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Lorentzian Lie n-algebras

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LORENTZIAN LIE n-ALGEBRAS

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ABSTRACT. We classify Lie n-algebras possessing an invariant lorentzian inner product.

CONTENTS

1. INTRODUCTION

It is natural to classify algebraic structures, especially those which do not arise in Algebra alone. Such is the case with Lie n-algebras (see below for a definition), which have made their appearance in several physical contexts independently from the initial purely algebraic aim of generalising the notion of a Lie algebra.

General classifications, however, are usually hard, unless one imposes further conditions. For example, the classification problem of Lie algebras is *not tame*, which in layman's terms translates as being hopeless. Standing on the shoulders of Killing and Élie Cartan, the semisimple Lie algebras can be classified in a relatively straightforward manner, whereas the Levi decomposition further reduces the problem to classifying solvable Lie algebras and their possible semidirect products with semisimple Lie algebras. However the classification of solvable Lie algebras except in very low dimension is not possible. Even nilpotent Lie algebras stop being tame in dimension ≥ 7.

One natural class of Lie algebras which extend the semisimple Lie algebras and which, due to their relative rarity, offer a hope of classification, are the metric Lie algebras: those possessing a (nondegenerate) ad-invariant inner product. It is a classical result that the metric Lie algebras for which the inner product is positive-definite are the reductive Lie algebras, which are generated under direct sum by the one-dimensional and the simple Lie algebras. For indefinite signature, things get a little more interesting. Medina and Revoy [1] (see also [2]) showed that metric Lie algebras of any signature are still generated by the

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same ingredients under direct sum and a new construction called a *double extension*. This construction has its origin in the classification of lorentzian Lie algebras, summarised as follows.

Theorem 1 ([3]). *A finite-dimensional indecomposable lorentzian Lie algebra is either one-dimensional, isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ or else isomorphic to $\mathbb{R}u \oplus \mathbb{R}v \oplus W$, with Lie bracket*

$$[u, x] = A(x) \qquad and \qquad [x, y] = \langle A(x), y \rangle \, v \ ,$$

where $\langle -, - \rangle$ is a positive-definite inner product on W, $A : W \to W$ is an invertible skewsymmetric endomorphism and we extend the inner product on W to all of V by declaring $u, v \perp W$, $\langle u, v \rangle = 1$ and $\langle u, u \rangle = \langle v, v \rangle = 0$.

Pushing this construction further allows a construction of all metric Lie algebras with signature $(2, p)$ by double extending the lorentzian Lie algebras [1]; however, the description of such metric Lie algebras as a double extension is ambiguous and it is not clear a priori which of the Lie algebras so obtained are isomorphic. A different approach which results in a classification of Lie algebras with signature $(2, p)$ is given by Kath and Olbrich in [4], following the announcement [5].

Lie n-algebras (for $n > 2$) are a natural generalisation of the notion of a Lie algebra, with which they coincide when $n = 2$. They arise in a number of unrelated problems in mathematical physics in areas such as integrable systems, supersymmetric gauge theories, supergravity and string theory. It is this which, to my mind, makes their classification into a *de facto* interesting problem.

A **Lie n-algebra** structure on a vector space V is a linear map $\Phi : \Lambda^n V \to V$, denoted simply by a n-ary bracket, with the property that for every $x_1, \ldots, x_{n-1} \in V$, the left multiplication $\mathrm{ad}_{x_1, \ldots, x_{n-1}} : y \mapsto [x_1, \ldots, x_{n-1}, y]$ is a derivation over the bracket. If $n = 2$ this latter property is precisely the Jacobi identity for a Lie algebra. For $n > 2$ we will call it the n-Jacobi identity. For $n > 2$, Lie n-algebras were introduced by Filippov [6] and are often referred to as Filippov n-algebras. Many of the structural results in the theory of Lie algebras have their analogue in the theory of Lie n-algebras, often with some refinement; although it seems that Lie n-algebras become more and more rare as n increases, due perhaps to the fact that the n-Jacobi identity imposes more and more conditions as n increases. From now on, whenever we write Lie n-algebra, we will assume that $n > 2$ unless otherwise stated.

