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The complex geometry of Teichmüller space

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The complex geometry of Teichmüller space

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Abstract

We study isometric maps between Teichmüller spaces and bounded symmetric domains in their *Kobayashi metric*. We prove that every totally geodesic isometry from a disk to Teichmüller space is either holomorphic or anti-holomorphic; in particular, it is a Teichmüller disk. However, we prove that in dimensions two or more there are no holomorphic isometric immersions between Teichmüller spaces and bounded symmetric domains and also prove a similar result for isometric submersions.

CONTENTS

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To my beloved parents, sister and nephew.

1. Introduction

In this dissertation we will study isometric maps between Teichmüller spaces $\mathcal{T}_{g,n} \subset \mathbb{C}^{3g-3+n}$ and bounded symmetric domains $B \subset \mathbb{C}^N$ in their *Kobayashi metric*.

In a nutshell, we will prove that every *totally geodesic isometry* $\Delta \cong \mathbb{CH}^1 \overset{f}{\hookrightarrow} \mathcal{T}_{g,n}$ is either *holomorphic* or *anti-holomorphic*. However, when $\dim_{\mathbb{C}} B, \dim_{\mathbb{C}} \mathcal{T}_{g,n} \geq 2$ we will show that there are *no holomorphic isometric immersions* $B \overset{f}{\hookrightarrow} \mathcal{T}_{g,n}$ or $\mathcal{T}_{g,n} \overset{f}{\hookrightarrow} B$ and similarly, we will show that there are *no holomorphic isometric submersions* $B \overset{g}{\twoheadrightarrow} \mathcal{T}_{g,n}$ or $\mathcal{T}_{g,n} \overset{g}{\twoheadrightarrow} B$. Furthermore, we will prove that there are *no complex linear isometries* $(\mathbb{C}^2, ||\cdot||_2) \overset{P}{\hookrightarrow} (Q(X), ||\cdot||_1)$ and, using the same methods, we will prove a birational generalization of Royden's theorem for complex projective varieties of general type.

Preliminaries. See § 2 for background material.

Let $\mathcal{T}_{g,n}$ denote the Teichmüller space of *marked* Riemann surfaces of genus g with n marked points. It is the orbifold universal cover of the moduli space of curves $\mathcal{M}_{g,n}$ and is naturally a complex manifold of dimension $3g - 3 + n$. It is known that Teichmüller space can be realized as a bounded domain $\mathcal{T}_{g,n} \subset \mathbb{C}^{3g-3+n}$, by the Bers embeddings. [Bers]

Let $B \subset \mathbb{C}^N$ denote a bounded domain. It is a bounded *symmetric* domain if every $p \in B$ is an *isolated fixed point* for a holomorphic involution $\sigma_p : B \to B$ with $\sigma_p^2 = \mathrm{id}_B$. It is known that Hermitian symmetric spaces of non-compact type can be realized as bounded symmetric domains $B \subset \mathbb{C}^N$, by the Harish-Chandra embeddings. [Hel]

Let $\mathbb{CH}^1 \cong \{\, z \in \mathbb{C} : |z| < 1 \,\}$ denote the complex hyperbolic line, realized as the unit disk with the Poincaré metric $\dfrac{|dz|}{1 - |z|^2}$ of constant curvature -4. Schwarz's lemma shows that every holomorphic map $f : \mathbb{CH}^1 \to \mathbb{CH}^1$ is non-expanding.

The *Kobayashi metric* of a bounded domain B is the largest *Finsler* metric such that every holomorphic map $f : \Delta \cong \mathbb{CH}^1 \to B$ is non-expanding: $||f'(0)||_B \leq 1$.

It determines both a norm $|| \cdot ||_B$ on the tangent bundle and a distance $d_B(\cdot, \cdot)$ on pairs of points [Ko], and has the fundamental property that every holomorphic map between complex domains is non-expanding.

Holomorphic rigidity. An important feature of the Kobayashi metric of Teichmüller spaces is that every holomorphic map $f : \mathbb{CH}^1 \hookrightarrow \mathcal{T}_{g,n}$ such that df is an isometry on tangent spaces is *totally geodesic*: it sends real geodesics to real geodesics preserving their length. Moreover, Teichmüller disks show that there are such holomorphic isometries through every point in every complex direction.

In section § 3, we will prove: [1]

Theorem 1.1. *Every totally geodesic isometry $f : \mathbb{CH}^1 \hookrightarrow \mathcal{T}_{g,n}$ for the Kobayashi metric is either holomorphic or anti-holomorphic. In particular, it is a Teichmüller disk.*

We will also deduce:

Corollary 1.2. *Every totally geodesic isometry $f : \mathcal{T}_{g,n} \hookrightarrow \mathcal{T}_{h,m}$ is either holomorphic or anti-holomorphic.*

We note that there are many holomorphic isometries $f : \mathcal{T}_{g,n} \hookrightarrow \mathcal{T}_{h,m}$ between Teichmüller spaces $\mathcal{T}_{g,n}, \mathcal{T}_{h,m}$ in their Kobayashi metric, given by pulling-back complex structures via covering maps of the underlying topological surfaces. [GL]

Symmetric spaces vs Teichmüller spaces. Like Teichmüller spaces there are also many holomorphic isometries $f : B \hookrightarrow \widetilde{B}$ between bounded symmetric domains $B, \widetilde{B}$ in their Kobayashi metric. See § 6.

However, we will show that in dimension two or more Teichmüller spaces and bounded symmetric domains *do not mix*. More precisely, we will prove:

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[1]Theorem 1.1 solves problem 5.3 from [FM].

Theorem 1.3. *Let B be a bounded symmetric domain and $\mathcal{T}_{g,n}$ be a Teichmüller space with $\dim_{\mathbb{C}} B$, $\dim_{\mathbb{C}} \mathcal{T}_{g,n} \geq 2$. There are no holomorphic isometric immersions*

$$B \stackrel{f}{\hookrightarrow} \mathcal{T}_{g,n} \quad or \quad \mathcal{T}_{g,n} \stackrel{f}{\hookrightarrow} B$$

such that df is an isometry for the Kobayashi norms on tangent spaces.

We will also prove a similar result for submersions:

Theorem 1.4. *Let B and $\mathcal{T}_{g,n}$ be as in Theorem 1.3. There are no holomorphic isometric submersions*

$$B \stackrel{g}{\twoheadrightarrow} \mathcal{T}_{g,n} \quad or \quad \mathcal{T}_{g,n} \stackrel{g}{\twoheadrightarrow} B$$

such that dg^* is an isometry for the dual Kobayashi norms on cotangent spaces.

As an application of Theorem 1.3, we have:

Theorem 1.5. *Let (V, g) be a complete Kähler manifold with $\dim_{\mathbb{C}} V \geq 2$ and with holomorphic sectional curvature at least -4. There is no holomorphic map $f : V \to \mathcal{T}_{g,n}$ such that df is an isometry on tangent spaces.*

Proof. The monotonicity of holomorphic sectional curvature under a holomorphic map and the existence of totally geodesic holomorphic isometries $\Delta \cong \mathbb{CH}^1 \hookrightarrow \mathcal{T}_{g,n}$ through every direction imply that V has constant holomorphic curvature -4. [Roy] Since V is Kähler, we have $V \cong \mathbb{CH}^N$, which is impossible when $N \geq 2$. $\square$

We also note the following immediate corollary.

Corollary 1.6. *There is no locally symmetric variety V isometrically immersed in the moduli space of curves $\mathcal{M}_{g,n}$, nor is there an isometric copy of $\mathcal{M}_{g,n}$ in V, for the Kobayashi metrics, so long as both have dimension two or more.*

Geometry of norms. The space holomorphic quadratic differentials $Q(X)$ is naturally identified with the cotangent space $T_X^* \mathcal{T}_{g,n}$ over $X \in \mathcal{T}_{g,n}$ and the norm $||q||_1 = \int_X |q|$ for $q \in Q(X)$ coincides with the *dual* Kobayashi norm. Similarly, the complex vector space $\mathbb{C}^2$ is naturally identified with the (co-) tangent space $T_p \mathbb{CH}^2$ over $p \in \mathbb{CH}^2$ and the Euclidean norm $|| \cdot ||_2$ coincides with the (*dual*) Kobayashi norm. See §5.

Refining Theorem 1.4, we will prove:

Theorem 1.7. *There is no complex linear isometry* $P : (\mathbb{C}^2, || \cdot ||_2) \hookrightarrow (Q(X), || \cdot ||_1)$.

We also have the following application complementary to Theorem 1.5:

Corollary 1.8. *Let (V, g) be a Hermitian manifold with $dim_{\mathbb{C}} V \geq 2$. There is no holomorphic map $g : \mathcal{T}_{g,n} \to V$ such that dg^* is an isometry cotangent spaces.*

Birational invariants. Let X be a complex projective variety. The m-pluricanonical spaces $H^0(X, mK_X)$ are classical birational invariants of X and they are equipped with natural (pseudo)-norms $||q||_{2/m} = \int_X |q|^{2/m}$, which depend only on the birational type of X.

Answering a conjecture of S.-T. Yau [2] [CY], we will prove:

Theorem 1.9. *Let X,Y be two complex projective varieties of general type such that the linear system $|mK_X|$ defines a birational embedding $X \dashrightarrow \mathbb{P}H^0(X, mK_X)^*$. Every linear isometry $P : (H^0(X, mK_X), || \cdot ||_{2/m}) \hookrightarrow (H^0(Y, mK_Y), || \cdot ||_{2/m})$ comes from a unique rational map $f : Y \dashrightarrow X$ such that $P = e^{i\theta} \cdot f^*$.*

Remark. When X and Y are hyperbolic Riemann surfaces in $\mathcal{T}_{g,n}$ and $m = 2$, we recover Royden's classical theorem as a special case.

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[2]A (weaker) version of the conjecture was also obtained by C.-Y. Chi. [CY]

Questions. We conclude with some open questions.

1. Let $f : \Delta \cong \mathbb{CH}^1 \to \mathcal{T}_{g,n}$ be a C^1 smooth map such that df is a (real) isometry for the Kobayashi norms on tangent spaces. Is f a Teichmüller disk?

Theorem 1.1 suggests that the answer to 1 is positive.

