An effective theory on the light shell

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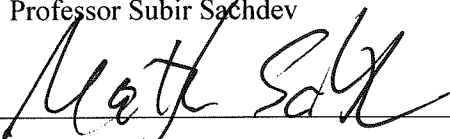
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An effective theory on the light shell

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An effective theory on the light shell

Abstract

We describe work on the construction of an effective field theory on a spherical light shell. The motivation arises from classical electromagnetism: If a collision produces charged particles with zero net charge emerging simultaneously from a point and instantaneously accelerating to the speed of light, then the electromagnetic fields due to these charges lie entirely on a spherical shell expanding at the speed of light. We show that this also applies to classical color radiation from high-energy collisions that produce colored particles. Specifically, the color fields produced in such a process are associated with a non-linear σ-model on the 2D light shell with specific symmetry-breaking terms. The quantum version of such a picture exhibits asymptotic freedom and should therefore be a useful starting point for a light-shell effective theory for QCD.

We start in the simplified context of zero-flavor scalar quantum electrodynamics. Our effective theory has 3 major ingredients: breaking down the fields into soft and hard sectors with the large energy of the hard fields in the radial direction scaled out, a special gauge called light-shell gauge in which the picture simplifies, and a gauge-invariant source defined on a spherical light shell having infinitesimal radius.

We match the fields between the effective theory and the full theory, meaning zero-flavor scalar QED. This allows us to compute the amplitude for the production of any number of scalars from the gauge-invariant source. We then find the tree-level

amplitude for the emission of a photon using our effective theory and show that our result agrees with the full theory.

To calculate loop effects in our effective theory, we need the photon propagator in light-shell gauge. We derive this propagator and use it to calculate the 1-loop correction to the amplitude for the production of a scalar and anti-scalar pair arising from virtual photon effects. This reduces to a pair of purely angular integrals in the effective theory and reproduces the familiar double logs of the full theory subject to an appropriate interpretation of an angular cutoff.

Contents

Citations to Previously Published Work

H. Georgi, G. Kestin, and A. Sajjad, "Color Fields on the Light-Shell," http://arxiv.org/abs/1004.1404arXiv:1004.1404 [hep-ph].

H. Georgi, G. Kestin, and A. Sajjad, "The Photon Propagator in Light-Shell Gauge," http://arxiv.org/abs/1312.1741arXiv:1312.1741 [hep-ph].

H. Georgi, G. Kestin, and A. Sajjad, "Towards an Effective Field Theory on the Light-Shell Effective," http://arxiv.org/abs/1401.7667arXiv:1401.7667 [hep-ph].

To my exceptional mother

Farida Sajjad

1   Introduction

This thesis is based on our work on Light Shell Effective Theory described in [1], [2], [3] and [4] co-authored with Howard Georgi and Gregory Kestin. In this introduction, I will briefly summarize the basic concept behind this work and give an outline of the important results presented in later chapters.

The basic motivation for our work comes from a picture in chapter 5 of Ed Purcell'classic textbook on electromagnetism [5] illustrating how a pulse of electromagnetic radiation emerges from a kink in the field of a charge that instantaneously accelerates from rest to a constant velocity. We believe that a similar picture may yield a useful starting point for an effective field theory description of very high energy collisions between hadrons.

In a collider experiment, colourless incoming particles come in and interact in a very small region of space-time, spewing away coloured particles going off at high energies in various directions. This is similar to the situation in classical electromagnetism where a collision can result in charged particles suddenly emerging from a single point from an initially neutral charge distribution. In the limit where the collision takes place instantaneously and the charged particles move out at the speed of light, a pulse of radiation is produced which moves outward along a spherical surface whose radius expands at the speed of light. The electromagnetic fields are then sharply peaked on this spherical surface, and zero both inside and outside it. We call this surface the light shell (as it is the equal t slice of the lightcone of the initial space-time event at $t = r = 0$).

We believe that this picture where all the physics resides on a spherical light shell expanding at the speed of light should also apply to the case of hadronic collisions. In this case, the initial collision involves hard QCD processes taking place at energies large compared to the QCD scale. This produces very high energy colored particles that fly apart at the speed of light and these particles, along with the color electric and magnetic fields they produce will be confined to an expanding light shell, just as in the case of

electromagnetism. We hope this picture may be useful to describe the physics for the range of times between the very short time scale of the initial collision and the "long" time scale of $1/\Lambda_{QCD}$ at which point confinement comes into play.

In chapter 2, this idea is fleshed out in detail by first giving the calculation of electromagnetic fields produced by a collection of charged particles with zero net charge instantaneously accelerating to the speed of light as mentioned above. One important finding in this regard is that not only the electromagnetic fields reside entirely on the light shell, but the potentials too can be expressed in a similar form where they are zero everywhere except on the $t = r$ sphere if we choose the gauge condition $v_\mu A^\mu = 0$, where v^μ is a light-like vector pointing away from the origin. This gauge, which we call the light-shell gauge (or LSG), will also be an important part of the construction of our effective theory.

The analysis is then extended to the non-abelian case by looking at classical color fields in the appropriate limit. Our analysis shows that the classical color electric fields on the light shell can be related to a non-linear σ-model on a static two dimensional sphere with the Goldstone bosons playing the role of the potential fields and with specific symmetry breaking related to the color charges of the high energy particles producing the fields.

Before going on to construct LSET, it is possible to have some fun connecting the classical electromagnetic fields in our physical set up with the tree-level quantum mechanical amplitude for the emission of a photon in QED/SQED (the amplitude is the same in both QED and SQED). This is done in detail in chapter 3. The argument can be made in two directions. In classical electromagnetism, the intensity is given as the sum of the squares of the E and B fields. This can be interpreted as a probability of photon emission and therefore connected with the tree-level quantum amplitude for photon emission in QED. Conversely, the quantum mechanical probability amplitude is the expectation value of the gauge field in momentum space. It is possible to show that its square, after some simple manipulation, can be expressed as the intensity of electromagnetic fields.

The remaining chapters present our work on developing the quantum effective theory

based on our classical picture. Our focus has thus far been in the simplified case of 0-flavor scalar quantum electrodynamics. In simple words, this is a limit in which we can ignore matter loops. This allows us to focus on the basic construction of the effective theory without being distracted by complicated loop effects. The ultimate goal is to extend this work beyond the zero-flavor case and all the way to QCD, but this should serve as a good starting point.

Chapter 2 describes the construction of our quantum effective theory for zero flavour scalar QED (also referred to as the full theory in this thesis). Translating the above-mentioned classical picture of charged particles emerging from a single point and accelerating instantaneously to the speed of light into quantum field theory suggests a gauge invariant source at $t = r = 0$. In our effective theory, however, we work in light-shell gauge which is not defined at the origin. We therefore define a source that is "spread out" around the origin on a light shell having infinitesimal radius. This is consistent with our classical picture where all the physics goes on the light shell in the high-energy limit. We therefore spread out our source onto a surface $r = s$ surrounding the origin, near the light shell, with $t - r \to 0$ as $E \to \infty$.

Moreover, the effective theory framework requires that we set an energy scale $\mathcal{E}$ to define what we mean by "high energy". In the spirit of HQET [6] (for a recent and comprehensive review see [7]) we scale out the large momenta associated with the energetic outgoing particles. We call the associated decomposition into fields above and below $\mathcal{E}$ the large radial energy (LRE) expansion, and refer to the fields with high energies as LRE fields. Specifically, the LRE fields correspond to high-energy particles produced by the source carrying large energy $E > \mathcal{E}$ outwards from $t = r = 0$ into the bulk space. We will see that to leading order in $1/E$, the direction $\hat{r}$ of propagation away from the origin is a classical variable and we can label the LRE fields by $\hat{r}$. But in the presence of interactions, the directions of the LRE fields cannot be specified precisely. So to each charged LRE field we assign an "angular size".

The LSET Lagrangian is obtained by applying the LRE expansion to sQED, and expanding in orders of $1/E$. An important result that emerges from this is that the gauge interactions at leading order (in $1/E$) are proportional to $v^\mu A_\mu$ and therefore vanish in light-shell gauge. With these interactions eliminated as gauge artifacts, all the physics at high energies is described in terms of a gauge invariant source at the origin of space-time. Because different configurations of LRE fields (different energies and directions) do not interfere, each such configuration is associated with its own sector, and the source in the EFT is a sum over all such sectors, separated by superselection rules.

Since the point of any effective theory is to reproduce the physics of a more fundamental theory in a certain limit, our theory must be able to do the same. This is the subject of chapter 5. The standard canonical quantization recipe is applied to the LRE theory, and the connection between LRE scalars and full theory fields is established by comparing the commutation relations of the creation/annihilation operators in both theories. This understanding of creation/annihilation operators for LRE fields is then also employed in the calculation of the tree-level amplitude for the emission of a hard photon. The photon production amplitude calculated from our effective theory is found to match the familiar result from the full theory.

In order to go beyond tree-level effects, we need the photon propagator in light-shell gauge. Calculating this turns out to be a non-trivial task because LSG is a non-covariant gauge, making it impossible for us to follow the well-known methods for covariant gauges [8]. Chapter 6 explains the calculation of the LSG photon propagator. We hope that the technique we have introduced for this purpose may also have some relevance for other non-covariant gauges, though this is not a question we have yet explored.

The ultimate goal of our effective theory is to get another handle on the IR and collinear divergences that afflict QED and QCD. While this is ongoing work, some of our initial results are very encouraging and are described in chapter 7. Specifically, in our calculation of the leading order virtual photon exchange process in our effective theory setting, we are able

to obtain a result that looks very similar to the corresponding full theory result with the familiar double logs. The only difference is that our result includes the log of an angular cutoff for which we do not yet have an obvious physical connection with the ratio of energies appearing in the corresponding result in the full theory. One very interesting feature of our theory, which is especially evident in this particular calculation, is that all our calculations can be described in terms of purely angular integrals.

The results summarized above (and described in subsequent chapters in detail) show that our theory certainly has some potential as a viable description for QED and QCD processes. Beyond the obvious (and broad) question of eventually extending this beyond zero flavor QED all the way to QCD, there are also several more specific questions that come up in the course of our analysis and require further research. Some of these are discussed in chapter 8.

In short, while we already have the well-developed Soft Collinear Effective Theory (SCET) [9, 10] which is already being successfully applied to QCD processes at high energies [11], the idea of an effective theory on the sphere is so appealing and different that we believe there is a real possibility that it could provide some new insights.

2 Classical motivation

2.1 The retarded potential calculation

We start by considering two oppositely charged particles with charges $\pm q$ created instantaneously at $t = 0$ at the origin and then moving apart in opposite directions at the speed of light with velocities $\pm \hat{n}$, respectively. The charge density can be written as

$$\rho(\vec{r}, t) = qe \left[\delta \left(\vec{r} - \vec{R}(t) \right) - \delta \left(\vec{r} + \vec{R}(t) \right) \right] \tag{1}$$

and the current density is

$$\vec{\mathcal{J}}(\vec{r}, t) = qe\,\hat{n} \left[\delta \left(\vec{r} - \vec{R}(t) \right) + \delta \left(\vec{r} + \vec{R}(t) \right) \right] \tag{2}$$

where

$$\vec{R}(t) = \hat{n} t \theta(t) \tag{3}$$

The scalar potential (in Lorentz-Heaviside units) is given by

$$\phi(\vec{r}, t) = \frac{1}{4\pi} \int \frac{\rho \left(\vec{r'}, t - \left| \vec{r} - \vec{r'} \right| \right)}{\left| \vec{r} - \vec{r'} \right|} \, d^3 r' \tag{4}$$

and the vector potential is

$$\vec{A}(\vec{r}, t) = \frac{1}{4\pi} \int \frac{\vec{\mathcal{J}} \left(\vec{r'}, t - \left| \vec{r} - \vec{r'} \right| \right)}{\left| \vec{r} - \vec{r'} \right|} \, d^3 r' \tag{5}$$

Now, focusing on the contribution of the positive charge, for every point $\vec{r}$ on or inside the sphere of radius t, we have a unique retarded position

$$\vec{r'} = \hat{n} \, r' \tag{6}$$

6

that contributes to the retarded potential and satisfies the condition

$$\vec{r'} = \vec{R}\left(t - \left|\vec{r} - \vec{r'}\right|\right) \tag{7}$$

Since both sides are in the $\hat{n}$ direction, this can be written as

$$r' = t - \left|\vec{r} - \hat{n}\, r'\right| \geq 0 \tag{8}$$

or equivalently,

$$t - r' = \left|\vec{r} - \hat{n}\, r'\right| \tag{9}$$

We will actually be able to do the d^3r' integral easily because of the delta functions in the charge and current distributions. Define the argument of the delta function

$$\vec{r''} \equiv \vec{r'} - \vec{R}\left(t - \left|\vec{r} - \vec{r'}\right|\right) \tag{10}$$

The Jacobian is then easily obtained by using (3) along with the chain rule:

$$\frac{\partial r''_j}{\partial r'_k} = \delta_{jk} - V_j\left(t - \left|\vec{r} - \vec{r'}\right|\right)\frac{r_k - r'_k}{\left|\vec{r} - \vec{r'}\right|} \tag{11}$$

where $\vec{V}$ is the velocity, and is given by $\partial R/\partial t = \hat{n}\,\theta(t)$. For the d^3r' integral, we need the inverse of the determinant of this Jacobian evaluated where $r'' = 0$. Calculating this is a straightforward exercise, and we get

$$\left|\frac{\partial r'}{\partial r''}\right|_{\vec{r''}=0} = \frac{1}{1 - \vec{V}\left(t - \left|\vec{r} - \vec{r'}\right|\right)\cdot\left(\vec{r} - \vec{r'}\right)\Big/\left|\vec{r} - \vec{r'}\right|} = \frac{1}{1 - \hat{n}\cdot\left(\vec{r} - \vec{r'}\right)\Big/\left|\vec{r} - \vec{r'}\right|} \tag{12}$$

7

Using (12) in (4) then gives the retarded potential

$$
\begin{aligned}
\phi_+ \left(\vec{r}, t\right) &= q\frac{e}{4\pi}\,\theta(t - r)\left(\frac{1}{\left|\vec{r} - \vec{r'}\right|}\right)\left(\frac{1}{1 - \hat{n}\cdot\left(\vec{r} - \vec{r'}\right)\Big/\left|\vec{r} - \vec{r'}\right|}\right) \\
&= q\frac{e}{4\pi}\,\theta(t - r)\frac{1}{\left|\vec{r} - \vec{r'}\right| - \hat{n}\cdot\left(\vec{r} - \vec{r'}\right)} \\
&= q\frac{e}{4\pi}\theta(t - r)\frac{1}{t - \hat{n}\cdot\vec{r}}
\end{aligned}
\tag{13}
$$

where in the last step, we have used (9). We can repeat the calculation for the negative charge by appropriately changing the sign of the velocity. So the total potential (due to both charges) is

$$
\phi\left(t, \vec{r}\right) = q\frac{e}{4\pi}\,\theta(t - r)\left(\frac{1}{t - \hat{n}\cdot\vec{r}} - \frac{1}{t + \hat{n}\cdot\vec{r}}\right)
\tag{14}
$$

Similarly, the vector potential is obtained by using (12) and (9) in (5):

$$
\vec{A}\left(t, \vec{r}\right) = q\frac{e}{4\pi}\,\hat{n}\,\theta(t - r)\left(\frac{1}{t - \hat{n}\cdot\vec{r}} + \frac{1}{t + \hat{n}\cdot\vec{r}}\right)
\tag{15}
$$

This looks rather nice, but we can simplify it even further by completely eliminating the potential inside the sphere by making a gauge transformation. We take

$$
\phi \to \phi + \frac{\partial\Lambda}{\partial t} \qquad \vec{A} \to \vec{A} - \vec{\nabla}\Lambda
\tag{16}
$$

and choose

$$
\Lambda\left(t, \vec{r}\right) = q\frac{e}{4\pi}\,\theta(t - r)\,\log\left(\frac{t + \hat{n}\cdot\vec{r}}{t - \hat{n}\cdot\vec{r}}\right)
\tag{17}
$$

The gauge transformed potentials are then

$$
A^0\left(t, \vec{r}\right) = \phi\left(t, \vec{r}\right) = q\frac{e}{4\pi}\,\delta(t - r)\,\log\left(\frac{t + \hat{n}\cdot\vec{r}}{t - \hat{n}\cdot\vec{r}}\right) = q\frac{e}{4\pi}\,\delta(t - r)\,\log\left(\frac{1 + \hat{n}\cdot\hat{r}}{1 - \hat{n}\cdot\hat{r}}\right)
\tag{18}
$$

and

$$\vec{A}(t, \vec{r}) = q \frac{e}{4\pi} \hat{r}\, \delta(t - r) \log\left(\frac{1 + \hat{n} \cdot \hat{r}}{1 - \hat{n} \cdot \hat{r}}\right) = \hat{r}\, \phi(t, \vec{r}) \tag{19}$$

which are determined by the single function ϕ. Note that these potentials satisfy the light-shell gauge condition mentioned in the introduction

$$v_\mu A^\mu = 0 \tag{20}$$

where

$$v^0 = 1 \ \text{ and } \ \vec{v} = \hat{r} \tag{21}$$

This gauge will be an important part of the construction of our effective theory as we shall see later. We call this the light-shell gauge (LSG).

It is straightforward to calculate the electric and magnetic fields from these potentials. We find that they are both parallel to the surface of the sphere and are given by

$$\vec{E}(t, \vec{r}) = -q \frac{e}{4\pi} \hat{r} \times (\hat{r} \times \hat{n})\, \delta(t - r) \left(\frac{1}{r - \hat{n} \cdot \vec{r}} + \frac{1}{r + \hat{n} \cdot \vec{r}}\right) \tag{22}$$

$$\vec{B}(t, \vec{r}) = q \frac{e}{4\pi} \hat{r} \times \hat{n}\, \delta(t - r) \left(\frac{1}{r - \hat{n} \cdot \vec{r}} + \frac{1}{r + \hat{n} \cdot \vec{r}}\right) \tag{23}$$

We have thus found that the electric and magnetic fields lie entirely on the surface of the sphere as we claimed in the introduction. It is also straightforward to see that this, as well as the finding that we can eliminate the fields inside the sphere by going to light-shell gauge, also holds for the case of more than 2 charges. Specifically, the potentials for the general charge case (with the total sum of the charges being 0) are given by

$$A^0(t, \vec{r}) = \phi(t, \vec{r}) = -\sum_j q_j \frac{e}{4\pi} \delta(t - r) \log\left(1 - \hat{n}_j \cdot \hat{r}\right) \tag{24}$$

9

and

$$\vec{A}(t, \vec{r}) = \hat{r}\,\phi(t, \vec{r}) \tag{25}$$

The E and B fields for the general charge case are then

$$\vec{E}(t, \vec{r}) = -\sum_j q_j \frac{e}{4\pi}\,\hat{r} \times (\hat{r} \times \hat{n}_j)\,\delta(t - r)\,\frac{1}{r - \hat{n}_j \cdot \vec{r}} \tag{26}$$

$$\vec{B}(t, \vec{r}) = \sum_j q_j \frac{e}{4\pi}\,\hat{r} \times \hat{n}_j\,\delta(t - r)\,\frac{1}{r - \hat{n}_j \cdot \vec{r}} \tag{27}$$

This result forms the basis of our pursuit of finding a 2-dimensional effective theory for QED on the $t = r$ sphere. For non-abelian theories, the situation is a bit more complicated as we are not able to do a similar retarded potential calculation, but we will soon show that the electric and magnetic fields corresponding to a non-abelian gauge theory also lie entirely on the light shell.

2.2 The non-abelian case

The non-Abelian case is more complicated, and it is not obvious how to write down and solve the relevant equations directly. Here we will adopt a less direct route by assuming a simple form for the gauge fields and imposing the physics of the collision. Specifically, we will start by assuming that the gauge fields are zero outside the $t = r$ sphere. We will then go on to construct the field strengths $\mathcal{F}_a^{\mu\nu}$, and impose the following two conditions:

1. In the extreme relativistic limit, we expect no energy/momentum density inside the light shell. Thus the field strengths must vanish for $r < t$, and lie entirely on the sphere.

