The Geometry of the Weil-Petersson Metric in Complex Dynamics

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The geometry of the Weil-Petersson metric in complex dynamics

A dissertation presented

by

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Abstract

In this work, we study an analogue of the Weil-Petersson metric on the space of Blaschke products of degree 2 proposed by McMullen. We show that the Weil-Petersson metric is incomplete and study its metric completion. Our work parallels known results for the Teichmüller space of a punctured torus.

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1. Introduction

In this work, we study an analogue of the Weil-Petersson metric on the space of Blaschke products of degree 2 proposed in [McM2]. We show that the Weil-Petersson metric is incomplete and study its metric completion. Our work parallels known results for the Teichmüller space of a punctured torus.

1.1. **The traditional Weil-Petersson metric.**

To set the stage, we recall the definition and basic properties of the Weil-Petersson metric on Teichmüller space. Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of marked Riemann surfaces of genus g with n punctures. For a Riemann surface $X \in \mathcal{T}_{g,n}$, let

$Q(X)$ be the space of holomorphic quadratic differentials with $\int_X |q| < \infty$ and

$M(X)$ be the space of measurable Beltrami coefficients satisfying $||\mu||_\infty < \infty$.

There is a natural pairing between quadratic differentials and Beltrami coefficients given by integration $\langle \mu, q\rangle = \int_X \mu q$. One has natural identifications $T^*_X\mathcal{T}_{g,n} \cong Q(X)$ and $T_X\mathcal{T}_{g,n} \cong M(X)/Q(X)^\perp$. We will discuss two natural metrics on Teichmüller space: the Teichmüller metric and the Weil-Petersson metric. For a quadratic differential $q \in Q(X)$, let $||q||_T = \int_X |q|$ and $||q||^2_{\mathrm{WP}} = \int_X \rho^{-2}|q|^2$ where ρ is the hyperbolic metric on X. The Teichmüller and Weil-Petersson lengths of tangent vectors are defined by duality, i.e. $||\mu||_T := \sup_{||q||_T=1}\left|\int_X \mu q\right|$ and $||\mu||_{\mathrm{WP}} := \sup_{||q||_{\mathrm{WP}}=1}\left|\int_X \mu q\right|$.

The Teichmüller and Weil-Petersson metrics are invariant under the mapping class group $\mathrm{Mod}_{g,n}$. Unlike the Teichmüller metric, the Weil-Petersson metric is not complete.

For the Teichmüller space of a punctured torus $\mathcal{T}_{1,1} \cong \mathbb{H}$, the mapping class group is $\mathrm{Mod}_{1,1} \cong \mathrm{SL}(2,\mathbb{Z})$. Let us denote the Weil-Petersson metric on $\mathcal{T}_{1,1}$ by $\omega_T(z)|dz|$. To describe the metric completion of $(\mathcal{T}_{1,1}, \omega_T)$, we need a system of disjoint horoballs. Let $B_{1/0}(\eta)$ denote the horoball $\{z : \mathrm{Im}\ y \geq 1/\eta\}$ that rests on $\infty = 1/0$ and $B_{p/q}(\eta)$ denote the horoball of Euclidean diameter η/q^2 that rests on p/q. For a fixed $\eta \geq 0$,

$\bigcup_{p/q\in\mathbb{Q}\cup\{\infty\}} B_{p/q}(\eta)$ is an $\mathrm{SL}(2,\mathbb{Z})$-invariant collection of horoballs. When $\eta = 1$, the horoballs have disjoint interiors but many mutual tangencies. We denote the boundary horocycles by $H_{p/q}(\eta) := \partial B_{p/q}(\eta)$ and $H_{1/0}(\eta) := \partial B_{1/0}(\eta)$.

Consider $\mathbb{H}$ with the usual topology. Extend this topology to $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ by further requiring $\{B_{p/q}(\eta)\}_{\eta\geq 0}$ to be open sets for $p/q \in \mathbb{Q} \cup \{\infty\}$. Let us also consider a family of incomplete $\mathrm{SL}(2,\mathbb{Z})$-invariant model metrics ρ_α on the upper half-plane: for $\alpha > 0$, let ρ_α be the unique $\mathrm{SL}(2,\mathbb{Z})$-invariant metric which coincides with the hyperbolic metric $|dz|/y$ on $\mathbb{H} \setminus \bigcup_{p/q\in\mathbb{Q}\cup\{\infty\}} B_{p/q}(1)$ and is equal to $|dz|/y^{1+\alpha}$ on $B_{1/0}(1)$.

Lemma 1.1. *For $\alpha > 0$, the metric completion of $(\mathbb{H}, \rho_\alpha)$ is homeomorphic to $\mathbb{H}^*$.*

Sketch of proof. To see that the irrational points are infinitely far away in the ρ_α metric, notice that the horoballs $B_{p/q}(2)$ cover the upper half-plane while by $\mathrm{SL}(2,\mathbb{Z})$-invariance, the distance between $H_{p/q}(2)$ and $H_{p/q}(3)$ is bounded below in the ρ_α metric. Therefore, any path γ that tends to an irrational number must pass through infinitely many protective shells $B_{p/q}(3) \setminus B_{p/q}(2)$. In fact, this argument shows that an incomplete path γ is trapped within some horoball $B_{p/q}(3)$, from which it follows that it must eventually enter arbitrarily small horoballs. By the form of ρ_α in $B_{p/q}(1)$, it is easy to see that the completion attaches only one point to the cusp at p/q. □

Theorem 1.1 (Wolpert). *The Weil-Petersson metric on $\mathcal{T}_{1,1}$ is comparable to $\rho_{1/2}$, i.e. $1/C \leq \omega_T/\rho_{1/2} \leq C$ for some $C \geq 0$.*

Corollary. *The metric completion of $(\mathcal{T}_{1,1}, \omega_T)$ is homeomorphic to $\mathbb{H}^*$.*

For background on Teichmüller theory and more information on the Weil-Petersson metric, we refer the reader to the books [Hub], [IT] and [Wol].

1.2. **Main results.** In this thesis, we replace the study of Fuchsian groups with complex dynamical systems on the unit disk $\mathbb{D} = \{z : |z| < 1\}$. Inspired by Sullivan's dictionary, we are interested in understanding the Weil-Petersson metric on the space

$$\mathcal{B}_2 = \left\{ \begin{array}{c} f : \mathbb{D} \to \mathbb{D} \text{ is a proper degree 2 map} \\ \text{with an attracting fixed point} \end{array} \right\} \Big/ \text{conjugacy by } \mathrm{Aut}(\mathbb{D}) \tag{1.1}$$

The multiplier at the attracting fixed point $a : f \to f'(p)$ gives a holomorphic isomorphism $\mathcal{B}_2 \cong \mathbb{D}$. By putting the attracting fixed point at the origin, we can parametrize $\mathcal{B}_2$ by

$$a \in \mathbb{D} : \qquad z \to f_a(z) = z \cdot \frac{z + a}{1 + \overline{a}z}. \tag{1.2}$$

All degree 2 Blaschke products are quasi-symmetrically conjugate to each other on the unit circle, and except for the special map $z \to z^2$, they are quasi-conformally conjugate on the entire disk. For this reason, it is somewhat simpler to work with $\mathcal{B}_2^\times := \mathcal{B}_2 \setminus \{z \to z^2\}$, the quasi-conformal moduli space $\mathcal{M}(f)$ of a rational map described in [MS].

Given a map $f \in \mathcal{B}_2^\times \cong \mathbb{D}^*$, an f-invariant Beltrami coefficient on the unit disk $\mu \in M(\mathbb{D})^f$ defines a tangent vector in $\mathcal{T}_f\mathcal{B}_2$. An f-invariant Beltrami coefficient descends to a Beltrami coefficient on the quotient torus of the attracting fixed point: $M(\mathbb{D})^f \cong M(T_f)$. According to [MS], μ defines a trivial deformation in $\mathcal{B}_2^\times$ if and only if it defines a trivial deformation of $T_f \in \mathcal{T}_{1,1}$. With this correspondence, $\mathcal{T}_{1,1}$ is naturally the universal cover of $\mathcal{B}_2^\times$. We can pullback the Weil-Petersson metric ω_B on $\mathcal{B}_2$ by $a(\tau) := e^{2\pi i\tau}$ to obtain a metric on $\mathcal{T}_{1,1} \cong \mathbb{H}$, which we also denote ω_B.

Conjecture. The metric ω_B on $\mathcal{T}_{1,1} \cong \mathbb{H}$ is comparable to $\rho_{1/4}$ on $\{\tau : \operatorname{Im}\tau < 1\}$. In particular, the metric completion of $(\mathcal{T}_{1,1}, \omega_B)$ is homeomorphic to $\mathbb{H}^*$.

In this thesis, we show that $1/4$ is the correct exponent in the conjecture above. More precisely, we show that:

Theorem 1.2. *The Weil-Petersson metric ω_B on $\mathcal{T}_{1,1} \cong \mathbb{H}$ satisfies:*

(*a*) $\omega_B \le C\rho_{1/4}$.

(*b*) *There exists $C_{\mathrm{small}} > 0$ such that on $\bigcup_{p/q\in\mathbb{Q}} B_{p/q}(C_{\mathrm{small}})$, $\omega_B \ge C\rho_{1/4}$.*

Corollary. *The Weil-Petersson metric on $\mathcal{B}_2$ is incomplete. In fact, the Weil-Petersson length of the line segment $e(p/q) \cdot [1-\delta, 1)$ is finite.*

Corollary. *The space $\mathbb{H}^*$ naturally embeds into the completion of $(\mathcal{T}_{1,1}, \omega_B)$.*

Remark. The cusp at infinity is somewhat special: for $y > 1$,

$$(1/C)e^{-y}|dz| \le w_B \le Ce^{-y}|dz|.$$

Along radial rays $a \to e(p/q)$, we have a more precise estimate:

Theorem 1.3. *For every rational number $p/q \in \mathbb{Q}$, there exists a constant $C_{p/q}$ such that as $\tau = p/q + it \to p/q$, $\omega_B/\rho_{1/4} \to C_{p/q}$.*

Conjecture. We conjecture that $C_{p/q}$ is a universal constant, independent of p/q.

1.3. **Properties of the Weil-Petersson metric.** In this section, we give a definition of the Weil-Petersson metric on $\mathcal{B}_2^\times \subset \mathcal{B}_2$ in the form most useful for our later work. In Section 1.6, we will give equivalent definitions which work on the entire space $\mathcal{B}_2$. For example, we will describe the Weil-Petersson metric as the second derivative of the Hausdorff dimension of certain Julia sets.

It is convenient to put the Beltrami coefficient on the exterior unit disk. For a Beltrami coefficient $\mu \in M(\mathbb{D})$, we let μ^+ denote the "reflection" of μ in the unit circle:

$$\mu^+ = \begin{cases} 0 & \text{for } z \in \mathbb{D} \\ (1/\overline{z})^*\mu & \text{for } z \in S^2 \setminus \mathbb{D} \end{cases} \tag{1.3}$$

Suppose $X \in \mathcal{T}_{g,n}$ is a Riemann surface and $\mu \in M(X)$ is a Beltrami coefficient. If $X \cong \mathbb{D}/\Gamma$, we can consider μ as Γ-invariant Beltrami coefficient on the unit disk. Let v be a solution of $\overline{\partial} v = \mu^+$. Since the set of all solutions is of the form $v + \mathrm{sl}(2, \mathbb{C})$, the third derivative v''' uniquely depends on μ^+. Since v''' is an infinitesimal version of the Schwarzian derivative, it is naturally a quadratic differential. In [McM2], McMullen observed that

$$||\mu||^2_{\mathrm{WP}} = \lim_{r \to 1^-} \frac{4}{3} \cdot \frac{1}{2\pi} \int_{|z|=r} \left| \frac{v'''(z)}{\rho(z)^2} \right|^2 d\theta. \tag{1.4}$$

Similarly, given a Blaschke product $f \in \mathcal{B}_2^{\times}$, we can solve the equation $\overline{\partial} v = \mu^+$ for $\mu \in M(\mathbb{D})^f$. As above, a solution v of the equation $\overline{\partial} v = \mu^+$ is well-defined up to adding a holomorphic vector field in $\mathrm{sl}(2, \mathbb{C})$, and so v''' is uniquely defined. Following [McM2], we *define* the Weil-Petersson metric $||\mu||^2_{\mathrm{WP}}$ using the integral average (1.4), provided that the limit exists. In Chapter 7, we will show that the limit exists for all degree 2 Blaschke products other than $z \to z^2$.

1.4. **A glimpse of incompleteness as $a \to 1$ radially.** In this section, we sketch the proof of the upper bound in Theorem 1.2. To establish the incompleteness of the Weil-Petersson metric, we consider "half-optimal" Beltrami coefficients $\mu_\lambda \cdot \chi_{\mathcal{G}(f_a)}$ which take up half the attracting torus, but are sparse near the unit circle.

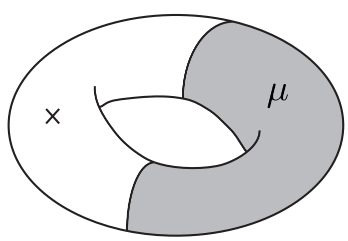

FIGURE 1. The support of the Beltrami coefficient takes up half of the quotient torus.

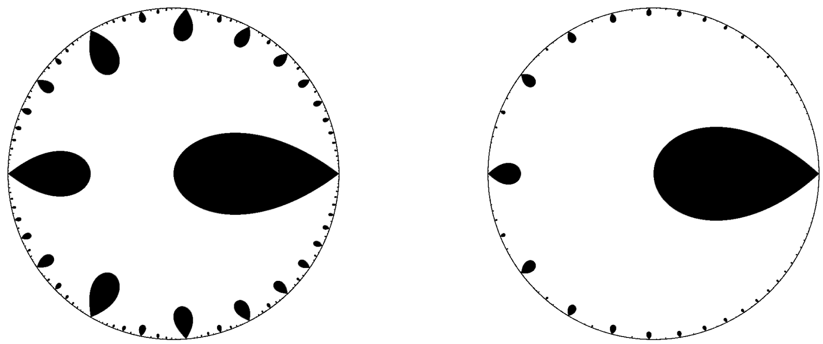

FIGURE 2. Gardens $\mathcal{G}(f_a)$ for the Blaschke products with $a = 0.5$ and 0.8.

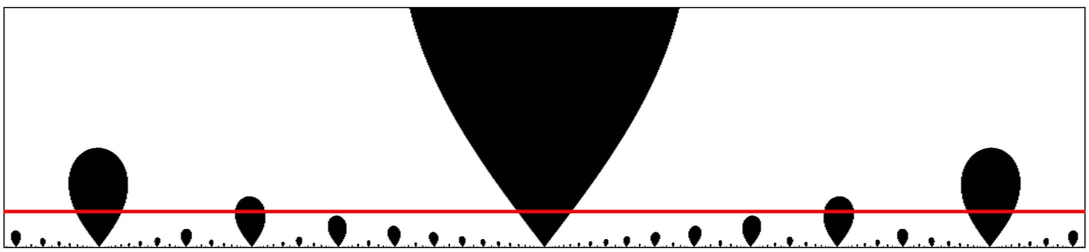

FIGURE 3. A blow-up of $\mathcal{G}(f_{0.5})$ near the boundary. A circle $\{z : |z| = r\}$ with r close to 1 meets $\mathcal{G}(f_{0.5})$ in small density.

The garden $\mathcal{G}(f_a) \subset \mathbb{D}$ is a certain invariant subset of the unit disk. To construct the garden $\mathcal{G}(f_a)$, we pick an annulus $A = \mathcal{G}(f_a)/f_a \subset T_a$ which takes up half of the Euclidean area of the quotient torus at the attracting fixed point. To give upper bounds for the Weil-Petersson metric, we will estimate the length of the intersection of $\mathcal{G}(f_a)$ with $S_r := \{z : |z| = r\}$. We will show that

$$\left(\frac{\omega_B}{\rho_{\mathbb{D}^*}}\right)^2 \leq C \cdot \limsup_{r\to 1} |\mathcal{G}(f_a) \cap S_r| \tag{1.5}$$

In order for the estimate (1.5) to be efficient, we take A to be a collar neighbourhood of the shortest p/q-geodesic in the quotient torus $T_a^\times$. To prove part (a) of Theorem 1.2, we will show that for $a = e^{2\pi i\tau}$ with $\tau \in H_{p/q}(\eta)$,

$$\limsup_{r\to 1} |\mathcal{G}(f_a) \cap S_r| = O(\eta^{1/2}). \tag{1.6}$$

Combining (1.5) and (1.6), we see that $\omega_B \leq C\rho_{1/4}$ as desired.

Remark. The trick of truncating the support of the Beltrami coefficient can be found in the proof of Corollary 1.3 in [McM1].

1.5. **A glimpse of the convergence** $\omega_B/\rho_{1/4} \to C_{p/q}$**.** In this section, we give a sketch of the proof of Theorem 1.3. To understand the behaviour of the Weil-Petersson metric as $a \to e(p/q)$ radially, we study the convergence of Blaschke products to vector fields. For example, as $a \to 1$ along the real axis, while the maps $f_a(z) = z \cdot \frac{z+a}{1+\bar{a}z}$ tend pointwise to the identity, the long-term dynamics tends to the flow of a holomorphic vector field $\kappa_1 = z \cdot \frac{z-1}{z+1} \cdot \frac{\partial}{\partial z}$. For the radial approach $a \to e(p/q)$, the maps $f_a(z) \to az$ converge pointwise to a rotation, and therefore $f_a^{\circ q}(z)$ tends to the identity. We can extract a limiting vector field $\kappa_{p/q}$ by taking limits of the high iterates of $f_a^{\circ q}$. It turns out that the limiting vector field $\kappa_{p/q}$ is a q-fold cover of the vector field κ_1.

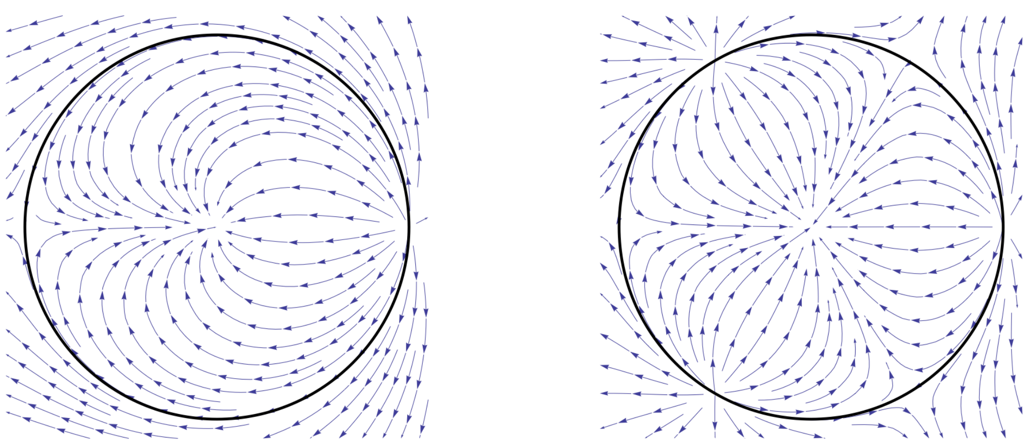

FIGURE 4. The vector fields κ_1 and $\kappa_{1/3}$.

From the convergence of Blaschke products to vector fields, it follows that the flowers that make up the gardens $\mathcal{G}(f_a)$ for $a \approx e(p/q)$ have nearly the same affine shape. We use this to show that $||\mu_\lambda \cdot \chi_{\mathcal{G}(f_a)}||^2_{\mathrm{WP}}$ is proportional to the "flower count" $\lim_{r\to 1} \frac{n(r,f_a)}{1-r}$ where $n(r, f_a)$ is the number of flowers that intersect the circle S_r. By renewal theory, $\lim_{r\to 1} \frac{n(r,f_a)}{1-r} \sim C'_{p/q} \cdot (1-|a|)^{1/2}$ as $a \to e(p/q)$.

Remark. Intuitively, for the integral average (1.4) to exist, when we replace $r = 1-\delta$ by $r = 1 - \delta/2$ say, we expect to intersect twice as many flowers to "replenish" the integral, i.e. we expect that the number of flowers is inversely proportional to δ.

1.6. **Notes and references.** In this section, we describe the space of Blaschke products of higher degree and equivalent definitions of the Weil-Petersson metric.

Blaschke products of higher degree. Similar to $\mathcal{B}_2$, we can define the space $\mathcal{B}_d$ of marked degree d Blaschke products which have an attracting fixed point modulo conformal conjugacy. By moving the attracting fixed point to the origin as before, we can parametrize $\mathcal{B}_d$ by

$$\{a_1, a_2, \dots, a_{d-1}\} \in \mathbb{D} : \qquad z \to f_{\mathbf{a}}(z) = z \cdot \prod_{i=1}^{d-1} \frac{z + a_i}{1 + \overline{a_i} z}. \tag{1.7}$$

We let $a = a_1 a_2 \cdots a_{d-1} = f'_{\mathbf{a}}(0)$ be the multiplier of the attracting fixed point. It is because the maps are *marked* that we can distinguish the conformal conjugacy classes of $\mathbf{a} = \{a_1, a_2, \dots, a_{d-1}\}$ and $\zeta \cdot \mathbf{a} = \{\zeta a_1, \zeta a_2, \dots, \zeta a_{d-1}\}$. See [McM3] for more on markings.