2. Is there a holomorphic map $f : (V, g) \to \mathcal{T}_{g,n}$ from a Hermitian manifold with $\dim_\mathbb{C} V \geq 2$ such that df is an isometry?

3. Is there a *round* two-dimensional complex slice in $Q(X)^* \cong T_X \mathcal{T}_{g,n}$?

Theorems 1.3 and 1.5 suggest that the answers to both questions 2 and 3 are negative.

Notes and References.

In his pioneering paper [Roy], H. Royden showed that the Kobayashi metric of $\mathcal{T}_{g,n}$ coincides with its classical Teichmüller metric and initiated the study of the norms $(Q(X), || \cdot ||_1)$ to show that, when $\dim_\mathbb{C} \mathcal{T}_g \geq 2$, the group of holomorphic automorphisms $\mathrm{Aut}(\mathcal{T}_g)$ is discrete. A proof that the group $\mathrm{Aut}(\mathcal{T}_{g,n})$ is discrete for all Teichmüller spaces with $\dim_\mathbb{C} \mathcal{T}_{g,n} \geq 2$ is given in [EM] and [Mar]. The proofs of Theorem 1.7 and Theorem 1.9 are based on ideas of V. Markovic from these two papers. See § 5.

In addition to the abundance of holomorphic isometries $f : \mathbb{CH}^1 \hookrightarrow \mathcal{T}_{g,n}$, there are many holomorphic isometric submersions $g : \mathcal{T}_{g,n} \twoheadrightarrow \mathbb{CH}^1$. [Kra], [Mc3, Theorem 5.1]. Both of these two properties are also true for bounded symmetric domains. [Ko], [Ku]. We also note that the Kobayashi metric of a bounded symmetric domain $B \subset \mathbb{C}^N$ does not coincide with its Hermitian symmetric metric, unless B has rank one and $B \cong \mathbb{CH}^N$.

The existence of Teichmüller curves, isometrically immersed curves in $\mathcal{M}_{g,n}$, have many applications to the study of the dynamics of billiards. [V], [Mc2] Corollary 1.6 shows that there are no higher dimensional, locally symmetric, analogues of Teichmüller curves.

2. Background in Teichmüller theory

In this section, we give a brief introduction to Teichmüller theory along with all the necessary background material we use in the rest the dissertation. For a more thorough introduction to Teichmüller theory we refer to [Hub] and [GL].

Teichmüller space. Let $S = S_{g,n}$ be a connected, oriented surface of genus g and n punctures and let X be a Riemann surface of finite type[3] homeomorphic to S. A marking of X by S is an orientation-preserving homeomorphism $f : S \to X$. Two marked Riemann surfaces $f : S \to X$, $g : S \to Y$ are equivalent if $g \circ f^{-1} : X \to Y$ is isotopic to a holomorphic isomorphism. The Teichmüller space of S, denoted by $\mathcal{T}_{g,n}$ or $\text{Teich}(S)$, is the set of equivalence classes of *marked* Riemann surfaces (X, f), where $f : S \to X$ is an orientation-preserving homeomorphism. When it is clear from the context we will simply denote the point corresponding to the equivalence class of (X, f) by X.

Teichmüller space $\mathcal{T}_{g,n}$ is the orbifold universal cover of the moduli space of curves $\mathcal{M}_{g,n}$ and is naturally a complex manifold of dimension $3g - 3 + n$. It is known that Teichmüller space can be realized as a bounded domain of holomorphy $\mathcal{T}_{g,n} \subset \mathbb{C}^{3g-3+n}$, by the Bers embeddings. [Bers]

Quadratic differentials. Let X be a Riemann surface. A quadratic differential q on X is a global holomorphic section of the line bundle K_X^2; it is locally given by $q = q(z)dz^2$, where $q(z)$ is holomorphic. If $q(p) \neq 0$, then we can find a local chart near $p \in X$ in which $q = dz^2$. If $q(p) = 0$ is a zero of order N, the we can find a local chart in which $q = z^N dz^2$.

A quadratic differential determines a flat metric $|q|$ on X and a (measured) foliation $\mathcal{F}(q)$ tangent to the vectors $v = v(z)\frac{\partial}{\partial z}$ with $q(v) < 0$. The foliation $\mathcal{F}(q)$ comes equipped with a transverse measure: a measure on the space of arcs transverse to

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[3]A Riemann surface X is of finite type if $X = \overline{X} \setminus E$ for some compact Riemann surface $\overline{X}$ and a finite set E.

the leaves of $\mathcal{F}(q)$, such that the natural maps between transverse arcs (following the leaves of the foliation) are measure-preserving.

For the differential $q = dz^2$ on $\mathbb{C}$, the metric is just the Euclidean metric and the foliation is by vertical lines with transverse measure $|\text{Re}(dz)|$.

Let γ be a simple closed curve, we denote its transverse length by $i(\gamma, \mathcal{F}(q))$, which by definition equals to $\inf \int_{\gamma'} |\text{Re}(\sqrt{q})|$, where the infimum is over all closed curves γ' homotopic to γ. The measured foliation $\mathcal{F}(-q)$ has leaves orthogonal to the leaves of $\mathcal{F}(q)$ and the product of their transverse measures gives the area form determined by q.

Teichmüller metric. The area of the flat metric $|q|$ is given by the norm $||q||_1 = \int_X |q|$. A quadratic differential q is *integrable* if $||q||_1 < +\infty$, which is equivalent to q having at worse simple poles at the punctures of X. The co-tangent space $T_X^* \mathcal{T}_{g,n}$ for $X \in \mathcal{T}_{g,n}$ is the space of integrable quadratic differentials $Q(X)$ on X and we denote the bundle of quadratic differentials over $\mathcal{T}_{g,n}$ by $Q\mathcal{T}_{g,n} \cong T^* \mathcal{T}_{g,n}$.

The tangent space $T_X \mathcal{T}_{g,n}$ for $X \in \mathcal{T}_{g,n}$ can be described by duality. Let $M(X)$ denote the space of bounded Beltrami differentials $\mu = \mu(z)\frac{\overline{dz}}{dz}$ equipped with the norm $||\mu||_\infty = \sup_{z \in X} |\mu(z)| < +\infty$ and denote by $M(X)_1$ its unit ball. There is a natural pairing $M(X) \times Q(X) \to \mathbb{C}$, given by $(\mu, q) \mapsto \int_x q\mu$ and the tangent space $T_X \mathcal{T}_{g,n} = M(X)/Q(X)^\perp \cong Q(X)^*$.

The space $Q(X)$ is equipped with a strictly convex, C^1 smooth, norm $||\cdot||_1$, which by (norm) duality induces a strictly convex, C^1 smooth, norm on $T_X \mathcal{T}_{g,n}$. The family of these norms defines a Finsler metric $||\cdot||_{\mathcal{T}_{g,n}}$ on $\mathcal{T}_{g,n}$ and an induced (inner) distance function $d_{\mathcal{T}_{g,n}}(\cdot, \cdot)$ on pairs of points, which is known as the Teichmüller metric.

The Teichmüller metric is complete and coincides with the Kobayashi metric on $\mathcal{T}_{g,n}$ as a complex manifold. [Roy] In particular, it has the following remarkable property: every holomorphic map $f : \Delta \cong \mathbb{CH}^1 \to \mathcal{T}_{g,n}$ is non-expanding $||f'(0)||_{\mathcal{T}_{g,n}} \leq 1$; where

$\mathbb{CH}^1 \cong \Delta = \{\, z \in \mathbb{C} \ : \ |z| < 1 \,\}$ is the unit disk, equipped with the Poincaré metric $\dfrac{|dz|}{1 - |z|^2}$ of constant curvature -4.

Measured foliations. A *measured foliation* $\mathcal{F}$ on the surface $S = S_{g,n}$ is a *singular* foliation - with singularities at the punctures of S, modeled on $\mathcal{F}(q)$'s for $q \in Q(X)$ - equipped with a (transverse) measure α on all transverse arcs to $\mathcal{F}$, such that the natural maps between transverse arcs (following leaves of the foliation) are measure-preserving. Two measured foliations $\mathcal{F}$ and $\mathcal{G}$ are equivalent if they differ by an isotopy of S, preserving their transverse measures, and a finite sequence of Whitehead moves. We denote by $\mathcal{MF}_{g,n}$ the space of equivalence classes of measured foliations. It is known that $\mathcal{MF}_{g,n}$ has the structure of a *piecewise linear* manifold, which is homeomorphic to $\mathbb{R}^{6g-6+2n} \setminus 0$. [FLP]

Geometric intersection pairing. The geometric intersection $i(\gamma, \delta)$ of two simple closed curves γ and δ on S is the minimum number of intersections $|\widetilde{\gamma} \cap \widetilde{\delta}|$, over all closed curves $\widetilde{\gamma}$ and $\widetilde{\delta}$ isotopic to γ and δ, respectively.

The intersection pairing $i(\mathcal{F}, \mathcal{G})$ extends to measured foliations $\mathcal{F}$ and $\mathcal{G}$ and depends only on their equivalence class in $\mathcal{MF}_{g,n}$. It is obtained by forming the measure given (locally) by the product of the transverse measures of $\mathcal{F}$ and $\mathcal{G}$ and minimizing its total mass over all representatives in their equivalence class. It is known that the geometric intersection pairing $i(\cdot, \cdot) : \mathcal{MF}_{g,n} \times \mathcal{MF}_{g,n} \to \mathbb{R}$ is continuous. [Bon]

Uniquely ergodic foliations. A measured foliation $\mathcal{F}$ on S is *uniquely ergodic* if it is minimal and admits a unique, up to scaling, transverse measure; in particular, $i(\gamma, \mathcal{F}) > 0$ for all simple closed curves γ. We note that this property depends only on the equivalence class of a measured foliation, hence it is well-defined on $\mathcal{MF}_{g,n}$.

Theorem 2.1. *([CCM]) For any $(X, q) \in Q\mathcal{T}_{g,n}$ the set of angles $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, for which $\mathcal{F}(e^{i\theta}q)$ is uniquely ergodic, has Hausdorff dimension one and in particular full Lebesgue measure.*

Extremal length. Let $\rho = \rho(z)|dz|^2$ be a conformal metric on X and let $\mathcal{F}$ be a measured foliation on X. We can form the measure given locally by the product of the transverse measure of $\mathcal{F}$ and ρ-length along the leaves of $\mathcal{F}$. We define the ρ-length $\ell_\rho(\mathcal{F})$ of $\mathcal{F}$ to equal the infimum, over all measured foliations equivalent to $\mathcal{F}$, of the (total) mass of this product measure.

