Aspects of Symmetry in de Sitter Space

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Aspects of Symmetry in de Sitter Space

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Aspects of Symmetry in de Sitter Space

Abstract

We study various aspects of symmetry in four-dimensional de Sitter space (dS_4). The asymptotic symmetry group at future null infinity ($\mathcal{I}^+$) of dS_4 is shown to be given by the group of three-dimensional diffeomorphisms acting on $\mathcal{I}^+$. However, for physics relevant to an eternal observatory in dS_4, we should instead impose unconventional future boundary conditions at $\mathcal{I}^+$. These boundary conditions violate conventional causality, but we argue the causality violations cannot be detected by any experiment in the observatory. As the next step, we study the relevant dynamics in quantum dS_4 by illuminating some previously inaccessible aspects of the dS/CFT dictionary in the context of the higher spin dS_4/CFT_3 correspondence relating Vasiliev's higher-spin gravity on dS_4 to a Euclidean $Sp(N)$ CFT_3. We found that CFT_3 states created by operator insertions are found to be dual to (anti) quasinormal modes (QNM) in the bulk. A R-norm is defined on the R^3 bulk Hilbert space and shown for the scalar case to be equivalent to both the Zamolodchikov and pseudounitary C-norm of the $Sp(N)$ CFT_3. The QNMs are found to lie in two complex highest-weight representations of the dS_4 isometry group and form a complete orthogonal basis with respect to the R-norm. The conventional Euclidean vacuum may be defined as the state annihilated by half of the QNMs, and the Euclidean Green function is obtained from a simple mode sum. Finally, as a step towards understanding non-linear dynamics of dS_4 we study both

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linear and non-linear deformations of dS$_4$ which leave the induced conformal metric and trace of the extrinsic curvature unchanged for a fixed hypersurface. These deformations are required to be regular at the future horizon of the static patch observer. When the slices are arbitrarily close to the cosmological horizon, the finite deformations are characterized by solutions to the incompressible Navier-Stokes equation.

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Citations to Previously Published Work

Most of Chapter 2 has appeared in:

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Chapter 1

Introduction and summary

1.1 Why de Sitter space?

There is perhaps no more exciting time than now! Since the discovery of the cosmic microwave background (CMB) roughly half a century ago, the field of observational cosmology has provided us with incredibly more precise snapshots of the early universe. Recent observational detection of the primordial gravitational waves[1], if confirmed, will lend strong support to the idea that the early universe underwent an inflationary phase, i.e. accelerating expansion. The simplest inflationary model includes a period of approximately de Sitter phase whose gravitational (quantum) fluctuations presumably imprinted in the photons observed in [1]. It is truly amazing that these data are telling us about the physics occurring roughly 10^{-36} seconds after the beginning of our universe, i.e. about 14 billion years ago!

Fast-forwarding to the present day (and perhaps some time in the future), as if in a deja vu, the universe appears to be expanding in a accelerating manner *again* [2, 3]. This surprisingly discovery was no small feat and led to the 2011 Nobel Prize in Physics awarded

to Perlmutter, Schmidt and Riess – the leaders of the Supernova Cosmology Project and the High-Z Supernova Search Team. This observation suggests the possibility that we live in a universe with a non-zero (but extraordinarily small) cosmological constant $\Lambda \approx 10^{-120}$ in natural units. Understanding the origin of this (perhaps unnaturally) small value of the cosmological constant has remained a challenge.

Given that de Sitter (dS) is relevant to the inflationary era as well as the future phase of our universe, understanding dS quantum gravity is therefore of great importance. dS is the most symmetric solution to Einstein's gravity with a positive cosmological constant Λ. The scale-invariant isometry of de Sitter space already manifests itself as the scale-invariant property of the CMB, while many on-going work attempt to extract interesting observational consequences out of understanding the breaking of the full de Sitter isometry group. It is therefore of immense observational interest to study quantum effects in de Sitter space.

To do so, we first need to propose what are the sharp questions which we want to answer. It is perhaps useful to first draw inspirations and lessons from the tremendous progress and successes in the studies of quantum gravity in flat space and anti de Sitter space. In the context of black hole physics, the Bekenstein-Hawking black hole entropy formula:

$$S_{BH} = \frac{kc^3 A}{4\hbar G} \tag{1.1.1}$$

is perhaps one of the most beautiful formulae in modern physics. In a single equation, it ties together many areas of modern physics ranging from quantum mechanics and gravity to statistical mechanics. Thus, understanding what black hole microstates are is of fundamental importance to unravel the quantum nature of spacetime. In the past few decades, tremendous breakthroughs in string theories have allowed for successful tour de force counting of black hole microstates[4]. Furthermore, the advent of the AdS/CFT correspondence

has enabled us to map the counting of black hole microstates to the counting of states in a CFT. We have also understood that the relevant dynamics for quantum gravy in flat space and AdS space are the flat space S-matrix and AdS boundary correlators, which are the observables in quantum gravity in the respective spacetime.

On the other hand, back to dS, there exists the de Sitter entropy formula, which in four dimension reads

$$S_{BH} = \frac{3\pi k c^3}{\hbar G \Lambda}. \tag{1.1.2}$$

Similar to the black hole entropy formula, this de Sitter entropy formula contains not only the important fundamental constants of nature, it also encodes information about the cosmological constant – one of the biggest challenge in the current study of our universe. In contrast to black hole entropy, the nature of dS entropy is very mysterious. It is not even clear what are the appropriate microstates (or degrees of freedom) one should count. Furthermore, perhaps not unrelated, the proper characterization of dynamics in asymptotically future de Sitter spacetimes - the analog of the S-matrix for asymptotically Minkowski spacetimes or boundary correlators for asymptotically anti-de Sitter (AdS) spacetimes - remains an open problem. These two big questions – what are the microstates in de Sitter entropy formula and what are the relevant observables in quantum de Sitter space – will be the central focus of this thesis. Our work provides some progress in answering these questions.

1.2   What is in this thesis?

In attacking any problem, it is important to understand what tools are available and suitable for the problem at hand. In this section, we will briefly summarize what is in our toolbox and then outline the main contents of this thesis and highlight how these tools will

be featured.

Before diving into the big questions about de Sitter quantum gravity or entropy, we first step back and understand the semi-classical structure of de Sitter space. The first tool that we are going to use is the asymptotic symmetry group (ASG) analysis. Since we will review this method in more details later on, here it is sufficient to comment that this particular tool has yielded rich and powerful insights into how quantum gravity works in AdS [5] and flat space [6, 7]. In AdS, the ASG analysis of [5] provided the first hint of AdS/CFT while the flat space ASG analysis of [6, 7] yielded the infinite-dimensional BMS group, whose consequences on the flat space S-matrix (and in particular the Weinberg's soft-graviton theorem) have recently been uncovered [8, 9, 10]. It is worth mentioning that the ASG study of the near-horizon geometry of extremal Kerr black holes has revealed interesting conformal structure, which might shed some light on the entropy counting of the extremal Kerr black holes [11]. Since the ASG analysis is closely linked to what boundary conditions are relevant to the dynamics that we are interested in, the study of appropriate boundary conditions is an important element of the ASG analysis. Thus, in carrying out the ASG analysis in de Sitter space, we will have to discuss appropriate boundary conditions in de Sitter space.

Next, we bravely embark on a journey to explore some quantum questions in de Sitter space. The tool that we need is holography. A framework where questions in dS quantum gravity could be sharply addressed is the dS/CFT correspondence[12, 13]. The dS/CFT proposal seeks to apply ideas and tools we have learnt from AdS/CFT to de Sitter space. Although many aspects of AdS/CFT and dS/CFT seem similar, there are fundamental differences which render some basic entries in the dS/CFT dictionary not well-understood.

For many years, the lack of microscopic realizations of dS/CFT has impeded progress to answering many questions in dS/CFT; fortunately, recent advancement in the higher-spin dS_4/CFT_3 correspondence [14] has provided us with a microscopic model to study issues in dS concretely. We are able to elucidate new entries in the dS_4/CFT_3 dictionary and understand that quasinormal modes play an essential and interesting role in the perturbative modes of scalar fields. Understanding how to quantize quasinormal modes in the bulk is then a needed step to further explore the structure of dS/CFT.

The last tool that we need is the recent idea which is called the fluid/gravity correspondence. This idea proposes in a very concrete way that black hole horizon dynamics is governed by the incompressible Navier-Stokes equation[15, 16, 17]. It provides a powerful way to study the nonlinear dynamics of black hole horizons. Given the striking similarities between the thermodynamics of a black hole horizon and a cosmological horizon, we will extend the fluid/gravity correspondence to the case of a cosmological horizon. Furthermore, we will see later that such fluid dynamics is closely tied to the de Sitter quasinormal modes, thus understanding such nonlinear dynamics might prove to be useful in the context of dS/CFT.

1.2.1   Structure of this thesis

This thesis consists of four main parts. Chapter 2 studies the asymptotic symmetry group (ASG) at future null infinity ($\mathcal{I}^+$) of asymptotically four-dimensional de Sitter spacetimes (dS_4). It is shown to be given by the group of three-dimensional diffeomorphisms acting on $\mathcal{I}^+$. Finite charges are constructed for each choice of ASG generator together with a two-surface on $\mathcal{I}^+$. A conservation equation is derived relating the evolution of the charges

with the radiation flux through $\mathcal{I}^+$.

In Chapter 3, we then consider asymptotically future de Sitter spacetimes endowed with an eternal observatory. In the conventional descriptions, the conformal metric at the future boundary $\mathcal{I}^+$ is deformed by the flux of gravitational radiation. We however impose an unconventional future "Dirichlet" boundary condition requiring that the conformal metric is flat everywhere except at the conformal point where the observatory arrives at $\mathcal{I}^+$. This boundary condition violates conventional causality, but we argue the causality violations cannot be detected by any experiment in the observatory. We show that the bulk-to-bulk two-point functions obeying this future boundary condition are not realizable as operator correlation functions in any de Sitter invariant vacuum, but they do agree with those obtained by double analytic continuation from anti-de Sitter space.

Having discussed issues related to symmetries and boundary conditions, in Chapter 4, we move on to aspects of holography in de Sitter space. In particular, we want to understand the details of dS/CFT dictionary. A recently conjectured microscopic realization of the dS_4/CFT_3 correspondence relating Vasiliev's higher-spin gravity on dS_4 to a Euclidean $Sp(N)$ CFT_3 is used to illuminate some previously inaccessible aspects of the dS/CFT dictionary. In particular it is argued that states of the boundary CFT_3 on S^2 are holographically dual to bulk states on geodesically complete, spacelike R^3 slices which terminate on an S^2 at future infinity. The dictionary is described in detail for the case of free scalar excitations. The ground states of the free or critical $Sp(N)$ model are dual to dS-invariant plane-wave type vacua, while the bulk Euclidean vacuum is dual to a certain mixed state in the CFT_3. CFT_3 states created by operator insertions are found to be dual to (anti) quasinormal modes in the bulk. A 'R-norm', which involves reflection across the equator of a spatial S^3 slice, is

defined on the R^3 bulk Hilbert space and shown for the scalar case to be equivalent to both the Zamolodchikov and pseudounitary C-norm of the $Sp(N)$ CFT$_3$.

To further explore interesting features of the quasinormal modes, in Chapter 5 we study a scalar field in dS$_4$ whose quasinormal modes are singular on the past horizon of the south pole and decay exponentially towards the future. These are found to lie in two complex highest-weight representations of the dS$_4$ isometry group $\mathsf{SO}(4,1)$. The Klein-Gordon norm cannot be used for quantization of these modes because it diverges. However the 'R-norm' for the quasinormal modes is nonsingular. The quasinormal modes are shown to provide a complete orthogonal basis with respect to the R-norm. Adopting the associated R-adjoint effectively transforms $\mathsf{SO}(4,1)$ to the symmetry group $\mathsf{SO}(3,2)$ of a 2+1-dimensional CFT. It is further shown that the conventional Euclidean vacuum may be defined as the state annihilated by half of the quasinormal modes, and the Euclidean Green function is obtained from a simple mode sum. Quasinormal quantization contrasts with some conventional approaches in that it maintains manifest dS-invariance throughout. The results are expected to generalize to other dimensions and spins.

Finally, in chapter 6, we switch gear and study nonlinear dynamics of de Sitter horizon to relate the quasinormal modes to the fluid modes in the fluid/gravity correspondence applied to de Sitter horizon. There are (at least) two surfaces of particular interest in eternal de Sitter space. One is the timelike hypersurface constituting the lab wall of a static patch observer and the other is the future boundary of global de Sitter space. We study both linear and non-linear deformations of four-dimensional de Sitter space which obey the Einstein equation. Our deformations leave the induced conformal metric and trace of the extrinsic curvature unchanged for a fixed hypersurface. This hypersurface is

either timelike within the static patch or spacelike in the future diamond. We require the deformations to be regular at the future horizon of the static patch observer. For linearized perturbations in the future diamond, this corresponds to imposing incoming flux solely from the future horizon of a single static patch observer. When the slices are arbitrarily close to the cosmological horizon, the finite deformations are characterized by solutions to the incompressible Navier-Stokes equation for both spacelike and timelike hypersurfaces. We then study, at the level of linearized gravity, the change in the discrete dispersion relation as we push the timelike hypersurface toward the worldline of the static patch. Finally, we study the spectrum of linearized solutions as the spacelike slices are pushed to future infinity and relate our calculations to analogous ones in the context of massless topological black holes in AdS_4.

Chapter 2

Asymptotic Symmetries and Charges

Diffeomorphisms are the basic symmetry of general relativity. In spacetimes with an asymptotic boundary, there is an interesting subgroup of the diffeomorphisms, often referred to as the asymptotic symmetry group (ASG), which "acts nontrivially" on the boundary data. The precise form of the ASG depends on the spacetime in question, the boundary conditions, the dynamics and the precise definition of "acts nontrivially". In some cases charges can be associated with the asymptotic symmetries.

The subject of asymptotic symmetries began with the seminal work of Arnowitt, Deser and Misner [18, 19] who showed that the ADM energy and momentum of an asymptotically Minkowski spacetime are associated with asymptotic translations at spatial infinity. Bondi, Metzner and Sachs (BMS) [6, 7, 20, 21] studied the asymptotic symmetries at future null infinity[1] ($\mathcal{I}^+$) and discovered an infinite-dimensional group known as the BMS group. The structure at $\mathcal{I}^+$ is much more complicated than the one at spatial infinity because radiation

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[1] Our terminology is such that future null infinity means the conformal boundary where null geodesics terminate while future timelike infinity is where timelike geodesics terminate.

can pass through $\mathcal{I}^+$. More recently [22, 23], integrable charges have been constructed which are parameterized by both a choice of generator of the BMS group and a two-surface on $\mathcal{I}^+$. These charges obey a conservation equation reflecting the possibility of radiation flux at $\mathcal{I}^+$[24, 25].

In this chapter, we give a definition of the ASG for four-dimensional spacetimes, denoted dS^+, which are asymptotically de Sitter.[2] Such spacetimes are of special interest because they may include the one we inhabit. The asymptotic region $\mathcal{I}^+$ of such spacetimes is a spacelike surface which coincides with future timelike infinity. Its structure is in some ways similar to Minkowskian $\mathcal{I}^+$ because there is in general a nonzero radiation flux, and an infinite-dimensional group is expected. We show that the ASG so defined is all of the three-dimensional diffeomorphisms acting on $\mathcal{I}^+$.[3]

A naive power-counting of falloffs of the relevant structures as $\mathcal{I}^+$ is approached indicates that the charges associated with the ASG for dS^+ should be divergent and depend unpleasantly on the manner in which $\mathcal{I}^+$ is approached. However, adapting the results and insights of a number of recent papers [23, 29, 30, 31], we find that finite charges which do not depend on the approach to $\mathcal{I}^+$ can in fact be constructed. The resulting expression for the charges (2.4.31) and their conservation equation (2.3.19) below are our main results.

Due to the fact that our solution space is radiative, a construction of Dirac brackets that generate the associated symmetries and a Dirac bracket algebra would require an analysis along the lines of [24] for the BMS charges (see also [32]).

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[2]Our analysis here is local and does not take into considerations the global or topological properties of $\mathcal{I}^+$. Interesting work related to such issues in asymptotically (global) de Sitter spacetimes includes references such as [26, 27, 28].

[3]Although the considerations here are purely classical, we note in passing that the ASG defined here is not necessarily a candidate symmetry group for a holographic dual for dS^+ quantum gravity as the latter may involve a different treatment of $\mathcal{I}^+$.

This chapter is organized as follows. In section 2.1, we discuss the asymptotic expansion of dS^+ spacetimes near future infinity. In section 2.2, we propose our definition for the ASG of dS^+. Finite charges for those diffeomorphims tangent to $\mathcal{I}^+$ comprising the ASG are constructed in section 2.3. These charges are refined using covariant phase space techniques in section 2.4.

2.1 Boundary Conditions

We are interested in solutions of the Einstein equations with a positive cosmological constant $\Lambda \equiv 3/\ell^2$:[4]

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\frac{3}{\ell^2}g_{\mu\nu} \; , \tag{2.1.1}$$

where $R_{\mu\nu}$ and R are the four-dimensional Ricci tensor and scalar. ℓ is also called the de Sitter radius. The de Sitter space is the simplest solution of (2.1.1). More complicated solutions often have singularities in either the future or the past. In this work, we are mainly interested in solutions which are asymptotically future de Sitter, which we will refer to as dS^+ spacetimes. That is, they may be singular in the past, but in some sense approach pure de Sitter in the future.

In order to define our notion of a dS^+ spacetime, we must specify boundary conditions. We wish to make the boundary conditions relatively tight to simplify the calculations, while still allowing for example the possibility of gravitational waves to reach $\mathcal{I}^+$. It was shown by Starobinsky [33] that a very general class of excited de Sitter spacetimes can at late times

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[4]We do not include additional matter in our discussions, although at the classical level the inclusion of light matter to the analysis is straightforward.

be put in the "Fefferman-Graham" (FG) form [34]

$$\frac{ds^2}{\ell^2} = -\frac{d\eta^2}{\eta^2} + \frac{dx^i dx^j}{\eta^2}\left(g_{ij}^{(0)} + \eta^2 g_{ij}^{(2)} + \eta^3 g_{ij}^{(3)} + \dots\right) . \tag{2.1.2}$$

Here $\mathcal{I}^+$ is $\eta \to 0$. Note that the term proportional to η^{-1} is set to zero in these coordinates using Einstein equations. For the purposes of this work, we take the definition of a dS^+ spacetime to be any solution of the Einstein equation (2.1.1) with an expansion of the form (2.1.2) with the $g^{(k)}$ smooth tensors on $\mathbb{R}^3$.

The Einstein equation imposes relations among the coefficients in the FG expansion. These include[5]

$$g_{ij}^{(2)} = R_{ij}[g^{(0)}] - \frac{1}{4}R[g^{(0)}]g_{ij}^{(0)} , \tag{2.1.3}$$

$$\nabla^j g_{ij}^{(3)} = \operatorname{tr} g_{ij}^{(3)} = 0 , \tag{2.1.4}$$

where the covariant derivative and trace are defined with respect to $g^{(0)}$. Moreover, the coefficients $g^{(k)}$ of η^{k-2} for $k > 3$ are then fully determined by $g^{(0)}$ and $g^{(3)}$. Hence, the data characterizing the spacetime is a boundary metric $g^{(0)}$ and a traceless conserved tensor $g^{(3)}$. Here, we note that $g^{(3)}$ has all the properties of a stress energy and we will show later (in section 2.3) that it is proportional to the Brown-York stress tensor.

2.1.1 Conformal Slicing Transformations

The precise forms of the boundary data $g^{(0)}$ and $g^{(3)}$ depend on the precise choice of slices labeled by constant η as $\mathcal{I}^+$ is approached. Consider an infinitesimal slicing transformation characterized by the diffeomorphism $\eta \to \eta - \eta\delta\sigma(x^i)$. In order to preserve the FG gauge

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[5]The coefficient $g_{ij}^{(2)}$ is called the Schouten tensor which is important in the study of conformal tensors. We thank the referee for pointing this out.

(2.1.2), this must be accompanied by an η-dependent diffeomorphism tangent to the slice. FG gauge-preserving slicing transformations are generated by the vector fields

$$\xi^{(\delta\sigma)} = \eta\delta\sigma(x^k)\partial_\eta + \ell^2 \left[\partial_j\delta\sigma(x^k)\right] \int^\eta \frac{d\eta'}{\eta'} g^{ij}(\eta', x^k)\partial_i \ . \tag{2.1.5}$$

These slicing transformations act as conformal transformations on the data at $\mathcal{I}^+$ according to

$$\delta g_{ij}^{(0)} \ = \ 2\delta\sigma g_{ij}^{(0)} \ , \tag{2.1.6}$$

$$\delta g_{ij}^{(3)} \ = \ -\delta\sigma g_{ij}^{(3)} \ . \tag{2.1.7}$$

We see that $g^{(0)}$ transforms with weight 2 while $g^{(3)}$ transforms with weight -1. Hence, the physical boundary data, which do not depend on a slicing choice near $\mathcal{I}^+$ are a conformal metric of conformal weight 2 and a traceless symmetric conserved tensor of weight -1. Henceforth, by boundary data we always mean physical boundary data in this sense.

2.2 The Asymptotic Symmetry Group

The ASG is defined as the quotient group

$$\text{ASG} \equiv \frac{\text{allowed diffeomorphisms}}{\text{trivial diffeomorphisms}} \ . \tag{2.2.8}$$

In the present case, an allowed diffeomorphism is any one which preserves the FG form of every dS^+ metric. We define the trivial diffeomorphisms to be those which leave the boundary data on $\mathcal{I}^+$ invariant.

The most general allowed diffeomorphism is of the form

$$\xi = \phi^i(x^k)\partial_i + \eta\delta\sigma(x^k)\partial_\eta + \frac{\eta^2}{2}g^{(0)ij}\partial_j\delta\sigma(x^k)\partial_i + \mathcal{O}(\eta^4) \ , \tag{2.2.9}$$

13

where here and hereafter i, j indices of $g_{ij}^{(k)}$ are raised and lowered with $g_{ij}^{(0)}$.

As seen above, the $\delta\sigma$-dependent terms do not change the boundary data. Moreover, the subleading η terms fall off so rapidly that they also do not change the boundary data. Hence, only the first term in (2.2.9) is non-trivial, and the ASG is generated by diffeomorphisms of the form

$$\xi^{\mathrm{ASG}} = \phi^i(x^k)\partial_i \ . \tag{2.2.10}$$

Hence, the ASG is simply the diffeomorphisms of $\mathbb{R}^3$.

The short and simple treatment we have given here parallels the one which originally led to the BMS group as the ASG for the Minkowski space $\mathcal{I}^+$. A somewhat unsatisfactory feature of this treatment is that it is not clear precisely what is non-trivial about the remaining $\mathbb{R}^3$ diffeomorphisms. The best answer to this question is (and will be given below) provided by a more elaborate discussion involving the construction of asymptotic charges transforming non-trivially under the ASG. The trivial diffeomorphisms are defined as those whose associated charges or generators vanish when the constraints are applied. However, when there are energy fluxes through the boundary, as is the case for $\mathcal{I}^+$ in either Minkowski or de Sitter spacetimes, the charges will not be conserved and are rather subtle to define. The definition of such charges for Minkowski $\mathcal{I}^+$[22, 23, 25] did not appear until several decades after the original work of BMS and requires considerably more technology. Using the vanishing of these charges as the definition of triviality, one indeed recovers the BMS group.

Fortunately, the hard work which went into the definition of asymptotic charges at Minkowski $\mathcal{I}^+$ along with the development of the covariant phase space formalism [29, 35] can be readily adapted to the dS^+ case. We will see in the following that defining a trivial

diffeomorphism as one whose associated charge vanishes happily leads back to the conclusion stated above that the dS^+ ASG is all of the diffeomorphisms of $\mathbb{R}^3$.

2.3 Brown-York Charges

In this section, we will use the Brown-York formalism [36] to compute the charges associated with the ASG generators (2.2.10) as well as their conservation laws.[6]

The Brown-York formalism as developed in [36] is relevant only for diffeomorphisms which do not move the boundary of a given hypersurface. They therefore cannot be used to associate charges or determine triviality of the more general allowed diffeomorphisms (2.2.9). This requires not only the more fully developed but also the more complicated covariant phase space formalism. We will see in the following section that in the end this leads back to the same expression for the charges found more simply in this section.

2.3.1 Charges

The Brown-York stress tensor associated to a three-dimensional hypersurface Σ, including possible counterterms required for finiteness at $\mathcal{I}^+$ of dS^+, is given by

$$T_{ij} = -\frac{1}{8\pi G}\left(K_{ij} - K\gamma_{ij} - c_1 \frac{2}{\ell}\gamma_{ij} - c_2 \ell G_{ij}\right) . \tag{2.3.11}$$

Here γ_{ij}, G_{ij} and K_{ij} are the induced intrinsic metric, Einstein tensor and extrinsic curvature respectively of Σ, while c_1 and c_2 are at this point arbitrary constants. This expression is derived [36, 38, 39] by taking a variational derivative of the Einstein-Hilbert action plus

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[6]For alternative definitions of charges in asymptotically de Sitter spacetimes, we refer the reader to [37] and references therein.

counterterms with respect to the induced metric at the boundary. The regularized action whose variation gives (2.3.11) is

$$S_{total} = \frac{1}{16\pi G} \int_{\mathcal{M}} d^4x \sqrt{-g} \left(R\left[g\right] - 6/\ell^2 \right) + \int_{\mathcal{I}^+} L_{GH} + \int_{\mathcal{I}^+} L_{ct} , \qquad (2.3.12)$$

where the Gibbons-Hawking term L_{GH} and the local boundary counterterms L_{ct} are give by

$$L_{GH} \equiv \frac{1}{8\pi G} \sqrt{\gamma} K\left[\gamma\right] d^3x, \quad L_{ct} \equiv \frac{1}{16\pi G} \sqrt{\gamma} \left(c_2 \ell^2 R[\gamma] - 4c_1/\ell \right) d^3x . \qquad (2.3.13)$$

Evaluating the on-shell Brown-York stress tensor at a surface of small constant η near $\mathcal{I}^+$ of an arbitrary dS^+ spacetime, one finds (along the lines of [40, 41, 42]):

$$
\begin{aligned}
T_{ij} &= -\frac{\ell}{8\pi G} \left[\frac{2}{\eta^2} (1 - c_1) g_{ij}^{(0)} + (3 - 2c_1) g_{ij}^{(2)} - \left(g^{(0)kl} g_{kl}^{(2)} \right) g_{ij}^{(0)} - c_2 G_{ij}[g^{(0)}] \right. \\
&\quad \left. + \eta \left(\frac{7}{2} - 2c_1 \right) g_{ij}^{(3)} \right] + \mathcal{O}(\eta^2) .
\end{aligned}
\qquad (2.3.14)
$$

In the hope of constructing finite charges at dS^+, we choose c_1 and c_2 to make this expression as small as possible for $\eta \to 0$. With the choice of

$$c_1 = c_2 = 1 , \qquad (2.3.15)$$

we arrive at

$$T_{ij} = -\frac{3\eta\ell}{16\pi G} g_{ij}^{(3)} \equiv \eta T_{(0)ij} , \quad T_i^i = \nabla^i T_{ij} = 0 , \qquad (2.3.16)$$

where we have defined the $\mathcal{O}(1)$ part $T_{(0)ij}$ for convenience. Charges are associated with a two-dimensional compact submanifold $\partial\Sigma$ of $\mathcal{I}^+$, which is a boundary of a noncompact three-volume Σ, together with a vector field ξ of the form (2.2.10) generating the ASG. We define these charges by working on a hypersurface of small constant η and using a submanifold $\partial\Sigma_\eta$ of Σ that approaches the desired $\partial\Sigma$ on $\mathcal{I}^+$ for $\eta \to 0$ and is the boundary of a noncompact hypersurface Σ_η in dS^+. The expression is

$$Q_\xi^{BY} \left[g^{(0)}, g^{(3)}, \partial\Sigma \right] = \lim_{\eta \to 0} \int_{\partial\Sigma_\eta} d^2x \sqrt{\sigma} n^i \xi^j T_{ij} . \qquad (2.3.17)$$
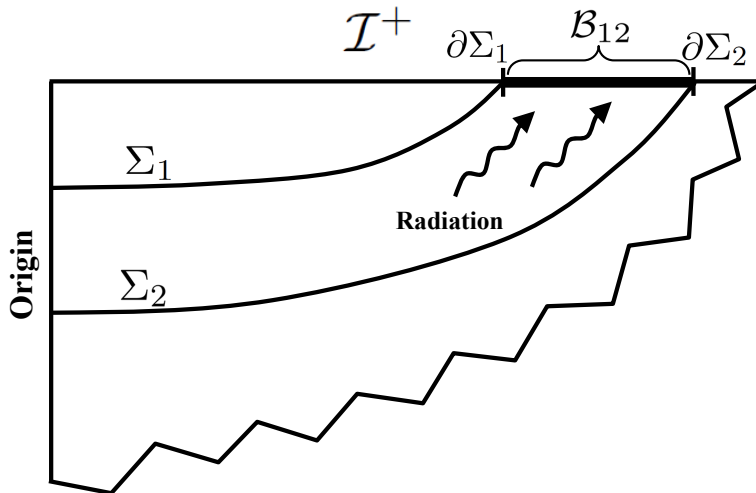
Fig. 2.1: Consider two spacelike slices Σ_1 and Σ_2 ending on $\partial\Sigma_1$ and $\partial\Sigma_2$. The difference between the Brown-York charge ΔQ_{BY} is given by the integral over the radiation flux F_ξ in the 3-volume ($\mathcal{B}_{12}$ in $\mathcal{I}^+$) bounded by $\partial\Sigma_1$ and $\partial\Sigma_2$. Here, the Penrose diagram depicts a spacetime which tends to de Sitter space in the far future. The spacelike jagged line represents the far past which could for example be the Big Bang.

n^i here is tangent to Σ_η and normal to $\partial\Sigma_\eta$, while σ is the induced metric on $\partial\Sigma_\eta$. Finiteness of the charges follows from the fact that $n^i \sim \eta$, $\sqrt{\sigma} \sim \eta^{-2}$ and $T_{ij} \sim \eta$. This expression is manifestly conformally invariant:

$$Q_\xi^{BY}\left[g_{ij}^{(0)}, g_{ij}^{(3)}\right] = Q_\xi^{BY}\left[\Omega^2 g_{ij}^{(0)}, \Omega^{-1} g_{ij}^{(3)}\right] \ . \tag{2.3.18}$$

This is equivalent to the statement that the charges are independent of the precise manner in which the surface $\partial\Sigma_\eta$ is taken to the boundary dS^+. When there are appropriate asymptotic Killing vectors, these Brown-York charges reduce to the conserved Abbott-Deser charges [43].

2.3.2 Conservation Equation

Consider two submanifolds $\partial\Sigma_1$ and $\partial\Sigma_2$ of $\mathcal{I}^+$ which bound two spacelike hypersurfaces Σ_1 and Σ_2. In general we do not expect that the charge associated to a generic vector field ξ will be the same for $\partial\Sigma_1$ and $\partial\Sigma_2$, in part because energy and other fluxes can leak out through the region $\mathcal{B}_{12}$ of $\mathcal{I}^+$ between them (see Fig. 2.1). Rather, we expect the difference in the charges to be related to a flux through this region. The formula for this is straightforward to obtain:

$$
\begin{aligned}
\Delta Q_{BY} &\equiv \int_{\partial\Sigma_2} d^2x \sqrt{\sigma}\, n^i \xi^j T_{ij} - \int_{\partial\Sigma_1} d^2x \sqrt{\sigma}\, n^i \xi^j T_{ij} \\
&= -\frac{3\ell}{32\pi G} \int_{\mathcal{B}_{12}} g^{(3)ij} \mathcal{L}_\xi g^{(0)}_{ij} \sqrt{g^{(0)}} d^3x \\
&\equiv \int_{\mathcal{B}_{12}} F_\xi \; ,
\end{aligned}
\tag{2.3.19}
$$

where we have integrated by parts and used $\nabla^j T_{ij} = 0$.

When ξ is an isometry of $g^{(0)}$, the flux vanishes. The form of (2.3.19) is reminiscent of the analogous case in Minkowski space where $F_\xi = -\frac{1}{32\pi G} N_{ab}\chi^{ab}\boldsymbol{\epsilon}^{(3)}$ with $\chi_{ab} \propto \mathcal{L}_\xi g_{ab}$ and N_{ab} the Bondi news tensor [22, 23, 44]. In the BMS case, the flux vanishes whenever the news tensor is zero, which is equivalent to the absence of radiation. In our context, we can interpret $g^{(3)}_{ij}$ as the 'de Sitter news tensor'. When $g^{(3)}_{ij} = 0$, as in pure de Sitter, there is no radiation and no flux. Indeed, adding a gravitational wave to pure de Sitter [45] gives rise to a non-zero $g^{(3)}_{ij}$ term which is recorded by the flux.

2.4 Covariant Phase Space Charges

In this section, we consider the charges given by the covariant phase space formalism [23, 29, 35]. One advantage of these charges is that, in contrast to the Brown-York charges,

they may be evaluated for diffeomorphisms that are not tangent to $\mathcal{I}^+$.[7]

2.4.1 The Symplectic Form

In this section, we construct a symplectic form $\omega\left[\delta_1 g, \delta_2 g\right]$ for the phase space of initial data on a hypersurface Σ ending on $\mathcal{I}^+$. The problem is to find an expression which is finite for all on-shell deformations which preserve the FG form of the metric. A mathematically very similar problem was considered by Compère and Marolf in [30], who defined a finite symplectic form for the case of Neumann, rather than the usual Dirichlet, boundary conditions on the metric in anti-de Sitter space.

