agreement with [21]).

**2.7. Summary.** In summary, we are interested in classifying or understanding how to construct *indecomposable metric* Lie 3-algebras admitting a *maximally isotropic centre*, *conformal derivations* and *isometric anti-automorphisms*. In the next sections we will prove a structure theorem for metric Lie 3-algebras, classify those with signature  $(2, p)$  and study their Lie algebras of (conformal) derivations. Finally in Section 6, we will revisit these conditions in light of the above results.

### 3. METRIC LIE 3-ALGEBRAS AND DOUBLE EXTENSIONS

Recall that a (finite-dimensional, real) Lie 3-algebra consists of a finite-dimensional real vector space  $V$  together with a linear map  $\Phi : \Lambda^3 V \rightarrow V$ , denoted simply as a 3-bracket, obeying a generalisation of the Jacobi identity. To define it, let us recall that an endomorphism  $D \in \text{End } V$  is said to be a **derivation** if

$$D[x_1, x_2, x_3] = [Dx_1, x_2, x_3] + [x_1, Dx_2, x_3] + [x_1, x_2, Dx_3] ,$$

for all  $x_i \in V$ . Then  $(V, \Phi)$  defines a **Lie 3-algebra** if the endomorphisms  $\text{ad}_{x_1, x_2} \in \text{End } V$ , defined by  $\text{ad}_{x_1, x_2} y = [x_1, x_2, y]$ , are derivations:

$$[x_1, x_2, [y_1, y_2, y_3]] = [[x_1, x_2, y_1], y_2, y_3] + [y_1, [x_1, x_2, y_2], y_3] + [y_1, y_2, [x_1, x_2, y_3]] , \quad (4)$$

for all  $y_i \in V$ . We call it the 3-Jacobi identity or, in the present context, the fundamental identity. The vector space of derivations is a Lie subalgebra of  $\mathfrak{gl}(V)$  denoted  $\text{Der } V$ . The derivations  $\text{ad}_{x_1, x_2} \in \text{Der } V$  span the ideal  $\text{ad } V \triangleleft \text{Der } V$  consisting of **inner derivations**.

We recall that a metric Lie 3-algebra is a triple  $(V, \Phi, b)$  consisting of a finite-dimensional real Lie 3-algebra  $(V, \Phi)$  together with a nondegenerate symmetric bilinear form  $b : S^2 V \rightarrow \mathbb{R}$ , denoted simply by  $\langle -, - \rangle$ , subject to the invariance condition of the inner product

$$\langle [x_1, x_2, y_1], y_2 \rangle = - \langle [x_1, x_2, y_2], y_1 \rangle , \quad (5)$$

for all  $x_i, y_i \in V$ .

Given two metric Lie 3-algebras  $(V_1, \Phi_1, b_1)$  and  $(V_2, \Phi_2, b_2)$ , we may form their **orthogonal direct sum**  $(V_1 \oplus V_2, \Phi_1 \oplus \Phi_2, b_1 \oplus b_2)$ , by declaring that

$$[x_1, x_2, y] = 0 \quad \text{and} \quad \langle x_1, x_2 \rangle = 0 ,$$

for all  $x_i \in V_i$  and all  $y \in V_1 \oplus V_2$ . The resulting object is again a metric Lie 3-algebra. A metric Lie 3-algebra is said to be **indecomposable** if it is not isomorphic to an orthogonal direct sum of metric Lie 3-algebras  $(V_1 \oplus V_2, \Phi_1 \oplus \Phi_2, b_1 \oplus b_2)$  with  $\dim V_i > 0$ . In order to classify the metric Lie 3-algebras, it is clearly enough to classify the indecomposable ones. In this section we will prove a structure theorem for indecomposable Lie 3-algebras. We will prove that they are constructed from the simple and the one-dimensional Lie 3-algebras by iterating two constructions: the orthogonal direct sum just defined and the “double extension” to be defined below.

**3.1. Basic notions and notation.** From now on let  $(V, \Phi)$  be a Lie 3-algebra. Given subspaces  $W_i \subset V$ , we will let  $[W_1 W_2 W_3]$  denote the subspace of  $V$  consisting of elements  $[w_1, w_2, w_3] \in V$ , where  $w_i \in W_i$ .

We will use freely the notions of subalgebra, ideal and homomorphisms as reviewed in [31]. In particular a **subalgebra**  $W \triangleleft V$  is a subspace  $W \subset V$  such that  $[WWW] \subset W$ , whereas an **ideal**  $I \triangleleft V$  is a subspace  $I \subset V$  such that  $[IVV] \subset I$ . A linear map  $\phi : V_1 \rightarrow V_2$  between Lie 3-algebras is a **homomorphism** if  $\phi[x_1, x_2, x_3] = [\phi(x_1), \phi(x_2), \phi(x_3)]$ , for all  $x_i \in V_1$ . An **isomorphism** is a bijective homomorphism. There is a one-to-one correspondence between ideals and homomorphisms and all the standard theorems hold. In particular, intersection and sums of ideals are ideals. An ideal  $I \triangleleft V$  is said to be **minimal** if any other ideal  $J \triangleleft V$  contained in  $I$  is either 0 or  $I$ . Dually, an ideal  $I \triangleleft V$  is said to be **maximal** if any other ideal  $J \triangleleft V$  containing  $I$  is either  $V$  and  $I$ . A Lie 3-algebra is said to be **simple** if it has no proper ideals and  $\dim V > 1$ .

**Lemma 1.** *If  $I \triangleleft V$  is a maximal ideal, then  $V/I$  is simple or one-dimensional.*

Simple Lie 3-algebras have been classified.

**Theorem 2** ([47, §3]). *A simple real Lie 3-algebra is isomorphic to one of the Lie 3-algebras defined, relative to a basis  $(\mathbf{e}_i)_{i=1,2,3,4}$ , by*

$$[\mathbf{e}_1, \dots, \widehat{\mathbf{e}}_i, \dots, \mathbf{e}_4] = (-1)^i \varepsilon_i \mathbf{e}_i , \quad (6)$$

where a hat denotes omission and where the  $\varepsilon_i$  are signs.

It is plain to see that simple real Lie 3-algebras admit invariant inner products of any signature. Indeed, the Lie 3-algebra in (6) leaves invariant the diagonal inner product with entries  $(\varepsilon_1, \dots, \varepsilon_4)$ . We will let  $S_{p,q}$  denote the simple metric Lie 3-algebra with signature  $(p, q)$ . There are, up to homothety, precisely three:  $S_{0,4}$ ,  $S_{1,3}$  and  $S_{2,2}$ , corresponding to euclidean, lorentzian and split signatures, respectively. The Lie 3-bracket of  $S_{p,q}$  is given relative to a basis  $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4)$  by (6) where the signs  $\varepsilon_i$  are given by  $(++++)$  for  $S_{0,4}$ ,  $(-+++)$  for  $S_{1,3}$  and  $(--++)$  for  $S_{2,2}$ . Implicit in the notation is a choice of invariant inner product, which is given by  $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = \lambda \varepsilon_i \delta_{ij}$ , for some  $\lambda > 0$ .

Complementary to the notion of (semi)simplicity is that of solvability. As shown by Kasymov [48], there are more than one notion of solvability for Lie 3-algebras. However we will use here the original notion introduced by Filippov [43]. Let  $I \triangleleft V$  be an ideal. We define inductively a sequence of ideals

$$I^{(0)} = I \quad \text{and} \quad I^{(k+1)} = [I^{(k)}I^{(k)}I^{(k)}] \subset I^{(k)}. \quad (7)$$

We say that  $I$  is **solvable** if  $I^{(s)} = 0$  for some  $s$ , and we say that  $V$  is **solvable** if it is solvable as an ideal of itself. If  $I, J \triangleleft V$  are solvable ideals, so is their sum  $I + J$ , leading to the notion of a maximal solvable ideal  $\text{Rad } V$ , known as the **radical** of  $V$ . A Lie 3-algebra  $V$  is said to be **semisimple** if  $\text{Rad } V = 0$ . Ling [47] showed that a semisimple Lie 3-algebra is isomorphic to the direct sum of its simple ideals. The following result is due to Filippov [43] and can be paraphrased as saying that the radical is a *characteristic* ideal.

**Theorem 3** ([43, Theorem 1]). *Let  $V$  be a Lie 3-algebra. Then  $D \text{Rad } V \subset \text{Rad } V$  for every derivation  $D \in \text{Der } V$ .*

We say that a subalgebra  $L < V$  is a **Levi subalgebra** if  $V = L \oplus \text{Rad } V$  as vector spaces. Ling showed that, as in the theory of Lie algebras, Lie 3-algebras admit a Levi decomposition.

**Theorem 4** ([47, Theorem 4.1]). *Let  $V$  be a Lie 3-algebra. Then  $V$  admits a Levi subalgebra.*

A further result of Ling's which we shall need is the following.

**Theorem 5** ([47, §2]). *Let  $V$  be a Lie 3-algebra. Then  $V$  is semisimple if and only if the Lie algebra  $\text{ad } V$  of inner derivations is semisimple and all derivations are inner, so that  $\text{Der } V = \text{ad } V$ .*

In turn this allows us to prove the following useful result.

**Proposition 6.** *Let  $0 \rightarrow A \rightarrow B \rightarrow \overline{C} \rightarrow 0$  be an exact sequence of Lie 3-algebras. If  $A$  and  $\overline{C}$  are semisimple, then so is  $B$ .*

*Proof.* Since  $A$  is semisimple, Theorem 5 says that  $\text{ad } A$  is semisimple.  $B$  is a representation of  $\text{ad } A$ , hence fully reducible. Since  $A$  is an  $\text{ad } A$ -subrepresentation of  $B$ , we have  $B = A \oplus C$ , where  $C$  is a complementary  $\text{ad } A$ -subrepresentation. Since  $A \triangleleft B$  is an ideal (being the kernel of a homomorphism),  $\text{ad } A(C) = 0$ , whence  $[AAC] = 0$ .

The subspace  $C$  is actually a subalgebra, since the component  $[CCC]_A$  of  $[CCC]$  along  $A$  is  $\text{ad } A$ -invariant by the 3-Jacobi identity and the fact that  $C$  is  $\text{ad } A$ -invariant. This means that  $[CCC]_A$  is central in  $A$ , but  $A$  is semisimple, whence it must vanish. Hence,  $[CCC] \subset C$ . Since the projection  $B \rightarrow \overline{C}$  maps  $C$  isomorphically to  $\overline{C}$ , we see that this isomorphism is one of Lie 3-algebras, hence  $C < B$  is semisimple and indeed  $[CCC] = C$ .

It remains to show that  $[ACC] = 0$ . For  $c_i \in C$ , the restriction of  $\text{ad}_{c_1, c_2}$  to  $A$  is a derivation of  $A$ , which belongs to  $\text{ad } A$  since for  $A$  semisimple, all derivations are inner. Since  $C$  is  $\text{ad } A$ -invariant, the 3-Jacobi identity says that  $\text{ad}_{c_1, c_2}$  is  $\text{ad } A$ -invariant, whence it belongs to the centre of  $\text{ad } A$ . However  $\text{ad } A$  is semisimple and hence its centre is 0.

In summary,  $B = A \oplus C$  as a Lie 3-algebra. Since  $A$  and  $C$  are semisimple and hence a sum of simple ideals, so is  $B$ .  $\square$

A useful notion that we will need is that of a representation of a Lie 3-algebra. A **representation** of Lie 3-algebra  $V$  on a vector space  $W$  is a Lie 3-algebra structure on the direct sum  $V \oplus W$  satisfying the following three properties:

- (1) the natural embedding  $V \rightarrow V \oplus W$  sending  $v \mapsto (v, 0)$  is a Lie 3-algebra homomorphism, so that  $[VVV] \subset V$  is the original 3-bracket on  $V$ ;
- (2)  $[V VW] \subset W$ ; and
- (3)  $[V WW] = 0$ .

The second of the above conditions says that we have a map  $\text{ad } V \rightarrow \text{End } W$  from inner derivations of  $V$  to linear transformations on  $W$ . The 3-Jacobi identity for  $V \oplus W$  says that this map is a representation of the Lie algebra  $\text{ad } V$ . Viceversa, any representation  $\text{ad } V \rightarrow \text{End } W$  defines a Lie 3-algebra structure on  $V \oplus W$  extending the Lie 3-algebra structure of  $V$  and demanding that  $[V WW] = 0$ . Taking  $W = V$  gives rise to the **adjoint representation**, whereas taking  $W = V^*$  gives rise to the **coadjoint representation**, where if  $\alpha \in V^*$  then

$$[v_1, v_2, \alpha] = \beta \in V^* \quad \text{where} \quad \beta(v) = -\alpha([v_1, v_2, v]) \quad , \quad (8)$$

for all  $v, v_i \in V$ .

Let us now introduce an inner product, so that  $(V, \Phi, b)$  is a metric Lie 3-algebra.

If  $W \subset V$  is any subspace, we define

$$W^\perp = \{v \in V \mid \langle v, w \rangle = 0, \forall w \in W\} \quad .$$

Notice that  $(W^\perp)^\perp = W$ . We say that  $W$  is **nondegenerate**, if  $W \cap W^\perp = 0$ , whence  $V = W \oplus W^\perp$ ; **isotropic**, if  $W \subset W^\perp$ ; and **coisotropic**, if  $W \supset W^\perp$ . Of course, in positive-definite signature, all subspaces are nondegenerate.

An equivalent criterion for decomposability is the existence of a proper nondegenerate ideal: for if  $I \triangleleft V$  is nondegenerate,  $V = I \oplus I^\perp$  is an orthogonal direct sum of ideals. For the proofs of the following results, the reader is asked to consult [31, §2.2].

**Lemma 7.** *Let  $I \triangleleft V$  be a coisotropic ideal of a metric Lie 3-algebra. Then  $I/I^\perp$  is a metric Lie 3-algebra.*

If  $I \triangleleft V$  is an ideal, the **centraliser**  $Z(I)$  is defined by the condition  $[Z(I)IV] = 0$ . Taking  $V$  as an ideal of itself, we arrive at the **centre**  $Z(V)$  of  $V$ .

**Lemma 8.** *Let  $V$  be a metric Lie 3-algebra. Then the centre is the orthogonal subspace to the derived ideal; that is,  $[VVV] = Z(V)^\perp$ .*

**Proposition 9.** *Let  $V$  be a metric Lie 3-algebra and  $I \triangleleft V$  be an ideal. Then*

- (1)  $I^\perp \triangleleft V$  is also an ideal;
- (2)  $I^\perp \triangleleft Z(I)$ ; and
- (3) if  $I$  is minimal then  $I^\perp$  is maximal.

**3.2. Structure of metric Lie 3-algebras.** We now investigate the structure of metric Lie 3-algebras. If a Lie 3-algebra  $V$  is not simple or one-dimensional, then it has a proper ideal and hence a minimal ideal. Let  $I \triangleleft V$  be a minimal ideal of a metric Lie 3-algebra. Then  $I \cap I^\perp$ , being an ideal contained in  $I$ , is either 0 or  $I$ . In other words, minimal ideals are either nondegenerate or isotropic. If nondegenerate,  $V = I \oplus I^\perp$  is decomposable. Therefore if  $V$  is indecomposable,  $I$  is isotropic. Moreover, by Proposition 9 (2),  $I$  is abelian and furthermore, because  $I$  is isotropic,  $[IIV] = 0$ .

It follows that if  $V$  is euclidean and indecomposable, it is either one-dimensional or simple, whence of the form (6) with all  $\varepsilon_i = 1$ . This result, originally due to Nagy [44] (see also [17, 18]), was conjectured in [49].