Mating. It is a remarkable fact that given two Blaschke products $f_{\mathbf{a}}, f_{\mathbf{b}}$, one can find a rational map $f_{\mathbf{a},\mathbf{b}}(z)$ – the *mating* of $f_{\mathbf{a}}, f_{\mathbf{b}}$ – whose Julia set is a quasi-circle $\mathcal{J}_{\mathbf{a},\mathbf{b}}$ which separates the Riemann sphere into two domains Ω_-, Ω_+ such that on one side $f_{\mathbf{a},\mathbf{b}}(z)$ is conformally conjugate to $f_{\mathbf{a}}$, and to $f_{\overline{\mathbf{b}}}$ on the other. The mating is unique up to conjugation by a Möbius transformation. One can prove the existence of a mating by quasi-conformal surgery (see [Mil2] for details) and that the mating $\mathcal{B}_d \times \mathcal{B}_d \to \mathrm{Rat}_d$ varies holomorphically with parameters. A natural way to put a complex structure on $\mathcal{B}_d$ is via the *Bers embedding* $\mathcal{B}_d \to \mathscr{P}_d$ which takes a Blaschke product and mates it with z^d to obtain a polynomial of degree d. Here the space $\mathscr{P}_d \cong \mathbb{C}^{d-1}$ is considered modulo affine conjugacy. The image of the Bers embedding is the generalized main cardioid in $\mathscr{P}_d$.

Question. What is the completion of $\mathcal{B}_d$ with respect to the Weil-Petersson metric? Are the additional points precisely the geometrically finite parameters on the boundary of the generalized main cardioid? What is the topology on $\overline{\mathcal{B}_d}$?

Remark. Wolpert showed that the metric completion of $(\mathcal{T}_{g,n}, \omega_T)$ is the augmented Teichmüller space $\overline{\mathcal{T}_{g,n}}$, the action of the mapping class group $\operatorname{Mod}_{g,n}$ extends isometrically to $(\overline{\mathcal{T}_{g,n}}, \omega_T)$ and the quotient $M_{g,n} = \overline{\mathcal{T}_{g,n}} / \operatorname{Mod}_{g,n}$ is the Deligne-Mumford compactification.

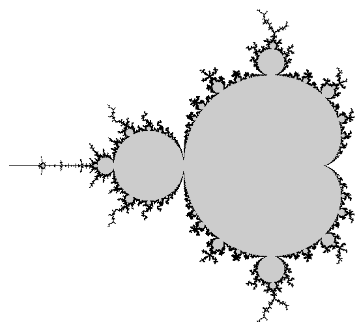

FIGURE 5. The Mandelbrot set

Equivalent definitions of the Weil-Petersson metric. For a smooth path $\{f_t\}$ in $\mathcal{B}_d$, one can form the vector field $v = dH_{0,t}/dt|_{t=0}$ where $H_{0,t} : \mathbb{D} \to \Omega_-(f_{0,t})$ is the conformal conjugacy between f_0 and $f_{0,t}$. For a Blaschke product other than $z \to z^d$, one can define $||\dot{f}_t||^2_{\mathrm{WP}}$ by the integral average (1.4), while for $z \to z^d$, one can use a more complicated integral average described in [McM2].

Remark. The definition of the Weil-Petersson metric via mating is slightly more general than the one via quasi-conformal conjugacy given earlier because quasi-conformal deformations do not exhaust the entire tangent space $T_f\mathcal{B}_d$ at the special parameters $f \in \mathcal{B}_d$ that have critical relations.

In [McM2], McMullen showed that

$$||\dot{f}_t||^2_{\mathrm{WP}} = \frac{\mathrm{Var}(\dot{\phi}, m)}{\int \log|\phi'|dm} = \frac{4}{3}\cdot \frac{d^2}{dt^2}\Big|_{t=0} \mathrm{H.dim}\,\mathcal{J}_{0,t} \tag{1.8}$$

$$= -\frac{1}{3}\cdot\frac{d^2}{dt^2}\Big|_{t=0}\mathrm{H.dim}(H_{t,t})_* m \tag{1.9}$$

where

$\mathcal{J}_{0,t}$ is the Julia set of $f_{0,t}$,

$H_{t,t} : S^1 \to S^1$ is the conjugacy between f_0 and f_t on the unit circle,

$(H_{t,t})_* m$ is the push-forward of the Lebesgue measure,

$\phi_t = \log|f'_{0,t}(H_{0,t}(z))|$,

$\int \log|\phi'|dm$ is the Lyapunov exponent,

$\mathrm{Var}(h, m) := \lim_{n\to\infty}\int|S_n h(x)|^2 dm$ denotes the "asymptotic variance" in the context of dynamical systems.

Remark. Since $\mathcal{J}_{0,t}$ is a Jordan curve, $\mathrm{H.dim}\,\mathcal{J}_{0,t} \geq 1$, so $\frac{d}{dt}|_{t=0}\,\mathrm{H.dim}\,\mathcal{J}_{0,t} = 0$ and $\frac{d^2}{dt^2}|_{t=0}\,\mathrm{H.dim}\,\mathcal{J}_{0,t} \geq 0$. Similarly, since $(H_{t,t})_* m$ is a measure supported on the unit circle, $\mathrm{H.dim}(H_{t,t})_* m \leq 1$, $\frac{d}{dt}|_{t=0}\,\mathrm{H.dim}(H_{t,t})_* m = 0$ and $\frac{d^2}{dt^2}|_{t=0}\,\mathrm{H.dim}(H_{t,t})_* m \leq 0$.

1.7. Related ideas and open questions.

Quasi-conformal geometry. The characterizations (1.8) and (1.9) of the Weil-Petersson metric are reflected in quasiconformal geometry in the duality between quasi-conformal expansion and quasi-symmetric compression.

Theorem 1.4 (Smirnov [S])**.** *For a k-quasi-conformal map $f : S^2 \to S^2$,*

$$\mathrm{H.dim}\, f(S^1) \leq 1 + k^2.$$

Remark. If the dilatation $\mu(z) = \frac{\overline{\partial f}}{\partial f}$ is supported on the exterior unit disk, one has the stronger estimate $\mathrm{H.dim}\, f(S^1) \leq 1 + \tilde{k}^2$ where $k = \frac{2\tilde{k}}{1+\tilde{k}^2}$.

Theorem 1.5 (Smirnov, Prause [PrS])**.** *For a k-quasi-conformal map $f : S^2 \to S^2$, symmetric with respect to the unit circle, one has* $\mathrm{H.dim}\, f_* m \ge 1 - k^2$.

From (1.8) and (1.9), it is easy to deduce *weaker* forms of the infinitesimal statements of Theorems 1.4 and 1.5 in the dynamical setting, i.e. $\mathrm{H.dim}\, f(S^1) \le 1 + Ck^2$ and $\mathrm{H.dim}\, f_* m \ge 1 - Ck^2$ with a constant $C > 1$. Conversely, using either Theorem 1.4 or Theorem 1.5, it is easy to see that:

Corollary. *The Weil-Petersson metric on $\mathcal{B}_2$ is bounded above by $\sqrt{1/6} \cdot \rho_{\mathbb{D}}$.*

Proof. For a map $f_a \in \mathcal{B}_2$, the Bers embedding β_{f_a} gives a holomorphic motion of the exterior unit disk $H_a : \mathcal{B}_2 \times \mathbb{D}^+ \to \mathbb{C}$ given by $H_a(b, z) := H_{b,a}(z)$. Note that the motion H_a is centered at a since $H_a(a, \cdot)$ is the identity. By the λ-lemma (e.g. see [AIM, Theorem 12.3.2]), one can extend H_a to a quasi-conformal motion $\tilde{H}_a$ of the Riemann sphere satisfying $||\mu_{\tilde{H}_a(b,\cdot)}||_\infty \le \frac{b-a}{1-\overline{a}b}$. Observe that as $\rho(b, a) \to 0$, $\frac{b-a}{1-\overline{a}b} \sim \frac{1}{2} \cdot \rho_{\mathbb{D}}(b, a)$. Since $\tilde{H}_a(b, \cdot)$ is conformal on the exterior unit disk, by the remark following Theorem 1.4, it follows that $||\dot{f}_t||^2_{\mathrm{WP}} \le \frac{1}{6} \cdot ||\dot{f}_t||^2_{\rho_{\mathbb{D}}}$ as desired. □

The pressure metric. In the context of complex dynamics, the expression

$$||\dot{\phi}||_P^2 := \frac{\mathrm{Var}(\dot{\phi}, m)}{\int \log |\phi'| dm}$$

appeared in the works [PUZ1], [PUZ2] which is based on the earlier work of Makarov [Ma] on the law of the iterated logarithm of harmonic measure. It was also studied on spaces of metric graphs in [PoS] and in higher Teichmüller theory in [BCLS].

Why degree 2 ? In this thesis, we stick to the degree 2 case for concreteness. Many arguments presented here extend almost verbatim to $\mathcal{B}_d$, or even to spaces of infinite degree maps – for example, to spaces of universal covering maps of finite complements (while the forward orbits of these infinite degree maps are very wild near the unit circle, backward iteration is nearly affine).

Some useful notation. Let $\mathcal{B}_{p/q}(\eta) := a(B_{p/q}(\eta))$. For small $\eta > 0$, $\mathcal{B}_{p/q}(\eta)$ is approximately a horoball in the unit disk of Euclidean diameter $2\pi\eta$ resting on $e(p/q)$. For $z_1, z_2 \in \mathbb{D}$, let $d_{\mathbb{D}}(z_1, z_2) = \inf \int_\gamma \rho$ denote the hyperbolic distance between z_1 and z_2, and $[z_1, z_2]$ denote the hyperbolic geodesic connecting z_1 and z_2. To compare quantities, we use:

* $A \lesssim B$ means that $A < \text{const} \cdot B$
* $A \sim B$ means that $A/B \to 1$
* $A \asymp B$ means that $C_1 \cdot B < A < C_2 \cdot B$ for some constants C_1, C_2
* $A \approx_\epsilon B$ means that $|A/B - 1| \lesssim \epsilon$

(For the convenience of the reader, we provide a full index of notation at the back of the thesis.)

2. Background in Analysis

In this chapter, we explain how to bound the integral average (1.4) in terms of the density of the support of a Beltrami coefficient. We also discuss a version of Koebe's distortion theorem for maps that preserve the unit circle.

2.1. **Teichmüller theory in the disk.** For a Beltrami coefficient μ, let $v(z) = v_\mu(z)$ be a solution of the equation $\overline{\partial} v = \mu$. The following formula is well-known (e.g. see [IT, Theorem 4.37]):

$$v'''(z)dz^2 = \left(-\frac{6}{\pi}\int_{\mathbb{C}} \frac{\mu(\zeta)}{(\zeta - z)^4}|d\zeta|^2\right)dz^2. \tag{2.1}$$

for $z \notin \text{supp}\,\mu$. To obtain upper bounds for the Weil-Petersson metric, we will use the following estimate:

Theorem 2.1. *Suppose μ is a Beltrami coefficient with $||\mu||_\infty < 1$ supported on the exterior of the unit disk. Then,*

$$\limsup_{r\to 1} \frac{1}{2\pi}\int_{|z|=r}\left|\frac{v_\mu'''(z)}{\rho(z)^2}\right|^2 d\theta \;\le\; \frac{9}{4}\cdot ||\mu||_\infty^2 \cdot \limsup_{r\to 1^+} \left|\text{supp}\,\mu \cap S_r\right|. \tag{2.2}$$

Lemma 2.1. *For $z_1, z_2 \in \mathbb{C}$ and $\gamma \in \text{Aut}(\mathbb{D})$,*

$$\frac{\gamma'(z_1)\cdot\gamma'(z_2)}{(\gamma(z_1)-\gamma(z_2))^2} = \frac{1}{(z_1 - z_2)^2}. \tag{2.3}$$

For a point z in the unit disk, let z^+ denote its mirror image with respect to S^1. From formula (2.1), it is easy to see that:

Theorem 2.2. *Suppose μ is a Beltrami coefficient with $||\mu||_\infty < 1$ supported on the exterior of the unit disk. Then,*

(a) $|v'''/\rho^2| \le 3/2 \cdot ||\mu||_\infty$.

(b) *If* $\rho(z^+,\ supp(\mu)) \ge R$ *then* $|(v'''/\rho^2)(z)| \lesssim e^{-R}$.

(c) v'''/ρ^2 *is uniformly continuous in the hyperbolic metric.*

Proof. By Möbius invariance (Lemma 2.1), it suffices to prove these assertions at the origin. Clearly,

$$|v'''(0)| \leq \frac{6}{\pi} \int_{|\zeta|>1} \frac{1}{|\zeta|^4} \cdot |d\zeta|^2 \leq 12 \int_1^\infty \frac{dr}{r^3} = 6.$$

Hence $|v'''/\rho^2(0)| \leq \frac{3}{2}$. This proves (a). For (b), recall that $\rho(0,z) = -\log(1-|z|) + O(1)$. Then,

$$|v'''(0)| \leq \frac{6}{\pi} \int_{1+Ce^{-R}>|\zeta|>1} \frac{1}{|\zeta|^4} \cdot |d\zeta|^2 \lesssim e^{-R}.$$

For (c), one needs to notice that the kernel $\frac{1}{(\zeta-z)^4}$ is uniformly continuous at $z=0$ for $\{\zeta : |\zeta| > 1\}$. □

We now prove Theorem 2.1:

Proof of Theorem 2.1. Let $V_\mu := \frac{6}{\pi} \int \frac{|\mu(z)|}{|\zeta - z|^4}$. The proof of part (a) of Theorem 2.2 shows that $|V_\mu/\rho^2| \leq 3/2 \cdot ||\mu||_\infty$. Define $\mu_\theta := |\mu(e^{-i\theta}z)|$ and $\mu_* := \frac{1}{2\pi}\int \mu_\theta(z) d\theta$. Since $||\mu_*||_\infty \leq ||\mu||_\infty \cdot\ \limsup_{r\to1^+}\bigl|\operatorname{supp}\mu \cap S_r\bigr|$,

$$\frac{1}{2\pi}\int_{|z|=r} \left|\frac{V_\mu(z)}{\rho(z)^2}\right| d\theta = \frac{1}{2\pi}\int_{|z|=r} \left|\frac{V_{\mu_*}(z)}{\rho(z)^2}\right| d\theta \leq \frac{3}{2}\cdot = ||\mu||_\infty \cdot\ \limsup_{r\to1^+}\bigl|\operatorname{supp}\mu\cap S_r\bigr|.$$

Equation (2.2) follows from the Cauchy-Schwarz inequality. □

2.2. **A distortion theorem.** The classical Koebe's distortion theorem says that if $h : B(0,1) \to \mathbb{C}$ is univalent, then $|h'(z) - 1| \lesssim |z|$. We will need a version of Koebe's distortion theorem for maps which preserve the real line or the unit circle:

Theorem 2.3. *Suppose $h : B(0,1) \to \mathbb{C}$ is a univalent function which satisfies $h(0) = 0$, $h'(0) = 1$ and takes real values on $(-1,1)$. For $t < t_0$ sufficiently small, on the ball $B(0,t)$, h is nearly an isometry in the hyperbolic metric, i.e. $h^*\rho_{\mathbb{H}} \approx_t \rho_{\mathbb{H}}$.*

Here "$A \approx_\epsilon B$" denotes that $|A/B - 1| \lesssim \epsilon$. For a set $E \subset B(0,t)$, we call a set of the form $h(E)$ a *t-nearly affine copy* of E.

Sketch of Proof. Write $z = x + iy$. By the classical version of Koebe's distortion theorem, we see that $|h'(x) - 1| \lesssim t$. Applying the classical Koebe's distortion again, but this time centered at x, we see that $h(x + iy) \approx_t h(x) + iy$. $\square$

For two points $z_1, z_2 \in \mathbb{H}$, let $d_{\mathbb{H}}(z_1, z_2) := \inf \int_\gamma \rho$ denote the hyperbolic distance between z_1 and z_2. We note two useful consequences of Theorem 2.3:

Lemma 2.2. *For two points $z_1, z_2 \in B(0, t) \cap \mathbb{H}$, $d_{\mathbb{H}}(z_1, z_2) = d_{\mathbb{H}}(h(z_1), h(z_2)) + O(t)$.*

Proof. To see this, consider the geodesic γ that connects z_1 and z_2. We partition γ into several pieces: $\gamma_n := \gamma \cap \{w : t/2^{n+1} \le \operatorname{Im} w < t/2^n\}$. Each γ_n consists of at most two geodesic segments of hyperbolic length $O(1)$. By Theorem 2.3,

$$\left| \int_{h(\gamma_n)} \rho - \int_{\gamma_n} \rho \right| = O(t/2^n).$$

Summing over $n = 0, 1, 2, \ldots$, we see that $d_{\mathbb{H}}(h(z_1), h(z_2)) < d_{\mathbb{H}}(z_1, z_2) + O(t)$. The reverse inequality may be obtained by applying this argument to h^{-1}. $\square$

Lemma 2.3. *The map h distorts the Euclidean area of a ball*

$$B_{\text{hyp}}(z, R) := \{w : d_{\mathbb{H}}(w, z) < R\}$$

contained in $B(0, t) \cap \mathbb{H}$ by a multiplicative factor of at most $1 + C(R) \cdot t$.

Remark. In the above lemma, we can replace "Euclidean area" with "hyperbolic area" or "area with respect to the volume form $|dz|^2/y$".

Suppose μ is a Beltrami coefficient supported on the half-ball $B(0, 1) \cap \mathbb{H}$. Set $\mu_h := h^*\mu = \mu(h(z)) \cdot \frac{\overline{h'(z)}}{h'(z)}$. It is easy to see that on the half-ball $B(0, t) \cap \mathbb{H}$, $|\mu_h(h(z)) - \mu(z)| \lesssim t \cdot ||\mu||_\infty$. Slightly less evident is the fact that:

Lemma 2.4. *On the lower half-ball $B(0,t) \cap \overline{\mathbb{H}}$, we have:*

$$\left| \frac{v'''_\mu}{\rho^2}(z) - \frac{v'''_{\mu_h}}{\rho^2}(h(z)) \right| \lesssim \phi_1(t) \cdot ||\mu||_\infty \tag{2.4}$$

where $\phi_1(t) \to 0$ as $t \to 0$.

Proof. By Lemma 2.3, for any $R > 0$, we can choose $t > 0$ sufficiently small so that h distorts the hyperbolic area on the ball $B_{\text{hyp}}(z, R)$ by an arbitrarily small multiplicative factor and h^{-1} distorts the hyperbolic area on the ball $B_{\text{hyp}}(h(z), R)$ by an arbitrarily small multiplicative factor. This observation implies equation (2.4) with μ replaced by $\mu \cdot \chi_{B_{\text{hyp}}(z,R)}$. However, by part (b) of Theorem 2.2, the contributions of $\mu \cdot \chi_{B_{\text{hyp}}(z,R)^c}$ and $\mu_h \cdot \chi_{B_{\text{hyp}}(h(z),R)^c}$ to $v_\mu/\rho^2(z)$ and $v_{\mu_h}/\rho^2(h(z))$ respectively are exponentially small in R. This completes the proof. □

Applications to Blaschke products. We will apply Koebe's distortion theorem to the inverse branches of Blaschke products. For a Blaschke product $f \in \mathcal{B}_d$, set $\delta_c := 1 - \max_c |c|$ where c ranges over the critical points. By the Schwarz lemma, for a point $\zeta \in S^1$, the ball $B(\zeta, \delta_c)$ is disjoint from the post-critical set, and therefore all possible inverse branches f^{-n} are well-defined univalent functions.

Define the "linearity zones" $U_t := \{z : 1 - t \cdot \delta_c \le |z| < 1\}$ for $t \le 1$. For Blaschke products, we have the following version of Lemma 2.4:

Lemma 2.5. *If μ is an invariant Beltrami coefficient supported on the exterior unit disk, and if the orbit $z \to f(z) \to \dots f^{\circ n}(z)$ is contained in some U_t with $t < t_0$ sufficiently small, then:*

$$\left| \frac{v'''_\mu}{\rho^2}(z) - \frac{v'''_\mu}{\rho^2}(f^{\circ n}(z)) \right| \lesssim \phi_2(t) \cdot ||\mu||_\infty \tag{2.5}$$

where $\phi_2(t) \to 0$ as $t \to 0$.

Lemma 2.5 follows from part (b) of Theorem 2.2 and Lemma 2.4.