2. The fields satisfy the non-abelian version of Maxwell's equations, which tell us how

the charges on the light shell produce the fields:

$$D_\nu \mathcal{F}^{\mu\nu} = 4\pi \mathcal{J}^\mu \qquad (28)$$

where $\mathcal{J}^\mu$ is a color current density.

For implementing this plan, we are ultimately interested in color gauge fields of the form

$$A_a^\mu(t, \vec{r}) = \xi_a^\mu(t, \vec{r}) \, \theta(t - r) \qquad (29)$$

which drop to zero discontinuously at the light shell. When we differentiate these gauge fields, we will find field strengths proportional to $\delta(t - r)$ — that is to say confined to the light shell. The basic idea is then to use (29) to construct the field strengths and see what the dynamics of classical QCD tells us about the field strengths on the light shell.

The form (29) is simple and appealing, and in the Abelian case, it is actually good enough to reproduce the results of a direct calculation using retarded potentials. However, as we will see, to understand the non-Abelian equations of motion, it is important to think about getting to this singular situation as a limit of smoother gauge fields. We want to understand when and how our results depend on the details of how we go to the discontinuous limit. So we will think about obtaining (29) as a limit of smooth gauge fields, $\mathcal{A}_a^\mu(\epsilon, t, \vec{r})$, such that

$$\lim_{\epsilon \to 0} \mathcal{A}_a^\mu(\epsilon, t, \vec{r}) = \xi_a^\mu(t, \vec{r}) \, \theta(t - r) \qquad (30)$$

To construct the field strengths, we will need derivatives of this as well as products of more than one such field with different non-abelian group indices. For the derivatives, we will use

$$\partial^\mu \theta(t - r) = v^\mu \delta(t - r) \qquad (31)$$

where $v^\mu = (1, \hat{r})^\mu$, as we defined in (21). This gives the relation

$$\lim_{\epsilon \to 0} \partial^\nu \mathcal{A}_a^\mu(\epsilon, t, \vec{r}) = \theta(t - r)\, \partial^\nu \xi_a^\mu(t, \vec{r}) + \delta(t - r)\, v^\nu\, \xi_a^\mu(t, \vec{r}) \tag{32}$$

For a product of two such fields without derivatives, we can write

$$\lim_{\epsilon \to 0} \mathcal{A}_a^\mu(\epsilon, t, \vec{r})\, \mathcal{A}_b^\nu(\epsilon, t, \vec{r}) = \xi_a^\mu(t, \vec{r})\, \xi_b^\nu(t, \vec{r})\, \theta(t - r) \tag{33}$$

In the field strength, the relations (32) and (33) are all we need, and we find for $\epsilon \to 0$

$$\mathcal{F}_a^{\mu\nu} = \partial^\mu \mathcal{A}_a^\nu - \partial^\nu \mathcal{A}_a^\mu + f_{abc} \mathcal{A}_b^\mu \mathcal{A}_c^\nu \tag{34}$$

$$\to \delta(t - r)\, (v^\mu \xi_a^\nu - v^\nu \xi_a^\mu) + \theta(t - r)\, (\partial^\mu \xi_a^\nu - \partial^\nu \xi_a^\mu + f_{abc} \xi_b^\mu \xi_c^\nu) \tag{35}$$

Note that we have normalized the gauge fields to behave simply under the non-Abelian gauge invariance, so that under a gauge transformation ($T_a = \lambda_a/2$ where the λ_a are the Gell-Mann matrices)

$$\mathcal{A}^\mu = \mathcal{A}_a^\mu T_a \to U \mathcal{A}^\mu U^\dagger - i U \partial^\mu U^\dagger \tag{36}$$

We now apply the 1st of the two conditions we listed above (namely that the field strengths vanish inside the sphere). This means that in equation (35), the coefficient of the theta functions must be zero:

$$\partial^\mu \xi_a^\nu - \partial^\nu \xi_a^\mu + f_{abc} \xi_b^\mu \xi_c^\nu = 0 \tag{37}$$

We then have field strengths only on the light shell

$$\mathcal{F}_a^{\mu\nu} \to F_a^{\mu\nu} = \delta(t - r)\, (v^\mu \xi_a^\nu - v^\nu \xi_a^\mu) \tag{38}$$

12

We now apply to this field strength the second condition, namely, that the fields satisfy the non-abelian Maxwell's equations) (28). While doing so, we will also get some terms containing derivatives of delta functions, and will assume that these must vanish.

On the left hand side of (28), we encounter two interesting things. In color components, it can be divided into four terms as follows.

$$\partial_\nu \left(\partial^\mu \mathcal{A}_a^\nu - \partial^\nu \mathcal{A}_a^\mu \right) + \partial_\nu \left(f_{abc} \mathcal{A}_b^\mu \mathcal{A}_c^\nu \right) + f_{ade} \mathcal{A}_{d\nu} \left(\partial^\mu \mathcal{A}_e^\nu - \partial^\nu \mathcal{A}_e^\mu \right) + f_{ade} \mathcal{A}_{d\nu} \left(f_{ebc} \mathcal{A}_b^\mu \mathcal{A}_c^\nu \right) \quad (39)$$

The $\epsilon \to 0$ limits of the second and fourth terms in (39) are straightforward, respectively

$$\partial_\nu \left(\theta(t-r) f_{abc} \xi_b^\mu \xi_c^\nu \right) \tag{40}$$

$$\theta(t-r) f_{ade} \xi_{d\nu} \left(f_{ebc} \xi_b^\mu \xi_c^\nu \right) \tag{41}$$

The first term can be written as a sum of three terms:

$$\partial^0 \left(\delta(t-r) v_\nu \left(v^\mu \xi_a^\nu - v^\nu \xi_a^\mu \right) \right) \tag{42}$$

$$+\delta(t-r) \left(v_0 \partial_\nu - \partial_0 v_\nu \right) \left(v^\mu \xi_a^\nu - v^\nu \xi_a^\mu \right) \tag{43}$$

$$+\partial_\nu \left(\theta(t-r) \left(\partial^\mu \xi_a^\nu - \partial^\nu \xi_a^\mu \right) \right) \tag{44}$$

The last of these, (44), combines with (40) to give zero by virtue of (37). The first must vanish if we are to avoid derivatives of δ-functions, which implies (because $v_\mu v^\mu = 0$)

$$\delta(t-r) v_\mu \xi^\mu = 0 \tag{45}$$

Comparing with (38), you can see that this is the condition that the color electric field on the light shell is tangent to the light shell, perpendicular to the direction of motion of the light shell, $\hat{r}$. We expected this on physical grounds, and we now see that it is necessary for

the consistency of the picture. Comparing (45) with (30) also tells us that the gauge field in the limit $\lim_{\epsilon \to 0} \mathcal{A}_a^\mu(\epsilon, t, \vec{r})$ satisfies the light-shell gauge condition at least on the sphere.

Finally, we consider the third term in (39). This term is problematic because it is **not determined** by the limiting value of $\mathcal{A}^\mu$. The total derivative of a product of $\mathcal{A}^\mu$s is determined,

$$\left(\mathcal{A}_b^\nu(\epsilon, t, \vec{r}) \, \partial^\lambda \mathcal{A}_a^\mu(\epsilon, t, \vec{r}) + \mathcal{A}_a^\mu(\epsilon, t, \vec{r}) \, \partial^\lambda \mathcal{A}_b^\nu(\epsilon, t, \vec{r})\right) \to \partial^\lambda \left(\theta(t-r) \, \xi_a^\mu(t, \vec{r}) \xi_b^\nu(t, \vec{r})\right) \quad (46)$$

However, for the product of one $\mathcal{A}$ with the derivative of another, the limit depends on the details of their shapes. In general we can write

$$\mathcal{A}_a^\mu(\epsilon, t, \vec{r}) \, \partial^\lambda \mathcal{A}_b^\nu(\epsilon, t, \vec{r}) \to$$
$$\theta(t-r) \, \xi_a^\mu(t, \vec{r}) \, \partial^\lambda \xi_b^\nu(t, \vec{r}) + \delta(t-r) \left(\frac{1}{2} v^\lambda \, \xi_a^\mu(t, \vec{r}) \, \xi_b^\nu(t, \vec{r}) + \kappa_{ab}^{\mu\lambda\nu}(t, \vec{r})\right) \quad (47)$$

where

$$\kappa_{ab}^{\mu\lambda\nu}(t, \vec{r}) = -\kappa_{ba}^{\nu\lambda\mu}(t, \vec{r}) \quad (48)$$

The κ term is the most general thing we can write down consistent with (46).[1] Using (47), we get for the third term in (39)

$$\theta(t-r) \, f_{abc} \, \xi_{b\nu} \left(\partial^\mu \xi_c^\nu - \partial^\nu \xi_c^\mu\right) + \delta(t-r) \, \kappa_a^\mu \quad (49)$$

where

$$\kappa_a^\mu = f_{abc} \, g_{\lambda\nu} \left(\kappa_{bc}^{\lambda\mu\nu} - \kappa_{bc}^{\lambda\nu\mu}\right) \quad (50)$$

and we have used (45) and the antisymmetry of f_{abc} to set

$$\delta(t-r) \, \frac{1}{2} \, f_{abc} \, \xi_{b\nu} \left(v^\mu \xi_c^\nu - v^\nu \xi_c^\mu\right) = 0 \quad (51)$$

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[1]Note that this ambiguity only appears in the non-Abelian theory because of the non-linearity of the equations of motion. There is no κ in E&M.

14

We will see later that something crucial happened in (49). The explicit non-linear dependence on ξ in (51) goes away, but the κ term remembers the non-linear form of the field equations. We will argue later that this extra κ term is necessary for the consistency of the picture. Putting all this together, again using (37), Maxwell's equations become

$$\delta(t-r)\left[(v_0\partial_\nu - v_\nu\partial_0)(v^\mu\xi^\nu - v^\nu\xi^\mu) + \kappa_a^\mu - 4\pi\sigma_a\,v^\mu\right] = 0 \tag{52}$$

We are interested in what these equations tell us about the fields on the light shell, so we will eliminate t and evaluate (whenever we can) the fields for $t = r$. Look for example at $\mu = 0$ in (52).

$$\delta(t-r)\left[\left(\vec{\nabla} + \hat{r}\partial_0\right)\cdot\left(\vec{\xi}_a(t,\vec{r}) - \hat{r}\xi_a^0(t,\vec{r})\right) + \kappa_a^0 - 4\pi\sigma_a\right] = 0 \tag{53}$$

Define "light-shell fields" which are functions only of $\vec{r}$ by setting $t = r$ to go onto the light shell:

$$\vec{e}_a(\vec{r}) \equiv \left.\left(\vec{\xi}_a(t,\vec{r}) - \hat{r}\,\xi_a^0(t,\vec{r})\right)\right|_{t=r} \tag{54}$$

Then because of (45), these fields are transverse,

$$\hat{r}\cdot\vec{e}_a(\vec{r}) = 0 \tag{55}$$

In terms of $\vec{e}$, (53) becomes

$$\delta(t-r)\left(\vec{\nabla}\cdot\vec{e}_a(\vec{r}) - 4\pi\sigma_a(\vec{r}) + \kappa_a^0(\vec{r})\right) = 0 \tag{56}$$

Notice that the derivatives of ξ with respect to $\vec{r}$ and t have conspired to give derivatives of the light-shell fields just with respect to $\vec{r}$. Because (56) is true for all t, we must have

$$\vec{\nabla}\cdot\vec{e}_a(\vec{r}) = 4\pi\sigma_a(\vec{r}) - \kappa_a^0(\vec{r}) \tag{57}$$

15

Thus $\vec{e}_a$ is a kind of electric field on the light shell, but (57) is true in a static 3D space.[2]

For the space components of (52), a similar manipulation gives

$$\vec{\nabla} \times (\hat{r} \times \vec{e}_a) = 4\pi\sigma_a\,\hat{r} - \vec{\kappa}_a \tag{58}$$

This is very reasonable. It says that the curl of the magnetic field on the light shell is related to the current and $\vec{\kappa}$. We can combine (58) and (57), to obtain

$$\hat{r} \times \left(\vec{\nabla} \times \vec{e}_a\right) = \left(\hat{r}\,\kappa_a^0 - \vec{\kappa}_a\right) \tag{59}$$

We will see shortly that this gives a constraint on $\vec{\kappa}$.

In E&M, in spite of the singularity of (38), we can give a direct physical meaning to the light-shell fields. $\vec{e}$ is the impulse per unit charge produced by the light shell as it passes by a stationary infinitesimal test charge. This is finite and independent of the detailed shape of the field as the shell width goes to zero. It is not so obvious that this concept makes sense in the non-Abelian case, because we cannot make an arbitrarily small test charge. It appears that to construct gauge invariant quantities that are finite in the $\epsilon \to 0$ limit, we have to take ratios. For example the surface energy density on the light shell goes to ∞ as $\epsilon \to 0$, but ratios of energy densities at different points should be finite.

Now let's look in more detail at the vanishing of the field in the interior and see what part of this we can write in terms of light-shell fields. We know from the vanishing of the field for $r < t$ that

$$\nabla^j \xi_a^k - \nabla^k \xi_a^j = f_{abc}\xi_b^j\xi_c^k \quad \text{and} \quad \partial^0 \xi_a^j + \nabla^j \xi_a^0 = -f_{abc}\xi_b^0\xi_c^j \tag{60}$$

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[2]You might wonder what becomes of the color gauge invariance, since it looks like the gauge field ξ is simply turning into the gauge invariant field strength, $\vec{e}_a$. The answer is that gauge transformations that preserve the form (29) of A^μ change the ξs inside but do not change the light-shell fields, $\vec{e}_a$ except for global color rotations, which of course remain.

We can combine these into light-shell fields as follows:

$$\left(\nabla^j + \hat{r}^j \partial^0\right)\left(\xi_a^k - \hat{r}^k\,\xi_a^0\right) - \left(\nabla^k + \hat{r}^k \partial^0\right)\left(\xi_a^j - \hat{r}^j\,\xi_a^0\right) = f_{abc}\left(\xi_b^j - \hat{r}^j\,\xi_b^0\right) \times \left(\xi_c^k - \hat{r}^k\,\xi_c^0\right) \quad (61)$$

so we can use (54) and set $t = r$ and conclude that the 3D theory of $\vec{e}_a$ satisfies

$$\nabla^j e_a^k - \nabla^k e_a^j = f_{abc}e_b^j e_c^k \quad \text{or} \quad \vec{\nabla} \times \vec{e}_a = \frac{1}{2}f_{abc}\,\vec{e}_b \times \vec{e}_c \quad (62)$$

(62), all by itself, has a number of consequences. Because the $\vec{e}_a$s are perpendicular to $\hat{r}$, their cross product must be in the $\hat{r}$ direction. Thus

$$\hat{r} \times \left(\vec{\nabla} \times \vec{e}_a\right) = 0 \quad (63)$$

But if we take the gradient of (55) and simplify, we get

$$\hat{r} \times \left(\vec{\nabla} \times \vec{e}_a\right) = -\frac{1}{r}\left(1 + \vec{r}\cdot\vec{\nabla}\right)\vec{e}_a \quad (64)$$

And on comparing this with (63), we see that $\vec{e}_a$ scales trivially,

$$\left(\vec{r}\cdot\vec{\nabla}\right)\vec{e}_a = -\vec{e}_a \quad (65)$$

Thus $\vec{e}_a$ is just $1/r$ times a vector function of $\hat{r}$. Again, this follows directly from (62) which in turn follows from the vanishing of the fields inside the light shell. (63) together with (59) also implies

$$\kappa_a^\mu = v^\mu \kappa_a \quad (66)$$

for some scalar function κ_a, so that like the current, $\kappa_a^\mu \propto v^\mu$. Thus in the limit, all the information from the non-Abelian Maxwell's equations is contained in (66) and the

following relations:

$$\vec{\nabla} \times \vec{e}_a = \frac{1}{2} f_{abc}\, \vec{e}_b \times \vec{e}_c \qquad \hat{r} \cdot \vec{e}_a = 0 \qquad \left(\vec{r} \cdot \vec{\nabla}\right) \vec{e}_a = -\vec{e}_a \qquad (67)$$

$$\vec{\nabla} \cdot \vec{e}_a = 4\pi\sigma_a - \kappa_a \equiv 4\pi\tilde{\sigma}_a \qquad (68)$$

Notice that the effective charge density $4\pi\tilde{\sigma}_a$ must scale like $1/r^2$ (consistent with charge conservation).

We can solve (67) for the $\vec{e}_a$ fields as follows:

$$\vec{e}_a T_a = -i\, U(\hat{r})^\dagger \vec{\nabla} U(\hat{r}) \qquad (69)$$

where $U^\dagger U = I$ is a special unitary matrix. Now trivial scaling and transversality are automatic because U depends only on $\hat{r}$.

It is worth mentioning that some of these results have a striking resemblance with the findings in [12], in which the classical equation for the gluon field is solved for the case in which the source is a delta function along the light-cone in the z direction as opposed to our set up with a distribution of charges moving spherically outward from the origin. Specifically, (68) and (69) closely resemble equations 11 and 16, respectively, in [12].

Because of (65), our picture is classically scale invariant and we could write the classical theory as a purely two dimensional theory on the light shell, and simply choose $r = 1$. Physically, however, it is sometimes convenient to think about the theory as we actually use it, in the full three dimensional space, but with the fields living on an expanding light shell of radius $r = t$.

Having dealt with (67), we now want to find a Lagrangian that gives (68) as the equation of motion so that we can eventually do quantum mechanics. We have now eliminated time and are in purely Euclidean space, so this is just the energy. We would expect a contribution proportional to $\text{Tr}(\vec{e}^2)$, which in terms of U can be written as (where B is some geometrical

constant that we do not know how to calculate at this point, and g is the dimensionless coupling constant)

$$\frac{B}{g^2}\,\text{Tr}\left(\vec{\nabla}U(\hat{r})^\dagger \cdot \vec{\nabla}U(\hat{r})\right) \tag{70}$$

This is the Lagrangian for a non-linear σ-model on the light shell and the U fields (which in some sense are the potentials associated with the electric fields) are Goldstone boson fields associated with the breaking of an $SU(3)_L \times SU(3)_R$, $U \to LUR^\dagger$ down to the diagonal $SU(3)$, $U \to VUV^\dagger$. The electric fields $\vec{e}_a$ are Noether currents associated with the $SU(3)_R$ symmetry, so if (70) were the whole story, $\vec{e}_a$ would be conserved, in agreement with (68) without sources, for $\tilde{\sigma} = 0$. This is a renormalizable theory in 2D, and Polyakov showed long ago that the coupling g exhibits asymptotic freedom [13].

In fact, field theorists have long studied the analogies between non-Abelian gauge theories in $3+1$ dimensions and non-linear σ-models in 2 dimension, making use of some the powerful tools available in the smaller number of dimensions (see for example,[14]). Our analysis however suggests this is not just an analogy and that the non-linear σ-model **IS** QCD in an appropriate limit. A similar claim was also made in [15] where a simplified effective theory for QCD is derived in the high-energy limit. While this effective theory is still (3+1)-dimensional, its interactions are described, to leading order, in terms of a 2-dimensional σ-model on the transverse plane.

Returning to our own analysis, what happens in the situation we have found is simple and interesting. Because the fields live on the light shell of radius r, the momenta in the theory are actually angular momenta divided by r. The $\ell = 0$ mode is absent because it gives no contribution to $\vec{e}$ when the total net charge on the light shell is zero. The momenta are bounded away from zero and quantized in units of $1/r$. The infrared divergence that one would expect in a flat 2D theory is cut off at r. Because all the momenta scale with $1/r$, it is appropriate to choose the renormalization scale to scale with $1/r$, so the coupling depends on the radius.