Let  $V$  be an indecomposable metric Lie 3-algebra. Then  $V$  is either simple, one-dimensional (provided the index of the inner product is  $< 2$ ) or possesses an isotropic proper minimal ideal  $I$  which obeys  $[IIV] = 0$ . The perpendicular ideal  $I^\perp$  is maximal and hence by Lemma 1,  $\bar{U} := V/I^\perp$  is simple or one-dimensional, whereas by Lemma 7,  $\bar{W} := I^\perp/I$  is a metric Lie 3-algebra. The inner product on  $V$  induces a nondegenerate pairing  $g : \bar{U} \otimes I \rightarrow \mathbb{R}$ . Indeed, let  $[u] = u + I^\perp \in \bar{U}$  and  $v \in I$ . Then we define  $g([u], v) = \langle u, v \rangle$ , which is clearly independent of the coset representative for  $[u]$ . In particular,  $I \cong \bar{U}^*$  is either one- or 4-dimensional. If the signature of the inner product of  $\bar{W}$  is  $(p, q)$ , that of  $V$  is  $(p + r, q + r)$  where  $r = \dim I = \dim \bar{U}$ .

There are two possibilities for  $\bar{U}$ : either it is one-dimensional or else it is simple. We will treat both cases separately.

**3.2.1.  $\bar{U}$  is one-dimensional.** If the quotient Lie 3-algebra  $\bar{U} = V/I^\perp$  is one-dimensional, so is the minimal ideal  $I$ . Let  $u \in V$  be such that  $u \notin I^\perp$ , whence its image in  $\bar{U}$  generates it. Because  $I \cong \bar{U}^*$  is induced by the inner product, there is  $v \in I$  such that  $\langle u, v \rangle = 1$ . The subspace spanned by  $u$  and  $v$  is therefore nondegenerate, and hence as a vector space we have an orthogonal decomposition  $V = \mathbb{R}(u, v) \oplus W$ , where  $W$  is the perpendicular complement of  $\mathbb{R}(u, v)$ . It is clear that  $W \subset I^\perp$ , and that  $I^\perp = I \oplus W$  as a vector space. Indeed, the projection  $I^\perp \rightarrow \bar{W}$  maps  $W$  isomorphically onto  $\bar{W}$ .

From Proposition 9 (2), it is immediate that  $[u, v, x] = 0 = [v, w_1, w_2]$ , for all  $w_i \in W$ , whence  $v$  is central. Metricity then implies that the only nonzero 3-brackets take the form

$$\begin{aligned} [u, w_1, w_2] &= [w_1, w_2] \\ [w_1, w_2, w_3] &= -\langle [w_1, w_2], w_3 \rangle v + [w_1, w_2, w_3]_W, \end{aligned} \tag{9}$$

which defines  $[w_1, w_2]$  and  $[w_1, w_2, w_3]_W$  and where  $w_i \in W$ . The 3-Jacobi identity is equivalent to the following two conditions:

- (1)  $[w_1, w_2]$  defines a Lie algebra structure on  $W$ , which leaves the inner product invariant due to the skewsymmetry of  $\langle [w_1, w_2], w_3 \rangle$ ; and
- (2)  $[w_1, w_2, w_3]_W$  defines a metric Lie 3-algebra structure on  $W$  which is invariant under the Lie algebra structure.

We will see below that this says that  $V$  is the *double extension* of the metric Lie 3-algebra  $W$  by the one-dimensional Lie 3-algebra  $\bar{U}$ .



3.2.2.  $\overline{U}$  is simple. Consider  $I^\perp$  as a Lie 3-algebra in its own right and let  $R = \text{Rad } I^\perp$  denote its radical. By Theorem 4,  $I^\perp$  admits a Levi subalgebra  $L < I^\perp$ . Since  $I^\perp \triangleleft V$  and  $R \triangleleft I^\perp$  is a characteristic ideal,  $R \triangleleft V$ . Indeed, for all  $x_i \in V$ ,  $\text{ad}_{x_1, x_2}$  is a derivation of  $I^\perp$  (since  $I^\perp \triangleleft V$ ) and by Theorem 3, it preserves  $R$ . Let  $M = V/R$ . Notice that

$$\overline{U} = V/I^\perp \cong (V/R)/(I^\perp/R) = M/L .$$

Since  $L$  and  $\overline{U}$  are semisimple, Proposition 6 says that so is  $M$  and moreover that  $M \cong L \oplus \overline{U}$ . This means that  $R$  is also the radical of  $V$ , whence  $M$  is a Levi factor of  $V$ . This discussion is summarised by the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & R & \xlongequal{\quad} & R & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I^\perp & \longrightarrow & V & \longrightarrow & \overline{U} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \parallel \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & \overline{U} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
& & 0 & & 0 & & 
\end{array}$$

The map  $M \rightarrow \overline{U}$  admits a section, so that  $M$  has a subalgebra  $\tilde{U}$  isomorphic to  $\overline{U}$  and such that  $M = \tilde{U} \oplus L$ . Then the vertical map  $V \rightarrow M$  also admits a section, whence there is a subalgebra  $U < V$  isomorphic to  $\overline{U}$  such that  $V = I^\perp \oplus U$  (as vector space). Furthermore, the inner product on  $V$  pairs  $I$  and  $U$  nondegenerately, whence  $I \oplus U$  is a nondegenerate subspace. Let  $W$  denote its perpendicular complement, whence  $V = W \oplus I \oplus U$ . Clearly  $I^\perp = W \oplus I$ , whence the canonical projection  $I^\perp \rightarrow \overline{W}$  maps  $W$  isomorphically onto  $\overline{W}$ .

Let us now write the possible 3-brackets for  $V = W \oplus I \oplus U$ . First of all, by Proposition 9 (2),  $[V, I^\perp, I] = 0$ . Since  $U < V$ ,  $[UUU] \subset U$  and since  $I$  is an ideal,  $[UUI] \subset I$ . Similarly, since  $W \subset I^\perp$  and  $I^\perp \triangleleft V$  is an ideal,  $[WWW] \subset W \oplus I$ . We write this as

$$[w_1, w_2, w_3] := [w_1, w_2, w_3]_W + \varphi(w_1, w_2, w_3) ,$$

where  $[w_1, w_2, w_3]_W$  defines an 3-bracket on  $W$ , which is isomorphic to the Lie 3-bracket of  $\overline{W} = I^\perp/I$ , and  $\varphi : \Lambda^3 W \rightarrow I$  is to be understood as an abelian extension. It remains to understand  $[UWW]$  and  $[UUW]$ . First of all, we notice that because  $W \subset I^\perp$  which is an ideal, *a priori*  $[UWW] \subset W \oplus I$  and  $[UUW] \subset W \oplus I$ . However,

$$\langle [UUW], U \rangle = -\langle [UUU], W \rangle = 0$$

whence the component of  $[UUW]$  along  $I$  vanishes, so that  $[UUW] \subset W$ . Furthermore, the 3-Jacobi identity makes  $W$  into an  $\text{ad } U$ -representation and  $\varphi$  into an  $\text{ad } U$ -equivariant map.

Similarly,

$$\langle [UWW], W \rangle = -\langle [WWW], U \rangle ,$$

whence the  $W$  component of  $[UWW]$  is determined by the map  $\varphi$  defined above; whereas the  $I$  component

$$\langle [UWW], U \rangle = \langle [UUW], W \rangle$$

is thus determined by the action of  $\text{ad } U$  on  $W$ .

In summary, we have the following nonzero 3-brackets

$$[UUU] \subset U \quad [UUI] \subset I \quad [UUW] \subset W \quad [UWW] \subset W \oplus I \quad [WWW] \subset W \oplus I ,$$

which we will proceed to explain. The first bracket is simply the fact that  $U < V$  is a subalgebra, whereas the second makes  $I$  into a representation of  $U$ . In fact,  $I \cong U^*$  is the coadjoint representation (8). The third bracket defines an action of  $\text{ad } U$  on  $W$  and this also determines the  $I$ -component of the fourth bracket. The  $W$ -component of the fourth bracket is determined by the  $I$ -component of the last bracket. The last bracket defines a Lie 3-algebra structure on  $W \oplus I$ , which is an abelian extension of the Lie 3-algebra structure on  $\overline{W}$  by a ‘‘cocycle’’  $\varphi : \Lambda^3 W \rightarrow I$ . The inner product is such that  $\langle W, W \rangle$  and  $\langle U, I \rangle$  are nondegenerate and the only other nonzero inner product is  $\langle U, U \rangle$  which can be any  $\text{ad } U$ -invariant symmetric bilinear form on  $U$ , not necessarily nondegenerate.

Similarly to the case when  $\overline{U}$  is one-dimensional, we will interpret  $V$  as the *double extension* of the metric Lie 3-algebra  $\overline{W}$  by the simple Lie 3-algebra  $U$ .

More generally we have the following definition.

**Definition 10.** Let  $W$  be a metric Lie 3-algebra and let  $U$  be a Lie 3-algebra. Then by the **double extension of  $W$  by  $U$**  we mean the metric Lie 3-algebra on the vector space  $W \oplus U \oplus U^*$  with the following nonzero 3-brackets:

- $[UUU] \subset U$  being the bracket of the Lie 3-algebra  $U$ ;
- $[UUU^*] \subset U^*$  being the coadjoint action of  $\text{ad } U$  on  $U^*$ ;
- $[UUW] \subset W$  being the action of  $\text{ad } U$  on  $W$ ;
- $[UWW] \subset W \oplus U^*$ , where the  $U^*$  component is related to the previous bracket by

$$\langle [u_1, w_1, w_2], u_2 \rangle = \langle [u_1, u_2, w_1], w_2 \rangle .$$

- $[WWW] \subset W \oplus U^*$ , where the  $W$  component is the bracket of the Lie 3-algebra  $W$  and the  $U^*$  component is related to the  $W$  component of the previous bracket by

$$\langle [w_1, w_2, w_3], u_1 \rangle = \langle u_1, w_1, w_2 \rangle, w_3 \rangle .$$

These brackets are subject to the 3-Jacobi identity. Two of these identities can be interpreted as saying that the bracket  $[UUW] \subset W$  defines a Lie algebra homomorphism  $\text{ad } U \rightarrow \text{Der}^0 W$ , where  $\text{Der}^0 W$  is the Lie algebra of skewsymmetric derivations of the Lie 3-algebra  $W$ , whereas the map  $\Lambda^3 W \rightarrow U^*$  defining the  $U^*$  component of the  $[WWW]$  bracket is  $\text{ad } U$ -equivariant. We have not found similarly transparent interpretations for the other Jacobi identities. The above 3-brackets leave invariant the inner product on  $V$  with components

- $\langle W, W \rangle$ , being the inner product on the metric Lie 3-algebra  $W$ ;

- $\langle U, U^* \rangle$ , being the natural dual pairing; and
- $\langle U, U \rangle$ , being any  $\text{ad } U$ -invariant symmetric bilinear form.

**Remark 11.** It can be shown that if  $U$  is simple and  $[UUW] = 0$  then the resulting double extension is decomposable. Indeed, if  $[UUW] = 0$ , then by Jacobis the  $U^*$  component in  $[WWW]$  would have to be invariant under  $\text{ad } U$ . If  $U$  is simple, then this means that this component is absent, whence  $W$  would be a subalgebra and indeed an ideal since the  $U^*$  component in  $[UWW]$  is also absent. But  $W$  is nondegenerate, whence it decomposes  $V$ .

In summary, we have proved the following result.

**Theorem 12.** *Every indecomposable metric Lie 3-algebra  $V$  is either one-dimensional, simple or else it is the double extension of a metric Lie 3-algebra  $W$  by a one-dimensional or simple Lie 3-algebra  $U$ .*

Any metric Lie 3-algebra will be an orthogonal direct sum of indecomposables, each one being either one-dimensional, simple or a double extension of a metric Lie 3-algebra, which itself is an orthogonal direct sum of indecomposables of strictly lower dimension. Continuing in this way, we arrive at the following characterisation.

**Corollary 13.** *The class of metric Lie 3-algebras is generated by the simple and one-dimensional Lie 3-algebras under the operations of orthogonal direct sum and double extension.*

It is clear that the subclass of euclidean metric Lie 3-algebras is generated by the simple and one-dimensional euclidean Lie 3-algebras under orthogonal direct sum, since double extension always incurs in indefinite signature. The lorentzian indecomposables admit at most one double extension by a one-dimensional Lie 3-algebra and are easy to classify [31]. The indecomposables of signature  $(2, p)$  will admit at most two double extensions by one-dimensional Lie 3-algebras. We will find that there are three kinds of such metric Lie algebras: a simple Lie 3-algebra, one which can be written as a double extension and one which is the result of iterating two double extensions.

#### 4. METRIC LIE 3-ALGEBRAS WITH SIGNATURE $(2, p)$

In [31] we classified the lorentzian Lie 3-algebras and in this section we will continue by classifying the metric Lie 3-algebras with signature  $(2, p)$ . Clearly any such metric Lie 3-algebra will be isomorphic to one of two types:

- $V_0 \oplus V_1 \oplus \dots$ , where  $V_0$  is an indecomposable metric Lie 3-algebra of signature  $(2, *)$  and  $V_{i \geq 1}$  are indecomposable euclidean Lie 3-algebras; or
- $V_1 \oplus V_2 \oplus V_3 \oplus \dots$ , where  $V_{1,2}$  are indecomposable lorentzian Lie 3-algebras and  $V_{i \geq 3}$  are indecomposable euclidean Lie 3-algebras.

The indecomposable euclidean Lie 3-algebras have been classified in [44] (see also [17, 18]) and the indecomposable lorentzian Lie 3-algebras have been classified in [31]. It remains to classify the indecomposable metric Lie 3-algebra of signature  $(2, *)$  and this is what the rest of this section is devoted to.

**4.1. Notation.** In this section we will use the following notation. Lie 3-algebras will be denoted by capital letters  $V, W, \dots$ , whereas Lie algebras will be denoted by lowercase fraktur letters  $\mathfrak{g}, \mathfrak{h}, \dots$

If  $\mathfrak{k}$  is a metric Lie algebra, we will let  $W(\mathfrak{k})$  denote the metric Lie 3-algebra  $W(\mathfrak{k}) = \mathbb{R}u \oplus \mathbb{R}v \oplus \mathfrak{k}$  with inner product which extends the ad-invariant inner product on  $\mathfrak{k}$  by declaring  $u, v$  perpendicular to  $\mathfrak{k}$  and, in addition,  $\langle u, u \rangle = 0 = \langle v, v \rangle$ ,  $\langle u, v \rangle = 1$ . The Lie 3-brackets of  $W(\mathfrak{k})$  are given in terms of the inner product and the Lie bracket of  $\mathfrak{k}$  by

$$[u, x, y] = [x, y] \quad \text{and} \quad [x, y, z] = -\langle [x, y], z \rangle v, \quad (10)$$

for all  $x, y, z \in \mathfrak{k}$ . This class of Lie 3-algebras was discovered independently in [20–22]. We will mostly be interested in the case where  $\mathfrak{k}$  is a reductive Lie algebra with a positive-definite inner product, so that  $W(\mathfrak{k})$  is lorentzian. In this case, if  $\mathfrak{k}$  is not semisimple, then  $W(\mathfrak{k})$  will be decomposable, since any abelian summands of  $\mathfrak{k}$  will factorise. Indeed, if we let  $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{a}$  with  $\mathfrak{s}$  semisimple and  $\mathfrak{a}$  abelian, then  $W(\mathfrak{s} \oplus \mathfrak{a}) = W(\mathfrak{s}) \oplus A$ , where  $A$  is the abelian Lie 3-algebra sharing the same underlying vector space as the abelian Lie algebra  $\mathfrak{a}$ .