For example, over the complex numbers there is up to isomorphism a unique simple Lie n-algebra for every $n > 2$, of dimension $n + 1$ and whose n-bracket is given relative to a basis $(\boldsymbol{e}_i)$ by

$$[\boldsymbol{e}_1, \ldots, \widehat{\boldsymbol{e}_i}, \ldots, \boldsymbol{e}_{n+1}] = (-1)^i \boldsymbol{e}_i \ ,$$

where a hat denotes omission. Over the reals, they are all given by attaching a sign ε_i to each $\boldsymbol{e}_i$ on the right-hand side of the bracket. This result is due to Ling [7], who also established a Levi decomposition of an arbitrary Lie n-algebra into a direct sum of a semisimple Lie n-algebra (a direct sum of its simple ideals) and a maximal solvable subalgebra. This again reduces the classification of Lie n-algebras to that of the solvable

Lie n-algebras and their semidirect products with semisimple Lie n-algebras, but classifying solvable Lie n-algebras seems to be as hard for $n > 2$ as it is for $n = 2$.

Taking a cue from the case of Lie algebras, and because it is this class of Lie n-algebras which seem to appear in nature, one can restrict to metric Lie n-algebras. Let $b \in S^2 V^*$ be an inner product (i.e., a nondegenerate symmetric bilinear form), denoted simply as $\langle -, - \rangle$. We say that a Lie n-algebra (V, Φ, b) is **metric** if the left multiplications $\mathrm{ad}_{x_1, \dots, x_{n-1}}$ for $x_i \in V$ are skewsymmetric relative to b. Metric Lie n-algebras seem to have appeared for the first time in [8], albeit tangentially, and more prominently in [9], which has attracted a great deal of attention recently in the mathematical/theoretical physics community.

Given two metric Lie n-algebras (V_1, Φ_1, b_1) and (V_2, Φ_2, b_2), we may form their **orthogonal direct sum** $(V_1 \oplus V_2, \Phi_1 \oplus \Phi_2, b_1 \oplus b_2)$, by declaring that

$$[x_1, x_2, y_1, \dots, y_{n-2}] = 0 \qquad \text{and} \qquad \langle x_1, x_2 \rangle = 0 ,$$

for all $x_i \in V_i$ and all $y_i \in V_1 \oplus V_2$. The resulting object is again a metric Lie n-algebra. A metric Lie n-algebra is said to be indecomposable if it is not isomorphic (see below) to an orthogonal direct sum of metric Lie n-algebras $(V_1 \oplus V_2, \Phi_1 \oplus \Phi_2, b_1 \oplus b_2)$ with $\dim V_i > 0$. In order to classify the metric Lie n-algebras, it is clearly enough to classify the indecomposable ones.

The orthogonal Plücker identities conjectured in [8] imply that indecomposable Lie n-algebras admitting a positive-definite inner product are either simple or one-dimensional. This conjecture was proved by Nagy in [10] and independently by Papadopoulos [11]. For the special case of $n = 3$ this result was rediscovered in [12, 13]. Lorentzian Lie 3-algebras have been classified in [14], where a one-to-one correspondence is established between indecomposable lorentzian Lie 3-algebras and semisimple euclidean Lie algebras. More precisely one has the following

Theorem 2 ([14]). *Let (V, Φ, b) be a finite-dimensional indecomposable lorentzian Lie 3-algebra. Then it is either one-dimensional, simple or else isomorphic to $\mathbb{R}u \oplus \mathbb{R}v \oplus W$ with 3-bracket*

$$[u, x, y] = [x, y] \qquad \text{and} \qquad [x, y, z] = -\langle [x, y], z \rangle v ,$$

where $[-, -] : \Lambda^2 W \to W$ is a Lie bracket making W into a compact semisimple Lie algebra and $\langle -, - \rangle$ is a positive-definite ad-invariant inner product, extended to all of V by declaring $u, v \perp W$, $\langle u, v \rangle = 1$ and $\langle u, u \rangle = \langle v, v \rangle = 0$.

This latter class of Lie 3-algebras were discovered independently in [15–17].