In the usual Einstein-Hilbert Lagrangian approach, one begins with the variation of the four-form Lagrangian

$$\delta L_{EH}\left[g\right] = d\Theta_{EH}\left[g, \delta g\right], \tag{2.4.20}$$

where the so-called presymplectic three-form $\Theta_{EH}\left[g, \delta g\right]$ is the remaining boundary term. An expression for Θ_{EH} can be found in (A.2.9). From this presymplectic three-form we can define the symplectic three-form ω_{EH} by

$$\omega_{EH}[\delta_1 g, \delta_2 g] = \delta_1\Theta_{EH}\left[g, \delta_2 g\right] - \delta_2\Theta_{EH}\left[g, \delta_1 g\right] . \tag{2.4.21}$$

The symplectic product associated to Σ is then

$$\langle\delta_1 g|\delta_2 g\rangle_{EH,\Sigma} = \int_\Sigma \omega_{EH}[\delta_1 g, \delta_2 g] . \tag{2.4.22}$$

From the explicit expression in appendix A.1 for ω_{EH}, one may readily verify that this expression however is not in general finite for on-shell variations respecting our boundary

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[7]In what follows, we evaluate all expressions on-shell.

conditions (2.1.2). To remedy this we note that Θ_{EH} is ambiguous up to the addition of an exact form dB. Finiteness can be restored by a judicious choice of B, which is in fact canonically associated to the boundary counterterms added in the previous section to ensure finiteness of the Brown-York stress tensor. The result is [30]

$$\omega_{mod}\left[\delta_1 g, \delta_2 g\right] = \omega_{EH}\left[\delta_1 g, \delta_2 g\right] + d\omega_{ct}\left[\delta_1 \gamma, \delta_2 \gamma\right], \tag{2.4.23}$$

where γ is the induced metric on the boundary of Σ and ω_{ct} is the symplectic form built out of the counterterm action. Explicit expressions are given in appendix A.1. Using these expressions, a computation essentially identical – up to a few sign changes – to the one in [30] shows that the associated symplectic products are finite. Note however that local conservation of the symplectic form (2.4.23) does not imply conservation of the symplectic product (2.4.22) since physical excitations leak through $\mathcal{I}^+$.

2.4.2   Integrable Charges

Given a finite symplectic product, and an allowed diffeomorphism ξ of the form (2.2.9), the covariant phase space formalism provides a canonical construction of the charge difference between two solutions which differ by an amount δg.[8] Define

$$k_\xi^{dS}[\delta g] = I_\xi \omega_{mod}\left[\mathcal{L}_\xi g, \delta g\right], \tag{2.4.24}$$

where the object I_ξ is a homotopy operator whose explicit form can be found in [29, 46]. The infinitesimal charge difference for a two-dimensional compact hypersurface $\partial\Sigma$ in $\mathcal{I}^+$ is then

$$\delta Q_\xi = \int_{\partial\Sigma} k_\xi^{dS}[\delta g] = \delta Q_\xi^{BY} - \frac{1}{2}\int_{\partial\Sigma} d^2 x \sqrt{g^{(0)}} n^k [g^{(0)}] \xi_k T_{(0)}^{ij} \delta g_{ij}^{(0)}. \tag{2.4.25}$$

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[8]For some details of the derivation, we refer the reader to appendix A.2 and [30].

For future convenience, we introduce the three-form $\Theta_{(0)}$ through

$$\int_{\partial\Sigma} i_\xi \Theta_{(0)}[g^{(0)}, \delta g^{(0)}] \equiv \frac{1}{2} \int_{\partial\Sigma} d^2x \sqrt{g^{(0)}} n^k[g^{(0)}] \xi_k T_{(0)}^{ij} \delta g_{ij}^{(0)} \, , \qquad (2.4.26)$$

where i_ξ is the interior product with respect to ξ^i. Notice that upon imposing Dirichlet boundary conditions, i.e. that the variation of the boundary metric $g^{(0)}$ vanishes, the charges are equivalent to those of the Brown-York formalism. However such a boundary condition is inappropriate at the boundary of dS^+ because it precludes, for example, gravity waves.

A second, related, issue concerning these charge differences is that they are not integrable: there is no unambiguous way to integrate the infinitesimal charge differences up to a finite one. The obstruction is visible in the non-vanishing commutator

$$(\delta_1\delta_2 - \delta_2\delta_1) Q_\xi = -\delta_1 \int_{\partial\Sigma} i_\xi \Theta_{(0)}[g^{(0)}, \delta_2 g^{(0)}] - (1 \leftrightarrow 2) \neq 0 \, . \qquad (2.4.27)$$

Exactly the same problem is encountered in defining BMS type charges at $\mathcal{I}^+$ in Minkowski space. In this context Wald and Zoupas [23] have proposed adding an additional boundary term to the charges

$$\delta Q_\xi^{WZ} = \delta Q_\xi + \int_{\partial\Sigma} i_\xi \Theta^{WZ}\left[g^{(0)}, \delta g^{(0)}\right] \, . \qquad (2.4.28)$$

The Θ^{WZ} boundary term is designed to precisely cancel the term leading to non-integrability in (2.4.27). We can immediately identify, exactly as in [23],

$$\Theta^{WZ} = \Theta_{(0)} \, . \qquad (2.4.29)$$

As discussed in [23], the Θ^{WZ} term is related to the flux through:

$$F_\xi = \Theta^{WZ}\left[g^{(0)}, \delta_\xi g^{(0)}\right] \, , \qquad (2.4.30)$$

in agreement with our earlier expression (2.3.19).

Adding this term to the covariant phase space charge gives finally

$$\delta Q_\xi^{WZ} = \delta Q_\xi^{BY} \quad \Longrightarrow \quad Q_\xi^{WZ} = \int_{\partial\Sigma} d^2x \sqrt{\sigma} n^i \xi^j T_{ij} \ . \tag{2.4.31}$$

Hence, the covariant phase space charges, after a lengthy analysis, reduce precisely to those of Brown and York. However, it can now be seen explicitly that the conformal slicing transformations are indeed trivial, as earlier anticipated.

Chapter 3

Future Boundary Conditions

In this chapter, we consider asymptotically de Sitter (dS) spacetimes and endow them with with an eternally-funded observatory whose fattened worldline is denoted $\mathcal{W}_O$. We use the word observatory rather than observer to emphasize that we are considering an object of finite extent in which arbitrary experiments can be performed and indefinitely repeated. We explore herein the premise that these experiments comprise the basic dS observables, and that correlators on the cylindrical boundary of $\mathcal{W}_O$ play a role in dS dynamics in some respects akin to the role played by correlators on the asymptotic cylindrical boundary of AdS in AdS dynamics.[1]

Of course dS has an asymptotic future boundary - $\mathcal{I}^+$- which in some ways resembles the asymptotic spatial boundary of AdS, with the role of space and time reversed. However there are several key differences between these boundaries. One is that correlators on $\mathcal{I}^+$of dS cannot be measured by any physical experiment because all points on $\mathcal{I}^+$are causally

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[1]This approach to dS has been advocated elsewhere including [47, 48, 49, 50, 51, 52]. One objection to it is that an observatory which can record and store all such information must have an infinite number of microstates, while de Sitter space itself might have only finitely many [53].

disconnected.[2] For this reason they are sometimes referred to as "metaobservables" [54]. A second difference is that Dirichlet-type boundary conditions on the metric and other fields can be imposed at the asymptotic boundary of AdS, ensuring that no radiation passes out of the boundary of the spacetime and the energy is conserved. Imposing such Dirichlet-type boundary conditions at dS $\mathcal{I}^+$would violate causality and lead to inconsistencies with the usual type of dS initial value formulation on complete spacelike slices. In general we expect radiation can pass through $\mathcal{I}^+$and the charges are not conserved. The asymptotic structure of dS was recently analyzed in [55], where it was shown that the asymptotic symmetry group (ASG) is all the diffeomorphisms tangent to $\mathcal{I}^+$, and the associated charges obey a conservation law relating their variation to the radiation flux through $\mathcal{I}^+$. This is in marked contrast to AdS where the ASG is the finite-dimensional (for $D > 3$) conformal group. The dS case resembles more the case of Minkowski $\mathcal{I}^+$whose ASG is the infinite-dimensional BMS group [6, 7].

In this work, however, we question the notion that one should think about dS dynamics in terms of imposing initial data (or a quantum state) on a complete spacelike slice and then evolving it to the future. While mathematically well-defined, this is highly unphysical, since such slices necessarily contain causally disconnected regions. Hence the resulting spacetime does not correspond to anything which could be physically measured.

Here we propose that initial data should instead be imposed on the boundary $\partial \mathcal{W}_O$ of the fattened observatory worldline. In the gravity sector, which is esentially all we consider here, this means specifying the intrinsic metric and extrinsic curvature, subject of course to the constraint equations, on the cylindrical timelike hypersurface $\partial \mathcal{W}_O$. Physically this

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[2]This of course would not apply to an early-time approximately dS inflationary phase of our universe–see below.

means we are characterizing the spacetime by what passes in and out of the observatory walls – clearly measurable data. To determine the bulk geometry, we must evolve radially outward rather than forward in time.

The radial evolution of this initial data on $\partial \mathcal{W}_O$ may fully determine the geometry within the $\mathcal{W}_O$ causal diamond but not on $\mathcal{I}^+$. We propose to fix the $\mathcal{I}^+$ geometry by imposing Dirichlet boundary conditions on the conformal metric everywhere except at the conformal point $\mathcal{P}_O$ where $\mathcal{W}_O$ reaches $\mathcal{I}^+$. This condition together with the $\partial \mathcal{W}_O$ initial data remaining data on $\mathcal{I}^+$- which turns out to be the conformal traceless part of the extrinsic curvature - plausibly determines the entire spacetime.[3] The main - if simple - point of this work is that, while such a boundary condition is manifestly acausal, the causality violations are apparently unobservable, *i.e.* they cannot be detected by any physical experiment in the observatory. We show explicitly at the linearized level that the future boundary condition imposes no acausal restriction on the initial data on $\partial \mathcal{W}_O$, and that these conditions together determine the full dS geometry. We expect these results to extend beyond the linearized level to a finite neighborhood of the vacuum dS geometry.

To understand why this is possible, consider a gravity wave produced at the observatory which passes through the future $\mathcal{W}_O$ horizon and reaches $\mathcal{I}^+$. Dirichlet boundary conditions will acausally reflect it backward in time, but the reflected wave remains outside the $\mathcal{W}_O$ causal diamond. Another way of thinking of this is that the boundary condition acausally places "de Sitter demons" outside the $\mathcal{W}_O$ causal diamond. Every time a wave comes out of the observatory, a de Sitter demon sends a finely-tuned wave to $\mathcal{I}^+$ which interferes destructively with the observatory wave so as to maintain the Dirichlet boundary condition

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[3]In the case of vanishing cosmological constant, the problem of data on a worldtube and initial null slice was formulated in a very similar fashion in [56].

on $\mathcal{I}^+$.

Going beyond pure gravity, we expect this type of boundary condition makes sense in theories with no massive particles or black holes which are absolutely stable. Everything must ultimately decay to massless particles. If a localized stable object reaches $\mathcal{I}^+$, the future boundary condition cannot be maintained by the mechanism discussed here - although there may be a generalization.

The theory of inflation proposes that our universe had a long era in which the geometry was very close to dS with a large cosmological constant. The considerations of this work do not directly apply to this era. We are metaobservers for this early dS phase: we can see events which would have been forever causally disconnected had there been no exit from inflation. Indeed the CMB and its fluctuations can be approximately thought of as the correlation functions on the would-be $\mathcal{I}^+$ of the early dS phase [57]. There is also no horizon or Bekenstein-Hawking entropy associated to the early phase once the exit from inflation into the present phase is taken into account. Clearly there are qualitative differences between an early-time and future asymptotic dS phases. It would be interesting to understand how the description of one goes to the other as the lifetime of the phase becomes infinite.

The considerations of this work are purely classical and we do not attempt to define a quantum theory consistent with the future boundary conditions. Nevertheless our observations may have implications for attempts to construct a holographic dual for dS quantum gravity. This is of course a wide open problem. It is not even clear where the best home for the dual is: $\mathcal{I}^+$, the horizon and $\partial \mathcal{W}_O$ are among the possibilities. One might expect the ASG for dS to be the symmetry group of the dual theory. Taken at face value, the result of [55] that the ASG is all diffeomorphisms of $\mathcal{I}^+$ suggests that the dual should itself

be a theory of gravity in one lower dimensions. This large ASG came from the absence of $\mathcal{I}^+$ boundary conditions in the usual approach. If we apply Dirichlet boundary conditions at $\mathcal{I}^+$, as in the present work, the structure becomes very similar to that of AdS. Indeed the dS two-point functions with these boundary conditions are precisely the analytic continuations (in the cosmological constant) of the AdS two-point correlation functions, and transform under the Euclidean conformal group $SO(D, 1)$. This suggests that the holographic dual is a conformal field theory without gravity, as envisioned in the dS/CFT correspondence [12, 13]. Hence the boundary conditions proposed herein brings the structure of dS much closer to that of AdS, and hopefully will be useful in adapting insights from AdS holography to the dS context.

Our results resonate with a recent paper [58] considering future boundary conditions for conformal gravity in dS. It was shown that they can be chosen to classically reduce dS conformal gravity to dS Einstein gravity. This reduction however requires future boundary conditions everywhere on $\mathcal{I}^+$ and might be ruined by the exclusion of the point $\mathcal{P}_O$. Nevertheless our observations may be relevant to a better understanding of the relation between conformal and Einstein gravity. Our picture may also bear some relation to Schrodinger's Z_2 antipodal identification of dS [59, 60, 61, 62] or black hole final state boundary conditions [63] and is in the spirit of black hole complementarity [64].

This chapter is organized as follows. In section 3.1, we consider as a warmup the case of light scalars in dS$_3$. Below a critical value of the mass these modes, like 4D gravitons, do not oscillate at $\mathcal{I}^+$ and can have a slow or fast exponential falloff. We show that, for any mode sourced in the southern causal diamond, demons located in the causally complementary northern diamond can excite a northern mode which will interfere with the southern mode

in such a way that the total mode has only the fast-falling component near $\mathcal{I}^+$. The phase of the northern demon mode depends on the mass, angular momentum and frequency of the southern mode. In section 3.2, we show that northern demons in dS$_4$ can similarly destroy the slow falling components of gravity waves produced in the southern diamond. This means that the conformal metric on $\mathcal{I}^+$ retains its round shape, although the conformal extrinsic curvature is altered by the wave. In section 3.3, we consider two-point functions in our setup, returning for simplicity to the case of light dS$_3$ scalars. We show that a unique symmetric two-point function is determined by demanding dS invariance, fast falloff at $\mathcal{I}^+$ and Hadamard form of the coincident-point singularity. We further show that this two-point function can *not* arise as the Wightman function of two scalar fields in any quantum state defined on complete spacelike slices in dS, but *does* result from double analytic continuation of the standard AdS scalar two-point function. Appendix B contains a construction of the dS$_4$ graviton modes in global coordinates, complementing the static patch analysis of section 3.2.

3.1   Warmup: light scalars in dS$_3$

In this section we study light scalars with masses less than the critical value $\mu^2 = \Lambda/3$. The $\mu^2 > \Lambda/3$ case was studied with similar conventions in [65]. The asymptotic behavior of such light scalars resembles 4D gravitons in that they exponentially decay rather than oscillate near $\mathcal{I}^+$. We use the metric in static patch coordinates:

$$\frac{ds^2}{\ell^2} = -(1 - r^2)dt^2 + \frac{dr^2}{(1 - r^2)} + r^2 d\varphi^2 \ . \tag{3.1.1}$$

where $\ell^2 = 3/\Lambda$. The southern causal diamond associated to an observatory at the south pole is described by $r \in [0, 1]$. The northern causal diamond which will be populated by demons is described by a second copy also with $r \in [0, 1]$. We take time to run forward in the southern diamond and backwards in the northern diamond so that ∂_t is the globally defined Killing vector. The future and past diamonds, containing $\mathcal{I}^+$ and $\mathcal{I}^-$ respectively, are described by $r \in [1, \infty]$. In the future diamond, the spacelike t-coordinate runs from north to south, whereas in the past diamond it runs south to north.

3.1.1   Northern and southern modes

The scalar field modes may be labeled by the angular momentum j in the ϕ direction and the frequency ω in time. The general solution of the scalar wave equation for mass $0 < \mu^2 \ell^2 < 1$ in the southern patch is then given by:

$$\phi^S(t, r, \varphi) = \sum_{j \in \mathbb{Z}, \omega > 0} \left(\kappa_{j\omega} \phi^S_{\omega j}(t, r, \varphi) + \kappa^*_{\omega j} \phi^{S*}_{\omega j}(t, r, \varphi) \right) , \tag{3.1.2}$$

where the static patch modes smooth at the origin[4] are:

$$\phi^S_{\omega j} = e^{-i\omega t + ij\varphi} \, r^{|j|}(1 - r^2)^{i\omega/2} F(a, b; c; r^2) , \tag{3.1.3}$$

and the arguments of the hypergeometric function $F(a, b; c; r^2)$ are:

$$a \equiv \frac{1}{2}\left(|j| + i\omega + h_+\right) , \quad b \equiv \frac{1}{2}\left(|j| + i\omega + h_-\right) , \quad c \equiv 1 + |j| . \tag{3.1.4}$$

with

$$h_{\pm} \equiv 1 \pm \sqrt{1 - \mu^2 \ell^2}. \tag{3.1.5}$$

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[4]Other types of behavior at the origin might be considered depending on the nature of the observatory stationed there.

Note that $h_\pm$ are both real and positive in the mass range under consideration. There is a similar expansion for the northern modes, since the northern patch is described by an identical coordinate system with time running backwards.

Near the cosmological horizon

We now study the behavior of the static patch modes near the cosmological horizon $r = 1$. In Kruskal coordinates:

$$r = \frac{1 + UV}{1 - UV} \ , \quad t = \frac{1}{2} \log\left(-\frac{U}{V}\right) \ , \tag{3.1.6}$$

the southern diamond is the region $U > 0$, $V < 0$ and the future (past) horizon is at $V = 0$ ($U = 0$). Using the hypergeometric identity:

$$\begin{aligned}
F\left(a, b; c; z\right) &= \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)} F\left(a, b; 1 + a + b - c; 1 - z\right) \\
&+ \frac{\Gamma(c)\Gamma(a + b - c)}{\Gamma(a)\Gamma(b)}(1 - z)^{c - a - b} F\left(c - a, c - b; c - a - b + 1; 1 - z\right) \ , \quad (3.1.7)
\end{aligned}$$

the near horizon behavior is

$$\phi_{\omega j}^S \sim e^{ij\varphi}\left[\alpha_{\omega j}(-V)^{i\omega} + \alpha_{-\omega j}U^{-i\omega}\right] \ , \tag{3.1.8}$$

with:

$$\alpha_{\omega j} \equiv \frac{\Gamma(1 + |j|)\Gamma(-i\omega)2^{i\omega}}{\Gamma\left(\frac{1}{2}(|j| - i\omega + h_+)\right)\Gamma\left(\frac{1}{2}(|j| - i\omega + h_-)\right)} \ . \tag{3.1.9}$$

3.1.2   Future and past modes

In the future diamond, we can build $\phi_{j\omega}^{out\pm}$ modes that behave as $\sim r^{-h_\pm}$ near $\mathcal{I}^+$. Explicitly we find for the fast-falling $out+$ modes:

$$\phi_{\omega j}^{out+} = e^{-i\omega t + ij\varphi} \, r^{-h_+} \left(1 - \frac{1}{r^2}\right)^{i\omega/2} F(a, 1 - a_-^*; h_+; \frac{1}{r^2}) \ , \tag{3.1.10}$$

where a_-^* is given by taking the expression for a^* and replacing h_+ with h_-. For the slow-falling *out−* modes we find:

$$\phi_{\omega j}^{out-} = e^{-i\omega t + ij\varphi} \; r^{-h_-} \left(1 - \frac{1}{r^2}\right)^{-i\omega/2} F(a_-^*, 1 - a; h_-; \frac{1}{r^2}) \; . \tag{3.1.11}$$

Near the cosmological horizon

Once again, we can expand the above expressions for the *out* modes near the cosmological horizon. The Kruskal coordinates are now given by:

$$r = \frac{1 + UV}{1 - UV} \; , \quad t = \frac{1}{2} \log \left(\frac{U}{V}\right) \; , \tag{3.1.12}$$

where $U > 0$ and $V > 0$ in the future diamond. The *out+* modes behave as:

$$\phi_{\omega j}^{out+} \sim e^{ij\varphi} \left[\beta_{\omega j} V^{i\omega} + \beta_{-\omega j} U^{-i\omega}\right] \; , \tag{3.1.13}$$

with:

$$\beta_{\omega j} = \frac{\Gamma(h_+)\Gamma(-i\omega)2^{i\omega}}{\Gamma\left(\frac{1}{2}(h_+ - |j| - i\omega)\right) \Gamma\left(\frac{1}{2}(h_+ + |j| - i\omega)\right)} \; . \tag{3.1.14}$$

Similarly, the expansion of the *out−* modes near the cosmological horizon is as above, but with $\beta_{\omega j}$ replaced by:

$$\gamma_{\omega j} = \frac{\Gamma(h_-)\Gamma(-i\omega)2^{i\omega}}{\Gamma\left(\frac{1}{2}(h_- + |j| - i\omega)\right) \Gamma\left(\frac{1}{2}(h_- - |j| - i\omega)\right)} \; . \tag{3.1.15}$$

3.1.3 Matching the flux

When we send out a wave from the northern or southern patch, it will generically contain both fast and slow-falling out modes. Matching the flux across the future horizon determines the Bogoliubov transformation relating the northern and southern modes to the $\pm out$

modes:[5]

$$\phi^S_{\omega j} = A^{11}_{\omega j} \phi^{out-}_{\omega j} + A^{12}_{\omega j} \phi^{out+}_{\omega j} \ , \tag{3.1.16}$$

$$\phi^N_{\omega j} = A^{21}_{\omega j} \phi^{out-}_{\omega j} + A^{22}_{\omega j} \phi^{out+}_{\omega j} \ . \tag{3.1.17}$$

with:

$$N_{\omega j} A^{11}_{\omega j} = \alpha_{-\omega j} \beta_{\omega j} \ , \tag{3.1.18}$$

$$N_{\omega j} A^{12}_{\omega j} = -\alpha_{-\omega j} \gamma_{\omega j} \ , \tag{3.1.19}$$

$$N_{\omega j} A^{21}_{\omega j} = -\alpha_{\omega j} \beta_{-\omega j} \ , \tag{3.1.20}$$

$$N_{\omega j} A^{22}_{\omega j} = \alpha_{\omega j} \gamma_{-\omega j} \ . \tag{3.1.21}$$

where $N_{\omega j} \equiv \beta_{\omega j} \gamma_{-\omega j} - \gamma_{\omega j} \beta_{-\omega j} = i\sqrt{1 - \mu^2 \ell^2}/\omega$.

3.1.4 Demonic interference

Now we would like to demonstrate that the slow-falling piece of any southern mode at $\mathcal{I}^+$ by a wave produced by a de Sitter demon in the causally disconnected northern diamond. More precisely, if the observatory excites a normalized southern mode $\phi^S_{\omega j}$, the coefficient of the $\phi^{out-}_{\omega j}$ mode at $\mathcal{I}^+$ will be $A^{11}_{\omega j}$. So the demon must excite a northern mode with Fourier coefficient $-A^{11}_{\omega j}/A^{21}_{\omega j}$, to cancel the slow falling component of the incoming southern mode. Then, the coefficient of the fast falling mode $out+$ becomes:

$$A^{12}_{\omega j} - \frac{A^{11}_{\omega j}}{A^{21}_{\omega j}} A^{22}_{\omega j} = \frac{\Gamma(1 + |j|)\Gamma\left(\frac{1}{2}(h_+ - |j| + i\omega)\right)}{\Gamma(h_+)\Gamma\left(\frac{1}{2}(h_- + |j| + i\omega)\right)} \ . \tag{3.1.22}$$

Alternatively, we can express the above as:

$$A^{12}_{\omega j} - \frac{A^{11}_{\omega j}}{A^{21}_{\omega j}} A^{22}_{\omega j} = A^{12}_{\omega j}\left(1 - e^{i\lambda_{\omega j}}\right) \ , \tag{3.1.23}$$

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[5]For heavy modes, with $\mu\ell > 1$, this can be found in [65].

where:

$$\tan\left(\lambda_{\omega j}/2\right) \equiv \frac{\sin\left(\pi\sqrt{1-\mu^2\ell^2}\right)\sinh\left(\pi\omega\right)}{(-1)^j + \cos\left(\pi\sqrt{1-\mu^2\ell^2}\right)\cosh\left(\pi\omega\right)} \ . \tag{3.1.24}$$

Thus, the full mode with no slow-falling behavior near $\mathcal{I}^+$ is given by:

$$\phi_{\omega j}^S - \frac{A_{j\omega}^{11}}{A_{j\omega}^{21}}\phi_{\omega j}^N \ , \tag{3.1.25}$$

where:

$$-\frac{A_{j\omega}^{11}}{A_{j\omega}^{21}} = \frac{\Gamma(\frac{1}{2}(h_- + j - i\omega))\Gamma(\frac{1}{2}(h_+ - j + i\omega))}{\Gamma(\frac{1}{2}(h_- + j + i\omega))\Gamma(\frac{1}{2}(h_+ - j - i\omega))} = e^{-i\delta_{\omega j}} \ , \tag{3.1.26}$$

and $\tan(\delta_{\omega j}/2) \equiv \tan\left(\frac{\pi}{2}\left(j - \sqrt{1-\mu^2\ell^2}\right)\right)\tanh(\pi\omega/2)$.

3.2   Linearized gravity in dS$_4$

We now consider the problem of linearized gravity in the static patch of dS$_4$, following the work of [66]. The 4D dS metric is :

$$\frac{ds^2}{\ell^2} = -(1-r^2)dt^2 + \frac{dr^2}{(1-r^2)} + r^2 d\Omega_2^2 \ , \tag{3.2.27}$$

The linearized gravitational excitations can be parametrized by a transverse vector spherical harmonic and a scalar spherical harmonic. Together, these constitute two degrees of freedom. There is no transverse-traceless tensorial spherical harmonic for a two-sphere. Since the computation is essentially identical for both types of harmonics, we only consider the vector harmonics in what follows.

3.2.1 Vector excitations

We can express [66] the vectorial perturbations in terms of a transverse vectorial spherical harmonic $\mathcal{V}_i$:

$$\delta g_{ij} = 2r^2 H_T(r,t)\mathcal{V}_{ij} , \quad i,j \in \{\theta, \phi\} , \tag{3.2.28}$$

$$\delta g_{ai} = r f_a \mathcal{V}_i , \quad a \in \{t, r\} . \tag{3.2.29}$$

with all other components of $\delta g_{\mu\nu}$ vanishing. We have further defined:

$$\mathcal{V}_{ij} \equiv -\frac{1}{2k_V} \left(D_i \mathcal{V}_j + D_j \mathcal{V}_i \right) . \tag{3.2.30}$$

The vectorial harmonics satisfy:

$$\left(\Delta_{S^2} + k_V^2 \right) \mathcal{V}_i = 0 , \quad D_j \mathcal{V}^j = 0 . \tag{3.2.31}$$

The eigenvalues are given by $k_V^2 = l(l+1) - 1$ with $l = 1, 2, \ldots$ being the angular momentum on the S^2. Thus, they constitute a single degree of freedom.

Upon defining a master variable $\Phi(r,t) \equiv r^{-1}\Omega(r,t)$ and $(f^a + r D^a H_T/k_V) \equiv r^{-1}\epsilon^{ab} D_b \Omega$, it is found in [66] that the equation satisfied by the master field Φ is given by:

$$\Box_{g^{(2)}} \Phi - \frac{V_V}{(1-r^2)}\Phi = 0 \implies -(1-r^2)\frac{d}{dr}\left((1-r^2)\frac{d\Phi}{dr} \right) + V_V \Phi = \omega^2 \Phi , \tag{3.2.32}$$

with effective potential:

$$V_V = \frac{(1-r^2)}{r^2} \left(k_V^2 + 1 \right) . \tag{3.2.33}$$

The box operator is the Laplacian corresponding to the two-dimensional metric g_{ab} with $a, b \in \{t, r\}$. We have further assumed an oscillatory time behavior $\Phi(r,t) = e^{-i\omega t}\varphi(r)$ for the modes. For convenience, the subscript labels ω and l for the fields have been suppressed.

3.2.2 Solution near the origin

The solution that is well behaved near $r = 0$ is:

$$\varphi^S(r) = r^{l+1}(1 - r^2)^{-i\omega/2}F\left(a, b; c; r^2\right) \ , \tag{3.2.34}$$

with

$$a = \frac{1}{2}\left(1 + l - i\omega\right) \ , \quad b = \frac{1}{2}\left(2 + l - i\omega\right) \ , \quad c = \frac{3}{2} + l \ . \tag{3.2.35}$$

Clearly, there are a set of northern modes defined in the northern patch which are equivalent to the above, except that t runs backwards.

Near the cosmological horizon

Using hypergeometric identities, we can express the above in a way that makes manifest its behavior near the cosmological horizon $r^2 = 1$. Once again, we exploit equation (3.1.7). Notice that in this case, $c - a - b = i\omega$ and thus we find a linear combination of ingoing and outgoing modes for $\varphi(r)$ near the cosmological horizon. We use the Kruskal coordinates (3.1.6) in the southern diamond, with $U > 0$ and $V < 0$. Near the horizon where $r \to 1$ (and $UV \to 0$):

$$\varphi^S(r)e^{-i\omega t} \sim \alpha_{\omega l}(-V)^{i\omega} + \alpha_{\omega l}^* U^{-i\omega} \ . \tag{3.2.36}$$

The coefficients $\alpha_{\omega l}$ are given by:

$$\alpha_{\omega l} \equiv \frac{\Gamma(3/2 + l)\Gamma(-i\omega)2^{i\omega}}{\Gamma\left(\frac{1}{2}(1 + l - i\omega)\right)\Gamma\left(\frac{1}{2}(2 + l - i\omega)\right)} \ . \tag{3.2.37}$$

3.2.3   Solution near $\mathcal{I}^+$

We can also build solutions which are smooth in the region $r \in [1, \infty]$ containing $\mathcal{I}^+$.

We find two linearly independent solutions:

$$\varphi^{out-} = (r^2 - 1)^{-i\omega/2} r^{i\omega} F\left(\frac{1}{2}(1 + l - i\omega), \frac{1}{2}(-l - i\omega); \frac{1}{2}; \frac{1}{r^2}\right) , \qquad (3.2.38)$$

$$\varphi^{out+} = (r^2 - 1)^{-i\omega/2} r^{-1+i\omega} F\left(\frac{1}{2}(1 - l - i\omega), \frac{1}{2}(2 + l - i\omega); \frac{3}{2}; \frac{1}{r^2}\right) .$$

$$(3.2.39)$$

Near $\mathcal{I}^+$ the solutions behave like $\varphi^{out-} \sim 1$ and $\varphi^{out+} \sim 1/r + \mathcal{O}(1/r^3)$ which implies

$\Omega \sim r$ and $\Omega \sim 1 + \mathcal{O}(1/r^2)$. This in turn implies that the falloffs of the graviton itself, i.e.

$r^2 H_T(r, t)$, are given by $\sim r^2$ and $\sim r^{-1}$. Thus, as expected, there is a slow falling and fast

falling mode in accordance with the Starobinskii expansion [33].

Near the cosmological horizon

Once again, using the same hypergeometric identity (3.1.7), we can expand our solutions

near the cosmological horizon to find a linear combination of ingoing and outgoing modes.

Near the cosmological horizon $UV \to 0$, in the Kruskal coordinates (3.1.12) with $U > 0$ and

$V > 0$ we find:

$$\varphi^{out-}(r)e^{-i\omega t} \sim \beta_{\omega l} V^{i\omega} + \beta_{\omega l}^* U^{-i\omega} , \qquad (3.2.40)$$

$$\varphi^{out+}(r)e^{-i\omega t} \sim \gamma_{\omega l} V^{i\omega} + \gamma_{\omega l}^* U^{-i\omega} . \qquad (3.2.41)$$

The coefficients $\beta_{\omega l}$ and $\gamma_{\omega l}$ are given by:

$$\beta_{\omega l} \equiv \frac{\Gamma(1/2)\Gamma(-i\omega)2^{i\omega}}{\Gamma\left(\frac{1}{2}(1 + l - i\omega)\right)\Gamma\left(\frac{1}{2}(-l - i\omega)\right)} , \qquad (3.2.42)$$

$$\gamma_{\omega l} \equiv \frac{\Gamma(3/2)\Gamma(-i\omega)2^{i\omega}}{\Gamma\left(\frac{1}{2}(2 + l - i\omega)\Gamma\left(\frac{1}{2}(1 - l - i\omega)\right)\right)} . \qquad (3.2.43)$$

3.2.4 Matching the flux

The Bogoliubov transformation between the northern and southern modes and the $out\pm$ modes near $\mathcal{I}^+$ can now be obtained by matching the flux across the future horizons of the two static patches. We find:

$$\begin{pmatrix} \Phi^S_{\omega l} \\ \Phi^N_{\omega l} \end{pmatrix} = \mathbf{B}_{\omega l} \begin{pmatrix} \Phi^{out+}_{\omega l} \\ \Phi^{out-}_{\omega l} \end{pmatrix} . \tag{3.2.44}$$

The matrix $\mathbf{B}_{\omega l}$ is given by:

$$\mathbf{B}_{\omega l} = \frac{1}{(\beta^*_{\omega l}\gamma_{\omega l} - \gamma^*_{\omega l}\beta_{\omega l})} \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = -2i\omega \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} , \tag{3.2.45}$$

with:

$$B_{11} = -\alpha^*_{\omega l}\beta_{\omega l} , \tag{3.2.46}$$

$$B_{12} = \alpha^*_{\omega l}\gamma_{\omega l} , \tag{3.2.47}$$

$$B_{21} = \alpha_{\omega l}\beta^*_{\omega l} , \tag{3.2.48}$$

$$B_{22} = -\alpha_{\omega l}\gamma^*_{\omega l} . \tag{3.2.49}$$

3.2.5 Demonic interference for gravitons

As in the case of the scalar fields, we can tune the demon modes from the northern patch to cancel the non-normalizable graviton modes coming from the southern observatory. In particular, suppose the southern observer sends a single southern mode $\varphi^S_{\omega l}$, then its non-normalizable component is $\sim B_{12}\Phi^{out-}$. The northern demon will send in a mode with coefficient $-B_{12}/B_{22} = (-1)^l$ to cancel out the non-normalizable piece. The resultant mode

will only contain the Φ^{out+} (normalizable) mode whose coefficient is given by:

$$B_{11} - \frac{B_{12}}{B_{22}} B_{21} = \frac{i\Gamma\left(\frac{3}{2}+l\right)\Gamma\left(\frac{1}{2}(1-l+i\omega)\right)}{\omega\sqrt{\pi}\Gamma\left(\frac{1}{2}(1+l+i\omega)\right)} = B_{11}\left(1 - \frac{B_{12}B_{21}}{B_{22}B_{11}}\right) = 2B_{11} . \qquad (3.2.50)$$

since $B_{12}\,B_{21}/\left(B_{22}\,B_{11}\right) = -1 + \frac{2\sin\left(l\pi\right)}{\sin\left(l\pi\right)+i\sinh\left(\pi\omega\right)} = -1$. The full mode with no growing behavior near $\mathcal{I}^+$ is given by

$$\phi^S + (-1)^l \phi^N \qquad (3.2.51)$$

We do not fully understand why this result is so much simpler than that for the light scalar in dS$_3$ studied in the previous section. The modes (3.2.51) are eigenmodes of the dS anitipodal map and therfore must decay at $\mathcal{I}^-$ as well as $\mathcal{I}^+$. In the appendix, we compute the linearized graviton in global coordinates and verify this is indeed the case.