3. Blaschke Products

In this chapter, we give background information on Blaschke products. We discuss the quotient torus at the attracting fixed point, and special repelling periodic orbits called "simple cycles" on the unit circle. In the next chapter, we will examine the interface between these two objects.

3.1. **Attracting tori.** The dynamics of forward orbits of a Blaschke product

$$f_a(z) = z \cdot \frac{z+a}{1+\overline{a}z} \tag{3.1}$$

is very simple: all points in the unit disk are attracted to the origin. If the multiplier of the attracting fixed point $a \neq 0$, near the origin, the linearizing coordinate $\varphi_a(z) := \lim_{n\to\infty} a^{-n} \cdot f_a^n(z)$ conjugates f_a to multiplication by a. This means that

$$\varphi_a(f_a(z)) = a \cdot \varphi_a(z). \tag{3.2}$$

In fact, (3.2) determines φ_a uniquely up to the normalization $\varphi_a'(0) = 1$.

Let $\Omega^\times$ denote the unit disk with the grand orbits of the attracting fixed and critical point removed. From the existence of the linearizing coordinate, it is easy to see that the quotient $\hat{\varphi}_a : \Omega^\times \to T_a^\times := \Omega/(f_a)$ is a torus with one puncture. We denote the underlying closed torus by T_a. Let $\pi_a : \mathbb{C}^* \to T_a \cong \mathbb{C}^*/(\cdot\, a)$ denote the intermediate covering map defined implicitly by $\hat{\varphi}_a = \pi_a \circ \varphi_a$.

Remark. For a Blaschke product $f_{\mathbf{a}} \in \mathcal{B}_d$ with $a = f'(0) \neq 0$, the quotient torus $T_{\mathbf{a}}^\times$ has at most $(d-1)$ punctures but there could be less if there are critical relations. We let $\mathcal{B}_d^\times \subset \mathcal{B}_d$ denote the space of Blaschke products for which $T_{\mathbf{a}}^\times \in \mathcal{T}_{1,d-1}$.

3.2. **Multipliers of simple cycles.** On the unit circle, a Blaschke product has many repelling periodic orbits or cycles. Since all Blaschke products of degree 2 are quasi-symmetrically conjugate on the unit circle, we can label the periodic orbits of f_a by the corresponding periodic orbits of $z \to z^2$.

A cycle is *simple* if f preserves its cyclic ordering. In this case, we say that $\langle \xi_1, \xi_2, \dots, \xi_q \rangle$ has *rotation number* p/q if $f(\xi_i) = \xi_{i+p \pmod q}$.

Examples of cycles of degree 2 Blaschke products:

- $(1,2)/3$ has rotation number $1/2$,
- $(1,2,4)/7$ has rotation number $1/3$,
- $(1,2,4,3)/5$ is not simple.

In degree 2, for every fraction $p/q \in \mathbb{Q}/\mathbb{Z}$, there is a unique simple cycle of rotation number p/q. We denote its multiplier by $m_{p/q} := (f^{\circ q})'(\xi_1)$ which is a positive real number since Blaschke products preserve the unit circle. It is sometimes more convenient to work with $L_{p/q} := \log(f^{\circ q})'(\xi_1)$ which is an analogue of the length of a closed geodesic of a hyperbolic Riemann surface.

To show lower bounds for the Weil-Petersson metric in small horoballs $B_{p/q}(C_{\text{small}})$, we will use the fact that the multiplier of the p/q-cycle changes at a "definite rate" when moving in a direction *transverse* to the horocycle $H_{p/q}(\eta)$:

Theorem 3.1. *There exists a constant $C_{\text{small}} > 0$ such that for $\tau \in H_{p/q}(\eta)$ with $\eta < C_{\text{small}}$,*

(i) $m_{p/q} - 1 \sim (2\pi\eta)/2$ *as* $\eta \to 0$,

(ii) $\frac{d}{dv} \log m_{p/q} \asymp \frac{d\eta}{dv}$ *where* $v \in T_\tau \mathbb{H}^2$ *is a vector orthogonal to* $H_{p/q}(\eta)$.

Theorem 3.1 is essentially found in [McM4]. For the convenience of the reader, we will give a sketch of the arguments in Chapter 9. The main idea is to compare the "petal correspondence" with the holomorphic index formula.

4. Petals and Flowers

In this chapter, we give an overview of petals, flowers and gardens. As suggested by the terminology, gardens are made of flowers, and flowers are made of petals. We begin this section by giving a general definition of gardens, but then we specify to "half-flower gardens" which will be used throughout this thesis.

In fact, for a Blaschke product $f_a \in \mathcal{B}_2^\times$, one can draw infinitely many half-flower gardens $\mathcal{G}_{p/q}(\log a^q)$ – one for every choice of rotation number p/q and a choice of logarithm $\tau_q := \log a^q$. However, for $a \in \mathcal{B}_{p/q}(C_{\text{small}}) := a(B_{p/q}(C_{\text{small}}))$, the "correct" garden is $\mathcal{G}(f_a) := \mathcal{G}_{p/q}(\tau_q)$ with $\tau_q \approx 0$ – it is for this choice of half-flower garden that an estimate of the form (1.6) holds.

For example, when studying radial degenerations with $a \to 1$, it is natural to use gardens where flowers have only one petal (see Figure 2). However, for other parameters, it is more natural to use gardens where the flowers have more petals (see Figure 6 below).

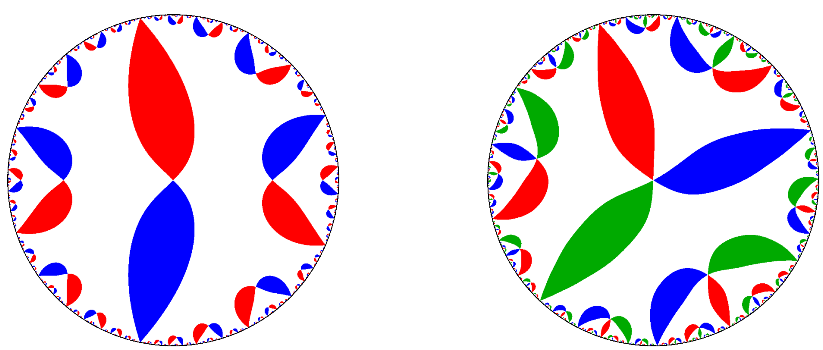

Figure 6. The gardens $\mathcal{G}_{1/2}(f_{-0.6})$ and $\mathcal{G}_{1/3}(f_{0.66 \cdot e^{2\pi i/3}})$.

4.1. **Curves on the quotient torus.** Inside the first homotopy group $\pi_1(T_a, *) \cong \mathbb{Z} \oplus \mathbb{Z}$, there is a canonical generator α represented by "counter-clockwise" loops $\hat{\varphi}_a(\epsilon \cdot e^{i\theta})$ with ϵ sufficiently small. By a *neutral* curve, we mean a curve whose homotopy class in $\pi_1(T_a, *)$ is an integral power of α. We classify all non-neutral curves as either *incoming* or *outgoing*.

A curve $\gamma \subset T_a$ is *outgoing* if every lift $\gamma_i^* = \pi_a^{-1}\gamma$ satisfies

$$\gamma_i^*(t+1) = (1/a)^q \cdot \gamma_i^*(t) \qquad \text{for some } q \geq 1.$$

In other words, a curve is outgoing if $\gamma^*(t) \to \infty$ as $t \to \infty$. A curve is *incoming* if the opposite holds, i.e. if $\gamma^*(t) \to 0$ as $t \to \infty$.

A complementary (out-going) generator β is canonically defined up to an integer multiple of α. In terms of the basis $\{\alpha, \beta\}$, we say that an out-going curve $(q-p)\alpha + p\beta$ has rotation number p/q. If we don't specify the choice of β, then p/q is only well-defined modulo 1.

4.2. **Lifting out-going curves.** Suppose γ is a simple closed outgoing curve in $T_a^\times$ of rotation number p/q mod 1. Its has q lifts to $\mathbb{C}^*$ under the projection $\pi_a : \mathbb{C}^* \to T_a$, which we denote $\gamma_1^*, \gamma_2^*, \dots \gamma_q^*$. The γ_i^* are "spirals" that join 0 to ∞. Each individual spiral is invariant under multiplication by a^q. We index the spirals so that multiplication by a sends γ_i^* to γ_{i+p}^*. Let $\tilde{\gamma}_i := \varphi_a^{-1}(\gamma_i^*)$ be (further) lifts in the unit disk emanating from the attracting fixed point.

Lemma 4.1. *Suppose γ is a simple closed outgoing curve in T_a of rotation number p/q. Then, $\tilde{\gamma}_i$ joins the attracting fixed point at the origin to a repelling periodic point $\xi_i \in S^1$ of of rotation number number p/q.*

Proof. Pick a point z_1 on $\tilde{\gamma}_i$, and approximate $\tilde{\gamma}_i$ by the backwards orbit of $f^{\circ q}$: $z_1 \leftarrow z_2 \leftarrow \cdots \leftarrow z_n \leftarrow \dots$ By the Schwarz lemma, the backwards orbit is eventually contained in some $U_t = \{z : 1 - t \cdot \delta_c \leq |z| < 1\}$, i.e. $z_n \in U_t$ for $n \geq N$. Since a Blaschke product is asymptotically affine, the hyperbolic distance between successive points $d_{\mathbb{D}}(z_n, z_{n+1})$ is bounded and hence z_n converges to some point ξ on the unit circle. The same argument shows that the hyperbolic length of the arc of $\tilde{\gamma}_i$ from z_n to z_{n+1} is bounded, and therefore $\tilde{\gamma}_i$ converges to ξ as well. Since $f(\tilde{\gamma}_i) = \tilde{\gamma}_{i+p}$, we see that $f(\xi_i) = \xi_{i+p}$. □

4.3. **Definitions of petals and flowers.** An annulus $A \subset T_a^\times$ homotopy equivalent in $T_a^\times$ to an out-going geodesic of rotation number p/q has q lifts in the unit disk emanating from the origin. We call these lifts *petals* and denote them $\mathcal{P}_{A_i}$, $i = 1, 2, \ldots, q$. Each petal connects the attracting fixed point to a repelling periodic point. A *flower* is the union of petals: $\mathcal{F} = \bigcup_{i=1}^q \mathcal{P}_{A_i}$. We refer to the attracting fixed point as the *center* of the flower and to the repelling periodic points as the *ends*. By construction, flowers are forward-invariant regions. The *garden* is the invariant region obtained by taking the union of all the repeated pre-images of the flower:

$$\mathcal{G} = \hat{\mathcal{F}} := \bigcup_{n=0}^{\infty} f_a^{-n}(\mathcal{F}).$$

We shall refer to (iterated) pre-images of petals and flowers as *pre-petals* and *pre-flowers* respectively. In degree 2, a flower has two pre-images: itself and an *immediate pre-flower* which we denote $\mathcal{F}_*$. Each pre-flower has two distinct pre-images. We define the centers and ends of pre-flowers as the pre-images of centers and ends of the flower. We typically label by a pre-petal by its end and a pre-flower by its center.

4.4. **Half-flower gardens.** An out-going homotopy class $[\gamma] \in \pi_1(T_a, *)$ determines a foliation of the quotient torus T_a by parallel lines. More precisely, we first foliate the punctured plane $\mathbb{C}^*$ by logarithmic spirals that are invariant under multiplication by a^q:

$$\gamma_\theta^* := \{e^{t \log a^q} \cdot e^{i\theta} : t \in [-\infty, \infty)\}$$

where the choice of $\log a^q$ is determined by $[\gamma]$. We then foliate the torus T_a by "lines" $\gamma_\theta := \pi_a(\gamma_\theta^*)$. By construction, $\gamma_\theta = \gamma_{\theta + 2\pi/q}$. We say that γ_θ is *regular* if it is contained in $T_a^\times$ and *singular* if it passes through a puncture. The singular lines partition the T_a into annuli; the lifts of which we call *whole petals*. (In degree 2, there is one singular line).

If a whole petal $\mathcal{P}^1$ consists of linearizing rays with arguments in $(\theta_1, \theta_2) = (\frac{x-y}{2}, \frac{x+y}{2})$, define the *α-petal* $\mathcal{P}^\alpha$ to consist of the linearizing rays with arguments in $(\frac{x-\alpha y}{2}, \frac{x+\alpha y}{2})$. By default, we take $\alpha = 1/2$ and we write $\mathcal{P} = \mathcal{P}^{1/2}$. We define the half-flower $\mathcal{F}$ as the union of all half-petals. It consists precisely of half the linearizing rays.

Remark. For the rest of the thesis, we use this system of flowers. When working with $a \approx e(p/q)$, we let $\mathcal{F} = \mathcal{F}_{p/q}$ denote the flower constructed from a foliation of the quotient torus by p/q-curves, arising from the choice of $\log a^q \approx \log 1 = 0$.

5. Quasiconformal Deformations

In this chapter, we describe the Teichmüller metric on $\mathcal{B}_2^\times$ and define pinching deformations. We also define the half-optimal Beltrami coefficients, which are supported on the half-flower gardens defined in the previous chapter.

For a Beltrami coefficient μ with $||\mu||_\infty < 1$, let w_μ be the quasi-conformal map fixing 0, 1, ∞ whose dilatation is μ. For a Beltrami coefficient supported on the unit disk (the exterior unit disk) define the symmetrized version w^μ to be the quasi-conformal map which has dilatation $\mu(z)$ on the unit disk (the exterior unit disk) and is symmetric with respect to inversion in the unit circle.

Given a rational map $f(z)$ and an invariant Beltrami coefficient $\mu \in M(S^2)^f$, we can form new rational maps by: $f_t = w_{t\mu} \circ f_0 \circ (w_{t\mu})^{-1}$. For a Blaschke product $f \in \mathcal{B}_d$, given $\mu \in M(\mathbb{D})^f$, we often use the symmetric deformations $f_t = w^{t\mu} \circ f_0 \circ (w^{t\mu})^{-1}$ so that $f_t \in \mathcal{B}_d$; however, the asymmetric deformations $f_{s,t} := w_{\mu_{s,t}} \circ f \circ (w_{\mu_{s,t}})^{-1}$ with $\mu_{s,t} := s\mu + (t\mu)^+$ are also useful. The formula for the variation of the multiplier of a rational map will play a fundamental role in this work:

Lemma 5.1 (e.g. Theorem 8.3 of [IT])**.** *Suppose $f_0(z)$ is a rational map with a fixed point at p_0 which is either attracting or repelling. If μ is an f-invariant Beltrami coefficient, $f_t = w^{t\mu} \circ f_0 \circ (w^{t\mu})^{-1}$ has a fixed $p_t = w_{t\mu}(p_0)$ and*

$$\frac{d}{dt}\Big|_{t=0} \log f_t'(p_t) = \pm \frac{1}{\pi} \cdot \int_{T_{p_0}} \frac{\mu(z)}{z^2} \cdot |dz|^2, \tag{5.1}$$

where T_{p_0} is the quotient torus at p_0. The sign is "+" in the repelling case and "−" in the attracting case.

Remark. Lemma 5.1 is a statement purely about the Teichmüller space $\mathcal{T}_1 \cong \mathbb{H}$ of the quotient torus. In fact, the right hand side of (5.1) is nothing more than the pairing $\langle \mu, \pm q\rangle$ where $q = \frac{1}{2\pi} \cdot \frac{dz^2}{z^2}$ is the unique quadratic differential on T_{p_0} satisfying $||q||_{T_{p_0}} = \int |q| = 1$.

5.1. **Teichmüller metric.** As noted in the introduction, $\mathcal{T}_{1,1}$ is the universal cover of $\mathcal{B}_2^\times$ arising from taking a Blaschke product to its quotient torus $T_a^\times \in \mathcal{T}_{1,1}$. The Teichmüller metric on $\mathcal{B}_2$ makes this correspondence a local isometry. More precisely, for a Beltrami coefficient $\mu \in M(\mathbb{D})^{f_a}$ representing a tangent vector in $T_{f_a}\mathcal{B}_2^\times$, we set

$$||\mu||_{T(\mathcal{B}_2)} := ||\hat{\varphi}(\mu)||_{T(\mathcal{T}_{1,1})}. \tag{5.2}$$

A well-known result of Royden says that the Teichmüller metric on $\mathcal{T}_{1,1}$ is equal to the Kobayashi metric; therefore, the Teichmüller metric on $\mathcal{B}_2^\times$ is half the hyperbolic metric on $\mathcal{B}_2^\times \cong \mathbb{D}^*$. (We use the convention that the hyperbolic metric on the unit disk is $\rho_{\mathbb{D}} = \frac{2|dz|}{1-|z|^2}$ while the Kobayashi metric is $\frac{|dz|}{1-|z|^2}$.)

Recall that for a tangent vector $v \in T_{T_a^\times}\mathcal{T}_{1,1}$, the *Teichmüller coefficient* associated to v is the unique Beltrami coefficient of minimal L^∞ norm which represents v. In particular, this implies that $||\mu||_T = ||\mu||_\infty$. It is well known that Teichmüller coefficients have the form $\lambda \cdot \overline{q}/|q|$ with $q \in Q(T_a^\times)$ where $Q(T_a^\times)$ is the set of integrable holomorphic quadratic differentials on the punctured torus $T_a^\times$.

Let $Q(T_a) \subset Q(T_a^\times)$ be the set of integrable holomorphic quadratic differentials on the closed torus T_a. If $\pi : \mathbb{C}^* \to \mathbb{C}^*/(\cdot\, a)$ denotes the projection map, then the Teichmüller coefficients on T_a are $\{\pi(\mu_\lambda^*),\ \lambda \in \mathbb{C}\}$ where $\mu_\lambda^* = \lambda \cdot \frac{w}{\overline{w}} \cdot \frac{d\overline{w}}{dw}$. Therefore, $\mu_\lambda := \varphi^*(\mu_\lambda^*)$ are invariant Beltrami coefficients on the unit disk. We refer to the μ_λ as the *optimal* Beltrami coefficients. Here, "optimal" is short for "multiplier-optimal" which refers to the fact that the fact that μ_λ maximizes $(d/dt)|_{t=0} \log a_t$ out all Beltrami coefficients with L^∞-norm λ (cf. Lemma 5.1).

Remark. For a degree 2 Blaschke product, the quotient torus $T_a^\times \in \mathcal{T}_{1,1}$ and so $Q(T_a) = Q(T_a^\times)$. For a Blaschke product $f_{\mathbf{a}} \in \mathcal{B}_d^\times$ of degree $d \geq 2$, the quotient torus has $(d-1)$ punctures, and so $Q(T_{\mathbf{a}}) \subsetneq Q(T_{\mathbf{a}}^\times)$. Therefore, the optimal Beltrami coefficients represent only a complex 1-dimensional set of directions in $T_{T_{\mathbf{a}}^\times}\mathcal{T}_{1,d-1}$.

Given a half-flower garden $\mathcal{G}(f_a)$, we define the *half-optimal Beltrami coefficient* to be $\mu_\lambda \cdot \chi_{\mathcal{G}}$. Using Lemma 5.1, is easy to see that:

Lemma 5.2. *The half-optimal pinching coefficients $\mu_\lambda \cdot \chi_{\mathcal{G}}$ are half as effective as the optimal pinching coefficients μ_λ, i.e. the map $f_t(\mu) := w^{t\mu} \circ f_0 \circ (w^{t\mu})^{-1}$ is conformally conjugate to $f_{2t}(\mu \cdot \chi_{\mathcal{G}}) := w^{2t\mu\cdot\chi_{\mathcal{G}}} \circ f_0 \circ (w^{t\mu\cdot\chi_{\mathcal{G}}})^{-1}$.*

5.2. **Pinching coefficients.** It is a natural to endow a closed torus $X \in \mathcal{T}_1$ with the flat (Euclidean) metric of area 1. Given a Euclidean geodesic $\gamma \subset X \in \mathcal{T}_1$, we define the pinching deformation $\{X_t\}_{t\geq 0}$ as "the most efficient deformation" that shrinks the Euclidean length of γ. More precisely, $X_t \in \mathcal{T}_1 \cong \mathbb{H}$ is the marked Riemann surface with $d_T(X, X_t) = t$ for which $\ell_{X_t}(\gamma)$ is minimal (where d_T is the Teichmüller distance in $\mathcal{T}_1$).

If we write $X = X_\tau = \mathbb{C}/\langle 1, \tau\rangle$ with $\tau \in \mathbb{H}$, a pinching deformation is a geodesic in $\mathcal{T}_{1,1} \cong \mathbb{H}$ which joins τ to a number $p/q \in \mathbb{Q} \cup \{\infty\}$ determined by $[\gamma]$. Alternatively, if we represent $X \cong \mathbb{C}/(\cdot\, a)$, then pinching is given by the Beltrami coefficients $\mu_{\text{pinch}} = t \cdot \lambda_{\text{pinch}} \cdot \frac{w}{\overline{w}} \cdot \frac{d\overline{w}}{dw}$ with $\lambda_{\text{pinch}} \in S^1$. In this model, $\lambda_{\text{pinch}} = \lambda_{\text{pinch}}(p/q, \tau_q)$ depends on a choice of p/q and $\tau_q = \log a^q$. It is possible but not necessary to compute λ_{pinch} explictly.