The extremal length of $\mathcal{F}$ on X is defined by $\lambda(\mathcal{F}, X) = \sup \frac{\ell_\rho(\mathcal{F})^2}{\text{area}(\rho)}$, where the supremum is with respect to all (Borel-measurable) conformal metrics ρ on X of finite area.

Theorem 2.2. *([HM];Hubbard-Masur) Let $X \in \mathcal{T}_{g,n}$. The map $q \mapsto \mathcal{F}(q)$ gives a homeomorphism $Q(X) \setminus \{0\} \cong \mathcal{MF}_{g,n}$. Moreover, $|q|$ is the unique extremal metric for $\mathcal{F}(q)$ and $\lambda(\mathcal{F}, X) = ||q||_1$.*

Teichmüller geodesics. Every unit-speed geodesic ray $\gamma : [0, \infty) \to \mathcal{T}_{g,n}$ is generated from a holomorphic quadratic differential $q \in Q(X)$ and has a unique lift to a Teichmüller geodesic ray $(X_t, q_t) \in Q_1\mathcal{T}_{g,n}$ with $||q_t||_1 = 1$. The map $q \mapsto (\mathcal{F}(q), \mathcal{F}(-q))$ gives an embedding

$$Q\mathcal{T}_{g,n} \hookrightarrow \mathcal{MF}_{g,n} \times \mathcal{MF}_{g,n}$$

which sends the Teichmüller geodesic (X_t, q_t) to a path of the form $(e^t\mathcal{F}(q), e^{-t}\mathcal{F}(-q))$. Moreover, it satisfies the relation $||q||_1 = i(\mathcal{F}(q), \mathcal{F}(-q))$ for all $q \in Q\mathcal{T}_{g,n}$.

Teichmüller disks. A totally geodesic holomorphic, or anti-holomorphic, isometry $\gamma_{\mathbb{C}} : \Delta \cong \mathbb{C}\mathbb{H}^1 \hookrightarrow \mathcal{T}_{g,n}$ is called a Teichmüller disk. It is known that every Teichmüller geodesic $\gamma : \mathbb{R} \hookrightarrow \mathcal{T}_{g,n}$ extends to a totally geodesic holomorphic isometry $\gamma_{\mathbb{C}} : \Delta \cong \mathbb{C}\mathbb{H}^1 \hookrightarrow \mathcal{T}_{g,n}$ such that $\gamma(t) = \gamma_{\mathbb{C}}(\tanh(t))$ for $t \in \mathbb{R}$.

Let $X \in \mathcal{T}_{g,n}$ and let $q \in Q(X)$ generate a Teichmüller geodesic γ with $\gamma(0) = X$. Then, the quadratic differential $e^{i\theta}q \in Q(X)$, for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, generates the Teichmüller geodesic given by $t \mapsto \gamma_{\mathbb{C}}(e^{-i\theta}\tanh(t))$ for $t \in \mathbb{R}$.

Holomorphic disks in Teichmüller space. The following theorem is a consequence of Slodkowski's theorem about holomorphic motions. [Sl] See [EKK] for a proof.

Theorem 2.3. *Let $f : \Delta \cong \mathbb{C}\mathbb{H}^1 \to \mathcal{T}_{g,n}$ be a holomorphic map with $||f'(0)||_{\mathcal{T}_{g,n}} = 1$, then f is a totally geodesic isometry for the Kobayashi metric. In particular, it is a Teichmüller disk.*

The following theorem is a consequence of Sullivan's rigidity theorem. See [Tan] for a proof and [Mc1], [Sh] for further applications and related ideas.

Theorem 2.4. *Let $\{f_t\}_{t\in\Delta}$ be a holomorphic family of holomorphic maps $f_t : \Delta \cong \mathbb{C}\mathbb{H}^1 \to \mathcal{T}_{g,n}$. If there is a set of $e^{i\theta} \in \partial\Delta$ of positive (Lebesgue) measure such that f_0 is unbounded along rays of the form $\lim_{r\to 1} f_0(re^{i\theta})$, then the family is trivial: $f_t = f_0$ for all $t \in \Delta$.*

3. Holomorphic rigidity

Let $\mathbb{CH}^1 \cong \Delta = \{\, z \in \mathbb{C} : |z| < 1 \,\}$ denote the complex hyperbolic line, realized as the unit disk with the Poincaré metric $\dfrac{|dz|}{1 - |z|^2}$ of constant curvature -4.

In this section we prove:

Theorem 3.1. *Every totally geodesic isometry $f : \mathbb{CH}^1 \hookrightarrow \mathcal{T}_{g,n}$ for the Kobayashi metric is either holomorphic or anti-holomorphic. In particular, it is a Teichmüller disk.*

The proof of the theorem uses the idea of *complexification* and leverages the following two facts. Firstly, every complete real (Kobayashi) geodesic for $\mathcal{T}_{g,n}$ is contained in a Teichmüller disk: a totally geodesic (holomorphic) isometry $\mathbb{CH}^1 \hookrightarrow \mathcal{T}_{g,n}$; and secondly, every family $\{f_t\}_{t \in \Delta}$ of *proper* holomorphic maps $f_t : \mathbb{CH}^1 \to \mathcal{T}_{g,n}$ is *trivial*: $f_t = f_0$ for all $t \in \Delta$ (Sullivan's rigidity theorem). See §2.

Outline of the proof. Let $\gamma \subset \mathbb{CH}^1$ be a complete real geodesic and denote by $\gamma_{\mathbb{C}} \subset \mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$ its *maximal* holomorphic extension to the bi-disk. We note that $\gamma_{\mathbb{C}} \cong \mathbb{CH}^1$ and we define $F|_{\gamma_{\mathbb{C}}}$ to be the *unique* holomorphic extension of $f|_\gamma$, which is a Teichmüller disk.

Applying this construction to all (real) geodesics in $\mathbb{CH}^1$, we will deduce that $f : \mathbb{CH}^1 \to \mathcal{T}_{g,n}$ extends to a *holomorphic* map $F : \mathbb{CH}^1 \times \overline{\mathbb{CH}^1} \to \mathcal{T}_{g,n}$ such that $f(z) = F(z, z)$ for $z \in \Delta \cong \mathbb{CH}^1$. Using that f is totally geodesic, we will show that F is *essentially* proper and hence, by Sullivan's rigidity theorem, we will conclude that either $F(z, w) = F(z, z)$ or $F(z, w) = F(w, w)$, for all $(z, w) \in \mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$. $\quad\square$

We also have the following:

Corollary 3.2. *Every totally geodesic isometry $f : \mathcal{T}_{g,n} \hookrightarrow \mathcal{T}_{h,m}$ is either holomorphic or anti-holomorphic.*

Proof of Corollary. The Kobayashi metric in the neighborhood of every point is bounded from above and below by Hermitian metrics. [Ko] It follows that the map f is locally Lipschitz and hence, it is differentiable at almost all points of $\mathcal{T}_{g,n}$.

Applying Theorem 3.1, we obtain that f sends Teichmüller disks in $\mathcal{T}_{g,n}$ to Teichmüller disks in $\mathcal{T}_{h,m}$. Let $X \in \mathcal{T}_{g,n}$ be a point such that the differential $df_X : T_X \mathcal{T}_{g,n} \to T_{f(X)} \mathcal{T}_{g,n}$ exists. It follows that the linear map df_X sends complex lines to complex lines, hence it is either a complex linear or complex anti-linear.

Since any two points of $\mathcal{T}_{g,n}$ are contained in a Teichmüller disk, we conclude that the linear maps df_X, for almost all $X \in \mathcal{T}_{g,n}$, are either complex linear or complex anti-linear simultaneously. In particular, we conclude that - up to conjugation - f is holomorphic almost everywhere as a distribution and the corollary follows from Weyl's lemma. [Kran] $\qquad\square$

The totally real diagonal. Let $\overline{\mathbb{CH}^1}$ be the complex hyperbolic line with its conjugate complex structure. The identity map is a *canonical* anti-holomorphic isomorphism $\mathbb{CH}^1 \cong \overline{\mathbb{CH}^1}$ and its graph is a totally real embedding $\delta : \mathbb{CH}^1 \hookrightarrow \mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$, given by $\delta(z) = (z, z)$ for $z \in \Delta \cong \mathbb{CH}^1$. We call $\delta(\mathbb{CH}^1)$ the *totally real diagonal.*

Geodesics and graphs of reflections. Let $\mathcal{G}$ denote the set of all real, unoriented, complete geodesics $\gamma \subset \mathbb{CH}^1$. In order to describe their *maximal* holomorphic extensions $\gamma_{\mathbb{C}} \subset \mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$, such that $\gamma_{\mathbb{C}} \cap \delta(\mathbb{CH}^1) = \delta(\gamma)$, it is convenient to parametrize $\mathcal{G}$ in terms of the set $\mathcal{R}$ of hyperbolic reflections of $\mathbb{CH}^1$ - or equivalently, the set of anti-holomorphic involutions of $\mathbb{CH}^1$. The map that associates a reflection $r \in \mathcal{R}$ with the set $\gamma = \text{Fix}(r) \subset \mathbb{CH}^1$ of its fixed points gives a bijection between $\mathcal{R}$ and $\mathcal{G}$.

Let $r \in \mathcal{R}$ and denote its graph by $\Gamma_r \subset \mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$; there is a natural holomorphic isomorphism $\mathbb{CH}^1 \cong \Gamma_r$, given by $z \mapsto (z, r(z))$ for $z \in \Delta \cong \mathbb{CH}^1$. We note that Γ_r is the *maximal* holomorphic extension $\gamma_{\mathbb{C}}$ of the geodesic $\gamma = \text{Fix}(r)$ to the bi-disk and it is *uniquely* determined by the property $\gamma_{\mathbb{C}} \cap \delta(\mathbb{CH}^1) = \delta(\gamma)$.

