Generic Global Rigidity in Complex and Pseudo-Euclidean Spaces

Generic Global Rigidity in Complex and Pseudo-Euclidean Spaces

Steven J. Gortler and Dylan P. Thurston

Abstract In this paper we study the property of generic global rigidity for frameworks of graphs embedded in d-dimensional complex space and in a d-dimensional pseudo-Euclidean space ($\mathbb{R}^d$ with a metric of indefinite signature). We show that a graph is generically globally rigid in Euclidean space iff it is generically globally rigid in a complex or pseudo-Euclidean space. We also establish that global rigidity is always a generic property of a graph in complex space, and give a sufficient condition for it to be a generic property in a pseudo-Euclidean space. Extensions to hyperbolic space are also discussed.

1 Introduction

The property of generic global rigidity of a graph in d-dimensional Euclidean space has recently been fully characterized [4, 7]. It is quite natural to study this property in other spaces as well. For example, recent work of Owen and Jackson [8] has studied the number of equivalent realizations of frameworks in $\mathbb{C}^2$. In this paper we study the property of generic global rigidity of graphs embedded in $\mathbb{C}^d$ as well as graphs embedded in a pseudo Euclidean space ($\mathbb{R}^d$ equipped with an indefinite metric signature).

We show that a graph Γ is generically globally rigid (GGR) in d-dimensional Euclidean space iff Γ is GGR in d-dimensional complex space. Moreover, for any metric signature, s, We show that a graph Γ is GGR in d-dimensional Euclidean space iff Γ is GGR in d-dimensional real space under the signature s. Combining this with results from [5] also allows us to equate this property with generic global rigidity in hyperbolic space.

In the Euclidean and complex cases, global rigidity can be shown to be a generic property: a given graph is either generically globally rigid, or generically globally flexible. In the pseudo Euclidean (and equivalently the hyperbolic) case, though, we do not know this to be true. In this paper we do establish that global rigidity in pseudo Euclidean spaces is a generic property for graphs that contain a large enough GGR subgraph (such as a d-simplex).

Steven J. Gortler
Havard School of Engineering and Applied Sciences, Department of Computer Sciences, 33 Oxford St., Cambridge, MA 02138 USAe-mail: sjg@cs.harvard.edu

Dylan P. Thurston
Department of Mathematics, Columbia University, New York, NY 10027-6940 USA e-mail: dpt@math.columbia.edu

2 Initial Definitions

Definition 1. We equip $\mathbb{R}^d$ with pseudo Euclidean metric in order to measure lengths. The metric is specified with a non negative integer s that determines how many of its coordinate directions are subtracted from the total. The squared length of a vector $\mathbf{w}$ is $|\mathbf{w}|^2 := -\sum_{i=1}^{s} \mathbf{w}_i^2 + \sum_{i=s+1}^{d} \mathbf{w}_i^2$. We will use the symbol $\mathbb{S}^d$ to denote the space $\mathbb{R}^d$ equipped with some fixed metric s. If $s = 0$, we have the Euclidean metric and the space may be denoted $\mathbb{E}^d$.

For complex space, The squared length of a vector $\mathbf{w}$ in $\mathbb{C}^d$ is $|\mathbf{w}|^2 := \sum_i \mathbf{w}_i^2$. Note here that we do not use conjugation, and thus vectors have complex squared lengths. (The use of conjugation would essentially reduce d-dimensional complex rigidity questions to 2d-dimensional Euclidean questions).

Definition 2. A *graph* Γ is a set of v vertices $\mathscr{V}(\Gamma)$ and e edges $\mathscr{E}(\Gamma)$, where $\mathscr{E}(\Gamma)$ is a set of two-element subsets of $\mathscr{V}(\Gamma)$. We will typically drop the graph Γ from this notation.

For $\mathbb{F} \in \{\mathbb{E}, \mathbb{S}, \mathbb{C}\}$, a *configuration* of the vertices $\mathscr{V}(\Gamma)$ of a graph in $\mathbb{F}^d$ is a mapping p from $\mathscr{V}(\Gamma)$ to $\mathbb{F}^d$. Let $C_{\mathbb{F}^d}(\mathscr{V})$ be the space of configurations in $\mathbb{F}^d$.

For $p \in C_{\mathbb{F}^d}(\mathscr{V})$ with $u \in \mathscr{V}(\Gamma)$, we write $p(u) \in \mathbb{F}^d$ for the image of u under p.

A *framework* $\rho = (p, \Gamma)$ of a graph is the pair of a graph and a configuration of its vertices. $C_{\mathbb{F}^d}(\Gamma)$ is the space of frameworks (p, Γ) with graph Γ and configurations in $\mathbb{F}^d$.

We may also write $\rho(u)$ for $p(u)$ where $\rho = (p, \Gamma)$ is a framework of the configuration p.

Definition 3. Two frameworks ρ and σ in $C_{\mathbb{F}^d}(\Gamma)$ are *equivalent* if for all $\{t, u\} \in \mathscr{E}$ we have $|\rho(t) - \rho(u)|^2 = |\sigma(t) - \sigma(u)|^2$.

Definition 4. Two configurations p and q in $C_{\mathbb{F}^d}(\mathscr{V})$ are *congruent* if for all vertex pairs, $\{t, u\}$, we have $|p(t) - p(u)|^2 = |q(t) - q(u)|^2$.

Two configurations p and q in $C_{\mathbb{F}^d}(\mathscr{V})$ are *strongly congruent* if they are related by a translation composed with an element of the orthogonal group of $\mathbb{F}^d$.

Remark 1. In $\mathbb{E}^d$, there is no difference between congruence and strong congruence. In other spaces, though, there can be some subtle differences. For the simplest example, in $\mathbb{C}^2$, the vectors $(0,0)$ and $(i,1)$ both have zero length, but are not related by a complex orthogonal transform. Such non-zero vectors with zero squared length are called *isotropic*. Thus the framework made up of a single edge connecting a vertex at the origin to a vertex at $(i,1)$ is congruent to the framework with both vertices at the origin, but the two frameworks are not strongly congruent.

Fortunately, these differences are easy to avoid; for example, congruence and strong congruence coincide for points with a d-dimensional affine span. These notions will also coincide when there are fewer than $d+1$ points, as long as the points are in affine general position. For more details, see Appendix 10.

We can now, finally, define global rigidity and flexibility.

Definition 5. A framework $\rho \in C_{\mathbb{F}^d}(\Gamma)$ is *globally rigid* in $\mathbb{F}^d$ if, for any other framework $\sigma \in C_{\mathbb{F}^d}(\Gamma)$ to which ρ is equivalent, we also have that ρ is congruent to σ. Otherwise we say that ρ is *globally flexible* in $\mathbb{F}^d$.

Definition 6. A configuration p in $C_{\mathbb{F}^d}(\mathscr{V})$ is generic if the coordinates do not satisfy any non-zero algebraic equation with rational coefficients. We call a framework generic if its configuration is generic. (See Appendix 9 for more background on (semi) algebraic sets and genericity).

Definition 7. A graph Γ is generically globally rigid (resp. flexible) in $\mathbb{F}^d$ if all generic frameworks in $C_{\mathbb{F}^d}(\Gamma)$ are globally rigid (resp. flexible). These properties are abbreviated GGR and GGF.

Definition 8. A property is *generic* if, for every graph, either all generic frameworks in $C_{\mathbb{F}^d}(\Gamma)$ have the property or none do. For instance, global rigidity in $\mathbb{E}^d$ is a generic property of a graph [7]. So in this case, if a graph is not GGR, it must be GGF.

3 Complex Generic Global Rigidity

Our main theorem in this section is

Theorem 1. *A graph Γ is generically globally rigid in $\mathbb{C}^d$ iff it is generically globally rigid in $\mathbb{E}^d$.*

Remark 2. This fully describes the generic situation for complex frameworks as it is easy to see that generic global rigidity in $\mathbb{C}^d$ is a generic property of a graph.

Recall that a complex algebraically *constructible set* is a finite Boolean combination of complex algebraic sets. Also, an irreducible complex algebraic set V cannot have two disjoint constructible subsets with the same dimension as V.

Chevalley's theorem states that the image under a polynomial map of a complex algebraically constructible set, all defined over $\mathbb{Q}$, is also a complex algebraically constructible set defined over $\mathbb{Q}$ [1, Theorem 1.22]. Chevalley's theorem allows one to apply elimination, effectively replacing all quantifiers in a Boolean-algebraic expression with algebraic equations and Boolean set operations.

Now, let us assume Γ is locally rigid in $\mathbb{C}^d$. We can partition $C_{\mathbb{C}^d}(\Gamma)$ such that in each part, P_n , all of the frameworks have the same number, n, of equivalent and non-congruent frameworks. In light of Chevalley's theorem, each of these parts is constructible. And exactly one of them, P_{n_0}, must be of full dimension. This part contains all of the generic points and represents the generic behavior of the framework. If $n_0 = 1$ then the graph is GGR, while if $n_0 > 1$ then it must be GGF.

3.1 $\Rightarrow$ of Theorem 1

The implication from Complex to Euclidean GGR follows almost directly from their definitions. For this argument we model each Euclidean framework ρ in $C_{\mathbb{E}^d}(\mathscr{V})$ as a Complex framework $\rho_{\mathbb{C}}$ in $C_{\mathbb{C}^d}(\mathscr{V})$ that happens to have all purely real coordinates. Clearly, for such configurations, the complex squared length measurement coincides with the Euclidean metric on real configurations.

Proof. Let ρ be a generic framework in $C_{\mathbb{E}^d}(\Gamma)$ and let $\rho_{\mathbb{C}}$ be its corresponding real valued framework in $C_{\mathbb{C}^d}(\Gamma)$. By our definitions, $\rho_{\mathbb{C}}$ is also generic when thought of as complex framework.

Since Γ is generically globally rigid in $\mathbb{C}^d$, $\rho_{\mathbb{C}}$ can have no equivalent and non-congruent framework in $C_{\mathbb{C}^d}(\Gamma)$, and thus it has no real valued, equivalent and non-congruent framework in $C_{\mathbb{C}^d}(\Gamma)$. Thus ρ has no equivalent and non-congruent framework in $C_{\mathbb{E}^d}(\Gamma)$.