It is also useful to define the notion of pinching deformations for annuli: given an annulus $A = A_0$, the pinching deformation $\{A_t\}$ is the deformation which shrinks the length of the core curve in A_0 the fastest (alternatively, the modulus of A_t grows as quickly as possible). For the annulus $A_{r,R} := \{z : r < |z| < R\}$, the pinching deformation is given by the Beltrami coefficient $\mu_{\text{pinch}} = -t \cdot \frac{w}{\overline{w}} \cdot \frac{d\overline{w}}{dw}$. It is easy to see that pinching a torus X with respect to geodesic γ is the same as pinching the annulus $A = X \setminus \gamma$.

6. Incompleteness: Special Case

In this section, we give a simple proof of the incompleteness of the Weil-Petersson metric in $\mathcal{B}_2$ when we take $a \to 1$ along the real axis. Our goal is not to give the most general argument, but to give the fastest route to the result. As noted in the introduction, to show that $\omega_B/\rho_{\mathbb{D}^*} \lesssim (1-|a|)^{1/4}$ on $(1/2, 1]$, it suffices to prove that:

Theorem 6.1. *For a Blaschke product $f_a \in \mathcal{B}_2$ with $a \in [1/2, 1)$, we have:*

$$\limsup_{r\to 1} \, |\mathcal{G}(f_a) \cap S_r| = O(\sqrt{1-|a|}). \tag{6.1}$$

We will deduce Theorem 6.1 from:

Theorem 6.2. *For a Blaschke product $f_a \in \mathcal{B}_2$ with $a \in [1/2, 1)$,*

- (a) *Every pre-petal lies within a bounded hyperbolic distance of a geodesic segment.*
- (b) *The hyperbolic distance between any two pre-petals exceeds $d_{\mathbb{D}}(0, a) - O(1)$.*

Recall that a horocycle connecting two points is exponentially longer than the geodesic: if $-x+iy, x+iy \in \mathbb{H}$, then the hyperbolic length of the horocycle joining them is $2 \cdot x/y$ while the length of the geodesic joining them is $\int_\theta^{\pi-\theta} \frac{d\theta}{\sin\theta} = 2\log(\cot(\theta/2))$ where $\cot\theta = x/y$. As $\cot\theta \approx 1/\theta$ for θ small, this is approximately $2\log(2 \cdot x/y)$. With this in mind, we argue as follows:

Proof of Theorem 6.1. By part (a) of Theorem 6.2, the hyperbolic length of the intersection of S_r with any single pre-petal is $O(1)$. By part (b) of Theorem 6.2, whenever the circle S_r intersects a pre-petal, an arc of hyperbolic length $O\Big(\sqrt{1-|a|}\Big)$ is disjoint from the other pre-petals. Therefore, only the $O(\sqrt{1-|a|})$-th part of S_r can be covered by pre-petals. $\square$

6.1. **Quasi-geodesic property.** We first verify the quasi-geodesic property for petals:

Lemma 6.1. *For $a \in [1/2, 1)$, the petal $\mathcal{P}(f_a)$ lies within a bounded hyperbolic neighbourhood of a geodesic ray.*

Proof. By symmetry, the linearizing ray $r_0 = \varphi_a^{-1}([0, \infty))$ is precisely the line segment $(0, 1)$ which lies within a bounded hyperbolic neighbourhood of a geodesic ray. It remains to show that the petal $\mathcal{P}(f_a)$ lies within a bounded hyperbolic neighbourhood of r_0. Suppose $z \in \mathcal{P}$ lies outside a small disk $D(0, \delta)$. Let F be the fundamental domain bounded by $\{z : |z| = \delta\}$ and its image under f_a. Under iteration, z eventually lands in F, e.g. $z_0 = f_a^{\circ N}(z) \in F$. Pick a point $x_0 \in r_0$ for which $d_{T_a^\times}(z_0, x_0) = O(1)$. (Here, we are using the fact that the limiting angle of the critical point is bounded away from 0, i.e. $\lim_{n\to\infty} \arg f^{\circ n}(c) \neq 0$. In fact, the forward orbit of the critical point lies on the segment $(-1, 1)$). Let $x = f^{-N}(x_0)$ be the N-th pre-image of x_0 along r_0. Clearly,

$$d_{\mathbb{D}}(z, x) \le d_{\Omega}(z, x) = d_{T_a^\times}(z_0, x_0) = O(1). \tag{6.2}$$

This completes the proof. □

6.2. **The structure lemma.** The quasi-geodesic property for pre-petals is an immediate consequence of the "structure lemma". The structure lemma says that the pre-petals are near-affine copies of the immediate pre-petal, while the immediate pre-petal is a near-Möbius copy of the petal – more precisely, $f : \mathcal{P}_{-1} \to \mathcal{P}$ is nearly the involution about the critical point: $m_{0\to c} \circ (-z) \circ m_{c\to 0}$.

Given a Blaschke product f, define its *critically-centered version* as $\tilde{f} = m_{c\to 0} \circ f \circ m_{0\to c}$ where $m_{0\to c} = \frac{z+c}{1+\bar{c}z}$ and $m_{c\to 0} = \frac{z-c}{1-\bar{c}z}$. We define the critically-centered versions of petals and pre-petals in the obvious way.

Lemma 6.2 (Structure lemma). *For $a \in [1/2, 1)$ on the real axis,*

(*i*) *The critically-centered petal $\tilde{\mathcal{P}} \subset B\Big(1, \mathrm{const} \cdot \sqrt{1-|a|}\Big)$.*

(*ii*) *The immediate pre-petal $\mathcal{P}_{-1} \subset B\Big(-1, \mathrm{const} \cdot (1-|a|)\Big)$.*

Proof. Part (*i*) follows from Lemma 6.1. To pin down the size and location of the immediate pre-petal, we use the fact that c is the hyperbolic midpoint of $[0, -a]$. In the critically-centered picture, the center of the petal is $m_{c\to 0}(0) = -c$ while the center of the immediate pre-petal is $m_{c\to 0}(-a) = c$. Therefore, part (*ii*) now follows from Koebe's distortion theorem. □

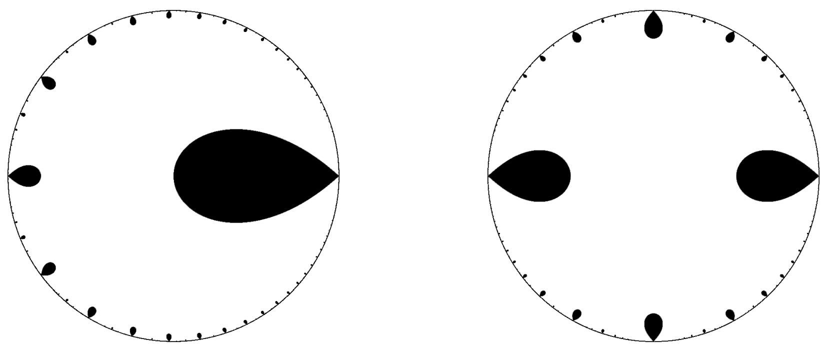

FIGURE 7. Half-petal families for the Blaschke products $f_{0.8}$ and $\tilde{f}_{0.8}$.

6.3. **Petal separation.** We now turn to showing that the petals are far apart:

Proof of part (*b*) *of Theorem 6.2.* Since the petal $\mathcal{P}$ is contained in a bounded hyperbolic neighbourhood of $(0, 1)$ and the immediate pre-petal $\mathcal{P}_{-1}$ is contained in a bounded hyperbolic neighbourhood of $(-1, -a)$, we see that $d_{\mathbb{D}}(\mathcal{P}, \mathcal{P}_{-1}) = d_{\mathbb{D}}(0, -a) - O(1)$. By the Schwarz lemma, given two pre-petals $\mathcal{P}_{\zeta_1}$ and $\mathcal{P}_{\zeta_2}$ with $f^{\circ n_1}(\zeta_1) = f^{\circ n_2}(\zeta_2) = 1$ with $n_1 \neq n_2$ (say $n_1 > n_2$),

$$d_{\mathbb{D}}(\mathcal{P}_{\zeta_1}, \mathcal{P}_{\zeta_2}) \le d_{\mathbb{D}}\Big(f^{\circ(n_1-1)}(\mathcal{P}_{\zeta_1}), f^{\circ(n_1-1)}(\mathcal{P}_{\zeta_2})\Big) \le d_{\mathbb{D}}(\mathcal{P}_{-1}, \mathcal{P}_1).$$

To complete the proof, it suffices to show that pre-petals $\mathcal{P}_{\zeta_1}$ and $\mathcal{P}_{\zeta_2}$ are far apart in the case that they have a common parent, e.g. when $f(\zeta_1) = f(\zeta_2) = \zeta$. This argument is topological. Observe that -1 and 1 separate the unit circle in two arcs, each of which is mapped to $S^1 \setminus \{1\}$ by f_a. Choose a curve γ contained in $\overline{\mathcal{P}_1^1} \cup \overline{\mathcal{P}_{-1}^1}$ that joins -1 and 1 – for example, the segment $(-1, 1)$ will do. Since ζ_1 and ζ_2 are located on the opposite sides of γ, any path from $\mathcal{P}_{\zeta_1}$ to $\mathcal{P}_{\zeta_2}$ must pass through γ. However, we already know that the distance between $\mathcal{P}_{\zeta_i}$ to either $\mathcal{P}_1$ and $\mathcal{P}_{-1}$ is greater than $d_{\mathbb{D}}(0, a) - O(1)$ which tells us that the hyperbolic $(\frac{1}{2} \cdot d_{\mathbb{D}}(0, a) - O(1))$–neighbourhood of γ is disjoint from $\mathcal{P}_{\zeta_1}$ and $\mathcal{P}_{\zeta_2}$. This completes the proof. $\square$

7. Renewal Theory

In this section, we show that for a Blaschke product other than $z \to z^d$, the integral average (1.4) defining the Weil-Petersson metric converges. The proof is based on renewal theory, which is the study of the distribution of repeated pre-images of a point. In the context of hyperbolic dynamical systems, this has been developed by Lalley [La]. We will apply his results to Blaschke products (thinking of them as maps from the unit circle to itself). Using an identity for the Green's function, we extend renewal theory to points inside the unit disk. Renewal theory will also be instrumental in giving bounds for the Weil-Petersson metric.

For a point x on the unit circle, let $n(x, R)$ denote the number of repeated pre-images y (i.e. $f^{\circ n}(y) = x$ for some $n \geq 0$) for which $\log |(f^{\circ n})'(z)| \leq R$. Also let $\mu_{R,x}$ be the probability measure on the unit circle which gives equal masses to each of the $n(R, x)$ pre-images. We show:

Theorem 7.1. *For a Blaschke product $f(z) \in \mathcal{B}_d$ other than $z \to z^d$, we have:*

$$n(x, R) \sim \frac{e^R}{\int \log |f'| dm} \qquad \text{as} \quad R \to \infty. \tag{7.1}$$

Furthermore, as $R \to \infty$, the measures $\mu_{R,x}$ tend weakly to the Lebesgue measure.

For a point $z \in \mathbb{D}$, let $\mathcal{N}(z, R)$ be the number of repeated pre-images of z that lie in the disk centered at the origin of hyperbolic radius R. Then,

Theorem 7.2. *Under the assumptions of Theorem 7.1, we have:*

$$\mathcal{N}(z, R) \sim \log \frac{1}{|z|} \cdot \frac{e^R}{\int \log |f'| dm} \qquad \text{as} \quad R \to \infty. \tag{7.2}$$

As before, when $R \to \infty$, the $\mathcal{N}(z, R)$ pre-images become equidistributed on the unit circle with respect to the Lebesgue measure.

7.1. **Green's function.** Let $G(z) = \log \frac{1}{|z|}$ be the Green's function of the disk with a pole at the origin. It is uniquely characterized by three properties:

(*i*) $G(z)$ is harmonic on the punctured disk,

(*ii*) $G(z)$ tends to 0 as $|z| \to 1$,

(*iii*) $G(z) - \log \frac{1}{|z|}$ is harmonic near 0.

Lemma 7.1. *For a Blaschke product $f \in \mathcal{B}_d$, we have:*

$$\sum_{f(w_i)=z} G(w_i) = G(z), \qquad z \in \mathbb{D}. \tag{7.3}$$

To see this, observe that $\sum_{f(w_i)=z} G(w_i)$ also satisfies the three properties above. From equation (7.3), it follows that the Lebesgue measure on the unit circle is invariant under f. Indeed, for a point $x \in S^1$, we apply the lemma to $z = rx$. Taking $r \to 1$, we see that $\sum_{f(y)=x} |f(y)|^{-1} = 1$ as desired. (Alternatively, one can apply $\frac{\partial}{\partial z}$ to both sides of equation (7.3) to obtain $\sum_{f(w)=z} \frac{f(w)}{wf'(w)} = 1$.)

In fact, the Lebesgue measure is ergodic. The argument is quite simple (see [SS] or [Ha]); for the convenience of the reader, we reproduce it here: if $E \subset S^1$ is an invariant set, we can form the harmonic extension $u_E(z) = \chi_E * P_z$. As $u_{f^{-1}E} = u_E \circ f$, we see that u_E is a harmonic function in the disk, invariant under f. But since 0 is an attracting fixed point, u_E must actually be constant, forcing E to have measure 0 or 1 as desired.

From the ergodicity of Lebesgue measure, it follows that conjugacies of distinct Blaschke products are not absolutely continuous.

7.2. **Weak mixing.** For the map $z \to z^d$, the pre-images come in packets and so $n(x, R)$ is a step function. Explicitly, $n(R, x) = 1 + d + d^2 + \cdots + d^{\lfloor \log R/\log d \rfloor}$. While $n(R, x)$ has exponential growth, due to the lack of mixing, some values of R are special. For all other Blaschke products, we do have the required mixing property and Theorem 7.1 follows from [La, Theorem 1 and formula (2.5)].

In the language of thermodynamic formalism, we must check that the potential $\phi_f(z) = -\log|f'(z)|$ is non-lattice, i.e. that there does not exist a bounded function γ such that $\phi = \psi + \gamma - \gamma \circ f$ with ψ valued in a discrete subgroup of $\mathbb{R}$. (To be honest, in [La], this equation holds not on S^1 but on the shift space $\Sigma = \{0, 1, \ldots, d-1\}^{\mathbb{N}}$ that codes the dynamics of f).

Sketch of Proof of Theorem 7.1. We consider the suspension flow $f : S^1 \to S^1$ by $\phi_f = \log|f'|$. If this flow is not weak-mixing, by [PP, Proposition 6.2], there exists a function w that is Hölder continuous on the shift space satisfying

$$w(f(x)) = e^{ia\phi_f(x)}w(x). \tag{7.4}$$

However, if we work *directly* on the unit circle and repeat the proof of [PP, Proposition 4.2], we see that we can find a function $w(x)$ satisfying (7.4) which is continuous on the unit circle. Since $w(x)$ is non-vanishing and has constant modulus, we can scale it by a constant if necessary so that $|w(x)| = 1$. Therefore, w admits a continuous branch of logarithm: $w(x) = e^{2\pi i v(x)}$. We obtain $v \circ f = a \cdot \phi_f + v + 2\pi M(x)$ where $M(x)$ is integer-valued. By continuity, we see that M is constant and therefore, ϕ_f is cohomologous to a constant.

This tells us that the Lebesgue measure m must also be the measure of maximal entropy. However, the measure of the maximum entropy is a topological invariant, thus if we have a conjugacy h between z^d and $f(z)$, the measure of the maximal entropy is $h_* m$. However, we know that the conjugacies of distinct Blaschke products are *not* absolutely continuous, therefore, we must have $f(z) = z^d$. □

7.3. **Computation of entropy.** Since the dimension of the unit circle is equal to 1, the entropy $h(f, m)$ of the Lebesgue measure coincides with the Lyapunov exponent $\frac{1}{2\pi}\int \log|f'(e^{i\theta})|d\theta$. We can compute the entropy using Jensen's formula:

Lemma 7.2. *The entropy of the Lebesgue measure for the Blaschke product $f_{\mathbf{a}}(z)$ with critical points $\{c_i\}$ is given by*

$$\text{(7.5)} \qquad \frac{1}{2\pi}\int \log|f'_{\mathbf{a}}(e^{i\theta})|d\theta = \sum G(c_i) - G(a) = \sum_{cp} G(c_i) - \sum_{zeros} G(z_i).$$

In particular, for degree 2 Blaschke products, as a tends to the unit circle, the entropy $h(f_a, m) \sim 1 - |c| \sim \sqrt{2(1-|a|)}$.

7.4. **Laminated area.** For a compact subset E in the disk, let $\hat{E}$ be its *saturation* under taking pre-images, i.e. $\hat{E} = \{\zeta : f^{\circ n}(\zeta) \in E \text{ for some } n \geq 0\}$. For a saturated set $\hat{E}$, define its *laminated area* as $\hat{\mathcal{A}}(\hat{E}) = \lim_{r\to 1^-} \frac{1}{2\pi}|E \cap S_r|$. We say that "$E$ *subtends* the $\hat{\mathcal{A}}(\hat{E})$-th part of the lamination." By Koebe's distortion theorem (see Section 2.2), we have the following useful estimate:

Lemma 7.3. *Suppose E is a subset of $U_t := \{z : 1 - t\cdot\delta_c \leq |z| < 1\}$ with $t < t_0$ sufficiently small. If E is is disjoint from all its pre-images, then*

$$\text{(7.6)} \qquad \hat{\mathcal{A}}(\hat{E}) \approx_{1/t} \frac{1}{2\pi h}\int_E \frac{1}{1-|z|}\cdot|dz|^2$$

(The notation "$A \approx_\epsilon B$" means that $|A/B - 1| \lesssim \epsilon$.)

Sketch of Proof. By breaking up E into little pieces, we can assume that $E \subset B(x,t)$ for some $x \in S^1$. We claim that $\int_E \frac{1}{1-|z|}\cdot|dz|^2 \approx_{1/t} \int_{f^{-n}(E)} \frac{1}{1-|z|}\cdot|dz|^2$ uniformly in n. Indeed, for each n-fold pre-image y (i.e. $f^{\circ n}(y) = x$), consider the t-affine copy E_y. By Lemma 2.3,

$$\int_{E_y} \frac{1}{1-|z|}\cdot|dz|^2 \approx_{1/t} |(f^{\circ n})'(y)|^{-1}\cdot\int_E \frac{1}{1-|z|}\cdot|dz|^2.$$

The claim follows in view of the the identity $\sum_{f^{\circ n}(y)=x} |(f^{\circ n})'(y)|^{-1} = 1$ (recall that the Lebesgue measure is invariant). Therefore, we may assume that $E \subset U_{t'}$ with $t' > 0$ arbitrarily small, i.e. we can pretend that f^{-1} is affine.

By approximation, it suffices to consider the case when $E = \mathscr{R}$ is a "rectangle" of the form

$$\left\{ z : 1 - |z| \in \Big((1 - \frac{\epsilon_1}{2})\delta, (1 + \frac{\epsilon_1}{2})\delta \Big),\ \arg z \in \Big(\theta_0 - \frac{\epsilon_2}{2}\delta, \theta_0 + \frac{\epsilon_2}{2}\delta \Big) \right\}$$

with ϵ_1, ϵ_2 small. For k large, the circle $S_{1-\delta/k} = \{z : |z| = 1 - \delta/k\}$ intersects $\approx \epsilon_1 k/h$ pre-images of $\mathscr{R}$. As the hyperbolic length of $S_{1-\delta/k}$ is $\sim 2\pi k/\delta$ and each pre-image has "horizontal" hyperbolic length of $\approx \epsilon_2$, the laminated area $\hat{\mathcal{A}}(\hat{\mathscr{R}}) \approx \frac{\epsilon_1 \epsilon_2}{2\pi h} \cdot \delta$ as desired. □

Recall from [McM2] that a continuous function $h : \mathbb{D} \to \mathbb{C}$ is *almost-invariant* if for any $\epsilon > 0$, there exists $r(\epsilon) < 1$, so that for any orbit $z \to f(z) \to \cdots \to f^{\circ n}(z)$ contained in $\{z : 1 - r \le |z| < 1\}$, we have $|h(z) - h(f^{\circ n}(z))| < \epsilon$. The argument above also tells us that:

Theorem 7.3. *Suppose f is a Blaschke product other than $z \to z^d$, and h is an almost-invariant function. Then the limit $\lim_{r \to 1^-} \frac{1}{2\pi} \int_{|z|=r} h(z) d\theta$ exists.*

Sketch of Proof. Let E be a backwards fundamental domain near the unit circle, e.g. take $E = f^{-1}(D(0,s)) \setminus D(0,r)$ with $s \approx 1$. Split E into many pieces on which h is approximately constant. By applying Lemma 7.3 to each piece and summing over the pieces, we see that as $r \to 1$, $\frac{1}{2\pi} \int_{|z|=r} h(z) d\theta$ oscillates within an arbitrarily small multiplicative factor. Hence, the limit converges. □

Applying the lemma to $h = |v'''/\rho^2|^2$ which is almost-invariant by Lemma 2.5, we see that:

Corollary. *Given a Blaschke product $f \in \mathcal{B}_d$ other than $z \to z^d$, the limit in the definition of the Weil-Petersson metric (1.4) exists for v associated to any tangent vector in $T_f \mathcal{B}_d$.*

8. Lower bounds for the Weil-Petersson metric

In this section, we explain how to obtain lower bounds for the Weil-Petersson metric using the (gradients of the) multipliers of the repelling periodic orbits on the unit circle. We first consider the "Teichmüller case" and then handle the "Blaschke case" by linear approximation. However, the approximation argument comes with a price: unlike in the Teichmüller case, to give a lower bound for the Weil-Petersson metric we must insist that the quotient torus of the repelling cycle changes at a definite rate in the Teichmüller metric.