The foliation by graphs of reflections. The union of the graphs of reflections $\bigcup_{r \in \mathcal{R}} \Gamma_r$ gives rise to a (singular) foliation of $\mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$ with holomorphic leaves Γ_r parametrized by the set $\mathcal{R}$. We have $\Gamma_r \cap \delta(\mathbb{CH}^1) = \delta(\mathrm{Fix}(r))$ for all $r \in \mathcal{R}$, and

$$(3.1) \qquad \qquad \Gamma_r \cap \Gamma_s = \delta(\mathrm{Fix}(r) \cap \mathrm{Fix}(s))$$

which is either empty or a single point for all $r, s \in \mathcal{R}$ with $r \neq s$. In particular, the foliation is smooth in the complement of the totally real diagonal $\delta(\mathbb{CH}^1)$.

We emphasize that the following simple observation plays a key role in the proof of the theorem. For all $r \in \mathcal{R}$:

$$(3.2) \qquad \qquad (z, w) \in \Gamma_r \iff (w, z) \in \Gamma_r$$

Geodesics and the Klein model. The Klein model gives a real-analytic identification $\mathbb{CH}^1 \cong \mathbb{RH}^2 \subset \mathbb{R}^2$ with an open disk in $\mathbb{R}^2$. It has the nice property that the hyperbolic geodesics are affine straight lines intersecting the disk. [Rat]

Remark. The holomorphic foliation by graphs of reflections defines a *canonical* complex structure in a neighborhood of the zero section of the tangent bundle of $\mathbb{RH}^2$.

The description of geodesics in the Klein model is convenient in the light of the following theorem of S. Bernstein.

Theorem 3.3. *([AhRo]; S. Bernstein) Let M be a complex manifold, $f : [0, 1]^2 \to M$ a map from the square $[0, 1]^2 \subset \mathbb{R}^2$ into M and $E \subset \mathbb{C}$ an ellipse with foci at $0, 1$. If there are holomorphic maps $F_\ell : E \to M$ such that $F_\ell|_{[0,1]} = f|_\ell$, for all vertical and horizontal slices $\ell \cong [0, 1]$ of $[0, 1]^2$, then f has a unique holomorphic extension in a neighborhood of $[0, 1]^2$ in $\mathbb{C}^2$.*

We use this to prove:

Lemma 3.4. *Every totally geodesic isometry $f : \mathbb{CH}^1 \hookrightarrow \mathcal{T}_{g,n}$ admits a unique holomorphic extension in a neighborhood of the totally real diagonal $\delta(\mathbb{CH}^1) \subset \mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$.*

Proof of 3.4. Using the fact that analyticity is a local property and the description of geodesics in the Klein model of $\mathbb{RH}^2$, we can assume - without loss of generality - that the map f is defined in a neighborhood of the unit square $[0,1]^2$ in $\mathbb{R}^2$ and has the property that its restriction on every horizontal and vertical line segment $\ell \cong [0,1]$ is a real-analytic parametrization of a Teichmüller geodesic segment. Moreover, we can also assume that the lengths of all these segments, measured in the Teichmüller metric, are uniformly bounded from above and from below away from zero.

Since every segment of a Teichmüller geodesic extends to a (holomorphic) Teichmüller disk in $\mathcal{T}_{g,n}$, there exists an ellipse $E \subset \mathbb{C}$ with foci at 0,1 such that the restrictions $f|_\ell$ extend to holomorphic maps $F_\ell : E \to \mathcal{T}_{g,n}$ for all horizontal and vertical line segments $\ell \cong [0,1]$ of $[0,1]^2$. Hence, the proof of the lemma follows from Theorem 3.3. $\qquad\square$

Proof of Theorem 3.1.

Let $f : \mathbb{CH}^1 \hookrightarrow \mathcal{T}_{g,n}$ be a totally geodesic isometry. Applying Lemma 3.4, we deduce that f has a *unique* holomorphic extension in a neighborhood of the totally real diagonal $\delta(\mathbb{CH}^1) \subset \mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$. We will show that f extends to a holomorphic map from $\mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$ to $\mathcal{T}_{g,n}$.

We start by defining a *new* map $F : \mathbb{CH}^1 \times \overline{\mathbb{CH}^1} \to \mathcal{T}_{g,n}$, satisfying:

1. $F(z,z) = f(z)$ for all $z \in \Delta \cong \mathbb{CH}^1$.

2. $F|_{\Gamma_r}$ is the *unique* holomorphic extension of $f|_{\mathrm{Fix}(r)}$ for all $r \in \mathcal{R}$.

Let $r \in \mathcal{R}$ be a reflection. There is a *unique* (holomorphic) Teichmüller disk $\phi_r : \mathbb{CH}^1 \hookrightarrow \mathcal{T}_{g,n}$ such that the intersection $\phi_r(\mathbb{CH}^1) \cap f(\mathbb{CH}^1) \subset \mathcal{T}_{g,n}$ contains the Teichmüller geodesic $f(\mathrm{Fix}(r))$ and $\phi_r(z) = f(z)$ for all $z \in \mathrm{Fix}(r)$.

We define F by $F(z, r(z)) = \phi_r(z)$ for $z \in \mathbb{CH}^1$ and $r \in \mathcal{R}$; equation (3.1) shows that F is well-defined and satisfies conditions (1) and (2) above.

We *claim* that $F : \mathbb{CH}^1 \times \overline{\mathbb{CH}^1} \to \mathcal{T}_{g,n}$ is the *unique* holomorphic extension of $f : \mathbb{CH}^1 \hookrightarrow \mathcal{T}_{g,n}$ such that $F(z,z) = f(z)$ for $z \in \mathbb{CH}^1$.

Proof of claim. We note that the restriction of F on the totally real diagonal $\delta(\mathbb{CH}^1)$ agrees with f and that there is a *unique* germ of holomorphic maps near $\delta(\mathbb{CH}^1)$ whose restriction on $\delta(\mathbb{CH}^1)$ coincides with f. Let us fix an element of this germ $\tilde{F}$ defined on a neighborhood $U \subset \mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$ of $\delta(\mathbb{CH}^1)$. For every $r \in \mathcal{R}$, the restrictions of F and $\tilde{F}$ on the intersection $U_r = U \cap \Gamma_r$ are holomorphic and equal along the real-analytic arc $U_r \cap \delta(\mathbb{CH}^1) \subset U_r$; hence they are equal on U_r. Since $\mathbb{CH}^1 \times \overline{\mathbb{CH}^1} = \bigcup_{r \in \mathcal{R}} \Gamma_r$, we conclude that $F|_U = \tilde{F}$ and, in particular, F is holomorphic near the totally real diagonal $\delta(\mathbb{CH}^1)$. Since, in addition to that, F is holomorphic along all the leaves Γ_r of the foliation, we deduce [4] that it is holomorphic at all points of $\mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$. $\qquad\square$

In order to finish the proof of the theorem, we use the *key* observation (3.2); which we recall as follows: the points (z,w) and (w,z) are always contained in the same leaf Γ_r of the foliation for all $z,w \in \Delta \cong \mathbb{CH}^1$. Using the fact that the restriction of F on every leaf Γ_r is a Teichmüller disk, we conclude that $d_{\mathcal{T}_{g,n}}(F(z,w), F(w,z)) = d_{\mathbb{CH}^1}(z,w)$.

Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, it follows that at least one of $F(\rho e^{i\theta}, 0)$ and $F(0, \rho e^{i\theta})$ diverges in Teichmüller space as $\rho \to 1$. In particular, there is a subset $I \subset \mathbb{R}/2\pi\mathbb{Z}$ with positive measure such that either $F(\rho e^{i\theta}, 0)$ or $F(0, \rho e^{i\theta})$ diverges as $\rho \to 1$ for all $\theta \in I$.

We assume first that the former of the two is true. Using that $F : \mathbb{CH}^1 \times \overline{\mathbb{CH}^1} \to \mathcal{T}_{g,n}$ is holomorphic, we can apply Theorem 2.4 (Sullivan's rigidity theorem) to the family $\{F(z, \overline{w})\}_{w \in \Delta}$ of holomorphic maps $F(\cdot, \overline{w}) : \Delta \cong \mathbb{CH}^1 \to \mathcal{T}_{g,n}$ for $w \in \Delta \cong \mathbb{CH}^1$, in order to deduce that $F(z, 0) = F(z, z) = f(z)$ for all $z \in \Delta$ and, in particular, f is holomorphic. If we assume that the latter of the two is true we - similarly - deduce that $F(0, z) = f(z)$ for all $z \in \Delta$ and, in particular, f is anti-holomorphic. $\qquad\square$

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[4]For a simple proof, using the power series expansion of F at $(0,0) \in \mathbb{CH}^1 \times \overline{\mathbb{CH}^1}$, see [Hör, Lemma 2.2.11].

4. Extremal length geometry

Let $\mathbb{CH}^2 \cong \{\, (z, w) \mid |z|^2 + |w|^2 < 1 \,\} \subset \mathbb{C}^2$ denote the complex hyperbolic plane, realized as the round unit ball with its Kobayashi metric. In this section we will use measured foliations and extremal length on Riemann surfaces to prove:

Theorem 4.1. *There is no holomorphic isometry $f : \mathbb{CH}^2 \hookrightarrow \mathcal{T}_{g,n}$ for the Kobayashi metric.*

Outline of the proof. Using the fact that the Kobayashi metric on $\mathcal{T}_{g,n}$ coincides with the Teichmüller metric and Theorem 2.3, we deduce that f would send real hyperbolic geodesics to Teichmüller geodesics. By Theorem 2.2, the set of Teichmüller geodesic rays from a point $X \in \mathcal{T}_{g,n}$ is naturally parametrized by the set of measured foliations $\mathcal{F} \in \mathcal{MF}_{g,n}$ with unit extremal length $\lambda(\mathcal{F}, X) = 1$. The proof leverages the fact that extremal length provides a link between the geometry of Teichmüller geodesics and the geometric intersection pairing for measured foliations.

In particular, assuming the existence of an isometry f, we consider pairs of measured foliations that parametrize orthogonal geodesic rays in the image of a (totally geodesic) real hyperbolic plane $\mathbb{RH}^2 \subset \mathbb{CH}^2$. We obtain a contradiction by computing their geometric intersection pairing in two different ways.

On the one hand, we use the geometry of complex hyperbolic horocycles and extremal length to show that the geometric intersection pairing does not depend on the choice of the real hyperbolic plane.