For completeness, we mention here that our result remains the same in the case of the scalar harmonic perturbations, since the effective equation V_S governing the scalar master function Φ_S [66] is equivalent to V_V (with $l = 0, 1, 2, \ldots$ in the scalar case).[6]

3.3 Analytic continuation AdS $\rightarrow$ dS

In this section we discuss the bulk-to-bulk two-point functions $G(x, x')$ consistent with our future boundary conditions. These are analogs of vacuum correlation functions, but since we have not explored herein how to define a quantum theory with acausal boundary conditions we can not realize $G(x, x')$ as $\langle 0\,|\phi(x)\phi(x')|\,0\rangle$. The allowed modes such as (3.1.25) are not a complete set on a spacelike slice so we cannot use them to define a state $|0\rangle$ on a such a slice. We regard this as a feature rather than a bug since, as we have argued, such a state is unphysical!

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[6]It would be interesting to understand whether such a demonic interference can elucidate the boundary conditions imposed at future infinity for the de Sitter-like spacetimes studied in [31, 67, 68].

Nevertheless a suitable two-point function $G(x, x')$ can be fully determined for $x \neq x'$ (i.e. up to an $i\varepsilon$ prescription not considered here) from general principles: the equation of motion for each argument, dS invariance, fast-falling boundary conditions at $\mathcal{I}^+$ and the Hadamard form of the short distance singularity. dS-invariance implies G can be written purely in terms of the quantity

$$P(x, x') = \cos \frac{d(x, x')}{\ell} \tag{3.3.52}$$

where the geodesic distance $d(x, x')$ between x and x' is imaginary for timelike separations. To be explicit in planar coordinates

$$ds_{dS}^2 = \frac{\ell^2}{\eta^2} \left(-d\eta^2 + dx_1{}^2 + dx_2^2 \right) \tag{3.3.53}$$

one has

$$P(x, x') = \frac{\eta^2 + \eta'^2 - (x_1 - x_1')^2 - (x_2 - x_2')^2}{2\eta\eta'} \ . \tag{3.3.54}$$

For the example of a scalar of mass μ in dS$_3$, G obeys:

$$(1 - P^2)\partial_P^2 G(P) - 3P\partial_P G(P) - \mu^2\ell^2 G(P) = 0 \ . \tag{3.3.55}$$

The solutions to the above equation are hypergeometric functions which generally involve both falloffs near $\mathcal{I}^+$. Choosing the solution which is only fast-falling near $\mathcal{I}^+$ gives us:

$$G(P) = N \left(\frac{2}{1 + P} \right)^{h_+} F \left(h_+, h_+ - \tfrac{1}{2}; 2h_+ - 1; \frac{2}{1 + P} \right) , \tag{3.3.56}$$

where N is a normalization factor. This has singularities at both the coincident point limit $P(x, x) = 1$ as well as the antipodal point limit $P(x, x_A) = -1$. The singularity for antipodally located points reflects the acausal character of our construction.

In the standard quantum formulation, the scalar Wightman function in the Euclidean vacuum, which is the only solution to (3.3.55) with no singularities at the antipodal point, is given by [65]:

$$G_E(P) = \frac{\Gamma(h_+)\Gamma(h_-)}{(4\pi)^{3/2}\Gamma(3/2)} F\left(h_+, h_-; \frac{3}{2}; \frac{1+P}{2}\right) . \qquad (3.3.57)$$

We can write our fast-falling two-point function G in terms of $G_E(P)$ and its antipodal cousin $G_E(-P)$ as:

$$G(P) = G_E(P) - e^{-i\delta}G_E(-P) , \qquad \delta \equiv \pi(1 - \sqrt{1 - \mu^2\ell^2}) \qquad (3.3.58)$$

provided we take the so-far undetermined normalization factor to be

$$N = -\frac{i2^{-2h_+}}{2\pi} . \qquad (3.3.59)$$

This clearly guarantees that the short distance singularity of G has the canonical Hadamard form.

The standard quantum formulation of a scalar in dS admits a one parameter family of dS-invariant vacua often referred to as α-vacua. With the exception of the Euclidean vacuum, the Wightman function for all of these vacua has antipodal singularities, and the short-distance singularity does not take the Hadamard form. Our Green function (3.3.58) is not the Wightman function in any of the dS α-vacua.

This is related to the observation [65, 69] that, although the double analytic continuation of AdS Wightman functions are some kind of dS two-point functions, they are not interpretable as Wightman functions in any dS-invariant state. Instead, they are precisely the two-point functions (3.3.58). To see this note that the Wightman function $G_{AdS}(x, x')$ for a scalar of mass μ_{AdS} in AdS$_3$ with radius ℓ_{AdS} obeys

$$(1 - P_{AdS}^2)\partial^2 G_{AdS}(P_{AdS}) - 3P_{AdS}\partial G(P_{AdS}) + \mu_{AdS}^2\ell_{AdS}^2 G_{AdS}(P_{AdS}) = 0 . \qquad (3.3.60)$$

with

$$P_{AdS} = \cos \frac{id_{AdS}}{\ell_{AdS}} \tag{3.3.61}$$

constructed from the geodesic distance d_{AdS} between two points (x, x') in AdS$_3$. Explicitly in Poincare coordinates,

$$ds_{AdS}^2 = \frac{\ell_{AdS}^2}{z^2} \left(dz^2 - dt^2 + dy^2 \right), \tag{3.3.62}$$

we have $P_{AdS}(x, x') = (z^2 + z'^2 - (t - t')^2 + (y - y')^2)/(2zz')$. Under double analytic continuation $z \to \eta, \ t \to x_1, \ y \to ix_2$ together with $\ell_{AdS} \to i\ell$, we have $ds_{AdS}^2 \to ds_{dS}^2$ and $P_{AdS} \to P$. Taking

$$P_{AdS} = P, \quad \ell_{AdS} = i\ell, \quad \mu_{AdS} = \mu, \quad G_{AdS} = G, \tag{3.3.63}$$

then (3.3.60) becomes exactly (3.3.55). The Hadamard-normalized solution picked out by the standard fast spatial falloff in AdS is then

$$G_{AdS}(P_{AdS}) = N \left(\frac{2}{1 + P_{AdS}} \right)^{h_+} F \left(h_+, h_+ - \tfrac{1}{2}; 2h_+ - 1; \frac{2}{1 + P_{AdS}} \right), \tag{3.3.64}$$

where here $h_+ = 1 + \sqrt{1 + \mu_{AdS}^2 \ell_{AdS}^2}$. Hence double analytic continuation maps the standard AdS Wightman function to the two-point function (3.3.58) consistent with future dS boundary conditions.

Note that both the dS and AdS two-point function have singularities at $P = -1$ which does not correspond to coincident (or null-separated) points. As discussed above, for the dS case this is an acausal singularity for spacelike antipodally separated points. In AdS, $P = -1$ corresponds to two noncoincident, timelike separated points connected by a light ray which is reflected off of the AdS boundary.

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Chapter 4

State/Operator Correspondence in Higher-Spin dS/CFT

The conjectured dS/CFT correspondence attempts to adapt the wonderful successes of the AdS/CFT correspondence to universes (possibly like our own) which exponentially expand in the far future. The hope [12, 70, 54, 57, 71] is to define bulk de Sitter (dS) quantum gravity in terms of a holographically dual CFT living at $\mathcal{I}^+$ of dS, which is the asymptotic conformal boundary at future null infinity. A major obstacle to this program has been the absence of any explicit microscopic realization. This has so far prevented the detailed development of the dS/CFT dictionary. This situation has recently been improved by an explicit proposal [14] relating Vasiliev's higher-spin gravity in dS$_4$ [72, 73] to the dual $Sp(N)$ CFT$_3$ described in [74]. In this chapter, we will use this higher-spin context to write some new entries in the dS/CFT dictionary.

The recent proposal [14] for a microscopic realization of dS/CFT begins with the duality relating the free (critical) $O(N)$ CFT$_3$ to higher-spin gravity on AdS$_4$ with Neumann

(Dirichlet) boundary conditions on the scalar field. Higher-spin gravity - unlike string theory [70] - has a simple analytic continuation from negative to positive cosmological constant Λ. Under this continuation, $\text{AdS}_4 \rightarrow \text{dS}_4$ and the (singlet) boundary CFT_3 correlators are simply transformed by the replacement of $N \rightarrow -N$. These same transformed correlators arise from the $Sp(N)$ models constructed from anticommuting scalars. It follows that the free (critical) $Sp(N)$ correlators equal those of higher-spin gravity on dS_4 with future Neumann (Dirichlet) scalar boundary conditions (of the type described in [75]) at $\mathcal{I}^+$.

This mathematical relation between the bulk dS and boundary $Sp(N)$ correlators may provide a good starting point for understanding quantum gravity on dS, but so far important physical questions remain unanswered. For example we do not know how to relate these physically *un*measurable correlators to a set of true physical observables or to the dS horizon entropy. These crucial entries in the dS/CFT dictionary are yet to be written.

As a step in this direction, in this chapter we investigate the relation between quantum states in the bulk higher-spin gravity and those in the boundary CFT_3. Bulk higher-spin gravity has fields of Φ^s with all even spins $s = 0, 2, ...$, which are dual to CFT_3 operators $\mathcal{O}^s$ with the same spins. In the CFT_3, we can also associate a state to each operator by the state-operator correspondence. One way to do this is to take the southern hemisphere of S^3, insert the operator $\mathcal{O}^s$ at the south pole, and then define a state $\Psi^s_{S^2}$ as a functional of the boundary conditions on the equatorial S^2. For every object in the CFT_3, we expect a holographically dual object in the bulk dS_4 theory. This raises the question: what is the bulk representation of the spin-s state $\Psi^s_{S^2}$?

In Lorentzian AdS_4 holography, the state created by a primary operator $\mathcal{O}$ in the CFT_3 on S^2 has, at weak coupling, a bulk representation as the single particle state of the field Φ

dual to $\mathcal{O}$ with a smooth minimal-energy wavefunction localized near the center of AdS$_4$. The form of the wavefunction is dictated by the conformal symmetry.

In dS$_4$ holography, the situation is rather different. States in dS$_4$ quantum gravity are usually thought of as wavefunctions on complete spacelike slices which are topologically S^3.[1] These do not seem to be good candidates for bulk duals to $\Psi^s_{S^2}$ because, among other reasons, they are not associated to any S^2 in $\mathcal{I}^+$. However, dS$_4$ also has everywhere spacelike and geodesically complete R^3 slices which end at an S^2 in $\mathcal{I}^+$. Here we propose a construction of the bulk version of $\Psi^s_{S^2}$ on these slices, again as single particle states whose form is dictated by the conformal symmetry. Interestingly, the classical wavefunction for the particle turns out to be the (anti) quasinormal modes for the static patch of de Sitter, as constructed in [76, 77].[2]

This relation between bulk and boundary states has a potentially profound nonperturbative consequence briefly mentioned in section 4.3.1 [79]. The operator $\mathcal{O}^0$ dual to a scalar Φ is bilinear in boundary fermions and hence obeys $(\mathcal{O}^0)^{\frac{N}{2}+1} = 0$. Under bulk-boundary duality this translates into an $\frac{N}{2}$-adicity relation for Φ: one cannot put more than $\frac{N}{2}$ bulk scalar quanta into the associated quasinormal mode. Further investigation of this dS exclusion principle is deferred to later work.

We also construct a norm for these bulk states and show that it is the Zamolodchikov norm on S^3 of the CFT$_3$ operator $\mathcal{O}^s$. Explicit formulae are exhibited only for the scalar $s = 0$ case but we expect the construction to generalize to all s.

This chapter is organized as follows. In section 4.1, we revisit the issue of the usual

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[1] As explained in [57, 71], such states do play an important role in dS/CFT, but as generating functions for correlators rather than as duals to CFT$_3$ states on S^2. The relation between the R^3 and S^3 bulk states in our example is detailed below.

[2] We are grateful to D. Anninos for pointing this out [78].

global dS-invariant vacua for a free massive scalar field, paying particular attention to the case of $m^2\ell^2 = 2$ (where ℓ is the de Sitter radius) arising in higher spin gravity. The invariant vacua include the familiar Bunch-Davies Euclidean vacuum $|0_E\rangle$, as well as a pair of $|0^{\pm}\rangle$ of in/out vacua with no particle production. As the scalar field acting on $|0^-\rangle$ ($|0^+\rangle$) obeys Dirichlet (Neumann) boundary conditions on $\mathcal{I}^+$, these are related to the critical (free) $Sp(N)$ model. Generically all dS-invariant vacua are Bogolyubov transformations of one another, but we find that at $m^2\ell^2 = 2$ the transformation is singular and the in/out vacua are non-normalizable plane-wave type states. In section 4.2, we use the conformal symmetries to find the classical bulk wavefunctions associated to an operator insertion on $\mathcal{I}^+$, and note the relation to (anti) quasinormal modes. The construction uses a rescaled bulk-to-boundary Green function defined with Neumann or Dirichlet $\mathcal{I}^+$ boundary conditions. We also show that the Klein-Gordon inner product of these wavefunctions agrees with the conformally-covariant CFT$_3$ operator two-point function. In section 4.3, we consider the Hilbert space on R^3 slices ending on an S^2 on $\mathcal{I}^+$. This Hilbert space was explicitly constructed in [80] for a free scalar on hyperbolic slices ending on $\mathcal{I}^+$. There are two such Hilbert spaces, which we denote the northern and southern Hilbert space, which live on spatial R^3 slices extending to the north or south of the S^2. The northern and southern slices add up to a global S^3. Hence the tensor product of the northern and southern Hilbert spaces is the global Hilbert space on S^3, much as the left and right Rindler Hilbert spaces tensor to the global Minkowski Hilbert space. We show that the global $|0^{\pm}\rangle$ vacua are simple tensor products of the northern and southern Dirichlet and Neumann vacua. We then use symmetries to uniquely identify the states of the southern Hilbert space with those of the free and critical $Sp(N)$ models on an S^2. This leads directly to the dS exclusion principle. We further

construct an inner product for the southern Hilbert space which agrees, for states dual to $\mathcal{I}^+$ operator insertions, to the conformal two-point function on $\mathcal{I}^+$. In section 4.4 we discuss the restriction of Euclidean vacuum to a southern state and recall from [80], that this is a mixed state which is thermal with respect to an $SO(3,1)$ Casimir. It would be interesting to relate this result to dS entropy in the present context. In section 4.5 we show that the standard CFT$_3$ state-operator correspondence maps the known pseudo-unitary C-norm of the $Sp(N)$ model to the Zamolodchikov two-point function. This completes the demonstration that the bulk states on R_S^3 have the requisite properties to be dual to the boundary $Sp(N)$ CFT$_3$ states on S^2. Speculations are made on the possible relevance of pseudo-unitarity to the consistency of dS/CFT in general. Appendix C.1 gives some explicit formulae for the $SO(4,1)$ Killing vectors of dS$_4$.

4.1   Global dS vacua at $m^2\ell^2 = 2$

In this section we describe the quantum theory of a free scalar field Φ in dS$_4$ with wave equation

$$(\nabla^2 - m^2)\Phi = 0, \tag{4.1.1}$$

and mass

$$m^2\ell^2 = 2. \tag{4.1.2}$$

This is the case of interest for Vasiliev's higher-spin gravity. While there have been many general discussions of this problem, peculiar singular behavior as well as simplifications appear at the critical value $m^2\ell^2 = 2$ which are highly relevant to the structure of dS/CFT. A parallel discussion of de Sitter vacua and scalar Green functions in the context of dS/CFT

was given in [65]. However that paper in many places specialized to the large mass regime $m^2\ell^2 > \frac{9}{4}$, excluding the region of current interest. The behavior in the region $m^2\ell^2 < \frac{9}{4}$ divides into three cases $m^2\ell^2 > 2$, $m^2\ell^2 = 2$ and $m^2\ell^2 < 2$. Much of the structure we describe below pertains to the entire range $m^2\ell^2 < \frac{9}{4}$ with an additional branch-cut prescription for the Green functions.

4.1.1 Modes

We will work in the dS_4 global coordinates

$$\frac{ds^2}{\ell^2} = -dt^2 + \cosh^2 t d^2\Omega_3 = -dt^2 + \cosh^2 t \left[d\psi^2 + \sin^2\psi\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right], \qquad (4.1.3)$$

where $\Omega^i \sim (\psi, \theta, \phi)$ are coordinates on the global S^3 slices. Following the notation of [65] solutions of the wave equation can be expanded in modes

$$\phi_{Lj}(x) = y_L(t)Y_{Lj}(\Omega) \qquad (4.1.4)$$

of total angular momentum L and spin labeled by the multi-index j. The spherical harmonics Y_{Lj} obey

$$
\begin{aligned}
Y_{Lj}^*(\Omega) &= (-)^L Y_{Lj}(\Omega) = Y_{Lj}(\Omega_A), \\
D^2 Y_{Lj}(\Omega) &= -L(L+2)Y_{Lj}(\Omega), \\
\int_{S^3} \sqrt{h} d^3\Omega Y_{Lj}^*(\Omega)Y_{L'j'}(\Omega) &= \delta_{L,L'}\delta_{j,j'}, \\
\sum Y_{Lj}^*(\Omega)Y_{Lj}(\Omega') &= \frac{1}{\sqrt{h}}\delta^3(\Omega - \Omega'),
\end{aligned}
\qquad (4.1.5)
$$

where $\sqrt{h}$ and D^2 are the measure and Laplacian on the unit S^3, Ω_A is the antipodal point of Ω, and here and hereafter $\sum$ denotes summation over all allowed values of L and j. The time dependence is then governed by the second order ODE

$$\partial_t^2 y_L + 3\tanh t \partial_t y_L + \left(m^2\ell^2 + \frac{L(L+2)}{\cosh^2 t}\right)y_L = 0. \qquad (4.1.6)$$

Neumann and Dirichlet modes

Eq. (4.1.6) has the *real* solutions

$$y_L^\pm = 2^{L+h_\pm+\frac{1}{2}}(L+1)^{\pm\frac{1}{2}}\cosh^L te^{-(L+h_\pm)t}F(L+\frac{3}{2}, L+h_\pm, h_\pm-\frac{1}{2}; -e^{-2t}) \tag{4.1.7}$$

where

$$h_\pm \equiv \frac{3}{2} \pm \sqrt{\frac{9}{4} - m^2\ell^2}. \tag{4.1.8}$$

We are interested in $m^2\ell^2 = 2$, which implies

$$h_- = 1, \quad h_+ = 2, \tag{4.1.9}$$

and

$$y_L^\pm = \frac{(-i)^{\frac{1}{2}\pm\frac{1}{2}}2^L}{\sqrt{1+L}}\cosh^L te^{-(L+1)t}\left[\frac{1}{(1-ie^{-t})^{2L+2}} \mp \frac{1}{(1+ie^{-t})^{2L+2}}\right]. \tag{4.1.10}$$

The modes behave near $\mathcal{I}^+$ as $e^{-h_\pm t}$

$$t \to \infty,$$
$$y_L^- \to (2(L+1)^{-\frac{1}{2}})e^{-t} + \mathcal{O}(e^{-3t}) \quad \text{Neumann},$$
$$y_L^+ \to (4(L+1)^{\frac{1}{2}})e^{-2t} + \mathcal{O}(e^{-4t}) \quad \text{Dirichlet}. \tag{4.1.11}$$

Accordingly we refer to the $+$ modes as Dirichlet and the $-$ modes as Neumann. We have normalized so that the Klein-Gordon inner product is

$$\langle\phi_{Lj}^+|\phi_{L'j'}^-\rangle_{S^3} \equiv i\int_{S^3} d^3\Sigma^\mu \phi_{Lj}^{+*}\overleftrightarrow{\partial_\mu}\phi_{L'j'}^- = i\delta_{LL'}\delta_{jj'}, \tag{4.1.12}$$

with $d^3\Sigma^\mu$ the induced measure times the normal to the S^3 slice.

Under time reversal

$$y_L^\pm(t) = \pm(-)^L y_L^\pm(-t), \tag{4.1.13}$$

so that

$$\phi_{Lj}^{\pm}(x) = \pm\phi_{Lj}^{\pm}(x_A) = (-)^L \phi_{Lj}^{\pm *}(x), \tag{4.1.14}$$

where the point x_A is antipodal to the point x. This implies that an incoming Dirichlet (Neumann) mode propagates to an outgoing Dirichlet (Neumann) mode. This is not the case for generic m^2 and, as will be seen below, allows for Dirichlet and Neumann vacua with no particle production.

Euclidean modes

Euclidean modes are defined by the condition that when dS_4 is analytically continued to S^4 they remain nonsingular on the southern hemisphere. In other words

$$y_L^E(t = -\frac{i\pi}{2}) = \text{nonsingular}. \tag{4.1.15}$$

One finds that the combination

$$y_L^E = \frac{y_L^- + iy_L^+}{\sqrt{2}} = \frac{2^{L+1}}{\sqrt{2L+2}} \frac{\cosh^L t\, e^{-(L+1)t}}{(1 - ie^{-t})^{2L+2}} \tag{4.1.16}$$

is nonsingular at $t = -i\pi/2$. Hence $y_L^\mp$ are simply the real and imaginary parts of the y_L^E. (4.1.13) and (4.1.14) imply the relations

$$y_L^{E*}(t) = (-)^{L+1} y_L^E(-t) \tag{4.1.17}$$

$$\phi_{Lj}^E(x_A) = (-)^{L+1} \phi_{Lj}^{E*}(x). \tag{4.1.18}$$

$$\langle \phi_{Lj}^E | \phi_{L'j'}^E \rangle_{S^3} = \delta_{LL'} \delta_{jj'} \tag{4.1.19}$$

4.1.2 Vacua

In the quantum theory Φ is promoted to an operator which we denote $\hat{\Phi}$ obeying the equal time commutation relation

$$[\hat{\Phi}(\Omega, t), \partial_t \hat{\Phi}(\Omega', t)] = \frac{i}{\sqrt{h}\cosh^3 t} \delta^3(\Omega - \Omega'). \tag{4.1.20}$$

Defining annihilation and creation operators

$$a_{Lj}^E = \langle \phi_{Lj}^E | \hat{\Phi} \rangle_{S^3}, \quad a_{Lj}^{E\dagger} = -\langle \phi_{Lj}^{E*} | \hat{\Phi} \rangle_{S^3}, \tag{4.1.21}$$

the global Euclidean (or Bunch-Davies) vacuum is defined by

$$a_{Lj}^E | 0_E \rangle = 0. \tag{4.1.22}$$

We normalize so that $\langle 0_E | 0_E \rangle = 1$. For any m^2 there is a family of dS-invariant vacua labeled by a complex parameter α. They are annihilated by the normalized Bogolyubov-transformed oscillators

$$a_{Lj}^\alpha = \frac{1}{\sqrt{1 - e^{\alpha + \alpha^*}}} \left(a_{Lj}^E - e^{\alpha^*} a_{Lj}^{E\dagger} \right). \tag{4.1.23}$$

We are interested in the vacua annihilated by the Dirichlet or Neumann modes for the case of $m^2\ell^2 = 2$, which correspond to $e^\alpha = \pm 1$. In that case the Bogolyubov transformation is singular. Nevertheless we can still construct non-normalizable plane-wave type vacua as follows.

The field operator may be decomposed as

$$\hat{\Phi} = \hat{\Phi}^+ + \hat{\Phi}^-, \tag{4.1.24}$$

where $\hat{\Phi}^\pm \sim e^{-h_\pm t}$ near $\mathcal{I}^+$. The squeezed states

$$|0^\pm\rangle = e^{\pm\frac{1}{2}\sum(-)^L (a_{Lj}^{E\dagger})^2} |0_E\rangle \tag{4.1.25}$$

then obey

$$\hat{\Phi}^- |0^-\rangle = 0 \qquad \text{Dirichlet}, \tag{4.1.26}$$

$$\hat{\Phi}^+ |0^+\rangle = 0 \qquad \text{Neumann}. \tag{4.1.27}$$

Since only Dirichlet (Neumann) modes act non-trivially on $|0^-\rangle$ ($|0^+\rangle$) we refer to it as the Dirichlet (Neumann) vacuum. These vacua are dS invariant. With the conventional norm, $\hat{\Phi}^\pm$ are hermitian and their eigenstates are non-normalizable. Generalized dS non-invariant plane-wave type Neumann states with nonzero eigenvalues for $\hat{\Phi}^+$

$$\hat{\Phi}^+ |\Phi^+\rangle = \Phi^+ |\Phi^+\rangle \tag{4.1.28}$$

are constructed as

$$|\Phi^+\rangle = e^{-\langle \Phi^+ | \hat{\Phi}^- \rangle_{S^3}} |0^+\rangle. \tag{4.1.29}$$

Φ^+ here is an arbitrary solution of the classical wave equation, which can be parameterized by an arbitrary function $\Phi^+(\Omega)$ on $\mathcal{I}^+$

$$t \to \infty, \quad \Phi^+(\Omega, t) \to \Phi^+(\Omega) e^{-h_+ t}. \tag{4.1.30}$$

The states are delta-functional normalizable with respect the usual inner product

$$\langle \Phi^+ | \Phi^{+\prime} \rangle = \delta \left(\Phi^+ - \Phi^{+\prime} \right), \tag{4.1.31}$$

where the delta function integrates to one with the measure

$$\mathcal{D}\Phi^+ \equiv \prod_{L,j} \frac{dc_{Lj}^+}{\sqrt{\pi}}, \quad \Phi^+(x) = \sum c_{Lj}^+ \phi_{Lj}^+(x). \tag{4.1.32}$$

The c_{Lj}^+ satisfies the reality condition $c_{Lj}^{+*} = c_{Lj}^+(-)^L$. One may similarly define generalized Dirichlet states obeying

$$\hat{\Phi}^- |\Phi^-\rangle = \Phi^- |\Phi^-\rangle. \tag{4.1.33}$$

The Euclidean vacuum can be expressed in terms of $|0^\pm\rangle$ as

$$|0_E> = \int \mathcal{D}\Phi^\pm e^{\mp\frac{1}{16}\int d^3\Omega d^3\Omega'\Phi^\pm(\Omega)\Delta_\mp(\Omega,\Omega')\Phi^\pm(\Omega')}|\Phi^\pm\rangle, \qquad (4.1.34)$$

where

$$
\begin{aligned}
\Delta_\pm(\Omega,\Omega') &= \mp\sum Y^*_{Lj}(\Omega)Y_{Lj}(\Omega')(2L+2)^{\pm 1} = \frac{1}{2^{2\mp 1}\pi^2}\frac{1}{(1-\cos\Theta_3)^{h_\pm}}, \\
\cos\Theta_3(\Omega,\Omega') &\equiv \cos\psi\cos\psi' + \sin\psi\sin\psi'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')).
\end{aligned}
$$

$$(4.1.35)$$

$\Delta_\pm$ are the (everywhere positive) two-point functions for a CFT$_3$ operator with $h_+ = 2$ and $h_- = 1$.[3] These satisfy

$$-\int\sqrt{h}d^3\Omega''\Delta_+(\Omega,\Omega'')\Delta_-(\Omega'',\Omega') = \frac{1}{\sqrt{h}}\delta^3(\Omega-\Omega'). \qquad (4.1.36)$$

We also have the relations

$$|\Phi^+\rangle = \frac{1}{N_0}\int\mathcal{D}\Phi^- e^{\langle\Phi^-|\Phi^+\rangle_{S^3}}|\Phi^-\rangle, \quad N_0 \equiv \prod_{L,j}\sqrt{2}, \qquad (4.1.37)$$

$$\langle\Phi^-|\Phi^+\rangle_{S^3} = \frac{1}{N_0}e^{\langle\Phi^-|\Phi^+\rangle_{S^3}}. \qquad (4.1.38)$$

In particular

$$\langle 0^-|0^+\rangle_{S^3} = \frac{1}{N_0}. \qquad (4.1.39)$$

The Wightman function in the Euclidean vacuum is

$$
\begin{aligned}
G_E(x;x') &= \sum\phi^E_{Lj}(x)\phi^{E*}_{Lj}(x') \\
&= \sum(-)^{L+1}\phi^E_{Lj}(x)\phi^E_{Lj}(x'_A)
\end{aligned}
$$

[3]Here, we regulate the expressions of $\Delta_\pm$ as sums over the spherical harmonics by introducing $e^{-L\epsilon}$ in each term in the sum and take the limit of $\epsilon \to 0$ at the end after the summation.

$$= \quad \tfrac{1}{2}\sum(-)^L\left(\phi^-_{Lj}(x)\phi^-_{Lj}(x') + \phi^+_{Lj}(x)\phi^+_{Lj}(x') + i\phi^+_{Lj}(x)\phi^-_{Lj}(x') - i\phi^-_{Lj}(x)\phi^+_{Lj}(x')\right).$$

$$(4.1.40)$$

In terms of the dS-invariant distance function

$$P(t,\Omega;t',\Omega') = \cosh t \cosh t' \cos\Theta_3(\Omega,\Omega') - \sinh t \sinh t', \qquad (4.1.41)$$

this becomes simply

$$G_E(x;x') = \frac{1}{8\pi^2}\frac{1}{1 - P(x;x')}, \qquad (4.1.42)$$

with the usual $i\epsilon$ prescription for the singularity.

4.2   Boundary operators and quasinormal modes

According to the $\mathrm{dS}_4/\mathrm{CFT}_3$ dictionary, for every spin zero primary CFT_3 operator $\mathcal{O}$ of conformal weight h there is a bulk scalar field Φ with mass $m^2\ell^2 = h(3-h)$. Boundary correlators of $\mathcal{O}$ are then related by a rescaling to bulk Φ correlators whose arguments are pushed to the boundary at $\mathcal{I}^+$. As in AdS/CFT, a particular classical bulk wavefunction of Φ can be associated to a boundary insertion of $\mathcal{O}$ (at the linearized level) by symmetries: it must scale with weight h under the isometry corresponding to dilations, obey the lowest-weight condition, and be invariant under rotations around the point of the boundary insertion. The resulting wavefunction is a type of bulk-to-boundary Green function. Interestingly[78], the (lowest) highest-weight modes can also be identified as (anti) quasinormal modes for the static patch of de Sitter, as constructed in [76, 77]. In this section we determine this wavefunction explicitly, regulate the singularities, generalize it to multi-particle insertions and define a symplectic product. In the following section we will then use these classical objects to construct the associated dual bulk quantum states and their inner products.

4.2.1 Highest and lowest weight wavefunctions

In this subsection we give expressions for the classical wavefunctions, associated to lowest (highest) weight primary operator insertions at the south (north) pole of $\mathcal{I}^+$ in terms of rescaled Green functions in the limit that one argument is pushed to $\mathcal{I}^+$. These wavefunctions each comes in a Neumann and a Dirichlet flavor, denoted $\Phi^{\pm}_{lw}(x)$ ($\Phi^{\pm}_{hw}(x)$) depending on whether the weight of the dual operator insertion is h_+ or h_-.

The relevant Green functions are[4]

$$G_{\pm}(x; x') \equiv G_E(x; x') \pm G_E(x; x'_A) = \frac{1}{8\pi^2} \left(\frac{1}{1 - P(x; x')} \pm \frac{1}{1 + P(x; x')} \right) \qquad (4.2.43)$$

with G_E the Wightman function for the Euclidean vacuum given in equation (4.1.42). G_- (G_+) obeys Neumann (Dirichlet) boundary conditions at $\mathcal{I}^+$ away from $x = x'$. These are for $m^2 \ell^2 = 2$ the Green functions with future boundary conditions as discussed in [75]. We have normalized them so that they have the Hadamard form for the short-distance singularity. In the Neumann case we begin with G_-, which is (using the mode decomposition (4.1.40)) given by

$$G_-(x; x') = \sum (-)^L \left(\phi^-_{Lj}(x)\phi^-_{Lj}(x') + i\phi^+_{Lj}(x)\phi^-_{Lj}(x') \right). \qquad (4.2.44)$$

From this we construct the rescaled Green function

$$\Phi^-_{lw}(x; t') = e^{h_- t'} G_-(x; t', \Omega_{SP}), \qquad (4.2.45)$$

in which the second argument is placed at the south pole Ω_{SP} where $\psi' = 0$. One may then check that (ignoring singularity prescriptions)

$$\Phi^-_{lw}(x) \equiv \lim_{t' \to \infty} \Phi^-_{lw}(x; t') = \frac{1}{2\pi^2(\sinh t - \cos \psi \cosh t)}. \qquad (4.2.46)$$

---

[4] Note that $G_+(G_-)$ is even (odd) under the antipodal map. Combinations of the Euclidean Green function with such properties have been previously studied in the context of elliptic Z_2-identification of de Sitter space [60, 61, 62] which may be related to our construction.