This might seem like a fairly minor detail, however it prevents us from showing that the completion of the Weil-Petersson metric on $\mathcal{B}_2$ attaches precisely the points $e(p/q) \in S^1$. We will show that in Teichmüller space, the Weil-Petersson length of a curve $X : [0,1] \to \mathcal{T}_{g,n}$ with $\ell_{X(0)}(\gamma) = m$ and $\ell_{X(0)}(\gamma) = M > m$ is bounded below by a definite constant $C_{g,n}$. However, we are unable to prove the analogous statement for the Weil-Petersson metric on $\mathcal{B}_d$ where we replace the "length of a hyperbolic geodesic" by "the (logarithm of the) multiplier of a periodic orbit."

8.1. **Lower bounds in Teichmüller space.** Consider the map $f(z) = \lambda z$ where λ is a positive real number not equal to 1. Suppose $\mu \in M(\mathbb{H})^f$ is a Beltrami coefficient supported on the upper half-plane. We can form the maps $f_t = w_{t\mu} \circ f_0 \circ (w_{t\mu})^{-1}$. Since we use the asymmetric deformations $w_{t\mu}$, $\lambda_t = f_t'(0)$ may no longer be real. We think of $v = dw_{t\mu}/dt$ as a holomorphic vector field on the lower half-plane. Let $\pi_\lambda : \mathbb{C} \to \mathbb{C}/(\cdot\,\lambda)$ be the quotient map.

Our goal is to give a lower bound for $|v'''/\rho^2|$ in terms of $||\pi_\lambda(\mu)||_{T(\mathcal{T}_1)} = |\dot{L}_0/(2L_0)|$ where $L_t = \log \lambda_t$ and $\dot{L}_t = (d/dt)|_{t=0} \log \lambda_t$. We first consider the case when μ is a *radial* Beltrami coefficient, i.e. μ is of the form

$$\mu(z) = k(\theta) \cdot \frac{z}{\overline{z}} \cdot \frac{d\overline{z}}{dz}. \tag{8.1}$$

Lemma 8.1. *For a radial Beltrami coefficient μ given by (8.1) and $z \in \overline{\mathbb{H}}$,*

$$v(z) = \frac{d}{dt}\Big|_{t=0} w^{t\mu}(z) = -\frac{1}{2\pi} z \log z \cdot \int k(\theta) d\theta \tag{8.2}$$

and therefore,

$$v'''(z) = \frac{1}{2\pi} \cdot \frac{1}{z^2} \cdot \int_0^\pi k(\theta) d\theta. \tag{8.3}$$

Proof. One computes:

$$\begin{aligned} v(z) &= \frac{1}{2\pi} \int \frac{z(z-1)}{\zeta(\zeta-1)(\zeta-z)} \cdot k(\theta) \cdot (\zeta/\overline{\zeta})|d\zeta|^2 \\ &= \frac{z}{2\pi} \int_0^\pi k(\theta) \int_0^\infty \frac{(z-1)e^{i\theta}}{(re^{i\theta}-1)(re^{i\theta}-z)} dr d\theta \\ &= \frac{z}{2\pi} \int_0^\pi k(\theta) \int_0^\infty e^{it}\left(\frac{1}{re^{i\theta}-1} - \frac{1}{re^{i\theta}-z}\right) dr d\theta \\ &= \frac{z}{2\pi} \int_0^\pi k(\theta) \int_0^\infty \left(\frac{1}{r-e^{-i\theta}} - \frac{1}{r-ze^{-i\theta}}\right) dr d\theta \\ &= \frac{z}{2\pi} \int_0^\pi k(\theta) \cdot (-\log z) d\theta. \end{aligned}$$

(Since we are working in $\mathbb{C} \setminus (-\infty, 0]$, the branch of the logarithm is well-defined). □

By Lemma 5.1, it follows that in the sector $\{z : \arg z \in (\pi/4, 3\pi/4)\}$, we have

$$\left|\frac{v'''(z)}{\rho^2}\right|^2 \asymp \left|\frac{(d/dt)|_{t=0} \log \lambda_t}{\log \lambda_0}\right|^2 \asymp ||\pi_\lambda(\mu)||_T^2. \tag{8.4}$$

Now suppose $\mu \in M(\mathbb{C})^f$ is an arbitrary Beltrami coefficient. While we don't have a *pointwise* lower bound, an averaged version of (8.4) suffices for our purposes. Suppose that $\mathscr{R} = S_{\theta_1,\theta_2} \cap F_{r_1,r_2}$ is an "annular rectangle" where

$$S_{\theta_1,\theta_2} = \{z : \arg z \in (\theta_1, \theta_2)\} \quad \text{and} \quad F_{r_1,r_2} = \{z : r_1 < |z| < r_2\}$$

with $(\theta_1, \theta_2) \subseteq (\pi/4, 3\pi/4)$ and $r_2/r_1 \geq \lambda_0$.

By averaging across radial rays and using the fact that the map $\mu \to v'''$ is linear, we see that:

$$\fint_{\mathscr{R}} \left| \frac{v'''(z)}{\rho^2} \right|^2 \cdot \rho^2 |dz|^2 \gtrsim \left| \frac{(d/dt)|_{t=0} \log \lambda_t}{\log \lambda_0} \right|^2 \asymp ||\pi_\lambda(\mu)||_T^2 \tag{8.5}$$

We apply this observation to the study of the Weil-Petersson metric in Teichmüller space. Suppose $X \in \mathcal{T}_{g,n}$ is a Riemann surface and $\gamma \subset X$ is a simple geodesic whose length is bounded above and below, e.g. $\lambda_1 < \ell_X(\gamma) < \lambda_2$. Let $p : \mathbb{H} \to X = \mathbb{H}/\Gamma$ be the universal covering map chosen so that the imaginary axis covers γ. By the collar lemma [Hub], there exists an annular rectangle $\mathscr{R}$ with $(r_1, r_2) = (1, e^{\ell_X(\gamma)})$ and $(\theta_1, \theta_2) = (\pi/2 - \epsilon_{\lambda_1,\lambda_2}, \pi/2 + \epsilon_{\lambda_1,\lambda_2})$ which has definite hyperbolic area, and for which $p|_{\mathscr{R}}$ is injective. It follows that for a Beltrami coefficient $\mu \in M(\mathbb{H})^\Gamma$, we have $||p(\mu)||_{\text{WP}} \gtrsim ||\pi_\lambda(\mu)||_T$.

8.2. **Lower bounds in complex dynamics.** We now return to complex dynamics. Recall that for a Blaschke product $f(z)$ and $\mu \in M(\mathbb{D})^f$, the asymmetric deformation is given by $f_{s,t} := w_{\mu_{s,t}} \circ f \circ (w_{\mu_{s,t}})^{-1}$ where $\mu_{s,t} := s\mu + t\mu^+$. Also recall that $L(\xi) = \log(f^{\circ q})'(\xi)$ denotes the logarithm of the multiplier of a periodic orbit $f^{\circ q}(\xi) = \xi$.

Theorem 8.1. *Suppose $f(z) \in \mathcal{B}_2$ is Blaschke product and $f^{\circ q}(\xi) = \xi$ is a repelling periodic point on the unit circle with $(f^{\circ q})'(\xi) < M_2$. If $\mu(z) \in M(\mathbb{D})^f$ is an f-invariant Beltrami coefficient such that $|\dot{L}_{0,t}(\xi)/L(\xi)| \asymp 1$, there exist a ball*

$$\mathscr{B} = B\Big(\xi \cdot (1 - c_1 \cdot \delta_c), c_2 \cdot \delta_c\Big) \quad \textit{for which} \quad \fint_{\mathscr{B}} \left| \frac{v'''}{\rho^2}(z) \right|^2 \cdot |dz|^2 \asymp 1. \tag{8.6}$$

Theorem 8.1 follows from the previous section and Koebe's distortion theorem. It is in the use of Koebe's distortion theorem that we need to know that $|\dot{L}_{0,t}(\xi)/L(\xi)| \asymp 1$. Theorem 8.1 produces *one* ball on which the quadratic differential $\fint_{\mathscr{B}} |v'''/\rho^2| \asymp 1$. However, by Lemma 2.5, the same estimate holds on the inverse images of $\mathscr{B}$.

Lemma 8.2. *If we additionally assume that* $M_1 < (f^{\circ q})'(\xi)$*, then*

$$\limsup_{r\to 1^-} \frac{1}{2\pi}\int_{|z|=r}\left|\frac{v'''(z)}{\rho(z)^2}\right|^2 d\theta \asymp 1. \tag{8.7}$$

To see this, notice that since the multiplier is bounded from below, we can choose c_1 and c_2 so that the (repeated) inverse images of $\mathscr{B}$ are disjoint from $\mathscr{B}$ (and thus from each other). By Lemma 7.3, the inverse images of $\mathscr{B}$ spread over a definite portion of the Riemann surface lamination (i.e. the Lebesgue measure of the intersection of $\hat{\mathscr{B}}$ with a circle $\{z : |z| = r\}$ for r sufficiently close to 1 is bounded below).

In Chapter 10, we will explain how to give effective lower bounds using a repelling periodic orbit whose multiplier is small $(f^{\circ q})'(\xi) < M_1$.

Remark. To give lower bounds for the Weil-Petersson metric, we used the gradient of the multiplier of a periodic orbit in the μ^+ direction. In view of the the identities

$$\begin{aligned}
(d/dt)|_{t=0} \log(f_{t,t}^{\circ q})'(\xi_{t,t}) &= 2 \cdot \mathrm{Re}(d/dt)|_{t=0} \log(f_{0,t}^{\circ q})'(\xi_{0,t}), \\
(d/dt)|_{t=0} \log(f_{it,it}^{\circ q})'(\xi_{it,it}) &= -2 \cdot \mathrm{Im}(d/dt)|_{t=0} \log(f_{0,t}^{\circ q})'(\xi_{0,t}),
\end{aligned}$$

we can also use the gradient of the multiplier in the Blaschke slice, i.e. in the $\mu + \mu^+$ or $i\mu + (i\mu)^+$ directions.

9. Multipliers of Simple Cycles

In this chapter, we prove Theorem 3.1. We first make some useful definitions. Let $T_{p/q}$ denote the quotient torus associated to the repelling periodic orbit of rotation number p/q and $T^{\text{in}}_{p/q} \subset T_{p/q}$ be the half of the torus which is associated to points inside the unit disk. Let $P_{p/q} \subset T^{\text{in}}_{p/q}$ be the footprint of $\mathcal{F}$ in $T^{\text{in}}_{p/q}$. The footprint of the whole flower $\mathcal{F}^1$ is then defined to be the part of $T^{\text{in}}_{p/q}$ filled by the whole flower $\mathcal{F}^1$. The proof of Theorem 3.1 is based on the following lemma:

Lemma 9.1. *There exists a constant $C_{\text{small}} > 0$, so that for $a \in \mathcal{B}_{p/q}(\eta)$, we have:*

- *(i) The footprint $P^1_{p/q}$ of the whole petal contains an angle of opening at least $0.99\,\pi$.*
- *(ii) The footprint $P_{p/q}$ of the half-petal is contained in a central angle of $0.51\,\pi$.*

9.1. **Conformal modulus of an annulus.** To prove Lemma 9.1, we need two preliminary facts. We begin with a formula for the conformal modulus of an annulus. We use the convention that the annulus $A_{r,R} := \{z : r < |z| < R\}$ has modulus $\frac{\log(R/r)}{2\pi}$, which is the extremal length of the curve family $\Gamma_\uparrow(A_{r,R})$ consisting of curves that join the two boundary components of $A_{r,R}$. We denote the dual curve family by $\Gamma_\circlearrowleft(A_{r,R})$, consisting of curves that separate the two boundary components.

If $B \subset A$ is an essential sub-annulus of A, we say that B is *round* in A if the pair (A, B) is conformally equivalent to a pair of concentric round annuli $(A_{r,R}, A_{r',R'})$. Alternatively, $B \subset A$ is round if the pinching deformations for A and B are compatible, i.e. $\mu_{\text{pinch}}(B) = \mu_{\text{pinch}}(A)|_B$.

Lemma 9.2. *Suppose $S^* = \{z = e^{i\theta} \cdot e^{\mathbb{R} \log \alpha} : \theta_1 < \theta < \theta_2\} \subset \mathbb{C}^*$ where $|\alpha| < 1$ and a branch of the logarithm $\log \alpha$ has been chosen. Then the annulus*

$$S^* \;/\; \{z \sim \alpha z\} \qquad \textit{has modulus} \qquad (\theta_2 - \theta_1) \cdot \operatorname{Re}\Big(\frac{1}{\log \alpha}\Big).$$

Proof. If we take apply the map $\log z / \log \alpha$, we see that $S^* \mathbin{/} \{z \sim \alpha z\}$ is conformally conjugate to a parallelogram with vertices $0, 1, \frac{i(\theta_2-\theta_1)}{\log \alpha}, 1 + \frac{i(\theta_2-\theta_1)}{\log \alpha}$, where the sides $[0, \frac{i(\theta_2-\theta_1)}{\log \alpha}]$ and $[1, 1 + \frac{i(\theta_2-\theta_1)}{\log \alpha}]$ are identified by parallel translation. Using a cut-and-paste surgery, we see that this parallelogram is conformally conjugate to the rectangle with vertices $0, 1, i \cdot \mathrm{Re}\, \frac{\theta_2-\theta_1}{\log \alpha}, 1 + i \cdot \mathrm{Re}\, \frac{\theta_2-\theta_1}{\log \alpha}$. Applying the map $z \to e^{2\pi i z}$, we find that this rectangle is conformally conjugate to the annulus of modulus $(\theta_2 - \theta_1) \cdot \mathrm{Re}\Big(\frac{1}{\log \alpha}\Big)$ as desired. ☐

Conversely, for a region $T^* \subset \mathbb{C}^*$ bounded by two Jordan curves γ_1, γ_2 that is invariant under multiplication by α, we define the *generalized angle* between γ_1 and γ_2 as

$$\beta := \frac{\mathrm{mod}(T^* \mathbin{/} \{z \sim \alpha z\})}{\mathrm{Re}\Big(\frac{1}{\log \alpha}\Big)}.$$

9.2. **Holomorphic index formula.** We now turn to the holomorphic index formula. Recall that if $g(z)$ is a holomorphic map with a fixed point $g(\zeta) = \zeta$, the *index* of ζ is defined as

$$I_\zeta := \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - g(z)} \tag{9.1}$$

where γ is a small counter-clockwise loop around ζ. If the multiplier $\lambda = g'(\zeta)$ is not 1, this expression reduces to $\frac{1}{1-\lambda}$. By the residue theorem, one has:

Theorem 9.1 (Holomorphic Index Formula)**.** *Suppose $R(z)$ is a rational function and $\{\zeta_i\}$ is the set of its fixed points. Then, $\sum I_{\zeta_i} = 1$.*

For a Blaschke product $f \in \mathcal{B}_d$, the holomorphic index formula says:

$$\sum \frac{1}{r_i - 1} = \frac{1 - |a|^2}{|1 - a|^2} \tag{9.2}$$

where the sum ranges over the repelling periodic points on the unit circle, and a is the multiplier of the attracting fixed point.

9.3. **Petal correspondence.** Since a whole petal joins the attracting fixed point to a repelling periodic point, it provides a conformal equivalence between an annulus $A^1 \subset T_a$ with $P^1_{p/q} \subset T_{p/q}$. As there are q whole petals at the attracting fixed point, we have:

$$\frac{\beta}{\log m_{p/q}} = \operatorname{Re} \frac{1}{q} \cdot \frac{2\pi}{\log(1/a^q)} \tag{9.3}$$

where β is the generalized angle representing the modulus of mod $P^1_{p/q}$. The holomorphic index formula gives a lower bound on $m_{p/q}$:

$$\frac{1}{m_{p/q} - 1} \le \frac{1}{q} \cdot \frac{1 - |a^q|^2}{|1 - a^q|^2} \tag{9.4}$$

Following [McM4], we compare the equations (9.3) and (9.4):

Proof of Lemma 9.1. Suppose $a \in \mathcal{H}_{p/q}(\eta)$. If $\eta > 0$ is small, then $a^q \in \mathcal{H}_1(\frac{\eta+\theta}{q})$ with $|\theta|$ small. On this "horocycle", Re $\frac{1}{\log(1/a^q)} \approx \frac{q}{2\pi\eta+\theta}$ while the Poisson kernel $\frac{1-|a^q|^2}{|1-a^q|^2} \approx \frac{2q}{2\pi\eta+\theta}$. Comparing (9.3) and (9.4), we see that if η is sufficiently small, then β can be made arbitrarily close to π. By the standard modulus estimates (Lemmas 9.3 and 9.4 below), it follows that the footprint $P^1_{p/q}$ must contain an angle of opening close to π. They also show that the footprint of the half-petal $P_{p/q}$ is contained in an angle of opening 0.51π. This proves (i) and (ii). □

We now prove Theorem 3.1:

Proof of Theorem 3.1. For (i), we plug in $\beta \approx \pi$ into (9.3) to obtain

$$1/\log m_{p/q} \approx 2/(2\pi\eta) \quad \text{or} \quad m_{p/q} \approx 1 + (2\pi\eta)/2.$$

Part (ii) is somewhat harder. Since the footprint of the whole petal $P^1_{p/q}$ contains a definite angle of size $> 0.51\pi$, it is easy to construct *some* Beltrami coefficient ν which effectively changes the multiplier of the repelling periodic orbit, i.e. $dm_{p/q}(f^{t\nu})/dt|_{t=0} \asymp 1$. As $\mathcal{B}_2$ is one-dimensional, we see that for any optimal Beltrami coefficient μ, we must have either

$$dm_{p/q}(f^{t\mu})/dt|_{t=0} \asymp 1 \quad \text{or} \quad dm_{p/q}(f^{it\mu})/dt|_{t=0} \asymp 1. \tag{9.5}$$

This is sufficient for applications to the Weil-Petersson metric; however, for completeness, we will show that the first alternative holds when $\mu = \mu_{\text{pinch}} \in M(\mathbb{D})$ is the optimal pinching coefficient built from the attracting torus.

As the dynamics of f is approximately linear near the repelling fixed point, $\mu = \mu_{\text{pinch}}$ descends to a Beltrami coefficient $\nu \in M(T_{p/q})$, with $\operatorname{supp} \nu \subset T^{\text{in}}_{p/q}$. Since $\mu|_{A^1}$ is the optimal pinching coefficient for A^1, $\nu|_{P^1_{p/q}}$ is the optimal pinching coefficient for the annulus $P^1_{p/q}$. By Lemma 9.1, when $\eta > 0$ is small, the footprint $P^1_{p/q}$ takes up most of $T^{\text{in}}_{p/q}$, and as $T^{\text{in}}_{p/q}$ is a round annulus in $T^{p/q}$, ν is approximately equal to the optimal pinching coefficient for $T_{p/q}$ on $T^{\text{in}}_{p/q}$. When we consider deformations $f^{t\mu}$ in the Blaschke slice, we use the Beltrami coefficient $\mu + \mu^+$, which corresponds to $\nu + \nu^+ \in M(T_{p/q})$. As $\nu + \nu^+ \in M(T_{p/q})$ is approximately equal to the optimal pinching coefficient for $T_{p/q}$ (at least away from the trace of the unit circle in $T_{p/q}$), it is clear that $dm_{p/q}/d\eta \asymp 1$ when η is sufficiently small. □

9.4. **Standard modulus estimates.** For convenience of the reader, we state the standard estimates for annuli that we have used in the proofs of Lemma 9.1. The key to these estimates is the fact that if $B \subset A = A_{r,R}$ is an essential sub-annulus, then $\operatorname{mod} B \ge \operatorname{Area}(B, \frac{|dz|}{2\pi|z|}) := \int_B (\frac{|dz|}{2\pi|z|})^2$ with equality if and only if $B \subset A$ is a round sub-annulus.