3.2 $\Leftarrow$ of Theorem 1

For the other direction of Theorem 1, we start with a complex version of a theorem by Connelly [4]:

Theorem 2. *Let ρ be a generic framework in $C_{\mathbb{C}^d}(\Gamma)$. If ρ has a complex equilibrium stress matrix of rank $v-d-1$, then Γ is generically globally rigid in $\mathbb{C}^d$.*

Proof. The proof of the complex version of this theorem follows identically to Connelly's proof of the Euclidean version. In particular, the proof shows that any framework with the same complex squared edge lengths as ρ must be strongly congruent, and thus congruent to it.

(The interested reader can see [4] for the definition of an equilibrium stress matrix).

Next, we recall a theorem from Gortler, Healy and Thurston [7]

Theorem 3. *Let ρ be a generic framework in $C_{\mathbb{E}^d}(\Gamma)$ with at least $d+2$ vertices. If ρ does not have a real equilibrium stress matrix of rank $v-d-1$, then Γ is generically globally flexible in $\mathbb{E}^d$. Moreover, there must be an even number of noncongruent frameworks with the same squared edge lengths as ρ in $\mathbb{E}^d$.*

And now we can prove this direction of our Theorem.

Proof. From Theorem 2, if Γ is not generically globally rigid in $\mathbb{C}^d$, there is no generic framework in $C_{\mathbb{C}^d}(\Gamma)$ that has a complex equilibrium stress matrix of rank $v-d-1$. Thus there can be no real valued and generic framework in $C_{\mathbb{C}^d}(\Gamma)$ with complex equilibrium stress matrix of rank $v-d-1$, and thus no generic framework in $C_{\mathbb{E}^d}(\Gamma)$ with a complex or real equilibrium stress matrix of rank $v-d-1$. Thus from Theorem 3, Γ is generically globally flexible in $\mathbb{E}^d$.

4 Pseudo Euclidean Generic Global Rigidity: Results

Our main theorem on pseudo Euclidean generic global rigidity is as follows:

Theorem 4. *For any pseudo Euclidean space $\mathbb{S}^d$, a graph Γ is generically globally rigid in $\mathbb{E}^d$ iff it is generically globally rigid in $\mathbb{S}^d$.*

Unfortunately we do not know if generic global rigidity is a generic property in $\mathbb{S}^d$. It is conceivable that there are some graphs that are not GGR in $\mathbb{S}^d$ but that do have *some* generic frameworks that are globally rigid in $\mathbb{S}^d$. We leave this as an open question. We do have the following partial result

Theorem 5. *If a graph Γ is not GGR in $\mathbb{S}^d$ and it has a GGR subgraph Γ_0 with $d+1$ or more vertices, then Γ must be GGF in $\mathbb{S}^d$.*

5 $\Rightarrow$ of Theorem 4

This argument is essentially identical to that of Section 3.1.

Definition 9. Given a pseudo Euclidean space $\mathbb{S}^d$ with signature s, we model each configuration $\rho \in C_{\mathbb{S}^d}(\mathcal{V})$ as a Complex configurations $\rho_{\mathbb{C}} \in C_{\mathbb{C}^d}(\mathcal{V})$ that happens to have the first s of its coordinates purely imaginary and the remaining $d-s$ of its coordinates purely real. We call this an *s-signature, real valued complex configuration*. We will shorten this to simply an *s-valued configuration*.

It is easy to verify that for such configurations, the complex squared length measurement coincides with the metric on $\mathbb{S}^d$.

And now we can prove this direction of our Theorem.

Proof. Let ρ be a generic framework in $C_{\mathbb{S}^d}(\Gamma)$. We model this with $\rho_{\mathbb{C}}$, an s-valued complex framework in $C_{\mathbb{C}^d}(\Gamma)$.

$\rho_{\mathbb{C}}$ must be a generic framework in $C_{\mathbb{C}^d}(\Gamma)$. For suppose there is a non-zero polynomial $\phi_{\mathbb{C}}$ with rational coefficients, that vanishes on $\rho_{\mathbb{C}}$. Then there is a polynomial ϕ with coefficients in $\mathbb{Q}(i)$ that vanishes on the real coordinates of ρ. Let $\bar{\phi}$ be the polynomial obtained by taking the conjugate of every coefficient in ϕ, and let $\psi := \phi * \bar{\phi}$. Then ψ is non zero and vanishes on ρ. Since ψ is fixed by conjugation, it has coefficients in $\mathbb{Q}$. This polynomial would make ρ non generic, leading to a contradiction.

Since Γ is generically globally rigid in $\mathbb{E}^d$, from Theorem 1 it is also generically globally rigid in $\mathbb{C}^d$. Thus $\rho_{\mathbb{C}}$ can have no equivalent and non-congruent framework in $C_{\mathbb{C}^d}(\Gamma)$, and thus it can have no s-valued, equivalent and non-congruent framework in $C_{\mathbb{C}^d}(\Gamma)$. Thus ρ can have no equivalent and non-congruent framework in $C_{\mathbb{S}^d}(\Gamma)$.

6 $\Leftarrow$ of Theorem 4

Remark 3. For this proof, we cannot apply the same reasoning as section 3.2, as many of the stress matrix arguments and conclusions from [7] simply do not carry over to pseudo Euclidean spaces. Indeed, Jackson and Owen [8] have found a graph, they call G_3, that is GGF in $\mathbb{E}^2$, but for which there is always an *odd* number of equivalent realizations in 2-dimensional Minkowski space. Moreover, it is not even clear that for general pseudo Euclidean spaces of dimension 3 or greater, the "number of equivalent realizations mod 2" is even a generic property.

For this direction, we will show the contrapositive: namely, if there is a generic Euclidean framework that is not globally rigid, then there must be a generic framework in $\mathbb{S}^d$ that is not globally rigid. To do this, we will apply a basic construction by Saliola and Whiteley [11] that takes a pair of equivalent Euclidean frameworks and produces a pair of equivalent frameworks in the desired space $C_{\mathbb{S}^d}(\Gamma)$. Whiteley refers to this recipe as a generalized Pogorelov map [11].

Definition 10. Let P be the map from pairs of frameworks in $C_{\mathbb{E}^d}(\Gamma)$ to pairs of frameworks in $C_{\mathbb{S}^d}(\Gamma)$ defined as follows:

Step 1: Let ρ and σ be two frameworks in $\mathbb{E}^d$. Take their average to obtain $a := \frac{\rho+\sigma}{2}$. Take their difference to obtain $f := \frac{\rho-\sigma}{2}$.

Step 2: Let $\tilde{a}$ be the framework in $C_{\mathbb{S}^d}(\Gamma)$ with the same (real) coordinates of a. Let $\tilde{f}$ be defined by negating the first s of the coordinates in f.

Step 3: Finally, set $P(\rho,\sigma) := (\tilde{\rho},\tilde{\sigma})$ where $\tilde{\rho} := \tilde{a}+\tilde{f}$ and $\tilde{\sigma} := \tilde{a}-\tilde{f}$.

The Pogorelov map is useful due to the following [11]:

Theorem 6. *Let ρ and σ be two equivalent frameworks in $C_{\mathbb{E}^d}(\Gamma)$. Then $P(\rho,\sigma)$ are a pair of equivalent frameworks in $C_{\mathbb{S}^d}(\Gamma)$.*

Proof. Using the notation of Definition 10 we see the following.

Step 1: From the *averaging principal* [3], a must be infinitesimally flexible with flex f.

Step 2: $\tilde{f}$ must be an infinitesimal flex for $\tilde{a}$ in $C_{\mathbb{S}^d}(\Gamma)$ [10].

Step 3: From the *flex-antiflex principal* [3] (also sometimes called the de-averaging principal), $\tilde{\rho}$ must be equivalent to $\tilde{\sigma}$ in $C_{\mathbb{S}^d}(\Gamma)$.

Remark 4. It is, perhaps, interesting to note that in our case, the map has the very simple form of "coordinate swapping". In particular, it is an easy calculation to see that $\tilde{\rho}$ will be made up of the first s coordinates of ρ and the remaining coordinates of σ, while $\tilde{\sigma}$ will be made up of the first s coordinates of σ and the remaining coordinates of ρ. It is also an simple calculation to directly verify, without using the averaging principle, that coordinate swapping will map pairs of equivalent Euclidean frameworks to pairs of equivalent frameworks in $C_{\mathbb{S}^d}(\Gamma)$.

Additionally, we can ensure that $\tilde{\rho}$ is not congruent to $\tilde{\sigma}$.

Lemma 1. *Let ρ and σ be two equivalent frameworks in $C_{\mathbb{E}^d}(\Gamma)$. And let $(\tilde{\rho}, \tilde{\sigma}) := P(\rho, \sigma)$. Then ρ and σ are congruent in $C_{\mathbb{E}^d}(\Gamma)$ iff $\tilde{\rho}$ and $\tilde{\sigma}$ are congruent in $C_{\mathbb{S}^d}(\Gamma)$.*

Proof. Congruence between configurations is the same as equivalence between complete graphs over these configurations. Thus this property must map across the Pogorelov map (which does not depend on the edge set), and its inverse.

6.1 Genericity

The main (annoyingly) difficult technical issue left is to show that this construction can create a *generic* framework in $C_{\mathbb{S}^d}(\Gamma)$ that is globally flexible. A priori, it is conceivable that the image of the Pogorelov map, acting on all pairs of equivalent and non-congruent Euclidean frameworks, can only produce pseudo Euclidean configurations that lie on some subvariety of $C_{\mathbb{S}^d}(\Gamma)$. In this section, we rule this possibility out.

In this discussion, we will assume that Γ is generically locally rigid (otherwise we are done), but that it is not GGR in $\mathbb{E}^d$.

Definition 11. Let E^+ ('E' for 'equivalent') be the algebraic subset of $C_{\mathbb{E}^d}(\Gamma) \times C_{\mathbb{E}^d}(\Gamma)$ consisting of pairs of equivalent tuples. Let C^+ ('C' for 'congruent') be the algebraic subset of $C_{\mathbb{E}^d}(\Gamma) \times C_{\mathbb{E}^d}(\Gamma)$ consisting of pairs of congruent tuples. Let π_1 be the projection from a pair of frameworks onto its first factor.