Up to this point, we believe that our analysis is quite robust. In the appropriate limit, we can describe the physics in terms of light-shell fields, and the condition that the field strengths vanish inside the light shell implies quite directly that these fields are described by a non-linear σ-model. We are on shakier ground from here on, where we discuss the dependence on the charges and currents of the high energy particles that are producing the fields. Here κ gets involved, and in our indirect approach to the limit, we do not know exactly what κ is. But we believe that a non-zero κ is necessary and have a guess for its form, and we will now discuss the reasons for the belief and the guess. Suppose first that $\kappa = 0$. Then the equation of motion for U would be (from (68)),

$$\vec{\nabla} \cdot \left(-i\, U^\dagger \vec{\nabla} U \right) = 4\pi\sigma \tag{71}$$

where the right hand side is independent of U. However, it is not possible to add to the Lagrangian (70) a term $F(U)$ that gives this equation of motion, because Noether's theorem requires that to get (71) from an infinitesimal symmetry transformation,

$$\delta U = U\, i\delta\zeta \tag{72}$$

we need

$$\delta F = 4\pi \operatorname{Tr}(\sigma \delta\zeta) \tag{73}$$

To see why this is a problem, write U in terms of unconstrained octet components, $U = e^{i\Pi_a T_a}$ so (72) is

$$\delta\zeta = O_a \delta\Pi_a \quad \text{where} \quad O_a \equiv -iU^\dagger \frac{\delta U}{\delta \Pi_a} \tag{74}$$

Thus we want

$$\frac{\delta F}{\delta \Pi_a} = \frac{B}{g^2} 4\pi\sigma\, O_a \tag{75}$$

20

But

$$\frac{\delta O_a}{\delta \Pi_b} - \frac{\delta O_b}{\delta \Pi_a} = -i\left[O_a \, , \, O_b\right] \neq 0 \tag{76}$$

which means that (75) is not consistent. In the presence of κ, there are additional terms in δF coming from the dependence of κ (and thus $\tilde{\sigma}$) on the Πs. One simple possibility is

$$4\pi\tilde{\sigma} = 4\pi\sigma - \kappa = 2\pi(\sigma U + U^\dagger \sigma) \tag{77}$$

which would emerge in the equation of motion from the Lagrangian

$$\frac{B}{g^2}\,\text{Tr}\left(\vec{\nabla}U^\dagger \cdot \vec{\nabla}U - 2\pi i(\sigma U - U^\dagger \sigma)\right) \tag{78}$$

This our guess for the structure of the effective theory on the light shell.

We believe that this analysis makes a very plausible case that very high energy collisions involving colored particles can be described by a light-shell effective field theory in which the dynamical fields are the Goldstone bosons of a non-linear σ-model on the light shell at $t = r$. To go further, we must go beyond our indirect arguments and see how to construct the light-shell effective theory directly from the underlying QCD theory. Then we should be able to do the perturbative matching onto the light-shell effective theory from the QCD physics of the original high-energy collision and better understand the physical meaning of our light-shell fields. However, before getting to QCD, we have first been focusing on the simpler case of QED, and the remaining chapters in this thesis describe some of our efforts in that direction.

3 An amusing calculation

(This chapter is not necessary for understanding the ideas developed in later chapters and can therefore be skipped by a disinterested reader.)

Before constructing our quantum effective theory, it is an amusing exercise to connect our classical picture for electromagnetism derived in section 2.1 with quantum electrodynamics by interpretting the intensity of the electric and magnetic fields on the sphere in as a quantum mechanical amplitude squared for photon emission. We can make this connection in two ways:

1. Understanding the square of the amplitude for photon emission in QED as the intensity of the electromagnetic fields.

2. Working in the opposite direction and interpretting the electromagnetic field intensities as a probability of photon production.

3.1 The amplitude squared in the quantum theory as an intensity

It is common knowledge that the amplitude for a process in which a photon is emited by a scalar/anti-scalar (or for that matter a fermion/anti-fermion pair) coming out of some collision can be expressed as a product of the amplitude for the process in which no photon is emited and an extra piece corresponding to the production of the photon. The latter factor is thus the probability amplitude for the emission of a photon and is also the tree-level expectation value of the gauge field in momentum space. In our physical set up, we want to focus on the limit where the charged particles have very large energy compared to the emitted photon. In the specific case of a particle/anti-particle pair[3] the amplitude of a

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[3]This argument does not depend on the spin of the particles

photon being emited is given by

$$A^\nu = -ie \left(\frac{p^\nu}{(pq)} - \frac{\bar{p}^\nu}{(\bar{p}q)} \right) \tag{79}$$

where p, $\bar{p}$ and q are the momenta of the particle, anti-particle and photon, respectively. This is related to the sphere in the simple way. The fields on the sphere in momentum space are

$$F^{\mu\nu} = (q^\mu A^\nu - q^\nu A^\mu) \, \delta(t - r) \tag{80}$$

where

$$q = (q^0, q^0 \, \hat{r}) \quad \text{and} \quad A^\mu \propto \xi^\mu \tag{81}$$

where we are using some of the notation defined in chapter 2.

Now, consider the intensity

$$E \propto \vec{E}^2 + \vec{B}^2 = (q^0 \vec{A} - \vec{q} A^0)^2 + (\vec{q} \times \vec{A})^2 \tag{82}$$

which is (using $q_\mu A^\mu = 0$)

$$|\vec{q}|^2 \left(2|\vec{A}|^2 + A^{0^2} - (\hat{q} \cdot \vec{A})^2 - 2A^0 \hat{q} \cdot \vec{A} \right) = 2|\vec{q}|^2 \left(|\vec{A}|^2 - A^{0^2} \right) \tag{83}$$

This is proportional to $-A_\mu A^\mu$, which in is also what we get when we square the amplitude and sum over photon polarizations. From (79) it is

$$-A_\mu A^\mu = e^2 \frac{2(p\bar{p})}{(pq)(\bar{p}q)} \tag{84}$$

So the energy on the sphere is proportional to the squared amplitude in QFT.

23

3.2 The classical fields as probability amplitudes

This can be formulated in terms of a fourier analysis of our classical electromagnetic fields on the sphere. To understand this, we first make a simple argument in terms of the energy carried by a plane wave and connect it with the probability of the number of photons. Starting with a plane wave argument allows us to focus on the concept without worrying about the features of spherical coordinates. We then restate the idea in terms of spherical waves. Lastly, we use our spherical wave formulae to perform the needed calculations to obtain the probability of the number of photons obtained from our classical electromagnetic field and compare it with the probability given by the quantum mechanical amplitude.

3.2.1 Energy in a plane wave

To get a sense of what is going on, first look at the energy per unit area in a plane wave in the x direction

$$\vec{E}(\vec{r}, t) = \vec{\varepsilon} f(x - t) \tag{85}$$

Nothing depends on y and z, so the energy per unit area is just

$$\propto \int |\vec{E}(x)|^2 \, dx \propto \int |\vec{\tilde{E}}(k)|^2 \, dk \propto \int |\tilde{f}(k)|^2 \, dk \tag{86}$$

where $\vec{\tilde{E}}(k)$ is the field Fourier transformed only in x (Fourier transforming in y and z would give momentum δ-functions that tell us that the photons are going in the right direction — we don't need this because we are looking at energy per unit area). So the energy per unit area per unit wave-length is proportional to $|\tilde{f}(k)|^2$. For small k (which is the limit we need to consider because in our physical set up, the charged particles have large energy), $\tilde{f}(k)$ goes to a constant proportional to

$$\int f(x) \, dx \tag{87}$$

This is great because it just depends on the total impulse that the wave imparts to a charged particle as the wave moves past - a finite and physically measurable quantity even when f goes to a δ function. So while the energy per unit area goes to infinity as the wave shrinks to a δ-function in $x - t$, the energy per unit area per unit k for k much smaller than one over the length of the pulse goes to a finite constant proportional to $|\tilde{f}(0)|^2$, which is independent of the shape of the pulse. Since each photon carries energy k, the number of photons is proportional to $|\tilde{f}(0)|^2/k$. Having explained the concept, we now translate this to the case of spherical waves.

3.2.2 Fourier transform for a spherical wave

A spherical wave has the form

$$\frac{1}{r}\exp(ikr) \tag{88}$$

To see why this makes sense, note that the square of this, which corresponds to the intensity, falls off as $1/r^2$. Now, if we have a function of the form

$$f(r) = \frac{1}{r}g(r), \tag{89}$$

Then we have

$$\tilde{f}(k) = \int_0^\infty dr \, r^2 \, f(r)\frac{1}{r}\exp(ikr) \tag{90}$$

$$= \int_0^\infty dr \, g(r) \, \exp(ikr) \tag{91}$$

So what's happening is that the $1/r$ in $\frac{1}{r}\exp(ikr)$ combines with the $1/r$ in $f(r)$ to give $1/r^2$, which is then cancelled out by the r^2 in the integration measure in spherical coordinates. With the $1/r$ factors out of the way, we are left to fourier transform $g(r)$.

We thus have

$$f(r) = \frac{1}{2\pi r}\int_{-\infty}^\infty dk \, \tilde{f}(k) \, \exp(ikr) \tag{92}$$

25

The energy per solid angle is thus:

$$dE/d\Omega = \int_0^\infty \|f(r)\|^2\, r^2\, dr \tag{93}$$

This is the energy/solid angle and not the energy per unit area because of the r^2 in the radial integral. Plugging (92) into this gives

$$dE/d\Omega = \frac{1}{(2\pi)^2} \int_0^\infty dr \int\int dk\, dk'\, \tilde{f}(k)\tilde{f}^*(k')\, \exp(i(k-k')r), \tag{94}$$

where the factors of $1/r$ have been canceled by the r^2 in the spherical coordinates measure.

This has the delta function $\delta(k-k')$ in the somewhat unfamiliar (and naively suspicious looking) form

$$\int_0^\infty \frac{dr}{2\pi} \exp(i(k-k')r) = \delta(k-k') \tag{95}$$

The familiar form of this fourier representation of the delta function is to have the integral over r from $-\infty$ to ∞ instead of 0 to ∞. However, (95) is correct in the context of our calculation, and the proof is given in appendix A.

We thus have

$$dE/d\Omega = \frac{1}{4\pi} \int\int dk\, dk'\, \tilde{f}(k)\tilde{f}(k')\, \delta(k-k') \tag{96}$$

$$= \frac{1}{2\pi} \int dk\, \|\tilde{f}(k)\|^2 \tag{97}$$

The energy per solid angle per unit k is then given by omitting dk from this:

$$\frac{1}{2\pi}\|\tilde{f}(k)\|^2 \tag{98}$$

For small k, however, $\tilde{f}(k)$ goes to a constant:

$$\tilde{f}(0) = \int_0^\infty g(r)dr. \tag{99}$$

26

Using this along with the fact that the energy of a photon is k, we can write the number of photons per solid angle per unit k as

$$\frac{1}{2\pi k}\|\tilde{f}(0)\|^2 \tag{100}$$

$$= \frac{1}{2\pi k}\|\int_0^\infty dr\, g(r)\|^2 \tag{101}$$

3.3 Applying to our classical theory

The energy density per unit volume is given by

$$E_v = \frac{1}{2}\left[(\vec{E})^2 + (\vec{B})^2\right] \tag{102}$$

And recalling (22 and (23), the E and B fields (in Heaviside-Lorentz units) for the case of two oppositely charged particles going in opposite directions are given by

$$\vec{E}(t,\vec{r}) = -e\delta(t-r)\frac{\hat{r}\times(\hat{r}\times\hat{n})}{4\pi r}\left(\frac{1}{1+\hat{r}\cdot\hat{n}} + \frac{1}{1-\hat{r}\cdot\hat{n}}\right) \tag{103}$$

$$= -2e\delta(t-r)\frac{\hat{r}\times(\hat{r}\times\hat{n})}{4\pi r\sin(\theta)} \tag{104}$$

and

$$\vec{B}(t,\vec{r}) = 2e\delta(t-r)\frac{\hat{r}\times\hat{n}}{4\pi r\sin(\theta)} \tag{105}$$

Here we have used $\hat{r}\cdot\hat{n} = \cos(\theta)$ to make the simplifications.

Thus for both E and B, the r-dependent part is

$$f(r) = \delta(t-r)\frac{1}{r}, \tag{106}$$

That is,

$$g(r) = \delta(t-r) \tag{107}$$

27

and the square of the angular part for both $\vec{E}$ and $\vec{B}$ is

$$\frac{e^2}{4\pi^2 \sin(\theta)} \tag{108}$$

Putting it all together by using (101) gives the no of photons per solid angle per unit k:

$$\frac{e^2}{8\pi^3 k \sin^2(\theta)} \| \int_0^\infty \delta(t-r)dr \|^2 \tag{109}$$

$$= \frac{e^2}{8\pi^3 k \sin^2(\theta)} \tag{110}$$

3.4   The QED calculation for comparison

In the quantum mechanical picture, the probability is given by the magnitude squared of the photon emission amplitude, which was given in (84). For the particular case when the scalar and anti-scalar are back to back, this can be written as

$$\|\mathcal{M}_{photon}\|^2 = -A^\mu A_\mu = \frac{2e^2}{E_q^2 \sin(\theta)} \tag{111}$$

To convert this into a probability of photons per solid angle per unit q, we simply need to multiply this by the factor

$$\frac{d^3q}{(2\pi)^3 2E_q} \tag{112}$$

With $E_q = q$, and $d3q = q^2 \, dq \, d\Omega$, this becomes

$$\frac{d^3q}{(2\pi)^3 2E_q} = \frac{qdqd\Omega}{16\pi^3} \tag{113}$$

We then get the differential probability:

$$\|\mathcal{M}_{photon}\|^2 \frac{d^3q}{(2\pi)^3 2E_q} = \frac{e^2}{8\pi^3 q \sin(\theta)} \tag{114}$$

28

And this matches the classical result in (110).

4 Constructing Our Effective Theory

We now begin the explicit construction of the effective field theory, incorporating the intuition gained from the classical picture by studying the quantum mechanics of particle production from a gauge invariant source at the origin of space-time. We will see how a gauge invariant product of scalar fields at the origin of space-time gives rise to an effective field theory of the high energy physics that depends only on the angles of the momenta of the high energy particles and fields. This 2-dimensional effective theory is our light-shell effective theory (LSET). Here we present it in the simplified venue of 0-flavor scalar quantum electrodynamics in which we can ignore matter loops (see C). This strips away most of the physics so that we can focus on the basic ingredients of our effective theory.

As with any effective theory, the soft physics is left unchanged, so we focus on the physics associated with hard particles, and we distinguish the part of the LSET Lagrangian involving hard particles by referring to it as $\mathcal{L}_{LRE}$.

For this, we introduce a field decomposition we will refer to as a large radial energy (LRE) expansion. We will shortly see that this expansion naturally suggests to light-shell gauge as a convenient gauge in which the leading order matter-photon interactions vanish. These two ingredients make up the LSET Lagrangian which we will construct in section 4.1. The physics associated with such interactions is then described entirely by a gauge invariant source around the origin, which we will discuss in section 4.2.

4.1 Constructing the light-shell effective theory Lagrangian

We start with the large radial energy expansion, which is reminiscent of the field decomposition of HQET [6] and LEET (the precursor of SCET [16] that sums soft logs but not collinear logs). We scale out the uninteresting large momenta associated with the energetic particles, but as its name suggests the LRE expansion involves scaling out by a spherical wave. In order to do this, we set an energy scale $\mathcal{E}$, that determines which fields are large

radial energy fields $\Phi_E^{(*)}$, and which fields are soft ϕ_s. The decomposition is

$$\phi = \phi_s + \sum_{E>\mathcal{E}} \left(\frac{e^{-iE(t-r)}}{\sqrt{2E}} \Phi_{E,+q} + \frac{e^{iE(t-r)}}{\sqrt{2E}} \Phi_{E,-q}^* \right)$$

$$\phi^* = \phi_s^* + \sum_{E>\mathcal{E}} \left(\frac{e^{-iE(t-r)}}{\sqrt{2E}} \Phi_{E,-q} + \frac{e^{iE(t-r)}}{\sqrt{2E}} \Phi_{E,+q}^* \right)$$

(115)

where $\Phi_{E,\pm q}$ ($\Phi_{E,\pm q}^*$) annihilates (creates) high energy outgoing scalars with charge $\pm q$. In the following, we will focus on the particles with charge $+q$ and drop the $\pm q$ subscripts to simplify the notation. As usual in such an effective field theory decomposition, the x dependence of the EFT field is assumed to be slow compared to the t and r dependence of the exponential factor $e^{iE(t-r)}$, and derivatives of Φ_E are assumed to be small compared to $\mathcal{E}$ in the effective theory.[4] The $1/\sqrt{2E}$ is a normalization, the reason for which will soon be apparent.

Applying this expansion, the LSET Lagrangian can be written as an expansion in the small parameter $(1/E)$, where E is the energy scaled out of the energetic field at hand. Let's begin to look at $\mathcal{L}_{LRE}$ by examining $\mathcal{L}_\phi$, the kinetic energy of our matter field, to leading order in $1/E$. Using our expansion (115), focusing on the LRE terms, and using

$$\frac{\partial(t-r)}{\partial x_\mu} = v^\mu$$

(116)

we get

$$(D^\mu \phi)^* D_\mu \phi \to \frac{1}{2E} \left((D^\mu + iEv^\mu)\Phi_E^* \right) \left((D^\mu - iEv^\mu)\Phi_E \right)$$

(117)

The cross terms are leading in the $1/E$ expansion, and have a factor of E from the derivatives

---

[4]This is a bit trickier than it sounds. See [17].

acting on the spherical wave, which cancels the normalization from (115), giving

$$= i\,\Phi_E^* \left(\partial_t + (\hat{r}\cdot\vec{\nabla} + \vec{\nabla}\cdot\hat{r})/2 \right)\Phi_E + \frac{1}{2Er^2}\Phi_E^*\,\tilde{L}^2\,\Phi_E \tag{118}$$

where the $\tilde{L}^2 = -r^2\,(\vec{\nabla}_\perp^T - iq\vec{A}_\perp)\cdot(\vec{\nabla}_\perp - iq\vec{A}_\perp)$ and we omit terms that vanish by the zeroth-order equations of motion. While the $\tilde{L}^2$ term is of order $1/E$, it also has rapid r dependence as $r \to 0$, which we do not want. We can make the following field redefinition to eliminate it:

$$\tilde{\Phi}_E(x) \equiv \exp\left[i\,\frac{\tilde{L}^2}{2Er} \right]\Phi_E(x) \tag{119}$$

Note that since the derivatives in $\tilde{L}^2$ are all covariant, $\tilde{\Phi}_E(x)$ transforms just like $\Phi_E(x)$ under gauge transformations. In terms of $\tilde{\Phi}_E(x)$, and ignoring interaction terms, the kinetic energy becomes

$$\mathcal{L}_\phi = i\,\tilde{\Phi}_E^* \left((\partial_\mu v^\mu + v^\mu \partial_\mu)/2 \right)\tilde{\Phi}_E = i\,\tilde{\Phi}_E^* \left(\partial_t + (\hat{r}\cdot\vec{\nabla} + \vec{\nabla}\cdot\hat{r})/2 \right)\tilde{\Phi}_E \tag{120}$$

The kinetic energy term (120) looks very much like the corresponding terms in HQET [6] and LEET [16], but there the analog of the vector v^μ is a constant, time-like in HQET and light-like in LEET. The fact that v^μ varies with $\hat{r}$ is responsible for unique properties of the LRE expansion. For example, the LRE decomposition (115) is invariant under rotations about the origin, not just covariant like HQET or LEET.

The $\tilde{\Phi}_E$ propagator associated with the kinetic energy term (120) is directional and has the form[5]

$$\left\langle 0\,\middle|\, T\,\tilde{\Phi}_E(x)\,\tilde{\Phi}_E^*(x')\,\middle|\,0 \right\rangle = \frac{1}{rr'}\,\theta(t - t')\,\delta(t - r - t' + r')\,\delta(\hat{r} - \hat{r}') \tag{121}$$

One can check (121) easily and it can be formally derived using canonical quantization, as

---

[5]When $\hat{r}$ appears as an argument, it refers to dependence on angles θ and ϕ. Likewise $\hat{r}_j$ refers to the angles θ_j and ϕ_j. So, here $\delta(\hat{r} - \hat{r}')$ is equal to $\delta(z - z')\delta(\phi - \phi')$, with $z = \cos(\theta)$.

we show in appendix D. The propagator (121) describes radially outgoing particles and this form establishes the connection between the spatial direction of the coordinate x and the direction of propagation of the particle, which determines the direction of the momentum of the LRE particle far away from the source. This connection between position space and momentum space for the high-energy particles is one of the crucial components of our construction. We will return to this and see the connection very explicitly in section 5.1. But while the connection is exact in the free theory, we would expect that quantum effects make it impossible to specify the momentum direction precisely. This expectation is reified in the calculation of quantum loops where specifying the directions precisely leads to divergences [4]. We assume that this is associated with the physical impossibility of measuring a jet direction exactly. Thus we associate an angular size with each LRE particle quantifying the uncertainty in direction.