Using (4.2.44) and the asymptotics (4.1.11) one finds that near $\mathcal{I}^+$ (not ignoring singularities)

$$
\begin{aligned}
\Phi_{lw}^-(x) &= 8\sum(-)^L\left(\frac{e^{-t}}{2L+2}Y_{Lj}(\Omega)Y_{Lj}(\Omega_{SP}) + ie^{-2t}Y_{Lj}(\Omega)Y_{Lj}(\Omega_{SP}) + \mathcal{O}(e^{-3t})\right) \\
&= 8e^{-t}\Delta_-(\Omega,\Omega_{SP}) + 8i\frac{e^{-2t}}{\sqrt{h}}\delta^3(\Omega-\Omega_{SP}) + \mathcal{O}(e^{-3t}) \qquad (4.2.47)
\end{aligned}
$$

Let us now confirm that $\Phi_{lw}^-(x)$ has the same symmetries as an insertion of a primary operator $\mathcal{O}(\Omega_{SP})$ at the south pole of $\mathcal{I}^+$. First we note that the choice of a point on $\mathcal{I}^+$ breaks $SO(4,1)$ to $SO(3) \times SO(1,1)$. Both $\Phi_{lw}^-(x)$ and $\mathcal{O}(\Omega_{SP})$ are manifestly invariant under the $SO(3)$ spatial rotations. The generator of $SO(1,1)$ dilations, denoted L_0, acts on $\mathcal{O}(\Omega_{SP})$ as

$$
[L_0, \mathcal{O}(\Omega_{SP})] = h_-\mathcal{O}(\Omega_{SP}). \qquad (4.2.48)
$$

In the bulk it is generated by the Killing vector field

$$
L_0 = \cos\psi\partial_t - \tanh t\sin\psi\partial_\psi, \qquad (4.2.49)
$$

where the south pole is $\psi = 0$. dS invariance implies

$$
(L_0 - L_0')G_-(x,x') = 0. \qquad (4.2.50)
$$

It follows from this together with the definition (4.2.46) that the wavefunction obeys

$$
L_0\Phi_{lw}^-(x) = h_-\Phi_{lw}^-(x). \qquad (4.2.51)
$$

By construction it obeys the wave equation

$$
(\nabla^2 - m^2)\Phi_{lw}^-(x) = 0. \qquad (4.2.52)
$$

Acting on $SO(3)$ invariant symmetric functions we have

$$
\ell^2\nabla^2 = -L_0(L_0 - 3) + M_{-k}M_{+k}, \qquad (4.2.53)
$$

where the 6 Killing vector fields $M_{\pm k}$ (given in Appendix C.1) are the raising and lowering operators for L_0 and we sum over k. It then follows that

$$M_{-k}M_{+k}\Phi_{lw}^-(x) = \left(m^2\ell^2 - h_-(3-h_-)\right)\Phi_{lw}^-(x) = 0, \qquad (4.2.54)$$

and hence

$$M_{+k}\Phi_{lw}^-(x) = 0. \qquad (4.2.55)$$

which corresponds to the lowest-weight condition for the $\mathcal{O}$

$$[M_{+k}, \mathcal{O}(\Omega_{SP})] = 0. \qquad (4.2.56)$$

It may be shown that these symmetries uniquely determine the solution. Hence Φ_{lw}^- is identified as the classical wavefunction associated to the insertion of the primary $\mathcal{O}$ at the south pole.

A parallel argument leads to the dual of a highest weight operator insertion at the north pole . The wavefunction is

$$\Phi_{hw}^-(x;t') = \lim_{t'\to\infty} e^{h_-t'}G_-(x;t',\Omega_{NP}). \qquad (4.2.57)$$

This obeys the relations

$$M_{-k}\Phi_{hw}^-(x) = 0, \qquad L_0\Phi_{hw}^-(x) = -h_-\Phi_{hw}^-(x), \qquad (4.2.58)$$

and has the asymptotic behavior

$$\Phi_{hw}^-(x) = 8e^{-t}\Delta_-(\Omega,\Omega_{NP}) + 8i\frac{e^{-2t}}{\sqrt{h}}\delta^3(\Omega - \Omega_{NP}) + \mathcal{O}(e^{-3t}). \qquad (4.2.59)$$

Similar formulae apply to the Dirichlet case by beginning with G_+ in the above construction and replacing $+ \leftrightarrow -$. For example

$$\Phi_{hw}^+(x) = -8e^{-2t}\Delta_+(\Omega,\Omega_{NP}) - 8i\frac{e^{-t}}{\sqrt{h}}\delta^3(\Omega - \Omega_{NP}) + \mathcal{O}(e^{-3t}). \qquad (4.2.60)$$

We see from the above that the highest-weight wavefunction is smooth on the future horizon of the southern static patch dS$_4$, and hence related to the quasinormal modes found in [76, 77]. The lowest quasinormal mode which is invariant under the $SO(3)$ of the static dS$_4$ is exactly the Φ_{hw}^- with $h_- = 1$ while the second lowest $SO(3)$-invariant quasinormal mode corresponds to Φ_{hw}^+ with $h_+ = 2$. Lowest weight states are smooth on the past horizon and hence related to anti-quasinormal modes.

4.2.2   General multi-operator insertions

In the preceding subsections we found the bulk duals of primary operators inserted at the north/south pole in the coordinates (4.1.3). This can be generalized to insertions at an arbitrary point on $\mathcal{I}^+$ with a general time slicing near $\mathcal{I}^+$. Let us introduce coordinates $x \sim (y^i, t)$ such that near $\mathcal{I}^+$

$$ds_4^2 \to -dt^2 + e^{2t}h_{ij}(y)dy^i dy^j, \quad i,j = 1,2,3. \tag{4.2.61}$$

The dual wavefunction is then the $t' \to \infty$ limit of the rescaled Green function, denoted by

$$\Phi_{y_1}^\pm(x) = \lim_{t' \to \infty} e^{h_\pm t'} G_\pm(x; t', y_1). \tag{4.2.62}$$

For the special cases of operator insertions at the north or south pole in global coordinates these reduce to our previous expressions. Note that coordinate transformations of the form $t \to t + f(y)$ induce a conformal transformation on $\mathcal{I}^+$

$$h_{ij} \to e^{2f(y)}h_{ij}, \quad \Phi_{y_1}^\pm(x) \to e^{h_\pm f(y)}\Phi_{y_1}(x), \tag{4.2.63}$$

as appropriate for a conformal field of weight $h_\pm$. Hence the relative normalization in (4.2.46) will depend on the conformal frame at $\mathcal{I}^+$.

One may also consider multi-operator insertions such as $\mathcal{O}(y_1)\mathcal{O}(y_2)$ in the CFT$_3$ at $\mathcal{I}^+$. At the level of free field theory considered here these are associated to a bilocal wavefunction in the product of two bulk scalar fields

$$\Phi_{y_1}(x_1)\Phi_{y_2}(x_2). \tag{4.2.64}$$

We will use Φ_Ω to denote these wavefunctions when working in global coordinates (4.1.3). We note that in such coordinates near $\mathcal{I}^+$ for an insertion at a general point

$$\Phi_{\Omega_1}^\pm(t,\Omega) = \mp 8e^{-h_\pm t}\Delta_\pm(\Omega,\Omega_1) \mp 8i\frac{e^{-h_\mp t}}{\sqrt{h}}\delta^3(\Omega - \Omega_1) + \mathcal{O}(e^{-3t}). \tag{4.2.65}$$

4.2.3 Klein-Gordon inner product

We wish to define an inner product between e.g. two Neumann wavefunctions $\Phi_{\Omega_1}^-$ and $\Phi_{\Omega_2}^-$. Later on we will compare this to the inner product on the CFT$_3$ Hilbert space and the two-point function of $\mathcal{O}$ on S^3. One choice is to take a global spacelike S^3 slice in the interior and define the Klein-Gordon inner product

$$\left\langle \Phi_{\Omega_1}^- | \Phi_{\Omega_2}^- \right\rangle_{S^3} \equiv i\int_{S^3} d^3\Sigma^\mu \Phi_{\Omega_1}^{-*}\overleftrightarrow{\partial_\mu}\Phi_{\Omega_2}^-. \tag{4.2.66}$$

This integral does not depend on the choice of S^3 which can be pushed up to $\mathcal{I}^+$. One may then see immediately from (4.2.65) that there are two nonzero terms proportional to Δ_- giving

$$\left\langle \Phi_{\Omega_1}^- | \Phi_{\Omega_2}^- \right\rangle_{S^3} = 16\Delta_-(\Omega_1 - \Omega_2) \tag{4.2.67}$$

One may also define an inner product not on global spacelike S^3 slices, but on a spacelike R^3 slice which ends on an S^2 on $\mathcal{I}^+$. The result is invariant under any deformation of the S^2 which does not cross the insertion point. To be definite, we take the S^2 to be the equator,

Ω_1 to be in the northern hemisphere and Ω_2 to be in the southern hemisphere, and the slice to be R_S^3 which intersects the south pole. One then finds, pushing R_S^3 up to the southern hemisphere of $\mathcal{I}^+$

$$\left\langle \Phi_{\Omega_1}^- | \Phi_{\Omega_2}^- \right\rangle_{R_S^3} \equiv i \int_{R_S^3} d^3\Sigma^\mu \Phi_{\Omega_1}^{-*} \overleftrightarrow{\partial_\mu} \Phi_{\Omega_2}^- = 8\Delta_-(\Omega_1 - \Omega_2). \tag{4.2.68}$$

Similarly, the inner product between two Dirichlet wavefunctions is given by

$$\left\langle \Phi_{\Omega_1}^+ | \Phi_{\Omega_2}^+ \right\rangle_{R_S^3} = -8\Delta_+(\Omega_1 - \Omega_2). \tag{4.2.69}$$

4.3 The southern Hilbert space

We now turn to the issue of bulk quantum states. Quantum states in dS are often discussed, as in section 4.1, in terms of a Hilbert space built on the global S^3 slices. The structure of the vacua and Green functions for such states was described in section 4.1. However dS has the unusual feature that there are geodesically complete topologically R^3 spacelike slices which end on an S^2 in $\mathcal{I}^+$, which we will typically take to be the equator. Examples of these are the hyperbolic slices, the quantization on which was studied in detail in [80]. We will see that the quantum states built on these R^3 slices are natural objects in dS/CFT. An S^2 in $\mathcal{I}^+$ is in general the boundary of a "northern" slice, denoted R_N^3 and a "southern" slice denoted R_S^3. The topological sum obeys $R_S^3 \cup R_N^3 = S^3$. Hence the relation of the southern and northern Hilbert spaces on R_S^3 and R_N^3 to that on S^3 is like that of the left and right Rindler wedges to that of global Minkowski space. It is also like the relation of the Hilbert spaces of the northern and southern causal diamonds to that of global dS. However the diamond Hilbert spaces in dS quantum gravity are problematic in quantum gravity with a fluctuating metric because it is hard to find sensible boundary conditions.

A strong motivation for considering the $R^3_{S,N}$ slices comes from the picture of a state in the boundary CFT_3. The state-operator correspondence in CFT_3 begins with an insertion of a (primary or descendant) operator $\mathcal{O}$ at the south pole of S^3, and then defines a quantum state as a functional of the boundary conditions on an S^2 surrounding the south pole. For every object in the CFT_3, we expect a holographically dual object in the bulk dS_4 theory. The dual bulk quantum state must somehow depend on the choice of S^2 in $\mathcal{I}^+$. Hence it is natural to define the bulk state on the R^3 slice which ends on this S^2 in $\mathcal{I}^+$. This is how holography works in AdS/CFT: CFT states live on the boundaries of the spacelike slices used to define the bulk states.

4.3.1 States

In order to define quantum states on R^3_S, we first note that modes of the scalar field operator $\hat{\Phi}(\Omega, t)$ are labeled by operators $\hat{\Phi}^{\pm}(\Omega)$ defined on $\mathcal{I}^+$ via the relation

$$\lim_{t \to \infty} \hat{\Phi}(\Omega, t) = e^{-h_+ t} \hat{\Phi}^+(\Omega) + e^{-h_- t} \hat{\Phi}^-(\Omega). \tag{4.3.70}$$

They satisfy the following commutation relation

$$\left[\hat{\Phi}^+(\Omega), \hat{\Phi}^-(\Omega') \right] = \frac{8i}{\sqrt{h}} \delta^3(\Omega - \Omega'). \tag{4.3.71}$$

We may then decompose these $\mathcal{I}^+$ operators as the sum of two terms

$$\hat{\Phi}^{\pm}(\Omega) = \hat{\Phi}^{\pm}_N(\Omega) + \hat{\Phi}^{\pm}_S(\Omega) \tag{4.3.72}$$

where the first (second) acts only on R^3_N (R^3_S). Defining the northern and southern Dirichlet and Neumann vacua by

$$\hat{\Phi}^{\pm}_N |0^{\pm}_N\rangle = 0, \quad \hat{\Phi}^{\pm}_S |0^{\pm}_S\rangle = 0, \tag{4.3.73}$$

it follows from the decomposition (4.3.72) that the global vacua have a simple product decomposition[5]

$$|0^\pm\rangle = |0_N^\pm\rangle|0_S^\pm\rangle. \tag{4.3.74}$$

Excited southern states may then be built by acting on one of these southern vacua with $\hat{\Phi}_S$. We wish to identify these states with those of the CFT_3 on S^2.

In the higher-spin dS/CFT correspondence there are actually two CFT_3's living on $\mathcal{I}^+$: the free $Sp(N)$ model, associated to Neumann boundary conditions, and the critical $Sp(N)$ model, associated to Dirichlet boundary conditions. Since the field operators $\hat{\Phi}_S$ acting on $|0_S^+\rangle$ ($|0_S^-\rangle$) obeys, according to equation (4.1.26), Neumann (Dirichlet) boundary conditions near the southern hemisphere of $\mathcal{I}^+$, it is natural to identify

$$
\begin{aligned}
|0_S^+\rangle &\sim \text{ free } Sp(N) \text{ vacuum} \\
|0_S^-\rangle &\sim \text{ critical } Sp(N) \text{ vacuum.}
\end{aligned}
\tag{4.3.75}
$$

Next we want to consider excited states and their duals. To be specific we consider the Neumann theory built on $|0_S^+\rangle$. Parallel formulae apply to the Dirichlet case. Operator versions of the classical wavefunctions $\Phi_\Omega^-(x)$ are constructed as

$$\hat{\Phi}_{\Omega_S}^- \equiv \left\langle \Phi_{\Omega_S}^- | \hat{\Phi} \right\rangle_{R_S^3}, \tag{4.3.76}$$

where Ω_S is presumed to lie on the southern hemisphere. We can make a quantum state

$$|\Omega_S^-\rangle \equiv \hat{\Phi}_{\Omega_S}^- |0_S^+\rangle = \hat{\Phi}^-(\Omega_S)|0_S^+\rangle, \tag{4.3.77}$$

---

[5]Of course a general quantum state on S^3 is a sum of products of northern and southern states, and reduces to a southern density matrix, not a pure state, after tracing over the northern Hilbert space. We shall see this explicitly below for the case of the Euclidean vacuum.

where in the last line we used (4.2.65). By construction this will be a lowest weight state, and we therefore identify it as the bulk dual to the CFT$_3$ state created by the primary operator $\mathcal{O}$ dual to the field Φ.

This connection leads to an interesting nonperturbative dS exclusion principle [79]. The operator $\mathcal{O}$ has a representation in the $Sp(N)$ theory as

$$\mathcal{O} = \Omega_{AB}\eta^A\eta^B, \quad A, B = 1, ...N, \tag{4.3.78}$$

where η^A are N anticommuting real scalars and Ω_{AB} is the quadratic form on $Sp(N)$. It follows that

$$\mathcal{O}^{\frac{N}{2}+1} = 0. \tag{4.3.79}$$

Bulk-boundary duality and the state-operator relation described above then implies the nonperturbative relation

$$\left[\hat{\Phi}^\pm(\Omega)\right]^{\frac{N}{2}+1} = 0. \tag{4.3.80}$$

Hence the quantum field operators $\hat{\Phi}^\pm(\Omega)$ are $\frac{N}{2}$-adic. One is not allowed to put more than $\frac{N}{2}$ quanta in any given quasinormal mode. This is similar to the stringy exclusion principle for AdS [81] and may be related to the finiteness of dS entropy. Nonperturbative phenomena due to related finite N effects in the $O(N)$ case have been discussed in [82]. We hope to investigate further the consequences of this dS exclusion principle.

4.3.2 Norm

Having identified the bulk duals of the boundary CFT$_3$ states, we wish to describe the bulk dual of the CFT$_3$ norm. The standard bulk norm is defined by $\Phi(x) = \Phi^\dagger(x)$. However this norm is not unique. It has been argued for a variety of reasons beginning in [54] that

it is appropriate to modify the norm in the context of dS – see also [65, 83]. Here we have the additional problem that this standard norm is divergent for states of the form (4.3.77). We now construct the modified norm for states on R_3^S by demanding that it is equivalent to the CFT$_3$ norm. The construction here generalizes to dS$_4$ the one given in [65] for dS$_3$.

The bulk action of dS Killing vectors $K_A^\mu \partial_\mu$ on a scalar field is generated by the integral over any global S^3 slice

$$\hat{\mathcal{L}}_A = \int_{S^3} d^3 \Sigma^\mu T_{\mu\nu} K_A^\nu, \tag{4.3.81}$$

where $T_{\mu\nu}$ is the bulk stress tensor constructed from the operator $\hat{\Phi}$. If we take $\hat{\Phi}^\dagger(x) = \hat{\Phi}(x)$, then $\hat{\mathcal{L}}_A = \hat{\mathcal{L}}_A^\dagger$ which is not what we want. The CFT$_3$ states are in representations of the $SO(3,2)$ conformal group. These arise from analytic continuation of the 10 $SO(4,1)$ conformal Killing vectors on S^3 which are the boundary restrictions of the bulk dS$_4$ Killing vectors $K_A^\mu \partial_\mu$. Usually, the standard CFT$_3$ norm has a self-adjoint dilation operator $\mathcal{L}_0$ generating $-i \sin \psi \partial_\psi$ as well as 3 self-adjoint $SO(3)$ rotation operators $\mathcal{J}_k$. The remaining 6 raising and lowering operators $\mathcal{L}_{\pm k}$ arising from the Killing vectors $iM_{\pm k}$ (described in the Appendix C.1) then obey $\mathcal{L}_{\pm k}^\dagger = \mathcal{L}_{\mp k}$ in the conventional CFT$_3$ norm.

To obtain an adjoint with the desired properties, we define the modified adjoint

$$\hat{\Phi}^\dagger(x) = \mathcal{R}\hat{\Phi}(x)\mathcal{R} = \hat{\Phi}(Rx), \tag{4.3.82}$$

where here and hereafter † denotes the bulk modified adjoint. The reflection operator $\mathcal{R}$ is the discrete isometry of S^3 which reflects through the S^2 equator $R(\psi, \theta, \phi) = (\pi - \psi, \theta, \phi)$ along with complex conjugation. In particular, it maps the south pole to the north pole while keeping the equator invariant. This implies that $\mathcal{L}_0$ (generating iL_0) and $\mathcal{J}_k$ are self

adjoint while

$$\mathcal{L}_{\pm k}^{\dagger} = -i \int_{S^3} d\Sigma^{\mu}(x) T_{\mu\nu}(Rx) M_{\pm k}^{\nu}(x) = -i \int_{S^3} d\Sigma^{\mu}(x) T_{\mu\nu}(x) M_{\pm k}^{\nu}(Rx) = \mathcal{L}_{\mp k}. \quad (4.3.83)$$

Hence we have constructed an adjoint admitting the desired $SO(3,2)$ action. We do not know whether or not it is unique.

The action of $\mathcal{R}$ maps an operator defined on the southern hemisphere to one defined on the southern hemisphere of $\mathcal{I}^+$ according to

$$\hat{\Phi}^{\pm\dagger}(\Omega) = \hat{\Phi}^{\pm}(\Omega_R), \quad (4.3.84)$$

Hence the action of $\mathcal{R}$ exchanges the northern and southern hemispheres, and maps a southern $\mathcal{I}^+$ state to a northern one. Therefore it cannot on its own define an adjoint within the southern Hilbert space. For this we must combine (4.3.82) with a map from the north to the south. Such a map is provided by the Euclidean vacuum. The global Euclidean bra state (constructed with the standard adjoint) can be decomposed in terms of a basis of northern and southern bra states

$$\langle 0_E| = \sum_{m,n} E_{mn} \langle m_S| \langle n_N|. \quad (4.3.85)$$

We then define the modified adjoint of an arbitrary southern state $|\Psi_S\rangle$ by

$$|\Psi_S\rangle^{\dagger} \equiv \langle 0_E|\mathcal{R}|\Psi_S\rangle. \quad (4.3.86)$$

We will denote the corresponding inner product by an S subscript

$$\langle \Psi_S'|\Psi_S\rangle_S \equiv (|\Psi_S'\rangle^{\dagger})|\Psi_S\rangle. \quad (4.3.87)$$

For example choosing the basis so that

$$\mathcal{R}|m_S\rangle = |m_N\rangle \quad (4.3.88)$$

we have

$$\langle m_S | n_S \rangle_S = E_{nm}. \tag{4.3.89}$$

In particular one finds

$$\langle 0_S^+ | 0_S^+ \rangle_S = \langle 0_E | (|0_S^+\rangle \mathcal{R} |0_S^+\rangle) = 1. \tag{4.3.90}$$

Let us now compute the norm of the southern state $|\Omega_S^-\rangle$ in (4.3.77). The action of $\mathcal{R}$ gives a northern state which we will denote $|R\Omega_S^-\rangle$. The norm is then

$$\langle \Omega_S^- | \Omega_S^- \rangle_S = \langle 0_E | (|\Omega_S^-\rangle |R\Omega_S^-\rangle) = \langle 0_E | \hat{\Phi}^-(\Omega_S) \hat{\Phi}^-(R\Omega_S) |0^+\rangle. \tag{4.3.91}$$

Using the relation

$$|0_E\rangle = N_0 e^{-\frac{1}{16} \int d^3\Omega d^3\Omega' \hat{\Phi}^+(\Omega) \Delta_-(\Omega,\Omega') \hat{\Phi}^+(\Omega')} |0^-\rangle \tag{4.3.92}$$

we find

$$\langle \Omega_S^- | \Omega_S^- \rangle_S = 8\Delta_-(\Omega_S, R\Omega_S). \tag{4.3.93}$$

This is proportional to the S^3 two-point function of a dimension h_- primary at the points Ω_S and $R\Omega_S$. The analogous computation in the Dirichlet theory gives

$$\langle \Omega_S^+ | \Omega_S^+ \rangle_S = -8\Delta_+(\Omega_S, R\Omega_S). \tag{4.3.94}$$

4.4 Boundary dual of the bulk Euclidean vacuum

In the preceding section we have argued that dS/CFT maps CFT$_3$ states on an S^2 in $\mathcal{I}^+$ to bulk states on the southern slice ending on the S^2. A generic state in a global dS slice does not restrict to a pure southern state. However we can always define a density matrix by tracing over the northern Hilbert space. In particular, such a southern density matrix ρ_S^E can be associated to the global Euclidean vacuum $|0_E\rangle$. The choice of an equatorial S^2 in $\mathcal{I}^+$

breaks the $SO(4,1)$ symmetry group down to $SO(3,1)$, which also preserves the hyperbolic slices ending on the S^2. ρ_E^S must be invariant under this $SO(3,1)$. In fact ρ_E^S follows from results in [80]. Writing the quadratic Casimir of $SO(3,1)$ as $C_2 = -(1+p^2)$, it was shown, in a basis which diagonalizes p, that

$$\rho_E^S = N_1 e^{-2\pi p}, \tag{4.4.95}$$

where N_1 is determined by $\mathrm{tr}\rho_E^S = 1$. It would be interesting to investigate this further and compute the entropy $S = -\mathrm{tr}\rho_E^S \ln \rho_E^S$ in the $Sp(N)$ model.

4.5 Pseudounitarity and the C-norm in the $Sp(N)$ **CFT**$_3$

In this section we consider the $Sp(N)$ model (where N is even) and compare the norms to those computed above. The action is

$$I_{Sp(N)} = \frac{1}{8\pi} \int d^3x \left[\delta^{ij}\delta_{ab}\partial_i\bar{\chi}^a\partial_j\chi^b + m^2\bar{\chi}\chi + \lambda\left(\bar{\chi}\chi\right)^2 \right], \tag{4.5.96}$$

where $\chi^a(a = 1, \ldots, \frac{N}{2})$ is a complex anticommuting scalar and $\bar{\chi}\chi \equiv \delta_{ab}\bar{\chi}^a\chi^b$. This has a global $Sp(N)$ symmetry and we restrict to $Sp(N)$ singlet operators.[6] For the free theory $m = \lambda = 0$ while the critical theory is obtained by flowing to a nontrivial fixed point λ_F. The $Sp(N)$ theory is not unitary in the sense that in the standard norm following from (4.5.96) one has that [74]

$$H \neq H^\dagger \tag{4.5.97}$$

and $\langle \Psi'|\Psi \rangle$ is not preserved. Nevertheless, as detailed in [74], there exists an operator C with the properties

$$C^\dagger C = C^2 = 1, \quad C\chi^\dagger C = \bar{\chi}, \quad CH^\dagger C = H, \quad C|0\rangle = |0\rangle. \tag{4.5.98}$$

---

[6]The $U(\frac{N}{2})$ theory has the same action but is restricted only to $U(\frac{N}{2})$ singlets.

To write it in real fields, for e.g., in the case of $Sp(2)$, writing the real and imaginary part of χ as η_1 and η_2, the action of C becomes $\eta_2 = C\eta_1^\dagger C$. One may then define a "pseudounitary" C-inner product

$$\langle\Psi'|\Psi\rangle_C \equiv \langle\Psi'|C|\Psi\rangle \tag{4.5.99}$$

which is preserved under hamiltonian time evolution. Such hamiltonians are pseudoher-mitian and are similar to those studied in [84]. We note that the norm is not positive definite.

Inserting an operator O_i constructed from χ^a at the south pole gives a functional of the boundary conditions on the equatorial S^2 which we define as the state $|O_i\rangle$. This is the standard state-operator correspondence. An inner product for such states associated to O_i and O_j can be defined by the two point function with O_i at one pole and O_j at the other. It follows from (4.5.98) that this is the C-inner product for the states $|O_i\rangle$ and $|O_j\rangle$:

$$\langle O_i|O_j\rangle_C = \langle O_i|C|O_j\rangle = \langle O_i^\dagger C O_j\rangle = \langle O_i O_j\rangle. \tag{4.5.100}$$

In the last line, we used the fact that the (singlet) currents in the $Sp(N)$ models satisfy $C O_i^\dagger C = O_i$ since $C(\bar\chi\chi)^\dagger C = \bar\chi\chi$. For primary operators of weight h_i we then have [14]

$$\langle O_i|O_i\rangle_C = -N\Delta_{h_i}(\Omega_{NP}, \Omega_{SP}). \tag{4.5.101}$$

Hence it is the C-norm which maps under the state-operator correspondence to the Zamolod-chikov norm defined as the Euclidean two point function on S^3. As seen in [14] this C-norm then agrees with the bulk inner product (4.3.93)-(4.3.94) of the dual state for the scalar case.[7] Moreover, as the bulk and CFT$_3$ norms assign the same hermiticity properties to the

---

[7]A factor of i^h explained in [14] relating operator insertions to bulk fields makes the two-point function (4.3.93) negative.

$SO(4,1)$ generators, this result will carry over to descendants of the primaries. A generalization of this construction to all spins seems possible.

One of the puzzling features of dS/CFT is that the dual CFT cannot be unitary in the ordinary sense. This is not a contradiction of any kind because unitarity of the Euclidean CFT is not directly connected to any spacetime conservation law. At the same time quantum gravity in dS – and its holographic dual – should have some good property replacing unitarity in the AdS case. It is not clear what that good property is. The appearance of a pseudounitary structure in the case of dS/CFT analyzed here is perhaps relevant in this regard.

Chapter 5

dS Quasinormal Mode Quantization

In this chapter, we present a new and potentially useful approach to an old problem: the quantization of a scalar field in four-dimensional de Sitter spacetime (dS_4), which has an $SO(4,1)$ isometry group. One standard approach begins with the spherical harmonics of the S^3 spatial sections, and proceeds by solving the wave equation for the time-dependent modes. Linear combinations of these modes that are nonsingular under a certain analytic continuation are then identified as the Euclidean modes and used to define the quantum Euclidean vacuum. The vacuum so constructed exhibits manifest $SO(4)$-invariance and can also be shown to possess the full $SO(4,1)$ symmetry of dS_4. Another common approach singles out the southern causal diamond and relies on a special Killing vector field, denoted L_0, which generates southern Killing time and whose corresponding eigenmodes have real frequency ω. This construction displays manifest $SO(3) \times SO(1,1)$ symmetry and again leads to the dS-invariant Euclidean vacuum. The modes employed in these and similar constructions are *not* in $SO(4,1)$ multiplets and hence $SO(4,1)$-invariance of the final expressions is nontrivial. For example, the action of the dS_4 isometries on the southern diamond L_0

eigenmodes shifts the frequency by *imaginary* integer multiples of $2\pi/\ell$ (where ℓ is the dS radius) while the usual southern diamond modes all have real frequencies.

It is natural to adopt scalar modes which lie in highest-weight representations of $\mathsf{SO}(4,1)$ and therefore boast manifest dS-invariance. These turn out to be nothing but the quasinormal modes of the southern diamond, which have complex L_0 eigenvalues and comprise four real or two complex highest-weight representations.[1] They are singular on the past horizon and decay exponentially towards the future, as opposed to the conventional southern diamond modes which oscillate everywhere. In order to quantize in a quasinormal mode basis, a norm is needed. The singularities on the past horizon render the Klein-Gordon norm singular, which is presumably why the quasinormal modes have not typically been used for quantization. However a variety of other equally suitable norms *have* been employed for various reasons in dS [61, 62, 54, 89, 65, 60, 90]. One of them – the so called R-norm [90] – differs from the Klein-Gordon norm by the insertion of a spatial reflection through the equator of the S^3 slice, thereby exchanging the north and south poles. We demonstrate that the R-norm is finite for quasinormal modes and hence suitable for quantization. We also show that the Euclidean vacuum has the simple and manifestly dS-invariant definition as the state annihilated by two of the four sets of quasinormal modes. Moreover the Euclidean Green function is shown, as anticipated in [77], to be obtainable from a simple sum over the quasinormal modes. We caution the reader that quasinormal modes have singularities on the past horizon which we regulate with an $i\epsilon$-prescription. Our statements about completeness and mode sums depend on taking the $\epsilon \to 0$ limit at the end of our calculations.

The real Killing vectors which generate the dS isometries have an $\mathsf{SO}(4,1)$ Lie bracket

---

[1]Interesting work on the normalizability and completeness of quasinormal modes for black holes can be found in [85, 86, 87, 88].

algebra and are antihermitian with respect to the Klein-Gordon norm. However they have mixed hermiticity under the R-norm. Multiplication by appropriate factors of i produces complex Killing vector fields which are antihermitian under the R-norm. The Lie algebra of these R-antihermitian vector fields turns out to generate $\mathsf{SO}(3,2)$, which is precisely the symmetry group of a 2+1-dimensional CFT, and the transformed notion of hermiticity is exactly the one conventionally employed when studying CFT_3 on the Euclidean plane [65]. Hence this $\mathsf{SO}(4,1) \to \mathsf{SO}(3,2)$ transformation, and the use of quasinormal modes, fits naturally within the $\mathrm{dS}_4/\mathrm{CFT}_3$ correspondence [54, 70, 12, 57, 71, 14].[2]

This chapter is organized as follows. In section 5.1, we begin by reviewing the standard global S^3 modes and the construction of Euclidean modes and Green functions. In section 5.2, we show that the quasinormal modes comprise the highest-weight modes and their descendants, specializing for simplicity to the case of conformal mass $m^2\ell^2 = 2$. Then in section 5.3, the modified R-norm and its properties are presented. Next, in section 5.4 we prove that half the quasinormal modes are Euclidean modes and demonstrate their completeness by deriving the Euclidean Green function from a quasinormal mode sum. In section 5.5, we generalize these results to the case of light scalars with $m^2\ell^2 \leq 9/4$.

Finally, in section 5.6, we isolate quasinormal modes that vanish in the northern or southern diamonds – the analogues of Rindler modes in Minkowski space – and present a complete set of quasinormal modes in the southern diamond. These might eventually be useful for understanding the thermal nature of physics in a single dS causal diamond, but we do not pursue this direction further herein.

---

[2]For every bulk scalar Φ one expects a dual operator $\mathcal{O}$ in the boundary CFT_3. The bulk state with one quantum in the lowest quasinormal mode is dual to the CFT_3 state associated to an $\mathcal{O}$ insertion at the north pole of the S^3 at $\mathcal{I}^+$, and the descendants fill out $\mathsf{SO}(3,2)$ representations on both sides of the duality [90].