Definition 12. Since Γ is not GGR in $\mathbb{E}^d$, $\dim(\pi_1(E^+ \backslash C^+)) = v*d$ and so E^+ must have at least one irreducible component E, with $\dim(\pi_1(E)) = v*d$ and such that it contains at least one tuple of non-congruent frameworks. We choose one such component and call it E. As per Remark 8, E must be defined over some algebraic extension of $\mathbb{Q}$. Thus if e is generic in E, then $\pi_1(e)$ is a generic framework in $C_{\mathbb{E}^d}(\Gamma)$.

Lemma 2. *Let $e := (\rho, \sigma) \in E$ be generic. Then ρ is not congruent to σ.*

Proof. Congruence is a relation that can be expressed with polynomials over $\mathbb{Q}$. By our assumptions on E, these polynomials do not vanish identically over E.

Lemma 3. *The (real) dimension of E is $v*d + \binom{d+1}{2}$. Moreover, if (ρ, σ) is generic in E, then for all σ^c in the congruence class $[\sigma]$, (ρ, σ^c) must be in E.*

Proof. We will pick a generic $e = (\rho, \sigma) \in E$, and look at the dimension of the fiber $\pi_1^{-1}(\rho)$ near this point e. (By considering only this neighborhood, we can avoid dealing with any non-smooth points of E, and thus can view this as a smooth map between manifolds). The dimension of E must be the sum of the dimension of the span of $\pi_1(E)$, which is $v*d$, and the dimension of this fiber.

Since e is generic in E, ρ must be generic in $C_{\mathbb{E}^d}(\Gamma)$. Thus, from Lemma 11 (below), σ must be locally rigid and with non degenerate affine span. Thus its congruence class has dimension $\binom{d+1}{2}$.

Since e is generic in E, from Lemma 24, all nearby points in E^+ must, in fact, lie in E. In particular, for $\sigma^c \in [\sigma]$ and close to σ, the point (ρ, σ^c) must be in E. Thus the dimension of the fiber in E near e must be $\binom{d+1}{2}$. This gives us the desired dimension.

Moreover, since E is algebraic, for any $\sigma^c \in [\sigma]$, the point (ρ, σ^c) must be in E. This follows from the fact that the (Zariski) closure of a subset must be a subset of the closure.

Corollary 1. *Let π_2 be the projection of a pair onto its second factor. The (real) dimension of $\pi_2(E)$ is $v*d$. And if e is generic in E, $\pi_2(e)$ is generic in $C_{\mathbb{E}^d}(\Gamma)$.*

To study the behavior of P on E, we move our discussion over to complex space.

Definition 13. Let $E^+_{\mathbb{C}}$ be the algebraic subset of $C_{\mathbb{C}^d}(\Gamma) \times C_{\mathbb{C}^d}(\Gamma)$ consisting of pairs of equivalent tuples. Let $E_{\mathbb{C}}$ be any component of $E^+_{\mathbb{C}}$ that includes E. (This can be done as the complexification of E must be irreducible - see Definition 28). From Corollary 2, below, we will also soon see that there is only one such component.

Lemma 4. *The (complex) dimension of $E_{\mathbb{C}}$ is $v*d + \binom{d+1}{2}$.*

Proof. $E_{\mathbb{C}}$ includes the complexification of E (see Definition 28), and so by assumption, the complex dimension of $\pi_1(E_{\mathbb{C}})$ must be at least $v*d$, and thus must be equal to $v*d$. We can then follow the proof of Lemma 3 to establish the complex dimension of the generic π_1 fibers of $E_{\mathbb{C}}$

Corollary 2. *$E_{\mathbb{C}}$ is* the *complexification of E. A generic point of E is generic in $E_{\mathbb{C}}$.*

Proof. By assumption, $E_{\mathbb{C}}$ is irreducible and contains E. Moreover the complex dimension of $E_{\mathbb{C}}$ equals the real dimension of E. Thus $E_{\mathbb{C}}$ cannot be larger than the complexification of E. Genericity carries across complexification (see Definition 28).

To study P, we will look at a complex Pogorelov map $P_{\mathbb{C}}$, that essentially reproduces the behavior of P when restricted to real input. In particular, this map will take real valued complex pairs, to s-valued complex pairs. We define $P_{\mathbb{C}}$ as the composition of some very simple maps.

Definition 14. Let $H_{\mathbb{C}}$, (a Haar like transform) be the invertible map from $(\rho_{\mathbb{C}}, \sigma_{\mathbb{C}})$, a pair of frameworks in $C_{\mathbb{C}^d}(\Gamma)$, to the pair $(\frac{\rho_{\mathbb{C}}+\sigma_{\mathbb{C}}}{2}, \frac{\rho_{\mathbb{C}}-\sigma_{\mathbb{C}}}{2})$.

Let $S_{\mathbb{C}}$ be the the invertible map that takes $(a_{\mathbb{C}}, f_{\mathbb{C}})$, a pair of frameworks in $C_{\mathbb{C}^d}(\Gamma)$, to the pair $(\tilde{a}_{\mathbb{C}}, \tilde{f}_{\mathbb{C}})$, where the $\tilde{a}_{\mathbb{C}}$ is obtained from $a_{\mathbb{C}}$ by multiplying its first s coordinates by i, while $\tilde{f}_{\mathbb{C}}$ is obtained from $f_{\mathbb{C}}$ by multiplying its first s coordinates by $-i$.

$H^{-1}_{\mathbb{C}}(\tilde{a}_{\mathbb{C}}, \tilde{f}_{\mathbb{C}})$, the inverse Haar map, is simply $(\tilde{a}_{\mathbb{C}} + \tilde{f}_{\mathbb{C}}, \tilde{a}_{\mathbb{C}} - \tilde{f}_{\mathbb{C}})$.

Given this, $P_{\mathbb{C}} := H^{-1}_{\mathbb{C}} \circ S_{\mathbb{C}} \circ H_{\mathbb{C}}$.

This complex Pogorelov map coincides with the real map described above. In particular suppose ρ and σ are in $C_{\mathbb{E}^d}(\Gamma)$, and suppose $\rho_{\mathbb{C}}$ and $\sigma_{\mathbb{C}}$ are the corresponding real valued frameworks in $C_{\mathbb{C}^d}(\Gamma)$. Let $(\tilde{\rho}, \tilde{\sigma}) := P(\rho, \sigma)$ and $(\tilde{\rho}_{\mathbb{C}}, \tilde{\sigma}_{\mathbb{C}}) := P_{\mathbb{C}}(\rho_{\mathbb{C}}, \sigma_{\mathbb{C}})$. Then $\tilde{\rho}_{\mathbb{C}}$ and $\tilde{\sigma}_{\mathbb{C}}$ are the s-valued complex representations of $\tilde{\rho}$ and $\tilde{\sigma}$.

Clearly $P_{\mathbb{C}}$ maps $E^+_{\mathbb{C}}$ to itself. But a priori, it might map the component $E_{\mathbb{C}}$ to some other component of $E^+_{\mathbb{C}}$, and this other component might project under π_1 and π_2 onto a subvariety of (non generic) frameworks $C_{\mathbb{C}^d}(\Gamma)$. Our goal will be to show that this does not happen; instead $E_{\mathbb{C}}$ maps to itself under $P_{\mathbb{C}}$. As this map preservers genericity, and generic points of $E_{\mathbb{C}}$ project under π_1 to generic frameworks in $C_{\mathbb{C}^d}(\Gamma)$, we will then be done. (See Figure 1).

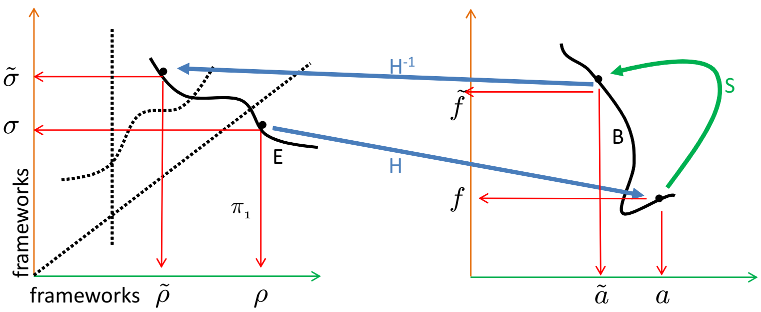

Fig. 1: Left: The space of pairs of complex frameworks. (All $\mathbb{C}$ subscripts are dropped for clarity). The locus of equivalent pairs, $E^+_{\mathbb{C}}$, is shown in solid and dotted black. At least one component, $E_{\mathbb{C}}$, shown in solid black, has the property that $\dim(\pi_1(E_{\mathbb{C}})) = v * d$. Right: The space of pairs of complex frameworks. The variety $B_{\mathbb{C}} := H_{\mathbb{C}}(E_{\mathbb{C}})$ is made up of some frameworks and their flexes. (The image under $H_{\mathbb{C}}$ of the other components of $E^+_{\mathbb{C}}$ is not shown). The map $S_{\mathbb{C}}$ maps $B_{\mathbb{C}}$ to itself, and thus the Pogorelov map is an automorphism of $E_{\mathbb{C}}$.

Definition 15. Let $B_{\mathbb{C}} := (H_{\mathbb{C}}(E_{\mathbb{C}}))$, ('B' for 'bundles' of flexes over frameworks). Since $B_{\mathbb{C}}$ is isomorphic to $E_{\mathbb{C}}$, it too must be an algebraic set. For any $(a_{\mathbb{C}}, f_{\mathbb{C}}) \in B_{\mathbb{C}}$, from the averaging principle, $f_{\mathbb{C}}$ is an infinitesimal flex for $a_{\mathbb{C}}$. $B_{\mathbb{C}}$ is irreducible (Lemma 22). And if $e_{\mathbb{C}}$ is generic in $E_{\mathbb{C}}$, $H_{\mathbb{C}}(e_{\mathbb{C}})$ (from Lemma 25) must be generic in $B_{\mathbb{C}}$.