**4.2. Structure of metric Lie 3-algebras with signature  $(2, p)$ .** Let  $V$  be a finite-dimensional indecomposable metric Lie 3-algebra of signature  $(2, p)$ . The results of Section 3.2 allow us to conclude that one of two situations can happen: either  $V$  is simple, whence isomorphic to  $S_{2,2}$ , or  $V$  will be the double extension by a one-dimensional Lie 3-algebra  $U$  of a lorentzian Lie 3-algebra  $W$ . In other words,  $V = \mathbb{R}(u, v) \oplus W$ , where  $\langle u, u \rangle = \langle v, v \rangle = 0$  and  $\langle u, v \rangle = 1$  and  $u, v \perp W$ , and with 3-brackets

$$[u, x, y] = [x, y] \quad \text{and} \quad [x, y, z] = [x, y, z]_W - \langle [x, y], z \rangle v, \quad (11)$$

for all  $x, y, z \in W$ , which defines  $[x, y]$  and where  $[x, y, z]_W$  are the 3-brackets of  $W$ . The 3-Jacobi identity is equivalent to  $[-, -] : \Lambda^2 W \rightarrow W$  being a Lie (2-)bracket which leaves invariant the Lie 3-bracket  $[-, -, -]_W$ . Furthermore, both the Lie algebra and the Lie 3-algebra structures on  $W$  preserve the lorentzian inner product.

$W$  is therefore simultaneously a lorentzian Lie algebra and a lorentzian Lie 3-algebra, relative to the same inner product. We shall denote it  $W$  as a Lie 3-algebra and as a vector space, but  $\mathfrak{w}$  as a Lie algebra. As Lie 3-algebra we may write it as  $W = W_0 \oplus W_1$ , where  $W_0$  is an indecomposable lorentzian Lie 3-algebra and  $W_1$  is a euclidean Lie 3-algebra. By the results of [31],  $W_0$  can be either one-dimensional, isomorphic to  $S_{1,3}$  or else isomorphic to  $W(\mathfrak{s})$ , where  $\mathfrak{s}$  is a euclidean semisimple Lie algebra, whereas by the results of [44] (see also [17, 18]),  $W_1 \cong A \oplus S_{0,4} \oplus \dots \oplus S_{0,4}$ , where  $A$  is an abelian euclidean Lie 3-algebra. In summary, we have the following possibilities for  $W$ , as a Lie 3-algebra:

- $W = A \oplus S_{0,4} \oplus \dots \oplus S_{0,4}$ , where  $A$  is a lorentzian abelian Lie 3-algebra;
- $W = A \oplus S_{1,3} \oplus S_{0,4} \oplus \dots \oplus S_{0,4}$ , where  $A$  is a euclidean abelian Lie 3-algebra; and
- $W = W(\mathfrak{k}) \oplus S_{0,4} \oplus \dots \oplus S_{0,4}$ , where  $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{a}$  is a euclidean reductive Lie algebra.

As a Lie algebra, the adjoint representation of  $\mathfrak{w}$  on  $W$  preserves both the inner product and the 3-algebra structure. It follows from the results of [31, §3.2] that the adjoint representation of  $\mathfrak{w}$  preserves the subspaces corresponding to the simple factors and hence

also their perpendicular complement. This means that each summand of the vector space  $W$  is a subrepresentation under the adjoint representation and hence an ideal. Furthermore, it also follows from the results of [31, §3.2], that any summands in  $W$  isomorphic to  $S_{0,4}$  factor out, decomposing  $V$  in the process. Hence for indecomposable  $V$ , they have to be absent. This means that the possibilities for  $W$  as a Lie 3-algebra become

- (I)  $W = A$ , where  $A$  is a lorentzian abelian Lie 3-algebra;
- (II)  $W = A \oplus S_{1,3}$ , where  $A$  is a euclidean abelian Lie 3-algebra; and
- (III)  $W = W(\mathfrak{s} \oplus \mathfrak{a})$ ;

whereas for  $W$  as a Lie algebra, we find that in the case (I),  $\mathfrak{w}$  is a lorentzian Lie algebra of dimension  $\dim A$ ; whereas in case (II),  $\mathfrak{w} = \mathfrak{g} \oplus \mathfrak{h}_0$ , where  $\mathfrak{g}$  is a euclidean Lie algebra of dimension  $\dim A$  and  $\mathfrak{h}_0$  is a four-dimensional lorentzian Lie algebra, hence either abelian or isomorphic to one of the following three Lie algebras:  $\mathfrak{so}(3) \oplus \mathbb{R}$ ,  $\mathfrak{so}(1,2) \oplus \mathbb{R}$  or the solvable Nappi–Witten Lie algebra [50], described as a central extension of the Lie algebra of euclidean motions of  $\mathbb{R}^2$  or, alternatively, as a double extension of an abelian two-dimensional euclidean Lie algebra by a one-dimensional Lie algebra [51]. Case (III) is somewhat different and will be treated in detail below.

Now we will show that case (II) always decomposes  $V$  and hence it cannot occur. The proof is analogous to the one in [31, §3.2] for the euclidean simple factors  $S_{0,4}$ . We only need to show that any of the nonabelian Lie algebras  $\mathfrak{h}_0$  can occur as the reduced Lie algebra associated to some  $x \in S_{1,3}$ ; that is, the one with Lie bracket  $[x, -, -]$ . As mentioned above, there are three possible nonabelian lorentzian four-dimensional Lie algebras:

- $\mathfrak{so}(3) \oplus \mathbb{R}$ , which has a timelike centre;
- $\mathfrak{so}(1,2) \oplus \mathbb{R}$ , which has a spacelike centre; and
- the solvable Nappi–Witten Lie algebra, which has a lightlike centre.

Given the causal characters of the centres, the following should perhaps not be too surprising.

**Proposition 14.** *Let  $0 \neq x \in S_{1,3}$  and let  $\mathfrak{h}_x$  denote the Lie algebra structure on the vector space  $S_{1,3}$  with Lie bracket  $[x, -, -]$ . Then*

- if  $x$  is timelike,  $\mathfrak{h}_x \cong \mathfrak{so}(3) \oplus \mathbb{R}$ ;
- if  $x$  is spacelike,  $\mathfrak{h}_x \cong \mathfrak{so}(1,2) \oplus \mathbb{R}$ ; and
- if  $x$  is lightlike,  $\mathfrak{h}_x$  is isomorphic to the Nappi–Witten Lie algebra.

*Proof.* The 3-brackets of  $S_{1,3}$ , relative to a pseudo-orthonormal basis  $(\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ , are given by

$$[\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2] = -\mathbf{e}_3 \quad [\mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_3] = +\mathbf{e}_2 \quad [\mathbf{e}_0, \mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_1 \quad [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_0 .$$

The automorphism group of these brackets is  $\text{SO}(1,3)$ , whence without loss of generality we can take  $x$  to be  $\mathbf{e}_0$ ,  $\mathbf{e}_3$  or  $\mathbf{e}_0 + \mathbf{e}_3$  in the timelike, spacelike and lightlike cases, respectively. We now discuss the reduced Lie algebras in each case.

- For  $x = \mathbf{e}_0$ , we have

$$[\mathbf{e}_1, \mathbf{e}_2] = -\mathbf{e}_3 \quad [\mathbf{e}_1, \mathbf{e}_3] = +\mathbf{e}_2 \quad [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_1 ,$$

and in addition  $\mathbf{e}_0$  central. The resulting Lie algebra is clearly isomorphic to  $\mathfrak{so}(3) \oplus \mathbb{R}$ .

- For  $x = \mathbf{e}_3$ , we have

$$[\mathbf{e}_0, \mathbf{e}_1] = +\mathbf{e}_2 \quad [\mathbf{e}_0, \mathbf{e}_2] = -\mathbf{e}_1 \quad [\mathbf{e}_1, \mathbf{e}_2] = -\mathbf{e}_0 .$$

with  $\mathbf{e}_3$  central. The resulting Lie algebra is clearly isomorphic to  $\mathfrak{so}(1, 2) \oplus \mathbb{R}$ .

- Let  $\mathbf{e}_\pm = \mathbf{e}_3 \pm \mathbf{e}_0$  and take  $x = \mathbf{e}_+$ , to obtain

$$[\mathbf{e}_1, \mathbf{e}_2] = -\mathbf{e}_+ \quad [\mathbf{e}_-, \mathbf{e}_1] = -2\mathbf{e}_2 \quad [\mathbf{e}_-, \mathbf{e}_2] = +2\mathbf{e}_1 ,$$

with  $\mathbf{e}_+$  central. We recognise this as the double extension of the the abelian Lie algebra spanned by  $\mathbf{e}_1, \mathbf{e}_2$  by the one-dimensional Lie algebra spanned by  $\mathbf{e}_-$  with dual  $\mathbf{e}_+$ . We can interpret  $\mathbf{e}_1, \mathbf{e}_2$  as the generators of translations in the plane and  $\mathbf{e}_-$  as the generator of rotations, and we are then centrally extending the translations by  $\mathbf{e}_+$ . In either of these descriptions, we see that the resulting Lie algebra is isomorphic to the Nappi–Witten Lie algebra. □

Essentially the same proof as that in [31, §3.2] for  $S_{0,4}$  now shows that  $S_{1,3}$  can be twisted out of  $V$ , decomposing it. In summary, case (2) cannot occur.

Finally, let us discuss case (III). We will let  $\mathfrak{k} = \mathfrak{s} \oplus \mathfrak{a}$  denote a generic reductive Lie algebra with a positive-definite invariant inner product. Under the adjoint representation,  $\mathfrak{w}$  gets mapped to the Lie algebra  $\text{Der}^0 W(\mathfrak{k})$  of skewsymmetric derivations of the Lie 3-algebra  $W(\mathfrak{k})$ , which we now determine.

**Proposition 15.** *Let  $\mathfrak{s}$  and  $\mathfrak{a}$  be a semisimple and abelian Lie algebras, respectively, with invariant positive-definite inner products. Then*

$$\text{Der}^0 W(\mathfrak{s} \oplus \mathfrak{a}) \cong (\text{ad } \mathfrak{s} \times \mathfrak{s}_{ab}) \oplus (\mathfrak{so}(\mathfrak{a}) \times \mathfrak{a}) ,$$

with  $\mathfrak{s}_{ab}$  and  $\mathfrak{a}$  acting as null rotations on the lorentzian vector space  $W(\mathfrak{s} \oplus \mathfrak{a})$ .

*Proof.* The most general skewsymmetric endomorphism of  $W(\mathfrak{s} \oplus \mathfrak{a})$  is given in terms of  $\alpha \in \mathbb{R}$ ,  $y, z \in \mathfrak{s}$ ,  $b, c \in \mathfrak{a}$ ,  $f \in \mathfrak{so}(\mathfrak{s})$ ,  $g \in \mathfrak{so}(\mathfrak{a})$  and  $\varphi : \mathfrak{s} \rightarrow \mathfrak{a}$  by

$$\begin{aligned} D\mathbf{e}_- &= \alpha\mathbf{e}_- + y + b & Dx &= -\langle z, x \rangle \mathbf{e}_- - \langle y, x \rangle \mathbf{e}_+ + f(x) + \varphi(x) \\ D\mathbf{e}_+ &= -\alpha\mathbf{e}_+ + z + c & Da &= -\langle c, a \rangle \mathbf{e}_- - \langle b, a \rangle \mathbf{e}_+ - \varphi^t(a) + g(a) , \end{aligned}$$

for all  $x \in \mathfrak{s}$  and  $a \in \mathfrak{a}$ . Demanding that  $D$  preserves the Lie 3-bracket we obtain the following extra conditions:

- From  $[\mathbf{e}_+, \mathbf{e}_-, x]$  we find  $[z, x]_{\mathfrak{s}} = 0$  for all  $x$ , but since  $\mathfrak{s}$  is semisimple and has trivial centre, we conclude that  $z = 0$ .
- From  $[\mathbf{e}_-, x_1, x_2]$  we find that  $\varphi$  must annihilate  $[\mathfrak{s}, \mathfrak{s}]_{\mathfrak{s}}$ , which implies that  $\varphi = 0$  since  $\mathfrak{s}$  is semisimple. One also finds that  $f + \alpha \text{id}$  is a derivation of  $\mathfrak{s}$  which, since all derivations are inner, allows us to conclude that  $f + \alpha \text{id} \in \text{ad } \mathfrak{s}$ .
- From  $[a, x_1, x_2]$  we find that  $\langle c, a \rangle = 0$  for all  $a$ , whence  $c = 0$ .
- From  $[x_1, x_2, x_3]$  we find that  $f$  must be skewsymmetric, which means that  $\alpha = 0$ , whence  $f \in \text{ad } \mathfrak{s}$ .



In summary, the most general  $D \in \text{Der}^0 W(\mathfrak{s} \oplus \mathfrak{a})$  is given in terms of  $g \in \mathfrak{so}(\mathfrak{a})$ ,  $c \in \mathfrak{a}$  and  $y, z \in \mathfrak{s}$  by

$$\begin{aligned} D\mathbf{e}_- &= z + c & Dx &= [y, x]_{\mathfrak{s}} - \langle z, x \rangle \mathbf{e}_+ \\ D\mathbf{e}_+ &= 0 & Da &= ga - \langle c, a \rangle \mathbf{e}_+ . \end{aligned}$$

The Lie bracket of  $\text{Der}^0 W(\mathfrak{s} \oplus \mathfrak{a})$  can be computed simply from the commutator of derivations, and we obtain

$$[(y_1, z_1, g_1, c_1), (y_2, z_2, g_2, c_2)] = ([y_1, y_2]_{\mathfrak{s}}, [y_1, z_2]_{\mathfrak{s}} - [y_2, z_1]_{\mathfrak{s}}, [g_1, g_2]_{\mathfrak{so}(\mathfrak{a})}, g_1 c_2 - g_2 c_1) ,$$

which is precisely the direct sum of  $\mathfrak{s} \times \mathfrak{s}_{\text{ab}}$  (with generic elements  $(y, z)$ ) and  $\mathfrak{so}(\mathfrak{a}) \times \mathfrak{a}$  (with generic elements  $(g, c)$ ).  $\square$

It follows that  $W(\mathfrak{s} \oplus \mathfrak{a})$  is not fully reducible as a representation of  $\text{Der}^0 W(\mathfrak{s} \oplus \mathfrak{a})$ . Indeed,  $\mathbf{e}_+$  spans an invariant subspace without a complementary subspace which is also invariant.

Proposition 15 restricts the possible Lie algebra structures  $\mathfrak{w}$  on the underlying vector space of  $W(\mathfrak{s} \oplus \mathfrak{a})$ , since for every  $w \in \mathfrak{w}$ ,  $\text{ad}_w \in \text{Der}^0 W(\mathfrak{s} \oplus \mathfrak{a})$ . In fact, we have the following

**Proposition 16.** *The most general Lie algebra  $\mathfrak{w}$  is given by*

$$\begin{aligned} [\mathbf{e}_-, a] &= Ja & [s_1, s_2] &= [\psi s_1, s_2]_{\mathfrak{s}} + \langle z, [s_1, s_2]_{\mathfrak{s}} \rangle \mathbf{e}_+ \\ [\mathbf{e}_-, s] &= [z, s]_{\mathfrak{s}} & [a_1, a_2] &= [a_1, a_2]_{\mathfrak{r}} + \langle Ja_1, a_2 \rangle \mathbf{e}_+ \end{aligned}$$

for all  $a, a_1, a_2 \in \mathfrak{a}$  and  $s, s_1, s_2 \in \mathfrak{s}$  and where  $J \in \mathfrak{so}(\mathfrak{a})$ ,  $z \in \mathfrak{s}$ ,  $[-, -]_{\mathfrak{r}}$  defines a reductive Lie algebra structure  $\mathfrak{r}$  on the vector space  $\mathfrak{a}$ ,  $J$  is a derivation over  $[-, -]_{\mathfrak{r}}$ , and  $\psi \in \text{End } \mathfrak{s}$  obeys

$$[\psi s_1, s_2]_{\mathfrak{s}} = [s_1, \psi s_2]_{\mathfrak{s}} \tag{12}$$

and

$$\psi[\psi s_1, s_2]_{\mathfrak{s}} = [\psi s_1, \psi s_2]_{\mathfrak{s}} , \tag{13}$$

for all  $s_1, s_2 \in \mathfrak{s}$ , and

$$[z, \psi s]_{\mathfrak{s}} = \psi[z, s]_{\mathfrak{s}} , \tag{14}$$

for all  $s \in \mathfrak{s}$ .