In addition, in Appendix C.1 we provide the explicit forms of dS$_4$ Killing vectors as well as their commutation relations. This is followed by Appendix C.2, which computes the norm of spherically symmetric descendants using the $\mathsf{SO}(4,1)$ algebra, and Appendix C.3, which provides details on the Euclidean two-point function evaluated on the south pole observer's worldline.

We expect our discussion to generalize to the case of heavy scalars with $m^2\ell^2 > 9/4$ as well as other dimensions and spin.

5.1 $\mathsf{SO}(4)$-invariant global mode decomposition

In this section we describe the standard dS$_4$ mode decomposition in terms of the spherical harmonics of the S^3 spatial sections. These modes are regular everywhere on dS$_4$ and sometimes referred to as 'global modes'.

We will work in the dS$_4$ global coordinates $x = (t, \psi, \theta, \phi)$ with line element

$$\frac{ds^2}{\ell^2} = -dt^2 + \cosh^2 t \, d\Omega_3^2 = -dt^2 + \cosh^2 t \left[d\psi^2 + \sin^2 \psi \left(d\theta^2 + \sin^2 \theta \, d\phi^2 \right) \right], \qquad (5.1.1)$$

where $\Omega = (\psi, \theta, \phi)$ are coordinates on the global S^3 slices. We denote the north and south pole by

$$\Omega_{SP} \sim \psi = 0, \qquad \Omega_{NP} \sim \psi = \pi. \qquad (5.1.2)$$

In this coordinate system, the dS-invariant distance function $P(x; x')$ is given by

$$P(t, \Omega; t', \Omega') = \cosh t \cosh t' \cos \Theta_3(\Omega, \Omega') - \sinh t \sinh t', \qquad (5.1.3)$$

where $\Theta_3(\Omega, \Omega')$ denotes the geodesic distance function on S^3 and

$$\cos \Theta_3(\Omega, \Omega') = \cos \psi \cos \psi' + \sin \psi \sin \psi' \left[\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \right]. \qquad (5.1.4)$$

Following the notation of [90], solutions of the wave equation

$$\left(\nabla^2 - m^2\right)\Phi = 0 \tag{5.1.5}$$

may be expanded in representations of the $\mathsf{SO}(4)$ rotations of the S^3 spatial slice at fixed t:

$$\Phi_{Lj}(x) = y_L(t)Y_{Lj}(\Omega). \tag{5.1.6}$$

These have total $\mathsf{SO}(4)$ angular momentum L and spin labeled by the multi-index j. The S^3 spherical harmonics Y_{Lj} obey the identities

$$
\begin{aligned}
Y_{Lj}^*(\Omega) &= (-)^L Y_{Lj}(\Omega) = Y_{Lj}(\Omega_A), \\
D^2 Y_{Lj}(\Omega) &= -L(L+2)Y_{Lj}(\Omega), \\
\int_{S^3} d^3\Omega\sqrt{h}\, Y_{Lj}^*(\Omega)Y_{L'j'}(\Omega) &= \delta_{L,L'}\delta_{j,j'}, \\
\sum Y_{Lj}^*(\Omega)Y_{Lj}(\Omega') &= \frac{1}{\sqrt{h}}\delta^3(\Omega - \Omega'),
\end{aligned}
\tag{5.1.7}
$$

where Ω_A denotes the antipodal point of Ω, while $\sqrt{h} = \sin^2\psi\sin\theta$ and D^2 are the measure and Laplacian on the unit S^3, respectively. Here and hereafter, $\sum$ denotes summation over all allowed values of L and the multi-index j. The time dependence $y_L(t)$ is then governed by the differential equation

$$\partial_t^2 y_L + 3\tanh t\,\partial_t y_L + \left[m^2\ell^2 + \frac{L(L+2)}{\cosh^2 t}\right]y_L = 0. \tag{5.1.8}$$

The general solution has the $\mathcal{I}^+$ falloff

$$y_L \to e^{-h_\pm t}, \qquad h_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - m^2\ell^2}. \tag{5.1.9}$$

For the time being, we restrict our attention to the case $m^2\ell^2 = 2$, which corresponds to a conformally coupled scalar with

$$h_+ = 2, \qquad h_- = 1. \tag{5.1.10}$$

The case of generic mass is qualitatively similar but with algebraic functions replaced by hypergeometric ones. We give the correspondingly more involved formulae in section 5.5.

The so-called Euclidean modes, which define the vacuum, are those which remain non-singular on the southern hemisphere when dS_4 is analytically continued to S^4. In other words, they are defined by the condition

$$y_L^E \left(t = -\frac{i\pi}{2} \right) = \text{nonsingular}. \tag{5.1.11}$$

Explicitly, these modes are [90]:

$$y_L^E = \frac{2^{L+1}}{\sqrt{2L+2}} \frac{\cosh^L t \, e^{-(L+1)t}}{(1 - ie^{-t})^{2L+2}}. \tag{5.1.12}$$

Note that they are singular on the northern hemisphere at $t = i\pi/2$. In terms of the Klein-Gordon inner product on global S^3 slices,

$$\langle \Phi_1, \Phi_2 \rangle_{KG} \equiv i \int_{S^3} d^3\Sigma^\mu \, \Phi_1^* \overleftrightarrow{\partial_\mu} \Phi_2, \tag{5.1.13}$$

we have normalized the modes such that

$$\left\langle \Phi_{Lj}^E, \Phi_{L'j'}^E \right\rangle_{KG} = \delta_{LL',jj'}. \tag{5.1.14}$$

Using these modes, one can define the Euclidean vacuum by the condition

$$\left\langle \Phi_{Lj}^E, \hat{\Phi} \right\rangle_{KG} |0_E\rangle = 0, \tag{5.1.15}$$

where $\hat{\Phi}$ is the quantum field operator. Since the modes Φ_{Lj}^E are not $\mathsf{SO}(4,1)$-invariant, it is not immediately obvious that the Euclidean vacuum is dS-invariant, but this can be checked explicitly. The Wightman function is

$$G_E(x; x') \equiv \langle 0_E | \hat{\Phi}(x) \hat{\Phi}(x') | 0_E \rangle = \sum \Phi_{Lj}^E(x) \Phi_{Lj}^{E*}(x'). \tag{5.1.16}$$

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Using the $i\epsilon$-prescription, this may be expressed in terms of the dS-invariant distance function $P(x; x')$ as

$$G_E(x; x') = \frac{1}{8\pi^2} \frac{1}{1 - P(x; x') + is(x; x')\epsilon}, \qquad (5.1.17)$$

where $s(x; x') > 0$ if x lies in the future of x' and $s(x; x') < 0$ otherwise.

If we rewrite $P(x; x')$ in terms of the coordinates X on the embedding 5D manifold with Minkowski spacetime metric η (in which dS$_4$ is just the hyperboloid $\eta_{\mu\nu} X^\mu X^\nu = \ell^2$), then we can represent $s(x; x')$ by [91]

$$s(X; Y) \equiv X^0 - Y^0. \qquad (5.1.18)$$

Note that this is exactly the same as sending $X^0 - Y^0 \to X^0 - Y^0 - i\epsilon$, since this latter choice of $i\epsilon$-prescription shifts $P(X; Y) = \eta_{\mu\nu} X^\mu Y^\nu / \ell^2$ as follows:

$$P(X; Y) \to P(X; Y) - i\epsilon(X^0 - Y^0). \qquad (5.1.19)$$

dS$_4$ has 10 real Killing vectors which, letting $k \in \{1, 2, 3\}$, we will refer to as the dilation L_0, the 3 boosts $M_k - M_{-k}$ and the 6 SO(4) rotation generators J_k and $M_k + M_{-k}$. Their explicit forms are given in Appendix C.1. The global modes indexed by L transform in the (L, L) representation of SO(4) with quadratic Casimir $L(L + 2)$, but they are not in definite SO(4, 1) representations. In particular, acting arbitrarily many times with the L_0 raising or lowering operators $M_{\pm k}$ gives a nonzero result, so they are in representations with unbounded L_0. In the next section we discuss a dS$_4$ mode decomposition using the highest-weight representations of SO(4, 1).

5.2 SO$(4, 1)$-invariant quasinormal modes

In this section we describe the SO$(4, 1)$-invariant mode decomposition in terms of (anti-) quasinormal modes. We begin by defining

$$
\begin{aligned}
G_\pm(x; x') &\equiv G_E(x; x') \pm G_E(x; x'_A) && (5.2.20) \\
&= \frac{1}{8\pi^2} \left[\frac{1}{1 - P(x; x') + i(X^0 - X'^0)\epsilon} \pm \frac{1}{1 + P(x; x') + i(X^0 + X'^0)\epsilon} \right],
\end{aligned}
$$

where x_A denotes the antipodal point of x. These Green functions fall off like $e^{-2h_\pm t}$ as both arguments are taken to $\mathcal{I}^+$. Next we introduce 'Ω-modes' as follows:

$$
\Phi_\Omega^\pm(x) \equiv \frac{\pi}{\sqrt{h_\pm}} \lim_{t \to \infty} e^{h_\pm t} G_\pm(x; \Omega, t). \tag{5.2.21}
$$

The normalization factor was chosen for future convenience.

In terms of the global coordinates, the Ω-modes take the explicit form

$$
\begin{aligned}
\Phi_\Omega^-(x) &= \frac{1}{2\pi} \frac{1}{[\sinh t - i\epsilon - \cosh(t)\cos\Theta_3(\Omega, x)]} && (5.2.22) \\
&= \frac{1}{2\pi} \frac{1}{[\sinh t - \cosh t \cos\Theta_3(\Omega, x)]} - \frac{i}{2\cosh t} \delta(\tanh t - \cos\Theta_3(\Omega, x)), \\
\Phi_\Omega^+(x) &= -\frac{1}{\sqrt{2}\pi} \frac{1}{[\sinh t - i\epsilon - \cosh(t)\cos\Theta_3(\Omega, x)]^2} && (5.2.23) \\
&= -\frac{1}{\sqrt{2}\pi} \frac{1}{[\sinh t - \cosh t \cos\Theta_3(\Omega, x)]^2} - \frac{i}{\sqrt{2}\cosh^2 t} \delta'(\tanh t - \cos\Theta_3(\Omega, x)).
\end{aligned}
$$

The delta-functions above are normalized as one-dimensional delta-functions, that is, such that $\int_{-\infty}^\infty dy\, \delta(y) = 1$. The Ω-modes can be expanded in terms of the Euclidean global SO(4) modes as follows:

$$
\begin{aligned}
\Phi_\Omega^-(x) &= \sqrt{8}\pi \sum \left[\frac{1}{\sqrt{L+1}} Y_{Lj}^*(\Omega) \right] \Phi_{Lj}^E(x), \\
\Phi_\Omega^+(x) &= -i\sqrt{16}\pi \sum \left[\sqrt{L+1}\, Y_{Lj}^*(\Omega) \right] \Phi_{Lj}^E(x). && (5.2.24)
\end{aligned}
$$

The lowest-weight and highest-weight modes are respectively given by [90]

$$\Phi_{lw}^{\pm}(x) \equiv \Phi_{\Omega_{SP}}^{\pm}(x), \qquad \Phi_{hw}^{\pm}(x) \equiv \Phi_{\Omega_{NP}}^{\pm}(x). \tag{5.2.25}$$

By construction, the modes $\Phi_{hw}^{\pm}$ are eigenfunctions of L_0 with eigenvalues $-h_{\pm}$ and are annihilated by M_{-k} for each $k \in \{1, 2, 3\}$. The descendants of the highest-weight modes are obtained by acting with the M_{+k}, for any $k \in \{1, 2, 3\}$ (see Appendix C.1):

$$M_{+K}\Phi_{hw}^{\pm}(x) \equiv M_{+k_1} \cdots M_{+k_n}\Phi_{hw}^{\pm}(x), \tag{5.2.26}$$

where K is a multi-index denoting the set $\{k_1, \ldots, k_n\}$.

The southern causal diamond (sometimes called the static patch) is the intersection of the causal past and future of the south pole. The highest-weight states are smooth everywhere in this diamond except for the past horizon where they are singular, and they decay exponentially towards the future. Therefore they, together with all their descendants appearing in (5.2.26) and their complex conjugates, comprise the quasinormal modes of the southern diamond. The lowest-weight states (with their descendants and complex conjugates) are singular on the future horizon and are the antiquasinormal modes of the southern diamond. To emphasize this we adopt the notation

$$\Phi_{QN}^{\pm}(x) \equiv \Phi_{hw}^{\pm}(x) = \Phi_{\Omega_{NP}}^{\pm}(x), \qquad \Phi_{AQN}^{\pm}(x) \equiv \Phi_{lw}^{\pm}(x) = \Phi_{\Omega_{SP}}^{\pm}(x). \tag{5.2.27}$$

At this point we have eight highest-weight representations of $\mathsf{SO}(4, 1)$, with elements

$$M_{+K}\Phi_{QN}^{+}, \quad M_{+K}\Phi_{QN}^{-}, \quad M_{+K}\Phi_{QN}^{+*}, \quad M_{+K}\Phi_{QN}^{-*},$$

$$M_{-K}\Phi_{AQN}^{+}, \quad M_{-K}\Phi_{AQN}^{-}, \quad M_{-K}\Phi_{AQN}^{+*}, \quad M_{-K}\Phi_{AQN}^{-*}. \tag{5.2.28}$$

We shall see below that this is an overcomplete set: only the first or second row of modes is needed to obtain a complete basis.

5.3 R-norm

We wish to expand the scalar field operator in the (anti-)quasinormal modes. Towards this end it is useful to introduce an inner product. The Klein-Gordon norms of the Ω-modes are

$$
\begin{aligned}
\left\langle \Phi_{\Omega_1}^{\pm}, \Phi_{\Omega_2}^{\pm} \right\rangle_{KG} &= \mp \frac{16\pi^2}{h_\pm} \Delta_\pm(\Omega_1, \Omega_2), \\
\left\langle \Phi_{\Omega_1}^{+}, \Phi_{\Omega_2}^{-} \right\rangle_{KG} &= - \left\langle \Phi_{\Omega_1}^{-}, \Phi_{\Omega_2}^{+} \right\rangle_{KG} = \frac{16\pi^2}{\sqrt{2}} \frac{i}{\sqrt{h}} \delta^3(\Omega_1 - \Omega_2),
\end{aligned}
\tag{5.3.29}
$$

where

$$
\Delta_\pm(\Omega, \Omega') = \frac{1}{2^{2\mp 1}\pi^2} \frac{1}{(1 - \cos\Theta_3)^{h_\pm}}
\tag{5.3.30}
$$

denote the two-point functions for a CFT$_3$ operators with dimensions $h_\pm$. These satisfy

$$
- \int d^3\Omega'' \sqrt{h} \, \Delta_+(\Omega, \Omega'') \Delta_-(\Omega'', \Omega') = \frac{1}{\sqrt{h}} \delta^3(\Omega - \Omega').
\tag{5.3.31}
$$

The norm of a highest-weight quasinormal mode is obtained by setting $\Omega_1 = \Omega_2 = \Omega_{NP}$, which is evidently divergent. Hence the Klein-Gordon norm is not suitable for quantization of the quasinormal modes.

Alternate norms have been employed in de Sitter spacetime for a variety of reasons [54, 89, 65, 60, 90]. Here, following [90], a useful 'R-norm' can be defined by inserting a reflection R on S^3 across the equator:

$$
\begin{aligned}
R : (\psi, \theta, \phi) &\rightarrow (\pi - \psi, \theta, \phi), \\
\left\langle \Phi_1, \Phi_2 \right\rangle_R &\equiv \left\langle \Phi_1, R\Phi_2 \right\rangle_{KG}.
\end{aligned}
\tag{5.3.32}
$$

With respect to this R-norm,

$$
\left\langle \Phi_{\Omega_1}^{\pm}, \Phi_{\Omega_2}^{\pm} \right\rangle_R = \mp \frac{16\pi^2}{h_\pm} \Delta_\pm(\Omega_1, R\Omega_2).
\tag{5.3.33}
$$

In particular, the norms of the highest-weight quasinormal modes are simply

$$\left\langle \Phi_{QN}^+, \Phi_{QN}^+ \right\rangle_R = -1, \qquad \left\langle \Phi_{QN}^-, \Phi_{QN}^- \right\rangle_R = 1, \qquad \left\langle \Phi_{QN}^+, \Phi_{QN}^- \right\rangle_R = 0, \qquad (5.3.34)$$

while the R-inner product between a quasinormal mode and the complex conjugate of any quasinormal mode vanishes.

Changing the norm affects the hermiticity properties of the 10 real Killing vector fields which generate the dS_4 isometries. Under the Klein-Gordon norm, their adjoints are

$$\begin{aligned} \langle L_0 f, g \rangle_{KG} &= \langle f, -L_0 g \rangle_{KG}, \quad \langle J_k f, g \rangle_{KG} = \langle f, -J_k g \rangle_{KG}, \\ \langle M_{\mp k} f, g \rangle_{KG} &= \langle f, -M_{\mp k} g \rangle_{KG}, \end{aligned} \qquad (5.3.35)$$

so that the Killing generators are all antihermitian. However, under the modified R-norm,

$$\langle L_0 f, g \rangle_R = \langle f, L_0 g \rangle_R, \qquad \langle J_k f, g \rangle_R = \langle f, -J_k g \rangle_R, \qquad \langle M_{\mp k} f, g \rangle_R = \langle f, M_{\pm k} g \rangle_R. \qquad (5.3.36)$$

To recover antihermitian generators in the R-norm, one must send $M_k + M_{-k} \to i(M_k + M_{-k})$ and $L_0 \to iL_0$ while keeping the rest of the generators the same. The Lie bracket algebra of the antihermitian vector fields is then $\mathsf{SO}(3,2)$ rather than $\mathsf{SO}(4,1)$. See Appendix C.1 for more details.

Interestingly $\mathsf{SO}(3,2)$ is the symmetry group of a CFT in $2+1$ dimensions. This suggests that the quantum states on which these generators act could belong to a $2+1$-dimensional CFT, which fits in nicely with the dS_4/CFT_3 conjecture.

Using (5.3.36) we can compute the norm of the descendants. For example, the norm of the first descendant is (not summing over k)

$$\left\langle M_{+k} \Phi_{QN}^\pm, M_{+k} \Phi_{QN}^\pm \right\rangle_R = \left\langle \Phi_{QN}^\pm, M_{-k} M_{+k} \Phi_{QN}^\pm \right\rangle_R = 2h_\pm \left\langle \Phi_{QN}^\pm, \Phi_{QN}^\pm \right\rangle_R. \qquad (5.3.37)$$

Observe that under this R-norm, the descendants of Φ_{QN}^+ are orthogonal to those of Φ_{QN}^-. For the $\mathsf{SO}(3)$-symmetric states, we provide the exact formula in Appendix C.2.

5.4 Completeness of quasinormal modes

In this section we show that the quasinormal modes

$$\left\{ M_{+K}\Phi_{QN}^-, \qquad M_{+K}\Phi_{QN}^{-*}, \qquad M_{+K}\Phi_{QN}^+, \qquad M_{+K}\Phi_{QN}^{+*} \right\} \tag{5.4.38}$$

form a complete set in the sense that the Euclidean Green function can be written as a simple sum over such modes. In particular, the antiquasinormal modes are not needed.

First we note from (5.2.24) that the quasinormal modes can be written as linear combinations of the global Euclidean modes, without using their complex conjugates. Therefore they are themselves Euclidean modes, and the Euclidean vacuum obeys

$$\left\langle M_{+K}\Phi_{QN}^\pm, \hat{\Phi} \right\rangle_R |0_E\rangle = 0. \tag{5.4.39}$$

Note that this relation, unlike the corresponding one for the global Euclidean modes, is manifestly dS-invariant because the quasinormal modes lie in representations of $\mathsf{SO}(4,1)$.

Let us now assume that we can expand the field operator in the presumably complete basis (5.4.38):

$$\hat{\Phi} = \sum_{K,K'} \left(N_{KK'}^+ \left\langle M_{+K}\Phi_{QN}^+, \hat{\Phi} \right\rangle_R M_{+K'}\Phi_{QN}^+ - N_{K'K}^+ \left\langle M_{+K}\Phi_{QN}^{+*}, \hat{\Phi} \right\rangle_R M_{+K'}\Phi_{QN}^{+*} \right.$$

$$\left. + N_{KK'}^- \left\langle M_{+K}\Phi_{QN}^-, \hat{\Phi} \right\rangle_R M_{+K'}\Phi_{QN}^- - N_{K'K}^- \left\langle M_{+K}\Phi_{QN}^{-*}, \hat{\Phi} \right\rangle_R M_{+K'}\Phi_{QN}^{-*} \right),$$

$$\tag{5.4.40}$$

where the $N_{KK'}^\pm$ are defined through

$$\sum_{K'} N_{KK'}^\pm \left\langle M_{+K'}\Phi_{QN}^\pm, M_{+L}\Phi_{QN}^\pm \right\rangle_R = \delta_{KL}. \tag{5.4.41}$$

Then, using (5.4.39), the quasinormal mode Green function is given by

$$G(x; x') = \sum_{K,K'} \Phi_K^+(x) \Phi_{K'}^{+*}(Rx') N_{KK'}^+ + \sum_{K,K'} \Phi_K^-(x) \Phi_{K'}^{-*}(Rx') N_{KK'}^-, \tag{5.4.42}$$

where $\Phi_K^\pm \equiv M_{+K} \Phi_{QN}^\pm$.

A demonstration that the function $G(x; x')$ so obtained is indeed the standard Euclidean Green function $G_E(x; x')$ implies that the quasinormal modes in (5.4.38) form a complete basis, in the sense that they satisfy

$$
\begin{aligned}
i\delta^3(\Omega - \Omega') &= \sqrt{\gamma} n^\mu \sum_{K,K'} N_{KK'}^+ \left[\Phi_K^{+*}(t, \Omega) \nabla_\mu \Phi_{K'}^+(t, R\Omega') - \Phi_K^+(t, \Omega) \nabla_\mu \Phi_{K'}^{+*}(t, R\Omega') \right] \\
&\quad + (+ \leftrightarrow -) \\
0 &= \sum_{K,K'} N_{KK'}^+ \left[\Phi_K^+(t, \Omega) \Phi_{K'}^{+*}(t, R\Omega') - \Phi_K^{+*}(t, \Omega) \Phi_{K'}^+(t, R\Omega') \right] + (+ \leftrightarrow -),
\end{aligned}
\tag{5.4.43}
$$

on a constant time slice with normal vector n^μ and induced metric $\gamma_{\mu\nu}$. Indeed, these two equations can be used to construct a retarded Green function, which in turn provides a solution to the wave equation with arbitrary initial data. Hence any suitably smooth solution to the wave equation can be decomposed on a Cauchy surface in terms of such a set of modes.

First, we would like to evaluate the sum (5.4.42) for the case $(x; x') = (t, \Omega_{SP}; t', \Omega_{SP})$ where both points lie on the south pole observer's worldline. The functions $\Phi_K^\pm(t, \Omega_{SP})$ are nonzero only for spherically symmetric descendants $L_{+1}^n \Phi_{QN}^\pm(t, \Omega)$ where $L_{\mp 1} \equiv \sum_{k=1}^3 M_{\mp k} M_{\mp k}$. The norm for such states is calculated in Appendix C.2 and is given by

$$\left\langle L_{+1}^n \Phi_{QN}^\pm, L_{+1}^m \Phi_{QN}^\pm \right\rangle_R = \frac{\Gamma(2 + 2n)\Gamma(2h_\pm + 2n - 1)}{\Gamma(2h_\pm - 1)} \left\langle \Phi_{QN}^\pm, \Phi_{QN}^\pm \right\rangle_R \delta_{nm}, \tag{5.4.44}$$

while the modes at $\Omega = \Omega_{SP}$ are given by

$$L_{+1}^n \Phi_{QN}^-(t, \Omega_{SP}) = \frac{\Gamma(2n + 2)}{2\pi} \frac{e^{-nt}}{(e^{+t} - i\epsilon)^{n+1}},$$

$$L_{+1}^n \Phi_{QN}^+(t, \Omega_{SP}) = -\frac{\Gamma(2n+3)}{2\sqrt{2\pi}} \frac{e^{-nt}}{(e^{+t} - i\epsilon)^{n+2}}. \tag{5.4.45}$$

Using

$$\left(L_{+1}^n \Phi_{QN}^-(t, R\Omega_{SP})\right)^* = -L_{+1}^n \Phi_{QN}^-(-t, \Omega_{SP})$$

$$\left(L_{+1}^n \Phi_{QN}^+(t, R\Omega_{SP})\right)^* = L_{+1}^n \Phi_{QN}^+(-t, \Omega_{SP}), \tag{5.4.46}$$

the full sum (5.4.42) is

$$
\begin{aligned}
G(t, \Omega_{SP}; t', \Omega_{SP}) &= -\frac{1}{4\pi^2} \sum_{k=0}^{\infty} \left\{ \frac{(2k+1)e^{-k(t-t')}}{\left[(e^{+t} - i\epsilon)(e^{-t'} - i\epsilon)\right]^{k+1}} + \frac{(2k+2)e^{-k(t-t')}}{\left[(e^{+t} - i\epsilon)(e^{-t'} - i\epsilon)\right]^{k+2}} \right\} \\
&= -\frac{1}{16\pi^2} \frac{1}{\sinh^2[(t-t')/2] - i\epsilon\tilde{s}(x;x')},
\end{aligned} \tag{5.4.47}
$$

where

$$\tilde{s}(x; x') \equiv \frac{\sinh t - \sinh t'}{1 + e^{t'-t}}. \tag{5.4.48}$$

Noting that for small ϵ, $\tilde{s}(x; x')$ is equivalent to $s(x; x')$ defined in (5.1.18), it follows that this Green function agrees with that in (5.1.17) on the south pole observer's worldline. Since the construction of our Green function is dS-invariant[3], agreement on this worldline implies that this Green function equals the Euclidean one on any two *timelike* separated points.

For *spacelike* separated points, we find from (5.4.42) that

$$G(t, \Omega_{SP}; t', \Omega_{NP}) = \frac{1}{8\pi^2 \left[1 + \cosh(t+t')\right]} = G_E(t, \Omega_{SP}; t', \Omega_{NP}). \tag{5.4.49}$$

By dS-invariance, we can extend this to any two spacelike-separated points. This concludes the proof that the quasinormal Green function (5.4.42) is indeed the Euclidean Green function.

---

[3]This follows from the fact that the Green function is just a position-space representation of the projection operator onto the highest-weight representation of the three-dimensional conformal group characterized by the highest-weight $-h$, as can be seen by writing out this projection as a sum over complete states of the representation and using the definition of $\mathsf{SO}(4,1)$ generators.

5.5   Results for general light scalars $(m^2 \ell^2 < 9/4)$

In the general case of a light scalar with $m^2 \ell^2 < 9/4$, we can write out the explicit form

of the Euclidean two-point function as (see for instance [65])

$$G_E(x; x') = \frac{\Gamma(h_+)\Gamma(h_-)}{16\pi^2} F\left[h_+, h_-, 2, \frac{1 + P(x; x') - is(x; x')\epsilon}{2}\right],$$
(5.5.50)

where

$$h_\pm = \frac{3}{2} \pm \mu, \qquad \mu = \sqrt{\frac{9}{4} - m^2 \ell^2}.$$
(5.5.51)

The asymptotic behaviors of the Euclidean Green function are:

$$
\begin{aligned}
\lim_{t' \to \infty} G_E(t, \Omega; t', \Omega_{NP}) &= \frac{\Gamma(h_- - h_+)\Gamma(h_+)}{2^{4-2h_+}\pi^2\Gamma(2 - h_+)} \frac{e^{-h_+ t'}}{(\sinh t - i\epsilon + \cosh t \cos \psi)^{h_+}} + (h_+ \leftrightarrow h_-), \\
\lim_{t' \to \infty} G_E(t, \Omega; -t', \Omega_{SP}) &= e^{-i\pi h_+} \frac{\Gamma(h_- - h_+)\Gamma(h_+)}{2^{4-2h_+}\pi^2\Gamma(2 - h_+)} \frac{e^{-h_+ t'}}{(\sinh t - i\epsilon + \cosh t \cos \psi)^{h_+}} \\
&\quad + (h_+ \leftrightarrow h_-).
\end{aligned}
$$
(5.5.52)

Note that in dealing with the branch-cut of $G_E(t, \Omega; t', \Omega')$, we go under (above) it when

$t > t'$ $(t < t')$ in accordance with the $i\epsilon$-prescription.

Let us define $G_\pm$ as

$$G_\pm(x; x') \equiv G_E(x; x') - e^{i\pi h_\mp} G_E(x; x'_A).$$
(5.5.53)

These satisfy the future boundary conditions in Ref. [75] in the region $P < -1$. Now, we

define the highest-weight modes as

$$\Phi_{QN}^\pm(x) \equiv \lim_{t' \to \infty} e^{h_\pm t'} G_\pm(t, \Omega; t', \Omega_{NP}).$$
(5.5.54)

The $\Phi_{QN}^\pm$ are explicitly given by

$$\Phi_{QN}^\pm(x) = \frac{1}{4\pi^{5/2}} \frac{\Gamma(\mp\mu)\Gamma(h_\pm)\left(1 - e^{\mp 2\pi i\mu}\right)}{[\sinh t - i\epsilon + \cosh t \cos \psi]^{h_\pm}} = \frac{1}{4\pi^{5/2}} \frac{\Gamma(\mp\mu)\Gamma(h_\pm)\left(1 - e^{\mp 2\pi i\mu}\right)}{\cosh^{h_\pm} t \left[\tanh(t - i\epsilon) + \cos \psi\right]^{h_\pm}}.$$
(5.5.55)

The asymptotic behavior of the modes as $t \to \infty$ is

$$\lim_{t\to\infty} \Phi_{QN}^{\pm}(t,\Omega) = \frac{2^{3-h_\pm}}{\sqrt{\pi}} \Gamma(\mp\mu)\Gamma(h_\pm) \left(1 - e^{\mp 2\pi i\mu}\right) \Delta_\pm(\Omega, \Omega_{NP}) e^{-h_\pm t} \mp \frac{4i}{\mu} \frac{\delta^3(\Omega - \Omega_{NP})}{\sqrt{h}} e^{-h_\mp t}.$$

(5.5.56)

We have defined[4]

$$
\begin{aligned}
\Delta_\pm(\Omega, \Omega') &= \frac{2^{3(h_\pm - 1)}}{\pi} \Gamma(2 - 2h_\pm) \sin\left(h_\pm \pi\right) \sum \frac{\Gamma(h_\pm + L)}{\Gamma(h_\mp + L)} Y_{Lj}(\Omega) Y_{Lj}^*(\Omega') \\
&= \frac{1}{2^{5 - 2h_\pm} \pi^2} \frac{1}{\left[1 - \cos\Theta_3(\Omega, \Omega')\right]^{h_\pm}},
\end{aligned}
$$

(5.5.58)

which satisfy

$$\frac{\pi^2}{8\cos^2\left(\pi\mu\right)\Gamma(2 - 2h_+)\Gamma(2 - 2h_-)} \int d^3\Omega'' \sqrt{h}\, \Delta_+(\Omega, \Omega'')\Delta_-(\Omega'', \Omega') = \frac{1}{\sqrt{h}}\delta^3(\Omega - \Omega').$$

(5.5.59)

The norm is easily evaluated at $\mathcal{I}^+$ to be

$$
\begin{aligned}
\left\langle \Phi_{\Omega_1}^{\pm}, \Phi_{\Omega_2}^{\pm} \right\rangle_R &= \frac{2^{5-h_\pm}}{\sqrt{\pi}} \Gamma(\mp\mu)\Gamma(h_\pm) \sin^2(\pi\mu)\Delta_\pm(\Omega_1, R\Omega_2), \\
\left\langle \Phi_{\Omega_1}^{\pm}, \Phi_{\Omega_2}^{\mp} \right\rangle_R &= \pm\frac{4i}{\mu} \left(1 - e^{\pm 2\pi i\mu}\right) \frac{\delta^3(\Omega_1 - R\Omega_2)}{\sqrt{h}}.
\end{aligned}
$$

(5.5.60)

As such, we find that the R-norms of the quasinormal modes are

$$\left\langle \Phi_{QN}^{\pm}, \Phi_{QN}^{\pm} \right\rangle_R = \Gamma(\mp\mu)\Gamma(h_\pm)\frac{\sin^2(\pi\mu)}{\pi^{5/2}}, \qquad \left\langle \Phi_{QN}^+, \Phi_{QN}^- \right\rangle_R = 0.$$

(5.5.61)

The rest of the discussion on the induced norms of the descendants carries over from the $m^2\ell^2 = 2$ case.

---

[4]Here, a useful identity on S^3 is

$$[1 - \cos\Theta_3(\Omega, \Omega')]^{-h} = 2^{2+h}\pi \sin(\pi h)\Gamma(2 - 2h) \sum \frac{\Gamma(L + h)}{\Gamma(3 + L - h)} Y_{Lj}(\Omega) Y_{Lj}^*(\Omega').$$

(5.5.57)

Next, we follow our previous strategy of showing that the mode sum and the Euclidean Green function agree on the south pole observer's worldline. Again, we evaluate

$$L_{+1}^n \Phi_{QN}^{\pm}(t, \Omega_{SP}) = \frac{\Gamma(\mp\mu)(1 - e^{\mp 2\pi i \mu})}{4\pi^{5/2}} \frac{\Gamma(2n+3)\Gamma(h_{\pm} + n)}{2\Gamma(n+2)} \frac{e^{-nt}}{(e^t - i\epsilon)^{n+h_{\pm}}}. \quad (5.5.62)$$

As before, the norm for such states is

$$\left\langle L_{+1}^n \Phi_{QN}^{\pm}, L_{+1}^n \Phi_{QN}^{\pm} \right\rangle_R = \frac{\Gamma(2+2n)\Gamma(2h_{\pm} + 2n - 1)}{\Gamma(2h_{\pm} - 1)} \left\langle \Phi_{QN}^{\pm}, \Phi_{QN}^{\pm} \right\rangle_R. \quad (5.5.63)$$

Note that

$$\left(L_{+1}^n \Phi_{QN}^{\pm}(t, R\Omega_{SP})\right)^* = e^{i\pi h_{\pm}} L_{+1}^n \Phi_{QN}^{\pm}(-t, \Omega_{SP}). \quad (5.5.64)$$

The quasinormal mode Green function (5.4.42) is then

$$G(t, \Omega_{SP}; t', \Omega_{SP}) = \sum_{n=0}^{\infty} \left[\frac{c_n^+ e^{-n(t-t')}}{\left[(e^t - i\epsilon)(e^{-t'} - i\epsilon)\right]^{n+h_+}} + \frac{c_n^- e^{-n(t-t')}}{\left[(e^t - i\epsilon)(e^{-t'} - i\epsilon)\right]^{n+h_-}} \right] \quad (5.5.65)$$

where

$$c_n^{\pm} = -\frac{e^{-i\pi h_{\pm}}}{2\pi^2 \sin(\pm\pi\mu)} \frac{\Gamma(\frac{3}{2}+n)\Gamma(h_{\pm}+n)}{\Gamma(1+n\pm\mu)\Gamma(1+n)}. \quad (5.5.66)$$

In Appendix C.3, we show that on the south pole observer's worldline, this Green function is equal to the Euclidean Green function. Thus by dS-invariance of both Green functions, they agree for any two timelike separated points in dS$_4$.