Lemma 5. *Let $b_{\mathbb{C}} \in B_{\mathbb{C}}$ be generic. Let $b'_{\mathbb{C}} := (a'_{\mathbb{C}}, f'_{\mathbb{C}})$ be a nearby tuple in $C_{\mathbb{C}^d}(\Gamma) \times C_{\mathbb{C}^d}(\Gamma)$ such that $f'_{\mathbb{C}}$ is an infinitesimal flex for $a'_{\mathbb{C}}$. Then $b'_{\mathbb{C}} \in B_{\mathbb{C}}$.*

Proof. The tuple, $e_{\mathbb{C}} := H^{-1}_{\mathbb{C}}(b_{\mathbb{C}})$, is generic in E. From the flex/antiflex principal, $(\rho'_{\mathbb{C}}, \sigma'_{\mathbb{C}}) := e'_{\mathbb{C}} := H^{-1}_{\mathbb{C}}(a'_{\mathbb{C}}, f'_{\mathbb{C}})$ must be an equivalent pair of frameworks and thus in $E^+_{\mathbb{C}}$, and $e'_{\mathbb{C}}$ must be near $e_{\mathbb{C}}$. From Lemma 24, all nearby points in $E^+_{\mathbb{C}}$ must, in fact, lie in $E_{\mathbb{C}}$. Thus $e'_{\mathbb{C}}$ must be in $E_{\mathbb{C}}$, and from our definitions, $H_{\mathbb{C}}(e'_{\mathbb{C}}) = b'_{\mathbb{C}}$ must be in $B_{\mathbb{C}}$.

Definition 16. Let $(a_{\mathbb{C}}, f_{\mathbb{C}}) = b_{\mathbb{C}}$ be a pair of framework in $C_{\mathbb{C}^d}(\Gamma)$. One can apply *coordinate scaling* to $b_{\mathbb{C}}$ by multiplying one chosen coordinate (out of the d coordinates in $\mathbb{C}^d$) of all the vertices in $a_{\mathbb{C}}$ by some complex scalar λ and the corresponding coordinate in all the vertices in $f_{\mathbb{C}}$ by $1/\lambda$.

Lemma 6. *The set $B_{\mathbb{C}}$ is invariant to coordinate scaling.*

Proof. Let $(a_{\mathbb{C}}, f_{\mathbb{C}}) = b_{\mathbb{C}} \in B_{\mathbb{C}}$ be generic. $f_{\mathbb{C}}$ is an infinitesimal flex for $a_{\mathbb{C}}$. Let us apply coordinate scaling to $b_{\mathbb{C}}$ with a scalar λ close to 1 and let us denote the result by $b'_{\mathbb{C}} = (a'_{\mathbb{C}}, f'_{\mathbb{C}})$. Looking at the effect of the rigidity matrix, we see that $f'_{\mathbb{C}}$ must be an infinitesimal flex for $a'_{\mathbb{C}}$, and from Lemma 5 must be in $B_{\mathbb{C}}$.

This means that $B_{\mathbb{C}}$ is invariant to nearly-unit coordinate scaling. Since $B_{\mathbb{C}}$ is algebraic, it must thus be invariant to all coordinate scaling. (This follows from the fact that the (Zariski) closure of a subset must be a subset of the closure).

Corollary 3. *$S_{\mathbb{C}}$ is an automorphism of $B_{\mathbb{C}}$. Thus $P_{\mathbb{C}}$ is an automorphism of $E_{\mathbb{C}}$. Thus if $e_{\mathbb{C}} \in E_{\mathbb{C}}$ is generic, then $P_{\mathbb{C}}(e_{\mathbb{C}})$ is generic in $E_{\mathbb{C}}$ and both $\pi_1(P_{\mathbb{C}}(e_{\mathbb{C}}))$ and $\pi_2(P_{\mathbb{C}}(e_{\mathbb{C}}))$ are generic in $C_{\mathbb{C}^d}(\Gamma)$.*

With this we can finish the proof of this direction of Theorem 4.

Proof. Assume that Γ is not GGR in $\mathbb{E}^d$. Pick a generic $(\rho, \sigma) \in E$ (Definition 12).

From Theorem 6, $P(\rho, \sigma) =: (\tilde{\rho}, \tilde{\sigma})$ is a pair of equivalent frameworks $C_{\mathbb{S}^d}(\Gamma)$ which are not congruent from Lemma 2.

Let $\rho_{\mathbb{C}}$ and $\sigma_{\mathbb{C}}$ be the real valued complex frameworks corresponding to ρ and σ. From Corollary 2, $(\rho_{\mathbb{C}}, \sigma_{\mathbb{C}})$ is generic in $E_{\mathbb{C}}$. Meanwhile, $P_{\mathbb{C}}(\rho_{\mathbb{C}}, \sigma_{\mathbb{C}}) = (\tilde{\rho}_{\mathbb{C}}, \tilde{\sigma}_{\mathbb{C}})$, where $\tilde{\rho}_{\mathbb{C}}$ is the s-valued, complex representation of $\tilde{\rho}$, and $\tilde{\sigma}_{\mathbb{C}}$ is the s-valued, complex representation of $\tilde{\sigma}$. From Corollary 3, $\tilde{\rho}_{\mathbb{C}}$ is generic in $C_{\mathbb{C}^d}(\Gamma)$. Therefore $\tilde{\rho}$ must be generic in $C_{\mathbb{S}^d}(\Gamma)$, and we can conclude that Γ is not GGR in $\mathbb{S}^d$.

7 Proof of Theorem 5

We will prove the theorem by first showing that the existence of a large enough GGR subgraph Γ_0 is sufficient to rule out any "cross-talk" between different real signatures. In particular, if we have an s-valued framework of Γ_0, then Γ_0 cannot have a congruent framework that is s'-valued where $s \neq s'$. Thus, if we have an s-valued framework of Γ, then Γ cannot have an equivalent framework that is s'-valued where $s \neq s'$. With such cross talk ruled out, we will be able to apply an algebraic degree argument to show that Γ is GGF in $\mathbb{S}^d$.

In this section we will model congruence classes of frameworks in $C_{\mathbb{C}^d}(\mathscr{V})$ using complex symmetric matrices of rank d or less. First we spell out some basic facts about these matrices, and their relationship to configurations, as well as the notions of congruence and equivalence.

Definition 17. Let $\mathscr{G}$ be the set of symmetric $v-1$ by $v-1$ complex matrices of rank d or less. This is a determinantal variety which is irreducible. Assuming that $v \geq d+1$, $\mathscr{G}$ is of complex dimension $v * d - \binom{d+1}{2}$, and any generic $\mathbf{M} \in \mathscr{G}$ will have rank d.

For any configuration $p \in C_{\mathbb{C}^d}(\mathscr{V})$ (or framework $\rho \in C_{\mathbb{C}^d}(\Gamma)$) we associate its *g-matrix* $\mathbf{G}(p) \in \mathscr{G}$ as follows. We first translate p so its first vertex is at the origin. For any two remaining vertices t, u, we define the corresponding matrix entry as

$$\mathbf{G}(p)_{t,u} := \sum_{i=1}^{d} p(t)_i \, p(u)_i \tag{1}$$

(This is like a Gram matrix, but there is no conjugation involved). Overloading this notation, if ρ is a framework with configuration p, we define $\mathbf{G}(\rho) := \mathbf{G}(p)$.

Definition 18. For any pair $\{t, u\}$, of distinct vertices in p, there is a linear map $\pi_{t,u}$ that computes the squared lengths between that pair using the entries in $\mathbf{G}(p)$. In the case where t is the first vertex (that was mapped to the origin), we have

$$\pi_{t,u}(\mathbf{G}(p)) = \mathbf{G}(p)_{u,u} \tag{2}$$

Otherwise, and in general,

$$\pi_{t,u}(\mathbf{G}(p)) = \mathbf{G}(p)_{t,t} + \mathbf{G}(p)_{u,u} - 2\mathbf{G}(p)_{t,u} \tag{3}$$

Applying this to all pairs of distinct vertices induces a linear map π_K from the set $\mathscr{G}$ to the set of symmetric v by v complex matrices with zeros on the diagonal.

Lemma 7. *The map π_K is injective.*

Proof. We just need to show that the kernel of π_K is 0. Let $\mathbf{M}$ be a matrix in the kernel of π_K. Starting with the first vertex at the origin, we find from Equation (2) that all of the diagonal entries, $\mathbf{M}_{u,u}$ must vanish. Then, from Equation (3), all the off diagonal entries of $\mathbf{M}$ must vanish as well.

Lemma 8. *p is congruent to q iff $\pi_K(\mathbf{G}(p)) = \pi_K(\mathbf{G}(q))$ and iff $\mathbf{G}(p) = \mathbf{G}(q)$.*

Proof. The first relation follows from the definition of congruence. The second follows from Lemma 7.

Corollary 4. *The map $\mathbf{G}$ acting on the quotient $C_{\mathbb{C}^d}(\mathscr{V})/\text{congruence}$ is injective.*

Lemma 9. *$\mathscr{G}$ is the Zariski closure of $\mathbf{G}(C_{\mathbb{C}^d}(\mathscr{V}))$. Moreover, if p is generic in $C_{\mathbb{C}^d}(\mathscr{V})$, then $\mathbf{G}(p)$ is generic in $\mathscr{G}$.*

Proof. Using Corollary 4, a dimension count verifies that the image $\mathbf{G}(C_{\mathbb{C}^d}(\mathscr{V}))$ must hit an open neighborhood of $\mathscr{G}$ (ie. a subset of full dimension). The results follow as $\mathscr{G}$ is irreducible.

Equivalence of frameworks can be defined through their g-matrices as well:

Definition 19. Let $\pi_{\mathscr{E}}$ be the linear mapping from $\mathscr{G}$ to $\mathbb{C}^e$ defined by applying $\pi_{t,u}$ to each of the edges in $\mathscr{E}(\Gamma)$.

ρ is equivalent to σ, iff $\pi_{\mathscr{E}}(\mathbf{G}(\rho)) = \pi_{\mathscr{E}}(\mathbf{G}(\sigma))$.

If ρ is generic in $C_{\mathbb{C}^d}(\Gamma)$, then (assuming $v \geq d+1$) $\pi_{\mathscr{E}}(\mathbf{G}(\rho))$ is generic in $\pi_{\mathscr{E}}(\mathscr{G})$.

The following Lemma will be useful when examining the cardinality of a fiber of $\pi_{\mathscr{E}}$.

Lemma 10. *Let $\mathbf{M}$ be any matrix in $\mathscr{G}$. If $\pi_{\mathscr{E}}(\mathbf{M})$ is real valued, there must be an even number of non real matrices in $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{M}))$.*

Proof. $\pi_{\mathscr{E}}$ is defined over $\mathbb{R}$ and thus if $\mathbf{M}_0$ is in $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{M}))$, so must its complex conjugate $\overline{\mathbf{M}}_0$. If such an $\mathbf{M}_0$ is not real, then it is not equal to its conjugate.