We now return to equation (117) to explore the consequences of the LRE expansion for $\mathcal{L}_{\text{int}}$. It can be written in the suggestive form

$$\frac{i}{2}\Big[(-\partial^\mu \Phi_E^\dagger)v_\mu \Phi_E + v^\mu \Phi_E^\dagger \partial_\mu \Phi_E\Big] \; + \; v_\mu A^\mu \Phi_E^\dagger \Phi_E \; + \; \frac{1}{2E}(D^\mu \Phi_E)^\dagger D_\mu \Phi_E \qquad (122)$$

In this form, it is clear that in LSG our interactions vanish at leading order. The removal of the gauge interactions with LRE scalars simplifies calculations, and it makes it clear that the essential physics of the high-energy particles is associated with the source at the origin. This is consistent with the expectation of a purely angular theory on the light shell.

Lastly, there is the kinetic energy term for the gauge field. We will show in section 6.2 that in LSG, it can be written in a matrix form as

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}^2 = -\frac{1}{2}\Big(A_r, \vec{A}_\perp^T\Big) \begin{pmatrix} (\partial_t + \vec{\nabla}\cdot\hat{r})(\partial_t + \hat{r}\cdot\vec{\nabla}) & (\partial_t + \vec{\nabla}\cdot\hat{r})\vec{\nabla}^T \\ \vec{\nabla}(\partial_t + \hat{r}\cdot\vec{\nabla}) & \vec{\nabla}\vec{\nabla}^T + \square I \end{pmatrix} \begin{pmatrix} A_r \\ \vec{A}_\perp \end{pmatrix} \qquad (123)$$

33

where

$$A_r \equiv \hat{r} \cdot \vec{A} \tag{124}$$

is the radial component of $\vec{A}$ (and is not to be confused with A_r in the covariant tensor form). In terms of this, the LSG condition is therefore $A^0 = A_r$. $\vec{A}_\perp$ is the transverse component of $\vec{A}$, and is treated here as a column vector. It is given by

$$\vec{A}_\perp = \vec{A} - \hat{r}\,\hat{r}^T \vec{A} = \vec{A} - \left(\hat{r} \cdot \vec{A}\right)\hat{r} \tag{125}$$

The temporal component of A^μ does not appear in (123) because in LSG, it is equal to A_r.

An LRE expansion, similar to that of the scalars, holds for the gauge field. This expansion is in terms of longitudinal and perpendicular components of the gauge field, which is appropriate in light-shell gauge. Again, the rescaling of each field is determined by the canonical form of the kinetic energy term.

$$\vec{A} = \vec{A}_s + \sum_{E>\mathcal{E}} \left(\frac{1}{\sqrt{2E}} e^{iE(t-r)} \vec{\mathcal{A}}_{E\perp}^* + \frac{1}{\sqrt{2E}} e^{-iE(t-r)} \vec{\mathcal{A}}_{E\perp} + e^{iE(t-r)} \mathcal{A}_{Er}^* \hat{r} + e^{-iE(t-r)} \mathcal{A}_{Er} \hat{r} + \cdots \right) \tag{126}$$

After applying this expansion to (123) and considering the LRE terms, we can redefine the gauge field as

$$\begin{pmatrix} \mathbb{A}_{Er} \\ \vec{\mathbb{A}}_{E\perp} \end{pmatrix} = \begin{pmatrix} 1 & (\partial_t + \hat{R} \cdot \vec{\nabla})^{-1} \vec{\nabla}^T / \sqrt{2E} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A_{Er} \\ \vec{A}_{E\perp} \end{pmatrix} \tag{127}$$

Where $(\partial_t + \vec{\nabla} \cdot \hat{R})^{-1}$ is the inverse of a differential operator that is non-local in space and time and given explicitly by (see appendix F.2 for derivation)

$$(\partial_t + \vec{\nabla} \cdot \hat{R})^{-1}(x, x') = \frac{1}{r^2} \theta(t - t')\, \delta(t - r - t' + r')\, \delta(\hat{r} - \hat{r}') \tag{128}$$

Operators of this type appear frequently in our LSET analysis, especially when we get to the computation of the LSG photon propagator and radiative corrections in chapters 6 and

34

7. These operators are treated on the same footing as linear operators, so for example, the first row of (127) could be written more explicitly as

$$
\mathbb{A}_{Er}(x) = A_{Er}(x) + \int \left((\partial_t + \hat{R} \cdot \vec{\nabla})^{-1}(x, x') \, \vec{\nabla}'^T / \sqrt{2E} \, \vec{A}_{E\perp}(x') \, d^4 x' \right)
\tag{129}
$$

In section (5) we will see that the quanta of the $\vec{\mathbb{A}}_\perp$ field can be directly related to those of the full theory. Also, this allows us to write the LRE photon kinetic energy in the following diagonal form.

$$
\mathcal{L}_{A,LRE} = \begin{pmatrix} \mathbb{A}_{Er}^* & \vec{\mathbb{A}}_{E\perp}^* \end{pmatrix} \begin{pmatrix} -(\partial_t + \vec{\nabla} \cdot \hat{r})(\partial_t + \hat{r} \cdot \vec{\nabla}) & 0 \\ 0 & (\partial_t + \hat{r} \cdot \vec{\nabla}/2 + \vec{\nabla} \cdot \hat{r}/2) \end{pmatrix} \begin{pmatrix} \mathbb{A}_{Er} \\ \vec{\mathbb{A}}_{E\perp} \end{pmatrix}
\tag{130}
$$

The final piece of the LSET Lagrangian is the source. This is where the interesting physics of our theory lies, and we describe it in the following section.

4.2 The LSET source

So far we have constructed the LSET Lagrangian by bringing the large radial energy expansion and light-shell gauge to the full theory. In doing so we have removed all of the interactions of the LRE particles except for those directly associated with the point source at the origin in the full theory. The full-theory source is proportional to a gauge invariant product of local fields at the origin. Thus we also expect the corresponding source in the EFT to be gauge invariant. The conventions for the gauge transformations of our fields are listed in appendix B.

While the full-theory source is at the origin, light-shell gauge is ill-defined there, so we begin by considering a source in the EFT that is "spread out" about the origin. We also expect from our classical picture that as the energy in the event goes to infinity, all of the physics goes onto the light shell, at $t = r$. Thus in our quantum version, we spread out our

source onto a surface $r = s$ surrounding the origin, near the light shell, with $t - r \to 0$ as $E \to \infty$. To understand the symmetry of the spread-out source, it is convenient to write $\varphi(x) = \varphi(t, r, \hat{r})$ and to let

$$\varphi(r, r, \hat{r}) \equiv \varphi(x)|_{t=r} \tag{131}$$

represent either an LRE field or a soft field on the light shell. When we eventually write down the full source, the LRE fields will be evaluated at particular values of $\hat{r}$, while the soft fields will be integrated over $\hat{r}$. But this notation will allow us to focus on the symmetries for both types of fields simultaneously. In this notation, a term in the source spread out over S appears as

$$\mathcal{O} \propto \prod_j \varphi_j^\dagger(s, s, \hat{r}_j) \tag{132}$$

This is not gauge invariant, but transforms as

$$\mathcal{O} \to \mathcal{O} \prod_j \exp\left[-i\, q_j\, \Lambda(x_j)|_{t_j=r_j=s}\right] \tag{133}$$

To maintain gauge invariance we construct a compensating exponential on the light shell

$$\exp\left(i\frac{e}{2\pi} \int \left(\sum_j \ell(\hat{r}, \hat{r}_j)\right) \partial_\mu A^\mu(x)\, dS\right) \tag{134}$$

where dS is our Lorentz covariant surface element on the small sphere.

$$dS = \theta(t)\, r\, \delta(r - s)\, \delta(r^2 - t^2)\, d^4x \tag{135}$$

and assuming zero net charge

$$\ell(\hat{r}, r_j) = q_j \log(1 - \hat{r}_j \cdot \hat{r}) \tag{136}$$

Putting all the pieces together, our gauge invariant source on the light shell, call it $\mathcal{S}$, is of

the form

$$
C \lim_{s \to 0} \int \exp\left(i\frac{e}{2\pi} \int \left(\sum_{j=1}^{m+n} \ell(\hat{r}, \hat{r}_j) \right) \partial_\mu A^\mu(x)\, dS \right)
$$

(137)

$$
\left(\prod_{j=1}^{m} r_j^{-1} \Phi_{j,E_j}^\dagger(x_j) \right) \left(\prod_{j=m+1}^{m+n} r_j^{-2} \phi_j^{(\dagger)}(x_j) \right) \left(\prod_{j=1}^{m+n} dS_j \right)
$$

where there are n soft scalars $\phi^{(\dagger)}$, m LRE scalars Φ, and dS_j refers to dS with $x^\mu \to x_j^\mu$. Also, each LRE scalar Φ will have an energy associated with it E_i, this is the energy scaled out by the LRE expansion. Notice that there is a constant C, which must be determined by matching.

Assuming gauge transformations on the light shell, our compensating exponential is unique. Also, one cannot help but notice the resemblance with the classical potentials (24) and (25). The measure dS fixes t and r, leaving us with purely angular dependence. This dependence solely on angles will persist for any process to any order in LSET.

5 Tree-level matching

5.1 Matching LRE scalars

The simplest non-trivial matching to consider is that of LRE scalars. For this, we will match the amplitude of a source creating a one-particle state in the full theory to the corresponding amplitude in the effective theory. Of course, for this source to be gauge invariant, the particle must be neutral. This allows us to focus on the LRE matching all by itself. In the process, we will define creation/annihilation operators in the EFT by relating them to the familiar creation/annihilation operators in the full theory. This construction can then be carried over trivially to interesting sources involving charged particles.

Let the matching condition be

$$\langle \vec{k} | \text{Full Source} | 0 \rangle \overset{\text{match}}{=} \langle \vec{k} | \text{EFT Source} | 0 \rangle \tag{138}$$

$\langle \vec{k} |$ is a one particle state for a scalar with momentum $k^\mu = (k, \vec{k})$ as defined in the full theory. This matching will connect the position space of the effective theory to the momentum space of the full theory, as well as fix the coefficient of the effective theory source. The full theory source is just $\phi(0)$. The EFT source for a high-energy particle, to leading order, has the form

$$c_1 \int d\Omega_1 \, s(\hat{r}_1) \Phi^\dagger_{1,E_1}(s(\hat{r}_1), s(\hat{r}_1), z_1, \phi_1) \tag{139}$$

where c_1 is the coefficient we will determine herein. The matching condition is then

$$\langle \vec{k} | \phi(0) | 0 \rangle \overset{\text{match}}{=} \left\langle \vec{k} \left| c_1 \int d\Omega_1 \, s(\hat{r}_1) \Phi^\dagger_{1,E_1}(s(\hat{r}_1), s(\hat{r}_1), z_1, \phi_1) \right| 0 \right\rangle \tag{140}$$

The LHS is 1, and the RHS of (140) is

$$\left\langle 0 \left| \sqrt{2k}\, a_k \int d\Omega_1\, c_1\, s(\hat{r}_1) \Phi^\dagger_{1,E_1}(s(\hat{r}_1), s(\hat{r}_1), z_1, \phi_1) \right| 0 \right\rangle \tag{141}$$

Making the commutation of operators involved above well defined requires a few steps. First, define a full theory annihilation (a_s) operator in spherical coordinates by relating it to a standard full theory operator. The familiar commutation relation is

$$\left[a_p, a^\dagger_{p'}\right] = (2\pi)^3\, \delta^{(3)}(\vec{p} - \vec{p}') \tag{142}$$

which can be expressed in spherical coordinates as

$$\left[a_p, a^\dagger_{p'}\right] = \frac{(2\pi)^3}{p^2}\, \delta(p - p')\, \delta(z - z')\, \delta(\phi - \phi') \tag{143}$$

The spherical creation (a_s) and annihilation $(a^\dagger_s)$ operators we define by

$$\left[a_p, a^\dagger_{p'}\right] \equiv \frac{(2\pi)^2}{pp'} \left[a_s(p, z, \phi), a^\dagger_s(p', z', \phi')\right] \tag{144}$$

Notice that we have

$$\left(\frac{p}{2\pi}a_p\right)\left(\frac{p'}{2\pi}a^\dagger_{p'}\right) - \left(\frac{p'}{2\pi}a^\dagger_{p'}\right)\left(\frac{p}{2\pi}a_p\right) = a_s(p, z, \phi)a^\dagger_s(p', z', \phi') - a^\dagger_s(p', z', \phi')a_s(p, z, \phi) \tag{145}$$

so the relations between conventional and spherical creation and annihilation operators are

$$a_s(p, z, \phi) = \frac{p}{2\pi}a_p \tag{146}$$

$$a^\dagger_s(p, z, \phi) = \frac{p}{2\pi}a^\dagger_p \tag{147}$$

In the EFT we can write our fields in terms of creation/annihilation operators as

$$\Phi^\dagger(t, r, \hat{r}) = \int e^{ik(t-r)} \frac{1}{r} a^\dagger_{LRE}(k, \hat{r}) \frac{dk}{2\pi} \tag{148}$$

which is described in detail in appendix D. Using the above two relations, (141) becomes

$$= \int d\Omega_1 \left\langle 0 \left| \sqrt{2k} \frac{2\pi}{k} a_s(k, \hat{k}) c_1 \int a^\dagger_{LRE}(k_1, \hat{r}_1) \frac{dk_1}{2\pi} \right| 0 \right\rangle \tag{149}$$

The final and crucial step is to notice that the commutation relations of a_s and a_{LRE} (in appendix D) look the same, but have two important differences: a_s involves angles in momentum space and the energy involved is the full energy k, whereas a_{LRE} involves angles in position space and the residual momentum k. So we identify the angles in momentum space and position space and set

$$a^\dagger_s(k_1 + k, \hat{k}) = a^\dagger_{LRE}(k_1, \hat{r}) \tag{150}$$

This allows us to turn our $a^\dagger_{LRE}$ into $a^\dagger_s$. We find that the RHS of (140) becomes

$$= 2\pi \sqrt{\frac{2}{k}} c_1 \tag{151}$$

So, we have $c_1 = \frac{1}{2\pi} \sqrt{\frac{k}{2}}$ and our full theory scalars relate to our LRE scalars as

$$\phi(0) = \frac{1}{2\pi} \sqrt{\frac{k}{2}} \int d\Omega_1 \; s(\hat{r}_1) \, \Phi^\dagger_{1, E_1}(s(\hat{r}_1), s(\hat{r}_1), \hat{r}_1) \tag{152}$$

c_1 is the contribution from one LRE scalar to C in our general source (137), but we will have contributions from matching the other LRE fields involved in the process as well. While this matching procedure is fairly simple, it is essential for connecting the objects in LSET, which are formulated in position space, to the momentum-space amplitudes one is accustomed

to calculating in the full theory. Relations for LRE photons operators, analogous to those introduced here, will be described in the following section.

5.2   Photon emission

We are now prepared to compare amplitudes in the full theory and effective theory. A relevant process to compare is one with the final state of an energetic photon, scalar (labelled by '-'), and anti-scalar (labelled by '+'). For this comparison we will focus on the transverse component. In the full-theory, we have

$$\langle \vec{k}\, \vec{p}_- \, \vec{p}_+ | \phi^*(0)\phi(0)|0\rangle = \frac{e}{|k|} \left(\frac{\hat{p}_- - \hat{k}(\hat{p}_- \cdot \hat{k})}{1 - \hat{k} \cdot \hat{p}_-} - \frac{\hat{p}_+ - \hat{k}(\hat{p}_+ \cdot \hat{k})}{1 - \hat{k} \cdot \hat{p}_+} \right) \tag{153}$$

On the LHS above, $\vec{k}$ refers to a transverse final photon state with momentum $\vec{k}$. In the effective theory we want to do the calculation with the same final state, but we now use our EFT source $\langle \vec{k}\, \vec{p}_- \, \vec{p}_+ | \mathcal{S}|0\rangle$. The dependence on scalar factors disappears. After integrating by parts and making use of the rescaling for LRE photons, we get

$$\left\langle \vec{k} \left| \frac{-ie}{2\pi} \int \left(\mathcal{A}_{Er}^*(x)\,(\partial_t + \hat{r} \cdot \vec{\nabla}) + \frac{1}{\sqrt{2E}} \vec{\mathcal{A}}_{E\perp}^*(x) \cdot \vec{\nabla} \right) \left(\sum_{j=+,-} \ell(\hat{r}, \hat{r}_j) \right) dS \right| 0 \right\rangle \tag{154}$$

Note that the exponential associated with the LRE expansion has gone away because of the $\delta(r^2 - t^2)$ in dS. Using the transformation that diagonalizes the kinetic energy (127)

gives

$$
= \left\langle \vec{k} \left| \frac{-ie}{2\pi} \int \left(\left(\delta(x-x') \mathrm{A}^*_{Er}(x') - \vec{\mathbb{A}}^*_{E\perp}(x') \cdot \vec{\nabla}' \left(\partial_t + \vec{\nabla} \cdot \hat{R} \right)^{-1}(x',x)/\sqrt{2E} \right) \left(\partial_t + \hat{r} \cdot \vec{\nabla} \right) \right. \right.
$$

$$
\left. \left. + \delta(x-x') \frac{1}{\sqrt{2E}} \vec{\mathbb{A}}^*_{E\perp}(x') \cdot \vec{\nabla}' \right) \left(\sum_{j=+,-} \ell(\hat{r}, \hat{r}_j) \right) dx' dS \right| 0 \right\rangle
$$

(155)

Where $\left(\partial_t + \vec{\nabla} \cdot \hat{R} \right)^{-1}$ is given in (128) and $\vec{\nabla}'$ involves derivatives with respect to x'. Since the final physical photon state is transverse, and the relevant propagator is diagonal, we can remove the term involving A^*_{Er}. Then simplifying and manipulating our differential operator gives

$$
= \left\langle \vec{k} \left| \frac{-ie}{2\pi} \int \left(\left(-\vec{\mathbb{A}}^*_{E\perp}(x') \cdot \vec{\nabla}' \left(\partial_t + \vec{\nabla} \cdot \hat{R} \right)^{-1}(x',x)/\sqrt{2E} \right) \left(\left(\partial_t + \vec{\nabla} \cdot \hat{r} \right) - \frac{2}{r} \right) \right. \right.
$$

$$
\left. \left. + \delta(x-x') \frac{1}{\sqrt{2E}} \vec{\mathbb{A}}^*_{E\perp}(x') \cdot \vec{\nabla}' \right) \left(\sum_{j=+,-} \ell(\hat{r}, \{\hat{r}_j\}) \right) dx' dS \right| 0 \right\rangle
$$

(156)

$$
= \left\langle \vec{k} \left| \frac{-ie}{\pi} \int \left(\vec{\mathbb{A}}^*_{E\perp}(x') \cdot \vec{\nabla}' \left(\partial_t + \vec{\nabla} \cdot \hat{R} \right)^{-1}(x',x) \frac{1}{\sqrt{2E}\,r} \right) \left(\sum_{j=+,-} \ell(\hat{r}, \{\hat{r}_j\}) \right) dx' dS \right| 0 \right\rangle
$$

(157)

Now, just as in (148) for scalars, we can write our transverse photon field as

$$
\vec{\mathbb{A}}^*_{E\perp}(t', r', \hat{r}') = \int e^{ik(t'-r')} \frac{1}{r'} \vec{\mathrm{a}}^\dagger_{E\perp}(k, \hat{r}') \frac{dk}{2\pi}
$$

(158)

Using this in (157) gives

$$
\begin{aligned}
= \Big\langle 0 \Big| \sqrt{2|k|}\, \vec{a}_{k\perp} \frac{-ie}{\pi\sqrt{2E}} \int \Big(e^{ik(t'-r')} \frac{1}{r'} \vec{\mathrm{a}}^{\dagger}_{E\perp} \cdot \vec{\nabla}' \left(\partial_t + \vec{\nabla}\cdot \hat{R}\right)^{-1}(x',x)\frac{1}{r} \Big) \\
\times \Big(\sum_{j=+,-} \ell(\hat{r},\{\hat{r}_j\}) \Big) \frac{dk}{2\pi}\, dx'dS \Big| 0 \Big\rangle
\end{aligned}
\tag{159}
$$