Lemma 9.3. *Suppose $A = A_{r,R}$ has modulus $\operatorname{mod} A < M$ and $B \subset A$ is an essential sub-annulus. For any $\epsilon > 0$, there exists a $\delta(\epsilon, M) > 0$ such that if*

$$\operatorname{mod} B \ge (1 - \delta) \operatorname{mod} A,$$

then B contains the "middle" annulus of modulus $(1 - \epsilon) \operatorname{mod} A$.

Proof. By symmetry, it suffices to show that if B avoids a curve γ that joins $z_1 \in S_R$ to $z_2 \in S_{R-(R-r)\epsilon}$, then B has modulus less than $(1 - C(M)\epsilon^2) \bmod A$. Giving an upper bound on the extremal length of $\Gamma_\uparrow(B)$ is equivalent to finding a lower bound on the extremal length of the curve family $\Gamma_\circlearrowleft(B)$. For this purpose, consider the metric

$$\rho = \begin{cases} \frac{|dz|}{2\pi|z|} & \text{on } A_{r,R} \setminus B(z_1, \frac{\epsilon}{3} \cdot (R-r)) \\ 0 & \text{on } A_{r,R} \setminus B(z_1, \frac{\epsilon}{3} \cdot (R-r)) \end{cases} \tag{9.6}$$

Observe that the ρ-length of any curve in $\Gamma_\circlearrowleft(B)$ is at least 1, but we have saved $C_1(M)\epsilon^2$ area as measured in the round metric $\frac{|dz|}{2\pi|z|}$, so

$$\lambda_{\Gamma_\circlearrowleft(B)} > \lambda_{\Gamma_\circlearrowleft(A)} + C_2(M)\epsilon^2. \tag{9.7}$$

As $\lambda_{\Gamma_\uparrow(B)} \cdot \lambda_{\Gamma_\circlearrowleft(B)} = 1$, we see that $\lambda_{\Gamma_\uparrow(B)} < (1 - C(M)\epsilon^2) \bmod A$ as desired. □

Essentially the same argument shows that:

Lemma 9.4. *Suppose $A = A_{r,R}$ has modulus $\bmod A < M$ and $B_1, B_2, B_3 \subset A$ are three essential disjoint annuli, with B_2 sandwiched between B_1 and B_3. For any $\epsilon > 0$, there exists a $\delta(\epsilon, M) > 0$ such that if*

$$\bmod B_2 \geq (1/2 - \delta) \bmod A \quad \textit{and} \quad \bmod B_1 + \bmod B_3 \geq (1/2 - \delta) \bmod A,$$

then B_2 is contained within the "middle" annulus of modulus $(1/2 + \epsilon) \bmod A$.

We leave the details to the reader.

10. Incompleteness: General Case

In this chapter, we prove Theorem 1.2. It suffices to show that for $a \in \mathcal{H}_{p/q}(\eta)$ with $\eta < C_{\text{small}}$, we have $||\mu \cdot \chi_{\mathcal{G}}||^2_{\text{WP}} \asymp \eta^{1/2}$. For $a \in \mathcal{B}_{p/q}(C_{\text{small}})$, the petals and flowers are still well-separated; however, they no longer satisfy the quasi-geodesic property. Nevertheless, we can still estimate the intersection of $\mathcal{G}(f_a)$ with a circle $\{z : |z| = r\}$ for r close to 1 using renewal theory. They gateway to our estimates is the following lemma:

Lemma 10.1 (Fundamental Lemma)**.** *Suppose that $\langle \xi_1, \xi_2, \dots, \xi_q \rangle$ is a repelling periodic orbit of a Blaschke product $f \in \mathcal{B}_2$ whose multiplier is $m < M_{\text{small}} := 1.01$. There exists a constant K sufficiently large such that the branch of $(f^{\circ q})^{-1}$ which takes ξ_i to itself, maps $B(\xi_i, R)$ strictly inside of itself, where $R := \frac{\delta_c}{K\sqrt{m-1}}$.*

Corollary. *In particular, for each $i = 1, 2, \dots, q$, the formula*

$$\varphi_{\xi_i}(z) := \lim_{n\to\infty} m^{-n} \cdot \Big((f_a^{\circ nq})^{-1}(z) - \xi_i \Big) \tag{10.1}$$

defines a univalent holomorphic function on $B(\xi_i, R)$ satisfying

$$\varphi_{\xi_i} \circ f = m^{-1} \cdot f, \quad \varphi_{\xi_i}(\xi_i) = 0, \quad (\varphi_{\xi_i})'(\xi_i) = 1.$$

By Koebe's distortion theorem, Lemma 10.1 implies that the dynamics of $f^{\circ q}$ is nearly linear in the balls $B(\xi_i, R)$, i.e. if $z, f^{\circ q}(z), f^{\circ 2q}(z), \dots, f^{\circ nq}(z) \in B(\xi_i, t \cdot R)$ with $t \geq 1$, then:

$$\left| \frac{|(f^{\circ nq})'(z)|}{m^n} - 1 \right| \lesssim 1/t \qquad \text{and} \qquad \arg(f^{\circ nq}(z) - \xi_i) - \arg(z - \xi_i) \lesssim 1/t. \tag{10.2}$$

Remark. Lemma 10.1 is only significant for repelling periodic orbits which have small multipliers. For $m > M_{\text{small}}$, we can apply Koebe's distortion theorem to the inverse branch $(f^{\circ q})^{-1}$ on $B(\xi_i, \delta_c)$ to see that there exists a constant K such that $(f^{\circ q})^{-1}$ maps the ball $B(\xi_i, \delta_c/K)$ inside of itself.

From Lemma 10.1, it follows that:

Theorem 10.1 (Flower bounds)**.** *For $f_a \in \mathcal{B}_2$ with $a \in \mathcal{B}_{p/q}(C_{\text{small}})$,*

$$\mathcal{F} \subset \bigcup_{i=1}^{q} S\Big(\xi_i, 0.52\,\pi, R\Big) \cup B\Big(0, 1 - 0.5 \cdot R\Big) =: \bigcup S_i \cup B. \tag{10.3}$$

Remark. We do not need to know *any* information about the behavior of the flower within the ball $B(0, 1 - 0.5 \cdot R)$.

Using Theorem 10.1, we extend the petal separation and structure lemmas to the wider class of parameters. Since the statements are interrelated, we state them as a single theorem:

Theorem 10.2. *For $a \in \mathcal{H}_{p/q}(\eta)$ with $\eta < C_{\text{small}}$,*

- (a) *The hyperbolic distance $d_{\mathbb{D}}(\mathcal{F}, c) \ge \frac{1}{2}\log\eta - O(1)$.*
- (b) *The hyperbolic distance $d_{\mathbb{D}}(\mathcal{F}, \mathcal{F}_*) \ge \log\eta - O(1)$.*
- (c) *The hyperbolic distance between any two pre-flowers exceeds $\log\eta - O(1)$.*
- (a') *The critically-centered flower $\tilde{\mathcal{F}} \subset B(-\hat{c}, const \cdot \eta^{1/2})$ where $\hat{c} := c/|c|$.*
- (b') *The immediate pre-flower $\mathcal{F}_*$ lies within $B(\hat{c}, const \cdot \delta_c \cdot \eta^{1/2})$.*

Using Theorems 10.1 and 10.2, it is easy to deduce Theorem 1.2. We give the details in Section 10.4.

10.1. **Preliminaries.** In this section, we collect some useful facts that will enable us to prove Lemma 10.1. We begin with a simple observation from hyperbolic geometry:

Lemma 10.2. *Suppose $z_1, z_2 \in \mathbb{D} \cap \{z : \operatorname{Re} z < 0\}$ are two points in the left half of the unit disk satisfying $|z_1 - (-1)| \asymp |z_2 - (-1)|$. Suppose $p \in (-1, 0)$. Then, $|m_{p\to 0}(z_1) - 1| \asymp |m_{p\to 0}(z_2) - 1|$ where $m_{p\to 0} = \frac{z+p}{1+\overline{p}z}$.*

To see Lemma 10.2, one simply needs to draw a picture of the geodesics orthogonal to $(-1, 1)$. Next, we recall a formula for the derivative of a Blaschke product:

Lemma 10.3 (Equation (3.1) of [McM4])**.** *Given a Blaschke product $f_{\mathbf{a}} \in \mathcal{B}_d$, for a point ζ on the unit circle, one has:*

$$|f'_{\mathbf{a}}(\zeta)| = 1 + \sum_{i=1}^{d-1} \frac{1-|a_i|^2}{|\zeta + a_i|^2} \tag{10.4}$$

From Lemma 10.3, it easily follows that:

Lemma 10.4. *Given a degree 2 Blaschke product $f \in \mathcal{B}_2$, for a point $\zeta \in S^1$ on the unit circle with $|f'(\zeta)| < M$, we have*

$$\Big|\zeta - (-a)\Big| \asymp \frac{\delta_c}{\sqrt{|f'(\zeta)| - 1}} \tag{10.5}$$

Given any constant $L > 0$, there exists a constant $K(M, L)$ such that the hyperbolic distance from the critical point c to the ball $B(\zeta, \frac{\delta_c}{K\sqrt{|f'(\zeta)|-1}})$ exceeds $\frac{1}{2}\log\frac{1}{|f'(\zeta)|-1} - L$.

From the bound on the $|f'(\zeta)|$, it is evident that $|\zeta - \hat{c}| = |\zeta - \hat{a}| \asymp |\zeta - a|$. To see the estimate on the hyperbolic distance, observe that $m_{c\to 0}$ maps the ball $B(\zeta, \frac{\delta_c}{K\sqrt{|f'(\zeta)|-1}})$ inside $B(-\hat{c}, \frac{C}{K\sqrt{|f'(\zeta)|-1}})$. We will also need a lemma from [McM5] which roughly says that away from the critical points, Blaschke products are close to hyperbolic isometries. For points a and b in the unit disk, we let $[a, b]$ denote the hyperbolic geodesic segment joining a and b.

Lemma 10.5 (Theorem 10.11 in [McM5])**.** *There is a constant $R > 0$ such that for any proper holomorphic map $f : \mathbb{D} \to \mathbb{D}$ of degree d,*

(1) If $d_{\mathbb{D}}([a, b], C(f)) > R$, then $d_{\mathbb{D}}(f(a), f(b)) = d_{\mathbb{D}}(a, b) + O(1)$.

(2) If $d_{\mathbb{D}}([a, b], f(C(f))) > R$, then $d_{\mathbb{D}}(f^{-1}(a), f^{-1}(b)) = d_{\mathbb{D}}(a, b) + O(1)$ where f^{-1} is any branch of the inverse chosen continuously along $[a, b]$.

In practice, we will use the following consequence of Lemma 10.5:

Lemma 10.6. *For $f \in \mathcal{B}_2$, let $\tilde{f}$ denote its critically-centered version, i.e.*

$$\tilde{f} = m_{c\to 0} \circ f \circ m_{0\to c}$$

where $m_{0\to c} = \frac{z+c}{1+\bar{c}z}$ and $m_{c\to 0} = \frac{z-c}{1-\bar{c}z}$. Given any point $\zeta \in S^1$, $\tilde{f}$ is injective on a ball $B(\zeta, C_{\text{inj}})$ of definite size.

Finally, we need an estimate on the derivative of a Blaschke product inside the unit disk:

Lemma 10.7 (Proposition 3.2 in [McM4])**.** *Given a Blaschke product $f \in \mathcal{B}_d$, for a point $\zeta \in S^1$, we have*

$$\max_{0\le r\le 1} |f'(r\zeta)| \le 4|f'(\zeta)|. \tag{10.6}$$

10.2. **Linearization at repelling periodic orbits.** With these preliminaries in mind, we show Lemma 10.1:

Proof of Lemma 10.1, when $q = 1$. We first prove the lemma in the special case when $q = 1$ as the computation in that case is slightly simpler. Let $\tilde{\xi} = m_{c\to 0}(\xi)$. Then, $m_{c\to 0}(B(\xi, R))$ is a ball inside $B(\tilde{\xi}, \frac{C\sqrt{m-1}}{K})$ of a comparable radius. By Lemma 10.6 and Koebe's distortion theorem, we see that on the ball $B(\tilde{\xi}, \frac{C\sqrt{m-1}}{K})$, we have $|\frac{\tilde{f}'(z)}{m} - 1| \le \frac{C_2\sqrt{m-1}}{K}$. In particular, it follows that $|\tilde{f}'(z) - mz| \le C_3/K \cdot (m-1)$. By choosing K sufficiently large, we can make $C_3/K \ll 1$, which tells us that $\tilde{f}^{-1}$ maps the ball $B(\tilde{\xi}, \frac{C\sqrt{m-1}}{K})$ into itself.

To check that $\tilde{f}^{-1}$ maps $m_{c\to 0}(B(\xi, R))$ into itself, we use the fact that $m_{c\to 0}(B(\xi, R))$ is ball inside $B(\tilde{\xi}, \frac{C\sqrt{m-1}}{K})$ of a comparable radius. When we contract the ball $m_{c\to 0}(B(\xi, R))$ by a factor of m with respect to $\tilde{\xi}$, and make an error of at most $C_3/K(m-1)$, we are still inside $m_{c\to 0}(B(\xi, R))$. □

In the general case, let $m = m_1 m_2 \cdots m_q$ where $m_i = |f'(\xi_i)|$. Set $\tilde{\xi}_i = m_{c\to 0}(\xi_i)$. Like in the $q = 1$ case, we first show that if K sufficiently large, then $(\tilde{f}^{\circ q})^{-1}$ maps $m_{c\to 0}(B(\xi_i, R))$ into itself. For this purpose, we show the following a priori estimate:

Lemma 10.8. *If K is sufficiently large, then for $k = 1, 2, \ldots, q$, we have*

$$f^{\circ k}(B(\xi_i, R)) \subset B(\xi_{i+k}, C_0 \cdot R). \tag{10.7}$$

Proof. Let us first check that $f^{\circ k}(B(\xi_i, R) \cap S^1) \subset B(\xi_{i+k}, 2R) \cap S^1$. If K is sufficiently large, then $|f'(\zeta)| < 1 + 2(m_i - 1)$ on $B(\xi_i, 2R) \cap S^1$. Thus, in one step, $B(\xi_i, R) \cap S^1$ can be bloated by a factor of at most $1 + 2(m_i - 1)$. Therefore in q steps, $B(\xi_i, R) \cap S^1$ can be bloated by a factor of at most $\prod 1 + (2(m_i - 1))$. Since $\prod m_i = m < 1.01$, this is less than 2. Equation (10.7) now follows from Lemma 10.7 with $C_0 = 8$. □

Proof of Lemma 10.1, general case. By the estimate on hyperbolic distance, we know that $m_{c\to 0}(B(\xi_i, R))$ is a ball contained in $\tilde{B}_i := B(\tilde{\xi}_i, \frac{C(m_i-1)}{K\sqrt{m-1}})$ where the radii of $m_{c\to 0}(B(\xi_i, R))$ and $\tilde{B}_i$ are comparable. Set $2C_0(M) \cdot \tilde{B}_i := B(\tilde{\xi}_i, 2C_0(M) \cdot \frac{C(m_i-1)}{K\sqrt{m-1}})$. In the critically-centered picture, the a priori estimate tells us that for $k = 1, 2, \ldots q$, $\tilde{f}(\tilde{B}_i) \in 2C_0(M) \cdot \tilde{B}_{i+k}$. By Koebe's distortion theorem, on $2C_0(M) \cdot \tilde{B}_i$, we have $|\tilde{f}'(z)/\mu_i - 1| \le \frac{C(m_i-1)}{K\sqrt{m-1}}$ where $\mu_i = \tilde{f}'(\tilde{\xi}_i)$. Since $\prod \mu_i = m$,

$$\left| \frac{(\tilde{f}^{\circ q})'(z)}{m} - 1 \right| \le C_2 \sum_i \frac{m_i - 1}{K\sqrt{m-1}} \le \frac{C_3\sqrt{m-1}}{K}.$$

It follows that $(\tilde{f}^{\circ q})^{-1}$ maps $\tilde{B}_i$ into itself. As in the $q = 1$ case, we can deduce that $(\tilde{f}^{\circ q})^{-1}$ maps $m_{c\to 0}(B(\xi_i, R))$ into itself. □

10.3. **Bounds on flowers.** Let $f_a \in \mathcal{B}_2$ be a Blaschke product with $a \in \mathcal{B}_{p/q}(C_{\text{small}})$. Denote the p/q-cycle by $\langle \xi_1, \xi_2, \ldots, \xi_q \rangle$. We have seen that if C_{small} is sufficiently small, then the footprint of the flower $\mathcal{F}$ in the quotient torus $T_{p/q}$ is contained in the central angle of width $0.51 \cdot \pi$. Since the dynamics of f_a is nearly linear within

$B(\xi_i, R)$, it follows that if K is sufficiently small, then

$$\mathcal{F} \cap B(\xi_i, R) \subset S(\xi, 0.52 \cdot \pi, R).$$

This proves Theorem 10.1.

Proof of Theorems 10.2. Part (a) from Theorem 10.1, from which (a') follows easily. Since $\tilde{f}^{-1}$ has an inverse branch on the ball $B(\hat{c}, 1)$ of definite size, by Koebe's distortion theorem, we see that $\tilde{\mathcal{F}}_*$ is a near-affine copy of $\tilde{\mathcal{F}}$. To pin down the size and location of the immediate pre-flower in the critically-centered picture, we use the fact that c is the hyperbolic midpoint of $[0, -a]$. It follows that in the critically-centered picture, the center of the flower is $m_{c\to 0}(0) = -c$ while the center of the immediate pre-flower is $m_{c\to 0}(-a) = c$. This proves (b') from which (b) is an easy consequence. Finally, (c) follows from the Schwarz lemma and the trick used in the proof of part (b) of Theorem 6.2. □

10.4. **Proof of the main theorem.** We are now ready to show that

$$||\mu \cdot \chi_{\mathcal{G}}||^2_{\text{WP}} \lesssim \eta^{1/2} \qquad \text{for } a \in \mathcal{H}_{p/q}(\eta) \text{ with } \eta < C_{\text{small}}.$$

When we reflect (10.1) about the critical point, we see that the immediate pre-flower $\mathcal{F}_*$ is contained in the union of the reflections $\bigcup S_i^* \cup B^*$. We claim that:

$$\int_{\mathcal{F}_*} \frac{|dz|^2}{1-|z|} \lesssim \delta_c \sqrt{m-1} \tag{10.8}$$

Assuming the claim, Lemma 7.3 tells us that $\hat{\mathcal{A}}(\mathcal{G}(f_a)) \lesssim \frac{\delta_c\sqrt{m-1}}{h(f_a,m)} \asymp \sqrt{m-1} \asymp \eta^{1/2}$, which tells us that $||\mu \cdot \chi_{\mathcal{G}}||^2_{\text{WP}} \lesssim \eta^{1/2}$ as desired. To prove the claim, we need to carefully reflect the petal about the critical point.

The reflection B^* of the ball $B(0, 1-0.5\cdot R)$ is contained in a horoball of diameter $\asymp \delta_c \cdot K\sqrt{m-1}$. Therefore, $\int_{B^*} \frac{|dz|^2}{1-|z|} \lesssim \delta_c\sqrt{m-1}$. Similar reasoning shows that the reflection S_i^* of S_i is contained in the sector $S(\xi_i^*, 0.53 \cdot \pi, R_i^*)$ with

$$R_i^* \asymp \delta_c \cdot \sqrt{m_i - 1} \cdot \frac{\sqrt{m_i - 1}}{\sqrt{m - 1}} = \delta_c \cdot \frac{m_i - 1}{\sqrt{m - 1}}. \tag{10.9}$$

The total contribution of these sectors to the integral (10.8) is roughly

$$\int_{\bigcup S_i^*} \frac{|dz|^2}{1 - |z|} \asymp \delta_c \sum \frac{m_i - 1}{\sqrt{m - 1}} \asymp \delta_c \sqrt{m - 1}. \tag{10.10}$$

This proves the upper bound. For the lower bound, observe that by Theorems 8.1 and 3.1, there exist balls

$$\mathscr{B}_i = B\biggl(\xi_i \cdot \Bigl(1 - c_1 \cdot \frac{\delta_c}{\sqrt{m - 1}}\Bigr),\ c_2 \cdot \frac{\delta_c}{\sqrt{m - 1}}\biggr) \tag{10.11}$$

lying in the sectors S_i on which $⨍_{\mathscr{B}_i} |v'''/\rho^2(z)|^2 \asymp 1$. The reflection $\mathscr{B}_i^*$ of $\mathscr{B}_i$ is essentially a ball of definite hyperbolic size whose Euclidean center is located at height $\asymp \delta_c \cdot \sqrt{m_i - 1} \cdot \frac{\sqrt{m_i - 1}}{\sqrt{m-1}} = \delta_c \cdot \frac{m_i - 1}{\sqrt{m-1}}$. Since the (repeated) pre-images of $\mathscr{B}_i^*$ are disjoint, and each repeated pre-image is a near-affine copy of $\mathscr{B}_i^*$, the laminated area of $\bigcup_i \hat{\mathscr{B}}_i^*$ is $\asymp \sum \frac{m_i - 1}{\sqrt{m-1}} \asymp \sqrt{m - 1} \asymp \eta^{1/2}$. Thus, the lower bounds match the upper bounds up to a multiplicative constant. This concludes the proof of Theorem 1.2.