The following lemma is useful above in the proof of Lemma 3.

Lemma 11. *Let Γ be generically locally rigid (in $\mathbb{C}^d$). Let ρ be generic in $C_{\mathbb{C}^d}(\Gamma)$. Let σ be equivalent to ρ. Then σ is infinitesimally rigid.*

Proof. If Γ has less than $d+2$ vertices and is generically locally rigid, it must be a simplex, and we are done.

From Corollary 4 and Lemma 9, the set of congruence classes of configurations has dimension $\dim(\mathscr{G})$, which is $v*d - \binom{d+1}{2}$. Due to local rigidity, its measurement set, $\pi_{\mathscr{E}}(\mathscr{G})$, has the same dimension.

Similarly, the set of frameworks with a degenerate affine span must map to g-matrices with rank no greater than $d-1$, and thus their measurement set must have dimension at most $v*(d-1) - \binom{d}{2}$. Thus such degenerate measurements are non generic in $\pi_{\mathscr{E}}(\mathscr{G})$.

Meanwhile, the set of infinitesimally flexible frameworks with non-degenerate span, is non generic in $C_{\mathbb{C}^d}(\mathscr{V})$, and so has dimension no larger than $v*d-1$. Its measurement set has dimension no larger than $v*d - 1 - \binom{d+1}{2}$. Thus the infinitesimally flexible measurements are non generic in the measurement set.

Thus a generic ρ cannot map under the edge squared-length map to any measurement arising from an infinitesimally flexible framework.

A real valued matrix in $\mathscr{G}$ corresponds with an s-valued configuration. At the heart of this correspondence is Sylvester's law of inertia.

Law 1 *Suppose* $\mathbf{M}$ *is a real valued symmetric matrix of size* $v-1$ *and rank* d*. Suppose that* $\mathbf{M} = \mathbf{B}^t\mathbf{D}\mathbf{B}$*, where* $\mathbf{B}$ *is a real non-singular matrix, and where* $\mathbf{D}$ *is a real diagonal matrix with* s *negative diagonal entries,* $d-s$ *positive diagonal entries, and* $v-1-d$ *zero diagonal entries. Let us call the triple* $(s, d-s, v-1-d)$ *the* signature *of* $\mathbf{D}$.

Then $\mathbf{M}$ *cannot be written as* $\mathbf{M} = \mathbf{B}'^t\mathbf{D}'\mathbf{B}'$*, where* $\mathbf{B}'$ *is real non-singular and* $\mathbf{D}'$ *is real diagonal with a different signature. Thus we can call* $(s, d-s, v-1-d)$ *the signature of* $\mathbf{M}$.

Since every real symmetric matrix has an orthogonal eigen-decomposition, it must have a signature.

Lemma 12. *Suppose some* $\mathbf{M} \in \mathscr{G}$ *has all real entries and has signature* $(s, d'-s, v-1-d')$ *for some* s *and* d' *(with* $d' \leq d$*). There exists an s-valued configuration* p *with an affine span of dimension* d' *and with* $\mathbf{G}(p) = \mathbf{M}$.

Proof. By assumption $\mathbf{M} = \mathbf{B}^t\mathbf{D}\mathbf{B}$ where $\mathbf{D}$ has signature $(s, d'-s, v-1-d')$. Wlog, let us assume that the entries in $\mathbf{D}$ appear in an order that matches the signature. Let us drop the last $v-1-d'$ rows of $\mathbf{B}$. Let us divide the jth row of $\mathbf{B}$ by $\sqrt{|\mathbf{D}_{j,j}|}$ to obtain an d' by $v-1$ matrix $\mathbf{P}'$. Then we can write $\mathbf{M} = \mathbf{P}'^t\mathbf{S}\mathbf{P}'$, where $\mathbf{S}$ is an d' by d' diagonal "signature" matrix with its first s diagonal entries of -1 and remaining $d'-s$ diagonal entries of 1. Since $\mathbf{B}$ is non-singular, $\mathbf{P}'$ has rank d'.

Multiplying the first s rows of $\mathbf{P}'$ by $\sqrt{1}$, we can write $\mathbf{M} = \mathbf{P}^t\mathbf{P}$. The columns of $\mathbf{P}$ (along with the origin) then give us the complex coordinates of an s-valued configuration $p \in C_{\mathbb{C}^d}(\mathscr{V})$ with $\mathbf{G}(p) = \mathbf{M}$.

Remark 5. When $d' < d$, this does not rule out the possibility of other frameworks with a different dimensional affine span, and different real metric signature. When $d' = d$, Corollary 5 (below) will in fact rule out any other signatures and span dimensions.

Lemma 13. *Let* $p \in C_{\mathbb{C}^d}(\mathscr{V})$ *be an s-valued configuration, then* $\mathbf{G}(p)$ *is real. If* p *has an affine span of dimension* $d' \leq d$*, then* $\mathbf{G}(p)$ *has rank no more than* d'*. Moreover, if* p *has an affine span of dimension* d*, then* $\mathbf{G}(p)$ *has signature* $(s, d-s, v-1-d)$.

Proof. Since p is s-valued, $\mathbf{G}(p)$ can be written in coordinates as $\mathbf{P}'^t\mathbf{S}\mathbf{P}'$, where $\mathbf{P}'$ is a d by $v-1$ real matrix. And $\mathbf{S}$ is a diagonal matrix with s entries of -1 and $d-s$ entries of 1. The rank of $\mathbf{G}(p)$ cannot exceed the rank of $\mathbf{P}'$ which is d'.

If the affine span of p has dimension d, then $\mathbf{P}'$ has rank d. Since the rows of $\mathbf{P}'$ are linearly independent, we can use those rows as the first d rows of a non singular $v-1$ by $v-1$ matrix $\mathbf{B}$. We can use $\mathbf{S}$ as the upper left block of a diagonal matrix $\mathbf{D}$ with the rest of the entries zeroed out. Then we can write $\mathbf{M} = \mathbf{B}^t\mathbf{D}\mathbf{B}$ giving us the stated signature.

Corollary 5. *Let* $p \in C_{\mathbb{C}^d}(\mathscr{V})$ *be an s-valued configuration with an affine span of dimension* d*. Let* $q \in C_{\mathbb{C}^d}(\mathscr{V})$ *be an s'-valued configuration that is congruent to* p*. Then* q *has an affine span of dimension* d *and* $s = s'$

Proof. From Lemma 13, $\mathbf{G}(p)$ has signature $(s, d-s, v-1-d)$. By the congruence assumption and Corollary 4, , $\mathbf{G}(p) = \mathbf{G}(q)$. As $\mathbf{G}(q)$ has rank d, q must have an affine span no less than d, and thus equal to d. From Lemma 13, $\mathbf{G}(q)$ must have signature $(s', d-s', v-1-d)$. Thus $s = s'$.

Now we can establish that when there is a GGR subgraph, the signature of all real matrices in a fiber of $\pi_{\mathscr{E}}$ is fixed.

Lemma 14. *Let Γ be a graph and Γ_0 a GGR subgraph with v_0 vertices where $v_0 \geq d+1$. Let ρ be an s-valued framework in $C_{\mathbb{C}^d}(\Gamma)$ for some s, with configuration p. Suppose also that the affine span of the vertices of Γ_0 in p is all of $\mathbb{C}^d$. Then all of the real matrices in the fiber $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{G}(\rho)))$ must have signature $(s, d-s, v-1-d)$.*

Proof. Wlog, let Γ_0 include the first vertex, and let its vertex set be $\mathscr{V}_0$. We denote by p_0 the configuration p restricted to $\mathscr{V}_0$. p_0, as a restriction of p, is s-valued.

Let $\mathbf{M}$ be any real matrix in the fiber, and let it have signature $(s', d'-s', v-1-d')$ for some s' and d'. From Lemma 12, there must be some q, an s'-valued configuration, with $\mathbf{G}(q) = \mathbf{M}$. When restricted to $\mathscr{V}_0$, the configuration q_0 must also be s'-valued. Since Γ_0 is complex GGR, p_0 must be congruent to q_0. Then from Corollary 5 q_0 must be s-valued and have affine span of dimension d. Thus $s = s'$. Since q, as a super-set of q_0, must have affine span of dimension d, then from Lemma 13, M must have signature $(s, d-s, v-1-d)$.

Definition 20. Let V and W be irreducible complex algebraic sets of the same dimension and $f : V \to W$ be a surjective (or just dominant) algebraic map, all defined over $\mathbf{k}$. Then the number of points in the fiber $f^{-1}(w)$ for any generic $w \in W$ is a constant. This constant is called the *algebraic degree* of f.

With this, we can complete the proof of Theorem 5 by applying a degree argument:

Proof. We will assume Γ is generically locally rigid, otherwise we are already done.

Let ρ be generic in $C_{\mathbb{E}^d}(\Gamma)$. From Lemma 13 $\mathbf{G}(\rho)$ is real with signature $(0, d, v-1-d)$ (ie. it is PSD). Because of the existence of a GGR subgraph, from Lemma 14, all of the real matrices in the fiber $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{G}(\rho)))$ must have the same signature. From Lemma 13 and Corollary 4, these matrices are in one to one correspondence with the congruence classes $[\rho_i]$ of equivalent frameworks in $C_{\mathbb{E}^d}(\Gamma)$. Since Γ is not GGR, from Theorem 3, there must be an even number of such classes and thus an even number of real matrices in the fiber.

From Lemma 10, there are an even number of non real matrices in the fiber and we see that the total cardinality of $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{G}(\rho)))$ is even. Since $\pi_{\mathscr{E}}(\mathbf{G}(\rho))$ is generic in the image $\pi_{\mathscr{E}}(\mathscr{G})$, this means that the *algebraic degree* of $\pi_{\mathscr{E}}$ must be even.

Now suppose σ is generic in $C_{\mathbb{S}^d}(\Gamma)$, which we model as a generic s-valued framework in $C_{\mathbb{C}^d}(\Gamma)$. $\mathbf{G}(\sigma)$ is real valued and has signature $(s, d-s, v-1-d)$. From Lemma 14 all of the real matrices in the fiber $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{G}(\sigma)))$ must have the same signature $(s, d-s, v-1-d)$.