For spacelike separated points, we consider the Euclidean Green function with one point at the south pole and the other point at the north pole. One notices that

$$G_E(t, \Omega_{SP}; t', \Omega_{NP}) = G_E(t, \Omega_{SP}; t', \Omega_{SP})|_{t' \to -t' + i\pi}. \quad (5.5.67)$$

Since these points are spacelike separated, we do not have to worry about the $i\epsilon$-prescription and the Green function is real. If we had evaluated the quasinormal mode Green function $G(t, \Omega_{SP}; t', \Omega_{NP})$, then we would have obtained the same sum as in (5.5.65), provided that we sent $t' \to -t'$ and removed the phase $e^{-i\pi h_\pm}$ from the coefficients (5.5.66). This is equivalent to sending $t' \to -t' + i\pi$ and hence by dS-invariance we have proved that for any two *spacelike* separated points, the quasinormal mode Green function is the Euclidean Green function.

5.6 Southern modes and T-norm

In this section we find quasinormal modes that vanish in the northern or southern diamonds – the analogues of Rindler modes in Minkowski space. We begin with the expression (5.5.55) for the lowest-weight mode

$$\Phi^\pm_{QN}(x) = \frac{1}{4\pi^{5/2}} \frac{\Gamma(\mp\mu)\Gamma(h_\pm)\left(1 - e^{\mp 2\pi i \mu}\right)}{[\sinh t - i\epsilon + \cosh t \cos \psi]^{h_\pm}}. \tag{5.6.68}$$

For generic mass $\Phi^\pm_{QN}(x)$ has a branch cut on the past horizon of the southern observer at $\tanh t = -\cos\psi$. We have chosen the phase convention so that the denominator is real above the past horizon. Crossing the past horizon gives an extra phase of $e^{i\pi h_\pm}$. It follows that the southern mode

$$\Phi^\pm_{QN,S}(x) \equiv \Phi^\pm_{QN}(x) + \Phi^{\pm *}_{QN}(x) \tag{5.6.69}$$

vanishes below the past horizon. Similarly the northern mode

$$\Phi^\pm_{QN,N}(x) \equiv e^{i\pi h_\pm}\Phi^\pm_{QN}(x) + e^{-i\pi h_\pm}\Phi^{\pm *}_{QN}(x). \tag{5.6.70}$$

vanishes above the past horizon. The R-norms between these modes are

$$
\begin{aligned}
\left\langle \Phi_{QN,S}^{\pm}, \Phi_{QN,S}^{\pm} \right\rangle_R &= \left\langle \Phi_{QN,N}^{\pm}, \Phi_{QN,N}^{\pm} \right\rangle_R = \left\langle \Phi_{QN,N}^{\pm}, \Phi_{QN,N}^{\mp} \right\rangle_R = \left\langle \Phi_{QN,S}^{\mp}, \Phi_{QN,S}^{\mp} \right\rangle_R = 0, \\
\left\langle \Phi_{QN,N}^{\pm}, \Phi_{QN,S}^{\pm} \right\rangle_R &= -2i \sin(\pi h_{\pm}) \left\langle \Phi_{QN}^{\pm}, \Phi_{QN}^{\pm} \right\rangle_R = -2i \sin(\pi h_{\pm}) \Gamma(\mp\mu) \Gamma(h_{\pm}) \frac{\sin^2(\pi\mu)}{\pi^{5/2}}.
\end{aligned}
$$

$$(5.6.71)$$

On the other hand, the R-norm for the global quasinormal modes is closely related to time-reflection:

$$
\left\langle f, \Phi_K^{\pm} \right\rangle_R = e^{-i\pi h_{\pm}} \left\langle f, T\Phi_K^{\pm*} \right\rangle_{KG}, \qquad T : t \to -t. \tag{5.6.72}
$$

While the R-norm has no analogue in the static patch, the T-norm is easily generalizable to the southern diamond as

$$
\left\langle \Phi_K^{\pm}, \Phi_{K'}^{\pm} \right\rangle_{T,B_S^3} \equiv \left\langle T\Phi_K^{\pm*}, \Phi_{K'}^{\pm} \right\rangle_{KG,B_S^3}, \tag{5.6.73}
$$

where B_S^3 denotes the integral over a complete slice in the southern diamond. We have

$$
\begin{aligned}
\left\langle \Phi_{QN,S}^{\pm}, T\Phi_{QN,S}^{\pm} \right\rangle_{T,B_S^3} &= \left\langle \Phi_{QN,S}^{\pm}, T\Phi_{QN,S}^{\mp} \right\rangle_{T,B_S^3} = \left\langle \Phi_{QN,S}^{\pm}, \Phi_{QN,S}^{\mp} \right\rangle_{T,B_S^3} = 0, \\
\left\langle \Phi_{QN,S}^{\pm}, \Phi_{QN,S}^{\pm} \right\rangle_{T,B_S^3} &= 2i \sin(\pi h_{\pm}) \Gamma(\mp\mu) \Gamma(h_{\pm}) \frac{\sin^2(\pi\mu)}{\pi^{5/2}}.
\end{aligned}
$$

$$(5.6.74)$$

Chapter 6

Fluid/Gravity Correspondence of dS Horizon

Any de Sitter static observers find themselves surrounded by a cosmological horizon whose size is set by the value of the cosmological constant. The nature of this horizon is rather enigmatic given that an observer can never reach her cosmological horizon – it is always at a finite fixed distance away from her. Thus, unlike a black hole, the static patch horizon is not localized in some finite region of space. On the other hand, it behaves classically, and to an extent quantum mechanically, very much like a black hole horizon. For instance there is a temperature and entropy associated with the cosmological horizon [47].

One of the main challenges confronted by theorists is to uncover the nature of the holographic principle in the context of asymptotically de Sitter universes. One may be inclined to propose that de Sitter holography should only describe a single patch, given that this is the region of space accessible to a single observer [51, 49, 83, 92, 53, 93, 94, 52, 48, 95, 75, 77]. In fact, the observer's worldline in the static patch resembles in many ways the boundary

of anti-de Sitter space (in the presence of an eternal black hole) and may constitute the ultraviolet regime of the fundamental description [48, 75, 77], perhaps in a way related to matrix theory. In such a case, the dynamics of the cosmological horizon would constitute the deep infrared behavior of the putative worldline theory very much like the near horizon dynamics of a black hole in anti-de Sitter space constitutes the deep infrared behavior of the dual conformal field theory. Such behavior rather universally takes the form of fluid dynamics.

There have been several attempts to relate general relativity to fluid mechanics dating back to the 1970s with the black hole membrane paradigm [96, 97, 98] (see [99] for an application to de Sitter space). The membrane paradigm focuses on the observation that the equations governing the dynamics of horizon surfaces in general relativity can be written in a form analogous to that of the Navier-Stokes equation of fluid mechanics. However, whilst finding a striking analogy, the central equation of the membrane paradigm is often referred to as the Damour-Navier-Stokes equation, highlighting the fact that it differs from the Navier-Stokes equation in key ways. Building on this, recent papers [15, 16, 17] constructed a setup where near horizon dynamics in gravity precisely relates the Einstein equation to the incompressible Navier-Stokes equation. These studies were also inspired by analyses of connections between gravity and fluid mechanics in the context of the AdS/CFT correspondence [100, 101, 102] and the low energy limit of the dual field theory. Given the striking similarities between the thermodynamics of a black hole horizon and a cosmological horizon, it is natural to extend such a fluid/gravity correspondence to include spacetimes with a cosmological horizon.

Yet another natural location for a definition of quantum gravity in de Sitter space is

the spacelike boundary at future infinity, known as $\mathcal{I}^+$ [103, 12, 70, 54, 57, 71, 55]. The 'metaobservables' on $\mathcal{I}^+$ are given by correlators between causally disconnected points. In a sense, we are metaobservers of the 'would be' $\mathcal{I}^+$ of the inflationary de Sitter era. Such metaobservables are conjectured to constitute the correlation functions of a lower dimensional non-unitary Euclidean conformal field theory. Recently, an exact example of this dS/CFT correspondence was proposed in [14]. It is not at all clear how the physics of the static patch observer is captured by the theory at $\mathcal{I}^+$. The static patch observer can at most observe a single point (or tiny region) of $\mathcal{I}^+$ where her worldline intersects the future boundary. Thus, from the static patch observer's point of view, fixing the geometry outside her future horizon is akin to fixing a gauge since it will never affect the physics she observes [75]. In particular, one might envision fixing the geometry near $\mathcal{I}^+$ by fixing the data on a spacelike slice in the future diamond and allowing only flux originating from a single static patch to come through. This is somewhat analogous to the boundary condition that there is no incoming flux from the past horizon of a black hole. It may then be speculated that the existence of a finite number of 'holographic reconstructions' of a static patch observer in the dual CFT at $\mathcal{I}^+$ would be a manifestation of the finiteness of the de Sitter entropy.[1]

After reviewing the classical geometry of de Sitter space, the first part of this chapter will explore some of the classical features of the cosmological horizon as viewed by an observer in a purely de Sitter universe – the static patch observer. We examine the Einstein equation both linearly and non-linearly and uncover that the solutions are characterized by solutions to the incompressible Navier-Stokes equation on a two-sphere.[2] This same equation recently

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[1] This may ultimately be related to the mysterious use of the Cardy formula for 'counting' the de Sitter entropy in some lower dimensional examples [65, 68].

[2] As in [17], we analyze the metric through the first three orders in a near-horizon expansion. A generalization of the all-orders proof of [104] might be possible in our case, but we will not attempt to do so

appeared in the context of the Schwarzschild black hole [17] and requires the velocity field

$v_i(\tau, \Omega^j)$ where $\Omega^i = \{\theta, \phi\}$ and the pressure $P(\tau, \Omega^j)$ to satisfy

$$\partial_\tau v^i + \nabla^i_{S^2} P + v_j \nabla^j_{S^2} v^i - \nu \left(\nabla^2_{S^2} v^i + R^i_j v^j \right) = 0 \ , \quad \nabla^i_{S^2} v_i = 0 \tag{6.0.1}$$

where ν is the viscosity. Indices are raised and lowered with respect to the round metric g_{ij}

on the S^2 of radius r_S for which $R_{ij}(= g_{ij}/r_S^2)$ is the Ricci tensor. At the linearized level,

this is done by imposing Dirichlet boundary conditions on a timelike surface arbitrarily

close to the cosmological horizon and the absence of incoming flux from the past horizon of

the static patch. These boundary conditions resemble the solipsistic boundary conditions

of [77], which allow for an examination of the isolated static patch dynamics, unperturbed

by external sources from the past horizon. We find that the linearized solutions must

obey the dispersion relation of the incompressible, linearized (pressureless) Navier-Stokes

equation (6.2.19). At the non-linear level, again in a near cosmological horizon expansion,

we impose (conformal) Dirichlet boundary conditions on a timelike slice and regularity

of the solutions as they approach the future horizon. By (conformal) Dirichlet boundary

conditions, we mean analysing perturbations which leave the induced geometry on a fixed

timelike hypersurface of constant extrinsic curvature unchanged up to a conformal factor.[3]

Then, we comment briefly on the possibilities of deforming this non-linear fluid by placing

a small black hole at the origin of the static patch. In an attempt to connect our fluid

dynamical modes to the analogous excitations of the worldline, which are the quasinormal

modes, we return to the linearized analysis to study how the linearized dispersion relation

varies as we push the surface from the cosmological horizon to the worldline.

herein.

[3]Henceforth, in the non-linear analysis, we will refer to these boundary conditions as Dirichlet boundary conditions.

In the second part of the chapter we make some mathematical observations about space-like slices foliating the region outside the future horizon of the static patch. We examine the behavior of linearized solutions to the Einstein equation near, but outside, the future cosmological horizon. Our solutions are subjected to Dirichlet boundary conditions on a fixed spacelike surface and to contain incoming flux solely from a single static patch observer. We find a discrete set of modes obeying the dispersion relation of the linearized Navier-Stokes equation, where the time coordinate has become the non-compact spacelike coordinate moving us along the spacelike slice. The non-linear solutions to the Einstein equation which satisfy Dirichlet boundary conditions on the spacelike slice and which are regular at the horizon from which flux is coming, are indeed characterized by solutions to the incompressible Navier-Stokes equation. The Navier-Stokes equation uncovered here on the spacelike slice is equivalent to that discussed in the context of the timelike surface, except that the sign of the viscosity is flipped. We end by noting that the setup of the problem in this future diamond of de Sitter space, and in particular the pole structure at $\mathcal{I}^+$, is connected by an analytic continuation to analogous problems in Lorentzian AdS$_4$ with hyperbolic slicing.

6.1 Geometry and Framework

In what follows we will study the geometries of several patches of de Sitter space pertinent to our analysis. Instead of the global patch of de Sitter space containing the past and future infinities, denoted by $\mathcal{I}^-$ and $\mathcal{I}^+$, we will focus on patches that are more suited to the description of local observers.

6.1.1 The Static Patch

The four-dimensional static patch metric solves the Einstein equation in the presence of a cosmological constant $\Lambda > 0$,

$$\mathcal{G}_{\mu\nu} \equiv G_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{6.1.2}$$

and is given by:

$$ds^2 = -\left(1 - (r/\ell)^2\right) dt^2 + \left(1 - (r/\ell)^2\right)^{-1} dr^2 + r^2 d\Omega_2^2 \ , \tag{6.1.3}$$

where $r \in [0, \ell]$, $t \in \mathbb{R}$ and $d\Omega_2^2$ is the round metric on S^2. The quantity ℓ is the de Sitter length and is related to the cosmological constant as $\Lambda = +3/\ell^2$. The above metric covers a quarter of the global de Sitter geometry, it describes the intersection of the future and past causal diamonds of a constant r worldline beginning at $\mathcal{I}^-$ and ending at $\mathcal{I}^+$. We call this the Southern patch of de Sitter space.

One notices that $r = \ell$ corresponds to a cosmological event horizon, beyond which events are forever out of causal contact from the Southern observer. The Killing vector ∂_t becomes null at $r = \ell$ and the above coordinate system breaks down.

The Southern patch can be smoothly connected to another region covering an additional quarter of de Sitter space, by continuing the above metric to $r \in [\ell, \infty]$. For $r > \ell$, t becomes a spacelike coordinate and r becomes timelike. We can consider gluing two such regions, one behind the past cosmological horizon, known as the past diamond containing $\mathcal{I}^-$, and the other beyond the future cosmological horizon, known as the future diamond containing $\mathcal{I}^+$.

The remaining quarter of the global de Sitter space is given by an additional static patch system known as the Northern patch. The Southern and Northern patches each intersects

$\mathcal{I}^{\pm}$ at a single point. In figure 6.1 we demonstrate the several patches discussed above in a Penrose diagram.
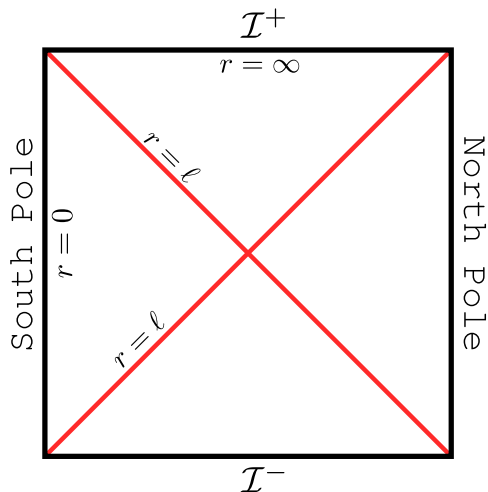
Fig. 6.1: Penrose diagram of de Sitter space indicating the various static patches and future/past diamonds.

6.1.2   Null Foliations

It will be convenient to introduce an additional coordinate system which smoothly covers both the Southern patch and the future diamond. This is achieved by the following coordinate transformation:

$$\ell du = dt - \frac{dr}{(1 - (r/\ell)^2)} \ , \qquad v = \frac{r}{\ell} \ , \tag{6.1.4}$$

leading to the metric

$$\frac{ds^2}{\ell^2} = -(1 - v^2)du^2 - 2dudv + v^2d\Omega_2^2 \ . \tag{6.1.5}$$

Up to a constant time shift we find $u\ell = t - \ell\tanh^{-1}r/\ell$. Constant u surfaces are null lines emanating from the origin at $v = 0$ and ending at $\mathcal{I}^+$ where $v = \infty$. The norm of the Killing vector ∂_u changes sign at $v = 1$.
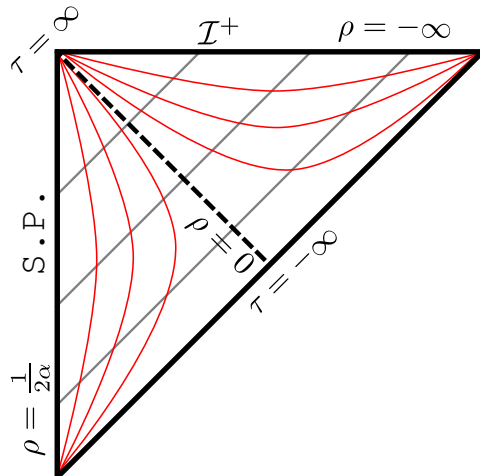
Fig. 6.2: Penrose diagram of de Sitter space indicating constant ρ (red) and τ (gray diagonal) slices.

6.1.3 Approaching the Horizon

Finally, we would like to introduce a dimensionless parameter $\alpha > 0$ allowing us to approach the cosmological horizon. In order to achieve this, we rescale time to $u = \tau/2\alpha$ and define $\rho = (1-v)/2\alpha$. As we take the limit $\alpha \to 0$, we redshift time and for any finite ρ, v will be forced to lie near the cosmological horizon. The metric is given by:

$$\frac{ds^2}{\ell^2} = \left(-\frac{\rho}{\alpha} + \rho^2\right) d\tau^2 + 2d\tau d\rho + (1 - 2\alpha\rho)^2 d\Omega_2^2 \,. \tag{6.1.6}$$

The coordinate range of ρ covering the Southern patch is given by $\rho \in [0, 1/2\alpha]$ and the norm of ∂_u vanishes at $\rho = 0$. The constant ρ and τ surfaces are shown in figure 6.2.

As opposed to the Schwarzschild case, where a similar expansion would continue for indefinite powers of α, the above expansion terminates at order $\mathcal{O}(\alpha^3)$. This is due to the absence of a term $\sim 2M/r$ in the $g_{\tau\tau}$ component. We could of course add such a term, which would correspond to introducing a small mass or black hole centered at the origin of the static patch.

6.2   Incompressible Fluids

Having specified the geometry relevant to our problem, we proceed to discuss the nature of perturbations solving the Einstein equation with positive Λ near the cosmological horizon. We begin with a linearized analysis.

6.2.1   Linearized Analysis

Linearized gravity about spherically symmetric spaces with non-zero cosmological constant was examined in [105, 106]. The two gravitational degrees of freedom transform as a (divergenceless) vector and a scalar under the $SO(3)$ symmetry of the S^2. There is no transverse-traceless tensorial spherical harmonic for a two-sphere. Let us work in a gauge where $\delta g_{ij} = 0$ for $x^i \in \{\Omega\}$. The metric vector perturbations can be expressed as:

$$\delta g_{it} \;=\; \mathcal{V}_i \times \left(1 - (r/\ell)^2\right)(1 + r\partial_r)\,\Phi_v \ , \tag{6.2.7}$$

$$\delta g_{ir} \;=\; \mathcal{V}_i \times \frac{r}{(1 - (r/\ell)^2)}\partial_t \Phi_v \ . \tag{6.2.8}$$

The vector spherical harmonic $\mathcal{V}_i$ satisfies the following relations on the unit two-sphere:

$$\left(\nabla_{S^2}^2 + k_V^2\right)\mathcal{V}_i = 0 \ , \quad \nabla_{S^2}^i \mathcal{V}_i = 0 \ , \tag{6.2.9}$$

with eigenvalues are $k_V^2 = l(l+1) - 1$ and $l = 1, 2, \ldots$ The master field Φ_v obeys the master equation:

$$\left(\nabla_{g^{(2)}}^2 - \frac{l(l+1)}{r^2}\right)\Phi_v = 0 \ , \tag{6.2.10}$$

where $g^{(2)}$ corresponds to the two-dimensional de Sitter static patch. A similar result holds for the scalar perturbations, which we discuss in appendix D.1.

The solutions to the above equation were analyzed in [75] and are found to be hypergeometric functions. For our purposes we would like to obtain the linearized solutions in the null coordinate system (6.1.5). Assuming a Fourier decomposition in time, $\Phi_v = e^{-2i\alpha\omega t(\tau,\rho)/\ell}\varphi_v(\rho)$, the equation of motion becomes:

$$\left(4\rho^2\left(1-\alpha\rho\right)^2\partial_\rho^2 + 4\rho\left(1-\alpha\rho\right)\left(1-2\alpha\rho\right)\partial_\rho + \right.$$
$$\left. \frac{4\alpha^2(1-2\alpha\rho)^2\omega^2 - 4\alpha\rho\left(1-\alpha\rho\right)\left(k_V^2+1\right)}{(1-2\alpha\rho)^2}\right)\varphi_v = 0 \ . \quad (6.2.11)$$

The two linearly independent solutions for $l > 1$ are given by:

$$\varphi_v^{out} = \rho^{-i\alpha\omega} \, {}_2F_1\left[a_1, b_1; c_1; \alpha\rho(-1+2\alpha\rho)^{-1}\right] (1-\alpha\rho)^{-i\alpha\omega} (1-2\alpha\rho)^{2i\alpha\omega} \ , (6.2.12)$$

$$\varphi_v^{in} = \rho^{+i\alpha\omega} \, {}_2F_1\left[a_2, b_2; c_2; \alpha\rho(-1+2\alpha\rho)^{-1}\right] (1-\alpha\rho)^{-i\alpha\omega} \ , \quad (6.2.13)$$

with:

$$a_1 = -l - 2i\alpha\omega \ , \quad b_1 = 1 + l - 2i\alpha\omega \ , \quad c_1 = 1 - 2i\alpha\omega \ ; \quad (6.2.14)$$

$$a_2 = -l \ , \quad b_2 = 1 + l \ , \quad c_2 = 1 + 2i\alpha\omega \ . \quad (6.2.15)$$

The superscripts 'out' and 'in' indicate that the mode is purely outgoing at the future horizon or purely incoming from the past horizon. The above expressions are linearly independent solutions for $(c_i - a_i - b_i) = 2i\alpha\omega \notin \mathbb{Z}$ (see [107]). In the case where $(c_i - a_i - b_i) = 2i\alpha\omega$ is an integer, logarithmic solutions will appear. Given that a_2 and b_2 are integers φ_v^{in} is in fact a finite polynomial for $2i\alpha\omega \notin \mathbb{Z}$, as it can be shown that the hypergeometric series terminates. For $l = 1$, the linearized perturbations become time independent and are like the introduction of a small amount of angular momentum (we discuss this case in appendix D.2).

The linearized purely outgoing metric components (6.2.7) in the (τ, ρ)-coordinate system become:

$$\delta g_{i\tau}^{out} = 2\mathcal{V}_i \times e^{-i\omega\tau} \, \rho^{i\alpha\omega+1}(1-\alpha\rho)^{-i\alpha\omega+1}\left(1 - \frac{(1-2\alpha\rho)}{2\alpha}\partial_\rho\right)\varphi_v^{out} \ , \qquad (6.2.16)$$

$$\delta g_{i\rho}^{out} = -2\alpha\mathcal{V}_i \times e^{-i\omega\tau}\left(\frac{1-\alpha\rho}{\rho}\right)^{-i\alpha\omega}\left[\left(1 - \frac{i\omega(1-2\alpha\rho)}{2\rho(1-\alpha\rho)}\right) - \frac{(1-2\alpha\rho)}{2\alpha}\partial_\rho\right]\varphi_v^{out} \ .$$

$$(6.2.17)$$

Both $\delta g_{i\tau}^{out}$ and $\delta g_{i\rho}^{out}$ are regular at the future horizon $\rho = 0$.

6.2.2   Linearized Fluid Modes

Having written down the linearized solutions, we now discuss the choice of boundary conditions. We impose Dirichlet boundary conditions on a given timelike hypersurface at some fixed ρ, and without loss of generality, we choose the location of this timelike hypersurface to be at $\rho = 1$. Taking $\alpha \to 0$ pushes this hypersurface arbitrarily near the cosmological horizon and thus allows us to probe the near horizon dynamics.

Our particular Dirichlet boundary condition, shown in figure 6.3, is that the linearized perturbations are purely outgoing and leave the intrinsic geometry of the $\rho = 1$ hypersurface unchanged.[4] Imposing $\delta g_{i\tau}^{out}(\rho = 1) = 0$ enforces a discrete dispersion relation, which to leading order in α is given by:

$$\omega_f = -i\left(l(l+1) - 2\right) \ , \quad l = 1, 2, \dots \qquad (6.2.18)$$

We interpret these linearized modes as fluid modes of the velocity field v^i of the incompress-

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[4]This is the simplest choice of Dirichlet boundary conditions and thus allows for a clear analysis. In general, we can choose more involved Dirichlet boundary conditions on the $\rho = 1$ hypersurface.
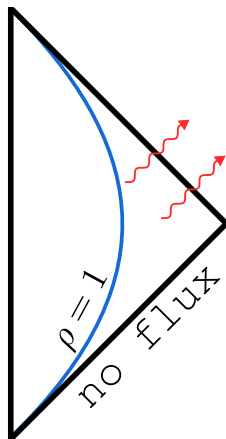
Fig. 6.3: Our boundary conditions for the linearized modes are such that the induced metric on the $\rho = 1$ slice is unchanged and there is no incoming flux from the past horizon.

ible, linearized (pressureless) Navier-Stokes equation on a sphere:

$$\partial_\tau v_i = \nu \left(\nabla^2_{S^2} v_i + R_{ij} v^j \right) \ , \quad \nabla^i_{S^2} v_i = 0 \ , \tag{6.2.19}$$

where the viscosity $\nu = 1$. The incompressibility of the fluid is equivalent to the vanishing divergence of $\mathcal{V}_i$, which can be seen by identifying $v_i \sim e^{-i\omega\tau} \mathcal{V}_i$. We further note that the explicit modes (6.2.16) with $\omega = \omega_f$ decay in time and are regular at the future horizon $\rho = 0$.[5]

By arguments similar to those in [15], one expects that the result $\nu = 1$ corresponds to a ratio of shear viscosity to entropy density which is $1/4\pi$. This suggests that the incompressible fluid we have found near the de Sitter horizon shares this feature with the fluids found near the Schwarzschild, Rindler and planar AdS black hole horizons [15, 16, 17, 108] (see also [99]).

There is also a decoupled set of scalar excitations which transform as scalars under

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[5]If these modes are taken back in time to $t \to -\infty$ they diverge and the perturbative solution is no longer reliable. As usual we only consider wavepackets of the linearized solutions which are finite for all asymptotia.

the $SO(3)$ of the S^2. However, the incompressibility condition implies that we could only consider the spherically symmetric scalar mode, which reduces the fluid vector field to a trivial one(see appendix D.1). We will not consider such modes in what follows and simply set them to zero in the linearized analysis.

6.2.3 Non-linear Analysis

Having analyzed the linearized case, we now turn to the question of non-linear deformations. The analysis follows directly the Schwarzschild case analyzed in [17].[6] In particular, in this non-linear analysis, we impose the (conformal) Dirichlet boundary conditions on the hypersurface as in [17].

To be precise, we will consider the following finite deformation of the static patch geometry as an expansion in α:[7]

$$
\begin{aligned}
\frac{ds^2}{\ell^2} = {} & -\frac{\rho}{\alpha} d\tau^2 \\
& + \rho^2 d\tau^2 + 2d\tau d\rho + d\Omega_2^2 + (1-\rho)\left[v^2 d\tau^2 - 2v_i d\tau dx^i\right] - 2\rho P d\tau^2 \\
& + \alpha\Big[(-4\rho + 2P)\,d\Omega_2^2 + (1-\rho)v_i v_j dx^i dx^j \\
& \quad - \left(\rho^2 - 1\right)\left(\nabla^2 v_i + R_{ji}v^j\right)d\tau dx^i - 2v_i d\rho dx^i + \left(v^2 + 2P\right)d\tau d\rho + 2\left(1 - \rho\right)\phi_i^{(\alpha)} d\tau dx^i\Big] \\
& + \alpha^2\left(4\rho^2 d\Omega_2^2 + 2g_{\rho i}^{(\alpha^2)} dx^i d\rho + g_{ij}^{(\alpha^2)} dx^i dx^j\right) + \dots.
\end{aligned}
\tag{6.2.20}
$$

The v^i, P and $\phi_i^{(\alpha)}$ are functions of (τ, Ω^i) only while the $g_{i\rho}^{(\alpha^2)}$ and $g_{ij}^{(\alpha^2)}$ are functions of (τ, ρ, Ω^i). We have chosen a gauge where $g_{\rho\rho} = 0$. As boundary conditions we require the

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[6]It should be noted that we have presented a more complete linearized analysis than would be possible for the Schwarzschild case, given the existence of exact linearized solutions in dS_4.

[7]When writing out the metric (6.2.20) we have omitted the metric components $g_{\tau\tau}^{(\alpha)}, g_{\tau\tau}^{(\alpha^2)}, g_{\rho\tau}^{(\alpha^2)}, g_{i\tau}^{(\alpha^2)}$ and higher order contributions since these do not affect the Einstein equation to the order that we consider.

perturbations to preserve the induced metric on the hypersurface $\rho = 1$

$$ds_{3d}^2 = \left(-\frac{1}{\alpha} + 1 \right) d\tau^2 + (1 - 2\alpha)^2 \, d\Omega_2^2 \,, \tag{6.2.21}$$

up to a conformal factor

$$1 + 2\alpha P + \mathcal{O}(\alpha^2) \,. \tag{6.2.22}$$

We also study perturbations such that this hypersurface has constant mean extrinsic curvature and that the solution is regular at the future horizon $\rho = 0$. These boundary conditions are the natural extension of the boundary conditions imposed on the linearized fluid modes of the last section.

We now examine the conditions on the deformation parameters imposed by the Einstein equation with a positive cosmological constant $\mathcal{G}_{\mu\nu} = 0$ up to and including $\mathcal{O}(\alpha^0)$. We further assume that the only excited field is the metric. The first non-trivial condition appears at $\mathcal{O}(\alpha^{-1})$. Here, for $\mathcal{G}_{\tau\tau} = 0$ to be satisfied, the velocity field v^i is required to be incompressible. At the next order $\mathcal{O}(\alpha^0)$, the non-trivial equations are $\mathcal{G}_{\tau\tau} = \mathcal{G}_{\tau i} = 0$. From the $\mathcal{G}_{\tau i} = 0$, it follows that the (v^i, P) need to satisfy the non-linear incompressible Navier-Stokes equation (6.0.1) on a unit S^2. Our result is in complete accordance with the linearized analysis.[8]

From the $\mathcal{G}_{\tau\tau} = 0$ Einstein equation at $\mathcal{O}(\alpha^0)$ we find the requirement

$$\nabla_{S^2}^i \phi_i^{(\alpha)} = 2\partial_\tau P + \partial_\tau(v^2) + \text{total derivatives on the 2-sphere.} \tag{6.2.23}$$

---

[8]We would not expect to see the pressure in the linearized analysis since at the linear level vector and scalar representations of $SO(3)$ decouple. This is no longer the case at second order in perturbation theory where we expect the equation for the vector representation to be affected by scalars as in the non-linear case. As in the linearized analysis, we have not considered sound modes, which would contribute to the divergence of v^i.

The above relation implies:

$$\partial_\tau \left(\int v^2 d\Omega_2 \right) = -2\partial_\tau \left(\int P d\Omega_2 \right) . \tag{6.2.24}$$

There is a similar condition between the velocity and pressure fields in the Schwarzschild case [17]. Such an integral relation can be compared to changes in the horizon area. This component of the Einstein equation also determines a scalar function involving $g_{\rho i}^{(\alpha^2)}, g_{ij}^{(\alpha^2)}$.

6.2.4 Deformations of the Fluid

A natural question to ask about the fluid is whether one can deform it. In this section, we discuss two simple examples of possible deformations of the fluid.

The first is given by adding a small non-rotating black hole of mass M at the origin of the static patch. This changes the $-g_{tt} = g^{rr}$ components of the metric (6.1.3) to $V(r) = 1 - (r/\ell)^2 - 2M/r$. For positive values of M, adding the black hole has the effect of pulling in the cosmological horizon and thus decreasing its size. For small $\varepsilon \equiv M/\ell$, the new position of the cosmological horizon is given by $r_{cos} = \ell(1 - \varepsilon)$ to leading order. In the analogous case of the Schwarzschild black hole, placing a mass at the center of the static patch corresponds to extracting some mass from the black hole, thus shrinking its horizon. The mass deformation we have described preserves the spherical symmetry of the background and thus the near horizon dynamics are expected to be governed by the Navier-Stokes equation on a sphere.