11. Limiting Vector Fields

In this chapter, we study the convergence of Blaschke products to vector fields. For a Blaschke product $f_{\mathbf{a}}(z) = z\prod_{i=1}^{d-1}\frac{z+a_i}{1+\overline{a_i}z}$, set $z_i := -a_i$. By a *radial degeneration*, we mean a sequence of Blaschke products $f_{\mathbf{a}} \in \mathcal{B}_d$ such that:

(1) The multiplier of the attracting fixed point tends *(asymptotically) radially* to $e(p/q)$, i.e. $\arg(e(p/q) - a) \to \arg(e(p/q))$.

(2) Each z_i converges to some point $e(\theta_i) \in S^1$.

(3) The limiting ratios of speeds at which the zeros escape are well-defined, i.e.

$$1 - |z_i| \sim \rho_i \cdot (1 - |a|)$$

with $\sum \rho_i = 1$.

To a radial degeneration, one can associate a natural measure μ on the unit circle which takes the escape rates into account: μ gives mass ρ_i/q to $e(\theta_i + j/q)$. (We use the convention that if some of the points coincide, we sum the masses.) We show:

Theorem 11.1. *One can compute:*

$$\kappa(z) = \lim_{a\to 1}\frac{f_{\mathbf{a}}^{\circ q}(z) - z}{1 - |a|} \to -z\int \frac{\zeta + z}{\zeta - z}d\mu_\zeta. \tag{11.1}$$

Furthermore,

$$f_{\mathbf{a}}^{\circ q}(z) - z - (1 - |a|)\kappa(z) = O\Big((1 - |a|)^2\Big) \tag{11.2}$$

uniformly in the closed unit disk away from supp μ.

Examples:

(1) As $a \to 1$ radially in $\mathcal{B}_2$, $f_a \to \kappa_1 := z \cdot \frac{z+1}{z-1} \cdot \frac{\partial}{\partial z}$.

(2) As $a \to e(p/q)$ radially in $\mathcal{B}_2$, $f_a^{\circ q} \to \kappa_{p/q} = ((-1)^{q+1} \cdot z^q)^* \kappa_1$.

Let $\{g^\eta\}_{0<\eta<1}$ be the semigroup generated by κ written in multiplicative notation, i.e. $g^{\eta_1} \circ g^{\eta_2} = g^{\eta_1 \eta_2}$, normalized so that $(g^\eta)'(0) = \eta$. Using (11.2), we promote the algebraic convergence in (11.1) to the dynamical convergence of the high-iterates of $f_{\mathbf{a}}$ to the flow generated by $\kappa(z)$:

Theorem 11.2. *For $0 < \eta < 1$, if we choose high iterates $T_{a,\eta}$ so that $(f_{\mathbf{a}}^{\circ q \cdot T_{a,\eta}})'(0) \to \eta$, then $f_{\mathbf{a}}^{\circ q \cdot T_{a,\eta}} \to g^\eta$ uniformly in the closed unit disk away from supp μ.*

For applications, it is convenient to use the convergence of linearizing coordinates:

Corollary. *As $a \to e(p/q)$ radially, the linearizing coordinates $\varphi_a : \mathbb{D} \to \mathbb{C}$ converge to the linearizing coordinate $\varphi_\kappa := \lim_{\eta \to 1^-} g^\eta(z)/\eta$ for (the semigroup generated by) the limiting vector field κ.*

Remark. More generally, one can consider *linear degenerations* where $a \to e(p/q)$ asymptotically along a linear ray, e.g. $a \approx e(p/q)(1 - \delta + \delta \cdot Ti)$. In this case, the limiting vector field takes the more general form:

$$\kappa(z) = \lim_{a \to 1} \frac{f_{\mathbf{a}}^{\circ q}(z) - z}{1 - |a|} \to -z \int \frac{\zeta + z}{\zeta - z} d\mu_\zeta + Ti \cdot z. \tag{11.3}$$

We call μ the *driving measure* and T the *rotational factor.*

11.1. **Blaschke vector fields.** Before proving Theorem 11.1, let us examine the vector fields that may be obtained by this process. Recall that for a holomorphic vector field κ, the poles of κ are the saddles of the vector field, while the zeros are sources if Re $\kappa'(z) > 0$ and sinks if Re $\kappa'(z) < 0$ (if Re $\kappa'(z) = 0$, then z is a “center” but in our case, it does not occur).

Observe that for $\zeta \in S^1$, the map $z \to \frac{\zeta+z}{\zeta-z}$ takes the disk to the right half-plane. Therefore, $\int \frac{\zeta+z}{\zeta-z} d\mu_\zeta$ is purely imaginary and monotone decreasing in $\arg z$ (except at the poles of κ). It follows that $\kappa = -z \int \frac{\zeta+z}{\zeta-z} d\mu_\zeta$ is tangent to the unit circle, has simple poles and in between any two poles has a unique zero. Since $a^q \to 1$ radially, it

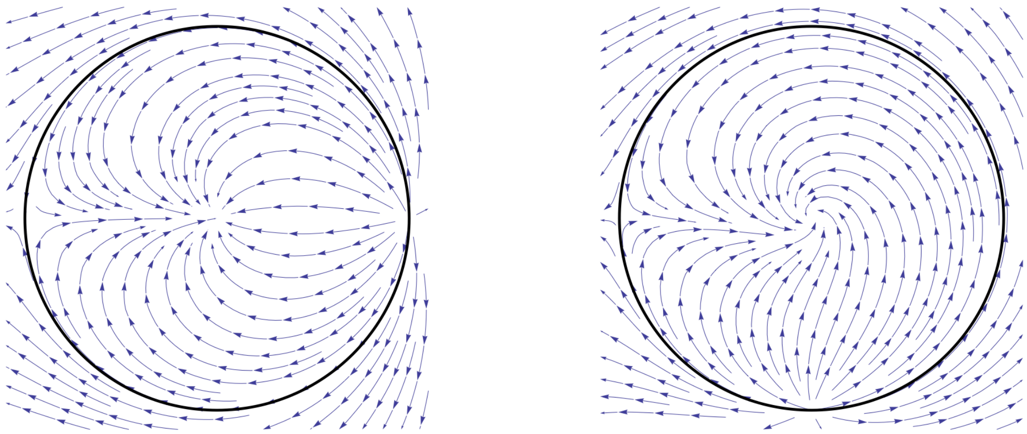

FIGURE 8. The vector fields $z \cdot \frac{z-1}{(z+1)} \cdot \frac{\partial}{\partial z}$ and $\Big(z \cdot \frac{z-1}{(z+1)} + iz\Big)\frac{\partial}{\partial z}$.

follows that $\kappa'(0) = -1$ and so 0 is a sink. It can be shown that the converse is true: any vector field which satisfies the above properties comes from a radial degeneration of Blaschke products, but we will not need this fact so we won't prove it here.

Lemma 11.1. *Let $M_a(z) = \frac{z+a}{1+\bar{a}z}$. Suppose $a \approx A \in S^1$ with $a = A(1 - \delta + \delta \cdot Ti)$ where δ. Then,*

$$\frac{M_a(z)/A - 1}{1 - |a|} = \Big(-\frac{A - z}{A + z} + Ti\Big) + O\Big((1 - |a|)^2\Big). \tag{11.4}$$

where the estimate is uniform for a in any non-tangential sector at A.

Proof. This is an exercise in differentiation. One simply needs to compute

$$\frac{\partial}{\partial\delta}\Big|_{\delta=0} \frac{1}{A} \cdot \frac{z + A(1 - \delta + \delta \cdot Ti)}{1 + (1/A)(1 - \delta - \delta \cdot Ti)z} = \frac{z - A}{z + A} + Ti$$

and use the fact that $1 - |a| \approx \delta$. □

We first prove Theorem 11.1 in the case when $a \to 1$. For a Blaschke product $f_{\mathbf{a}}(z) = z \prod_{i=1}^{d-1} \frac{z+a_i}{1+\overline{a_i}z}$, let $A_i = \hat{a}_i$, $A = \hat{a}$ and $T = T(f_{\mathbf{a}}) = -i \cdot \frac{A-1}{1-|a|}$. The idea is to compare $f_{\mathbf{a}}(z)$ to the vector field $\kappa(f_{\mathbf{a}})$ given by (11.3) with driving measure $\mu(f_{\mathbf{a}}) = \sum \frac{1-|a_i|}{1-|a|} \cdot \delta_{-A_i}$ and rotational factor $T(f_{\mathbf{a}})$:

Lemma 11.2. *We have the estimate:*

$$f_{\mathbf{a}}(z) - z - (1-|a|)\kappa(z) = O\Big((1-|a|)^2\Big) \tag{11.5}$$

uniformly in the closed unit disk away from supp μ.

Proof. Using that $\frac{z+a_i}{1+\overline{a_i}z} \approx A_i$, $\prod A_i = 1$ and $\prod(1+\delta_i) = 1 + \sum \delta_i + O(\max|\delta_i|^2)$,

$$f_{\mathbf{a}}(z) - z = z\left(\prod \frac{z+a_i}{1+\overline{a_i}z} - \prod A_i\right) + z(A-1) \approx z\sum\Big(\frac{1}{A_i}\frac{z+a_i}{1+\overline{a_i}z} - 1\Big) + z(A-1).$$

Therefore,

$$\frac{f_{\mathbf{a}}(z)-z}{1-|a|} \approx -z\sum \rho_i \cdot \left(\frac{A_i - z}{A_i + z}\right) + Ti\cdot z = -z\int \frac{\zeta+z}{\zeta-z}d\mu_\zeta + Ti\cdot z$$

as desired. □

Theorem 11.1 now follows in the case when $a \to 1$ since for radial degenerations, the rotational factor $T(f_{\mathbf{a}}) \to 0$.

Radial degenerations with $a \to e(p/q)$. As noted above, for a radial degeneration with $a \to e(p/q)$, we consider the limiting vector field of $f_{\mathbf{a}}^{\circ q}$ rather than of $f_{\mathbf{a}}$. To show that $f_{\mathbf{a}}^{\circ q}$ converges to a vector field κ whose driving measure gives mass ρ_i/q to each point $e(\theta_i + j/q)$, it suffices to analyze the zero set of $f_{\mathbf{a}}^{\circ q}$.

Let us first consider the case of a generic radial degeneration (i.e. when the points $e(\theta_i + j/q)$ are all different). The zero set of $f_{\mathbf{a}}^{\circ q}$ consists of the zeros of $f_{\mathbf{a}}$ and their $1, 2, \ldots, (q-1)$-fold pre-images. We omit the trivial zero at the origin and split the remaining zeros of $f_{\mathbf{a}}^{\circ q}$ into two groups: the *dominant zeros* and *subordinate zeros*. The dominant zeros are the zeros $z_i = z_{i,0}$ of $f_{\mathbf{a}}(z)$ and their shadows $z_{i,j}$ near $z_i \cdot e(-j\cdot p/q)$. We will refer to all other zeros as the subordinate zeros. From formula (7.3), it follows the heights of the subordinate zeros are insignificant compared to the heights of the dominant zeros. Thus, only the dominant zeros contribute to the limiting vector field.

Let us now consider the general case. For a point $z \in \mathbb{D}$ with $|z| \geq a$, say that w is a *dominant pre-image* of z under $f_{\mathbf{a}}$ if it is located near $e(-p/q)\hat{z}$ and is a *subordinate pre-image* otherwise. By a *dominant zero* of $f_{\mathbf{a}}^{\circ q}$, we mean a point $z \in \mathbb{D}$ which is the k-fold dominant pre-image of z_i for some $0 \leq k \leq q-1$. To show that the driving measure μ has the desired expression, it suffices to show that the subordinate zeros have negligible height. We prove this in two lemmas:

Lemma 11.3. *Suppose $f_{\mathbf{a}}(z) = z \prod \frac{z+a_i}{1+\overline{a_i}z}$ is a Blaschke product with $|a| = |f'(0)| \approx 1$. For K sufficiently large, in the ball $B(0, 1-K\sqrt{1-|a|})$, the map $f_{\mathbf{a}}$ is close to rotation by $\hat{a}$. More precisely, $\rho_{\mathbb{D}}(f(z), \hat{a}\cdot z) < C(K)$ with $C(K) \to 0$ as $K \to \infty$.*

Proof. The map $z \to \frac{z+a_i}{1+\overline{a_i}z}$ takes the ball $B(0, 1-K\sqrt{1-|a|})$ inside the ball

$$B\Big(\hat{a}_i, (C_1/K)\cdot\sqrt{1-|a|}\cdot\frac{1-|a_i|}{1-|a|}\Big).$$

Therefore, $|f_{\mathbf{a}}(z) - \hat{a}z| \leq (C_2/K)\cdot\sqrt{1-|a|}$ as desired. □

Lemma 11.4. *Suppose w satisfies $f(w) = z$ yet $|\hat{w} - e(-p/q)\hat{z}| \geq \epsilon$. Then,*

$$\frac{G(w)}{G(z)} = O_\epsilon(1-|a|). \tag{11.6}$$

Proof. Consider the hyperbolic geodesic $[0, w]$. Set $w_0 := (1-K\sqrt{1-|a|})\cdot w$ and write $[0,w] = [0,w_0]\cup[w_0,w]$. Since $f_{\mathbf{a}}$ restricted to the first segment $[0,w_0]$ is nearly rotation by $e(p/q)$, we see that during the first part of the journey from $f(0) = 0$ to $f(w) = z$ along $f([0,w])$, we have moved in the wrong direction, i.e.

$$d_{\mathbb{D}}(f(w_0), f(w)) = d_{\mathbb{D}}(0, w_0) + d_{\mathbb{D}}(0, f(z)) - O_\epsilon(1).$$

Since a Blaschke product is a contraction in the hyperbolic metric, we must have $d_{\mathbb{D}}(w_0, w) \geq d_{\mathbb{D}}(f(w_0), f(w))$ to make up for this detour. □

11.2. **Asymptotic semigroups.** By an *asymptotic semigroup*, we mean a family of holomorphic maps $\{f_t\}_{t\geq 0}$ converging uniformly to the identity map as $t \to 0$ on compact subsets of a domain Ω, such that

$$\left| f_t(z) - f_{t_1}(f_{t_2}(z)) \right| \leq O_K(t^2), \qquad (t = t_1 + t_2), \tag{11.7}$$

where the notation O_K denotes that the implicit constant is uniform on a compact subsets of Ω. It turns out that (11.7) is equivalent to the apparently stronger condition that there exists a holomorphic vector field κ on Ω satisfying

$$f_t = z + t \cdot \kappa(z) + O_K(t^2). \tag{11.8}$$

In this section, we will show that the condition (11.7) implies that the short term iteration of f_t approximates the flow of κ:

Theorem 11.3. *For $z \in B(z_0, R)$ compactly contained in Ω, for small time t, the limit*

$$g_t(z) := \lim_{\max t_i \to 0} f_{t_n}(f_{t_{n-1}}(\cdots(f_{t_1}(z))\cdots)) \tag{11.9}$$

over all possible partitions $t_1 + t_2 + \cdots + t_n = t$ exists, and defines a holomorphic function.

Remark. By uniqueness, $\{g_t\}$ satisfies $g_s \circ g_t = g_{s+t}$ as long as g_{s+t} is well-defined. We can recover κ be the generator of $\{g_t\}$.

Proof. Choose two balls $B(z_0, R'') \supset B(z_0, R') \supset B(z_0, R)$ compactly contained in Ω. We will first show that if t is sufficiently small, then for $z \in B(z_0, R)$, all intermediate computations of (11.9) stay within $B(z, R')$.

Now we make the following "partitioning" argument: we first consider very simple partitions with $n = 2^k$ and all the $t_i = t/2^k$. We imagine that we begin with one interval of length t. We split this interval in half and pay the cost of $C \cdot t^2$. We now have two intervals of size $t/2$. We split both of those intervals in half and pay the cost $C \cdot (t/2)^2$ for each splitting. We continue doing this until we have intervals of length $t/2^k$. We see that the total splittings that occur at a j-th step cost $C \cdot t^2/2^{j-1}$. Thus the total cost of all splittings that are used to form our subdivision is bounded by $2C \cdot t^2$. Clearly, this argument also applies to any "balanced" subdivision where all $\epsilon \le t_i \le 2\epsilon$ (with a larger constant). However, for any "unbalanced" subdivision, we can pay the cost $O(\max t_i)$ to make it balanced: namely, we keep splitting intervals in half whose size exceeds twice the smallest interval. Thus any subdivision (balanced or otherwise) costs $O(t^2)$.

Since $f_t(z)$ converges uniformly to the identity on $B(z_0, R'')$, it is easy to see that when $t > 0$ is sufficiently small, all the intermediate compositions $f_{t_k} \circ \cdots \circ f_{t_1}(z)$ stay in $B(z_0, R')$. Therefore,

$$d_{B(z_0,R'')}\Big(z, f_{t_n} \circ \cdots \circ f_{t_1}(z)\Big) \le \sum_{k=1}^{n} d_{B(z_0,R'')}\Big(f_{t_k} \circ \cdots \circ f_{t_1}(z),\ f_{t_{k-1}} \circ \cdots \circ f_{t_1}(z)\Big).$$

To combine the "costs," we use the fact that on $B(z_0, R')$, the hyperbolic metric $\rho_{B(z_0,R'')}$ is comparable to the Euclidean metric. Therefore, the partitioning argument above shows that the limit (11.9) converges. □

Theorem 11.2 is a special case of Theorem 11.3, where $\Omega = \mathbb{C} \setminus \mathcal{P}(\kappa)$ is the complement of the set of poles of $\kappa(z)$. By the Schwarz lemma, inside the unit disk, $g^t(z)$ can be defined for *all* time, where as on the unit circle, we can only define $g^t(z)$ until we hit a pole of $\kappa(z)$.

12. Asymptotics of the Weil-Petersson metric

In this chapter, we show Theorem 1.3 which says that as $a \to e(p/q)$ radially in $\mathcal{B}_2$, the ratio $\omega_B/\rho_{\mathbb{D}^*} \to C'_{p/q}(1-|a|)^{1/4}$. As noted in the introduction, the key to this result is the convergence of Blaschke products to vector fields. By the corollary to Theorem 11.2, it follows that:

Lemma 12.1. *As $a \to e(p/q)$ radially,*

(*i*) *The flowers $\mathcal{F}_{p/q}(f_a) \to \mathcal{F}_{p/q}(\kappa_{p/q})$ in the Hausdorff topology.*

(*ii*) *The optimal Beltrami coefficients $\mu_\lambda(f_a) = \varphi_a^*(\lambda \cdot z/\overline{z} \cdot d\overline{z}/dz)$ converge uniformly to $\varphi^*_{\kappa_{p/q}}(\lambda \cdot z/\overline{z} \cdot d\overline{z}/dz)$ on compact subsets of $\mathbb{D}^*$.*

Together with Lemma 10.1, this implies:

Lemma 12.2 (Quasi-geodesic property)**.** *As $a \to e(p/q)$ radially, each petal $\mathcal{P}_{\xi_i(f_a)}(f_a)$ lies within a bounded distance of the geodesic ray $[0, \xi_i(f_a)]$. Alternatively, the flower $\mathcal{F}(f_a)$ lies within a bounded neighbourhood of the hyperbolic convex hull of the origin and the ends $\xi_i(f_a)$.*

Since the flowers of the maps f_a (with a close to $e(p/q)$) have nearly the same shape, it follows that the pre-flowers of all f_a must also have nearly the same affine shape. Let $n(r, f_a)$ denote the number of pre-flowers that intersect the circle $S_r = \{z : |z| = r\}$ and μ_r be the probably measure which gives equal mass to the $n(r, f_a)$ pre-images of the repelling fixed point. Using renewal theory (Chapter 7), it is easy to see that:

Theorem 12.1. *As $a \to e(p/q)$ radially,*

(*a*) *The limit $c(f_a) = \lim_{r\to 1} \frac{n(r,f_a)}{1-r}$ exists.*

(*b*) *As $a \to e(p/q)$, $c(f_a) \sim C'_{p/q}(1-|a|)^{1/2}$.*

(*c*) *The measures μ_r tend weakly to the Lebesgue measure on the circle.*

Proof. To see part (*b*), observe that the size of the immediate pre-flower decays like $\sim (1-|a|)$. As the entropy $h(f_a) \sim \sqrt{1-|a|}$, it follows that $c(f_a) \sim \sqrt{1-|a|}$. $\square$

Flower counting hypothesis. To prove Theorem 1.3, we will show that as $a \to e(p/q)$ radially, $||\mu \cdot \chi_{\mathcal{G}}||^2_{\mathrm{WP}} \sim c(f_a)$. Intuitively, for r close to 1, the circle S_r intersects pre-flowers at "hyperbolically random" locations. However, we must be slightly careful since the pre-flowers whose size is less than $1-r$ (i.e. ones which do not intersect the circle S_r) still contribute to the integral average (1.4). To justify the intuition, we must show three things:

- The contributions of the pre-flowers are more or less independent.
- All pre-flowers of the same size contribute roughly the same amount.
- Most of the integral $\int_{|z|=1-r} |v'''/\rho^2|^2 d\theta$ comes from pre-flowers whose size is $\asymp 1-r$.