Since $\mathbf{G}(\sigma)$ is real, then so is $\pi_{\mathscr{E}}(\mathbf{G}(\sigma))$ so from Lemma 10 there must be an even number of non real matrices in the fiber $\pi_{\mathscr{E}}^{-1}(\pi_{\mathscr{E}}(\mathbf{G}(\sigma)))$, and thus an even number of real matrices in the fiber, all with signature $(s, d-s, v-1-d)$.

From Lemma 13 and Corollary 4, these are in one to one correspondence with the congruence classes $[\sigma_i]$ of equivalent s-valued frameworks in $C_{\mathbb{C}^d}(\Gamma)$. Thus Γ is generically globally flexible in $\mathbb{S}^d$.

Remark 6. The reasoning in the above proof does not hold when Γ does not have the required GGR subgraph. In particular, the non-GGR graph G_3 of Jackson and Owen [8] generically has an *odd* number (namely 45) of equivalent complex realizations in $\mathbb{C}^2$.

8 Extension to Hyperbolic Space

Combining ideas from the previous section with results from Connelly and Whiteley [5], we can transfer the property of generic global rigidity to hyperbolic space $\mathbb{H}^d$ as well.

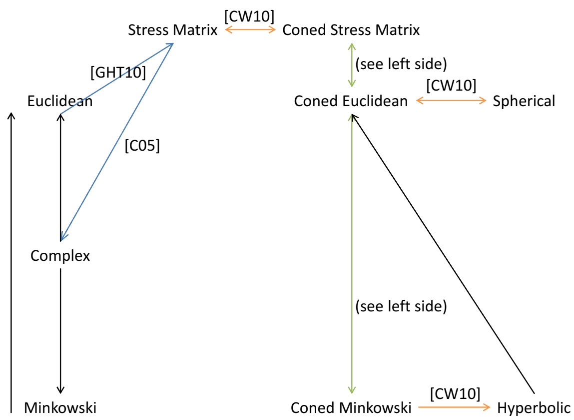

Fig. 2: Implications between generic global rigidity in various spaces. Black lines show implications proven in this paper.

Corollary 6. *A graph Γ is generically globally rigid in $\mathbb{E}^d$ iff it is generically globally rigid in $\mathbb{H}^d$.*

This can be done using the coning operation explored in [5], and the proof is developed below.

Definition 21. Given a graph Γ and a new vertex u, the *coned graph* $\Gamma * \{c\}$ is the graph obtained starting with Γ, adding the vertex c and adding an edge connecting c to each vertex in Γ.

Theorem 7 (Connelly and Whiteley [5]). *A graph Γ is generically globally rigid in $\mathbb{E}^d$ iff $\Gamma * \{c\}$ is generically globally rigid in $\mathbb{E}^{d+1}$.*

(This theorem is proven using an argument about equilibrium stress matrices. See Figure 2).

By modeling spherical d-space within a Euclidean d+1 space, Connelly and Whiteley then show the equivalence between Euclidean GGR of $\Gamma * \{c\}$ and spherical GGR of Γ.

In a similar manner, one can model hyperbolic space $\mathbb{H}^d$ within the d+1 dimensional pseudo Euclidean space that has one negative coordinate in its signature. We denote this *Minkowski space* as $\mathbb{M}^{d+1}$. In particular, we model $\mathbb{H}^d$ as the subset of vectors $\mathbf{v} \in \mathbb{M}^{d+1}$ such that $|\mathbf{v}|^2 = -1$ under the Minkowski metric, and such that $\mathbf{v}_1 > 0$, where $\mathbf{v}_1$ is the first coordinate of $\mathbf{v}$. For two vectors $\mathbf{v}$ and $\mathbf{w}$ on this "hyperbolic locus", their distance in $\mathbb{H}^d$ corresponds to the arcosh of their Minkowski inner product.

8.1 *Proof of Corollary* $\Rightarrow$

We begin with a hyperbolic lemma that mirrors a spherical lemma in [5].

Lemma 15. *Let ρ and σ be two equivalent and non congruent frameworks of Γ in $\mathbb{H}^d$, then there is a corresponding pair $(\rho''_{\mathbb{M}}, \sigma''_{\mathbb{M}})$ of equivalent and non congruent frameworks of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$. Moreover, if ρ (or σ) is generic in $\mathbb{H}^d$, then we can find a corresponding $\rho''_{\mathbb{M}}$ (or $\sigma''_{\mathbb{M}}$) that is generic in $\mathbb{M}^{d+1}$.*

Proof. Given ρ and σ, we model these as $\rho_{\mathbb{M}}$ and $\sigma_{\mathbb{M}}$, two frameworks of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$, with the cone vertex c at the origin and the rest of the vertices on the hyperbolic locus. For each vertex

$t \in \mathscr{V}(\Gamma)$, we pick a generic positive scale α_t and multiply all of the $d+1$ coordinates of $\rho_{\mathbb{M}}(t)$ and $\sigma_{\mathbb{M}}(t)$ by this α_t. Let us call the resulting pair, $\rho'_{\mathbb{M}}(t)$ and $\sigma'_{\mathbb{M}}(t)$. As in [5], $\rho'_{\mathbb{M}}(t)$ and $\sigma'_{\mathbb{M}}(t)$ are equivalent and non congruent in $\mathbb{M}^{d+1}$. By translating these frameworks by some generic offset, we obtain the desired pair $\rho''_{\mathbb{M}}$ and $\sigma''_{\mathbb{M}}$.

Proof (Proof of corollary $\Rightarrow$). Suppose a graph Γ is not GGR in $\mathbb{H}^d$ then from Lemma 15, $\Gamma * \{c\}$ is not GGR in $\mathbb{M}^{d+1}$. Then From Theorem 4, $\Gamma * \{c\}$ is not GGR in $\mathbb{E}^{d+1}$. Then from Theorem 7, Γ is not GGR in $\mathbb{E}^d$. See Figure 2.

8.2 Proof of Corollary $\Leftarrow$

In order to prove the other direction we restrict ourselves to Minkowski frameworks that can be moved to the hyperbolic locus using positive scaling.

Definition 22. We say that a framework ρ of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$ is *upper coned* if for all vertices $t \in \mathscr{V}(\Gamma)$, we have $|\rho(t) - \rho(c)|^2 < 0$ and $(\rho(t) - \rho(c))_1 > 0$. We say that ρ is *lower coned* if for all vertices $t \in \mathscr{V}(\Gamma)$, we have $|\rho(t) - \rho(c)|^2 < 0$ and $(\rho(t) - \rho(c))_1 < 0$.

The following lemma is the needed partial converse of Lemma 15.

Lemma 16. *Let ρ and σ be two equivalent and non congruent frameworks of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$. And let us also assume that ρ and σ are upper coned. Then there is a corresponding pair $(\rho_{\mathbb{H}}, \sigma_{\mathbb{H}})$ of equivalent and non congruent frameworks of Γ in $\mathbb{H}^d$. Moreover, if ρ (or σ) is generic in $\mathbb{M}^{d+1}$, then $\rho_{\mathbb{H}}$ (or $\sigma_{\mathbb{H}}$) is generic in $\mathbb{H}^d$.*

Proof. Given ρ and σ, we first translate the frameworks, moving the cone vertex, c, to the origin in $\mathbb{M}^{d+1}$. Let us call the resulting pair ρ' and σ'. For each vertex $t \in \mathscr{V}(\Gamma)$, we then divide all of the $d+1$ coordinates of $\rho'(t)$ and $\sigma'(t)$ by the positive quantity, $-|\rho(t) - \rho(c)|^2$ (which is the same as $-|\sigma(t) - \sigma(c)|^2$). Let us call the resulting pair, ρ'' and σ''. Due to our upper coned assumption, these vertices all lie on the hyperbolic locus and correspond to a pair of frameworks $\rho_{\mathbb{H}}$ and $\sigma_{\mathbb{H}}$ of Γ in $\mathbb{H}^d$. As in [5], the resulting frameworks, $\rho_{\mathbb{H}}$ and $\sigma_{\mathbb{H}}$, of Γ are equivalent, non congruent, and generic in $\mathbb{H}^d$.

In order to ultimately get upper coned Minkowski frameworks, we also define the following special framework classes.

Definition 23. We say that a framework ρ of $\Gamma * \{c\}$ in $\mathbb{E}^{d+1}$ is *spiky* if for one vertex $t_0 \in \mathscr{V}(\Gamma)$, we have $|\rho(t_0) - \rho(c)| > 2$ and for all edges $(t,u) \in \mathscr{E}(\Gamma)$, we have $|\rho(t) - \rho(u)| < \frac{1}{v}$.

Definition 24. We say that a framework ρ of $\Gamma * \{c\}$ in $\mathbb{F}^{d+1}$ is *upper cylindrical* if for all vertices $t \in \mathscr{V}(\Gamma)$, we have $(\rho(t) - \rho(c))_1 > 1$ and $\sum_{i=2}^{d+1} (\rho(t) - \rho(c))_i^2 < 1$.

Lemma 17. *Let Γ be connected. If a framework ρ of $\Gamma * \{c\}$ in $\mathbb{E}^{d+1}$ is spiky, then it is congruent to a framework which is upper cylindrical.*

Proof. We can find a rotation that moves $\rho(t_0) - \rho(c)$ onto the first axis, with a first coordinate greater than 2. Since Γ is connected, it has diameter no larger than v. From the triangle inequality, all of the coordinates of all of the vertices must satisfy the upper cylindrical conditions.

Lemma 18. *Let ρ and σ be two upper cylindrical frameworks of $\Gamma * \{c\}$ in $\mathbb{E}^{d+1}$. Then the resulting frameworks from the Pogorelov map to $\mathbb{M}^{d+1}$, $(\tilde{\rho},\tilde{\sigma}) := P(\rho,\sigma)$, are both upper cylindrical.*

Proof. This follows from directly the "coordinate swapping" interpretation of the Pogorelov map from Remark 4.

Lemma 19. *If a framework ρ of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$ is upper cylindrical, then it is upper coned.*

Proof. By definition, the first coordinates of all vertices have the required sign. Moreover, for any $t \in \mathscr{V}(\Gamma)$,

$$|\rho(t)-\rho(c))|^2 = -(\rho(t)-\rho(c)))_1^2 + \sum_{i=2}^{d+1} (\rho(t)-\rho(c)))_i^2 < 0. \tag{4}$$

And thus it is upper coned.

With these simple facts established, we can now apply the machinery from Section 6 to the problem at hand.