*Proof.* In this proof,  $W := W(\mathfrak{s} \oplus \mathfrak{a})$ . We determine the most general form of the Lie brackets using Proposition 15 and the fact that  $[x, y]$  can be interpreted both as  $\text{ad}_x y$  or as  $-\text{ad}_y x$ , which are the actions of  $\text{ad}_x \in \text{Der}^0 W$  on  $y$  and of  $\text{ad}_y \in \text{Der}^0 W$  on  $x$ , for all  $x, y \in W$ .

Since every derivation  $D \in \text{Der}^0 W$  annihilates  $\mathbf{e}_+$ , we see that  $\mathbf{e}_+$  is central in  $\mathfrak{w}$ .

Now consider the bracket  $[\mathbf{e}_-, a]$  for  $a \in \mathfrak{a}$ . From Proposition 15, we see that  $[\mathbf{e}_-, a] = \text{ad}_{\mathbf{e}_-} a \in \mathfrak{a} \oplus \mathbb{R}\mathbf{e}_+$ , whereas  $[\mathbf{e}_-, a] = \text{ad}_{-a} \mathbf{e}_- \in \mathfrak{a} \oplus \mathfrak{s}$ , whence  $[\mathbf{e}_-, a] \in \mathfrak{a}$  and hence  $\text{ad}_{\mathbf{e}_-} : \mathfrak{a} \rightarrow \mathfrak{a}$  defines a skewsymmetric endomorphism we call  $J$ . Similarly,  $[\mathbf{e}_-, s]$  for  $s \in \mathfrak{s}$ , belongs to  $\mathfrak{s}$ , whence  $\text{ad}_{\mathbf{e}_-} : \mathfrak{s} \rightarrow \mathfrak{s}$  defines a skewsymmetric endomorphism, which by Proposition 15 is actually an inner derivation (relative to the Lie bracket of  $\mathfrak{s}$ ), whence  $[\mathbf{e}_-, s] = [z, s]_{\mathfrak{s}}$  for some  $z \in \mathfrak{s}$ .

Now consider the bracket  $[a, s]$  for  $a \in \mathfrak{a}$  and  $s \in \mathfrak{s}$ . From Proposition 15,  $\text{ad}_a s \in \mathfrak{s} \oplus \mathbb{R}e_+$ , whereas  $\text{ad}_s a \in \mathfrak{a} \oplus \mathbb{R}e_+$ , whence  $[a, s] \in \mathbb{R}e_+$ . However, metricity shows that this has to vanish, since  $\langle [s, a], e_- \rangle = \langle s, [a, e_-] \rangle = 0$ , since  $[a, e_-] \in \mathfrak{a}$  and  $\mathfrak{a} \perp \mathfrak{s}$ .

Now consider the bracket  $[s_1, s_2]$  for  $s_i \in \mathfrak{s}$ . Since  $[s_1, e_-] = [s_1, z]$  and using metricity, we find that  $[s_1, s_2] = [\psi s_1, s_2] + \langle z, [s_1, s_2] \rangle e_+$ , for some endomorphism  $\psi \in \text{End } \mathfrak{s}$ , *not* necessarily a Lie algebra homomorphism.

Finally, we consider the bracket  $[a_1, a_2]$  for  $a_i \in \mathfrak{a}$ . Since  $[a_1, e_-] = -Ja_1$  and again using metricity, we find that  $[a_1, a_2] = g(a_1)a_2 + \langle Ja_1, a_2 \rangle e_+$ , where  $g : \mathfrak{a} \rightarrow \mathfrak{so}(\mathfrak{a})$ .

There are some conditions that we have to impose on these brackets: skewsymmetry, Jacobi identity and that the map  $\text{ad} : \mathfrak{w} \rightarrow \text{Der}^0 W$  is a Lie algebra homomorphism. These conditions are straight-forward to impose and can be summarised as follows. Skewsymmetry of the bracket and the Jacobi identity

$$[a_1, [a_2, a_3]] = [[a_1, a_2], a_3] + [a_1, [a_2, a_3]] .$$

imply that the bracket  $[a_1, a_2]_{\mathfrak{r}} := g(a_1)a_2$  defines a reductive (since  $\text{ad}^{\mathfrak{r}} a \in \mathfrak{so}(\mathfrak{a})$ ) Lie algebra structure, say  $\mathfrak{r}$ , on the vector space  $\mathfrak{a}$ . The skewsymmetry of the Lie bracket on  $\mathfrak{w}$  and the Jacobi identity

$$[e_-, [a_1, a_2]] = [[e_-, a_1], a_2] + [a_1, [e_-, a_2]] ,$$

say that  $J \in \text{Der}^0 \mathfrak{r}$ .

Condition (12) on  $\psi$  follows from the skewsymmetry of the Lie bracket and (14) from the Jacobi identity

$$[e_-, [s_1, s_2]] = [[e_-, s_1], s_2] + [s_1, [e_-, s_2]] .$$

Indeed, expanding the above Jacobi identity we see that

$$[z, [\psi s_1, s_2]_{\mathfrak{s}}]_{\mathfrak{s}} = [\psi [z, s_1]_{\mathfrak{s}}, s_2]_{\mathfrak{s}} + [\psi s_1, [z, s_2]_{\mathfrak{s}}]_{\mathfrak{s}} ,$$

which, upon using the Jacobi identity for  $[-, -]_{\mathfrak{s}}$  on the left-hand side and (12) on the first term of the right-hand side, becomes

$$[[z, \psi s_1]_{\mathfrak{s}}, s_2]_{\mathfrak{s}} = [[z, s_1]_{\mathfrak{s}}, \psi s_2]_{\mathfrak{s}} = [\psi [z, s_1]_{\mathfrak{s}}, s_2]_{\mathfrak{s}} .$$

Since  $\mathfrak{s}$  is semisimple and has trivial centre, we see that this implies  $[z, \psi s_1]_{\mathfrak{s}} = \psi [z, s_1]_{\mathfrak{s}}$ , which is (14). Finally, the Jacobi identity

$$[s_1, [s_2, s_3]] = [[s_1, s_2], s_3] + [s_1, [s_2, s_3]]$$

is equivalent to (13). Indeed, the above Jacobi identity expands to

$$[\psi s_1, [\psi s_2, s_3]_{\mathfrak{s}}]_{\mathfrak{s}} = [\psi [\psi s_1, s_2]_{\mathfrak{s}}, s_3]_{\mathfrak{s}} + [\psi s_2, [\psi s_1, s_3]_{\mathfrak{s}}]_{\mathfrak{s}}$$

up to central terms which vanish due to the Jacobi identity for  $[-, -]_{\mathfrak{s}}$ . Using the Jacobi identity of  $[-, -]_{\mathfrak{s}}$  on the above relation we find

$$[[\psi s_1, \psi s_2]_{\mathfrak{s}} - \psi [\psi s_1, s_2]_{\mathfrak{s}}, s_3]_{\mathfrak{s}} = 0 ,$$

which, using that  $\mathfrak{s}$  has trivial centre, becomes (13).  $\square$

**Remark 17.** Notice that (13) says that the image of  $\psi$  is a Lie subalgebra and that on its image,  $\psi$  commutes with the restriction there of  $\text{ad}^{\mathfrak{s}}$ .

It turns out that (12) implies (13). This will be useful later because (12) is linear in  $\psi$ .

**Lemma 18.** *Let  $\mathfrak{s}$  be a metric Lie algebra and let  $\psi \in \text{End } \mathfrak{s}$  obey (12) for all  $s_1, s_2 \in \mathfrak{s}$ . The new bracket  $[s_1, s_2] := [\psi s_1, s_2]_{\mathfrak{s}}$  obeys the Jacobi identity.*

*Proof.* The Jacobi identity of  $[s_1, s_2] := [\psi s_1, s_2]_{\mathfrak{s}}$  is equivalent to

$$\mathcal{C}_{1,2,3} [[s_1, s_2], s_3] = \mathcal{C}_{1,2,3} [\psi[\psi s_1, s_2]_{\mathfrak{s}}, s_3]_{\mathfrak{s}} = 0 ,$$

for all  $s_i \in \mathfrak{s}$ , and where  $\mathcal{C}$  indicates cyclic permutations of the indices. This is clearly equivalent to

$$\mathcal{C}_{1,2,3} \langle [[s_1, s_2], s_3], s_4 \rangle = 0 ,$$

for all  $s_i \in \mathfrak{s}$ . We now manipulate this expression using invariance of the inner product, equation (12) and the Jacobi identity for  $[-, -]_{\mathfrak{s}}$ :

$$\begin{aligned} \mathcal{C}_{1,2,3} \langle [\psi[\psi s_1, s_2]_{\mathfrak{s}}, s_3]_{\mathfrak{s}}, s_4 \rangle &= \mathcal{C}_{1,2,3} \langle [[\psi s_1, s_2]_{\mathfrak{s}}, \psi s_3]_{\mathfrak{s}}, s_4 \rangle \\ &= \mathcal{C}_{1,2,3} \langle [\psi s_1, s_2]_{\mathfrak{s}}, [\psi s_3, s_4]_{\mathfrak{s}} \rangle \\ &= \mathcal{C}_{1,2,3} \langle [\psi s_1, s_2]_{\mathfrak{s}}, [s_3, \psi s_4]_{\mathfrak{s}} \rangle \\ &= \mathcal{C}_{1,2,3} \langle [[\psi s_1, s_2]_{\mathfrak{s}}, s_3]_{\mathfrak{s}}, \psi s_4 \rangle \\ &= \mathcal{C}_{1,2,3} (\langle [\psi s_1, [s_2, s_3]_{\mathfrak{s}}]_{\mathfrak{s}}, \psi s_4 \rangle + \langle [[\psi s_1, s_3]_{\mathfrak{s}}, s_2]_{\mathfrak{s}}, \psi s_4 \rangle) . \end{aligned}$$

The last term on the RHS can be further rewritten as

$$\begin{aligned} \mathcal{C}_{1,2,3} \langle [[\psi s_1, s_3]_{\mathfrak{s}}, s_2]_{\mathfrak{s}}, \psi s_4 \rangle &= \mathcal{C}_{1,2,3} \langle [[s_1, \psi s_3]_{\mathfrak{s}}, s_2]_{\mathfrak{s}}, \psi s_4 \rangle \\ &= \mathcal{C}_{1,2,3} \langle [[s_2, \psi s_1]_{\mathfrak{s}}, s_3]_{\mathfrak{s}}, \psi s_4 \rangle \\ &= - \mathcal{C}_{1,2,3} \langle [[\psi s_1, s_2]_{\mathfrak{s}}, s_3]_{\mathfrak{s}}, \psi s_4 \rangle , \end{aligned}$$

where we have used cyclicity and (12). Therefore we see, using (12) again and the invariance of the inner product, that

$$\begin{aligned} \mathcal{C}_{1,2,3} \langle [\psi[\psi s_1, s_2]_{\mathfrak{s}}, s_3]_{\mathfrak{s}}, s_4 \rangle &= \frac{1}{2} \mathcal{C}_{1,2,3} \langle [\psi s_1, [s_2, s_3]_{\mathfrak{s}}]_{\mathfrak{s}}, \psi s_4 \rangle \\ &= - \mathcal{C}_{1,2,3} \langle [s_2, s_3]_{\mathfrak{s}}, [\psi s_1, \psi s_4]_{\mathfrak{s}} \rangle \\ &= \mathcal{C}_{1,2,3} \langle [s_1, [s_2, s_3]_{\mathfrak{s}}]_{\mathfrak{s}}, \psi^2 s_4 \rangle , \end{aligned}$$

which vanishes by the Jacobi identity of  $[-, -]_{\mathfrak{s}}$ .  $\square$

**Remark 19.** In the lemma we assumed only that  $\mathfrak{s}$  is a metric Lie algebra. However in order to relate the Jacobi identity of the bracket  $[\psi s_1, s_2]_{\mathfrak{s}}$  to (13) we did use, in addition, that  $\mathfrak{s}$  had trivial centre.

**Proposition 20.** *Let  $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_p$  be a semisimple Lie algebra with  $\mathfrak{s}_i$  its simple ideals, and let  $\psi \in \text{End } \mathfrak{s}$  obey (12) for all  $s_1, s_2 \in \mathfrak{s}$ . Then  $\psi = \sum_{i=1}^p \lambda_i \Pi_i$ , where  $\Pi_i$  is the orthogonal projection onto  $\mathfrak{s}_i$  and where  $\lambda_i \in \mathbb{R}$ .*

*Proof.* We observe that if  $\psi \in \text{End } \mathfrak{s}$  obeys condition (12), then so does any power  $\psi^n$ . Furthermore, condition (12) is clearly linear in  $\psi$ , whence any linear combination of endomorphisms satisfying condition (12) will again satisfy (12). Hence in particular the exponential  $\exp(t\psi)$  satisfies (12) and hence, by Lemma 18, also equation (13). By Remark 17,  $\exp(t\psi)$  commutes with the adjoint representation on its image, which being invertible is all of  $\mathfrak{s}$ . Since  $\mathfrak{s}$  is semisimple, Schur's lemma says that  $\exp(t\psi)$  is a scalar matrix on each simple factor; that is,  $\exp(t\psi) = \sum_i \theta_i(t) \Pi_i$ , with  $\Pi_i$  the orthogonal projection onto  $\mathfrak{s}_i$ . Differentiating at  $t = 0$ , we find  $\psi = \sum_i \lambda_i \Pi_i$ , where  $\lambda_i = \theta'_i(0)$ .  $\square$

In summary, we have proved the following refined version of Proposition 16.

**Proposition 21.** *The most general compatible Lie algebra structure  $\mathfrak{w}$  on the vector space  $W(\mathfrak{s} \oplus \mathfrak{a})$  is given by*

$$\begin{aligned} [e_-, a] &= Ja & [s_1, s_2] &= [\psi s_1, s_2]_{\mathfrak{s}} + \langle z, [s_1, s_2]_{\mathfrak{s}} \rangle e_+ \\ [e_-, s] &= [z, s]_{\mathfrak{s}} & [a_1, a_2] &= [a_1, a_2]_{\mathfrak{t}} + \langle Ja_1, a_2 \rangle e_+ \end{aligned}$$

for all  $a, a_1, a_2 \in \mathfrak{a}$  and  $s, s_1, s_2 \in \mathfrak{s}$  and where  $z \in \mathfrak{s}$ ,  $[-, -]_{\mathfrak{t}}$  defines a reductive Lie algebra structure on  $\mathfrak{a}$ ,  $J \in \mathfrak{so}(\mathfrak{a}) \cap \text{Der } \mathfrak{t}$ , and  $\psi \in \text{End } \mathfrak{s}$  is given by  $\psi = \sum_i \lambda_i \Pi_i$ , where  $\lambda_i \in \mathbb{R}$  and  $\Pi_i$  are the orthogonal projections onto the simple factors of  $\mathfrak{s}$ .