The purpose of this short note is to classify lorentzian Lie n-algebras for $n > 3$. The classification borrows from the techniques in [14], which themselves are inspired by [1, 2]. The main result is contained in Theorem 9. The classification of metric Lie n-algebras will not be attempted here, but see [18] for the case $n = 3$ and signature $(2, p)$.

Wolfgang Hein and Nils-Peter Skoruppa of the Fachbereich Mathematik der Universität Siegen for making available a copy of [7].

2. Metric Lie n-algebras

We will require the basic terminology of Lie n-algebras, as in [6] or [7] and of metric Lie n-algebras. Many of the proofs will be omitted, since they can be read *mutatis mutandis* from the ones for $n = 3$ in [14].

2.1. **Some structure theory.** Let (V, Φ) be a Lie n-algebra. Given subspaces $W_i \subset V$, we will let $[W_1, \ldots, W_n]$ denote the subspace of V consisting of elements $[w_1, \ldots, w_n] \in V$, where $w_i \in W_i$.

A subspace $W \subset V$ is a **subalgebra**, written $W < V$, if $[W, \ldots, W] \subset W$. A subalgebra $W < V$ is said to be **abelian** if $[W, \ldots, W] = 0$.

If V, W are Lie n-algebras, then a linear map $\phi : V \to W$ is a **homomorphism** if

$$\phi[x_1, \ldots, x_n] = [\phi(x_1), \ldots, \phi(x_n)] \ ,$$

for all $x_1, \ldots, x_n \in V$. If ϕ is also a vector space isomorphism, we say that it is an **isomorphism** of Lie n-algebras. An isomorphism $V \to V$ is called an **automorphism**. A subspace $I \subset V$ is an **ideal**, written $I \lhd V$, if $[I, V, \ldots, V] \subset I$. It follows that there is a one-to-one correspondence between ideals and kernels of homomorphisms. If $I \lhd V$ and $J \lhd V$, then $I \cap J \lhd V$ and $I + J \lhd V$. We will say that an ideal $I \lhd V$ is **minimal** if any other ideal $J \lhd V$ contained in I is either 0 or I. Dually, an ideal $I \lhd V$ is **maximal** if any other ideal $J \lhd V$ containing I is either I or V.

A Lie n-algebra is **simple** if it is not one-dimensional and every ideal $I \lhd V$ is either 0 or V.

Lemma 3. *If $I \lhd V$ is a maximal ideal, then V/I is simple or one-dimensional.*

Simple Lie n-algebras have been classified.

Theorem 4 ([7]). *A simple real Lie n-algebra is isomorphic to one of the $(n + 1)$-dimensional Lie n-algebras defined, relative to a basis $\boldsymbol{e}_i$, by*

$$[\boldsymbol{e}_1, \ldots, \widehat{\boldsymbol{e}_i}, \ldots, \boldsymbol{e}_{n+1}] = (-1)^i \varepsilon_i \boldsymbol{e}_i \ , \tag{1}$$

where a hat denotes omission and where the ε_i are signs.

It is plain to see that simple real Lie n-algebras admit invariant metrics of any signature. Indeed, the Lie n-algebra in (1) leaves invariant the diagonal metric with entries $(\varepsilon_1, \ldots, \varepsilon_{n+1})$.

The subspace $[V, \ldots, V]$ is an ideal called the **derived ideal** of V. Another ideal is provided by the **centre** Z, defined by the condition $[Z, V, \ldots, V] = 0$. More generally the **centraliser** $Z(W)$ of a subspace $W \subset V$ is the subalgebra defined by $[Z(W), W, V, \ldots, V] = 0$.

From now on let (V, Φ, b) be a metric Lie n-algebra. If $W \subset V$ is any subspace, we define

$$W^\perp = \{v \in V | \langle v, w \rangle = 0 \ , \forall w \in W\} \ .$$

Notice that $(W^\perp)^\perp = W$. We say that W is **nondegenerate**, if $W \cap W^\perp = 0$, whence $V = W \oplus W^\perp$; **isotropic**, if $W \subset W^\perp$; and **coisotropic**, if $W \supset W^\perp$. Of course, in positive-definite signature, all subspaces are nondegenerate.

A metric Lie n-algebra is said to be **indecomposable** if it is not isomorphic to a direct sum of orthogonal ideals or, equivalently, if it does not possess any proper nondegenerate ideals: for if $I \lhd V$ is nondegenerate, $V = I \oplus I^\perp$ is an orthogonal direct sum of ideals.