A slightly more involved deformation corresponds to placing a small rotating mass at the origin of the static patch. This will cause the cosmological horizon itself to rotate. The function determining the positions of the horizons is now given by:

$$V(r) = \left(1 + (a/r)^2\right) \left(1 - (r/\ell)^2\right) - 2M/r . \tag{6.2.25}$$

As with the mass term, adding angular momentum shrinks the cosmological horizon. To lowest order in small ε and small $\upsilon \equiv (a/\ell)^2$ we find that:

$$r_{cos} = \ell \left(1 - \varepsilon(1 - \upsilon) - \frac{3}{2}\varepsilon^2 + \mathcal{O}(\varepsilon^3, \varepsilon^2\upsilon, \varepsilon\upsilon^2, \upsilon^3) \right) \ . \tag{6.2.26}$$

The angular momentum of the space-time becomes $\mathcal{Q}_{\partial_\phi} = -aM/\left(1 + (a/\ell)^2\right)$. It should be noted that a finite deformation with angular momentum will also deform the sphere into a spheroidal surface. Thus we lose spherical symmetry and it might be possible that the near horizon dynamics is no longer governed by the Navier-Stokes equation on the round metric of S^2.

6.3 Pushing the Timelike Surface

So far we have analyzed the behavior near the cosmological horizon. Another timelike surface of interest in the static patch is given by the observer's worldline at $r = 0$.[9] Returning to the analysis of linearized gravity, if we impose Dirichlet boundary conditions leaving the worldline unperturbed for purely outgoing modes, we obtain another set of discrete modes known as quasinormal modes (see for example [76]). In the original static patch coordinates (6.1.3) these are given (for the vector modes) by:

$$\omega_n \ell = -i\left(n + l + 1\right) \ , \quad n = 0, 1, 2, \ldots \tag{6.3.27}$$

Due to the fact that $l \geq 1$, a gapless mode is absent in the above spectrum of quasinormal modes. This is in contrast to the fluid modes (6.2.18) which have $\omega_f = 0$ at $l = 1$. The

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[9]Due to its resemblance with the boundary of AdS in the presence of an eternal black hole, recent work has emphasized the potential importance of the worldline as a candidate for the ultraviolet (holographic) description of the static patch [48, 77]. Although such a holographic duality is far from clear, one expects that the infrared behavior must give rise to the Navier-Stokes equation described in the former section.

gapless mode is absent due to the fact that the $\omega = 0$, $l = 1$ perturbation diverges on the worldline, as shown in appendix D.2. It reappears in the spectrum as soon as we 'puff up' the thickness of the worldline.

6.3.1 'Flowing' the Dispersion Relations

Our aim is to study the behavior of perturbative data on constant r surfaces as we push them from the horizon toward the worldline. There is a clear distinction between the lowest $n = 0$ quasinormal modes (6.3.27) and the fluid modes (6.2.18). Given a constant r slice at some position $r = r_c$ we impose Dirichlet boundary conditions leaving the induced metric on the $r = r_c$ unchanged. This constant r surface is directly analogous to the timelike hypersurface at $\rho = 1$ considered above. Furthermore, we require that the modes are purely outgoing. As before, these two conditions will only be satisfied for a discrete set of modes, but the dispersion relation will now depend on the dimensionless parameter $x = r_c/\ell$. For the surface near the horizon we have $x \to 1$ and as we approach the worldline we have $x \to 0$. For general x, the problem cannot be approached analytically and we must resort to numerics.

The dispersion relation corresponds to the pole structure of the Green's function of the vector modes on the particular cutoff surface. Thus, naturally, a flow of the dispersion relation corresponds to a flow of the Green's function itself. For an incompressible fluid on an S^2 described by (6.0.1) we can readily obtain the tree level retarded Green's function of v^i (see for example [109]).

To perform the analysis, it is in fact more convenient to use the (τ, ρ)-coordinate system introduced in (6.1.6). To study different timelike hypersurfaces we fix $\rho = 1$ and tune α from

the horizon at 0 to the worldline at $1/2$. We must then study for what values of (complex) ω the purely outgoing solutions $\delta g_{i\tau}^{out}$ in (6.2.16) vanish at the $\rho = 1$ hypersurface. It is relatively straight forward to compute the corrections to the dispersion relation perturbatively in α. For instance, to linear order in α we find:[10]

$$\nu = 1 + \frac{\alpha}{2}\left(5 + 3k_V^2\right) \ . \tag{6.3.28}$$

Expression (6.3.28) is only reliable for $l^2 \lesssim 2/3\alpha$.

6.3.2 Numerical Results
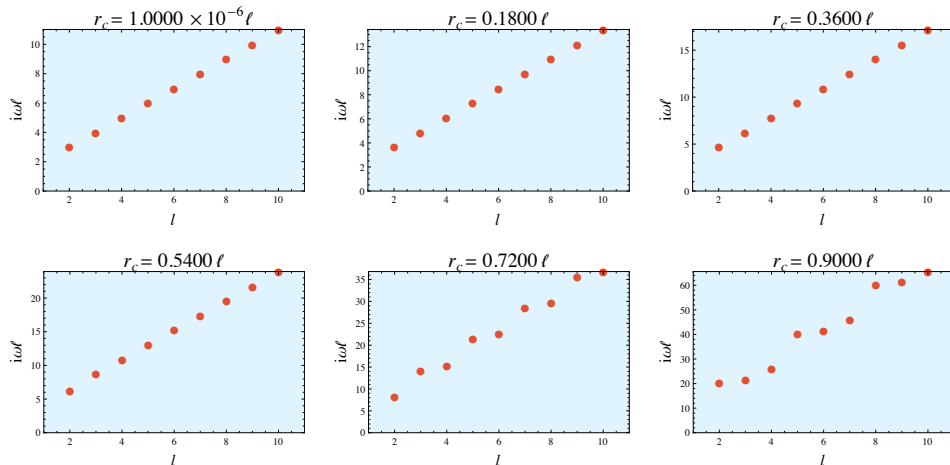
Fig. 6.4: Flow of frequency spectrum $i\omega\ell$ vs l as we move away from $r_c \approx 0$ toward the cosmological horizon.

Since we impose Dirichlet boundary conditions at $\rho = 1$, α parametrizes the location of our cutoff surface r_c with respect to the cosmological horizon. The relation is given by

$$\alpha = \frac{\ell - r_c}{2\ell} = \frac{1 - x}{2} \ . \tag{6.3.29}$$

---

[10]It is amusing to note that such a correction could be obtained by adding a suitable forcing term to our incompressible Navier-Stokes equation [109].

As we move r_c away from the cosmological horizon, we expect to deviate from our quadratic dispersion relation (6.2.18). Generically when searching for zeros of $\delta g_{i\tau}^{out}(\rho = 1)$ in (6.2.16) for arbitrary but fixed r_c and l, one runs the risk of finding any one of a tower of such zeros (see (6.3.27) for example).
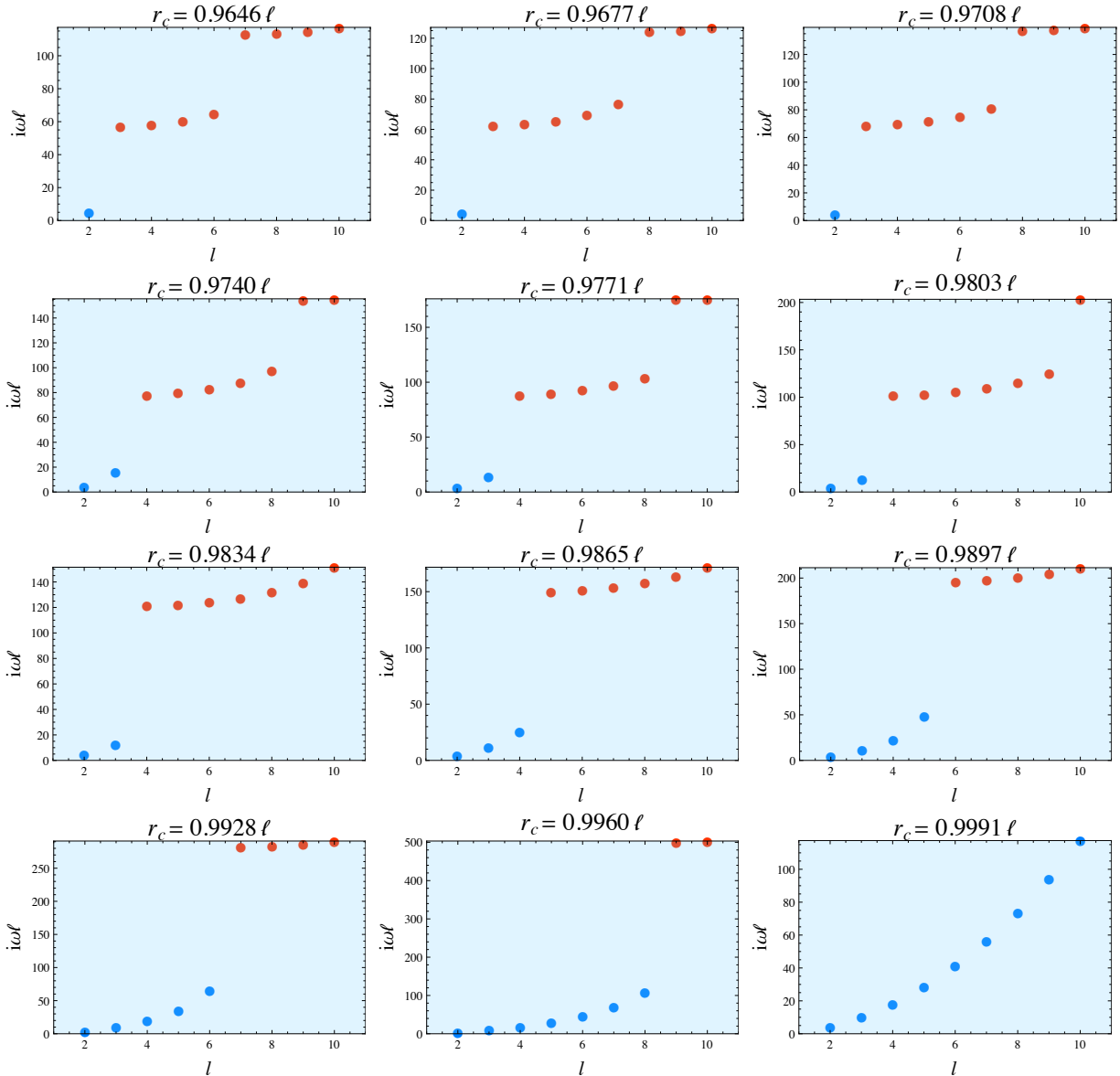
Fig. 6.5: Flow of frequency spectrum $i\omega\ell$ vs l as we approach $r_c = \ell$. Points obeying the fluid dispersion relation are represented in blue.

In order to make our analysis clear, we perform a search for the lowest lying zeros at a given r_c and l and do not present the rest of the tower. In the plots in figures 6.4 and 6.5 we have not searched for the $l = 1$ mode as this is where $\omega = 0$ and therefore our solutions φ_v^{out} and φ_v^{in} degenerate. We cover this case in appendix D.2. In what follows we will only examine the case of pure (negative) imaginary ω given that both the quasinormal modes (6.3.27) and the fluid modes (6.2.18) obey this property. It would be interesting to extend the analysis to general ω in the lower complex plane.

We now describe how the spectrum behaves as we approach $r_c = \ell$. As a reference, we also present the results for r_c close to the pole $r_c = 0$ in figure 6.4 where the linear dispersion relation is apparent. As r_c is increased, we start to observe a deviation from the linear behavior and the poles start to cluster into staircase-like behavior. For r_c close to the horizon, starting from the plot at the top left corner of figure 6.5, we note that there are (at least) three distinct sets of modes separated by large gaps. The $l = 2$ mode lies on the fluid dispersion relation line (meaning that it satisfies the dispersion relation given by (6.2.18) with quartic corrections as in (6.3.28)), whereas the rest do not. As we move r_c closer to the horizon, we see that the non-fluid modes get pushed higher and in the fourth plot, the $l = 3$ mode is plucked down to the fluid line. This happens once again for the $l = 4$ mode near $r_c = 0.9865\ell$ whereas the non-fluid modes keep getting pulled higher. The reason for these jumps is that the lowest lying zeroes of $\delta g_{i\tau}^{out}$ are modified discontinuously as a function of r_c as is visually depicted in figure D.1 of appendix D.3. Finally, we find that arbitrarily close to the horizon the dispersion relation lies entirely on the fluid dispersion relation (6.2.18) computed analytically.

As a check of our analysis in the previous section, in figure 6.6 we show that the correction
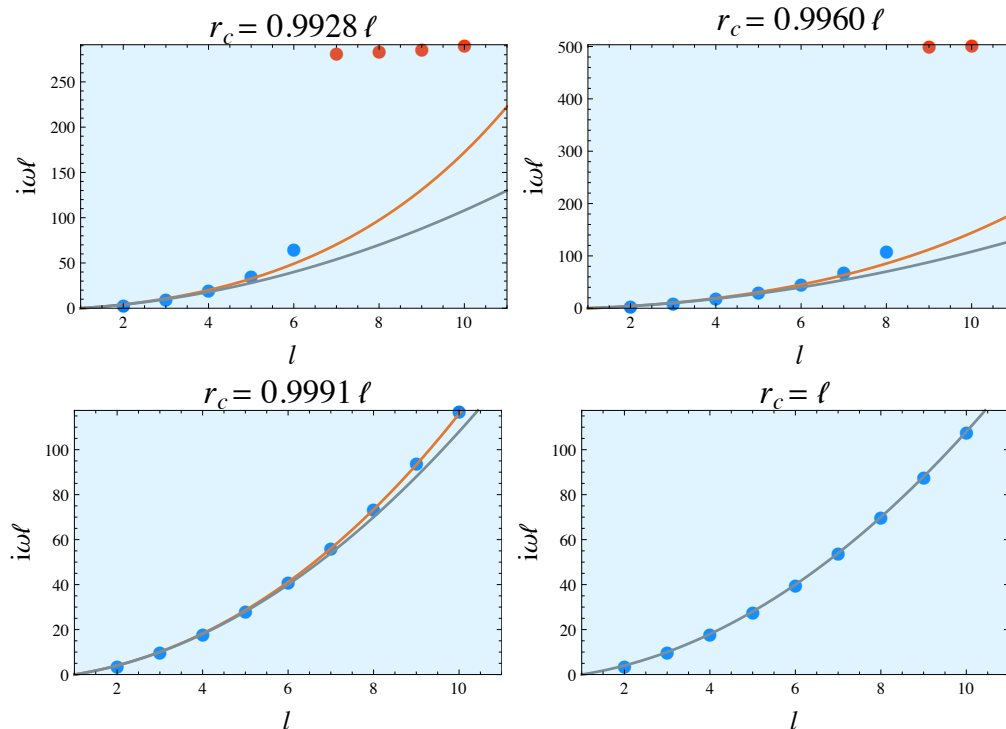
Fig. 6.6: Flow of fluid mode frequencies with fit lines. The gray line is given by $i\omega\ell = (l(l+1)-2)$ while the orange line is given by $i\omega\ell = \nu(\alpha)\,(l(l+1)-2)$ with $\nu(\alpha)$ given by (6.3.28). Note that the orange line fits the data better away from $r_c = \ell$ and the gray and orange lines coincide at the horizon, as expected.

at $\mathcal{O}(\alpha)$ of the fluid viscosity gives the correct behavior for the fluid mode frequencies for small but nonzero α. Notice that the orange line in figure 6.6 fits the data better than the gray line for r_c parametrically displaced from the cosmological horizon. The orange line is precisely the dispersion relation corrected to $\mathcal{O}(\alpha)$ in our analysis of linear perturbations of the background metric of the static patch. The gray line does not include $\mathcal{O}(\alpha)$ corrections.

It is interesting to note that all observed spectra of $i\omega\ell$ are monotonically increasing functions of l. This is a feature that we may expect by continuity away from the modes analytically computed at $r_c = \ell$, but holds as we push r_c a finite amount from the surface, even when jumps occur.

6.4 Incompressible Fluids on Spacelike Slices?

So far we have discussed several aspects of the static patch, which is the region accessible to a single observer. We would now like to briefly discuss some aspects of the future diamond. After all, future infinity is clearly a viable candidate location for the non-perturbative definition of an asymptotically de Sitter universe. Ordinarily, we would not associate the dual theory on $\mathcal{I}^+$ with the static patch observer. On the other hand, if we impose boundary conditions where there is no incoming flux from the Northern diamond, such that all data reaching $\mathcal{I}^+$ is coming from a single static patch one might conceive of such a relation.[11] In this section we will make some simple mathematical observations about the behavior of metric deformations on a spacelike slice just outside the cosmological horizon.

6.4.1 Linearized Analysis

Our aim is to solve the linearized equations in the future diamond, imposing Dirichlet boundary conditions on a constant r surface arbitrarily close to the cosmological horizon. In order to choose an appropriate near horizon coordinate system we begin with the future diamond, described by (6.1.3) with $r \in [\ell, \infty]$. As before, we introduce the following coordinate transformation:

$$\frac{t}{\ell} = \frac{1}{2\alpha}\tilde{\tau} - \frac{1}{2}\log\left(-\tilde{\rho}\left(1 - \alpha\tilde{\rho}\right)^{-1}\right) \ , \quad r = \ell(1 - 2\alpha\tilde{\rho}) \ . \tag{6.4.30}$$

In what follows we will drop the tildes. The metric becomes:

$$\frac{ds^2}{\ell^2} = \left(-\frac{\rho}{\alpha} + \rho^2\right)d\tau^2 + 2d\tau d\rho + (1 - 2\rho\alpha)^2 d\Omega_2^2 \ . \tag{6.4.31}$$

---

[11]Such 'holographic projections' of the static patch observer were also considered in the rotating Nariai geometry [110, 31, 67]. In that case a near cosmological horizon limit allowed for an isolated space-time whose (spacelike) boundary is of the same type as the spacelike slice we are discussing here.

Taking $\alpha \to 0$ with $\alpha > 0$, constant ρ slices for $\rho < 0$ now correspond to spacelike slices just above the cosmological horizon. This slice receives data from the future horizons of *both* the Southern and Northern patches. If we are to isolate one of the observers, say the Southern observer, we must impose that the incoming flux from the Northern static patch vanishes, as shown in figure 6.7.
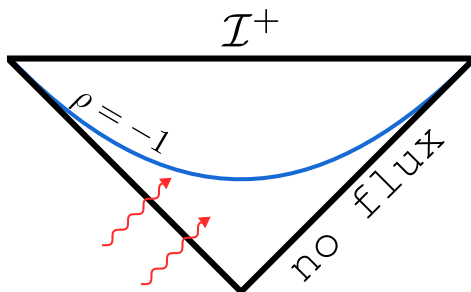
Fig. 6.7: Our boundary conditions for the linearized modes are such that the induced metric on the $\rho = -1$ slice is unchanged and there is no incoming flux from the Northern static patch.

The vector mode with vanishing flux from the Northern patch is generated by a master field $\Phi_v^S = e^{-2i\alpha\omega t(\tau,\rho)/\ell}\varphi_v^S$, with:

$$\varphi_v^S = (-\rho)^{-i\alpha\omega}{}_2F_1\left[a_1, b_1; c_1; \alpha\rho(-1+2\alpha\rho)^{-1}\right](1-\alpha\rho)^{-i\alpha\omega}(1-2\alpha\rho)^{2i\alpha\omega}, \qquad (6.4.32)$$

where

$$a_1 = -l - 2i\alpha\omega, \quad b_1 = 1 + l - 2i\alpha\omega, \quad c_1 = 1 - 2i\alpha\omega. \qquad (6.4.33)$$

The linearized metric deformation is given by:

$$\delta g_{i\tau}^S = 2\mathcal{V}_i \times e^{-i\omega\tau}(-\rho)^{i\alpha\omega+1}(1-\alpha\rho)^{-i\alpha\omega+1}\left(1 - \frac{(1-2\alpha\rho)}{2\alpha}\partial_\rho\right)\varphi_v^S, \qquad (6.4.34)$$

$$\delta g_{i\rho}^S = -2\alpha\mathcal{V}_i \times e^{-i\omega\tau}\left(\frac{1-\alpha\rho}{-\rho}\right)^{-i\alpha\omega}\left[\left(1 - \frac{i\omega(1-2\alpha\rho)}{2\rho(1-\alpha\rho)}\right) - \frac{(1-2\alpha\rho)}{2\alpha}\partial_\rho\right]\varphi_v^S. \qquad (6.4.35)$$

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Demanding that the above vector modes vanish at the spacelike $\rho = -1$ hypersurface in the limit where $\alpha \to 0$ gives the discrete relation:

$$\omega_{sf} = +i \left(l(l+1) - 2 \right) \ , \quad l = 1, 2, \ldots \tag{6.4.36}$$

This is precisely the same dispersion relation that was found earlier in (6.2.18) for the Lorentzian hypersurface but with $\nu_{sf} = -1$.[12] To linear order in α we find:

$$\nu_{sf} = -1 + \frac{\alpha}{2} \left(5 + 3k_V^2 \right) \ . \tag{6.4.37}$$

It is not hard to show perturbatively in α that $\nu_{sf}(\alpha) = -\nu(-\alpha)$. Upon evaluating the linearized metric modes on the 'spacelike fluid' frequencies (6.4.36) we find that the vector modes are regular as we approach the $\rho \to 0$ horizon.

6.4.2   Non-linear Analysis

We would now like to perform a non-linear analysis of the metric deformations in an α expansion, again in the context of the spacelike slices. The cutoff surface will now be at $\rho = -1$. Non-linear deformations analogous to those presented for timelike slicings are:

$$
\begin{aligned}
\frac{ds^2}{\ell^2} =& -\frac{\rho}{\alpha} d\tau^2 \\
& + \rho^2 d\tau^2 + 2d\tau d\rho + d\Omega_2^2 + (1+\rho) \left[v^2 d\tau^2 - 2v_i d\tau dx^i \right] + 2\rho P d\tau^2 \\
& - \alpha \Big[(4\rho + 2P) \, d\Omega_2^2 - (1+\rho) v_i v_j dx^i dx^j \\
& + \left(\rho^2 - 1 \right) \left(\nabla^2 v_i + R_{ji} v^j \right) d\tau dx^i + 2v_i d\rho dx^i - \left(v^2 + 2P \right) d\tau d\rho + 2 \left(1 + \rho \right) \phi_i^{(\alpha)} d\tau dx^i \Big] \\
& + \alpha^2 \left(4\rho^2 d\Omega_2^2 + 2g_{\rho i}^{(\alpha^2)} dx^i d\rho + g_{ij}^{(\alpha^2)} dx^i dx^j \right) + \ldots
\end{aligned}
\tag{6.4.38}
$$

---

[12]We note that a time-reversal $\tau \to -\tau$ transformation leads to $v^i \to -v^i$ and thus introduces a sign change to the viscosity term ν_{sf} in the non-linear Navier-Stokes equation (6.0.1). Thus, one can interpret a negative viscosity fluid as a time-reversed version of a positive viscosity fluid. We thank R. Loganayagam for pointing this out to us.

On the spacelike slice at $\rho = -1$, the internal geometry is conformally equal to

$$ds^2_{3d} = \left(\frac{1}{\alpha} + 1 \right) d\tau^2 + (1 + 2\alpha)^2 \, d\Omega^2_2 \, , \tag{6.4.39}$$

with a conformal factor

$$1 - 2\alpha P + \mathcal{O}(\alpha^2) \, . \tag{6.4.40}$$

Similarly to the timelike case, for this metric to solve the Einstein equation through $\mathcal{O}(\alpha^0)$, (v_i, P) are required to solve the incompressible Navier-Stokes equation:

$$\partial_\tau v^i + v_j \nabla^j_{S^2} v^i + \nabla^i_{S^2} P - \nu_{sf} \left(\nabla^2_{S^2} v^i + R^i_j v^j \right) = 0 \tag{6.4.41}$$

with

$$\nu_{sf} = -1 \, . \tag{6.4.42}$$

Note again that the viscosity ν_{sf} has changed sign from the fluid on the timelike slices. By integrating the $\mathcal{G}_{\tau\tau} = 0$ condition at $\mathcal{O}(\alpha^0)$ over the sphere, we again find the constraint that

$$\partial_\tau \left(\int v^2 d\Omega_2 \right) = -2\partial_\tau \left(\int P d\Omega_2 \right) \, . \tag{6.4.43}$$

Hence the structure of the Einstein equation with positive cosmological constant on the timelike slice near the cosmological horizon with the specified boundary conditions is closely mimicked in this spacelike context.

6.4.3   Pushing the Spacelike Slice to $\mathcal{I}^+$

We now wish to push the spacelike slice all the way to $\mathcal{I}^+$ and study the constraints on ω. We impose fast-falling Dirichlet boundary conditions at $\mathcal{I}^+$ and no incoming flux from the Northern patch.

Reverting to static patch (r, t)-coordinates, the solutions $\Phi_v(r, t) = e^{-i\omega t}\varphi_v(r)$ analogous to (6.2.12) and (6.2.13) near $\mathcal{I}^+$ were computed in [75] and are given by:

$$\varphi_v^- = \left(\frac{r^2}{\ell^2} - 1\right)^{-i\omega\ell/2} \left(\frac{r}{\ell}\right)^{i\omega} {}_2F_1\left(a_1; b_1; c_1; \frac{\ell^2}{r^2}\right) , \tag{6.4.44}$$

$$\varphi_v^+ = \left(\frac{r^2}{\ell^2} - 1\right)^{-i\omega\ell/2} \left(\frac{r}{\ell}\right)^{-1+i\omega\ell} {}_2F_1\left(a_2; b_2; c_2; \frac{\ell^2}{r^2}\right) , \tag{6.4.45}$$

with

$$a_1 = \frac{1}{2}(1 + l - i\omega\ell) , \quad b_1 = \frac{1}{2}(-l - i\omega\ell) , \quad c_1 = \frac{1}{2} ; \tag{6.4.46}$$

$$a_2 = \frac{1}{2}(1 - l - i\omega\ell) , \quad b_2 = \frac{1}{2}(2 + l - i\omega\ell) , \quad c_2 = \frac{3}{2} . \tag{6.4.47}$$

As we approach $\mathcal{I}^+$ in the limit $r \to \infty$ we find $\varphi_v^- \sim 1$ and $\varphi_v^+ \sim \ell/r$. Note that $(c_2 - a_2 - b_2) = i\omega\ell$. In order to eliminate deformations of the conformal metric at $\mathcal{I}^+$ we switch off the slow falling mode φ_v^-.

Our task is to eliminate the incoming Northern flux upon turning on φ_v^+. It is not hard to see that this will require $(c_2 - a_2 - b_2) = i\omega\ell$ to be an integer. We must further ensure that metric fluctuations are analytic for $i\omega\ell \in \mathbb{Z}^+$. To achieve this, we must eliminate the logarithmic term in the hypergeometric identity (D.4.10). This implies that either

$$a_2 = -n_1 \quad \text{or} \quad b_2 = -n_2 , \quad n_i = 0, 1, 2, \ldots \tag{6.4.48}$$

It turns out that of the two possibilities, only the second one is sufficient to eliminate the Northern incoming flux. In the first case, we have to impose a further inequality $n_1 \geq l$, whose origin is explained in appendix D.4. Hence, defining $n_1 \equiv l + \tilde{n}_1$, $\tilde{n}_1 = 0, 1, 2, \ldots$ and imposing no further condition on the integer n_2, we obtain the following conditions:

$$\omega_n^{\mathcal{I}^+}\ell = -i(2\tilde{n}_1 + 1 + l) , \quad \tilde{n}_1 = 0, 1, 2, \ldots \tag{6.4.49}$$

$$\omega_n^{\mathcal{I}^+}\ell = -i(2n_2 + 2 + l) , \quad n_2 = 0, 1, 2, \ldots \tag{6.4.50}$$

which combines into one single tower of modes

$$\omega_n^{\mathcal{I}^+}\ell = -i(n+l+1) , \quad n = 0,1,2,\ldots \tag{6.4.51}$$

Curiously, and perhaps interestingly, this is exactly the same set of modes as the quasinormal mode spectrum (6.3.27) in the Southern patch.

6.4.4 Topological Black Holes in AdS$_4$

In fact, the above calculation is mathematically equivalent to the computation of quasinormal modes for the massless topological black hole in AdS$_4$ [111, 112, 113] (see also [114]) upon continuing $k_V^2 \to -k_V^2$. This is due to the fact that the metric of the massless topological AdS$_4$:

$$ds^2 = -\left(-1 + \left(\frac{\tilde{r}}{\ell_{AdS}}\right)^2\right)d\tilde{t}^2 + \left(-1 + \left(\frac{\tilde{r}}{\ell_{AdS}}\right)^2\right)^{-1}d\tilde{r}^2 + \tilde{r}^2 d\mathcal{H}_2^2 \tag{6.4.52}$$

is related to the static patch metric by an analytic continuation. The two-dimensional space: $d\mathcal{H}_2^2 = (d\xi^2 + \sinh^2\xi d\tilde{\phi}^2)$ is the standard metric on the hyperbolic two-manifold. The analytic continuation from the static patch metric (6.1.3) to the above metric is given by:

$$\ell \to i\ell_{AdS} , \quad t \to i\tilde{t} , \quad r \to i\tilde{r} , \quad \theta \to i\xi , \quad \phi \to \tilde{\phi} . \tag{6.4.53}$$

An observer in the massless topological AdS$_4$ geometry observes a Hawking temperature given by $T = 1/2\pi\ell_{AdS}$. If one considers compact quotient of $\mathcal{H}_2$, there is a finite entropy proportional to $(\ell_{AdS}/\ell_P)^2$ associated with the horizon at $\tilde{r} = \ell_{AdS}$. We also expect such mathematical similarities to hold between the boundary correlators near the boundary of topological AdS$_4$ black holes and those near $\mathcal{I}^+$ (using the boundary conditions discussed above). It is also interesting to note that at the non-linear level one can add negative

energy to (6.4.52) and create spherically symmetric asymptotically AdS_4 topological black holes. This occurs up to a critical negative mass, for which one finds a solution interpolating between AdS_4 near the boundary and $\text{AdS}_2 \times \mathcal{H}^2$ [111]. Similarly, adding sufficient mass to the static patch leads to the Nariai solution which interpolates between dS_4 near $\mathcal{I}^+$ and $\text{dS}_2 \times S^2$. We hope to further uncover this map in future work, as it may provide insight into the nature of the static patch observer.

Appendix A

The Symplectic Form and Covariant Phase Space Charges

A.1 The Symplectic Form

The symplectic form for the Einstein-Hilbert Lagrangian is given by[1]

$$\omega^{\mu}_{EH}\left[\delta_1 g, \delta_2 g\right] = -P^{\mu\nu\beta\gamma\epsilon\zeta}\left(\delta_2 g_{\beta\gamma}\nabla_{\nu}\delta_1 g_{\epsilon\zeta} - (1 \leftrightarrow 2)\right) \ , \tag{A.1.1}$$

where the tensor density $P^{\mu\nu\beta\gamma\epsilon\zeta}$ is given by:

$$P^{\mu\nu\alpha\beta\gamma\delta} = \frac{\partial}{\partial g_{\gamma\delta,\alpha\beta}}\frac{\delta\mathcal{L}_{EH}}{\delta g_{\mu\nu}} \ , \tag{A.1.2}$$

where $\mathcal{L}_{EH} \equiv \sqrt{-g}R[g]/16\pi G$. More explicitly, $P^{\mu\nu\alpha\beta\gamma\delta}$ is given by:

$$P^{\mu\nu\alpha\beta\gamma\delta} = \frac{\sqrt{-g}}{32\pi G}\left(g^{\mu\nu}g^{\gamma(\alpha}g^{\beta)\delta} + g^{\mu(\gamma}g^{\delta)\nu}g^{\alpha\beta} + g^{\mu(\alpha}g^{\beta)\nu}g^{\gamma\delta}\right.$$
$$\left. -g^{\mu\nu}g^{\gamma\delta}g^{\alpha\beta} - g^{\mu(\gamma}g^{\delta)(\alpha}g^{\beta)\nu} - g^{\mu(\alpha}g^{\beta)(\gamma}g^{\delta)\nu}\right) \ . \tag{A.1.3}$$

---

[1]For asymptotically de Sitter spacetimes obeying the Fefferman-Graham form, one can show that the Iyer-Wald [23] and Barnich-Brandt constructions [29] of the symplectic form are equivalent.