Imposing that the resulting metric Lie 3-algebra  $V$  in (11) be indecomposable will further restrict the form of  $\mathfrak{w}$ , as will the fact that  $\mathfrak{w}$  is by construction a lorentzian Lie algebra and these have been classified. We turn to this now in order to finish the classification of indecomposable metric Lie 3-algebras of signature  $(2, p)$ .

**4.3. Indecomposable metric Lie 3-algebras of signature  $(2, p)$ .** We saw above that there are two types of indecomposable Lie 3-algebras  $V$  of signature  $(2, p)$ , characterised by a lorentzian  $p$ -dimensional vector space  $W$  on which we have both a metric Lie 3-algebra structure, also denoted  $W$ , and a metric Lie algebra structure, denoted  $\mathfrak{w}$ . These two possible  $W$ s, said to be of types I and III above, are the following:

- (I)  $W = A$ , abelian and  $\mathfrak{w}$  any lorentzian Lie algebra; that is,  $\text{ad } \mathfrak{w} < \mathfrak{so}(\mathfrak{w})$ ; and
- (III)  $W = W(\mathfrak{s} \oplus \mathfrak{a})$  and  $\mathfrak{w}$  given by Proposition 21, with some further restrictions to be explicated below.

We remark that although type I is the special case of type III with  $\mathfrak{s} = 0$ , it nevertheless pays to consider it separately.

**4.3.1. Type I.** The indecomposable Lie 3-algebra with  $W$  of type I are such that  $V = \mathbb{R}u \oplus \mathbb{R}v \oplus A$ , with  $A$  an abelian lorentzian Lie 3-algebra, with inner product which extends the one in  $A$  by declaring that  $u, v \perp A$ ,  $\langle u, u \rangle = 0 = \langle v, v \rangle$  and  $\langle u, v \rangle = 1$ . The Lie 3-brackets are

$$[u, x, y] = [x, y] \quad \text{and} \quad [x, y, z] = -\langle [x, y], z \rangle v, \quad (15)$$

with  $[x, y]$  the Lie brackets of a lorentzian Lie algebra  $\mathfrak{w}$ , which may be decomposable. The most general lorentzian Lie algebra is given by

$$\mathfrak{w} = \mathfrak{g}_0 \oplus \mathfrak{t} \oplus \mathfrak{s},$$

where  $\mathfrak{g}_0$  is an indecomposable lorentzian Lie algebra,  $\mathfrak{t}$  is an abelian euclidean Lie algebra and  $\mathfrak{s}$  is a euclidean semisimple Lie algebra. It follows from (15) that  $\mathfrak{t}$  cannot appear, for otherwise it decomposes  $V$ . Hence,  $\mathfrak{w} = \mathfrak{g}_0 \oplus \mathfrak{s}$ . The indecomposable lorentzian Lie algebras  $\mathfrak{g}_0$  have been classified.

**Theorem 22** ([52]). *A finite-dimensional indecomposable lorentzian Lie algebra is either one-dimensional, isomorphic to  $\mathfrak{so}(1, 2)$  or else isomorphic to the solvable Lie algebra  $\mathfrak{m}_J$ , defined on the vector space  $\mathbb{R}\mathbf{e}_- \oplus \mathbb{R}\mathbf{e}_+ \oplus E$  by the Lie bracket*

$$[\mathbf{e}_-, x] = Jx \quad \text{and} \quad [x, y] = \langle Jx, y \rangle \mathbf{e}_+,$$

for all  $x, y \in E$  and where  $\langle -, - \rangle$  is a positive-definite inner product on  $E$ ,  $J \in \mathfrak{so}(E)$  is invertible, and we extend the inner product on  $E$  to all of  $V$  by declaring  $\mathbf{e}_\pm \perp E$ ,  $\langle \mathbf{e}_+, \mathbf{e}_- \rangle = 1$  and  $\langle \mathbf{e}_\pm, \mathbf{e}_\pm \rangle = 0$ .

We cannot take  $\mathfrak{g}_0$  to be one-dimensional, since this decomposes  $V$ , whence  $\mathfrak{w} = \mathfrak{so}(1, 2) \oplus \mathfrak{s}$  or  $\mathfrak{w} = \mathfrak{m}_J \oplus \mathfrak{s}$ , which we will call **type Ia** and **type Ib**, respectively.

In summary, (indecomposable) type Ia metric Lie 3-algebras of signature  $(2, p)$  are constructed as follows. The initial data consists of a semisimple Lie algebra  $\mathfrak{s}$  with a positive-definite ad-invariant inner product (which is implicit in the notation) and a choice of invariant inner product on  $\mathfrak{so}(1, 2)$ , which comes down to a positive real number which multiplies (the negative of) the Killing form. The corresponding indecomposable type Ia metric Lie 3-algebra is denoted  $V_{\text{Ia}}(\mathfrak{s})$ . The underlying vector space is  $\mathbb{R}(u, v) \oplus \mathfrak{so}(1, 2) \oplus \mathfrak{s}$  with  $\langle u, u \rangle = \langle v, v \rangle = 0$ ,  $\langle u, v \rangle = 1$ , and all  $\oplus$ s orthogonal. The nonzero Lie 3-brackets are given by

$$\begin{aligned} [u, x, y] &= [x, y]_{\mathfrak{so}(1,2)} & [x, y, z] &= -\langle [x, y]_{\mathfrak{so}(1,2)}, z \rangle v \\ [u, s_1, s_2] &= [s_1, s_2]_{\mathfrak{s}} & [s_1, s_2, s_3] &= -\langle [s_1, s_2]_{\mathfrak{s}}, s_3 \rangle v, \end{aligned} \tag{16}$$

for all  $x, y, z \in \mathfrak{so}(1, 2)$  and  $s_i \in \mathfrak{s}$ .

Similarly, (indecomposable) type Ib metric Lie 3-algebras of signature  $(2, p)$  are constructed as follows. The initial data consists of a triple  $(E, J, \mathfrak{s})$  consisting of an even-dimensional euclidean space  $E$  with a nondegenerate skewsymmetric endomorphism  $J$ , and a semisimple Lie algebra  $\mathfrak{s}$  with a positive-definite ad-invariant inner product (which is implicit in the notation). The corresponding indecomposable type Ib metric Lie 3-algebra is denoted  $V_{\text{Ib}}(E, J, \mathfrak{s})$ . The underlying vector space is  $\mathbb{R}(u, v) \oplus \mathbb{R}(\mathbf{e}_+, \mathbf{e}_-) \oplus E \oplus \mathfrak{s}$  with  $\langle u, u \rangle = \langle v, v \rangle = \langle \mathbf{e}_\pm, \mathbf{e}_\pm \rangle = 0$ ,  $\langle u, v \rangle = 1$ ,  $\langle \mathbf{e}_+, \mathbf{e}_- \rangle = 1$  and all  $\oplus$ s orthogonal. The nonzero Lie 3-brackets are given by

$$\begin{aligned} [u, s_1, s_2] &= [s_1, s_2]_{\mathfrak{s}} & [u, \mathbf{e}_-, x] &= Jx \\ [s_1, s_2, s_3] &= -\langle [s_1, s_2]_{\mathfrak{s}}, s_3 \rangle v & [u, x, y] &= \langle Jx, y \rangle \mathbf{e}_+ \\ & & [\mathbf{e}_-, x, y] &= -\langle Jx, y \rangle v, \end{aligned} \tag{17}$$

for all  $x, y \in E$ ,  $s_i \in \mathfrak{s}$ .

As discussed in §6, a sufficient condition for the decoupling of negative-norm states from the Bagger–Lambert lagrangian, is that the Lie 3-algebra should admit a maximally isotropic centre. In signature  $(2, p)$  this means a two-dimensional isotropic centre. This

indeed happens in type Ib, since both  $e_+$  and  $v$  are central and span an isotropic subspace. Due to the simplicity of  $\mathfrak{so}(2, 1)$ , this does not happen in type Ia.

4.3.2. *Type III.* In this case  $W = W(\mathfrak{s} \oplus \mathfrak{a})$  and  $\mathfrak{w}$  is a lorentzian Lie algebra structure on  $W$  with brackets given by Proposition 21. Indecomposability forces the following condition: the reductive Lie algebra structure  $\mathfrak{r}$  on the vector space  $\mathfrak{a}$  must be semisimple, since any element in the centre of  $\mathfrak{r}$  is central in  $V$  and of positive norm, whence it spans a nondegenerate ideal.

At the same time, being lorentzian, the Lie algebra  $\mathfrak{w}$  can be one of the following:

- $\mathfrak{w} = \mathfrak{t} \oplus \mathfrak{k}$ , where  $\mathfrak{t}$  is lorentzian abelian and  $\mathfrak{k}$  is euclidean semisimple;
- $\mathfrak{w} = \mathfrak{so}(1, 2) \oplus \mathfrak{t} \oplus \mathfrak{k}$ , with  $\mathfrak{t}$  and  $\mathfrak{k}$  euclidean abelian and semisimple, respectively; and
- $\mathfrak{w} = \mathfrak{m}_J \oplus \mathfrak{t} \oplus \mathfrak{k}$ , with  $\mathfrak{m}_J$  defined in Theorem 22 and again  $\mathfrak{t}$  and  $\mathfrak{k}$  euclidean abelian and semisimple, respectively.

Again we must exclude the “middle third,” this time because  $\text{ad } \mathfrak{so}(1, 2) \cong \mathfrak{so}(1, 2)$  and there is no  $\mathfrak{so}(1, 2)$  subalgebra of  $\text{Der}^0 W(\mathfrak{s} \oplus \mathfrak{a})$ . This leaves the first and the third cases, which we will call **type IIIa** and **type IIIb**, respectively.

In type IIIa, we see that  $e_- \in \mathfrak{t}$  is central, whence in the notation of Proposition 21,  $z = 0$  and  $J = 0$ , whence we have Lie algebra structures on  $\mathfrak{a}$  and  $\mathfrak{s}$  which leave invariant the euclidean inner product. This means that the corresponding Lie algebras are reductive. From Proposition 20, it follows that the Lie algebra structure on  $\mathfrak{s}$  is isomorphic to an ideal of  $\mathfrak{s}$ . We already observed that the Lie algebra structure  $\mathfrak{r}$  on  $\mathfrak{a}$  has to be semisimple, for any abelian factors would decompose  $V$ , assumed indecomposable. Therefore the data describing such a Lie 3-algebra is  $(\mathfrak{s}, \mathfrak{r}, \lambda_1, \dots, \lambda_p)$ , where  $\mathfrak{s} = \mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_p$  is a semisimple Lie algebra decomposed into its simple ideals,  $\lambda_i \in \mathbb{R}$  and  $\mathfrak{r}$  is a semisimple Lie algebra. Both  $\mathfrak{s}$  and  $\mathfrak{r}$  are equipped with positive-definite invariant inner products. We let  $\psi = \sum_{i=1}^p \lambda_i \Pi_i$ , with  $\Pi_i : \mathfrak{s} \rightarrow \mathfrak{s}_i$  denoting the orthogonal projection on  $\mathfrak{s}_i$ . The corresponding indecomposable Lie 3-algebra is denoted  $V_{\text{IIIa}}(\mathfrak{s}, \mathfrak{k}, \lambda_i)$ . The underlying vector space is  $\mathbb{R}(u, v) \oplus \mathbb{R}(e_+, e_-) \oplus \mathfrak{s} \oplus \mathfrak{r}$  with  $\langle u, u \rangle = \langle v, v \rangle = \langle e_\pm, e_\pm \rangle = 0$ ,  $\langle u, v \rangle = 1$ ,  $\langle e_+, e_- \rangle = 1$  and all  $\oplus$ s orthogonal. The nonzero Lie 3-brackets are given by

$$\begin{aligned} [u, s_1, s_2] &= [\psi s_1, s_2]_{\mathfrak{s}} & [s_1, s_2, s_3] &= -\langle [s_1, s_2]_{\mathfrak{s}}, s_3 \rangle e_+ - \langle [\psi s_1, s_2]_{\mathfrak{s}}, s_3 \rangle v \\ [u, a_1, a_2] &= [a_1, a_2]_{\mathfrak{r}} & [a_1, a_2, a_3] &= -\langle [a_1, a_2]_{\mathfrak{r}}, a_3 \rangle v . \\ [e_-, s_1, s_2] &= [s_1, s_2]_{\mathfrak{s}} \end{aligned} \tag{18}$$

Finally, let us consider case IIIb. In this case,  $\mathfrak{w} \cong \mathfrak{m}_J \oplus \mathfrak{t} \oplus \mathfrak{k}$ , where  $\mathfrak{k}$  is semisimple,  $\mathfrak{t}$  is abelian, and  $\mathfrak{m}_J$  is one of the indecomposable lorentzian Lie algebras in Theorem 22. Consider the Lie algebra  $\mathfrak{w}$  in Proposition 21. It will be convenient to split  $\mathfrak{s}$  and  $\mathfrak{a}$  further as follows. Let  $\mathfrak{s} = \mathfrak{g} \oplus \mathfrak{h}$ , where  $\mathfrak{h} = \ker \psi$  and  $\mathfrak{g} = \text{im } \psi$ . This split corresponds to partitioning the simple ideals  $\mathfrak{s}_i$  of  $\mathfrak{s}$  into two sets, depending on whether  $\lambda_i$  is or is not zero. As a result both  $\mathfrak{g}$  and  $\mathfrak{h}$  are semisimple and commute with each other. Similarly let us decompose the reductive Lie algebra  $\mathfrak{r}$  (with underlying vector space  $\mathfrak{a}$ ) into  $\mathfrak{r} = \mathfrak{l} \oplus \mathfrak{b}$ ,



where  $\mathfrak{l}$  is semisimple and  $\mathfrak{b}$  is abelian. Relative to such splits, the Lie bracket of  $\mathfrak{w}$  becomes

$$\begin{aligned} [\mathbf{e}_-, s] &= [z, s]_{\mathfrak{s}} & [h_1, h_2] &= \langle z, [h_1, h_2]_{\mathfrak{h}} \rangle \mathbf{e}_+ \\ [\mathbf{e}_-, a] &= Ja & [\ell_1, \ell_2] &= [\ell_1, \ell_2]_{\mathfrak{l}} + \langle J\ell_1, \ell_2 \rangle \mathbf{e}_+ \\ [g_1, g_2] &= [\psi g_1, g_2]_{\mathfrak{g}} + \langle z, [g_1, g_2]_{\mathfrak{g}} \rangle \mathbf{e}_+ & [b_1, b_2] &= \langle Jb_1, b_2 \rangle \mathbf{e}_+ , \end{aligned}$$

for all  $a \in \mathfrak{a}$ ,  $s \in \mathfrak{s}$ ,  $g_i \in \mathfrak{g}$ ,  $h_i \in \mathfrak{h}$ ,  $\ell_i \in \mathfrak{l}$  and  $b_i \in \mathfrak{b}$ .

The first observation is that because  $\mathfrak{l}$  and  $\mathfrak{g}$  are semisimple, they do not admit nontrivial central extensions. This means that we can eliminate the component of  $J$  in  $\text{End } \mathfrak{l}$  and the component of  $z$  in  $\mathfrak{g}$  via an isometry, as we will now see. But first a preliminary result.

**Lemma 23.** *Under the decomposition  $\mathfrak{r} = \mathfrak{l} \oplus \mathfrak{b}$ , with  $\mathfrak{l}$  semisimple and  $\mathfrak{b}$  abelian,  $J \in \text{Der}^0 \mathfrak{r}$  decomposes as*

$$J = \begin{pmatrix} J_{\mathfrak{l}} & 0 \\ 0 & J_{\mathfrak{b}} \end{pmatrix} ,$$

where  $J_{\mathfrak{l}} \in \text{Der}^0 \mathfrak{l}$  and  $J_{\mathfrak{b}} \in \mathfrak{so}(\mathfrak{b})$ .