Lemma 5. *Let $I \lhd V$ be a coisotropic ideal of a metric Lie n-algebra. Then $I/I^\perp$ is a metric Lie n-algebra.*

Lemma 6. *Let V be a metric Lie n-algebra. Then the centre is the orthogonal subspace to the derived ideal; that is, $[V, \ldots, V] = Z^\perp$.*

Proposition 7. *Let V be a metric Lie n-algebra and $I \lhd V$ be an ideal. Then*

(1) *$I^\perp \lhd V$ is also an ideal;*
(2) *$I^\perp \lhd Z(I)$; and*
(3) *if I is minimal then $I^\perp$ is maximal.*

2.2. **Structure of metric Lie n-algebras.** We now investigate the structure of metric Lie n-algebras. If a Lie n-algebra is not simple or one-dimensional, then it has a proper ideal and hence a minimal ideal. Let $I \lhd V$ be a minimal ideal of a metric Lie n-algebra. Then $I \cap I^\perp$, being an ideal contained in I, is either 0 or I. In other words, minimal ideals are either nondegenerate or isotropic. If nondegenerate, $V = I \oplus I^\perp$ is decomposable. Therefore if V is indecomposable, I is isotropic. Moreover, by Proposition 7 (2), I is abelian and furthermore, because I is isotropic, $[I, I, V, \ldots, V] = 0$.

It follows that if V is euclidean and indecomposable, it is either one-dimensional or simple, whence of the form (1) with all $\varepsilon_i = 1$. This result, originally due to [10], was conjectured in [8].

Let V be an indecomposable metric Lie n-algebra. Then V is either simple, one-dimensional, or possesses an isotropic proper minimal ideal I which obeys $[I, I, V, \ldots, V] = 0$. The perpendicular ideal $I^\perp$ is maximal and hence by Lemma 3, $U := V/I^\perp$ is simple or one-dimensional, whereas by Lemma 5, $W := I^\perp/I$ is a metric Lie n-algebra.

The inner product on V induces a nondegenerate pairing $g : U \otimes I \to \mathbb{R}$. Indeed, let $[u] = u + I^\perp \in U$ and $v \in I$. Then we define $g([u], v) = \langle u, v \rangle$, which is clearly independent of the coset representative for $[u]$. In particular, $I \cong U^*$ is either one- or $(n + 1)$-dimensional. If the signature of the metric of W is (p, q), that of V is $(p + r, q + r)$ where $r = \dim I = \dim U$. So that if V is to have lorentzian signature, $r = 1$ and W must be euclidean; although not necessarily indecomposable.

A lorentzian Lie n-algebra decomposes into one lorentzian indecomposable factor and zero or more indecomposable euclidean factors. As discussed above, the indecomposable euclidean Lie n-algebras are either one-dimensional or simple. On the other hand, an indecomposable lorentzian Lie n-algebra is either one-dimensional, simple or else possesses a one-dimensional isotropic minimal ideal. It is this latter case which remains to be treated and we do so now.

The quotient Lie n-algebra $U = V/I^\perp$ is also one-dimensional. Let $u \in V$ be such that $u \notin I^\perp$, whence its image in U generates it. Because $I \cong U^*$, there is $v \in I$ such that $\langle u, v \rangle = 1$. Complete it to a basis (v, x_a) for $I^\perp$. Then (u, v, x_a) is a basis for V, with (x_a) spanning a subspace isomorphic to $W = I^\perp/I$ and which, with a slight abuse of notation, we will also denote W. It is possible to choose u so that $\langle u, u \rangle = 0$ and such that $\langle u, x \rangle = 0$ for all $x \in W$. Indeed, given any u, the map $x \mapsto \langle u, x \rangle$ defines an element in the dual W^*. Since the restriction of the inner product to W is nondegenerate, there is some $z \in W$ such that $\langle u, x \rangle = \langle z, x \rangle$ for all $x \in W$. We let $u' = u - z$. This still obeys $\langle u', v \rangle = 1$ and now also $\langle u', x \rangle = 0$ for all $x \in W$. Finally let $u'' = u' - \frac{1}{2} \langle u', u' \rangle v$, which still satisfies $\langle u'', v \rangle = 1$, $\langle u'', x \rangle = 0$ for all $x \in W$, but now satisfies $\langle u'', u'' \rangle = 0$ as well.