12.1. **Decay of correlations.** In this section, we use "flower" to mean either a flower or a pre-flower. Write the half-optimal coefficient $\mu = \sum_{\mathcal{F}} \mu_{\mathcal{F}}$ with $\mu_{\mathcal{F}}$ supported on $\mathcal{F}$. For a flower $\mathcal{F}$, set

$$v'''_{\mathcal{F}}(z) = \int_{\mathcal{F}} \frac{\mu(\zeta)}{(\zeta - z)^4}\, dA_\zeta.$$

Then $v'''(z) = \sum_{\mathcal{F}} v'''_{\mathcal{F}}(z)$. We wish to show that the integral average in (1.4) is proportional to the flower count. The main difficulty is that (1.4) features the L^2 norm so we have "correlations" $\sum_{\mathcal{F}_1 \neq \mathcal{F}_2} \int \frac{v'''_{\mathcal{F}_1}(z)}{\rho^2} \cdot \frac{\overline{v'''_{\mathcal{F}_2}(z)}}{\rho^2}$. We claim that these correlations are insignificant compared to the main term $\sum_{\mathcal{F}} \int \left|\frac{v'''_{\mathcal{F}}(z)}{\rho^2}\right|^2$.

For a point $z \in \mathbb{D}$, let $\mathcal{F}_z$ be the flower which is closest to z in the hyperbolic metric and $\mathcal{R}_z$ be the union of all the other flowers. The integral average (1.4) splits as follows:

$$\int \left|\frac{v'''_{\mathcal{F}_z}(z)}{\rho^2}\right|^2 + \frac{v'''_{\mathcal{F}_z}(z)}{\rho^2} \cdot \frac{\overline{v'''_{\mathcal{R}_z}(z)}}{\rho^2} + \frac{v'''_{\mathcal{R}_z}(z)}{\rho^2} \cdot \frac{\overline{v'''_{\mathcal{F}_z}(z)}}{\rho^2} + \left|\frac{v'''_{\mathcal{R}_z}(z)}{\rho^2}\right|^2$$

By the lower bounds established in Chapter 10, the first term is bounded below by the flower count which decays roughly like $\sim \sqrt{1-|a|}$. Each of three other terms contribute on the order of $O(1-|a|)$, and so are negligible. Take for instance the second term. By the triangle inequality, for any $z \in \mathbb{D}$,

$$\frac{v'''_{\mathcal{F}_z}(z)}{\rho^2} \cdot \frac{\overline{v'''_{\mathcal{R}_z}(z)}}{\rho^2} \lesssim e^{-d_{\mathbb{D}}(z,\mathcal{F}_z)} \cdot e^{-d_{\mathbb{D}}(z,\mathcal{R}_z)} \leq e^{-d_{\mathbb{D}}(\mathcal{F}_z,\mathcal{R}_z)}$$

This is bounded by $e^{-\rho(0,a)} \sim (1-|a|)$. The estimate for the other two error terms is similar.

12.2. **Convergence of Beltrami differentials.** For a Blaschke product with $a \approx e(p/q)$, define the idealized flower as $\mathcal{F}^{\text{id}}(f_a) := \mathcal{F}(g^\eta)$. Define the idealized immediate pre-flower $\mathcal{F}^{\text{id}}_*(f_a)$ as the Möbius involution of $\mathcal{F}(g^\eta)$ about $c(f_a)$. For all the other pre-flowers, let $\mathcal{F}^{\text{id}}_z(f_a)$ be an affine copy of $\mathcal{F}^{\text{id}}_*(f_a)$ centered at z. We define the idealized half-optimal Beltrami coefficient in a similar manner: on $\mathcal{F}^{\text{id}}(f_a)$, we let $\mu_{\text{id}} \cdot \chi_{\mathcal{F}^{\text{id}}}$ be the half-optimal Beltrami coefficient for the limiting vector field; while on the pre-flowers, we define $\mu_{\text{id}} \cdot \chi_{\mathcal{F}^{\text{id}}_z}$ by scaling $\mu_{\text{id}} \cdot \chi_{\mathcal{F}^{\text{id}}}$ appropriately. Let us denote the genuine half-optimal Beltrami coefficient by $\mu_{\text{half}} := \mu \cdot \chi_{\mathcal{G}}$. We claim that:

Lemma 12.3. *The difference*

$$\lim_{r\to 1} \int_{|z|=r} |v_{\mu_{\text{id}}}/\rho^2|^2 d\theta - \lim_{r\to 1} \int_{|z|=r} |v_{\mu_{\text{half}}}/\rho^2|^2 d\theta \leq \epsilon(a)\sqrt{1-|a|}$$

where $\epsilon(a) \to 0$ *as* $a \to e(p/q)$.

There are two sources of error. First, the pre-flowers don't quite match up with their idealized counterparts. Secondly, since the linearizing maps φ_a and φ_κ are slightly different, the Beltrami coefficients μ_{half} and μ_{ideal} themselves are slightly different.

Estimating the Symmetric Difference. Let us examine the symmetric difference between the (pre-)flowers and their idealized counterparts. For this purpose, given $\delta > 0$, we split the flower $\mathcal{F}^\alpha$ into three parts: the core $\mathcal{C}^\alpha_\delta$, the body $\mathcal{B}^\alpha_\delta$ and the ends $\mathcal{E}^\alpha_\delta = \bigcup_i \mathcal{E}^{\alpha,i}_\delta$:

$$\begin{aligned}\mathcal{C}^\alpha_\delta &= \mathcal{F}^\alpha \cap \{z : |z| < \delta\},\\ \mathcal{B}^\alpha_\delta &= \mathcal{F}^\alpha \cap \{z : \delta < |z| < 1-\delta\},\\ \mathcal{E}^\alpha_\delta &= \mathcal{F}^\alpha \cap \{z : 1-\delta < |z|\}.\end{aligned}$$

When a is sufficiently close to $e(p/q)$, the symmetric difference of $\mathcal{F}(f_a)$ and $\mathcal{F}^{\mathrm{id}}(f_a)$ is contained in

$$S(\mathcal{F}) := \mathcal{C}(f_a) \cup \mathcal{C}(g^\eta) \;\cup\; \Big(\mathcal{B}^{1/2+\epsilon}(g^\eta) \setminus \mathcal{B}^{1/2-\epsilon}(g^\eta)\Big) \;\cup\; \mathcal{E}(f_a) \cup \mathcal{E}(g^\eta). \tag{12.1}$$

We can define the core, body and ends of a pre-flower $\mathcal{F}_z$ as the pre-image of the corresponding part of $\mathcal{F}$. Similarly, we construct a set $S(\mathcal{F}_z)$ which contains the symmetric difference between the pre-flowers and their idealized versions. Let $S = \bigcup S(\mathcal{F}_z)$.

Write $\mu_{\mathrm{half}} = \mu_{\mathrm{half}} \cdot \chi_S + \mu_{\mathrm{half}} \cdot \chi_{S^c}$ and $v'''_{\mathrm{half}} = v'''_{\mu_{\mathrm{half}}\cdot\chi_S} + v'''_{\mu_{\mathrm{half}}\cdot\chi_{S^c}}$. The triangle inequality tells us that

$$|v'''_{\mu_{\mathrm{half}}\cdot\chi_S}|^2 - |v'''_{\mu_{\mathrm{half}}\cdot\chi_{S^c}}|^2 \le \Big|v'''_{\mu_{\mathrm{half}}\cdot\chi_S} + v'''_{\mu_{\mathrm{half}}\cdot\chi_{S^c}}\Big|^2 \le |v'''_{\mu_{\mathrm{half}}\cdot\chi_S}|^2 + |v'''_{\mu_{\mathrm{half}}\cdot\chi_{S^c}}|^2 \tag{12.2}$$

By Theorem 2.1, the integral average over $|v'''_{\mu_{\mathrm{half}}\cdot\chi_{S^c}}|^2$ is insignificant as the proportion

$$\limsup_{r\to 1} \frac{|\operatorname{supp}\mu_{\mathrm{half}} \cap S^c \cap \{z : |z| = r\}|}{|\operatorname{supp}\mu_{\mathrm{half}} \cap \{z : |z| = r\}|} \le \epsilon(\delta), \tag{12.3}$$

with $\epsilon(\delta) \to 0$ as $\delta \to 0$. The same trick allows us to replace μ_{id} by $\mu_{\mathrm{id}} \cdot \chi_S$.

Estimating the Difference between Beltrami Coefficients. The other source of error comes from the fact that the Beltrami coefficients μ_{half} and μ_{id} are slightly different as the linearizing maps φ_a and $\varphi(g^\eta)$ are slightly different. However, if a is sufficiently close to $e(p/q)$, the L^∞ norm of $|\mu_{\text{half}} - \mu_{\text{id}}|$ is arbitrarily small on $(\mathcal{B}^{\text{id}})^{1/2+\epsilon}(f_a) := \mathcal{B}^{1/2+\epsilon}(g^\eta)$. Hence, the same is true for pre-flowers. Since the difference $|\mu_{\text{half}} - \mu_{\text{id}}|$ is small in L^∞ sense, by part (a) of Theorem 2.2, the difference $\int |v'''_{\text{half}} - v'''_{\text{id}}|^2 \, d\theta$ is small. Using the triangle inequality as before completes the proof.

12.3. **Flowers: large and small.** We now show that for r sufficiently close to 1, most of the integral average $\fint_{S_r} |v'''/\rho^2|^2 d\theta$ comes from petals whose size is $\asymp (1-r)$. By mixing, for any $\epsilon > 0$, we can find an $r_{\text{mix}} = r_{\text{mix}}(\epsilon) < 1$ such that for $r \in [r_{\text{mix}}, 1)$, $\frac{n(r,f_a)}{(1-r)} \approx_\epsilon c(f_a)$. For a point z with $|z| = r$, write:

$$v'''(z)/\rho^2 = v'''_{\text{small}}(z)/\rho^2 + v'''_{\text{med}}(z)/\rho^2 + v'''_{\text{large}}(z)/\rho^2 + v'''_{\text{huge}}(z)/\rho^2 \tag{12.4}$$

where

$$\begin{cases} \text{small flowers} & \text{have size } s \le (1-r)/k \\ \text{medium flowers} & \text{have size } (1-r)/k \le s \le k(1-r) \\ \text{large flowers} & \text{have size } k(1-r) \le s \le 1 - r_{\text{mix}} \\ \text{huge flowers} & \text{have size } s \ge 1 - r_{\text{mix}} \end{cases}$$

From the the lower bound, it follows that the integral average $v'''_{\text{med}}(z)/\rho^2$ over only the medium flowers is $\asymp c(f_a)$. We claim that if we choose the "tolerance" k sufficiently large, then the other flowers contribute at most $\epsilon_2(k) \cdot c(f_a)$ where $\epsilon_2(k)$ can be made arbitrarily close to 0. By inspecting the proof of Theorem 2.1, it is easy to show that the small flowers contribute $\lesssim c(f_a)/k$ to the integral average. Since there are finitely many huge flowers and they satisfy the quasi-geodesic property, their contribution decays to 0 as $r \to 1$. Finally, the large flowers also satisfy the quasi-geodesic property, and by Theorem 2.1, they also contribute at most $\lesssim c(f_a)/k$. This completes the proof of the claim and therefore, the theorem.

12.4. **An alternate route.** In this section, we give a slightly different approach to Theorem 1.3. For a set $K \subset \mathbb{D}$, let $K(R) := \{z : d_{\mathbb{D}}(z, K) < R\}$ and $K(R_1, R_2) := \{z : R_1 \le d_{\mathbb{D}}(z, K) < R_2\}$. We will show that most of the integral average comes from $\hat{\mathcal{G}}(R)$. In particular, this tells us that we can use renewal theory to estimate the integral average (1.4):

Theorem 12.2. *For any $\epsilon > 0$, there exists an $R > 0$ such that when a is sufficiently close to $e(p/q)$,*

$$\lim_{r\to 1} \frac{1}{2\pi} \int_{|z|=r} |v'''/\rho^2|^2 d\theta \approx_\epsilon \lim_{r\to 1} \frac{1}{2\pi} \int_{S_r \cap \mathcal{G}(R)} |v'''/\rho^2|^2 d\theta \tag{12.5}$$

$$\approx_\epsilon \frac{1}{2\pi h(f_a)} \int_{\mathcal{F}_*(R)} \frac{|v'''/\rho^2|^2}{1-|z|} \cdot |dz|^2. \tag{12.6}$$

To prove the above theorem, we need the following simple observation:

Lemma 12.4. *The hyperbolic distance $d_{\mathbb{D}}(\mathcal{G}, \widehat{\{c\}}) \ge d_{\mathbb{D}}(0, c) - O(1)$.*

Proof. The proof follows from the Schwarz lemma. Suppose we want to estimate the hyperbolic distance $d_{\mathbb{D}}(\mathcal{F}_z, c')$ from a (pre-)flower to a (pre-)image of a critical point. Then either $d_{\mathbb{D}}(\mathcal{F}_z, c') \ge d_{\mathbb{D}}(\mathcal{F}, c)$ or $d_{\mathbb{D}}(\mathcal{F}_z, c') \ge d_{\mathbb{D}}(\mathcal{F}_*, B(0, |c|))$. In either case, $d_{\mathbb{D}}(\mathcal{F}_z, c') \ge d_{\mathbb{D}}(0, c) - O(1)$ as desired. □

Proof of Theorem 12.2. By Lemma 2.5, for any $\epsilon > 0$, we can choose $t(\epsilon) > 0$ so that the dynamics in $U_t := \{z : 1 - t \cdot \delta_c \le |z| < 1\}$ is sufficiently affine to guarantee that:

$$\lim_{r\to 1} \int_{S_r \cap \widehat{\mathcal{F}^1_* \cap U_t}} |v'''/\rho^2|^2 d\theta \approx_\epsilon \frac{1}{2\pi h(f_a)} \int_{\mathcal{F}^1_* \cap U_t} \frac{|v'''/\rho^2|^2}{1-|z|} \cdot |dz|^2 \tag{12.7}$$

It remains to estimate the error terms.

By Lemma 12.4, on the saturation of $E_1 = \mathcal{F}^1_* \cap U_t^c$, we have $|v'''/\rho^2|^2 = O(\delta_a)$. Therefore, the part of the integral average over $\hat{E}_1$ is insignificant compared to the main term. Write:

$$E_2 = (\mathcal{F}^1_* \cap U_t) \setminus \mathcal{F}_*(R) = \bigcup_{n=0}^{\infty} E_2^n = \mathcal{F}_*(R+n, R+n+1) \cap U_t. \tag{12.8}$$

Observe that on $\hat{E_2^n}$, we have $|v'''/\rho^2|^2 \lesssim e^{-2(R+n)}$, while by Lemma 7.3, the laminated area $\hat{\mathcal{A}}(\hat{E_2^n}) = O(e^{R+n} \cdot \delta_c)$. Therefore,

$$\limsup_{r\to 1} \int_{S_r \cap \hat{E_2^n}} |v'''/\rho^2|^2 d\theta \lesssim e^{-R} \cdot \delta_c. \tag{12.9}$$

Summing n from 0 to infinity, we see that $\hat{\mathcal{A}}(\hat{E}_2) = O(e^{-R} \cdot \delta_c)$ as desired. □

Index of Notation

* $\mathcal{T}_{g,n}$ – Teichmüller space of Riemann surfaces of genus g with n punctures
* $\text{Mod}_{g,n}$ – the mapping class group
* $||\mu||_{\text{WP}}$, $||\mu||_T$ – Weil-Petersson and Teichmüller norms of a Beltrami coefficient
* $Q(X)$ – holomorphic quadratic differentials with at worst simple poles on $X \in T_{g,n}$
* $M(X)$ – bounded measurable Beltrami coefficients on $X \in T_{g,n}$
* $M_1(X)$ – unit ball in $M(X)$, i.e. Beltrami coefficients with $||\mu||_\infty < 1$
* $\rho = \frac{2|dz|}{1-|z|^2}$ – the hyperbolic metric on the unit disk
* ρ_α – for $\alpha > 0$, incomplete model metrics on the upper half-plane invariant under $\text{SL}(2,\mathbb{Z})$
* $d_{\mathbb{D}}(z_1, z_2)$ – the hyperbolic distance between z_1 and z_2
* $[z_1, z_2]$ – for $z_1, z_2 \in \mathbb{D}$, the geodesic connecting z_1 and z_2 in the hyperbolic metric
* $\mathcal{B}_d$ – the space of degree d Blaschke products
* $A \lesssim B$ means that $A < \text{const} \cdot B$
* $A \sim B$ means that $A/B \to 1$
* $A \asymp B$ means that $C_1 \cdot B < A < C_2 \cdot B$ for some constants C_1, C_2
* $A \approx_\epsilon B$ means that $|A/B - 1| \lesssim \epsilon$
* S_r – the circle $\{z : |z| = r\}$
* $S(\zeta, \theta, R) = \{z : \arg(z/\zeta - 1) \in (\pi - \frac{\theta}{2}, \pi + \frac{\theta}{2})\} \cap B(\zeta, R)$ – sector at $\zeta \in S^1$
* $B_{p/q}(\eta)$ – a horoball in the upper half-plane resting at $e(p/q)$ of diameter η/q^2
* $H_{p/q}(\eta) := \partial B_{p/q}(\eta)$ – a horocycle in the upper half-plane
* $\mathcal{B}_{p/q}(\eta/q^2) = a(B_{p/q}(\eta/q^2)), \mathcal{H}_{p/q}(\eta/q^2) = a(H_{p/q}(\eta/q^2))$ where $a(\tau) = e^{2\pi i\tau}$
* $\delta_a = 1 - |a|$ for $f \in \mathcal{B}_2$, more generally, $\delta_a = \inf_i(1 - |a_i|)$ for $f \in \mathcal{B}_d$
* c – the critical point of a Blaschke product $f \in \mathcal{B}_2$ in the unit disk

* $\delta_c = 1 - |c|$ for $f \in \mathcal{B}_2$, more generally, $\delta_c = \inf_i(1 - |c_i|)$ for $f \in \mathcal{B}_d$
* $U_t := \{z : 1 - t \cdot \delta_c \le |z| < 1\}$ where $t < 1$ – linearity zone
* $\tilde{f} = m_{c\to 0} \circ f \circ m_{0\to c}$ – critically-centered version of a Blaschke product $f \in \mathcal{B}_2$
* $\varphi_a(z) := \lim_{n\to\infty} a^{-n} \cdot f_a^n(z)$ – the linearizing map of a Blaschke product f_a
* $\mathcal{G}(f_a), \mathcal{F}(f_a), \{\mathcal{P}_i(f_a)\}$ – the garden, flower, petals of a map $f \in \mathcal{B}_2$
* $\mathcal{F}_*(f_a), \mathcal{P}_*(f_a)$ – immediate pre-flower, immediate pre-petal
* $m_{c\to 0}(z) = \frac{z+c}{1+\bar{c}z}$, $m_{0\to c}(z) = \frac{z-c}{1-\bar{c}z}$.
* $\tilde{\mathcal{G}}(f_a), \tilde{\mathcal{F}}(f_a), \{\tilde{\mathcal{P}}_i(f_a)\}$ – critically-centered versions of the garden, flower and petals
* $m_{p/q} = |(f^{\circ q})'(\xi_1)|$ – the multiplier of the p/q-cycle of a Blaschke product $f \in \mathcal{B}_2$
* $G(z) = \log \frac{1}{|z|}$ – the Green's function of the unit disk with a pole at the origin
* $\hat{E} = \{\zeta : f^{\circ n}(\zeta) \in E \text{ for some } n \ge 0\}$ – where $E \subset \mathbb{D}$ is a set
* $\hat{z} = z/|z|$ – when $z \in \mathbb{C}^*$ is a point
* $z^+ = 1/\overline{z}$ is the reflection of a point z in the unit circle
* $\mu^+ = (1/\overline{z})^*\mu$ is the reflection of a Beltrami coefficient in the unit circle
* $\kappa_1(z) = z \cdot \frac{z+1}{z-1} \cdot \frac{\partial}{\partial z}$ and $\kappa_{p/q}(z) = ((-z)^q)^*\kappa_1$
* $\{g^\eta\}$ with $\eta \in (0,1)$ – semigroup generated by a vector field: $g^{\eta_1\eta_2} = g^{\eta_1} \circ g^{\eta_2}$
* $\varphi_\kappa(z) := \lim_{\eta\to 1^-} g^\eta(z)/\eta$ – the linearizing map of (the semigroup generated by a) radial Blaschke vector field κ

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