Lemma 20. *Let $\Gamma * \{c\}$ be generically locally rigid in $\mathbb{E}^{d+1}$. Suppose $\Gamma * \{c\}$ is not GGR in $\mathbb{E}^{d+1}$, then $\Gamma * \{c\}$ has an pair of generic frameworks in $\mathbb{M}^{d+1}$, that are equivalent, non congruent, and upper coned.*

Proof. The proof follows that of Section 6. The only issue is ensuring the upper coned-ness of the result.

When picking the component E (see Definition 12) we choose a component of E^+ such that E contains some non-congruent pair, $\dim(\pi_1(E)) = v * d$, and such that $\pi_1(E)$ contains a framework ρ that is spiky.

Since the set of frameworks that are spiky is of dimension $v * d$, and by assumption, $\Gamma * \{c\}$ is not GGR in $\mathbb{E}^{d+1}$, and thus GGF in $\mathbb{E}^{d+1}$, the projection $\pi_1(E^+ \backslash C^+)$ must include a set of spiky frameworks with dimension $v * d$. Thus, at least one component with the stated properties must exist. We will chose one such component and will call it E.

Pick an $e := (\rho,\sigma) \in E$ in the fiber above ρ. Since ρ is spiky, and spikiness only depends on edge lengths, σ must be spiky as well. Next, we perturb e in E to get $e' =: (\rho',\sigma')$ that is generic in E. Since spikiness is an open property, for small enough perturbations, both ρ' and σ' will still be spiky.

Since $\Gamma * \{c\}$ is generically locally rigid in $\mathbb{E}^{d+1}$, Γ must be connected. Thus from Lemma 17, we can choose an upper cylindrical σ'^c that is congruent to σ' and an upper cylindrical ρ'^c that is congruent to ρ'. From Lemma 3, since e' is generic in E the point $e'^c := (\rho'^c,\sigma'^c)$ must be in E as well.

Next we perturb e'^c within E to get $e'^{c\prime} =: (\rho'^{c\prime},\sigma'^{c\prime})$ which is generic in E. Since upper cylindricality is an open property, for small enough perturbations, both $\rho'^{c\prime}$ and $\sigma'^{c\prime}$ will still be upper cylindrical.

Now when we apply the Pogorelov map, $(\widetilde{\rho'^{c\prime}},\widetilde{\sigma'^{c\prime}}) := P(e'^{c\prime})$. As in the proof of Theorem 4, $\widetilde{\rho'^{c\prime}}$ and $\widetilde{\sigma'^{c\prime}}$ are equivalent, non congruent and generic frameworks in $\mathbb{M}^{d+1}$. From Lemma 18 both $\widetilde{\rho'^{c\prime}}$ and $\widetilde{\sigma'^{c\prime}}$ must be upper cylindrical, and from Lemma 19, both $\widetilde{\rho'^{c\prime}}$ and $\widetilde{\sigma'^{c\prime}}$ must be upper coned,

Proof (Proof of corollary ¡=). Suppose a graph Γ is not GGR in $\mathbb{E}^d$ then from Theorem 7, $\Gamma * \{c\}$ is not GGR in $\mathbb{E}^{d+1}$. Then from Lemma 20, $\Gamma * \{c\}$ has an pair of generic frameworks in $\mathbb{M}^{d+1}$ that are equivalent, non congruent, and upper coned. Then from Lemma 16, Γ is not GGR in $\mathbb{H}^d$.

Remark 7. In Section 7 of [5], there is a brief sketch describing how to directly use a Pogorelov type map to equate Euclidean GGR and hyperbolic GGR. That discussion does not go into the details showing that their construction hits an open neighborhood of frameworks (ie. a generic framework), which is the main technical contribution of our Theorem 4.

8.3 Hyperbolic GGF

Using coning, we can also prove a hyperbolic version of Theorem 5, namely:

Corollary 7. *If a graph Γ is not GGR in $\mathbb{H}^d$, and it has a GGR subgraph Γ_0 with $d+1$ or more vertices, then Γ must be GGF in $\mathbb{H}^d$.*

Proof. Having established that generic global rigidity transfers between Pseudo Euclidean spaces and through coning, we know that $\Gamma * \{c\}$, is not GGR in $\mathbb{M}^{d+1}$. Likewise, it has a coned subgraph with at least $d+2$ vertices, $\Gamma_0 * \{c\}$, that is GGR in $\mathbb{M}^{d+1}$. Thus, from Theorem 5, $\Gamma * \{c\}$ must be GGF in $\mathbb{M}^{d+1}$.

Let ρ be a framework of Γ in $\mathbb{H}^d$. We model this as $\rho_{\mathbb{M}}$, a framework of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$, with the cone vertex c at the origin and the rest of the vertices on the hyperbolic locus. For each vertex $t \in \mathscr{V}(\Gamma)$, we pick a generic positive scale α_t and multiply all of the $d+1$ coordinates of $\rho_{\mathbb{M}}(t)$ by this α_t. Let us call the resulting framework $\rho'_{\mathbb{M}}(t)$. By translating this frameworks by some generic offset, we obtain $\rho''_{\mathbb{M}}$, a generic framework of the coned graph in $\mathbb{M}^{d+1}$. Since the α_t are all positive, $\rho''_{\mathbb{M}}$ must be upper coned.

Since $\Gamma * \{c\}$ is GGF in $\mathbb{M}^{d+1}$, $\rho''_{\mathbb{M}}$ must have an equivalent and non-congruent framework, $\sigma''_{\mathbb{M}}$. From Lemma 21 (below), we can choose $\sigma''_{\mathbb{M}}$ to be upper coned. Then from Lemma 16, there must be a framework, σ, of Γ in $\mathbb{H}^d$, that is equivalent and non congruent to ρ.

Lemma 21. *Let Γ be a connected graph. Let (ρ, σ) be a pair of equivalent frameworks of $\Gamma * \{c\}$ in $\mathbb{M}^{d+1}$. Let us also assume that ρ is in general position. If ρ is upper coned, then either σ is upper coned or it is lower coned.*

Proof. Let t and u be two vertices of $\mathscr{V}(\Gamma)$ that are connected by an edge in Γ. Along with the edges $\{t, c\}$ and $\{u, c\}$, this defines a triangle T, which is a subgraph of $\Gamma * \{c\}$. Since σ is equivalent to ρ, these frameworks when restricted to T, must be, by definition, congruent.

Since ρ is in general position, from Corollary 8 these two frameworks of T must be strongly congruent. Thus, there is an orthogonal transform of $\mathbb{M}^{d+1}$ mapping $(\rho(t) - \rho(c))$ to $(\sigma(t) - \sigma(c))$ and mapping $(\rho(u) - \rho(c))$ to $(\sigma(u) - \sigma(c))$. An orthogonal transform either maps the entire upper cone to the upper cone, or it maps the entire upper cone to the lower cone. Since Γ is connected, this makes σ either upper coned or lower coned. (Moreover, by negating all of the coordinates in σ we can always obtain an upper coned equivalent framework).

9 Algebraic Geometry Background

We start with some preliminaries from real and complex algebraic geometry, somewhat specialized to our particular case. For a general reference, see, for instance, the book by Bochnak, Coste, and Roy [2]. Much of this is adapted from [7].

Definition 25. An affine, real (resp. complex) *algebraic set* or *variety* V defined over a field $\mathbf{k}$ contained in $\mathbb{R}$ (resp. $\mathbb{C}$) is a subset of $\mathbb{R}^n$ (resp $\mathbb{C}^n$) that is defined by a set of algebraic equations with coefficients in $\mathbf{k}$.

An algebraic set is closed in the Euclidean topology.

An algebraic set is *irreducible* if it is not the union of two proper algebraic subsets defined over $\mathbb{R}$ (resp $\mathbb{C}$). Any reducible algebraic set V can be uniquely described as the union of a finite number of maximal irreducible subsets called the *components* of V.

A real (resp. complex) algebraic set has a real (resp. complex) *dimension* $\dim(V)$, which we will define as the largest t for which there is an open subset of V, in the Euclidean topology, that is isomorphic to $\mathbb{R}^t$ (resp. $\mathbb{C}^t$). Any algebraic subset of an irreducible algebraic set must be of strictly lower dimension.

A point x of an irreducible algebraic set V is *smooth* (in the differential geometric sense) if it has a neighborhood that is smoothly isomorphic to $\mathbb{R}^{\dim(V)}$ (resp. $\mathbb{C}^{\dim(V)}$). (Note that in a real variety, there may be points with neighborhoods isomorphic to $\mathbb{R}^n$ for some $n < \dim(V)$; we will not consider these points to be smooth.)

Definition 26. Let $\mathbf{k}$ be a subfield of $\mathbb{R}$. A *semi-algebraic set* S defined over $\mathbf{k}$ is a subset of $\mathbb{R}^n$ defined by algebraic equalities and inequalities with coefficients in $\mathbf{k}$; alternatively, it is the image of a real algebraic set (defined only by equalities) under an algebraic map with coefficients in $\mathbf{k}$. A semi-algebraic set has a well defined (maximal) dimension t.

The real *Zariski closure* of S is the smallest real algebraic set defined over $\mathbb{R}$ containing it. (Loosely speaking, we can get an algebraic set by keeping all algebraic equalities and dropping the inequalities. We may need to enlarge the field to cut out the smallest algebraic set containing S but a finite extension will always suffice.)

We call S *irreducible* if its real Zariski closure is irreducible. An irreducible semi-algebraic set S has the same real dimension as its real Zariski closure.

A point on S is smooth if it has a neighborhood in S smoothly isomorphic to $\mathbb{R}^{\dim(S)}$.

Lemma 22. *The image of an irreducible real algebraic or semi-algebraic set under a polynomial map is an irreducible semi-algebraic set. The image of an irreducible complex algebraic set under a polynomial map is an irreducible complex algebraic set, possibly with a finite number of subvarieties cut out from it.*

We next define genericity in larger generality and give some basic properties.

Definition 27. A point in a (semi-)algebraic set V defined over $\mathbf{k}$ is *generic* if its coordinates do not satisfy any algebraic equation with coefficients in $\mathbf{k}$ besides those that are satisfied by every point on V.

Almost every point in an irreducible (semi) algebraic set V is generic.