From Proposition 7 (2), it is immediate that $[u, v, x_1 \ldots, x_{n-2}] = 0 = [v, x_1 \ldots, x_{n-1}]$, whence v is central. Metricity then implies that the only nonzero n-brackets take the form

$$
\begin{aligned}
[u, x_1, \ldots, x_{n-1}] &=: [x_1, \ldots, x_{n-1}] \\
[x_1, \ldots, x_n] &= (-1)^n \langle [x_1, \ldots, x_{n-1}], x_n \rangle v + [x_1, \ldots, x_n]_W .
\end{aligned}
\tag{2}
$$

The n-Jacobi identity is equivalent to the following two conditions:

(1) $[x_1, \ldots, x_{n-1}]$ defines a Lie $(n-1)$-algebra structure on W, which leaves the inner product invariant due to the skewsymmetry of $\langle [x_1, \ldots, x_{n-1}], x_n \rangle$; and

(2) $[x_1, \ldots, x_n]_W$ defines a euclidean Lie n-algebra structure on W which is invariant under the $(n-1)$-algebra structure.

We will show below that for V indecomposable, the Lie n-algebra structure on W is abelian.

Let (V, Φ) be a Lie n-algebra. It was already observed in [6] that every $z \in V$ defines an $(n-1)$-bracket $\Phi_z : \Lambda^{n-1} V \to V$, denoted simply as $[\ldots]_z$, and defined by

$$
[x_1, \ldots, x_{n-1}]_z := [x_1, \ldots, x_{n-1}, z] ,
\tag{3}
$$

which obeys the $(n-1)$-Jacobi identity as a consequence of the n-Jacobi identity. Thus Φ_z defines on V a Lie $(n-1)$-algebra structure for which z is a central element.

If (V, Φ, b) is a metric Lie n-algebra, then each of the Lie $(n-1)$-algebras (V, Φ_z, b) is a metric Lie $(n-1)$-algebra. Let V be a simple euclidean Lie n-algebra given relative to a basis e_i by (1) with all $\varepsilon_i = 1$. Moreover, such a basis is orthogonal, but not necessarily orthonormal. Thus there is a one parameter family of such metric Lie n-algebras, distinguished by the scale of the inner product. We will denote the simple Lie n-algebra with the above n-brackets by $\mathfrak{s}^{(n)}$. Fixing any nonzero $x \in \mathfrak{s}^{(n)}$, the Lie $(n-1)$-algebra Φ_x is isomorphic to $\mathfrak{s}^{(n-1)} \oplus \mathbb{R}$, where $\mathfrak{s}^{(n-1)}$ is the simple euclidean Lie $(n-1)$-algebra structure on the perpendicular complement of the line containing x.

3. LORENTZIAN LIE n-ALGEBRAS

We are now ready to classify the indecomposable lorentzian Lie n-algebras and hence all lorentzian Lie n-algebras. For $n = 3$ the classification of lorentzian Lie 3-algebras is given in [14] and is summarised in Theorem 2, whereas for $n = 2$, the lorentzian Lie algebras are classified in [3] and summarised in Theorem 1. Hence from now on we will take $n > 3$.

We have previously shown that $V = \mathbb{R}u \oplus \mathbb{R}v \oplus W$, with the n-brackets given by (2). We will now show that if V is indecomposable, then as a Lie n-algebra, W is necessarily abelian.

Since W is a euclidean Lie n-algebra, it is given by

$$W \cong \mathfrak{a} \oplus \mathfrak{s}_1^{(n)} \oplus \cdots \oplus \mathfrak{s}_q^{(n)}$$

where $\mathfrak{a}$ is a p-dimensional abelian Lie n-algebra. The inner product is such that the above direct sums are orthogonal, and the inner product on each of the factors is positive-definite.