On the other hand, by a direct geometric argument we show that this is impossible. More precisely, we have:

Proposition 4.2. *Let $q \in Q_1\mathcal{T}_{g,n}$ and $\mathcal{G} \in \mathcal{MF}_{g,n}$. There exist $v_1, \ldots, v_N \in \mathbb{C}^*$ such that $i(\mathcal{F}(e^{i\theta}q), \mathcal{G}) = \sum_{i=1}^{N} |Re(e^{i\theta/2}v_i)|$ for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.*

The proof of the proposition is given at the end of the section. See § 2 for background material in Teichmüller theory and notation. $\square$

Complex hyperbolic geodesics. Let $\mathbb{CH}^2 \cong \{ (z, w) \mid |z|^2 + |w|^2 < 1 \} \subset \mathbb{C}^2$ denote the complex hyperbolic plane, realized as the round unit ball with its Kobayashi metric, which is a Kaëhler metric with constant holomorphic curvature -4. [Ko]

Complex affine lines in $\mathbb{C}^2$, intersecting the unit ball, give totally geodesic holomorphic isometries $\mathbb{CH}^1 \stackrel{\phi}{\hookrightarrow} \mathbb{CH}^2$ and all such isometries ϕ arise in this way. Moreover, up to a holomorphic automorphism of $\mathbb{CH}^2$, ϕ has the form $\phi(z) = (z, 0)$ for $z \in \Delta \cong \mathbb{CH}^1$. In particular, $\gamma(t) = (\tanh(t), 0)$ is a real unit-speed geodesic in $\mathbb{CH}^2$.

Complex hyperbolic horocycles. Let $\gamma : [0, \infty) \to \mathbb{CH}^2$ be a unit-speed geodesic ray. Associated with it is a pair of transverse foliations of $\mathbb{CH}^2$, one by geodesics asymptotic to γ and another by horocycles asymptotic to γ.

Let $p \in \mathbb{CH}^2$ be a point. There exists a *unique* geodesic $\gamma_p : \mathbb{R} \to \mathbb{CH}^2$ and a *unique* constant $d_p \in \mathbb{R}$ such that $\gamma_p(0) = p$ and $\lim_{t \to \infty} d_{\mathbb{CH}^2}(\gamma(t), \gamma_p(t + d_p)) \to 0$. For $d \in \mathbb{R}$ the sets $H(\gamma, e^{-d})$, which consist of points $p \in \mathbb{CH}^2$ with $d_p = d$, define the foliation of $\mathbb{CH}^2$ by *complex hyperbolic horocycles* asymptotic to γ.

Teichmüller geodesics. Let $\gamma : [0, \infty) \to \mathcal{T}_{g,n}$ be a unit-speed Teichmüller geodesic ray. It has a unique lift to $\widetilde{\gamma}(t) = (X_t, q_t) \in Q_1 \mathcal{T}_{g,n}$ with $||q_t||_1 = 1$.

The map $q \mapsto (\mathcal{F}(q), \mathcal{F}(-q))$ gives an embedding

$$Q\mathcal{T}_{g,n} \hookrightarrow \mathcal{MF}_{g,n} \times \mathcal{MF}_{g,n}$$

which sends the lifted Teichmüller geodesic (X_t, q_t) to a path of the form: $(e^t \mathcal{F}(q), e^{-t} \mathcal{F}(-q))$ and satisfies $||q||_1 = i(\mathcal{F}(q), \mathcal{F}(-q))$. See § 2.

Extremal length horocycles. The description of a Teichmüller geodesic $(X_t, q_t) \in Q_1 \mathcal{T}_{g,n}$ in terms of measured foliations shows that the extremal length of $\mathcal{F}(q_t)$ along the geodesic equals $\lambda(\mathcal{F}(q_t), X_s) = e^{2(t-s)}$ for all $t, s \in \mathbb{R}$, which motivates the following definition.

Given $\mathcal{F} \in \mathcal{MF}_{g,n}$ the *extremal length horocycles* asymptotic to $\mathcal{F}$ are the level-sets of extremal length $H(\mathcal{F}, s) = \{\ X \in \mathcal{T}_{g,n} \mid \lambda(\mathcal{F}, X) = s\ \}$ for $s \in \mathbb{R}_+$.

Uniquely ergodic measured foliations. In general, the definition of complex hyperbolic horocycles in terms of geodesics has no analogous interpretation for extremal length horocycles. Nonetheless, the following theorem of H. Masur shows that the analogy works for uniquely ergodic measured foliations.

Theorem 4.3. *([Mas]; H. Masur) Let (X_t, q_t) and (Y_t, p_t) be two Teichmüller geodesics and $\mathcal{F}(q_0) \in \mathcal{MF}_{g,n}$ be uniquely ergodic.* [5] *Then $\lim_{t \to \infty} d_{\mathcal{T}_{g,n}}(X_t, Y_t) \to 0$ if and only if $\mathcal{F}(q_0) = \mathcal{F}(p_0)$ and $\lambda(\mathcal{F}(q_0), X_0) = \lambda(\mathcal{F}(p_0), Y_0)$.*

Proof of Theorem 4.1. Let $f : \mathbb{CH}^2 \hookrightarrow \mathcal{T}_{g,n}$ be a holomorphic isometry for the Kobayashi metric. We summarize the proof in the following three steps:

1. *Asymptotic behavior of geodesics determines the extremal length horocycles.*

2. *The geometry of horocycles determines the geometric intersection pairing.*

3. *Get a contradiction by a direct computation of the geometric intersection pairing.*

Step 1. Let $X = f((0,0)) \in \mathcal{T}_{g,n}$ and $q, p \in Q_1(X)$ unit area quadratic differentials generating the Teichmüller geodesic rays $f(\gamma_1), f(\gamma_2)$, where γ_1, γ_2 are two orthogonal geodesic rays in the image of the (totally geodesic) real hyperbolic plane $\mathbb{RH}^2 \subset \mathbb{CH}^2$, given by $\gamma_1(t) = (\tanh(t), 0)$ and $\gamma_2(t) = (0, \tanh(t))$.

Up to a holomorphic automorphism of $\mathbb{CH}^2$ and due to Theorem 2.1, there is no loss of generality to assume that the vertical measured foliations $\mathcal{F}(q)$ and $\mathcal{F}(p)$ are uniquely ergodic and minimal. In particular, we can apply Theorem 4.3.

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[5]By definition, a uniquely ergodic measured foliation is assumed to be minimal. See § 2 and compare with the definition given in [Mas].

We recall that the horocycle $H(\gamma_1, 1)$ is characterized by the property that for all points $P \in H(\gamma_1, 1)$ the geodesic distance between $\gamma_P(t)$ and $\gamma_1(t)$ tends to zero, where $\gamma_P(t)$ is the unique unit-speed geodesic through P asymptotic to γ_1. Applying Theorem 4.3, we conclude that:

$$(4.1) \qquad f(\mathbb{CH}^2) \cap H(\mathcal{F}(q), 1) = f(H(\gamma_1, 1))$$

Step 2. Let δ be the (unique) complete real geodesic in $\mathbb{CH}^2$ which is asymptotic to γ_1 in the positive direction and to γ_2 in the negative direction i.e. its endpoints are $(1, 0), (0, 1) \in \mathbb{C}^2$ in the ball model of $\mathbb{CH}^2$. Let P_1 and P_2 be the two points where δ intersects the horocycles $H(\gamma_1, 1)$ and $H(\gamma_2, 1)$, respectively.

The image of δ under the map f is a Teichmüller geodesic which is parametrized by a pair of measured foliations $\mathcal{F}, \mathcal{G} \in \mathcal{MF}_{g,n}$ with $i(\mathcal{F}, \mathcal{G}) = 1$ and its unique lift to $Q_1 \mathcal{T}_{g,n}$ is given by $(e^t \mathcal{F}, e^{-t} \mathcal{G})$. Let $\widetilde{P_i} = (e^{t_i} \mathcal{F}, e^{-t_i} \mathcal{G})$, for $i = 1, 2$, denote the lifts of P_1, P_2 along the geodesic δ. Then, the distance between the two points is given by $d_{\mathbb{CH}^2}(P_1, P_2) = t_2 - t_1$. Applying Step 1, we conclude that $e^{t_1} \mathcal{F} = \mathcal{F}(q)$ and $e^{-t_2} \mathcal{G} = \mathcal{F}(p)$. In particular, we have $i(\mathcal{F}(q), \mathcal{F}(p)) = e^{t_1 - t_2}$.

Remark. A simple calculation in the ball model shows that $t_2 - t_1 = \log(2)$ and therefore $i(\mathcal{F}(q), \mathcal{F}(p)) = \frac{1}{2}$.

Step 3. The automorphism given by $\phi(z, w) = (e^{-i\theta} z, w)$ is an isometry of $\mathbb{CH}^2$. It sends $H(\gamma_i, 1)$ to $H(\phi(\gamma_i), 1)$, for $i = 1, 2$, and the distance between P_1 and P_2 is clearly the same as the distance between $\phi(P_1)$ and $\phi(P_2)$. Moreover, the Teichmüller geodesic $f(\phi(\gamma_1))$ is generated by $e^{i\theta} q$, whereas the Teichmüller geodesic $f(\phi(\gamma_2))$ is still generated by $p \in Q(X)$. Applying Step 2, Theorem 2.1 and the continuity of the geometric intersection pairing, we conclude that $i(\mathcal{F}(e^{i\theta} q), \mathcal{G}) = \frac{1}{2}$ for all $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. But this contradicts Proposition 4.2. $\qquad \square$
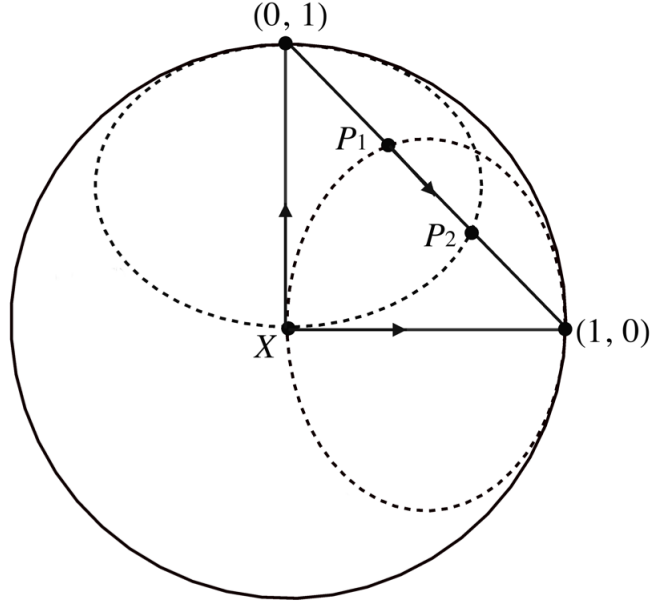
FIGURE 1. The real slice of $\mathbb{C}\mathbb{H}^2$ coincides with the Klein model of $\mathbb{R}\mathbb{H}^2$ with constant curvature -1.