*Proof.* Since  $J$  is a derivation,  $J[\ell_1, \ell_2] = [J\ell_1, \ell_2] + [\ell_1, J\ell_2]$ , which, since  $\mathfrak{b}$  commutes with  $\mathfrak{l}$ , shows that  $J[\ell_1, \ell_2] \in \mathfrak{l}$ . Since  $\mathfrak{l}$  is semisimple, this shows that  $J\ell \in \mathfrak{l}$  for all  $\ell \in \mathfrak{l}$ . Since  $\mathfrak{l}$  and  $\mathfrak{b}$  are orthogonal, we have for all  $\ell \in \mathfrak{l}$  and  $b \in \mathfrak{b}$ ,  $0 = \langle J\ell, b \rangle = -\langle \ell, Jb \rangle$ , whence  $Jb \in \mathfrak{b}$ . Defining  $J_{\mathfrak{l}}$  and  $J_{\mathfrak{b}}$  the restrictions of  $J$  to  $\mathfrak{l}$  and  $\mathfrak{b}$ , respectively, we see that  $J$  is as shown.  $\square$

Let  $J_{\mathfrak{l}} \in \text{Der}^0 \mathfrak{l}$  denote the restriction of  $J$  to  $\mathfrak{l}$ . Since  $\ell$  is semisimple,  $J_{\mathfrak{l}}(\ell) = [\ell_0, \ell]$  for some  $\ell_0 \in \mathfrak{l}$ . Let  $f_1 : \mathfrak{w} \rightarrow \mathfrak{w}$  be the isometry which sends  $\ell \mapsto \ell + \langle \ell_0, \ell \rangle \mathbf{e}_+$  and  $\mathbf{e}_- \mapsto \mathbf{e}_- - \ell_0 - \frac{1}{2}|\ell_0|^2 \mathbf{e}_+$  and is the identity elsewhere. Then  $J_{\mathfrak{l}}$  does not appear in the Lie brackets of  $f_1(\mathfrak{w})$ . Now let  $f_2 : \mathfrak{w} \rightarrow \mathfrak{w}$  denote the isometry which sends  $g \mapsto g + \langle \psi^{-1}z, g \rangle \mathbf{e}_+$  and  $\mathbf{e}_- \mapsto \mathbf{e}_- - \psi^{-1}z_{\mathfrak{g}} - \frac{1}{2}|\psi^{-1}z_{\mathfrak{g}}|^2 \mathbf{e}_+$  (and is the identity elsewhere), where  $z_{\mathfrak{g}}$  is the projection of  $z$  onto  $\mathfrak{g}$  along  $\mathfrak{h}$ . Then  $z_{\mathfrak{g}}$  does not appear in the Lie brackets of  $f_2(f_1(\mathfrak{w}))$ .

We may therefore take  $z \in \mathfrak{h}$  and  $J \in \text{End } \mathfrak{b}$  without loss of generality. The resulting nonzero Lie brackets for  $\mathfrak{w}$  are then

$$\begin{aligned} [\mathbf{e}_-, h] &= [z, h]_{\mathfrak{h}} & [h_1, h_2] &= \langle z, [h_1, h_2]_{\mathfrak{h}} \rangle \mathbf{e}_+ \\ [\mathbf{e}_-, b] &= Jb & [\ell_1, \ell_2] &= [\ell_1, \ell_2]_{\mathfrak{l}} \\ [g_1, g_2] &= [\psi g_1, g_2]_{\mathfrak{g}} & [b_1, b_2] &= \langle Jb_1, b_2 \rangle \mathbf{e}_+ , \end{aligned}$$

for all  $g_i \in \mathfrak{g}$ ,  $h, h_i \in \mathfrak{h}$ ,  $\ell_i \in \mathfrak{l}$  and  $b, b_i \in \mathfrak{b}$ , and where  $J \in \mathfrak{so}(\mathfrak{b})$  and  $z \in \mathfrak{h}$ . It follows from these brackets that if  $b \in \ker J \cap \mathfrak{b}$ , then it is central in  $V$  and has positive norm, whence  $V$  becomes decomposable. Therefore for indecomposability of  $V$  we require  $J$  to be nondegenerate when restricted to  $\mathfrak{b}$ , whence  $\mathfrak{b}$  must be even-dimensional.

As we now show, we can put  $z = 0$  without loss of generality. The following 3-brackets are the only ones in the type IIIb Lie 3-algebra  $V$  which involve  $z$  and  $\mathfrak{h}$ :

$$\begin{aligned} [u, \mathbf{e}_-, h] &= [z, h]_{\mathfrak{h}} & [\mathbf{e}_-, h_1, h_2] &= [h_1, h_2]_{\mathfrak{h}} - \langle z, [h_1, h_2]_{\mathfrak{h}} \rangle v \\ [u, h_1, h_2] &= \langle z, [h_1, h_2]_{\mathfrak{h}} \rangle \mathbf{e}_+ & [h_1, h_2, h_3] &= -\langle [h_1, h_2]_{\mathfrak{h}}, h_3 \rangle \mathbf{e}_+ , \end{aligned}$$

where  $h_i \in \mathfrak{h}$ . Consider the isometry  $f : V \rightarrow V$  mapping  $h \mapsto h - \langle z, h \rangle v$  and  $u \mapsto u + z - \frac{1}{2}|z|^2 v$  and equal to the identity elsewhere. The induced brackets in  $f(V)$  are formally the same, but with  $z = 0$ . We will therefore put  $z = 0$  from now on.

Doing so in the Lie brackets for  $\mathfrak{w}$  we obtain

$$\begin{aligned} [e_-, b] &= Jb & [g_1, g_2] &= [\psi g_1, g_2]_{\mathfrak{g}} \\ [b_1, b_2] &= \langle Jb_1, b_2 \rangle e_+ & [\ell_1, \ell_2] &= [\ell_1, \ell_2]_{\mathfrak{l}} \end{aligned}$$

As expected, the resulting Lie algebra  $\mathfrak{w}$  is isomorphic to  $\mathfrak{m}_J \oplus (\mathfrak{g} \oplus \mathfrak{l}) \oplus \mathfrak{h}$ , where  $\mathfrak{g} \oplus \mathfrak{l}$  is semisimple,  $\mathfrak{h}$  is abelian, and where the euclidean space  $E$  in  $\mathfrak{m}_J$  is  $\mathfrak{b}$ .

It follows that the data defining a type IIIb indecomposable metric Lie 3-algebra of signature  $(2, p)$  is the following: three semisimple Lie algebras  $\mathfrak{g}$ ,  $\mathfrak{h}$  and  $\mathfrak{l}$  each with a choice of euclidean inner product, an *invertible* endomorphism  $\psi = \sum_i \lambda_i \Pi_i$  of  $\mathfrak{g}$ , where  $0 \neq \lambda_i \in \mathbb{R}$  and  $\Pi_i$  are the orthogonal projections onto the simple ideals of  $\mathfrak{g}$ , and an even-dimensional euclidean vector space  $E$  with a nondegenerate  $J \in \mathfrak{so}(E)$ . The resulting type IIIb Lie 3-algebra, denoted  $V_{\text{IIIb}}(E, J, \mathfrak{l}, \mathfrak{h}, \mathfrak{g}, \psi)$ , has as underlying vector space  $\mathbb{R}(u, v) \oplus \mathbb{R}(e_+, e_-) \oplus E \oplus \mathfrak{l} \oplus \mathfrak{h} \oplus \mathfrak{g}$  with  $\langle u, u \rangle = \langle v, v \rangle = \langle e_{\pm}, e_{\pm} \rangle = 0$ ,  $\langle u, v \rangle = 1 = \langle e_+, e_- \rangle$  and all  $\oplus$ s orthogonal. The nonzero Lie 3-brackets are given by

$$\begin{aligned} [u, e_-, x] &= Jx & [u, g_1, g_2] &= [\psi g_1, g_2]_{\mathfrak{g}} \\ [u, x, y] &= \langle Jx, y \rangle e_+ & [e_-, g_1, g_2] &= [g_1, g_2]_{\mathfrak{g}} \\ [e_-, x, y] &= -\langle Jx, y \rangle v & [g_1, g_2, g_3] &= -\langle [g_1, g_2]_{\mathfrak{g}}, g_3 \rangle e_+ - \langle [\psi g_1, g_2]_{\mathfrak{g}}, g_3 \rangle v \\ [e_-, h_1, h_2] &= [h_1, h_2]_{\mathfrak{h}} & [u, \ell_1, \ell_2] &= [\ell_1, \ell_2]_{\mathfrak{l}} \\ [h_1, h_2, h_3] &= -\langle [h_1, h_2]_{\mathfrak{h}}, h_3 \rangle e_+ & [\ell_1, \ell_2, \ell_3] &= -\langle [\ell_1, \ell_2]_{\mathfrak{l}}, \ell_3 \rangle v \end{aligned} \tag{19}$$

where  $x, y \in E$ ,  $h, h_i \in \mathfrak{h}$ ,  $g_i \in \mathfrak{g}$  and  $\ell_i \in \mathfrak{l}$ .

We can recognise several subalgebras among the above 3-brackets. First of all we recognise two decomposable subalgebras: one isomorphic to  $W(\mathfrak{l} \oplus \mathbb{R}^{1,1})$ , where the  $\mathbb{R}^{1,1}$  is the abelian two-dimensional lorentzian Lie algebra spanned by  $e_{\pm}$ , and one isomorphic to  $W(\mathfrak{h} \oplus \mathbb{R}^{1,1})$ , with  $u, v$  spanning the  $\mathbb{R}^{1,1}$ . Then we have  $V(E, J)$ , defined by

$$[u, e_-, x] = Jx \quad [u, x, y] = \langle Jx, y \rangle e_+ \quad [e_-, x, y] = -\langle Jx, y \rangle v, \tag{20}$$

for all  $x, y \in E$ . It is an indecomposable metric Lie 3-algebra with signature  $(2, *)$ . We then have  $V(\mathfrak{g}, \psi)$ , with 3-brackets

$$\begin{aligned} [u, g_1, g_2] &= [\psi g_1, g_2]_{\mathfrak{g}} \\ [e_-, g_1, g_2] &= [g_1, g_2]_{\mathfrak{g}} \\ [g_1, g_2, g_3] &= -\langle [g_1, g_2]_{\mathfrak{g}}, g_3 \rangle e_+ - \langle [\psi g_1, g_2]_{\mathfrak{g}}, g_3 \rangle v \end{aligned} \tag{21}$$

If  $\psi = \lambda \text{id}$  is a scalar endomorphism,  $V(\mathfrak{g}, \psi)$  is decomposable. Indeed, in this case define

$$\hat{u} = \frac{1}{2}(\lambda^{-1}u + e_-) \quad \hat{v} = \lambda v + e_+ \quad \hat{e}_+ = \frac{1}{2}(\lambda v - e_+) \quad \hat{e}_- = \lambda^{-1}u - e_- .$$

Then  $[\hat{e}_\pm, \mathfrak{g}, \mathfrak{g}] = 0$ , whence  $V(\mathfrak{g}, \psi) \cong W(\mathfrak{g} \oplus \mathbb{R}^{1,1})$ . Furthermore, if  $\mathfrak{h}$  and  $\mathfrak{l}$  are absent, then if  $\psi$  is again a scalar a short calculation shows that  $V_{\text{IIIb}}(E, J, \mathfrak{o}, \mathfrak{o}, \mathfrak{g}, \lambda \text{id}) \cong V_{\text{IIIb}}(E, -\lambda^{-1}J, \mathfrak{o}, \mathfrak{g}, \mathfrak{o}, 0)$ .

Finally let us remark that type Ib is the special case of type IIIb where  $\mathfrak{h}$  and  $(\mathfrak{g}, \psi)$  are absent; and, similarly, type IIIa is the special case of IIIb where  $\mathfrak{h}$  and  $(E, J)$  are absent. In summary, we have proved the following

**Theorem 24.** *Let  $V$  be an indecomposable metric Lie 3-algebra with signature  $(2, p)$ . Then it is isomorphic to either  $V_{\text{Ia}}(\mathfrak{s})$  or  $V_{\text{IIIb}}(E, J, \mathfrak{l}, \mathfrak{h}, \mathfrak{g}, \psi)$ , which have been defined in (16) and (19), respectively. If the centre of  $V$  contains a maximally isotropic plane, then it is of type IIIb, otherwise it is of type Ia.*

The type IIIb algebras actually encapsulate a large class of metric Lie 3-algebras depending on which of the data  $(E, J)$ ,  $\mathfrak{k}$ ,  $\mathfrak{l}$  or  $(\mathfrak{g}, \psi)$  is present. It is worth listing the possible cases which can occur, because they do tend to have different properties and, as explained in Section 2, many physically desirable properties of the Bagger–Lambert model can be translated into properties of the Lie 3-algebra. Ignoring the trivial case, where none of the structures are present, we have a priori 15 different types of metric Lie 3-algebras. However some of these types are isomorphic and need not be counted as different. To see this, we notice that  $V(E, J)$  admits an automorphism  $(u, v, \mathbf{e}_-, \mathbf{e}_+) \mapsto (-\mathbf{e}_-, -\mathbf{e}_+, u, v)$  (and the identity on  $E$ ) which preserves both the 3-brackets and the inner product. Under this map, the subalgebras  $W(\mathfrak{h}) \oplus \mathbb{R}^{1,1}$  and  $W(\mathfrak{l}) \oplus \mathbb{R}^{1,1}$  of  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{l}, \mathfrak{g}, \psi)$  get mapped to each other, and  $V(\mathfrak{g}, \psi)$  is mapped to  $V(-\psi\mathfrak{g}, -\psi^{-1})$ , where  $-\psi\mathfrak{g}$  means the Lie algebra with bracket  $[x, y]_{-\psi\mathfrak{g}} := [-\psi x, y]_{\mathfrak{g}}$ . This means that type IIIb splits into 11 different types:

- (1)  $V(E, J)$ ;
- (2)  $W(\mathfrak{h}) \oplus \mathbb{R}^{1,1}$ , which is in the same class as  $W(\mathfrak{l}) \oplus \mathbb{R}^{1,1}$  and  $V(\mathfrak{g}, \psi)$  with  $\psi = \lambda \text{id}$  a scalar;
- (3)  $V(\mathfrak{g}, \psi)$ ,  $\psi$  not a scalar;
- (4)  $V_{\text{IIIb}}(E, J, \mathfrak{o}, \mathfrak{o}, \mathfrak{g}, \psi)$ ,  $\psi$  again not a scalar;
- (5)  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{o}, \mathfrak{o}, 0)$ , which is in the same class as  $V_{\text{IIIb}}(E, J, \mathfrak{o}, \mathfrak{o}, \mathfrak{g}, \psi)$  for  $\psi$  a scalar and  $V_{\text{IIIb}}(E, J, \mathfrak{o}, \mathfrak{l}, \mathfrak{o}, 0)$ ;
- (6)  $V_{\text{IIIb}}(0, 0, \mathfrak{h}, \mathfrak{l}, \mathfrak{o}, 0)$ ;
- (7)  $V_{\text{IIIb}}(0, 0, \mathfrak{h}, \mathfrak{o}, \mathfrak{g}, \psi)$ , which is in the same class as  $V_{\text{IIIb}}(0, 0, \mathfrak{o}, \mathfrak{l}, \mathfrak{g}, \psi)$ ;
- (8)  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{l}, \mathfrak{o}, 0)$ ;
- (9)  $V_{\text{IIIb}}(0, 0, \mathfrak{h}, \mathfrak{l}, \mathfrak{g}, \psi)$ ;
- (10)  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{o}, \mathfrak{g}, \psi)$ , which is in the same class as  $V_{\text{IIIb}}(E, J, \mathfrak{o}, \mathfrak{l}, \mathfrak{g}, \psi)$ ; and
- (11) the full  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{l}, \mathfrak{g}, \psi)$ .