Lemma 8. *The Lie $(n-1)$-algebra structure on W is such that the adjoint representation $\mathrm{ad} : \Lambda^{n-2}W \to \mathrm{End}\, W$ preserves the above orthogonal decomposition, whence each of the orthogonal summands are actually $(n-1)$-ideals.*

Proof. We will show that each of the $\mathfrak{s}_i^{(n)}$ are submodules. Since the $(n-1)$-adjoint action of W preserves the inner product, this means that so is $\mathfrak{a}$ and hence the claim, since submodules of the $(n-1)$-adjoint representation are $(n-1)$-ideals.

Consider $\mathfrak{s}_1^{(n)}$, say, and let $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n+1})$ be a basis. For every $\boldsymbol{x} := x_1 \wedge \cdots \wedge x_{n-2} \in \Lambda^{n-2}W$,

$$\mathrm{ad}_{\boldsymbol{x}}\, \boldsymbol{e}_i := [x_1, \ldots, x_{n-2}, \boldsymbol{e}_i] = y_i + z_i \ ,$$

where $y_i \in \mathfrak{s}_1^{(n)} \cap \boldsymbol{e}_i^{\perp}$ and $z_i \in \left(\mathfrak{s}_1^{(n)} \right)^{\perp}$. We cannot have a component along $\boldsymbol{e}_i$ because invariance of the inner product says that $\mathrm{ad}_{\boldsymbol{x}}\, \boldsymbol{e}_i \perp \boldsymbol{e}_i$. Consider $\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_{n+1}$. This being essentially the n-bracket in $\mathfrak{s}_1^{(n)}$, it is also preserved under $\mathrm{ad}_{\boldsymbol{x}}$, whence

$$\begin{aligned}
0 &= \mathrm{ad}_{\boldsymbol{x}}\, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_{n+1} + \cdots + \boldsymbol{e}_1 \wedge \cdots \wedge \mathrm{ad}_{\boldsymbol{x}}\, \boldsymbol{e}_{n+1} \\
&= (y_1 + z_1) \wedge \cdots \wedge \boldsymbol{e}_{n+1} + \cdots + \boldsymbol{e}_1 \wedge \cdots \wedge (y_{n+1} + z_{n+1}) \\
&= z_1 \wedge \cdots \wedge \boldsymbol{e}_{n+1} + \cdots + \boldsymbol{e}_1 \wedge \cdots \wedge z_{n+1} \ .
\end{aligned}$$

But each of these terms is independent, whence $z_i = 0$ and $\mathrm{ad}_{\boldsymbol{x}}$ preserves $\mathfrak{s}_1^{(n)}$. $\square$

As a Lie $(n-1)$-algebra, $W = W_0 \oplus W_1 \oplus \cdots \oplus W_q$, where W_0 is a p-dimensional euclidean Lie $(n-1)$-algebra and $W_{i>0}$ are $(n+1)$-dimensional euclidean Lie $(n-1)$-algebras. Euclidean Lie $(n-1)$-algebras are orthogonal direct sums of an abelian Lie $(n-1)$-algebras (possibly zero-dimensional) and zero or more copies of the n-dimensional euclidean simple Lie $(n-1)$-algebra $\mathfrak{s}^{(n-1)}$. In particular, on dimensional grounds, each $W_{i>0}$ is either abelian or isomorphic to $\mathfrak{s}^{(n-1)} \oplus \mathbb{R}$.

We will now show that every $\mathfrak{s}^{(n)}$ summand in the Lie n-algebra W factorises in V, contradicting the assumption that V is indecomposable.

Consider one such $\mathfrak{s}^{(n)}$ summand, say $\mathfrak{s}_1^{(n)}$. The corresponding Lie $(n-1)$-algebra W_1 is either abelian or isomorphic to $\mathfrak{s}^{(n-1)} \oplus \mathbb{R}$. If W_1 is abelian, so that the $(n-1)$-brackets vanish, then it follows from (2) that for any $x \in \mathfrak{s}_1^{(n)}$, $[u, x, V, \ldots, V] = 0$ and $[x, y, V, \ldots, V] = 0$ for any $y \in W$ perpendicular to $\mathfrak{s}_1^{(n)}$. Hence $\mathfrak{s}_1^{(n)} \lhd V$ is a nondegenerate ideal, contradicting the indecomposability of V.