Again, for the gauge fields' creation/annihilation operators, we can use relations analogous to those introduced in the previous section for scalars,

$$
\vec{a}_s(p,z,\phi) = \frac{p}{2\pi}\vec{a}_p
\tag{160}
$$

Identifying

$$
\vec{a}^{\dagger}_s(k+E,\hat{k}) = \vec{\mathrm{a}}^{\dagger}_E(k,\hat{r})
\tag{161}
$$

allows us to have creation/annihilation operators with well-defined commutation relations. Also, note that $E = |\vec{k}|$, and (159) becomes

$$
= \Big\langle 0 \Big| \vec{a}_{s\perp}(E,\hat{r}_k) \frac{-2ie}{E} \int \Big(e^{ik(t'-r')} \frac{1}{r'}\, \vec{a}^{\dagger}_{s\perp}(k+E,\hat{r}') \cdot \vec{\nabla}' \left(\partial_t + \vec{\nabla}\cdot \hat{R}\right)^{-1}(x',x)\frac{1}{r} \Big)
\tag{162}
$$

$$
\times \Big(\sum_{j=+,-} \ell(\hat{r},\{\hat{r}_j\}) \Big) \frac{dk}{2\pi}\, dx'dS \Big| 0 \Big\rangle
\tag{163}
$$

The relevant commutation relation is

$$
\Big[\vec{a}_{s\perp}(p,z,\phi)\,,\ \vec{a}^{\dagger}_{s\perp}(p',z',\phi') \Big] = 2\pi\, P_{\perp}\, \delta(p-p')\, \delta(z-z')\, \delta(\phi-\phi')
\tag{164}
$$

where $P_{\perp}$ is a projection operator for the perpendicular components. Using this we obtain

$$
= \frac{-2ie}{E} \int \Big(\delta(\hat{r}_k - \hat{r}')e^{ik(t'-r')}\frac{1}{r'}\vec{\nabla}'_{\perp} \left(\partial_t + \vec{\nabla}\cdot \hat{R}\right)^{-1}(x',x)\frac{1}{r} \Big) \Big(\sum_{j=+,-} \ell(\hat{r},\{\hat{r}_j\}) \Big) dx'dS
\tag{165}
$$

43

which involves

$$r\vec{\nabla}_{\perp}\left(\sum_{j=+,-}\ell(\hat{r},\{\hat{r}_j\})\right)=\sum_{j=+,-}q_j\left(\frac{\hat{r}_j-\hat{r}(\hat{r}\cdot\hat{r}_j)}{1-\hat{r}\cdot\hat{r}_j}\right) \tag{166}$$

Using this along with integrating over dx' and dS in (165) gives

$$-\frac{ie}{E}\sum_{j=+,-}q_j\left(\frac{\hat{r}_j-\hat{r}_k(\hat{r}_k\cdot\hat{r}_j)}{1-\hat{r}_k\cdot\hat{r}_j}\right) \tag{167}$$

(167) has the same absolute magnitude as (153), confirming the structure of the effective

theory.

6 The Light-shell gauge propagator

In order to go beyond the tree-level matching calculations discussed in section 5, we need the photon propagator in LSG. Typically, we perform calculations in gauge theories while working in covariant gauges, for which the procedure has been well established [8]. But LSG is a non-covariant gauge and therefore the usual technique does not work.

One non-covariant gauge that shares some characteristics with LSG is radial (Fock-Schwinger) gauge [18] which is defined by the condition

$$x_\mu A^\mu = 0, \tag{168}$$

and has found widespread use in QCD sum-rules [19]. Shared characteristics between LSG and radial gauge include breaking translational invariance by choosing an origin and **coordinate dependent gauge condition**. As a result, it is often convenient to use a position space formulation rather than momentum space formulation. While these gauges share some characteristics, only LSG guarantees zero field strength off of the light shell [1] and allows for simplification of calculations in LSET [3]. Another important difference is that the radial gauge condition is invariant under homogeneous Lorentz transformations, while LSG is only invariant under rotations about the origin.

Since we are at such an early stage (the first, as far as we know) in exploring LSG, we restrict our analysis to QED where we can avoid complications that come with non-abelian theories.[6] Even in QED, we cannot use standard techniques for calculating propagators in non-covariant gauges, such as LSG. We therefore, along the road to the LSG propagator, present a different derivation which we hope may prove useful in other gauges as well.

The basic outline of our derivation is as follows. We begin by writing the photon lagrangian in LSG in the matrix form stated in (123) in chapter 4.1. In particular, this

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[6]We hope to eventually extend this work to QCD and in the process describe attributes avoided herein (e.g. ghosts).

form was

$$\mathcal{L} = -\frac{1}{2} \begin{pmatrix} A_r & \vec{A}_\perp^T \end{pmatrix} M \begin{pmatrix} A_r \\ \vec{A}_\perp \end{pmatrix} \tag{169}$$

In chapter 4.1, we wrote out the matrix explicitly without proof, but we will calculate it in 6.2. After that, we will show how from M we are able to construct the LSG propagator. This is not simply a matter of inverting M because $\vec{A}_\perp$ does not have a radial component. What we therefore need to compute is the inverse of M restricted to the subspace from which we have projected out this (non-existent) radial component. We will see that doing so turns out to be non-trivial since M does not commute with the projection operator in the radial direction. As a result, we cannot express M in a diagonal basis and simply take the inverse on the relevant subspace to obtain the propagator. We therefore need to follow a slightly more involved procedure. Our technique, we hope, may also be applicable to some other non-covariant gauges.

The first step is to calculate the matrix M in equation (169). This will involve some vector derivatives, and it will be convenient to use a bit of special notation to separate the radial and transverse components of the ∇ operator. Let us therefore start by defining this notation.

6.1   Notation for vector derivatives

We will consider the radial derivatives and the transverse part of $\vec{\nabla}$ separately. For the radial part, we will have two forms of derivatives:

$$\hat{r} \cdot \vec{\nabla} = \partial_r \tag{170}$$

and

$$\vec{\nabla} \cdot \hat{r} = \frac{1}{r^2} \partial_r r^2 \tag{171}$$

Coming to the transverse part, we will define $\vec{\nabla}_\perp$ as the angular part of the del operator.

46

That is, when acting as the angular part of the gradient of some scalar f, it will be given by

$$\vec{\nabla}_\perp f = \left(\vec{\nabla} - \hat{r}(\hat{r} \cdot \nabla) \right) f \tag{172}$$

On the other hand, as the angular part of the divergence of a vector $\vec{V}$, $\vec{\nabla}_\perp$ will be given by

$$\vec{\nabla}_\perp \vec{V} = \left(\vec{\nabla} - (\nabla \cdot \hat{r})\hat{r} \right) \cdot \vec{V} f \tag{173}$$

Lastly, the angular part of the Laplacian is the angular divergence of the angular gradient. Just as the full Laplacian is written as $\nabla^2 f = \vec{\nabla} \cdot (\vec{\nabla} f)$, we can write the angular part of the Laplacian as

$$\nabla_\perp^2 f = \vec{\nabla}_\perp \cdot (\vec{\nabla}_\perp f) = \left(\nabla^2 - (\vec{\nabla} \cdot \hat{r})(\hat{r} \cdot \vec{\nabla}) \right) f \tag{174}$$

That is,

$$\nabla_\perp^2 = \left(\nabla^2 - (\vec{\nabla} \cdot \hat{r})(\hat{r} \cdot \vec{\nabla}) \right) = -\frac{L^2}{r^2} \tag{175}$$

6.2   The Kinetic Energy Matrix in LSG

(The disinterested reader can skip the derivation and jump to equation (206) near the end of this section.)

We start with the photon Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(\vec{\nabla} A^0 + \partial_t \vec{A})^2 - \frac{1}{2}(\vec{\nabla} \times \vec{A})^2 \tag{176}$$

We then plug in the LSG condition $A^0 = A_r$, giving

$$\mathcal{L} = \frac{1}{2}\left(\nabla A_r + \partial_t \vec{A} \right)^2 - \frac{1}{2}\left(\nabla \times \vec{A} \right)^2 \tag{177}$$

Focusing on the first piece, we get

$$(\partial_t \vec{A} + \vec{\nabla} A_r)^2 = (\hat{r}(\partial_t + \hat{r} \cdot \vec{\nabla})A_r + \vec{\nabla}_\perp A_r + \partial_t \vec{A}_\perp)^2 \tag{178}$$

$$= ((\partial_t + \hat{r} \cdot \vec{\nabla})A_r)^2 + (\vec{\nabla}_\perp A_r)^2 + (\partial_t \vec{A}_\perp)^2 + 2(\vec{\nabla}_\perp A_r \partial_t A_\perp) \tag{179}$$

Integrating this by parts gives

$$= -A_r(\partial_t + \vec{\nabla} \cdot \hat{r})(\partial_t + \hat{r} \cdot \vec{\nabla})A_r - A_r \vec{\nabla}^2 A_r + A_r \left(\vec{\nabla} \cdot \hat{r}\right)\left(\hat{r} \cdot \vec{\nabla}\right) A_r$$

$$-\vec{A}_\perp \cdot \partial_t^2 \vec{A}_\perp - A_r \partial_t \vec{\nabla}_\perp \cdot A_\perp - \vec{A}_\perp \cdot \vec{\nabla}_\perp \partial_t A_r \tag{180}$$

For the $(\vec{\nabla} \times \vec{A})^2$ term, we can write

$$(\vec{\nabla} \times \vec{A})^2 = (\vec{\nabla} \times A_r \hat{r} + \vec{\nabla} \times \vec{A}_\perp)^2 \tag{181}$$

We can now work out the rr, $r\perp$ and $\perp\perp$ terms in this separately by writing all the cross products explicitly in terms of cartesian indices. Using the identity

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = a_j b_k c_j d_k - a_j b_k c_k d_j \tag{182}$$

we get

$$(\vec{\nabla} \times \hat{r} A_r)^2 = (\hat{r} \times \vec{\nabla} A_r)^2 = (\hat{r}_j \nabla_k A_r)(\hat{r}_j \nabla_k A_r) - (\hat{r}_j \nabla_k A_r)(\hat{r}_k \nabla_j A_r) \tag{183}$$

$$= (\vec{\nabla} A_r)^2 - (\hat{r}_k \nabla_k A_r)(\hat{r}_j \nabla_j A_r) \tag{184}$$

$$= (\vec{\nabla} A_r)^2 - (\hat{r} \cdot \vec{\nabla} A_r)^2 \tag{185}$$

On integrating this by parts, we get

$$\rightarrow -A_r \nabla^2 A_r + A_r (\vec{\nabla} \cdot \hat{r})(\hat{r} \cdot \vec{\nabla}) A_r \tag{186}$$

Similarly, the $r \perp$ term is given by

$$(\hat{r} \times \vec{\nabla} A_r)(\vec{\nabla} \times \vec{A}_\perp) = (\hat{r}_j \nabla_k A_r)(\nabla^j \vec{A}_\perp^k) - (\hat{r}_j \nabla_k A_r)(\nabla^k \vec{A}_\perp^j) \tag{187}$$

$$= (\nabla_k A_r)(\hat{r}_j \nabla^j \vec{A}_\perp^k) - (\hat{r}_j \nabla_k A_r)(\nabla^k \vec{A}_\perp^j) \tag{188}$$

Integrating this by parts gives

$$\rightarrow -A_r \nabla_k (\hat{r}_j \nabla^j \vec{A}_\perp^k) + (\nabla_k \hat{r}_j \nabla^k A_r) \vec{A}_\perp^j \tag{189}$$

$$= -A_r \hat{r}_j \nabla^j \nabla_k \vec{A}_\perp^k - A_r (\nabla_k \hat{r}_j) \nabla^j \vec{A}_\perp^k + (\nabla_k \hat{r}_j)(\nabla^k A_r) \vec{A}_\perp^j + \hat{r}_j (\nabla_k \nabla^k A_r) \vec{A}_\perp^j \tag{190}$$

The last line is zero as $\hat{r} \cdot \vec{A}_\perp = 0$.

$$= -A_r \hat{r}_j \nabla^j \nabla_k \vec{A}_\perp^k - A_r (\nabla_k \hat{r}_j) \nabla^j \vec{A}_\perp^k + (\nabla_k \hat{r}_j)(\nabla^k A_r) \vec{A}_\perp^j \tag{191}$$

Integrate the second term by parts:

$$= -A_r \hat{r}_j \nabla^j \nabla_k \vec{A}_\perp^k + \vec{A}_\perp^k \nabla_j (A_r (\nabla^k \hat{r}^j)) + (\nabla_k \hat{r}_j)(\nabla^k A_r) \vec{A}_\perp^j \tag{192}$$

$$= -A_r \hat{r}_j \nabla^j \nabla_k \vec{A}_\perp^k + \vec{A}_\perp^k (\nabla_j A_r)(\nabla_k \hat{r}^j) + \vec{A}_\perp^k A_r (\nabla_j \nabla_k \hat{r}^j) + (\nabla_k \hat{r}_j)(\nabla^k A_r) \vec{A}_\perp^j \tag{193}$$

In the third term, use $(\nabla_j \nabla_k \hat{r}_j) = \frac{2}{r^2} \hat{r}_k$. We then get zero as $\hat{r} \cdot \vec{A}_\perp = 0$. The second and fourth terms are equal since $\nabla_j \hat{r}_k = \nabla_k \hat{r}_j$. We then have

49

$$= -A_r \hat{r}_j \nabla^j \nabla_k \vec{A}_\perp^k + 2(\nabla_j A_r)(\nabla_k \hat{r}^j)\vec{A}_\perp^k \tag{194}$$

In the second term, use $\nabla_k \hat{r}^j = \frac{1}{r}(\delta_k^j - \hat{r}^j \hat{r}_k)$.

We then have

$$= -A_r \hat{r}_j \nabla^j \nabla_k \vec{A}_\perp^k + 2(\nabla_j A_r)\frac{1}{r}(\delta_k^j - \hat{r}_j \hat{r}_k)\vec{A}_\perp^k \tag{195}$$

$$= -A_r \hat{r}_j \nabla^j \nabla_k \vec{A}_\perp^k + 2(\nabla_j A_r)\frac{1}{r}\vec{A}_\perp^j \tag{196}$$

Integrate the second term by parts:

$$= -A_r \hat{r}_j \nabla^j \nabla_k \vec{A}_\perp^k - 2A_r \frac{1}{r}\nabla_j \vec{A}_\perp^j - 2A_r (\nabla_j \frac{1}{r})\vec{A}_\perp^j \tag{197}$$

The last term is zero since $\nabla_j \frac{1}{r} = -\frac{1}{r^2}\hat{r}_j$ and its dot product with $\vec{A}_\perp$ is zero.

$$= -A_r \hat{r}_j \nabla^j \nabla_k \vec{A}_\perp^k - 2A_r \frac{1}{r}\nabla_j \vec{A}_\perp^j \tag{198}$$

Lastly, noting that $\hat{r} \cdot \vec{\nabla} + 2/r = \vec{\nabla} \cdot \hat{r}$, we get

$$= -A_r \vec{\nabla} \cdot \hat{r}\vec{\nabla} \cdot \vec{A}_\perp \tag{199}$$

It is also possible to show through two integrations by parts that this is equal to

$$= -(\vec{A}_\perp \cdot \vec{\nabla})(\hat{r} \cdot \vec{\nabla})A_r \tag{200}$$

We have thus derived the result that

$$(\hat{r} \times \vec{\nabla}A_r)(\vec{\nabla} \times \vec{A}_\perp) = -A_r \vec{\nabla}_j \cdot \hat{r}\vec{\nabla} \cdot \vec{A}_\perp = -(\vec{A}_\perp \cdot \vec{\nabla})(\hat{r} \cdot \vec{\nabla})A_r \tag{201}$$

50

But in the Lagrangian we have $2(\hat{r} \times \vec{\nabla} A_r)(\vec{\nabla} \times \vec{A}_\perp)$, so we can write

$$2(\hat{r} \times \vec{\nabla} A_r)(\vec{\nabla} \times \vec{A}_\perp) = -A_r \vec{\nabla}_j \cdot \hat{r} \vec{\nabla} \cdot \vec{A}_\perp - (\vec{A}_\perp \cdot \vec{\nabla})(\hat{r} \cdot \vec{\nabla}) A_r \tag{202}$$

Lastly, the $\perp\perp$ term is

$$(\vec{\nabla} \times \vec{A}_\perp) \cdot (\vec{\nabla} \times \vec{A}_\perp) = (\nabla^j A_\perp^k)(\nabla^j \vec{A}_\perp^k) - (\nabla^j \vec{A}_\perp^k)(\nabla^k \vec{A}_\perp^j) \tag{203}$$

$$= -\vec{A}_\perp \nabla^2 \vec{A}_\perp + (\vec{A}_\perp \cdot \vec{\nabla})(\vec{\nabla} \cdot \vec{A}_\perp) \tag{204}$$

We have got all the pieces now. So, combining (180), (186), (202) and (204) gives our matrix. That is,

$$\mathcal{L} = -\frac{1}{2} \begin{pmatrix} A_r & \vec{A}_\perp^T \end{pmatrix} M \begin{pmatrix} A_r \\ \vec{A}_\perp \end{pmatrix} \tag{205}$$

where we now know the matrix M is given by

$$M = \begin{pmatrix} (\partial_t + \vec{\nabla} \cdot \hat{r}) \\ \vec{\nabla} \end{pmatrix} \begin{pmatrix} (\partial_t + \hat{r} \cdot \vec{\nabla}) & \vec{\nabla}^T \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & I \, \square \end{pmatrix} \tag{206}$$

Now things get a little complicated. The 4×4 matrix differential operator M is invertible, but its inverse is not the propagator we want. The LSG propagator is the inverse of M restricted to the subspace from which we have projected out the (non-existent) radial component of $\vec{A}_\perp$. Let P be the projection operator onto the radial direction of $\vec{A}$. Then the inverse we are looking for is the operator D satisfying

$$P\,D = D\,P = 0$$
$$(I - P)\,M\,(I - P)\,D = D\,(I - P)\,M\,(I - P) = (I - P) \tag{207}$$

51

Because P does not commute with M, we cannot simply invert M and then project onto the relevant subspace. Instead, we will use a 2-step procedure. We will first show how the linear algebra of this 2-step procedure works in general, and then apply it to the LSG propagator in particular.

6.3 Inversion on a subspace

Our aim is to take an invertible matrix M, and find its inverse restricted to the subspace projected onto by $(I - P)$, where P is a projection operator onto a subspace and I is the identity matrix. That is, we wish to find the matrix D satisfying (207). There are two steps. Step one (which, for LSG, we will put off until later and relegate to an appendix) is to find the inverse of M^{-1} on the space projected onto by P. That is, we find an operator ν satisfying

$$\nu\, P = P\, \nu = \nu \qquad \nu\, P\, M^{-1}\, P = P\, M^{-1}\, P\, \nu = P \tag{208}$$

Then in step two we consider the following operator:

$$D = M^{-1} - M^{-1}\, \nu\, M^{-1} = M^{-1} - M^{-1}\, P\, \nu\, P\, M^{-1}, \tag{209}$$

It is straightforward to apply (208) to see that D satisfies (207), and thus it is the desired inversion of M on the subspace projected by $(I - P)$.

6.4 Returning to the LS gauge propagator

We now show how we can apply (209) to find the LSG propagator. In this and the following sections we will use an operator notation (discussed in more detail in appendix E) in which differential operators, their inverses, and ordinary functions of coordinates are all treated as linear operators acting on the tensor product space of our 4-component index space and the space of functions of the coordinates.