Proposition 4.2. *Let* $q \in Q_1 \mathcal{T}_{g,n}$ *and* $\mathcal{G} \in \mathcal{MF}_{g,n}$. *There exist* $v_1, \ldots, v_N \in \mathbb{C}^*$ *such that* $i(\mathcal{F}(e^{i\theta}q), \mathcal{G}) = \sum_{i=1}^N |Re(e^{i\theta/2}v_i)|$ *for all* $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Proof of Proposition 4.2. Let $q \in Q(X)$ be a unit area quadratic differential. We assume first that q has no poles and that $\mathcal{G}$ is an isotopy class of simple closed curves. The metric given by $|q|$ is flat with conical singularities of negative curvature at its set of zeros and hence the isotopy class of simple closed curves $\mathcal{G}$ has a unique geodesic representative, which is a finite union of saddle connections of q. In particular, we can readily compute $i(\mathcal{F}(e^{i\theta}q), \mathcal{G})$ by integrating $|Re(\sqrt{e^{i\theta}q})|$ along the union of these saddle connections. It follows that:

$$(4.2) \qquad i(\mathcal{F}(e^{i\theta}q), \mathcal{G}) = \sum_{i=1}^N |Re(e^{i\theta/2}v_i)| \quad \text{for all} \quad \theta \in \mathbb{R}/2\pi\mathbb{Z}$$

where N denotes the number of the saddles connections and $\{v_i\}_{i=1}^N \subset \mathbb{C}^*$ are their holonomy vectors.

We note that when q has simple poles, there need not be a geodesic representative in $\mathcal{G}$ anymore. Nevertheless, equation (4.2) is still true by applying the argument to a sequence of length minimizing representatives.

Finally, we observe that the number of saddle connections N is bounded from above by a constant that depends only on the topology of the surface. Combining this observation with the fact that any $\mathcal{G} \in \mathcal{MF}_{g,n}$ is a limit of simple closed curves and that the geometric intersection pairing $i(\cdot, \cdot) : \mathcal{MF}_{g,n} \times \mathcal{MF}_{g,n} \to \mathbb{R}$ is continuous, we conclude that equation (4.2) is true in general. □

5. Geometry of norms

The cotangent space $T_X^* \mathcal{T}_{g,n}$ over $X \in \mathcal{T}_{g,n}$ is naturally identified with the space of integrable holomorphic quadratic differentials $Q(X)$, which are global sections of the line bundle K_X^2, and the *dual* Kobayashi norm coincides with the norm $||q||_1 = \int_X |q|$ for $q \in Q(X)$. See §2. Similarly, $||q||_{2/m} = \int_X |q|^{2/m}$ define (pseudo)-norms for sections of the line bundles K_X^m for $m > 0$. In this section, we prove two theorems about the geometry of these norms for Riemann surfaces and complex projective varieties of general type.

No round two dimensional slices. Let $|| \cdot ||_2$ denote the standard Hermitian (*round*) norm on $\mathbb{C}^N$ for $N > 0$. The restriction of the $|| \cdot ||_1$ norm on every complex line $\mathbb{C} \subset Q(X)$ is always *round*. However, it is known that as long as $\dim_{\mathbb{C}} Q(X) \geq 2$ the $|| \cdot ||_1$ norm on $Q(X)$ is never *round*.

In this section, we prove:

Theorem 5.1. *There is no complex linear isometry $P : (\mathbb{C}^2, || \cdot ||_2) \hookrightarrow (Q(X), || \cdot ||_1)$.*

The proof leverages the geometry of the rational function on X obtained from the pencil of bi-canonical divisors in the image of P and a theorem of W. Rudin from functional analysis.

Norms and birational invariants. Let X and Y be two closed Riemann surfaces of genus $g \geq 2$. It is known that every linear isometry $P : Q(X) \tilde{\to} Q(Y)$ comes from a (unique) isomorphism $f : Y \tilde{\to} X$ such that $P = e^{i\theta} \cdot f^*$. [Roy] In this section, we prove a generalization of this in the context of birational geometry.

Let X be a complex projective variety, the m-pluricanonical spaces $H^0(X, mK_X)$, which are holomorphic sections of the line bundle K_X^m for $m > 0$, are classical birational invariants of X and they are equipped with natural (pseudo)-norms $||q||_{2/m} = \int_X |q|^{2/m}$, which depend only on the birational type of X.

When X is of general type we can choose m so that the linear system of m-canonical divisors $|mK_X|$ defines a birational embedding $\phi_{|mK_X|} : X \dashrightarrow \mathbb{P}H^0(X, mK_X)^*$. [GH] Answering a conjecture of S.-T. Yau [CY], we prove that the norm $||\cdot||_{2/m}$ determines the birational type of X.

Theorem 5.2. *Let X and Y be two complex projective varieties such that X is of general type and $|mK_X|$ defines a birational embedding $X \dashrightarrow \mathbb{P}H^0(X, mK_X)^*$. Every complex linear isometry $P : (H^0(X, mK_X), ||\cdot||_{2/m}) \hookrightarrow (H^0(Y, mK_Y), ||\cdot||_{2/m})$ comes from a unique rational map $f : Y \dashrightarrow X$ such that $P = e^{i\theta} \cdot f^*$.*

The proofs. The main tool that we use in the proof of both theorems is the following:

Theorem 5.3. *([Ru];W. Rudin) Let μ and ν be two finite positive measures and $p \in \mathbb{R}_+ \setminus 2\mathbb{Z}$. If there exist elements $f_1, \ldots, f_n \in L^p(\mu)$ and $g_1, \ldots, g_n \in L^p(\nu)$ such that*

$$\int |1 + t_1 \cdot f_1 + \ldots + t_n \cdot f_n|^p d\mu = \int |1 + t_1 \cdot g_1 + \ldots + t_n \cdot g_n|^p d\nu$$

for all n-tuples of complex numbers $(t_1, \ldots, t_n) \in \mathbb{C}^n$, then we have an equality of measures $F_*(\mu) = G_*(\nu)$ on $\mathbb{C}^n$, where $F_*(\mu)$ and $G_*(\nu)$ are the push-forwards of μ and ν via the maps $F = (f_1, \ldots, f_n)$ and $G = (g_1, \ldots, g_n)$, respectively.

Remark. The idea of applying Rudin's theorem in the study of Teichmüller spaces is due to V. Markovic and appears in [Mar], [EM]. In particular, our proof of Theorem 5.2 follows closely these two papers.

Proof of 5.1. Assume that there are $p, q \in Q(X)$ such that

(5.1)
$$\int_X |ap + bq| = \sqrt{|a|^2 + |b|^2} \quad \text{for all} \quad (a, b) \in \mathbb{C}^2$$

Let $B^2 \subset \mathbb{C}^2$ denote the round unit ball and note that the group of unitary matrices preserves the unit ball B^2 as well as the standard (Hermitian) volume form on $\mathbb{C}^2$. In particular, using a change of variables given by a unitary transformation, we conclude that there is a constant $c > 0$ such that:

(5.2)
$$c \cdot \int_{B^2} |az + bw||dz|^2|dw|^2 = \sqrt{|a|^2 + |b|^2} \quad \text{for all} \quad (a, b) \in \mathbb{C}^2$$

We will get a contradiction by applying Theorem 5.3 to equations (5.1) and (5.2).

Let $\mu = |p|$ and $\nu = c \cdot |z| \cdot |dz|^2|dw|^2$ be two measures on X and B^2 and let $f = \frac{q}{p}$ and $g(z, w) = \frac{w}{z}$ be two rational functions on X and B^2, respectively.

Then, applying Theorem 5.3, for every (Borel) measurable set $E \subset \widehat{\mathbb{C}}$ we have:

(5.3)
$$\int_{f^{-1}(E)} \mu = \int_{g^{-1}(E)} \nu$$

We will show that this is impossible by comparing the growth rates of small disks for both sides of equation (5.3).

Let $\Delta(a, R)$ be the disk with center $a \in \mathbb{C}$ and radius $R > 0$ in the Euclidean metric of the coordinate chart $\mathbb{C} \subset \widehat{\mathbb{C}}$.

We claim that for every disk $\Delta(a, R)$ the RHS of (5.3) is of order $O(R^2)$ as $R \to 0$. Indeed, since $g^{-1}(\Delta(a, R)) \subset \{ (z, w) \in \mathbb{C}^2 : |z| < 1 , |w - az| < R \}$ the measure ν is bounded above by a constant multiple of the standard Euclidean measure on $\mathbb{C}^2$. The claim follows by Fubini's theorem and the fact that the (Euclidean) area of a disk of radius R is $O(R^2)$.

Contrary to that, we prove that there is a disk $\Delta(a, R) \subset \mathbb{C}$ such that the LHS of (5.3) grows at a slower than quadratic rate.

Given a point $x \in X$, we denote by $v_x(p)$ the order of vanishing of p and by $r_x(f)$ the ramification index of f at the point x.

Using that the degree of a meromorphic quadratic differential on a closed Riemann surface is $4g - 4$ along with Riemann-Hurwitz's formula for f we have:

$$\sum_{x \in X} v_x(p) = 4g - 4 < 4g - 4 + 4 \deg f = 2 \cdot \left(\sum_{x \in X} (r_x(f) - 1) \right)$$

which shows that there is a point $a \in X$ with $v_a(p) < 2 \cdot (r_a(f) - 1) \iff \frac{v_a(p) + 2}{r_a(f)} < 2$.