If $W_1 \cong \mathfrak{s}^{(n-1)} \oplus \mathbb{R}$, it has a one-dimensional centre spanned by, say, $x \in W_1$. Multiplying x by a scalar if necessary, we can assume that the Lie $(n-1)$-structure on W_1 is given by the $(n-1)$-bracket $[x, \ldots]_W$ induced from the Lie n-algebra structure on $\mathfrak{s}_1^{(n)}$; that is, for all $y_1, \ldots, y_{n-1} \in W_1$, $[y_1, \ldots, y_{n-1}] = [x, y_1, \ldots, y_{n-1}]_W$.

This allows us to "twist" $\mathfrak{s}_1^{(n)}$ into a nondegenerate ideal of V. Indeed, define now a vector space isomorphism $\varphi : V \to V$ by

$$\begin{aligned}
\varphi(v) &= v \\
\varphi(u) &= u - x - \tfrac{1}{2}|x|^2 v \\
\varphi(y) &= y + \langle y, x \rangle v \\
\varphi(z) &= z \ ,
\end{aligned} \tag{4}$$

for all $y \in \mathfrak{s}_1^{(n)}$ and $z \in W \cap \left(\mathfrak{s}_1^{(n)}\right)^{\perp}$. It follows from (2) that, since v is central, for all $y_i \in \mathfrak{s}_1^{(n)}$,

$$\begin{aligned}
[\varphi(u), \varphi(y_1), \ldots, \varphi(y_{n-1})] &= [u - x, y_1, \ldots, y_{n-1}] \\
&= [y_1, \ldots, y_{n-1}] - [x, y_1, \ldots, y_{n-1}]_W - (-1)^n \langle [x, y_1, \ldots, y_{n-2}], y_{n-1} \rangle v \\
&= \langle [y_1, \ldots, y_{n-2}, y_{n-1}], x \rangle v = 0 \ ,
\end{aligned}$$

since x, being central, is perpendicular to the derived $(n-1)$-ideal $[W_1, \ldots, W_1]$ by Lemma 6. Finally, let $y_1, \ldots, y_n \in \mathfrak{s}_1^{(n)}$, and since v is central, we have

$$\begin{aligned}
[\varphi(y_1), \ldots, \varphi(y_n)] &= [y_1, \ldots, y_n] \\
&= (-1)^n \langle [y_1, \ldots, y_{n-1}], y_n \rangle v + [y_1, \ldots, y_n]_W \\
&= - \langle [y_1, \ldots, y_{n-1}, x]_W, y_n \rangle v + [y_1, \ldots, y_n]_W \\
&= \langle [y_1, \ldots, y_{n-1}, y_n]_W, x \rangle v + [y_1, \ldots, y_n]_W \\
&= \varphi([y_1, \ldots, y_n]_W) \ .
\end{aligned}$$

In other words, $\varphi(\mathfrak{s}_1^{(n)})$ is a subalgebra. Since it commutes with $\varphi(u)$, $\varphi(v)$ and $\varphi(z)$ for $W \ni z \perp \mathfrak{s}_1^{(n)}$, we see that $\varphi(\mathfrak{s}_1^{(n)}) \lhd V$.

It remains to show that φ preserves the inner product. If $y \in \mathfrak{s}_1^{(n)}$, then

$$\langle \varphi(u), \varphi(y) \rangle = \langle u - x - \tfrac{1}{2}|x|^2 v, y + \langle x, y \rangle v \rangle = \langle x, y \rangle \langle u, v \rangle - \langle x, y \rangle = 0 \ ,$$

it is clear that $\langle \varphi(u), \varphi(v) \rangle = 1$ and that $\langle \varphi(v), \varphi(v) \rangle = 0$, whence we need only check

$$\langle \varphi(u), \varphi(u) \rangle = \langle u - x - \tfrac{1}{2}|x|^2 v, u - x - \tfrac{1}{2}|x|^2 v \rangle = -|x|^2 \langle u, v \rangle + \langle x, x \rangle = 0 \ .$$

In other words, we conclude that $\varphi(\mathfrak{s}_1^{(n)}) \lhd V$ is a degenerate ideal, contradicting again the fact that V is indecomposable.