52

In this language, the projection operator P is

$$P = \begin{pmatrix} 0 & 0 \\ 0 & \hat{R}\hat{R}^T \end{pmatrix} \tag{210}$$

Since the formula (209) for the inverse on a subspace involves the inverse of M on the full space, we must begin by finding M^{-1}. For this purpose, it is convenient to note that M can be written in terms of a diagonal matrix M_d and a triangular matrix T as (where I_n is the $n \times n$ identity operator)

$$M = T M_d T^\dagger, \tag{211}$$

where

$$M_d = \begin{pmatrix} \left(\partial_t + \vec{\nabla}^T \hat{R}\right)\left(\partial_t + \hat{R}^T \vec{\nabla}\right) & 0 \\ 0 & \Box \end{pmatrix}, \tag{212}$$

$$T = \begin{pmatrix} 1 & 0 \\ \vec{\nabla}\left(\partial_t + \vec{\nabla}^T \hat{R}\right)^{-1} & I_3 \end{pmatrix} \tag{213}$$

and

$$T^\dagger = \begin{pmatrix} 1 & \left(\partial_t + \hat{R}^T \vec{\nabla}\right)^{-1} \vec{\nabla}^T \\ 0 & I_3 \end{pmatrix} \tag{214}$$

This makes inverting M straightforward, and we get for M^{-1}

$$\begin{pmatrix} \left(\partial_t + \hat{R}^T \vec{\nabla}\right)^{-1}\left(1 + \vec{\nabla}^T \Box^{-1} \vec{\nabla}\right)\left(\partial_t + \vec{\nabla}^T \hat{R}\right)^{-1} & -\left(\partial_t + \hat{R}^T \vec{\nabla}\right)^{-1} \vec{\nabla}^T \Box^{-1} \\ -\Box^{-1} \vec{\nabla}\left(\partial_t + \vec{\nabla}^T \hat{R}\right)^{-1} & \Box^{-1} \end{pmatrix} \tag{215}$$

The next ingredient we need is the inverse of M^{-1} restricted to the subspace. Here it is

53

useful to avoid the matrix structure and define a linear operator μ, as

$$\mu = \begin{pmatrix} 0 & \hat{R}^T \end{pmatrix} M^{-1} \begin{pmatrix} 0 \\ \hat{R} \end{pmatrix} \tag{216}$$

whence ν in (208) is given by

$$\nu = \begin{pmatrix} 0 & 0 \\ 0 & \hat{R}\mu^{-1}\hat{R}^T \end{pmatrix} \tag{217}$$

Now we can just use (209) and put the pieces together to formally compute the LSG propagator. Doing so and simplifying gives the following results:

$$D_{rr} = \left(\partial_t + \hat{R}^T\vec{\nabla}\right)^{-1}\left(1 + \vec{\nabla}^T C \vec{\nabla}\right)\left(\partial_t + \vec{\nabla}^T\hat{R}\right)^{-1} \tag{218}$$

In this, the first term only involves inverses of derivatives of t and r, and is therefore directional (i.e. it has a trivial angular dependence given by angular delta functions). The second term is not directional.

$$D_{r\perp} = -\left(\partial_t + \hat{R}^T\vec{\nabla}\right)^{-1}\vec{\nabla}^T C \tag{219}$$

$$D_{\perp r} = -C\vec{\nabla}\left(\partial_t + \vec{\nabla}^T\hat{R}\right)^{-1} \tag{220}$$

$$D_{\perp\perp} = C \tag{221}$$

where C is given by

$$C = \square^{-1} - \square^{-1}\hat{R}\mu^{-1}\hat{R}^T\square^{-1} \tag{222}$$

Note that from this form we can see that C is transverse; that is, if we act with the projection operator for the transverse subspace on either side of C, we get C. What remains to be done is to derive an explicit form for C, which is done in detail in appendix G, with the

result

$$C = -R\vec{\nabla}_\perp \,\Box^{-1}\,L^{-2}\,R\,\vec{\nabla}_\perp^T + \vec{L}\,\Box^{-1}\,L^{-2}\,\vec{L}^T \tag{223}$$

where

$$R \equiv |\vec{R}|. \tag{224}$$

Since this involves L^{-2}, we must show that this is well defined. We show in appendix H that because of the operators that appear on either side of L^{-2} in (223), the L^{-2} operator never acts on an $L = 0$ state, and the expression (223) makes sense.

Putting (223) into (218-221) gives (for details, see appendix I)

$$)D_{rr} = \left(\partial_t + \hat{R}^T\vec{\nabla}\right)^{-1}\left(1 - R^{-1}\,L^2\,\Box^{-1}\,R^{-1}\right)\left(\partial_t + \vec{\nabla}^T\hat{R}\right)^{-1} \tag{225}$$

$$D_{r\perp} = \left(\partial_t + \hat{R}^T\vec{\nabla}\right)^{-1}R^{-1}\,\Box^{-1}\,\vec{\nabla}_\perp^T R \tag{226}$$

$$D_{\perp r} = R\vec{\nabla}_\perp\,\Box^{-1}\,R^{-1}\left(\partial_t + \vec{\nabla}^T\hat{R}\right)^{-1} \tag{227}$$

$$D_{\perp\perp} = -R\vec{\nabla}_\perp\,\Box^{-1}\,L^{-2}\,R\,\vec{\nabla}_\perp^T + \vec{L}\,\Box^{-1}\,L^{-2}\,\vec{L}^T \tag{228}$$

We can also combine these into a 3×3 matrix form, call it D_3, appropriate for unconstrained $\vec{A}$ fields:

$$\hat{R}\left(\partial_t + \hat{R}^T\vec{\nabla}\right)^{-1}\left(\partial_t + \vec{\nabla}^T\hat{R}\right)^{-1}\hat{R}^T + \vec{L}\,\Box^{-1}\,L^{-2}\,\vec{L}^T$$

$$-\left(R\vec{\nabla}_\perp L^{-2} - \hat{R}\left(\partial_t + \hat{R}^T\vec{\nabla}\right)^{-1}R^{-1}\right)L^2\,\Box^{-1}\left(L^{-2}R\vec{\nabla}_\perp^T - R^{-1}\left(\partial_t + \vec{\nabla}^T\hat{R}\right)^{-1}\hat{R}^T\right) \tag{229}$$

7 Radiative corrections

The simplest loop order effect arises in the exclusive production of a hard scalar/anti-scalar pair with no photons. The amplitude for this process is described in our effective theory in terms of the source acting on the vacuum to produce the relevant final state:

$$\langle \vec{p}_- \, \vec{p}_+ \, |\mathcal{S}| \, 0 \rangle \tag{230}$$

where the final scalar and anti-scalar have momenta $\vec{p}_+$ and $\vec{p}_-$, respectively. Recall that our effective theory source (137) consists of a product of LRE fields, soft scalars and an an exponential containing the photon field. For the process under consideration, however, we will not have any soft fields since the scalar and anti-scalar are both hard. The tree-level contribution to the amplitude is therefore given in terms of a pair of LRE fields in the source acting on the vacuum as discussed in section 5. This amplitude is however renormalized by effects arising from the exponential (in the source) involving the gauge fields. The full amplitude thus nicely factorizes into the scalar part which is equal to unity as in the full theory by matching, and the part corresponding to the exponential sandwiched between photon vacuum states:

$$\left\langle 0 \left| \exp\left(i\frac{e}{2\pi} \int \left(\sum_{j=1}^{m+n} \ell(\hat{r}, \hat{r}_j) \right) \partial_\mu A^\mu(x) \, dS \right) \right| 0 \right\rangle \tag{231}$$

As we noted in section (4.2), the measure dS in the exponential fixes t and r so that we are on the sphere, leaving us with purely angular integrals. The one-loop term in this is the order e^2 contribution and is given by

$$\frac{1}{2}\left(\frac{ie}{2\pi}\right)^2 \sum_{j,k} q_j q_k \, \langle 0| \, \partial_\mu A^\mu(x_1) \, \partial_\nu A^\nu(x_2) \, |0\rangle \ln(1 - \hat{r}_j \cdot \hat{r}_1) \ln(1 - \hat{r}_2 \cdot \hat{r}_k) \, dS_1 dS_2 \tag{232}$$

In order to proceed further, we need the 2-point function of $\partial_\mu A^\mu$ on the sphere which

we will discuss next.

7.1   The relevant LSG Green's function on the sphere

We showed how to calculate the LSG photon propagator in section 6. We now use our results in that section to compute the 2-point function of $\partial_\mu A^\mu$ on the sphere, which we need to calculate the loop contribution given in (232).

It will be more convenient to use the results for the LSG propagator given in terms of the operator C in equations (218-221). The reason for this is that when we put together all the pieces, we will actually not need the explicit form for C. Instead, C will only appear in the form $\vec{\nabla}^T C \vec{\nabla}$, which has the simpler form (see appendix I for derivation)

$$\vec{\nabla}^T C \vec{\nabla} = -L^2 R^{-1} \Box^{-1} R^{-1} \tag{233}$$

Now, start by recalling that in LSG, $\partial_\mu A^\mu = (\partial_t + \vec{\nabla} \cdot \hat{r})A_r + \vec{\nabla} \cdot \vec{A}_\perp$. We therefore have

$$
\begin{aligned}
r_1^2 r_2^2 \, &\langle 0|\partial_\mu A^\mu(x_1) \, \partial_\nu A^\nu(x_2)|0\rangle \\
&= iR^2 \left((\partial_t + \vec{\nabla} \cdot \hat{R}) \, D_{rr} \, (\partial_t + \hat{R} \cdot \vec{\nabla}) + (\partial_t + \vec{\nabla} \cdot \hat{R}) D_{r\perp} \vec{\nabla} \right. \\
&\quad \left. + \vec{\nabla}^T D_{\perp r}(\partial_t + \hat{R} \cdot \vec{\nabla}) + \vec{\nabla}^T D_{\perp\perp} \vec{\nabla} \right) R^2
\end{aligned}
\tag{234}
$$

We now look at each of the pieces explicitly. For the D_{rr} term, we get

$$iR^2(\partial_t + \vec{\nabla}\cdot\hat{R})D_{rr}(\partial_t + \hat{R}\cdot\vec{\nabla})R^2 = iR^2(\partial_t + \vec{\nabla}\cdot\hat{R})(\partial_t + \hat{R}\cdot\vec{\nabla})^{-1}(\partial_t + \vec{\nabla}\cdot\hat{R})^{-1}(\partial_t + \hat{R}\cdot\vec{\nabla})R^2 \tag{235}$$

$$+iR^2(\partial_t + \vec{\nabla}\cdot\hat{R})(\partial_t + \hat{R}\cdot\vec{\nabla})^{-1}\vec{\nabla}^T C\vec{\nabla}(\partial_t + \vec{\nabla}\cdot\hat{R})^{-1}(\partial_t + \hat{R}\cdot\vec{\nabla})R^2 \tag{236}$$

It turns out that the contribution of the first term (i.e. the part not involving C) goes to zero when we take the $r \to 0$ limit at both points. This is clear from a simple power

57

counting analysis of the operators and factors of r.

This leaves us with the contribution given in (236). To simplify this, we rewrite the derivative operators by noting that $\vec{\nabla} \cdot \hat{r} = \hat{r} \cdot \vec{\nabla} + 2/r$ and get

$$
\begin{aligned}
& iR^2(\partial_t + \vec{\nabla} \cdot \hat{R})D_{rr}(\partial_t + \hat{R} \cdot \vec{\nabla})R^2 \\
& \rightarrow iR^2\left(1 + \frac{2}{R}(\partial_t + \hat{R} \cdot \nabla)^{-1}\right)\left(1 + \vec{\nabla}^T C \vec{\nabla}\right)\left(1 - (\partial_t + \vec{\nabla} \cdot \hat{R})^{-1}\frac{2}{R}\right)R^2
\end{aligned} \tag{237}
$$

Continuing in the same manner, we see that the off-diagonal entries of the propagator give the following contributions

$$
iR^2(\partial_t + \vec{\nabla} \cdot \hat{R})D_{r\perp}\vec{\nabla}R^2 = -iR^2\left(1 + \frac{2}{R}(\partial_t + \hat{R} \cdot \vec{\nabla})^{-1}\right)\vec{\nabla}^T C \vec{\nabla} R^2 \tag{238}
$$

and

$$
iR^2\vec{\nabla}^T D_{\perp r}(\partial_t + \hat{R} \cdot \vec{\nabla})R^2 = -iR^2\vec{\nabla}^T C \vec{\nabla}\left(1 - (\partial_t + \vec{\nabla} \cdot \hat{R})^{-1}\right)R^2 \tag{239}
$$

Lastly, the $D_{\perp\perp}$ term simply gives

$$
iR^2\vec{\nabla}^T D_{\perp\perp}\vec{\nabla}R^2 = iR^2\vec{\nabla}^T C \vec{\nabla} R^2 \tag{240}
$$

Adding (237),(238), (239) and (240) together and simplifying, we see that everything cancels except

$$
r_1^2 r_2^2 \left\langle 0\right| \partial_\mu A^\mu(x_1) \partial_\nu A^\nu(x_2) \left|0\right\rangle = -4iR(\partial_t + \hat{R} \cdot \vec{\nabla})^{-1}\vec{\nabla}^T C \vec{\nabla}(\partial_t + \vec{\nabla} \cdot \hat{R})^{-1}R \tag{241}
$$

$$
= 4iR(\partial_t + \hat{R} \cdot \vec{\nabla})^{-1} L^2 R^{-1} \Box^{-1} R^{-1}(\partial_t + \vec{\nabla} \cdot \hat{R})^{-1}R \tag{242}
$$

where in the last step, we have used (233). Since everything in this except $\Box^{-1}$ is directional, we can pull the L^2 to the left

$$
= 4i L^2 R (\partial_t + \hat{R} \cdot \vec{\nabla})^{-1} R^{-1} \Box^{-1} R^{-1}(\partial_t + \vec{\nabla} \cdot \hat{R})^{-1}R \tag{243}
$$

To proceed further, we use the explicit expressions for the various inverse operators in this equation which are given in appendices E and F.2. The general expression gets rather complicated, but in our limit where both r_1 and r_2 go on the light shell, the result turns out to be miraculously simple (see appendix J for details)

$$= \frac{1}{2\pi^2} L^2 \frac{1}{1 - \hat{r}_1 \cdot \hat{r}_2 + i\epsilon} \tag{244}$$

Plugging this into (232) and simplifying the coefficients, we get the 1-loop correction

$$-\frac{1}{64\pi^4} \sum_{j,k} q_j q_k \int \ln(1 - \hat{r}_j \cdot \hat{r}_1) \ln(1 - \hat{r}_2 \cdot \hat{r}_k) \, L^2 \frac{1}{1 - \hat{r}_1 \cdot \hat{r}_2 + i\epsilon} \, d\Omega_1 d\Omega_2 \tag{245}$$

7.2   Computing the integral

It is convenient to move the angular derivatives in L^2 around by integrating by parts so that each of the L operators in L^2 acts on one of the logs. Using $L \ln(1 - \hat{r}_1 \cdot \hat{r}_j) = \frac{\hat{r}_1 \times \hat{r}_j}{1 - \hat{r}_1 \cdot \hat{r}_j}$, and introducing a notation in which $z_{12} \equiv \hat{r}_1 \cdot \hat{r}_2$ etc, we get

$$-\frac{1}{64\pi^4} \sum_{j,k} q_j q_k \int \frac{(\hat{r}_1 \times \hat{r}_j) \cdot (\hat{r}_2 \times \hat{r}_k)}{(1 - z_{1j})(1 - z_{12} + i\epsilon)(1 - z_{2k})} \, d\Omega_1 d\Omega_2 \tag{246}$$

Looking at the form of this integral, we can see that it has infinities where the dot products in the denominators approach 1. We therefore need to introduce a regularization method. One obvious way to do so is to insert an angular cutoff by taking $1 - z \to 1 - z + \lambda$.

7.2.1   Computing the off-diagonal ($j \neq k$) integral

We first focus on the $j \neq k$ integral. For this case, we will find that we need to introduce an angular cutoff only for $1 - z_{12}$ as the other infinities are integrable. The $i\epsilon$ on the other hand can be dropped since we will not be doing any contour integrals, and we can take

$\lambda \gg |\epsilon|$. The $j \neq k$ integral is thus

$$-\frac{1}{64\pi^4} \sum_{j,k,j\neq k} q_j q_k \int \frac{(\hat{r}_1 \times \hat{r}_j) \cdot (\hat{r}_2 \times \hat{r}_k)}{(1 - z_{1j})(1 - z_{12} + \lambda)(1 - z_{2k})} d\Omega_1 d\Omega_2 \qquad (247)$$

To do the integral, we will now rewrite the numerator in a way that separates the finite and infinite pieces and focus on the terms that contribute to the double logs. It is convenient to first focus on one of the integrals, say the $d\Omega_1$ one:

$$\int \frac{\hat{r}_1 \times \hat{r}_j}{(1 - z_{1j})(1 - z_{12} + \lambda)} = c(z_{2j}) \, \hat{r}_2 \times \hat{r}_j \qquad (248)$$

where we have written the RHS based on the symmetry structure of the integral on the LHS. To find $c(z_{2j})$, we can dot both sides by $\hat{r}_2 \times \hat{r}_j$. Since $(\hat{r}_2 \times \hat{r}_j) \cdot (\hat{r}_1 \times \hat{r}_j) = z_{12} - z_{1j} z_{2j}$, this gives

$$\int \frac{z_{12} - z_{1j} z_{2j}}{(1 - z_{1j})(1 - z_{12} + \lambda)} d\Omega_1 = (1 - z_{2j}^2) \, c(z_{2j}) \qquad (249)$$

$$= \int \frac{(1 - z_{1j}^2) - (1 - z_{12} + z_{1j}(z_{2j} - z_{1j}))}{(1 - z_{1j})(1 - z_{12} + \lambda)} d\Omega_1 \qquad (250)$$

The second term is completely finite and therefore does not contribute to the double logs. We therefore drop it and focus on the second one:

$$\int \frac{(1 - z_{1j}^2)}{(1 - z_{1j})(1 - z_{12} + \lambda)} d\Omega_1 = \int \frac{1 + z_{1j}}{1 - z_{12} + \lambda} d\Omega_1 = \int \frac{(1 + z_{2j}) + (z_{1j} - z_{2j})}{1 - z_{12} + \lambda} d\Omega_1 \qquad (251)$$

Again the second term is finite, so we drop it, and the interesting term is

$$= \int \frac{(1 + z_{2j})}{1 - z_{12} + \lambda} d\Omega_1 = 2\pi (1 + z_{2j}) \log \frac{2}{\lambda} \qquad (252)$$

Thus the part of the function c that is relevant for the double logs is

$$c(z_{2j}) = \frac{2\pi}{1 - z_{2j}} \log \frac{2}{\lambda} \qquad (253)$$

60

Now, taking this along with (248) and (247) gives

$$-\frac{1}{32\pi^3}\sum_{j,k,j\neq k}q_jq_k\log\frac{2}{\lambda}\int\frac{(\hat{r}_2\times\hat{r}_j)\cdot(\hat{r}_2\times\hat{r}_k)}{(1-z_{2j})(1-z_{2k})}d\Omega_2 \tag{254}$$

or

$$-\frac{1}{32\pi^3}\sum_{j,k,j\neq k}q_jq_k\log\frac{2}{\lambda}\int\frac{z_{jk}-z_{2j}z_{2k}}{(1-z_{2j})(1-z_{2k})}d\Omega_2 \tag{255}$$

We show how to do the remaining integral in appendix K. The result for the term that gives the double logs is

$$-\frac{1}{8\pi^2}\sum_{j,k,j\neq k}q_jq_k\log\frac{2}{\lambda}\log\frac{1}{1-z_{jk}} \tag{256}$$

This is our result for the $j\neq k$ integral.

7.2.2 The $j=k$ integral

The calculation for the $j=k$ integrals is much more delicate and depends on the details of the angular cut-off around the $\hat{r}_+$ and $\hat{r}_-$ directions. For this reason, we cannot just take $\hat{r}_j=\hat{r}_k$ and $z_{jk}=1$ in (246). Instead, we need to work out the $j=k$ integral just like the $j\neq k$ case (i.e. explicitly treat $\hat{r}_j$ and $\hat{r}_k$ as separate directions), and then take the directions in the result to be nearly the same up to an angular cutoff. That is, we take the result for $j\neq k$ in (256) and replace $1-z_{jk}\to 1-z_{jj}$ by angular cutoffs $\lambda_j=\theta_j^2/2$ (around the $\hat{r}_j$ and $\hat{r}_k$ directions) instead of taking $1-z_{jk}\to 1$.