We notice that all but the second and sixth cases are indecomposable.

## 5. THE LIE ALGEBRA OF DERIVATIONS

In this section we will consider the Lie algebras of derivations of the metric Lie 3-algebras classified in Section 4.3. In particular we will deconstruct the type IIIb Lie 3-algebra found

in the previous section and discuss some natural special cases. Let us make some generic remarks about automorphisms and derivations.

Given a metric Lie 3-algebra  $V$  there are several groups of automorphisms which are of interest. The largest of such groups is the group  $\text{Aut } V$  consisting of all automorphisms of  $V$ :

$$\text{Aut } V = \{ \varphi \in \text{GL}(V) \mid \varphi[x, y, z] = [\varphi x, \varphi y, \varphi z], \forall x, y, z \in V \} . \quad (22)$$

Because  $V$  possesses an inner product, it is natural to restrict ourselves to the subgroup  $\text{Aut}^c V$  of  $\text{Aut } V$  consisting of automorphisms which rescale the inner product:

$$\text{Aut}^c V = \{ \varphi \in \text{Aut } V \mid \langle \varphi x, \varphi y \rangle = \mu \langle x, y \rangle, \forall x, y \in V, \exists \mu \in \mathbb{R} \} , \quad (23)$$

which will be denoted **conformal automorphisms**. Similarly, we can restrict to automorphisms which preserve the inner product, namely the **orthogonal automorphisms**

$$\text{Aut}^0 V = \{ \varphi \in \text{Aut } V \mid \langle \varphi x, \varphi y \rangle = \langle x, y \rangle, \forall x, y \in V \} . \quad (24)$$

Finally, we have the so-called **inner automorphisms**, which are obtained by exponentiating the inner derivations. We will call this group  $\text{Ad } V$ . It is clear that we have the following chain of inclusions

$$\text{Ad } V < \text{Aut}^0 V < \text{Aut}^c V < \text{Aut } V .$$

Their Lie algebras are, respectively, the Lie algebras of derivations, conformal derivations, skewsymmetric derivations and inner derivations, giving rise to a similar chain of inclusions

$$\text{ad } V \triangleleft \text{Der}^0 V < \text{Der}^c V < \text{Der } V ,$$

with  $\text{ad } V$  the ideal of inner derivations, which generate the gauge transformations in the Bagger–Lambert theory. This allows us to think of  $\text{Ad } V$  as the gauge group of the theory.

**5.1. Type Ia.** The 3-brackets of  $V_{\text{Ia}}(\mathfrak{s})$  are given in (16), making  $V_{\text{Ia}}(\mathfrak{s}) \cong W(\mathfrak{so}(1, 2) \oplus \mathfrak{s})$ , whose automorphisms were determined in [31, Proposition 11], which we recall here for convenience.

**Proposition 25.** *Every 3-algebra automorphism  $\varphi \in \text{Aut } V_{\text{Ia}}(\mathfrak{s})$  is given by*

$$\begin{aligned} \varphi(v) &= \beta^{-3}v \\ \varphi(u) &= \beta u + \gamma v + t \\ \varphi(x) &= \beta^{-1}a(x) - \beta^{-2} \langle t, a(x) \rangle v , \end{aligned}$$

for all  $x \in \mathfrak{so}(1, 2) \oplus \mathfrak{s}$  and where  $\beta \in \mathbb{R}^\times$ ,  $\gamma \in \mathbb{R}$ ,  $t \in \mathfrak{so}(1, 2) \oplus \mathfrak{s}$  and  $a \in \text{SO}(1, 2) \times \text{Aut}^0 \mathfrak{s}$ , where  $\text{Aut}^0 \mathfrak{s}$  denotes the subgroup of automorphisms of  $\mathfrak{s}$  which preserve the inner product.

Restricting to those automorphisms which preserve the inner product up to a homothety, we find that  $\gamma$  is fixed in terms of  $\beta$  and the norm of  $t$ :

**Proposition 26.** *Every 3-algebra conformal automorphism  $\varphi \in \text{Aut}^c V_{\text{Ia}}(\mathfrak{s})$  is given by*

$$\begin{aligned} \varphi(v) &= \beta^{-3}v \\ \varphi(u) &= \beta u - \frac{1}{2}\beta|t|^2v + t \end{aligned}$$

$$\varphi(x) = \beta^{-1}a(x) - \beta^{-2} \langle t, a(x) \rangle v ,$$

for all  $x \in \mathfrak{so}(1, 2) \oplus \mathfrak{s}$  and where  $t \in \mathfrak{so}(1, 2) \oplus \mathfrak{s}$  and  $a \in \text{SO}(1, 2) \times \text{Aut}^0 \mathfrak{s}$ . The inner product is rescaled by  $\beta^{-2}$ :  $\langle \varphi x, \varphi y \rangle = \beta^{-2} \langle x, y \rangle$ .

Restricting to automorphisms which preserve the inner product we find [31, Proposition 12] that  $\beta = 1$ .

**Proposition 27.** *Every 3-algebra automorphism  $\varphi \in \text{Aut}^0 V_{Ia}(\mathfrak{s})$  preserving the inner product is given by*

$$\varphi(v) = v \quad \varphi(u) = u - \frac{1}{2}|t|^2 v + t \quad \varphi(x) = a(x) - \langle t, a(x) \rangle v ,$$

for all  $x \in \mathfrak{so}(1, 2) \oplus \mathfrak{s}$  and where  $t \in \mathfrak{so}(1, 2) \oplus \mathfrak{s}$  and  $a \in \text{SO}(1, 2) \times \text{Aut}^0 \mathfrak{s}$ .

As shown in [31], the connected component of  $\text{Aut}^0 V_{Ia}(\mathfrak{s})$  is  $\text{Ad } V_{Ia}(\mathfrak{s})$ , consisting of the inner automorphisms obtained by exponentiating the inner derivations of the Lie 3-algebra  $V_{Ia}(\mathfrak{s})$ . This gives the following.

**Proposition 28.** *Let  $\text{Der}^0 V_{Ia}(\mathfrak{s})$  denote the Lie algebra of skewsymmetric derivations of the Lie 3-algebra  $V_{Ia}(\mathfrak{s})$ . Then*

$$\text{Der}^0 V_{Ia}(\mathfrak{s}) \cong (\mathfrak{so}(1, 2) \times \mathbb{R}^3) \oplus (\mathfrak{s} \times \mathfrak{s}_{ab}) \cong \text{ad } V_{Ia}(\mathfrak{s}) .$$

The Lie algebra  $\text{Der } V_{Ia}(\mathfrak{s})$  of  $\text{Aut } V_{Ia}(\mathfrak{s})$  consists of derivations of  $V$ . It is isomorphic to the real Lie algebra with generators  $D, S, L_x$  and  $T_x$  for  $x \in \mathfrak{so}(1, 2) \oplus \mathfrak{s}$ , subject to the following nonzero Lie brackets:

$$[D, S] = -4S , \quad [D, T_x] = -2T_x , \quad [L_x, L_y] = L_{[x,y]} \quad \text{and} \quad [L_x, T_y] = T_{[x,y]} .$$

If we let  $\mathfrak{a}$  denote the two-dimensional solvable Lie subalgebra spanned by  $D$  and  $S$ , then we find that  $\text{Der } V_{Ia}(\mathfrak{s})$  has the following structure

$$\text{Der } V_{Ia}(\mathfrak{s}) \cong \mathfrak{a} \times \text{ad } V_{Ia}(\mathfrak{s}) .$$

**5.2. Type IIIb.** Finally let us consider  $V := V_{\text{IIIb}}(E, J, \mathfrak{l}, \mathfrak{h}, \mathfrak{g}, \psi)$ , a general type IIIb Lie 3-algebra as defined in (19). As mentioned at the end of Section 4.3.2, this type consists of 9 different types of indecomposable Lie 3-algebras, depending on which of the four ingredients  $(E, J)$ ,  $\mathfrak{l}$ ,  $\mathfrak{h}$  or  $(\mathfrak{g}, \psi)$  are present.

Our strategy will be the following. We will first write down the most general endomorphism of  $V$  taking into account that derivations preserve the centre and the first derived ideal  $V' = [VVV]$ . In particular, elements that are not in  $V'$  can not appear in the image of elements of  $V'$ . Then we will impose the derivation property to derive constraints which do not depend on which of the ingredients are present. Finally we will consider those constraints which depend on the presence of a particular ingredient. We will omit the routine details and simply list the results.

For all type IIIb algebras,  $e_+, v$  are central, whereas in some cases  $e_-$  or  $u$  are central as well. Those cases, however, are decomposable and we shall ignore them in this section.

Similarly we observe that  $e_-$  or  $u$  do not belong to the first derived ideal. The most general  $D \in \text{End } V$  that preserves the the centre and the first derived ideal is given by

$$\begin{aligned}
De_+ &= \alpha e_+ + \beta v \\
Dv &= \gamma e_+ + \delta v \\
De_- &= ae_- + bu + x_- + h_- + \ell_- + g_- + \eta e_+ + \xi v \\
Du &= ce_- + du + x_u + h_u + \ell_u + g_u + \theta e_+ + \omega v \\
Dx &= \varphi_{EE}(x) + \varphi_{E\mathfrak{h}}(x) + \varphi_{E\mathfrak{l}}(x) + \varphi_{E\mathfrak{g}}(x) + A_1(x)e_+ + C_1(x)v \\
Dh &= \varphi_{\mathfrak{h}E}(h) + \varphi_{\mathfrak{h}\mathfrak{h}}(h) + \varphi_{\mathfrak{h}\mathfrak{l}}(h) + \varphi_{\mathfrak{h}\mathfrak{g}}(h) + A_2(h)e_+ + C_2(h)v \\
D\ell &= \varphi_{\mathfrak{l}E}(\ell) + \varphi_{\mathfrak{l}\mathfrak{h}}(\ell) + \varphi_{\mathfrak{l}\mathfrak{l}}(\ell) + \varphi_{\mathfrak{l}\mathfrak{g}}(\ell) + A_3(\ell)e_+ + C_3(\ell)v \\
Dg &= \varphi_{\mathfrak{g}E}(g) + \varphi_{\mathfrak{g}\mathfrak{h}}(g) + \varphi_{\mathfrak{g}\mathfrak{l}}(g) + \varphi_{\mathfrak{g}\mathfrak{g}}(g) + A_4(g)e_+ + C_4(g)v,
\end{aligned} \tag{25}$$

where  $\alpha, \beta, \gamma, \delta, a, b, c, d, \eta, \xi, \theta, \omega \in \mathbb{R}$ ,  $A_i, C_i$  are in  $E^*$ ,  $\mathfrak{h}^*$ ,  $\mathfrak{l}^*$  and  $\mathfrak{g}^*$ , respectively for  $i = 1, \dots, 4$ ,  $x_-, x_u \in E$ ,  $h_-, h_u \in \mathfrak{h}$ ,  $\ell_-, \ell_u \in \mathfrak{l}$ ,  $g_-, g_u \in \mathfrak{g}$  and  $\varphi_{V_1 V_2} : V_1 \rightarrow V_2$  are linear maps.

Notice that from the fact that elements on the different subspaces  $\mathfrak{h}, \mathfrak{l}, \mathfrak{g}$  and  $E$  never appear together on a 3-bracket and  $u, e_-$  only appear together with elements in  $E$ , we find that if  $V_1 \neq V_2$  then  $\varphi_{V_1 V_2} = 0$ . Also, from the vanishing brackets  $[u, h_1, h_2] = 0$  and  $[e_-, \ell_1, \ell_2] = 0$  we find  $\ell_- = h_u = 0$ .

We apply now the derivation  $D$  to all other brackets and obtain the following general map. It is implicit that if  $V$  does not include  $E$ , then  $x_- = x_u = 0$  and  $Dx$  does not appear. Similarly, if  $\mathfrak{h}$  was not there, then  $h_- = h_u = 0$  and  $Dh$  does not appear and so forth. After some calculation we obtain the following.

**Proposition 29.** *The most general derivation of the general  $V_{IIIb}(E, J, \mathfrak{l}, \mathfrak{h}, \mathfrak{g}, \psi)$  is given by*

$$\begin{aligned}
De_+ &= \alpha e_+ + \beta v \\
Dv &= \gamma e_+ + \delta v \\
De_- &= ae_- + bu + x_- + h_- + g_- + \eta e_+ + \xi v \\
Du &= ce_- + du + x_u + \ell_u + \psi g_- + \theta e_+ + \omega v \\
Dx &= \varphi(x) + \frac{1}{2}(\alpha + a)x - \langle x_-, x \rangle e_+ - \langle x_u, x \rangle v \\
Dh &= [h_D, h] - ah - \langle h_-, h \rangle e_+ \\
D\ell &= [\ell_D, \ell] - c\ell - \langle \ell_u, \ell \rangle v \\
Dg &= [g_D, g] - (a + b\psi)g - \langle g_-, g \rangle e_+ - \langle \psi g_-, g \rangle v,
\end{aligned}$$

where  $h_D \in \mathfrak{h}$ ,  $\ell_D \in \mathfrak{l}$ ,  $g_D \in \mathfrak{g}$ , and  $\varphi \in \mathfrak{so}(E)$  and commutes with  $J$ . (In other words,  $\varphi \in \mathfrak{u}(E, J)$ , the unitary Lie algebra of orthogonal endomorphisms of  $E$  which commute with  $J$ .) In addition, when  $E$  is present, we must impose the following

$$a + d = 0 \quad \beta = -c \quad \gamma = -b \quad \alpha + a = \delta + d; \tag{26}$$

when  $\mathfrak{h}$  is present, we impose the following

$$c = 0 \quad \beta = 0 \quad \alpha = -3a; \tag{27}$$



when  $\mathfrak{l}$  is present, we impose the following

$$b = 0 \quad \gamma = 0 \quad \delta = -3c ; \quad (28)$$

and finally when  $\mathfrak{g}$  is present, we impose the following

$$a + b\psi = d + c\psi^{-1} \quad \alpha + 3a + (\gamma + 3b)\psi = 0 \quad \beta + (\delta + 3a)\psi + 3b\psi^2 = 0 . \quad (29)$$

*Proof.* The proof is largely routine, except possibly for one thing. The condition on  $\varphi_{EE}$  in (25), says that

$$\varphi_{EE} \circ J - J \circ \varphi_{EE} = (a + d)J \quad \text{and} \quad \varphi_{EE} + \varphi_{EE}^t = (\alpha + a) \text{id} .$$

Consider now the exponential  $M(\tau) := \exp(\tau\varphi_{EE})$  for  $\tau \in \mathbb{R}$ . Then the first of the above equations says that

$$M(\tau)JM(\tau)^{-1} = \exp(\tau[\varphi_{EE}, -])J = e^{\tau(a+d)}J .$$

Taking determinants, using that  $J$  is nondegenerate, we see that  $e^{2n\tau(a+d)} = 1$ , where  $2n = \dim E$ . This being true for all  $\tau \in \mathbb{R}$  implies that  $a + d = 0$ . Thus  $\varphi_{EE}$  commutes with  $J$ . We break it into a skewsymmetric part (denoted  $\varphi$  above) and symmetric part, which by the second of the above equations on  $\varphi_{EE}$  is  $\frac{1}{2}(\alpha + a) \text{id}$ .  $\square$

We are interested in those derivations  $D$  which preserve the conformal class of the inner product:  $\langle Dv_1, v_2 \rangle + \langle v_1, Dv_2 \rangle = 2\mu \langle v_1, v_2 \rangle$ .