By symmetry of the argument with respect to p and q and the fact that $r_a(f) = r_a(1/f)$, we can assume that $a \notin f^{-1}(\infty) \iff f(a) \in \mathbb{C}$. Let $k = v_a(p)$ and $l = r_a(f)$; it follows that

$$\int_{f^{-1}(\Delta(f(a), R))} |p| \gtrsim \int_{\Delta(0, R^{1/l})} |z|^k |dz|^2 \gtrsim R^{\frac{k+2}{l}}$$

which contradicts the previous quadratic bound when $R \to 0$. $\qquad \square$

Proof of 5.2. Let X be a complex projective variety of general type such that the linear system of m-canonical divisors $|mK_X|$ defines a birational embedding

$$\phi = \phi_{|mK_X|} : X \dashrightarrow \mathbb{P} H^0(X, mK_X)^* \cong \mathbb{P}^N$$

with $N = \dim_{\mathbb{C}} H^0(X, mK_X) - 1$. Similarly, we have a rational map

$$\phi_{|mK_Y|} : Y \dashrightarrow \mathbb{P} H^0(Y, mK_Y)^*$$

Let $P : (H^0(X, mK_X), ||\cdot||_{2/m}) \hookrightarrow (H^0(Y, mK_Y), ||\cdot||_{2/m})$ be a linear isometry and denote by $p_0, \ldots, p_N$ a basis of $H^0(X, mK_X)$ and $q_i = P(p_i)$ for $i = 0, \ldots, N$.

Let $\mathbb{P}^N = \{ [z_0 : \ldots : z_N] \mid (z_0, \ldots, z_N) \in \mathbb{C}^{N+1} \setminus \{0\} \}$. Using the choice of basis from above we get rational maps $\phi : X \dashrightarrow \mathbb{P}^N$ and $\psi : Y \dashrightarrow \mathbb{P}^N$, where the latter is given by post-composing $\phi_{|mK_Y|}$ with $[P^*]$. Explicitly, they are given by

$\phi(x) = [\ p_0(x) : \ldots : p_N(x)\ ]$ and $\psi(y) = [\ q_0(y) : \ldots : q_N(y)\ ]$ for generic points $x \in X$ and $y \in Y$, respectively. [GH]

Denote by Φ_0 and Ψ_0 the restrictions of $\phi(X)$ and $\psi(Y)$ on the affine open set $\{z_0 \neq 0\} \cong \mathbb{C}^N \subset \mathbb{P}^N$, respectively. Let $X_0 \subset X$ denote the subset of $\phi^{-1}(\Phi_0)$ outside the base locus of $\phi = \phi_{|mK_X|}$ and similarly, let $Y_0 \subset Y$ denote the subset of $\psi^{-1}(\Psi_0)$ outside the base locus of ψ.

Since P is a linear isometry we can apply Theorem 5.3 to the measures $\mu = |p_0|^{2/m}$ and $\nu = |q_0|^{2/m}$ and the integrable (rational) functions $f_i = \frac{p_i}{p_0}$ and $g_i = \frac{q_i}{q_0}$, for $i = 1, \ldots, N$, on X_0 and Y_0, respectively.

We note that the rational maps ϕ and ψ are undefined only on sets of measure zero which are contained in a divisor of X and Y, respectively. In particular, using that P is an isometry and Theorem 5.3, we have:

$$||p_0||_{2/m} = \int_{\phi^{-1}(\Phi_0)} |p_0|^{2/m} = \int_{\psi^{-1}(\Phi_0)} |q_0|^{2/m} \leq ||q_0||_{2/m} = ||p_0||_{2/m}$$

We conclude that intersection $\Phi_0 \cap \Psi_0$ has full measure in Φ_0 and Ψ_0 for the measures μ and ν, respectively. Since it is a quasi-projective variety, this cannot be possible unless $\Phi_0 \cap \Psi_0$ shares a Zariski open subset with both Φ_0 and Ψ_0. In particular, Φ_0 and Ψ_0 are birationally equivalent and $\psi : Y \dashrightarrow X$ is the *unique* rational map which satisfies the conditions of the theorem. $\qquad\square$

Let $\mathcal{T}_{g,n} \subset \mathbb{C}^{3g-3+n}$ be a Teichmüller space and $B \subset \mathbb{C}^N$ a bounded symmetric domain equipped with their *Kobayashi* metrics. In this section, we use the results established in the previous sections to prove the following:

Theorem 6.1. *Let $B \subset \mathbb{C}^N$ be a bounded symmetric domain and $\mathcal{T}_{g,n}$ be a Teichmüller space with $dim_{\mathbb{C}} B, dim_{\mathbb{C}} \mathcal{T}_{g,n} \geq 2$. There are no holomorphic isometric immersions*

$$B \overset{f}{\hookrightarrow} \mathcal{T}_{g,n} \quad or \quad \mathcal{T}_{g,n} \overset{f}{\hookrightarrow} B$$

such that df is an isometry for the Kobayashi norms on tangent spaces.

We also prove a similar result for submersions.

Theorem 6.2. *Let B and $\mathcal{T}_{g,n}$ be as in Theorem 6.1. There are no holomorphic isometric submersions*

$$B \overset{g}{\twoheadrightarrow} \mathcal{T}_{g,n} \quad or \quad \mathcal{T}_{g,n} \overset{g}{\twoheadrightarrow} B$$

such that dg^* is an isometry for the dual Kobayashi norms on cotangent spaces.

Outline of the proofs. The proofs that $B \not\hookrightarrow \mathcal{T}_{g,n}$ and $\mathcal{T}_{g,n} \not\hookrightarrow B$ would follow from Theorem 4.1 and Theorem 5.1, respectively. The new ingredient we introduce in this section is a comparison of the *roughness* of Kobayashi metric for bounded symmetric domains and Teichmüller spaces, which we will use to prove that $\mathcal{T}_{g,n} \not\twoheadrightarrow B$ and $B \not\twoheadrightarrow \mathcal{T}_{g,n}$. $\qquad\qquad\square$

Preliminaries on symmetric spaces.

First, we give a quick review of symmetric spaces from the perspective of complex analysis; we refer to [Hel], [Sat] for a more thorough introduction.

We recall that a bounded domain $B \subset \mathbb{C}^N$ is *symmetric* if every point $p \in B$ is an *isolated* fixed point of a holomorphic involution $s_p : B \to B$. It follows from this

that the group of holomorphic automorphisms $\mathrm{Aut}(B)$ acts transitively on B. It is known that every Hermitian symmetric space of non-compact type can be realized as a bounded symmetric domain $B \subset \mathbb{C}^N$, by the Harish-Chandra embedding.

The Harish-Chandra embedding. Let $B \subset \mathbb{C}^N$ be a bounded symmetric domain and $p \in B$. There is a holomorphic embedding $B \overset{i}{\hookrightarrow} \mathbb{C}^N$ such that $i(B) \subset \mathbb{C}^N$ is *convex* and $i(p) = 0$. Moreover, this embedding is *unique* up to post-composition with a linear map; it is the Harish-Chandra realization of B centered at $p \in B$.

It is known that the Harish-Chandra realization of $B \subset \mathbb{C}^N$ has the following nice description: there is a finite dimensional (linear) subspace $V_B \subset M_{n,m}(\mathbb{C})$ of complex $n \times m$ matrices such that $B \cong \{V \in V_B : ||V||_B < 1\}$ is the *unit ball*, where $||V||_B = \sup_{||\xi||_2=1}||V(\xi)||_2$ is the operator norm for $V \in M_{n,m}(\mathbb{C})$. We note that $T_0 B \cong V_B \cong \mathbb{C}^N$. [Sat]

The Kobayashi metric on symmetric domains. Let $B \subset \mathbb{C}^N$ be a bounded symmetric domain. Using the description of B as the unit ball $\{ ||V||_B < 1 \} \subset V_B$, we obtain that the Kobayashi norm on $T_0 B$ coincides with the operator norm $||V||_B$ for $V \in T_0 B \cong V_B$. Moreover, the Kobayashi distance is given by $d_B(0, V) = \frac{1}{2} \log(\frac{1+||V||_B}{1-||V||_B})$ for $V \in B$. [Ku]

Symmetric domains in dimension one and two. When $\dim_\mathbb{C} B = 1$, we have $B \cong \mathbb{CH}^1$; we recall that $\mathbb{CH}^1$ is the unit disk $\Delta \subset \mathbb{C}$ equipped with the Poincaré metric of constant curvature -4. When $\dim_\mathbb{C} B = 2$, up to holomorphic isomorphism, we have that either $B \cong \mathbb{CH}^2$ or $B \cong \mathbb{CH}^1 \times \mathbb{CH}^1$.

The Kobayashi metric on $\mathbb{CH}^2 \cong \{ (z,w) \mid |z|^2 + |w|^2 < 1 \} \subset \mathbb{C}^2$ coincides with the unique invariant Kaëhler metric with constant holomorphic curvature -4. In particular, $||\xi||_{\mathbb{CH}^2} = ||\xi||_2 = \sqrt{|\xi_1|^2 + |\xi_2|^2}$ for $\xi = (\xi_1, \xi_2) \in T_{(0,0)}\mathbb{CH}^2 \cong \mathbb{C}^2$.

The Kobayashi metric on $\mathbb{CH}^1 \times \mathbb{CH}^1 \subset \mathbb{C}^2$ coincides with the sup-metric $||\xi||_{\mathbb{CH}^1 \times \mathbb{CH}^1} = ||\xi||_\infty = \sup\{|\xi_1|, |\xi_2|\}$ for $\xi = (\xi_1, \xi_2) \in T_{(0,0)}(\mathbb{CH}^1 \times \mathbb{CH}^1) \cong \mathbb{C}^2$. We also note

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that the *dual* Kobayashi metric is given by $||\phi||_{\mathbb{CH}^1 \times \mathbb{CH}^1} = ||\phi||_1 = |\phi_1| + |\phi_2|$ for $\phi = (\phi_1, \phi_2) \in T^*_{(0,0)}(\mathbb{CH}^1 \times \mathbb{CH}^1) \cong \mathbb{C}^2$. [Ko]

It is known that, when $\dim_{\mathbb{C}} B \geq 2$, there are holomorphic maps f and g such that: Either,

$$(6.1) \qquad\qquad \mathbb{CH}^2 \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} \mathbb{CH}^2 \qquad \text{and} \quad g \circ f = \mathrm{id}_{\mathbb{CH}^2}$$

Or,

$$(6.2) \qquad \mathbb{CH}^1 \times \mathbb{CH}^1 \overset{f}{\hookrightarrow} B \overset{g}{\twoheadrightarrow} \mathbb{CH}^1 \times \mathbb{CH}^1 \qquad \text{and} \quad g \circ f = \mathrm{id}_{\mathbb{CH}^1 \times \mathbb{CH}^1}$$

We note that, in both cases, df and dg^* are isometries for the Kobayashi norm on tangent spaces and the dual Kobayashi norm on co-tangent spaces, respectively.