This not only makes sense from the point of view of matching the results onto the full theory as we will shortly see, but can also be motivated based on grounds of physical consistency and symmetry. Specifically, we can make the following 3 arguments in this regard:

1. The λ and λ_j dependence must disappear as the hard emission becomes neutral. For example, if we have two oppositely charged hard particles in the same direction with $\theta_+\approx\theta_-\approx\theta\ll 1$, the λ dependence should cancel as $\theta_{+-}\to\theta$, because in this limit,

we have two small, oppositely charged and equal-sized jets sitting right on top of one another to the level of accuracy to which we know their directions.

2. We expect that the $j = k$ contributions should depend on θ_j or θ_k etc, but not both, and should be symmetrical in the two.

3. Lastly, log squared θ dependence is not possible, because it cannot be simultaneously dependent on θ_j and θ_j separately, symmetrical in $\theta_j \leftrightarrow \theta_k$ and also cancel when there are two oppositely charged particles in the same directions with $\theta_j \approx \theta_k \approx \theta$.

Based on these arguments, we conclude that the result for the diagonal integral is

$$ -\frac{1}{8\pi^2} q_j^2 \log \frac{2}{\lambda} \, \log \frac{2}{\theta_j^2} \tag{257} $$

7.3 Combining the results and comparing with the full theory

Having obtained the integrals for both diagonal and off-diagonal cases, we are now ready to put the results together and compare with the corresponding calculation in the full theory. We will focus on the case of two equal and opposite charges $q = \pm 1$.

For the total contribution of the off-diagonal integrals to the amplitude, we take (256) along with a factor of 2 arising from $\sum_{j,k} q_j q_k$. From the diagonal integrals we get (257) summed over both charges. We can combine the diagonal terms from the two charges in a single piece as $\ln(\theta_+^2/2) + \ln(\theta_-^2/2) = 2\ln(\theta_+\theta_-/2)$.

Lastly, since this is a part of the amplitude for the process without any real photons, the probability requires taking the magnitude squared. This gives another factor of 2^7. The final result for the double log part of the probability amplitude is therefore

$$ -\frac{e^2}{2\pi^2} \ln(\lambda) \ln\left(\frac{\theta_-\theta_+/2}{1 - z_{+-}} \right) \tag{258} $$

---

[7]Specifically, the amplitude has the form $1 - e^2 A + O(e^4)$, so the magnitude squared is $1 - 2e^2 A + O(e^4)$

Since our effective theory automatically disallows soft and collinear emission, this is the only double-log contribution that one should expect for the process at hand. This is different from the full theory where we must combine the 1-loop calculation with real soft and collinear emission to achieve the same result. The familiar result of the corresponding full theory calculation is

$$\frac{e^2}{2\pi^2} \log\left(\frac{\mathcal{E}}{\sqrt{E_1 E_2}}\right) \log\left(\frac{\theta_- \theta_+}{1 - z_{+-}}\right) \tag{259}$$

Comparing (258) and (259), we see that our EFT gives the same result as the full theory if we make the identification

$$\lambda = \frac{\mathcal{E}}{\sqrt{E_1 E_2}} \tag{260}$$

While we do not yet have an obvious justification for this connection, the similarity between the results in the two theories is clearly striking and suggests that our effective theory has some promise. It goes without saying that we wish to have a better understanding that allows us to make a physical connection between our angular cutoff λ and $\mathcal{E}$ along with the jet energies.

8 Some open questions

The above calculations and results show that LSET certainly has some potential as an effective theory for QED in the ultra-relativistic regime. It is not only able to reproduce the full theory result for the probability amplitude for the emission of a real photon, but also gives double logs with a striking resemblance with the corresponding calculation in standard QED. All this suggests that our effective theory is on to something. That said, there are several issues that we would like to understand better in future research. Given below is a brief summary of such questions.

8.1 Finding an exact 2-dimensional theory

Our original motivation was to find an exactly 2-dimensional theory on the light shell. The theory we have constructed here (recall chapter 4) is not exactly a 2-dimensional theory. The LRE Lagrangian and the photon kinetic energy terms are still in (3+1) dimensions. Only the source (recall section 4.2) lies entirely on the sphere. Due to the form of the exponential in the source involving the photon field, the integrals in the radiative corrections 7 were purely angular. But the photon propagator on the sphere had to be obtained by taking the LSG photon propagator in the full (3+1) dimensions and taking both points to be on the sphere. This is different from defining a purely 2-dimensional theory in which a field lies completely on the sphere and contributes to some physical effects.

The σ-model obtained in chapter 2 may potentially be a part of such a purely 2-dimensional theory. However, while the classical physics arguments in that section seem appealing, more work needs to be done in order to connect the σ-model with the quantum theory in regular QED and QCD.

8.2 Understanding the σ-model on the sphere

There are several interesting puzzles involving the σ-model on the sphere. First, before even going to the non-abelian theory and sticking to the simpler case of QED, we need to understand what it physically means to have a theory on the sphere. Specifically, recall that in regular quantum field theory, we carry out canonical quantization by using the equal time commutation relations between the fields and their conjugate momentum fields. That is, if we have a field Φ with the Lagrangian $\mathcal{L}$, then the momentum is given by

$$\Pi = \frac{\Delta \mathcal{L}}{\Delta \partial_t \Phi} \tag{261}$$

But if our theory is purely 2-dimensional in which the only coordinates are θ and ϕ, we cannot have a derivative with respect to t. This means that we cannot do usual quantum field mechanics with the σ-model unless perhaps if we are somehow able to extract the conjugate momentum fields on the sphere from the full theory just as we were able to derive the σ-model from the original QCD theory in (3+1) dimensions. Without figuring out how to do this, we cannot express the fields in terms of creation and annihilation operators and discuss how they create/annihilate particle states. Alternatively, we need to drop the idea of canonical quantization altogether and find a completely different way to interpret the 2-dimensional theory on the sphere in terms of the underlying physics.

8.3 Multiple photon emissions with large energies

In section 5.2 we showed that LSET reproduces the tree-level amplitude for the emission of a single photon. The full amplitude for a process involving some hard scalars and the emission of a real photon factorizes nicely into the amplitude for the process in which only the scalars are produced, and a factor corresponding to the probability of emitting the photon. This is a well-known feature of the full theory, and it is promising that our

effective theory reproduces it.

For the production of more than one real photon along with some hard scalars, our effective theory recipe is to expand the exponential (involving the gauge field) to the required order in e. This means that the multiple photon emission amplitude (calculated in LSET) also factorizes just as was the case for the single photon process. However, in the full theory, this works only if the photon momenta are much smaller than the momenta of the scalar lines from which they are emitted, suggesting that we are missing something in our effective theory.

As a specific example, the two photon emission amplitude in the full theory gets contributions from diagrams in which both photons are emitted from the same scalar. These contributions are proportional to terms like

$$\frac{1}{(p + k + k')^2 + i\epsilon} \frac{1}{(p + k')^2 + i\epsilon} \tag{262}$$

where p is the scalar's momentum, and k and k' are the momenta of the two photons. If k' is much larger than p and $p + k$, then this clearly does not factorize into a piece being equal to the amplitude of the process in which the k' photon is not emitted and the other factor associated with the emission of this additional photon.

Our effective theory in its current form is unable to reproduce such contributions to multiple photon processes in which the photons have large momenta. This suggests that we need some additional terms in our source on the sphere.

8.4 The meaning of angular cutoffs and jets

One big question is how to relate the angular cutoffs which we encountered in the LSET loop calculation in section 7 with energies. Ultimately, we want to have a picture in which we can understand running of the scale. This is a rather tricky issue since our effective theory is in position space and the connection with energies and momenta is not clear. At

the same time, the manner in which the loop calculation produces double logs that very closely resemble the corresponding result in the full theory suggests that the theory is most probably capturing the right physics if only we can find a way to make the connection suggested in (260).

Another thing we would like to understand better is the diagonal integral in the loop calculation. We gave some fairly appealing physical arguments in section 7.2.2 on this. However, we would still like to understand this better.

Since both these questions ultimately relate to jets, it is possible that we might not be able to answer them unless we start thinking in terms of jet algorithms. In either case, more work is needed on this front.

8.5   Understanding LSET in terms of holography

One tempting idea (and certainly a strongly held wish) is that perhaps we could derive the effective theory on the sphere as a realization of the holographic principle. Unfortunately, as far as we know, there is no work in the existing literature that makes headway in exploring this, and it is not obvious to us how to proceed in this direction. A simple idea is to start by trying to express the theory on the 2-dimensional sphere in terms of a KK mode expansion of the fields in the full, 4-d theory, but this proves to be exceedingly difficult even for the simple case of a scalar field.

9 Conclusion

We have constructed light-shell effective theory in the simplified example of zero flavor scalar QED in which matter loops are ignored. The ingredients of the theory include the large radial energy expansion, light-shell gauge and the LSET source. The large radial energy expansion involves breaking apart the scalar and photon fields into soft and hard fields, and scaling out the large radial energy-momenta of the latter by a spherical wave. The light-shell gauge emerges from this expansion as a special gauge in which the interactions of the large radial energy scalars with the photon fields vanish. The leading order physics associated with the source is then described entirely by a gauge invariant source on a spherical light shell having infinitesimal radius.

Our effective theory successfully reproduces the tree-level amplitude for the emission of a photon and the familiar double logs that arise at one-loop order in the full theory, subject to an appropriate choice of angular cutoffs which we would like to understand better. These results are very encouraging and suggest that our effective theory indeed contains some of the important physics of the full theory.

Thus far our analysis has been in the somewhat unrealistic context of zero flavor SQED, but we hope to eventually relax this restriction and incorporate effects arising from matter loops as well as fermionic matter. We also hope to extend this to non-abelian gauge theories, thus having an QCD on the light shell.

The ultimate goal of an effective theory is to provide a simplified description of physics from a more fundamental theory in a particular limit by omitting the unimportant degrees of freedom. The hope is that once all the symmetries and relevant degrees of freedom are identified, the effective theory automatically falls into place. No theory for general high-energy collisions achieves this so simply. SCET's derivation, for example, takes place in a particular subset of gauges [20]. We are also not there yet with LSET since, for example, we are still using the LSG gauge propagator off the light shell as well as on. However, our

results, which are described in this thesis, are very encouraging, and we hope that LSET can eventually provide new insights into the physics of the infrared and collinear effects that plague gauge theories.

A The strange looking fourier transform in spherical coordinates

In section 3.2.2, we used a somewhat unusual looking fourier representation of the delta function without proving that it is valid for our calculation. This was given in (263) and is reproduced here

$$\int_0^\infty \frac{dr}{2\pi} \exp(i(k-k')r) = \delta(k-k') \tag{263}$$

We now proove that this is the correct form in the context of our calculation.

First recall the standard form of a complex fourier series for a function defined on the interval $0 \le x \le L$:

$$f(x) = \sum_{n=-\infty}^{\infty} f_n \exp(2n\pi i x/L) \tag{264}$$

where

$$f_n = \frac{1}{L} \int_0^L dx\, f(x)\, \exp(-2n\pi I x/L) \tag{265}$$

Taking the $L \to \infty$ limit and converting the sum over n into an integral then gives the fourier transform representation for a function $f(x)$ defined on the domain $0 \le x \le \infty$. To see how this works, note that

$$k = 2n\pi/L \tag{266}$$

and therefore,

$$\Delta k = 2\pi \Delta n/L \tag{267}$$

or

$$\Delta n/L = \frac{\Delta k}{2\pi} \tag{268}$$

For $L \to \infty$, $\Delta k \to dk$, and since $\Delta n = 1$, the sum changes into an integral as

$$\sum_n \frac{1}{L} \to \int \frac{dk}{2\pi} \tag{269}$$

Thus for a function $f(x)$ defined on the domain $0 \leq x \leq \infty$, we get

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \; \tilde{f}(k) \; \exp(ikx) \tag{270}$$

where

$$\tilde{f}(k) = \int_{0}^{\infty} dx \; f(x) \; \exp(-ikx) \tag{271}$$

To derive the fourier representation of the delta function when x goes only from 0 to ∞, we can start with the following equation for the $0 \leq x \leq L$ case:

$$\int_{0}^{L} dx \sum_{n=-\infty}^{\infty} \frac{1}{L} \exp(2n\pi i x/L) = 1 \tag{272}$$

This works because for all $n \neq 0$, $\exp(2n\pi x/L)$ gives 0 when integrated over the whole period. This leaves the $n = 0$ term, for which $\exp(2n\pi i x/L) = 1$, and the integral over x from 0 to L gives L, which cancels out the $1/L$ to give unity.

Taking $L \to \infty$ and using (269) to convert the sum into an integral, we get

$$\frac{1}{2\pi} \times \int_{0}^{\infty} dx \int_{-\infty}^{\infty} \exp(ikx) = 1 \tag{273}$$

and therefore

$$\int_{0}^{\infty} \frac{dx}{2\pi} \exp(i(k - k')x) = \delta(k - k') \tag{274}$$

We have thus derived (263).

B   Conventions

We use the metric

$$
g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ & \\ 0 & -I \end{pmatrix}
\tag{275}
$$

Our conventions for gauge transformations are as follows:

$$
D^\mu = \partial^\mu - ie\,A^\mu
\tag{276}
$$

$$
A^\mu \to A^\mu + \frac{1}{e}\,\partial^\mu\Lambda
\tag{277}
$$

$$
\phi(x) \to e^{i\,\Lambda(x)}\,\phi(x)
\tag{278}
$$

The classical electromagnetism calculations have been done in Heaviside-Lorentz units.

C   0-Flavor Scalar Quantum Electrodynamics

We consider sQED with a gauge invariant source of scalars with various charges and flavor quantum numbers at the origin of space and time with a coefficient that goes to infinity as the number of flavors goes to zero so that there is a well-defined limit in which we can ignore matter loops. The details do not matter very much, but for example we could have k scalar fields ϕ_{j,α_j} for $j = 1$ to k with charges q_j and with the flavor label α_j running from 1 to n_j describing n_j (identical) flavors. Our source could then look like

$$\lambda \lim_{\{n \to 0\}} \frac{1}{K(\{n\})} \sum_{\{\alpha\}} \kappa_{\alpha_1 \cdots \alpha_k} \, \phi^*_{1,\alpha_1} \cdots \phi^*_{k,\alpha_k} \tag{279}$$

where

$$\sum_{\{\alpha\}} |\kappa_{\alpha_1 \cdots \alpha_k}|^2 = K(\{n\}) \tag{280}$$

and $K(\{n\}) \to 0$ if any of the n_js vanish. This describes the production of k "0-flavor" scalars. A trivial example is to have two fields with opposite charges and the same flavor and take $\kappa = \delta_{\alpha_1 \alpha_2}$ and $K = n$. We go through this song and dance to assure the reader that we can ignore matter loops in a mathematically consistent limit without otherwise putting important restrictions on the physics. The important thing about such a source is that it produces the charged particles with charges q_j, and henceforth we will drop the flavor indices and just ignore the matter loops.

D   LRE Canonical Quantization

The leading order Lagrangian for LRE scalars is

$$\mathcal{L} = i\Phi_E^\dagger \left(\partial_t + (\hat{r} \cdot \vec{\nabla} + \vec{\nabla} \cdot \hat{r})/2 \right) \Phi_E \tag{281}$$

If we take Φ_E as the canonical field,

$$\Pi_E = i\Phi_E^\dagger \tag{282}$$

and we can write

$$\Phi_E(t, r, \hat{r}) = \int e^{-ik(t-r)} \frac{1}{r} \, a_{LRE}(k, \hat{r}) \frac{dk}{2\pi} \tag{283}$$

with

$$\left[a_{LRE}(k, z, \phi), a_{LRE}^\dagger(k', z', \phi') \right] = 2\pi \, \delta(k - k') \, \delta(z - z') \, \delta(\phi - \phi') \tag{284}$$

so that the a_{LRE}s are annihilation operators and the $a_{LRE}^\dagger$s are creation operators. We can use the properties of the creation and annihilation operators to compute the propagator, (121).

Here Φ_E is a linear combination of annihilation operators. The positive sign of the i in (281) is required to give the right sign of the commutation relation (284). This, in turn is related to having pulled out a factor of $e^{-iE(t-r)}$ from ϕ and a factor of $e^{iE(t-r)}$ from ϕ^*, so we have built in the fact that Φ annihilates the vacuum.

For the LRE analysis, it is crucial to note that k is not a positive energy. It is a residual energy that can have either sign (of course it satisfies $|k| \ll \mathcal{E}$). Because of this, we only need the single term in (283) to give the canonical commutation relations. It is not possible to interpret (284) with only positive energies (at least, not without introducing negative-norm states).

E Operator Notation

Throughout this work, we have used a notation that involves local and non-local operators. For example, when a local operator, such as R^{-1} appears, it is

$$R^{-1}(x_1, x_2) = \frac{1}{r_1}\delta(x_1 - x_2) \tag{285}$$

and when not written, the delta function and integrations over the arguments are implicit. We also come across the operators $\Box^{-1}$, $\left(\partial_t + \hat{R} \cdot \vec{\nabla}\right)^{-1}$ and $\left(\partial_t + \vec{\nabla} \cdot \hat{R}\right)^{-1}$ and their products. We know that $\Box^{-1}$ is the position space propagator for a massless scalar and is given by

$$\Box^{-1}(x - y) = -\frac{i}{4\pi^2}\frac{1}{(x-y)^2 - i\epsilon} \tag{286}$$

As for the other operators, we find their functional forms in the next appendix.

F   Functional forms for inverse operators

Here we will derive the functional forms for the inverse operators $\left(\partial_t + \hat{R} \cdot \vec{\nabla}\right)^{-1}$, $\left(\partial_t + \vec{\nabla} \cdot \hat{R}\right)^{-1}$. This is somewhat subtle because the inverses are not uniquely defined and depend on boundary conditions.

We are first tempted to try canonical quantization just as we did for the LRE scalar. However, it turns out that this does not work well. We will therefore switch to the method of trying an ansatz. Nevertheless, it is somewhat amusing to see why canonical quantization does not work the way it nicely did for the LRE case. We therefore describe this problem in section F.1 (which the disinterested reader can skip) and return to deriving the inverses via the ansatz method in section F.2.

F.1   Trying canonical quantization

(The disinterested reader may skip to section F.2 as this approach does not succeed.)

Consider the theory with the Lagrangian

$$\mathcal{L} = i\Phi^* \left(\partial_t + \hat{r} \cdot \vec{\nabla}\right) \Phi \tag{287}$$

The general solution for Φ is

$$\Phi(t, r, \hat{r}) = \int \frac{dk}{2\pi} a_k(\hat{r}) \exp(-ik(t - r)) \tag{288}$$

The conjugate momentum is

$$\Pi = \partial \mathcal{L} / \partial(\partial_t \Phi) = i\Phi^* = i \int \frac{dk}{2\pi} a_k^\dagger(\hat{r}) \exp(ik(t - r)) \tag{289}$$

These need to satisfy the commutation relation

$$[\Phi(t,\vec{r}),\Pi(t,\vec{r'})] = i\delta^3(\vec{x}-\vec{x'}) \qquad (290)$$

This requires that

$$\frac{1}{r^2}\delta(r-r')\,\delta(z-z')\,\delta(\phi-\phi') = \int \frac{dk}{2\pi}\frac{dk'}{2\pi}\exp(-ik(t-r))\,\exp(ik'(t-r'))\,[a_k(\hat{r}),a_{k'}(\hat{r'})] \quad (291)$$

There is clearly a mismatch between the powers of r on the two sides of this equation, rendering this approach for finding the inverse of $\left(\partial_t + \hat{r}\cdot\vec{\nabla}\right)$ problematic.

We run into the same difficulty if we try to find the inverse of $\left(\partial_t + \vec{\nabla}\cdot\hat{r}\right)$ by considering the theory with the Lagrangian

$$\mathcal{L} = i\Phi^*\left(\partial_t + \vec{\nabla}\cdot\hat{r}\right)\Phi \qquad (292)$$

The general solution for Φ for the theory with this Lagrangian is

$$\Phi(t,r,\hat{r}) = \int \frac{dk}{2\pi r^2}a_k(\hat{r})\,\exp(-ik(t-r)) \qquad (293)$$

and with $\Pi = i\Phi^*$ again, requiring that Φ and Π satisfy the desired commutation relations, we get the condition

$$\frac{1}{r^2}\delta(r-r')\,\delta(z-z')\,\delta(\phi-\phi') = \int \frac{dk}{2\pi r^2}\frac{dk'}{2\pi r'^2}\exp(-ik(t-r))\,\exp(ik'(t-r'))\,[a_k(\hat{r}),a_{k'}(\hat{r'})]$$

$$(294)$$

where there is again a mismatch between the powers of r and r' on the two sides.