The symplectic form for the counterterm action S_{ct} in four-dimensions can be expressed in terms of the three-dimensional symplectic form of the Einstein-Hilbert Lagrangian as follows:

$$\omega_{ct}\left[\delta_1\gamma, \delta_2\gamma\right] = \omega_{EH}^{(3)}\left[\delta_1\gamma, \delta_2\gamma\right] \; . \tag{A.1.4}$$

In the FG gauge, variations of the metric reduce to variations of the induced metric γ at the boundary and the pullback of the $(d+1)$-dimensional symplectic structure ω_{EH} to the spatial slice Σ can be expressed in terms of the d-dimensional symplectic structure $\omega_{EH}^{(d)}$ as follows:

$$\omega_{EH}\left[\delta_1 g, \delta_2 g\right] = \frac{1}{\eta}\omega_{EH}^{(d)}\left[\delta_1\gamma, \delta_2\gamma\right] \; . \tag{A.1.5}$$

Expanding $\omega_{EH}^{(d)}$ on-shell using the FG expansion (2.1.2) yields

$$\omega_{EH}^{(d)}\left[\delta_1\gamma, \delta_2\gamma\right] = \eta^{2-d}\omega_{(0)EH}\left[\delta_1 g^{(0)}, \delta_2 g^{(0)}\right] +$$

$$\eta^{4-d}\omega_{(2)EH}\left[\delta_1 g^{(0)}, \delta_2 g^{(0)}\right] + \mathcal{O}\left(\eta^{5-d}\right) \; . \tag{A.1.6}$$

A.2 Covariant Phase Space Charges

In this appendix, we provide some details on the charges. We begin with the expression for the charge

$$\delta Q_\xi \equiv \int_{\partial\Sigma} k_\xi^{dS}[\delta g] \; , \tag{A.2.7}$$

where $k_\xi^{dS}[\delta g]$ was given in (2.4.24). In direct analogy to [30], we find

$$\delta Q_\xi = -\delta \int_{\partial\Sigma} K_\xi\left[g\right] + \int_{\partial\Sigma} i_\xi\Theta_{EH}[g, \delta g] + \int_{\partial\Sigma}\omega_{ct}[\mathcal{L}_\xi g, \delta g] \; , \tag{A.2.8}$$

where $\partial\Sigma$ is a compact two-dimensional submanifold of $\mathcal{I}^+$ and

$$\Theta_{EH}\left[g, \delta g\right] \equiv \frac{\sqrt{-g}}{96\pi G}\left[g^{\mu\nu}\nabla^\beta\delta g_{\beta\nu} - g^{\alpha\beta}\nabla^\mu\delta g_{\alpha\beta}\right]\epsilon_{\mu\lambda\rho\sigma}dx^\lambda dx^\rho dx^\sigma \; . \tag{A.2.9}$$

The $K_\xi[g]$ term is the usual Komar term [115]:

$$K_\xi[g] \equiv -\frac{\sqrt{-g}}{64\pi G} \left[\nabla^\mu \xi^\nu - \nabla^\nu \xi^\mu \right] \epsilon_{\mu\nu\beta\gamma} dx^\beta dx^\gamma \ . \qquad \text{(A.2.10)}$$

The counter terms are related to Θ_{EH} by the following condition:

$$\Theta_{EH}[g, \delta g]|_{\mathcal{I}^+} = \delta L_{GH} + \delta \gamma_{ij} \frac{\delta L_{ct}}{\delta \gamma_{ij}} - \tfrac{1}{2}\sqrt{g^{(0)}} T_{(0)}^{ij} \delta g_{ij}^{(0)} d^3 x \ . \qquad \text{(A.2.11)}$$

We further define the pre-symplectic form $\Theta_{ct}[\gamma, \delta\gamma]$ which is related to the counterterm symplectic structure ω_{ct} and counterterm Lagrangian as:

$$d\Theta_{ct} \equiv \delta L_{ct} - \delta \gamma_{ij} \frac{\delta L_{ct}}{\delta \gamma_{ij}} \ , \quad \omega_{ct}[\delta_1 \gamma, \delta_2 \gamma] = \delta_1 \Theta_{ct}[\gamma, \delta_2 \gamma] - \delta_2 \Theta_{ct}[\gamma, \delta_1 \gamma] \ . \qquad \text{(A.2.12)}$$

Combining the above, we obtain

$$\delta Q_\xi = \delta \int_{\delta\Sigma} \left(-K_\xi + i_\xi L_{GH} + i_\xi L_{ct} - \Theta_{ct}[\gamma, \delta_\xi \gamma] \right) - \int_{\partial\Sigma} i_\xi \Theta_{(0)}[g^{(0)}, \delta g^{(0)}] \ . \qquad \text{(A.2.13)}$$

It then follows from a straightforward calculation that (A.2.13) leads to

$$\delta Q_\xi = \delta Q_{BY} - \int_{\partial\Sigma} i_\xi \Theta_{(0)}[g^{(0)}, \delta g^{(0)}] \ . \qquad \text{(A.2.14)}$$

Appendix B

Graviton in Global Coordinates

We consider the global patch of dS$_4$:

$$\frac{ds^2}{\ell^2} = -d\tau^2 + \cosh^2 \tau d\Omega_3^2 \ , \tag{B.0.1}$$

covering the full space. In global conformal coordinates one has:

$$\frac{ds^2}{\ell^2} = \frac{1}{\cos^2 T} \left[-dT^2 + d\Omega_3^2 \right] \ . \tag{B.0.2}$$

We take the parametrization of the three-sphere to be

$$d\Omega_3^2 = \frac{1}{(1-r^2)} dr^2 + r^2 d\Omega_2^2 \ , \quad r \in [0,1] \ . \tag{B.0.3}$$

and defined $\cos T \equiv (\cosh \tau)^{-1}$. Gravitational perturbations about this background has been analysed in [116]. In their notation, $a(T) \equiv (\cos T)^{-1}$.

We focus on the tensor harmonics $Y_{(T)ij}$ of the three-sphere, obeying the transverse-traceless condition. These give rise to 2 independent degrees of freedom. The equation to be solved is given by [116]:

$$\delta R_{ij} - \frac{3}{\ell^2} \delta g_{ij} = 0 \ . \tag{B.0.4}$$

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Parametrizing the perturbation as $\delta g_{ij} = 2a(T)^2 H(T) Y_{(T)ij}$, the linearized Einstein's equation becomes:

$$0 \;=\; H'' + 2\Omega H' + 2\Omega' H + 4\Omega^2 H + \left(k_T^2 + 6\right) H - 6a^2 H \tag{B.0.5}$$

$$\Rightarrow 0 \;=\; H'' + 2\tan T H' + \left(2 + k_T^2\right) H \tag{B.0.6}$$

with $\Omega(T) \equiv a'(T)/a(T)$ and $k_T^2 = l\,(l+2) - 2$, $l = 1, 2, \dots$

The two independent solutions are then given by:

$$H^{(0)} \;=\; F\left(-\tfrac{1}{2} - K, -\tfrac{1}{2} + K; -\frac{1}{2}; \frac{1}{\cosh^2 \tau}\right) \tag{B.0.7}$$

$$H^{(3)} \;=\; \frac{1}{\cosh^3 \tau} F\left(1 - K, 1 + K; \frac{5}{2}; \frac{1}{\cosh^2 \tau}\right) \;, \tag{B.0.8}$$

where:

$$K \equiv \frac{\sqrt{k_T^2 + 3}}{2} \;. \tag{B.0.9}$$

Recall that $\delta g_{ij} \sim a^2 H(\tau) = (\cosh \tau)^2 H(\tau)$, we note the following Starobinskii fall-offs [33] near $\mathcal{I}^+$:

$$a(\tau)^2 H^{(0)} \;\sim\; e^{2\tau} + \dots \;, \tag{B.0.10}$$

$$a(\tau)^2 H^{(3)} \;\sim\; e^{-\tau} + \dots \;, \tag{B.0.11}$$

while at $\mathcal{I}^-$ a similar behavior is observed:

$$a(\tau)^2 H^{(0)} \;\sim\; e^{2\tau} + \dots \;, \tag{B.0.12}$$

$$a(\tau)^2 H^{(3)} \;\sim\; e^{-\tau} + \dots \tag{B.0.13}$$

The first mode is growing *both* at $\mathcal{I}^+$ and $\mathcal{I}^-$ and the second mode is decaying at *both* boundaries. This is due to the fact that these solutions only depend on τ through $\cosh \tau$ which is invariant under time reversal $\tau \to -\tau$. Hence the decay behavior near one boundary is the same as the behavior near the other boundary.

Appendix C

Killing Vectors and Massive Green's Function

C.1   dS$_4$ Killing vectors

In global coordinates (t, ψ, θ, ϕ), the dS$_4$ metric reads

$$ds^2 = -dt^2 + \cosh^2 t \, d\Omega_3^2, \quad d\Omega_3^2 \equiv d\psi^2 + \sin^2 \psi \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \tag{C.1.1}$$

with 10 Killing vectors:

$$
\begin{aligned}
L_0 &= \cos \psi \, \partial_t - \tanh t \sin \psi \, \partial_\psi, \\
M_{\mp 1} &= \pm \sin \psi \sin \theta \sin \phi \, \partial_t + (1 \pm \tanh t \cos \psi) \sin \theta \sin \phi \, \partial_\psi \\
&\quad + (\cot \psi \pm \tanh t \csc \psi)(\cos \theta \sin \phi \, \partial_\theta + \csc \theta \cos \phi \, \partial_\phi), \\
M_{\mp 2} &= \pm \sin \psi \sin \theta \cos \phi \, \partial_t + (1 \pm \tanh t \cos \psi) \sin \theta \cos \phi \, \partial_\psi \\
&\quad + (\cot \psi \pm \tanh t \csc \psi)(\cos \theta \cos \phi \, \partial_\theta - \csc \theta \sin \phi \, \partial_\phi), \\
M_{\mp 3} &= \pm \sin \psi \cos \theta \, \partial_t + (1 \pm \tanh t \cos \psi) \cos \theta \, \partial_\psi - (\cot \psi \pm \tanh t \csc \psi) \sin \theta \, \partial_\theta,
\end{aligned}
$$

$$
\begin{aligned}
J_1 &= \cos\phi\,\partial_\theta - \sin\phi\cot\theta\,\partial_\phi, \\
J_2 &= -\sin\phi\,\partial_\theta - \cos\phi\cot\theta\,\partial_\phi, \\
J_3 &= \partial_\phi.
\end{aligned}
\tag{C.1.2}
$$

Their non-zero commutators are:

$$
\begin{aligned}
{[J_i, J_j]} &= \sum_{k=1}^{3}\epsilon_{ijk}J_k, & [J_i, M_{\pm j}] &= \sum_{k=1}^{3}\epsilon_{ijk}M_{\pm k}, \\
{[L_0, M_{\pm i}]} &= \mp M_{\pm i}, & [M_{+i}, M_{-j}] &= 2L_0\delta_{ij} + 2\sum_{k=1}^{3}\epsilon_{ijk}J_k.
\end{aligned}
\tag{C.1.3}
$$

As expected, these are the relations which define the $\mathsf{SO}(4,1)$ algebra. The commutators on the first line indicate that the J_i generate an $\mathsf{SO}(3)$ subalgebra, under which the M_{+i} and the M_{-i} transform as vectors. The second line implies that for each $i \in \{1, 2, 3\}$, the Killing vectors $M_{\pm i}$ and L_0 form an $\mathsf{SO}(2,1)$ subalgebra satisfying (not summing over i)

$$
[M_{+i}, M_{-i}] = 2L_0, \qquad [L_0, M_{\pm i}] = \mp M_{\pm i}.
\tag{C.1.4}
$$

The scalar Laplacian is a Casimir operator. It reads:

$$
\ell^2 \nabla^2 = -L_0(L_0 - 3) + \sum_{i=1}^{3} M_{-i}M_{+i} + J^2.
\tag{C.1.5}
$$

The convention is that $J^2 = -L(L+1)$ on the spherical harmonics Y_{Lj}. The conformal Killing vectors of the S^3 are given by the restriction of dS$_4$ Killing vectors on $\mathcal{I}^+$:

$$
\begin{aligned}
L_0 &= -\sin\psi\,\partial_\psi, \\
M_{\mp 1} &= (1 \pm \cos\psi)\sin\theta\sin\phi\,\partial_\psi + (\cot\psi \pm \csc\psi)(\cos\theta\sin\phi\,\partial_\theta + \csc\theta\cos\phi\,\partial_\phi), \\
M_{\mp 2} &= (1 \pm \cos\psi)\sin\theta\cos\phi\,\partial_\psi + (\cot\psi \pm \csc\psi)(\cos\theta\cos\phi\,\partial_\theta - \csc\theta\sin\phi\,\partial_\phi), \\
M_{\mp 3} &= (1 \pm \cos\psi)\cos\theta\,\partial_\psi - (\cot\psi \pm \csc\psi)\sin\theta\,\partial_\theta, \\
J_1 &= \cos\phi\,\partial_\theta - \sin\phi\cot\theta\,\partial_\phi,
\end{aligned}
$$

$$
J_2 = -\sin\phi\,\partial_\theta - \cos\phi\cot\theta\,\partial_\phi,
$$

$$
J_3 = \partial_\phi. \tag{C.1.6}
$$

To relate the above de Sitter generators to the embedding coordinates X defined by

$$
\eta_{\mu\nu}X^\mu X^\nu = \ell^2, \tag{C.1.7}
$$

where η has signature $(4,1)$ and the usual Lorentz generators are given by

$$
M_{\mu\nu} = X_\mu\,\partial_\nu - X_\nu\,\partial_\mu, \tag{C.1.8}
$$

with commutators

$$
[M_{\mu\nu}, M_{\alpha\beta}] = \eta_{\alpha\nu}M_{\mu\beta} - \eta_{\alpha\mu}M_{\nu\beta} - \eta_{\beta\nu}M_{\mu\alpha} + \eta_{\beta\mu}M_{\nu\alpha}, \tag{C.1.9}
$$

we have for $i,j,k \in \{1,2,3\}$

$$
L_0 = M_{40},
$$

$$
M_{\mp k} = M_{4k} \mp M_{0k},
$$

$$
J_i = -\epsilon_{ijk}M_{jk}. \tag{C.1.10}
$$

The standard Klein-Gordon adjoint acts as:

$$
M_{\mu\nu}^\dagger = -M_{\mu\nu}, \tag{C.1.11}
$$

where the adjoint is defined in the standard way as $\langle f, M^\dagger g\rangle_{KG} \equiv \langle Mf, g\rangle_{KG}$. The action of R on the Killing vectors is:

$$
L_0 \to -L_0, \qquad J_k \to J_k, \qquad M_{\pm k} \to -M_{\mp k}, \tag{C.1.12}
$$

or equivalently, for $j,k \in \{1,2,3\}$,

$$
M_{40} \to -M_{40}, \qquad M_{4k} \to -M_{4k}, \qquad M_{0k} \to M_{0k}, \qquad M_{jk} \to M_{jk}. \tag{C.1.13}
$$

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In the R-norm, if we define $M^{\dagger_R}$ as $\left\langle f, M^{\dagger_R} g \right\rangle_R \equiv \left\langle Mf, g \right\rangle_R$ then

$$M_{40}^{\dagger_R} = M_{40}, \qquad M_{4k}^{\dagger_R} = M_{4k}, \qquad M_{0k}^{\dagger_R} = -M_{0k}, \qquad M_{jk}^{\dagger_R} = -M_{jk}. \qquad \text{(C.1.14)}$$

With respect to the R-norm, the antihermitian generators are iM_{40}, iM_{4k}, M_{0k} and M_{jk}. On the other hand, we have from (C.1.9) that for $i, j, k \in \{1, 2, 3\}$, XXX while the M_{jk} obey the usual $\mathsf{SO}(3)$ algebra. Now, if we sent $M_{40} \to iM_{40}$ and $M_{4k} \to iM_{4k}$, we would get the same algebra but with $\eta_{44} \to -\eta_{44}$. This demonstrates that the insertion of R in the norm transforms $\mathsf{SO}(4,1)$ into $\mathsf{SO}(3,2)$.

C.2 Norm for spherically symmetric states

Consider the operator $L_{\mp 1} \equiv \sum_{k=1}^{3} M_{\mp k} M_{\mp k}$, which evidently satisfies $[J_k, L_{\mp 1}] = 0$. Defining $|h + n\rangle \equiv L_{+1}^n |h\rangle$, where $|h\rangle$ is the spherically symmetric highest-weight state with $L_0 |h\rangle = -h |h\rangle$ and $J^2 |h\rangle = 0$, we have

$$[L_{+1}, L_{-1}] |h + n\rangle = 4L_0 \left(2L_0^2 + 2\nabla^2 - 3 \right) |h + n\rangle. \qquad \text{(C.2.15)}$$

The Casimir is

$$\nabla^2 = -L_0(L_0 - 3) + M_{-k}M_{+k} + J^2 = -L_0(L_0 + 3) + M_{+k}M_{-k} + J^2 \qquad \text{(C.2.16)}$$

and $\nabla^2 |h + n\rangle = -h(h - 3) |h + n\rangle$. Then using

$$[L_{-1}, L_{+1}] |h + n\rangle = 4(h + 2n)(8n^2 + 8nh + 6h - 3) |h + n\rangle, \qquad \text{(C.2.17)}$$

it is straightforward to show that

$$\begin{aligned} \langle h| L_{-1}^n L_{+1}^n |h\rangle &= 4n(n + h - 1)(2n + 1)(2n + 2h - 3) \langle h| L_{-1}^{n-1} L_{+1}^{n-1} |h\rangle \\ &= \frac{\Gamma(2 + 2n)\Gamma(2h + 2n - 1)}{\Gamma(2h - 1)} \langle h|h\rangle. \end{aligned} \qquad \text{(C.2.18)}$$

C.3 Green function at the south pole

We wish to evaluate the sum (5.4.42) over the quasinormal modes for the massive case,

$$
\begin{aligned}
G(t, \Omega_{SP}; t', \Omega_{SP}) &= \left[\left(e^t - i\epsilon \right) \left(e^{-t'} - i\epsilon \right) \right]^{-h_+} \sum_n c_n^+ \left[\frac{e^{-(t-t')}}{\left(e^t - i\epsilon \right) \left(e^{-t'} - i\epsilon \right)} \right]^n \\
&+ \left[\left(e^t - i\epsilon \right) \left(e^{-t'} - i\epsilon \right) \right]^{-h_-} \sum_n c_n^- \left[\frac{e^{-(t-t')}}{\left(e^t - i\epsilon \right) \left(e^{-t'} - i\epsilon \right)} \right]^n
\end{aligned}
\tag{C.3.19}
$$

where

$$
c_n^\pm = \pm \frac{e^{-i\pi h_\pm}}{(-2\pi^2) \sin(\pi\mu)} \frac{\Gamma(\frac{3}{2} + n)\Gamma(h_\pm + n)}{\Gamma(1 + n \pm \mu)\Gamma(1 + n)}.
\tag{C.3.20}
$$

Each sum combines into a hypergeometric function with argument shifted by ϵ

$$
\begin{aligned}
&2\sin(\pi\mu) G(t, \Omega_{SP}; t', \Omega_{SP}) \\
&= \frac{e^{-i\pi h_+}}{\left[\left(e^t - i\epsilon \right) \left(e^{-t'} - i\epsilon \right) \right]^{h_+}} \frac{\Gamma\left(\frac{3}{2} \right) \Gamma(h_+)}{(-\pi^2)\Gamma(1 + \mu)} F\left[h_+, \frac{3}{2}; 1 + \mu; \frac{e^{-(t-t')}}{\left(e^t - i\epsilon \right) \left(e^{-t'} - i\epsilon \right)} \right] \\
&\quad - \frac{e^{-i\pi h_-}}{\left[\left(e^t - i\epsilon \right) \left(e^{-t'} - i\epsilon \right) \right]^{h_-}} \frac{\Gamma\left(\frac{3}{2} \right) \Gamma(h_-)}{(-\pi^2)\Gamma(1 - \mu)} F\left[h_-, \frac{3}{2}; 1 - \mu; \frac{e^{-(t-t')}}{\left(e^t - i\epsilon \right) \left(e^{-t'} - i\epsilon \right)} \right].
\end{aligned}
\tag{C.3.21}
$$

Kummer's quadratic transformation

$$
F\left[\alpha, \beta, 2\beta, \frac{4z}{(1+z)^2} \right] = (1+z)^{2\alpha} F\left[\alpha, \alpha - \beta + \frac{1}{2}, \beta + \frac{1}{2}, z^2 \right]
\tag{C.3.22}
$$

with

$$
z \equiv \frac{e^{-(t-t')/2}}{\left[\left(e^t - i\epsilon \right) \left(e^{-t'} - i\epsilon \right) \right]^{1/2}}
\tag{C.3.23}
$$

allows us to rewrite the hypergeometric functions in the more recognizable form

$$
2\sin(\pi\mu) G(t, \Omega_{SP}; t', \Omega_{SP}) = e^{-i\pi h_+} R^{-h_+} H_+(P_\epsilon) - e^{-i\pi h_-} R^{-h_-} H_-(P_\epsilon)
\tag{C.3.24}
$$

where $H_\pm$ are analytical continuations of the Green functions $G_\pm$ we defined in (5.5.53) to the region $P > 1$:

$$
H_\pm(x; x') = G_E(x; x') - e^{-i\pi h_\mp} G_E(x; x'_A).
\tag{C.3.25}
$$

They are explicitly given by

$$H_\pm(P_\epsilon) = \frac{\Gamma(\mp\mu)\Gamma(h_\pm)\sin(\pm\pi\mu)}{2^{1+2h_\pm}\pi^{5/2}} \left(\frac{2}{1+P}\right)^{h_\pm} F\left[h_\pm, h_\pm - 1; 2(h_\pm - 1), \frac{2}{1+P_\epsilon}\right] \quad (C.3.26)$$

with the argument

$$\frac{2}{1+P_\epsilon} \equiv \frac{4z}{(1+z)^2} \quad (C.3.27)$$

while R is some correction factor

$$R \equiv z\left(e^t - i\epsilon\right)\left(e^{-t'} - i\epsilon\right) = e^{-(t-t')/2}\left(e^t - i\epsilon\right)^{1/2}\left(e^{-t'} - i\epsilon\right)^{1/2}. \quad (C.3.28)$$

Note that when $\epsilon \to 0$, this correction $R \to 1$ and $z \to e^{-(t-t')}$.

Also, observe that (C.3.27) implies that

$$P_\epsilon = \frac{1}{2}\left(z + \frac{1}{z}\right) = \frac{1}{2}\left\{\frac{e^{-(t-t')/2}}{[(e^t - i\epsilon)(e^{-t'} - i\epsilon)]^{1/2}} + \frac{[(e^t - i\epsilon)(e^{-t'} - i\epsilon)]^{1/2}}{e^{-(t-t')/2}}\right\}. \quad (C.3.29)$$

Away from the singularity at $t = t'$, we can set ϵ to zero so that

$$G(t, \Omega_{SP}; t', \Omega_{SP}) = \frac{e^{-i\pi h_+} H_+(P) - e^{-i\pi h_-} H_-(P)}{2\sin(\pi\mu)} = G_E(t, \Omega_{SP}; t', \Omega_{SP}). \quad (C.3.30)$$

The singularity structure for G can be also analyzed from (C.3.24). The correction factor R is regular near the singularity, while the $G_\pm(P)$ have poles when P approaches 1. Expanding (C.3.29) to first order in ϵ yields

$$P_\epsilon = \cosh(t - t') - i\epsilon\sinh(t - t')\left(\frac{e^{-t} + e^{t'}}{2}\right) = P - i\epsilon\hat{s}(x, x') \quad (C.3.31)$$

with

$$\hat{s}(x, x') \equiv \sinh(t - t')\left(\frac{e^{-t} + e^{t'}}{2}\right). \quad (C.3.32)$$

The singularity structure therefore is the same as in definition (5.5.50) for the Euclidean Green function up to redefinition of ϵ by some positive function.

Appendix D

dS Fluid/Gravity

D.1 Scalar Perturbations

Gravitational perturbations in static dS_4 consist of a scalar and vector mode of the $SO(3)$ of the two-sphere. There are no tensor modes in four dimensions. Scalar harmonic perturbations can be reduced to a single Ishibashi-Kodama master field [105, 106], which obeys the same effective equation as that of the vector perturbations except that the angular number l begins at $l = 0$ (instead of $l = 1$ in the vector case).

An incompressible fluid requires a divergenceless velocity field v_i. The scalar harmonic allows only the possibility $v_i \propto \mathcal{S}_i \equiv -\nabla_i \mathcal{S}$. $\mathcal{S}$ is the scalar harmonic on the sphere which satisfies:

$$\left(\nabla^2_{S^2} + k_S^2\right)\mathcal{S} = 0 \ , \quad k_S^2 = l(l+1) \ , \quad l = 0, 1, 2, \ldots \tag{D.1.1}$$

Imposing incompressibility leads to

$$\nabla^i_{S^2} v_i \propto \nabla^2_{S^2}\mathcal{S} = -k_S^2 \mathcal{S} \tag{D.1.2}$$

which vanishes for $k_S = 0$, i.e. the spherically symmetric mode $l = 0$. In this case $v_i = 0$ and we are left with a trivial fluid.

Thus, in the case of an incompressible fluid, the scalar mode only consists of trivial fluids. It would be interesting to investigate the case of a compressible fluid which allows for sound modes.[1]

D.2   The $l = 1$ Vector Perturbation

Gravitational vector perturbations with $l = 1$ differ from $l > 2$. We follow the discussion in [105, 106]. In addition to equation 6.2.9, spherical vector harmonics satisfy the following equation:

$$\left(\nabla_{S^2}^2 + k_V^2 - 3\right) \mathcal{V}_{ij} = 0 \quad , \quad \mathcal{V}_{ij} \equiv -\frac{1}{2k_V} \left[D_i \mathcal{V}_j + D_j \mathcal{V}_i\right] \ , \tag{D.2.3}$$

where $D_i \equiv (\nabla_{S^2})_i$. For $k_V^2 - 3 \leq 0$ it can be shown that $\mathcal{V}_{ij}$ vanishes and therefore, $\mathcal{V}^i$ must be a Killing vector on the sphere. In this case, we parametrize the perturbations as:

$$\delta g_{ai} = r f_a \mathcal{V}_i, \tag{D.2.4}$$

where $x^a = \{t, r\}$ and $x^i = \{\theta, \phi\}$. Given that $\mathcal{V}_{ij} = 0$ implies $\delta g_{ij} = 0$, we can no longer fix the gauge freedom by imposing $\delta g_{ij} = 0$. Instead, we will fix the gauge $f_r = 0$. From [106], using the only gauge invariant object F_{ab}:

$$r^{-1} F_{ab} = D_a \left(\frac{f_b}{r}\right) - D_b \left(\frac{f_a}{r}\right) \quad , \quad D_a \equiv \left(\nabla_{g^{(2)}}\right)_a \tag{D.2.5}$$

the Einstein equation imply that F_{ab} is:

$$F_{ab} = \frac{3J}{r^3} \epsilon_{ab} \tag{D.2.6}$$

---

[1]In [15], it was shown for planar horizons that the speed of sound for the scalar sound mode goes to infinity and effectively decouples from the non-relativistic fluid sector.

with J constant. In the case of a perturbation about a spherically symmetric black hole, this corresponds to giving it a small amount of angular momentum [106].

We can study the r dependence of the metric components. Given that the connection term will drop out due to anti-symmetrization, we can replace the covariant derivatives with ordinary derivatives and find:

$$-\frac{3J}{r^4} = \partial_r \left(\frac{f_t}{r} \right) \; . \tag{D.2.7}$$

The above integrates to

$$f_t(r) = J \left(\frac{1}{r^2} - \frac{r}{r_c^3} \right) \; , \tag{D.2.8}$$

where we have set Dirichlet boundary conditions on δg_{it} at $r = r_c$. Hence, the perturbed metric reads

$$ds^2 = ds_0^2 + J \left[\frac{1}{r} - \frac{r^2}{r_c^3} \right] \mathcal{V}_i(\theta, \phi) dt dx^i \tag{D.2.9}$$

with $\mathcal{V}_i$ a Killing vector on S^2.

D.3   Mind the Gap

Looking at the last two plots in figure 6.5, we see that as we move the cutoff surface away from the cosmological horizon, the zeroes for $l = 9$ and $l = 10$ modes jump discontinuously. We would like to describe how this behavior arises. Recall that our goal was to find the lowest lying zero of $\delta g_{i\tau}^{out}(\rho = 1)$ as a function of ω on the negative imaginary axis as we move r_c. Notice in figure D.1 that as we move r_c to smaller values, the lowest lying zero disappears and the new lowest lying zero is at a finite distance away in frequency space.
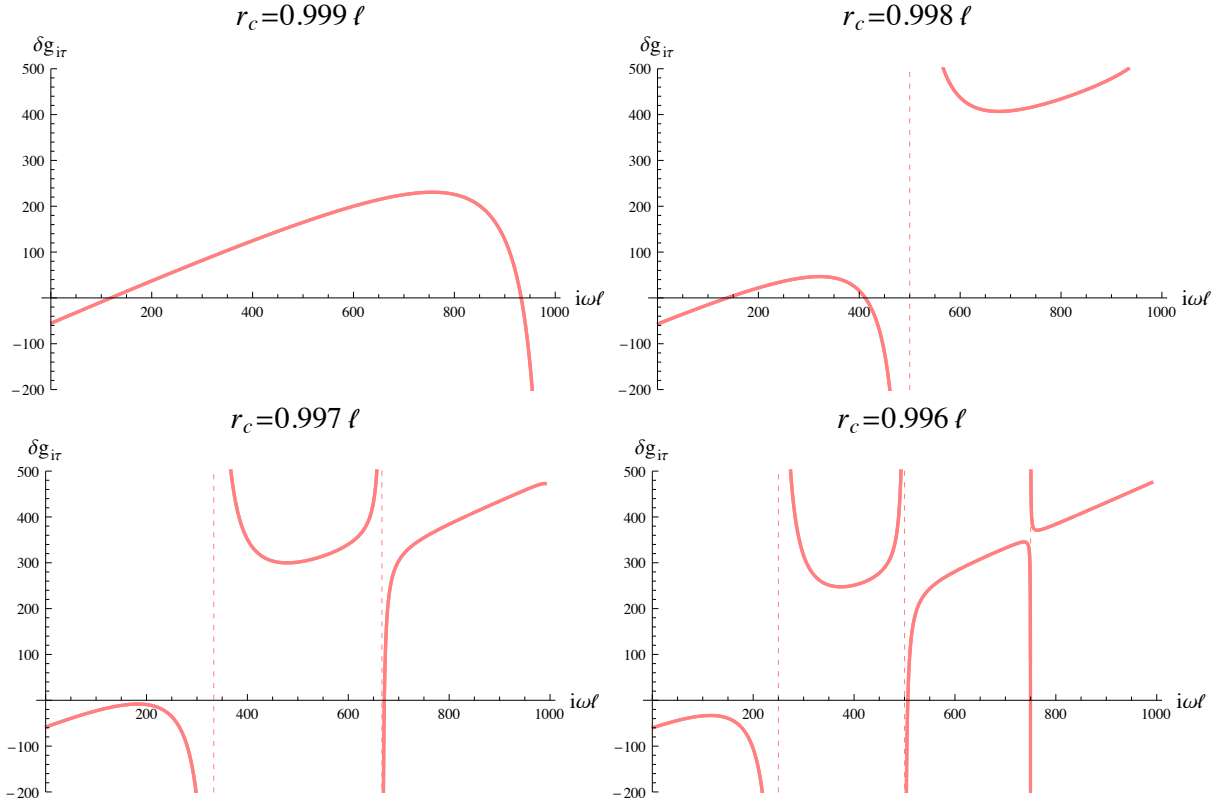
Fig. D.1: Flow of $\delta g_{i\tau}^{out}(\rho = 1)$ for $l = 10$ as we move r_c away from the cosmological horizon. We wish to find frequencies that make this function vanishes. As we vary r_c the zero jumps.

D.4   Hypergeometric Gymnastics

For $(c - a - b)$ positive integer the following relation holds [117]:

$$
{}_2F_1\left(a; b; c; z\right) = \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - b)\Gamma(c - a)} \sum_{n=0}^{c-a-b-1} \frac{(a)_n (b)_n}{(1 + a + b - c)_n n!}(1 - z)^n
$$

$$
+ (z - 1)^{c-a-b}\frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(c - b)_n (c - a)_n}{n!(n + c - a - b)!} \left[k_n - \log(1 - z)\right](1 - z)^n , \quad \text{(D.4.10)}
$$

where $(a)_n \equiv \Gamma(a + n)/\Gamma(a)$ are the Pochhammer symbols and

$$
k_n = \psi(n + 1) + \psi(n + 1 + c - a - b) - \psi(n + c - a) - \psi(n + c - b) \qquad \text{(D.4.11)}
$$

with $\psi(z) = d \log \Gamma(z)/dz$.

We treat the case covered by equation (D.4.10) as it is relevant to the text. To eliminate the log terms, we require $1/\Gamma(a)\Gamma(b) = 0$. In the case of $a = -n_1$, $n_1 = 0, 1, 2, \ldots$, if $c - b > 0$ (which in the spacelike case becomes $n_1 \geq l$), then the whole second sum vanishes and we get [118]:

$$_2F_1\left(a; b; c; z\right) = \frac{\Gamma(c - a - b)\Gamma(c)}{\Gamma(c - b)\Gamma(c - a)} {}_2F_1\left(a; b; 1 - c; 1 - z\right) \tag{D.4.12}$$

which goes to a constant as $r \to 1$ (here $z = \ell^2/r^2$) and translates into the modes with no incoming flux from the Northern patch. So, these are the modes we want.[2]

In the other regime $c - b \leq 0$, the first sum vanishes completely due to the gamma function's poles. Naively, we would think that the whole expression is zero, however, the second term contains a divergent term $\psi(n + c - b)$ in the sum that will cancel out the divergence in $1/\Gamma(a)$ for $n \leq -(c - b)$. then we actually have

$$\frac{\psi(n + c - b)}{\Gamma(a)} \sim \Gamma(-a + 1) \tag{D.4.13}$$

leading to[118]:

$$_2F_1\left(a; b; c; z\right) \sim \frac{\Gamma(c)\Gamma(-a + 1)}{(c - a - b)!\Gamma(b)} (1 - z)^{c - a - b} {}_2F_1\left(c - b; c - a; 1 + c; 1 - z\right). \tag{D.4.14}$$

This implies that the φ_v^+ tends to $(r^2 - 1)^{i\omega/2}$ (as $r \to 1$) which is an incoming wave from the Northern patch and should be excluded.

---

[2]In the case of $b = -n_2$, $n_2 = 0, 1, 2, \ldots$, the analogous inequality in the first regime is $c - a > 0$. However, our parameters imply that $c - a > 0$ is always true. Thus we are always in this first regime which gives the modes that we want and there is no further restriction on n_2.

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