Consequently there can be no $\mathfrak{s}^{(n)}$ summands in W, whence as a Lie n-algebra, W is abelian. As a Lie $(n-1)$-algebra it is euclidean, whence isomorphic to $\mathfrak{b} \oplus \mathfrak{s}_1^{(n-1)} \oplus \cdots \oplus \mathfrak{s}_m^{(n-1)}$, where $\mathfrak{b}$ is now an abelian Lie $(n-1)$-algebra. However the abelian summand centralises u and its perpendicular complement in W, hence it is central and nondegenerate in V, again

contradicting the fact that it is indecomposable. Therefore as a Lie $(n-1)$-algebra W is euclidean semisimple: $W \cong \mathfrak{s}_1^{(n-1)} \oplus \cdots \oplus \mathfrak{s}_m^{(n-1)}$. In particular it has dimension mn.

In summary, we have proved the following

Theorem 9. *Let (V, Φ, b) be a finite-dimensional indecomposable lorentzian Lie n-algebra. Then it is either one-dimensional, simple, or else $V = \mathbb{R}u \oplus \mathbb{R}v \oplus W$, with u, v complementary null directions perpendicular to W, such that the nonzero n-brackets take the form*

$$[u, x_1, \ldots, x_{n-1}] = [x_1, \ldots, x_{n-1}] \qquad and \qquad [x_1, \ldots, x_n] = (-1)^n \langle [x_1, \ldots, x_{n-1}], x_n \rangle v ,$$

where $[x_1, \ldots, x_{n-1}]$ makes W into a euclidean Lie $(n-1)$-algebra isomorphic to m copies of the simple euclidean Lie $(n-1)$-algebra $\mathfrak{s}^{(n-1)}$. In particular, $\dim V = mn + 2$.

Corollary 10. *Let (V, Φ, b) be a lorentzian Lie n-algebra. Then it is isomorphic to*

$$V = V_0 \oplus A \oplus \underbrace{\mathfrak{s}^{(n)} \oplus \cdots \oplus \mathfrak{s}^{(n)}}_{q} ,$$

where A is a $(p \geq 0)$-dimensional euclidean abelian Lie n-algebra, $q \geq 0$, and V_0 is either one-dimensional with a negative-definite inner product, a lorentzian simple Lie n-algebra, or an indecomposable lorentzian Lie n-algebra with brackets given in Theorem 9.

It will not have escaped the reader's attention that Theorems 1, 2 and 9 are very similar. In all cases an indecomposable lorentzian Lie n-algebra is either one-dimensional, isomorphic to a unique simple Lie n-algebra or else it is obtained by a "double extension" from a euclidean semisimple Lie $(n-1)$-algebra, where the rôle of a semisimple Lie 1-algebra is played by an invertible skewsymmetric endomorphism. Skew-diagonalising the endomorphism, we see that it too is a direct sum of irreducibles. Hence in all cases except for $n = 2$, there is a unique simple euclidean n-Lie algebra. We will not formalise the notion of double extension here, except to note that it clearly extends to Lie n-algebras for $n > 2$. In the lorentzian examples, we double extend by a one-dimensional Lie n-algebra, but it is possible to double extend by other Lie n-algebras as well. For example, let $\mathfrak{s}$ be a simple Lie n-algebra and consider the vector space $\mathfrak{s} \oplus \mathfrak{s}^*$ with the following n-bracket. If $x_i \in \mathfrak{s}$ and $\alpha \in \mathfrak{s}^*$, we define $[x_1, \ldots, x_n] = [x_1, \ldots, x_n]_{\mathfrak{s}}$ to be the n-bracket in $\mathfrak{s}$ and $[x_1, \ldots, x_{n-1}, \alpha] =: \beta \in \mathfrak{s}^*$ be defined by $\beta(x_n) = -\alpha([x_1, \ldots, x_n]_{\mathfrak{s}})$. This is a metric Lie n-algebra of signature $(n+1, n+1)$, relative to the inner product consisting of any invariant inner product on $\mathfrak{s}$, the dual pairing between $\mathfrak{s}$ and $\mathfrak{s}^*$ and declared to be identically zero on $\mathfrak{s}^*$.

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