This means that the LRE Lagrangian is a special case for which we get the powers of r to match on both sides, allowing us to use canonical quantization. We therefore need to try a different approach for obtaining the functional forms for $\left(\partial_t + \hat{R}\cdot\vec{\nabla}\right)^{-1}$ and $\left(\partial_t + \vec{\nabla}\cdot\hat{R}\right)^{-1}$.

F.2 Using a different approach

First, for $\left(\partial_t + \vec{\nabla} \cdot \hat{R}\right)^{-1}$, recall that $\vec{\nabla} \cdot \hat{r} = \frac{1}{r^2}\partial_r r^2$. We can therefore write an equation of the form

$$\left(\partial_t + \vec{\nabla} \cdot \hat{r}\right)\frac{1}{r}\,\theta(t - t')\,\delta(t - r - t' + r')\,\delta(z - z')\,\delta(\phi - \phi') \tag{295}$$

$$= \frac{1}{r^2}\,\delta(t - t')\,\delta(t - r - t' + r')\,\delta(z - z')\,\delta(\phi - \phi') = \delta^4(x - x') \tag{296}$$

We therefore conclude that our inverse is

$$\left(\partial_t + \vec{\nabla} \cdot \hat{R}\right)^{-1} = \frac{1}{r^2}\,\theta(t - t')\,\delta(t - r - t' + r')\,\delta(z - z')\,\delta(\phi - \phi') \tag{297}$$

Since $\left(\partial_t + \hat{R}\vec{\nabla}\right)^{-1}$ is the adjoint of $\left(\partial_t + \vec{\nabla} \cdot \hat{R}\right)^{-1}$, we get the former's explicit form by writing the adjoint version of (295). The result is

$$(\partial_t + \hat{R} \cdot \vec{\nabla})^{-1} = -\frac{1}{r'^2}\theta(t' - t)\,\delta(t - r - t' + r')\delta(z - z')\delta(\phi - \phi') \tag{298}$$

We use (298) and (297) as the functional forms for these inverse operators.

As an aside, these are not uniquely defined, and we can also write other forms for these inverses by taking different boundary conditions. However, we use (298) and (297) on grounds of consistency. If we instead take

$$\left(\partial_t + \vec{\nabla} \cdot \hat{R}\right)^{-1} = -\frac{1}{r^2}\,\theta(t' - t)\,\delta(t - r - t' + r')\,\delta(z - z')\,\delta(\phi - \phi') \tag{299}$$

and

$$(\partial_t + \hat{R} \cdot \vec{\nabla})^{-1} = \frac{1}{r'^2}\theta(t - t')\,\delta(t - r - t' + r')\delta(z - z')\delta(\phi - \phi'), \tag{300}$$

we are not able to obtain a well-defined product of these operators. Doing so is important because the product is the directional term in the LSG propagator. Specifically, when we multiply (300) and (299) in terms of the required convolution, we get an integral of the

78

form dr/r^2 which blows up at $r = 0$. In contrast, if we take the product of (298) and (297), we find that there are no such problems since the region near $r = 0$ is excluded by the boundary conditions. We therefore drop (300) and (299) and use (298) and (297) as our inverses.

G Derivation of C

We can find C by brute force, but here we will instead use a slicker approach, which will take advantage of (207). Using the formula (206) for M and our result for the propagator (218)-(221), it is straightforward to see that $(I_4 - P)\, M\, (I_4 - P)\, D$ is

$$
\begin{pmatrix}
1 & 0 \\
\vec{\nabla}_\perp \left(\partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} - (I_3 - P_3)\; \square\; C\, \vec{\nabla} \left(\partial_t + \vec{\nabla}^T \hat{R} \right)^{-1} & (I_3 - P_3)\; \square\; C
\end{pmatrix}
\tag{301}
$$

where $P_3 = \hat{R}\,\hat{R}^T$, and we have used $(I_3 - P_3)\, C = C$. For D to be the LSG propagator, we want the 2nd row entries of $(I - P)\, M\, (I - P)\, D$ to be 0 and $I_3 - P_3$. Both these requirements are satisfied if

$$
(I_3 - P_3)\; \square\; C = I_3 - P_3
\tag{302}
$$

We will now use this condition to find an explicit expression for C. Our approach will involve first finding a basis for the space perpendicular to $\hat{R}$, and then acting on (302) with various operators to find the components of C in this basis. We begin by identifying the proper basis. Notice that

$$
R\, \vec{\nabla}_\perp^T = i \left(\vec{L} \times \hat{R} \right)^T
\tag{303}
$$

So $\vec{L}$ and $R\, \vec{\nabla}_\perp$ are both orthogonal to $\hat{R}$ and orthogonal to one another, therefore forming our basis. We can express $(I_3 - P_3)$ in terms of them. First note that from (303) it follows that

$$
R\, \vec{\nabla}_\perp^T \vec{\nabla}_\perp R = -L^2
\tag{304}
$$

80

so with proper normalization we have

$$(I_3 - P_3) = \vec{L} \, L^{-2} \, \vec{L}^T - \vec{\nabla}_\perp \, R \, L^{-2} \, R \, \vec{\nabla}_\perp^T \tag{305}$$

Now we want to find the components of C. The first, and easiest component to find is computed by acting on (302) with $\vec{L}$ on both sides to give

$$\vec{L}^T \, (I_3 - P_3) \, \Box \, C \vec{L} = \vec{L}^T \, (I_3 - P_3) \, \vec{L} \tag{306}$$

This is easy because $\vec{L}$ commutes with $\Box$, so we get

$$\vec{L}^T \, C \, \vec{L} = \Box^{-1} \, L^2 \tag{307}$$

Acting on the left of (302) with $\vec{L}^T$ and on the right with $\vec{\nabla}_\perp \, R$ as follows

$$\vec{L}^T \, (I_3 - P_3) \, \Box \, C \vec{\nabla}_\perp \, R = \vec{L}^T \, (I_3 - P_3) \, \vec{\nabla}_\perp \, R \tag{308}$$

works similarly once we observe $\vec{L}^T \, \vec{\nabla}_\perp \, R = 0$, giving

$$\vec{L}^T \, C \, \vec{\nabla}_\perp \, R = 0 \tag{309}$$

The final two matrix elements require the commutator

$$\left[R \vec{\nabla}_\perp^T , \, \Box \, \right] = 2R^{-2} \, L^2 \, \hat{R}^T \tag{310}$$

We now take a detour to demonstrate this commutator relation. We can write

$$\Box = \partial_t^2 - \left(\vec{\nabla}^T \hat{R} \right) \left(\hat{R}^T \vec{\nabla} \right) + L^2 \, R^{-2} \tag{311}$$

The middle term in (311) can be written

$$\left(\vec{\nabla}^T \hat{R} \right) \left(\hat{R}^T \vec{\nabla} \right) = \left(\vec{\nabla}^T \vec{R} \right) R^{-2} \left(\vec{R}^T \vec{\nabla} \right)$$

$$= \left(\vec{R}^T \vec{\nabla} \right) R^{-2} \left(\vec{R}^T \vec{\nabla} \right) + 3 R^{-2} \left(\vec{R}^T \vec{\nabla} \right) = R^{-2} \left(\left(\vec{R}^T \vec{\nabla} \right)^2 + \left(\vec{R}^T \vec{\nabla} \right) \right)$$

(312)

We chose this particular form because $\left(\vec{R}^T \vec{\nabla} \right)$ is a scaling operator that counts the total powers R or $1/\vec{\nabla}$.[8] So this term commutes with $R \vec{\nabla}_\perp^T$ and the only term in $\square$ that fails to commute is $L^2 R^{-2}$.

The factors of R commute with both $\vec{\nabla}_\perp^T$ and L^2, so we just need to consider

$$\left[R \vec{\nabla}_\perp^T, L^2 \right]$$

(313)

Using (303), we can write this in components, as

$$\left[i \, \epsilon_{abc} L_b \hat{R}_c, L_d L_d \right]$$

(314)

$$= i \, \epsilon_{abc} L_b \left(L_d \left[\hat{R}_c, L_d \right] + \left[\hat{R}_c, L_d \right] L_d \right)$$

(315)

$$= -\epsilon_{abc} \epsilon_{cde} L_b \left(L_d \hat{R}_e + \hat{R}_e L_d \right)$$

(316)

$$= -L_b \left(\left[L_a, \hat{R}_b \right] + 2 \hat{R}_b L_a - 2 L_b \hat{R}_a - \left[\hat{R}_a, L_b \right] \right)$$

(317)

The first and fourth terms in (317) cancel each another. The second term vanishes because $\vec{L} \cdot \hat{R} = 0$. The third term gives

$$\left[R \vec{\nabla}_\perp^T, L^2 \right] = 2 L^2 \, \hat{R}^T$$

(318)

---

[8]Note also that the last form is trivial to remember because it vanishes for r^a with $a = 0$ or -1 as it should.

or

$$\left[R \vec{\nabla}_\perp^T, \, \Box \, \right] = 2 R^{-2} \, L^2 \, \hat{R}^T \tag{319}$$

which is (310).

We now return to the derivation of C, but note that (319) vanishes when acting on C. So, acting with $R\vec{\nabla}_\perp^T$ on the left and $\nabla_\perp R$ on the right gives

$$R \vec{\nabla}_\perp^T \, \Box \, C \vec{\nabla}_\perp R = \Box \, R \vec{\nabla}_\perp^T C \vec{\nabla}_\perp R = -L^2 \tag{320}$$

implying

$$R \vec{\nabla}_\perp^T C \vec{\nabla}_\perp R = - \Box^{-1} L^2 \tag{321}$$

In the same way we can see that the last component is zero

$$R \vec{\nabla}_\perp^T \, \Box \, C \vec{L} = \Box \, R \vec{\nabla}_\perp^T C \vec{L} = 0 \tag{322}$$

Combining (307), (309), (321) and (322) with (305) gives

$$C = -R \vec{\nabla}_\perp \, \Box^{-1} \, L^{-2} \, R \vec{\nabla}_\perp^T + \vec{L} \, \Box^{-1} \, L^{-2} \, \vec{L}^T \tag{323}$$

H Does L^{-2} make sense?

The derivation of C (in appendix G) formally involves the inverse of L^2, and of course this makes no sense on $L = 0$ states. But all we actually need is for (305) to make sense acting on arbitrary functions, so that

$$(I_3 - P_3)\, \vec{f}(\vec{r}) = \vec{L}\, L^{-2}\, \vec{L}^T \vec{f}(\vec{r}) - \vec{\nabla}\, R\, L^{-2}\, R\, \vec{\nabla}^T_{\perp} \vec{f}(\vec{r}) \tag{324}$$

This is perfectly well-defined, because if either the $\vec{L}^T \vec{f}(\vec{r})$ or $R\, \vec{\nabla}^T_{\perp} \vec{f}(\vec{r})$ component has zero angular momentum, then that component itself is zero. This can been seen by first noting that if L^2 acting on either of these components is zero, then the component must be a function of the radius only, call it $g(r)$. If we integrate $g(r)$ over $d\Omega$, we get $4\pi g(r)$, but at the same time we see that integrating either component over $d\Omega$ must be zero because in both cases we are integrating a total derivative over a closed surface. Therefore $g(r)$, which denotes either component, is necessarily zero.

I Useful results involving C

Here we compute $\vec{\nabla}^T C \vec{\nabla}$, $\vec{\nabla}^T C$ and $C \vec{\nabla}$, which we need to get our final results for the propagator. Recall that C was given in (223), which we reproduce here for convenience

$$C = -R \vec{\nabla}_\perp \, \Box^{-1} \, L^{-2} \, R \, \vec{\nabla}_\perp^T + \vec{L} \, \Box^{-1} \, L^{-2} \, \vec{L}^T \tag{325}$$

We first take

$$\vec{\nabla}^T C \vec{\nabla} = - \vec{\nabla}^T R \vec{\nabla}_\perp \, \Box^{-1} \, L^{-2} \, R \, \vec{\nabla}_\perp^T \vec{\nabla} + \vec{\nabla}^T \vec{L} \, \Box^{-1} \, L^{-2} \, \vec{L}^T \vec{\nabla} \tag{326}$$

For the first term, it is a straightforward exercise to see that

$$\vec{\nabla}^T R \vec{\nabla}_\perp = \vec{\nabla} \cdot r \vec{\nabla}_\perp = r \vec{\nabla}_\perp^2 = -\frac{L^2}{r} \tag{327}$$

and

$$\vec{\nabla}^T \vec{\nabla} = \nabla_\perp^2 = -\frac{L^2}{r^2} \tag{328}$$

For the second term, we note that

$$\vec{\nabla}^T \vec{L} = \vec{\nabla} \cdot \vec{L} = \vec{\nabla} \cdot (\vec{r} \cdot \vec{\nabla}) = 0 \tag{329}$$

And likewise,

$$\vec{L}^T \vec{\nabla} = (\vec{r} \times \vec{\nabla}) \cdot \vec{\nabla}) = 0 \tag{330}$$

Plugging (327), (328), (329) and (330) into (326) then gives

We then have

$$\vec{\nabla}^T C \vec{\nabla} = -\frac{1}{R} L^2 \, \Box^{-1} \, L^{-2} \, R \, \frac{L^2}{R^2} \tag{331}$$

$$= -L^2 \, R^{-1} \, \Box^{-1} \, R^{-1} \tag{332}$$

Likewise, inserting (327) and (329) into (325) gives

$$\vec{\nabla}^T C = R^{-1} \, \Box^{-1} \, \vec{\nabla}_\perp^T R \tag{333}$$

and similarly, inserting (328) and (330) into (325) gives

$$C \vec{\nabla} = R \vec{\nabla}_\perp \, \Box^{-1} \, R^{-1} \tag{334}$$

J Last step for the LSG propagator on the sphere

Here we derive the result (244) from (243). Using (298), we get

$$R\left(\partial_t + \hat{R}\cdot\vec{\nabla}\right)^{-1} R^{-1} = -\theta(t_2 - t_1)\,\delta(t_1 - r_1 - t_2 + r_2)\,\delta(z_1 - z_2)\,\delta(\phi_1 - \phi_2)\frac{r_1}{r_2^3} \qquad (335)$$

Now, if the point on the left goes on the $t_1 = r_1 = s$ sphere, this becomes

$$\to -\theta(t_2 - s)\,\delta(r_2 - t_2)\,\delta(z_1 - z_2)\,\delta(\phi_1 - \phi_2)\frac{s}{r_2^3} \qquad (336)$$

Likewise, using (297), we get

$$R^{-1}\left(\partial_t + \vec{\nabla}\cdot\hat{R}\right)^{-1} R = \frac{r_4}{r_3^3}\theta(t_3 - t_4)\,\delta(t_3 - r_3 - t_4 + r_4)\,\delta(z_3 - z_4)\,\delta(\phi_3 - \phi_4) \qquad (337)$$

and if the point on the right goes on the $t_4 = r_4 = s$ sphere, this goes to

$$\to \frac{s}{r_3^3}\theta(t_3 - s)\,\delta(t_3 - r_3)\,\delta(z_3 - z_4)\,\delta(\phi_3 - \phi_4) \qquad (338)$$

The last piece in (243) is $\Box^{-1}$, which is given by

$$\Box^{-1}(x_2 - x_3) = -\frac{i}{4\pi^2}\frac{1}{(x_2 - x_3)^2 - i\epsilon} = -\frac{i}{4\pi^2}\frac{1}{x_2^2 - x_3^2 - i\epsilon} \qquad (339)$$

Now, putting (336), (338) and (339) together, we get

$$4i\,L^2\,R\,(\partial_t + \hat{R}\cdot\vec{\nabla})^{-1}\,R^{-1}\,\Box^{-1}\,R^{-1}\,(\partial_t + \vec{\nabla}\cdot\hat{R})^{-1}R \qquad (340)$$

$$\to -\frac{1}{\pi^2}\,L^2\int d^4x_2\,d^4x_3\,\delta(r_2 - t_2)\,\delta(t_3 - r_3)\,\theta(t_2 - s)\,\theta(t_3 - s) \qquad (341)$$

$$\times \frac{s}{r_2^3}\frac{1}{(x_2 - x_3)^2 - i\epsilon}\frac{s}{r_3^3} \qquad (342)$$

Due to the delta functions, we get $(x_2-x_3)^2 = x_2^2 + x_3^2 - 2x_2 \cdot x_3 \to -2x_2 \cdot x_3 \to -2r_2 r_3 (1 - z_{23})$.

We therefore get

$$= \frac{1}{2\pi^2} L^2 \int dr_2 \, dr_3 \, \theta(r_2 - s) \, \theta(r_3 - s) \times \frac{s^2}{r_2^2 r_3^2} \frac{1}{1 - z_{23}} \tag{343}$$

Lastly, integrating over r_2 and r_3 gives

$$= \frac{1}{2\pi^2} L^2 \frac{1}{1 - z_{23}} \tag{344}$$

K The last integral in the loop calculation

This integral was

$$\int \frac{z_{jk} - z_{2j}z_{2k}}{(1 - z_{2j})(1 - z_{2k})} d\Omega_2 \tag{345}$$

This can be computed using a number of methods. For example, we can rewrite the numerator as

$$z_{jk} - z_{2j}z_{2k} = -(1 - z_{2j})(1 - z_{2k}) + (1 - z_{2j}) + (1 - z_{2k}) + (z_{jk} - 1) \tag{346}$$

In the denominator, we can take $1 - z + \lambda_{jk}$. This is only as a computational device and not as a cutoff insertion since we will see that the final result does not depend on λ_{jk}.

The first term in the numerator completely cancels the denominator, so its integral gives a constant and therefore it does not contribute to the double logs. The second and third terms involve cancellation of one of the denominators and we therefore get a trivial integral

$$\int \frac{1}{1 - z_{2k} + \lambda_{jk}} d\Omega_2 = -2\pi \ln(\lambda_{jk}/2) \tag{347}$$

So the 2nd and 3rd terms together give a total of $-4\pi \ln(\lambda_{jk})$.

For the 4th term, we can combine denominators:

$$
\begin{aligned}
(z_{jk} - 1) & \int \frac{1}{(1 - z_{2j} + \lambda_{jk})(1 - z_{2k} + \lambda_{jk})} d\Omega_2 \\
& = (z_{jk} - 1) \int d\Omega_2 \, d\alpha \frac{1}{[\alpha(1 - z_{2j} + \lambda_{jk}) + (1 - \alpha)(1 - z_{2k} + \lambda_{jk})]^2}
\end{aligned}
\tag{348}
$$

$$= (z_{jk} - 1) \int d\alpha \, d\Omega_2 \frac{1}{[1 - \hat{r}_2 \cdot (\alpha \hat{r}_j + (1 - \alpha)\hat{r}_k) + \lambda_{jk}]^2} \tag{349}$$

Doing the angular integral gives

$$= 4\pi(z_{jk} - 1)\int d\alpha \frac{1}{(1 + \lambda_{jk})^2 - (\alpha\hat{r}_j + (1 - \alpha)\hat{r}_k)^2} \tag{350}$$

which is equal to

$$= 2\pi(z_{jk} - 1)\int d\alpha \frac{1}{(\alpha - \alpha^2)(1 - z_{jk}) + \lambda_{jk} + \lambda_{jk}^2/2} \tag{351}$$

Next, integrating over α and extracting the log term gives

$$= 4\pi \ln \frac{\lambda_{jk}}{1 - z_{jk}} \tag{352}$$

The $\ln(\lambda_{jk})$ part in this cancels the result from the second and third terms in the numerator, and the logarithmic part of our integral's result is

$$\int \frac{z_{jk} - z_{2j}z_{2k}}{(1 - z_{2j})(1 - z_{2k})} d\Omega_2 = 4\pi \ln \frac{1}{1 - z_{jk}} + \text{non-logarithmic} \tag{353}$$

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