**Proposition 30.** *The most general conformal derivation of the general  $V_{IIIb}(E, J, \mathfrak{l}, \mathfrak{h}, \mathfrak{g}, \psi)$  is given by*

$$\begin{aligned} De_+ &= (2\mu - a)e_+ - cv \\ Dv &= -be_+ + (2\mu - d)v \\ De_- &= ae_- + bu + x_- + h_- + g_- - \theta v \\ Du &= ce_- + du + x_u + \ell_u + \psi g_- + \theta e_+ \\ Dx &= \varphi(x) + \mu x - \langle x_-, x \rangle e_+ - \langle x_u, x \rangle v \\ Dh &= [h_D, h] - ah - \langle h_-, h \rangle e_+ \\ D\ell &= [\ell_D, \ell] - c\ell - \langle \ell_u, \ell \rangle v \\ Dg &= [g_D, g] - (a + b\psi)g - \langle g_-, g \rangle e_+ - \langle \psi g_-, g \rangle v , \end{aligned}$$

where  $h_D \in \mathfrak{h}$ ,  $\ell_D \in \mathfrak{l}$ ,  $g_D \in \mathfrak{g}$ ,  $\varphi \in \mathfrak{u}(E, J)$  and where, if  $\mathfrak{g}$  is present  $\mu = -a - b\psi$  in addition to (29), if  $\mathfrak{h}$  is present  $a = -\mu$  in addition to (27), and if  $\mathfrak{l}$  is present  $c = -\mu$  in addition to (28). The skewsymmetric derivations are obtained setting  $\mu = 0$ .

## 6. IMPOSING THE PHYSICAL CONSTRAINTS

We will now conclude by revisiting the 3-algebraic criteria set out in Section 2 in light of our structural results of Section 3 and our classification of metric Lie 3-algebras with signature  $(2, p)$  in Section 4. We will select those  $(2, p)$  signature Lie 3-algebras which satisfy the criteria and indicate how to go about constructing more general metric Lie 3-algebras satisfying the criteria. We will focus on three specific criteria:

- decoupling of negative-norm states, which translates into the existence of a maximally isotropic centre;
- absence of scale, which translates into the existence of automorphisms which rescale the inner product; and
- parity invariance of the lagrangian, which translates into the existence of isometric anti-automorphisms.

**6.1. Decoupling of negative-norm states.** As discussed in Section 2.4, the existence of the shift symmetry used in [33,36] in order to decouple the negative-norm states present in the case of metric Lie 3-algebras of indefinite signature, translates into the existence of a maximally isotropic centre. As noted in Theorem 24, for  $(2, p)$  signature only case IIIb admits a maximally isotropic plane in its centre so that  $F_{v_1ABC} = 0 = F_{v_2ABC}$  relative to the basis defined in Section 2.1. Case Ia, however, has a non-vanishing  $F_{u_1u_2v_2a}$  component.

The results of Section 3 allow us to make a more general statement. As stated in Corollary 13, every metric Lie 3-algebra can be constructed out of the one-dimensional and simple Lie 3-algebras iterating the operations of double extension and orthogonal direct sum. It is clear from the structure of a double extension that double extending by a simple Lie 3-algebra  $U$  cannot result in a maximally isotropic subspace of the centre, for maximally isotropy means that  $U \oplus U^*$  should already contain a maximally isotropic subspace of the centre, yet for  $U$  simple,  $U \oplus U^*$  has trivial centre. This means that any double extension must be by a one-dimensional algebra. Since we only double extend by a one-dimensional algebra, the results of [31] show that any simple factor of the algebra we are double extending can be factored out, resulting in a decomposable Lie 3-algebra. Hence we conclude that the only indecomposable metric Lie 3-algebras admitting a maximally isotropic subspace of the centre are the ones constructed out of the one-dimensional Lie 3-algebra iterating the operations of orthogonal direct sum and double extension. This does not mean, however, that all such algebras have a maximally isotropic centre. The example of type Ia above shows that it is also necessary to impose the condition that the Lie algebra structure on the subspace corresponding to the metric Lie 3-algebra we are double extending, should also contain a maximally isotropic centre. Such metric Lie algebras have been studied in [53].

These remarks give *in principle* a prescription for the construction of such metric Lie 3-algebras. We start with a euclidean abelian Lie 3-algebra  $A_1$  and we double extend to  $\mathfrak{D}(A_1) = \mathbb{R}(u_1, v_1) \oplus A_1$ , with nonzero brackets

$$[u_1, x_1, y_1] = [x_1, y_1]_1 \quad \text{and} \quad [x_1, y_1, z_1] = -\langle [x_1, y_1]_1, z_1 \rangle v_1 ,$$

where  $[-, -]_1$  defines on  $A_1$  a metric Lie algebra structure  $\mathfrak{w}_1$ . The most general such Lie algebra is reductive, whence a direct sum of semisimple and abelian. It is, in fact, isomorphic to  $W(\mathfrak{k} \oplus \mathfrak{t})$ , where  $\mathfrak{k}$  is compact semisimple and  $\mathfrak{t}$  is abelian. This will be indecomposable if  $\mathfrak{t} = 0$ , otherwise it is decomposable. The most general lorentzian Lie 3-algebra built out of one-dimensional Lie 3-algebras is therefore isomorphic to  $W_2 := W(\mathfrak{k}) \oplus A_2$  for some compact semisimple Lie algebra  $\mathfrak{k}$  and where  $A_2$  is an abelian Lie

3-algebra. Of course,  $\mathfrak{k} = 0$ , in which case  $W(\mathfrak{k}) \oplus A_2 = A'_2$  is abelian. This yields all possible lorentzian Lie 3-algebras without simple factors [31].

We now consider the double extension of  $W_2$  by the one-dimensional algebra:  $\mathfrak{D}(A_2) = \mathbb{R}(u_2, v_2) \oplus W_2$ , with nonzero brackets

$$[u_2, x_2, y_2] = [x_2, y_2]_2 \quad \text{and} \quad [x_2, y_2, z_2] = [x_1, y_2, z_2]_2 - \langle [x_2, y_2]_2, z_2 \rangle v_2 ,$$

where  $[-, -]_2$  is a lorentzian Lie algebra structure  $\mathfrak{w}_2$  on  $W_2$  which leaves invariant the Lie 3-algebra brackets  $[-, -, -]_2$  of  $W_2$  and which has a maximally isotropic centre. In particular,  $\mathfrak{w}_2$  is a Lie subalgebra of  $\text{Der}^0 W_2$ . One must now determine the possible Lie subalgebras of  $\text{Der}^0 W_2$ , as was done in Section 4 except that we only allow those with a maximally isotropic centre. This yields a list of possible metric Lie 3-algebras with signature  $(2, *)$  and with maximally isotropic centre. Let  $W_3 = \mathfrak{D}(W_2) \oplus A_3$  be one such algebra. We must now consider all the possible double extensions  $\mathfrak{D}(W_3) = \mathbb{R}(u_3, v_3) \oplus W_3$  by a one-dimensional subalgebra, which has brackets

$$[u_3, x_3, y_3] = [x_3, y_3]_3 \quad \text{and} \quad [x_3, y_3, z_3] = [x_1, y_3, z_3]_3 - \langle [x_3, y_3]_3, z_3 \rangle v_3 ,$$

for some Lie algebra structure  $\mathfrak{w}_3$  with brackets  $[-, -]_3$  on  $W_3$  which is compatible with the 3-brackets on  $W_3$  and with maximally isotropic centre. This requires classifying the possible Lie subalgebras  $\mathfrak{w}_3 < \text{Der}^0 W_3$ , etc... There is a classification [53] of metric Lie algebras with signature  $(2, *)$ , whence it ought to be a matter of patience to classify the metric Lie 3-algebras of signature  $(3, *)$ . Going beyond this requires knowing the metric Lie algebras of signature  $(3, *)$  which is still open. Nevertheless, even if shy of a classification, the above procedure gives a way of constructing examples.

**6.2. Conformal automorphisms and the coupling constant.** As discussed in Section 2.5, the absence of scale in the Bagger–Lambert model is guaranteed by the existence of an automorphism in the Lie 3-algebra which rescales the inner product. Infinitesimally, such automorphisms are generated by derivations  $D \in \text{Der} V$  such that  $\langle Dx, y \rangle + \langle x, Dy \rangle = 2\mu \langle x, y \rangle$  for some  $\mu \in \mathbb{R}^\times$  and for all  $x, y \in V$ . Such derivations exist for a large class of  $(2, p)$ -signature Lie 3-algebras, as we now show.

Several types of metric Lie 3-algebras in  $(2, p)$  signature we have found admit conformal automorphisms that are completely analogous to the one noted in Section 2.5 that was used to fix the coupling constant in the lorentzian case. For instance, Proposition 26 for the type Ia algebras shows that the appropriate conformal automorphism here would be generated by the parameter  $\beta$ , with the same powers as in the lorentzian case. However, we have determined this class already to be physically uninteresting on the grounds that it does not obey the shift symmetry criterion noted above.

As noted at the end of Section 4.3.2, there are 9 classes of indecomposable type IIIb algebras, denoted  $V(E, J)$ ,  $V(g, \psi)$  for  $\psi$  not a scalar,  $V_{\text{IIIb}}(E, J, \mathfrak{o}, \mathfrak{o}, \mathfrak{g}, \psi)$  for  $\psi$  again not a scalar,  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{o}, \mathfrak{o}, 0)$ ,  $V_{\text{IIIb}}(0, 0, \mathfrak{h}, \mathfrak{o}, \mathfrak{g}, \psi)$ ,  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{l}, \mathfrak{o}, 0)$ ,  $V_{\text{IIIb}}(0, 0, \mathfrak{h}, \mathfrak{l}, \mathfrak{g}, \psi)$ ,  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{o}, \mathfrak{g}, \psi)$ , and the general  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{l}, \mathfrak{g}, \psi)$ .

It is straight-forward to determine which of these algebras possess conformal derivations with  $\mu \neq 0$ , by the use of Proposition 30.

**Proposition 31.** *The following indecomposable type IIIb Lie 3-algebras admit nontrivial conformal derivations:  $V(E, J)$ ,  $V(\mathfrak{g}, \psi)$  for  $\psi$  not a scalar,  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{o}, \mathfrak{o}, 0)$  and  $V_{\text{IIIb}}(0, 0, \mathfrak{h}, \mathfrak{o}, \mathfrak{g}, \psi)$  for any  $\psi$ .*

For each case in turn we will now exhibit a conformal derivation and the automorphism to which it exponentiates. In all cases, the automorphism is a simple rescaling of the basis elements.

For  $V(E, J)$  we have the following conformal derivation

$$D\mathbf{e}_+ = \mu\mathbf{e}_+ \quad Dv = 3\mu v \quad D\mathbf{e}_- = \mu\mathbf{e}_- \quad Du = -\mu u \quad Dx = \mu x ,$$

for all  $x \in E$ . This clearly exponentiates to the following conformal automorphism

$$(\mathbf{e}_+, v, \mathbf{e}_-, u, x) \mapsto (e^\mu \mathbf{e}_+, e^{3\mu} v, e^\mu \mathbf{e}_-, e^{-\mu} u, e^\mu x) ,$$

for all  $x \in E$ .

For  $V(\mathfrak{g}, \psi)$  and  $\psi$  not a scalar, we find the following conformal derivation

$$D\mathbf{e}_+ = 3\mu\mathbf{e}_+ \quad Dv = \mu v \quad D\mathbf{e}_- = -\mu\mathbf{e}_- \quad Du = -\mu u \quad Dg = \mu g ,$$

for all  $g \in \mathfrak{g}$ , which exponentiates to the following conformal automorphism

$$(\mathbf{e}_+, v, \mathbf{e}_-, u, g) \mapsto (e^{3\mu} \mathbf{e}_+, e^\mu v, e^{-\mu} \mathbf{e}_-, e^{-\mu} u, e^\mu g) ,$$

for all  $g \in \mathfrak{g}$ .

For  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{o}, \mathfrak{o}, 0)$ , we find the following conformal derivation

$$D\mathbf{e}_+ = 3\mu\mathbf{e}_+ \quad Dv = \mu v \quad D\mathbf{e}_- = -\mu\mathbf{e}_- \quad Du = -\mu u \quad Dx = \mu x \quad Dh = \mu h ,$$

for all  $x \in E$  and  $h \in \mathfrak{h}$ , which exponentiates to the following conformal automorphism

$$(\mathbf{e}_+, v, \mathbf{e}_-, u, x, h) \mapsto (e^{3\mu} \mathbf{e}_+, e^\mu v, e^{-\mu} \mathbf{e}_-, e^{-\mu} u, e^\mu x, e^\mu h) ,$$

for all  $x \in E$  and  $h \in \mathfrak{h}$ .

Finally for  $V_{\text{IIIb}}(0, 0, \mathfrak{h}, \mathfrak{o}, \mathfrak{g}, \psi)$  and any  $\psi$ , we find the following conformal derivation

$$D\mathbf{e}_+ = 3\mu\mathbf{e}_+ \quad Dv = \mu v \quad D\mathbf{e}_- = -\mu\mathbf{e}_- \quad Du = -\mu u \quad Dg = \mu g \quad Dh = \mu h ,$$

for all  $g \in \mathfrak{g}$  and  $h \in \mathfrak{h}$ , which exponentiates to the following conformal automorphism

$$(\mathbf{e}_+, v, \mathbf{e}_-, u, g, h) \mapsto (e^{3\mu} \mathbf{e}_+, e^\mu v, e^{-\mu} \mathbf{e}_-, e^{-\mu} u, e^\mu g, e^\mu h) ,$$

for all  $g \in \mathfrak{g}$  and  $h \in \mathfrak{h}$ .

**6.3. Parity invariance.** As discussed in Section 2.6, parity invariance of the Bagger–Lambert action demands the existence of an isometric anti-automorphism of the Lie 3-algebra. It is easy to find isometric anti-automorphisms for all four types of indecomposable Lie 3-algebras in Proposition 31 admitting nontrivial conformal automorphisms. They are given by  $\gamma : V \rightarrow V$  defined as follows:

- For  $V(E, J)$ ,

$$\gamma : (\mathbf{e}_+, v, \mathbf{e}_-, u, x) \mapsto (v, \mathbf{e}_+, u, \mathbf{e}_-, x) ,$$

for all  $x \in E$ ;

- For  $V(\mathfrak{g}, \psi)$ ,

$$\gamma : (\mathbf{e}_+, v, \mathbf{e}_-, u, g) \mapsto (-\mathbf{e}_+, -v, -\mathbf{e}_-, -u, g) ,$$

for all  $g \in \mathfrak{g}$ ;

- For  $V_{\text{IIIb}}(E, J, \mathfrak{h}, \mathfrak{o}, \mathfrak{o}, 0)$ ,

$$\gamma : (\mathbf{e}_+, v, \mathbf{e}_-, u, x, h) \mapsto (\mathbf{e}_+, -v, \mathbf{e}_-, -u, x, -h) ,$$

for all  $x \in E$  and  $h \in \mathfrak{h}$ ; and

- $V_{\text{IIIb}}(0, 0, \mathfrak{h}, \mathfrak{o}, \mathfrak{g}, \psi)$ ,

$$\gamma : (\mathbf{e}_+, v, \mathbf{e}_-, u, h, g) \mapsto (-\mathbf{e}_+, -v, -\mathbf{e}_-, -u, h, g) ,$$

for all  $g \in \mathfrak{g}$  and  $h \in \mathfrak{h}$ .

A fuller investigation of the Bagger–Lambert models associated to these four classes of Lie 3-algebras will be the subject of a forthcoming preprint.

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