Roughness of the Kobayashi metric on symmetric domains. Let $B \subset \mathbb{C}^N$ be a bounded symmetric domain and $p \in B$. We will use the description of B as the unit ball of a vector space of $n \times m$ matrices to prove the following:

Proposition 6.3. *Let $V : (-1, 1) \to B$ be a real-analytic path with $V(0) \neq p$. There is an integer $K > 0$ and an $\epsilon > 0$ such that $d_B(p, V(\cdot)) : [0, \epsilon) \to B$ is a real-analytic function of $t^{1/K}$ for $t \in [0, \epsilon)$.*

Proof. Let $B = \{ ||V||_B < 1 \} \subset V_B \subset M_{n,m}(\mathbb{C})$, the *convex* Harish-Chandra realization centered at $p \in B$, and denote by $\lambda_i(t)$ the eigenvalues of the positive $n \times n$ matrix $V(t)^*V(t)$, for $i = 1, \ldots, n$, counted with multiplicities.

The eigenvalues of $V(t)^*V(t)$ are the zeros of a polynomial, the coefficients of which are real-analytic functions of $t \in (-1, 1)$. Therefore, the points $(t, \lambda_i(t)) \in \mathbb{C}^2$, for $i = 1, \ldots, n$ and $t \in (-1, 1)$, lie on an algebraic curve $C = \{(t, \lambda) \in \mathbb{C}^2 : P(t, \lambda) = 0\}$, which is equipped with a finite-degree branched cover over $\mathbb{C}$, given by $t \in \mathbb{C}$.

Since the operator norm is given by $||V(t)||_B = \sup\{|\lambda_i(t)|^{1/2}\}_{i=1}^n$, the proof of the proposition follows by considering the Puiseux expansions of the $\lambda_i(t)$'s and the fact that $d_B(0, V(t)) = \frac{1}{2}\log(\frac{1+||V(t)||_B}{1-||V(t)||_B})$. $\qquad\square$

Roughness of the Kobayashi metric on Teichmüller spaces. In the proofs below, we will use the following two theorems of M. Rees:

Theorem 6.4. *([Rs1]; M. Rees) The Teichmüller distance $d_{\mathcal{T}_{g,n}} : \mathcal{T}_{g,n} \times \mathcal{T}_{g,n} \to \mathbb{R}_{\geq 0}$ is C^2 smooth on the complement of the diagonal $d_{\mathcal{T}_{g,n}}^{-1}(0)$.*

Theorem 6.5. *([Rs2]; M. Rees) When $\dim_{\mathbb{C}}\mathcal{T}_{g,n} \geq 2$, the Teichmüller distance $d_{\mathcal{T}_{g,n}} : \mathcal{T}_{g,n} \times \mathcal{T}_{g,n} \to \mathbb{R}_{\geq 0}$ is not $C^{2+\epsilon}$ for any $\epsilon > 0$.*

More precisely, let $X, Y \in \mathcal{T}_{g,n}$ be two distinct points, that lie on a Teichmüller geodesic that is generated by a quadratic differential $q \in Q(X)$ which has either a zero of order two or number of poles less than n. Then, there is a real analytic path $X(t) : (-1, 1) \to \mathcal{T}_{g,n}$ with $X(0) = X$ and such that the distance $d_{Teich}(X(t), Y)$ is not C^{2+h} smooth at $t = 0$ for every gauge function $h(t)$ with $\lim_{t\to 0}\frac{h(t)}{1/log(1/|t|)} = 0$.

The proofs of Theorems 6.1 and 6.2.

Let $B \subset \mathbb{C}^N$ be a bounded symmetric domain and $\mathcal{T}_{g,n}$ be a Teichmüller space with $\dim_{\mathbb{C}}B, \dim_{\mathbb{C}}\mathcal{T}_{g,n} \geq 2$.

$(B \not\hookrightarrow \mathcal{T}_{g,n})$

Theorem 4.1 shows that there is no holomorphic isometry $f : \mathbb{C}\mathbb{H}^2 \to \mathcal{T}_{g,n}$. Moreover, Theorem 2.4 (Sullivan's rigidity theorem) shows that there is no holomorphic isometry $f : \mathbb{C}\mathbb{H}^1 \times \mathbb{C}\mathbb{H}^1 \to \mathcal{T}_{g,n}$. Since $\dim_{\mathbb{C}}B \geq 2$, equations (6.1) and (6.2) show that there cannot be a holomorphic isometric[6] immersion $f : B \hookrightarrow \mathcal{T}_{g,n}$.

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[6]We recall that, by Theorem 2.3, a holomorphic isometric immersion between a bounded symmetric domain and a Teichmüller space is totally geodesic. In particular, it preserves the Kobayashi distance between points.

$(\mathcal{T}_{g,n} \not\twoheadrightarrow B)$

Theorem 5.1 shows that there is no holomorphic map $f : \mathcal{T}_{g,n} \twoheadrightarrow \mathbb{CH}^2$ such that df^* is an isometry for the *dual* Kobayashi norms. If there was a holomorphic map $f : \mathcal{T}_{g,n} \twoheadrightarrow \mathbb{CH}^1 \times \mathbb{CH}^1$, with df^* being an isometry for the *dual* Kobayashi norms, it would give a linear isometry $(\mathbb{C}^2, || \cdot ||_1) \hookrightarrow (Q(X), || \cdot ||_1)$, which is impossible since the norm $(\mathbb{C}^2, || \cdot ||_1)$ is not C^1 smooth, whereas $(Q(X), || \cdot ||_1)$ is C^1 smooth. [Roy, Lemma 1] Since $\dim_{\mathbb{C}} B \geq 2$, equations (6.1) and (6.2) show that there cannot be a holomorphic isometric submersion $g : \mathcal{T}_{g,n} \twoheadrightarrow B$.

$(\mathcal{T}_{g,n} \not\hookrightarrow B)$

Let us assume that there is a holomorphic isometric immersion $f : \mathcal{T}_{g,n} \hookrightarrow B$. We recall that, by Theorem 2.3, f would be totally geodesic and would preserve the Kobayashi distance between points.

Since $\dim_{\mathbb{C}} \mathcal{T}_{g,n} \geq 2$, we can choose two distinct points $X, Y \in \mathcal{T}_{g,n}$, which lie on a Teichmüller geodesic that is generated by a quadratic differential $q \in Q(X)$ which has either a zero of order two or number of poles less than n. Then, by Theorem 6.5, there is a real analytic path $X(t) : (-1, 1) \to \mathcal{T}_{g,n}$ with $X(0) = X$ and such that the distance $d_{Teich}(X(t), Y)$ is not C^{2+h} smooth at $t = 0$ for every gauge function $h(t)$ with $\lim_{t \to 0} \frac{h(t)}{1/log(1/|t|)} = 0$.

Let $p = f(Y) \in B$ and $V(\cdot) : (-1, 1) \to B$ be the real analytic path given by $V(t) = f(X(t))$ for $t \in (-1, 1)$. Using Theorem 6.4 and Proposition 6.3, we conclude that $d_B(p, V(t))$ is C^2 smooth and real analytic in $t^{1/K}$ for some integer $K > 0$ and all sufficiently small $t \geq 0$. Hence, it is $C^{2+\frac{1}{K}}$ smooth, which contradicts the construction of $X(t)$, from Theorem 6.5, by considering the gauge function $h(t) = t^{1/K}$ for $t \geq 0$.

$(B \not\twoheadrightarrow \mathcal{T}_{g,n})$

Assume that there is a holomorphic isometric submersion $g : B \twoheadrightarrow \mathcal{T}_{g,n}$ and let $p \in B$ and $X = g(p) \in \mathcal{T}_{g,n}$. Then, the co-derivative $P = dg_p^* : Q(X) \cong T_X^* \mathcal{T}_{g,n} \hookrightarrow T_p^* B$ is a linear isometry for the *dual* Kobayashi norms.

Using the Harish-Chandra realization of B centered at p, we can identify $T_p B \cong V_B \subset M_{n,m}(\mathbb{C})$ as a linear subspace of $n \times m$ matrices; moreover, using the natural Hermitian pairing on $M_{n,m}(\mathbb{C})$, given by $(A, B) = \text{Trace}(B^* A)$, we can identify $T_p^* B$ with V_B, such that the *dual* operator norm coincides with the trace-class norm $||V||_1 = \text{Trace}(|V|)$ for $V \in M_{n,m}(\mathbb{C})$, where $|V|$ is the *unique* positive square root of the positive $n \times n$ matrix $V^* V$. Concretely, $||V||_1 = \sum_{i=1}^{n} |\lambda_i|^{1/2}$, where λ_i denote the eigenvalues of the $n \times n$ matrix $V^* V$, for $i = 1, \ldots, n$, counted with multiplicities.

Let $p \in Q(X)$ be a quadratic differential with one double zero[7], all remaining zeros simple and with n simple poles at the marked points of X; and, let $q \in Q(X)$ be a *generic* quadratic differential with only simple zeros and n simple poles.

Let $V(t) = P(q) + t P(q) \in T_p^* B$ for $t \geq 0$. Using the same argument as in the proof of Proposition 6.3, we conclude that there is an integer $K > 0$ and an $\epsilon > 0$ such that the $||V(t)||_1 = \text{Trace}(|V(t)|)$ is a real analytic function of $t^{1/K}$ for $t \in [0, \epsilon)$.

However, this contradicts the fact that P is a linear isometry, since it is known that the power series expansion of $||p + t \cdot q||_1$ at 0, for sufficiently small $t \geq 0$, is of the form $a_0 + a_1 \cdot t + a_2 \cdot t^2 \log(\frac{1}{t}) + o(t^2 \log(\frac{1}{t}))$, with $a_i \in \mathbb{R}$, for $i = 0, 1, 2$, and $a_2 \neq 0$. [Roy, Lemma 1], [Rs2] $\qquad\qquad\square$

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[7]When $X \in \mathcal{T}_{0,5}$, there is no such p, in this case we consider p having only simple zeros and with number of (simple) poles less than n.

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