Remark 8. Note that the defining field might change when we take the real Zariski closure V of a semi-algebraic set S. For example, in $\mathbb{R}^1$, the single point $\sqrt{2}$ can be described using equalities and inequalities with coefficients in $\mathbb{Q}$, and thus it is semi-algebraic and defined over $\mathbb{Q}$. But as a real variety, the defining equation for this single-point variety requires coordinates in $\mathbb{Q}(\sqrt{2})$. Indeed, the smallest variety that contains the point $\sqrt{2}$ and that is defined over $\mathbb{Q}$ must also include the point $-\sqrt{2}$. However, this complication does not matter for the purposes of genericity.

Specifically, if $\mathbf{k}$ is a finite algebraic extension of $\mathbb{Q}$ and x is a generic point in an irreducible semi-algebraic set S defined over $\mathbf{k}$, then x is also generic in V, the real Zariski closure of S, defined over an appropriate field. This follows from a three step argument. First, a dimensionality argument shows that V must be a component of $V_{\mathbf{k}}^{+}$, the smallest real algebraic variety that is defined over $\mathbf{k}$

and contains S. Second, it is a standard algebraic fact that if a real (resp. complex) variety W^+ is defined over $\mathbf{k}$, a subfield of $\mathbb{R}$ (resp. $\mathbb{C}$), then any of its components is defined over some field $\mathbf{k}'$, a subfield of $\mathbb{R}$ (resp. $\mathbb{C}$), which is a finite extension of $\mathbf{k}$. Finally, from Lemma 23 (below), any non generic point $x \in V$ (ie. satisfying some algebraic equation with coefficients in $\mathbf{k}'$) must also satisfy some algebraic equation with coefficients in $\mathbf{k}$ (or even $\mathbb{Q}$) that is non-zero over V.

Lemma 23. *Let $\mathbf{k}'$ be some algebraic extension of $\mathbb{Q}$. Let V be an irreducible algebraic set defined over $\mathbf{k}'$. Suppose a point $x \in V$ satisfies an algebraic equation ϕ with coefficients in $\mathbf{k}'$ that is non-zero over V, then x must also satisfy some algebraic equation ψ with coefficients in $\mathbb{Q}$ that is non-zero over V.*

Proof. Let H be the Galois group of the (normal closure of) $\mathbf{k}'$ over $\mathbb{Q}$. For $h_i \in H$, denote $h_i(\phi)$ to be the polynomial where h_i is applied to each coefficient in ϕ. Let A be the (possibly empty) "annihilating set", such that $\forall h_i \in A$, $h_i(\phi)$ vanishes identically over V.

Let

$$\phi^\Sigma := \phi + \sum_{h_i \in A} \lambda_i h_i(\phi) \tag{5}$$

(Where the $\lambda_i \in \mathbb{Q}$ are simply an additional set of blending weights).

ϕ^Σ has the following properties:

- $\phi^\Sigma(x) = 0$.
- (For almost every λ), for any $h \in H$, $h(\phi^\Sigma)$ does not vanish identically over V. This follows since $h(\phi^\Sigma)$ is made up of a sum of $|A|+1$ polynomials, where no more than $|A|$ of them can vanish identically over V. Under almost any blending weights λ, their sum will not cancel.

Let

$$\psi := \prod_{h_i \in H} h_i(\phi^\Sigma) \tag{6}$$

ψ has the following properties:

- $\psi(x) = 0$.
- ψ does not vanish over V.
- $h(\psi) = \psi$. Thus ψ has coefficients in the fixed field, $\mathbb{Q}$.

The following propositions are standard [7]:

Proposition 1. *Every generic point of a (semi-)algebraic set is smooth.*

Lemma 24. *Let V^+ be a (semi) algebraic set, not necessarily irreducible, defined over* $\mathbf{k}$. *Let V be a component of V^+. Let x be generic in V. Then x does not lie on any other component of V^+. Moreover, any point $x' \in V^+$ that is sufficiently close to x cannot lie on any other component of V^+.*

Proof. As per Remark 8 any component must be defined over an algebraic extension of $\mathbf{k}$. The defining equations of any other component would produce an equation obstructing the genericity of x in V. Since a variety is a closed set in the Euclidean topology, no other component of V^+ can approach x.

Lemma 25. *Let V and W be (semi) algebraic sets with V irreducible, and let $f : V \to W$ be a surjective (or just dominant) algebraic map (ie. where each of the coordinates of $f(x)$ is a some polynomial expression in the coordinates of x), all defined over* $\mathbf{k}$. *Then if $x \in V$ is generic, $f(x)$ is generic inside W.*

Definition 28. The *complexification* V^* of a real variety V is the smallest complex variety that contains V [12]. The complex dimension of V^* is equal to the real dimension of V. If V is irreducible, then so is V^*. If V is defined over $\mathbf{k}$, so is V^*. A generic point in V is also generic in V^*.

10 Congruence

The following material is standard and is included here for completeness. This presentation is adapted from [6,9].

In *all* discussions in this section, we will assume that we have first translated any configuration, say $p \in C_{\mathbb{C}^d}(\mathscr{V})$ so that its first vertex lies at the origin. We then treat the rest of the vertices as vectors in $\mathbb{C}^d$, and call them the *vectors of p*.

Definition 29. We define the symmetric bilinear form $\beta(\mathbf{v},\mathbf{w})$ over pairs of vectors, $\{\mathbf{v},\mathbf{w}\}$ in $\mathbb{C}^d$ as $\beta(\mathbf{v},\mathbf{w}) := \mathbf{V}^t\mathbf{W}$ where $\mathbf{V}$ is the d by 1 (canonical) coordinate vector of $\mathbf{v}$. (No conjugation is used here). If O is an orthogonal transformation on $\mathbb{C}^d$, we have $\beta(\mathbf{v},\mathbf{w}) = \beta(O(\mathbf{v}),O(\mathbf{w}))$.

β is non degenerate: there is no non-zero vector, $\mathbf{v}$, such that $\beta(\mathbf{v},\mathbf{w}) = 0$ for all $\mathbf{w} \in \mathbb{C}^d$.

The squared length of a vector $\mathbf{v}$ is simply $\beta(\mathbf{v},\mathbf{v})$

With this notation, the $v-1$ by $v-1$ g-matrix has entries $\mathbf{G}(p)_{t,u} = \beta(\overrightarrow{p(t)},\overrightarrow{p(u)})$.

For the case of the pseudo Euclidean space $\mathbb{S}^d$ we define $\beta(\mathbf{v},\mathbf{w}) := \mathbf{V}^t\mathbf{S}\mathbf{W}$, where $\mathbf{S}$ is the d by d diagonal "signature matrix" having its first s diagonal entries -1, and the remaining diagonal entries 1.

Lemma 26. *Let p_0 be a configuration of $d+1$ points in $\mathbb{C}^d$, with affine span of dimension d. Then $\mathbf{G}(p_0)$ has rank d. The same is true in a pseudo Euclidean space $\mathbb{S}^d$.*

Proof. The matrix $\mathbf{G}(p_0)$ represents the form β, over all of $\mathbb{C}^d$, expressed in the basis defined by the vectors of p_0. Since β is a non-degenerate form, $\mathbf{G}(p_0)$ must have rank d.

Lemma 27. *Let p_0 and q_0 be two congruent configurations of $a+1$ points in $\mathbb{C}^d$, both with affine span of dimension a. Then p_0 is strongly congruent to q_0. The same is true in a pseudo Euclidean space $\mathbb{S}^d$.*

Proof. Since the vectors of p_0 and q_0 are in general linear position, we can find an invertible linear transform O_0 such that, for all of the vectors of p_0 and q_0, indexed by a vertex t, we have $\overrightarrow{q(t)} = O_0(\overrightarrow{p(t)})$. (The action of O_0 is uniquely defined between $span(p_0)$ and $span(q_0)$, the a-dimensional linear spaces spanned by the vectors of p_0 and the vectors of q_0.)

The matrix $\mathbf{G}(p_0)$ represents the form β, restricted to $span(p_0)$, expressed in the basis defined by the vectors of p_0, while $\mathbf{G}(q_0)$ represents β, restricted to $span(q_0)$, expressed in the basis defined by the vectors of q_0, Since $\mathbf{G}(p_0) = \mathbf{G}(q_0)$, the map O_0 must act as an isometry between all of $span(p_0)$ and $span(q_0)$.

If $a = d$ we are done. Otherwise, from Witt's theorem (see [9]), the isometric action of O_0 between $span(p_0)$ and $span(q_0)$ can be can be extended to an isometry, O, acting on all of $\mathbb{C}^d$. Thus p_0 and q_0 must be strongly congruent.

Lemma 28. *Let p and q be two congruent configurations of v points in $\mathbb{C}^d$, both with affine span of dimension a. Suppose also that $\mathbf{G}(p) = \mathbf{G}(q)$ has rank a. Then p is strongly congruent to q. The same is true in a pseudo Euclidean space $\mathbb{S}^d$.*

Proof. Since $\mathbf{G}(p)$ has rank a, it must have some a by a non-singular principal submatrix, associated with a subset of a vertices. The vertices in this subset must have a linear span of dimension a in both p and q. We denote by p_0 the configuration p restricted to the $a+1$ vertices comprised of this subset together with the first vertex (at the origin). And likewise for q_0. From Lemma 27 there must be an isometry O of $\mathbb{C}^d$, such that for any vertex t in p_0, we have $\overrightarrow{q_0(t)} = O(\overrightarrow{p_0(t)})$.

For any vertex $u \in \mathscr{V}$, by our assumption on the dimension of the affine span of p and q, we have $\overrightarrow{p(u)} \in span(p_0)$ and $\overrightarrow{q(u)} \in span(q_0)$. Since $\mathbf{G}(p_0) = \mathbf{G}(q_0)$ is invertible, the coordinates of $\overrightarrow{p(u)}$ with respect to the basis p_0, can be determined from the appropriate entries in $\mathbf{G}(p)$. Likewise, the coordinates of $\overrightarrow{q(u)}$ with respect to the basis q_0, can be determined from $\mathbf{G}(q)$. Since $\mathbf{G}(p) = \mathbf{G}(q)$ these coordinates must be the same. Thus $\overrightarrow{q(u)} = O(\overrightarrow{q(u)})$, and p and q are strongly congruent.

Corollary 8. *Let p and q be two congruent configurations of $v \geq d+1$ points in $\mathbb{C}^d$, both with a d-dimensional affine span. Then p is strongly congruent to q. If $v < d+1$, and p and q are in general position, then p is strongly congruent to q.*

The same is true in a pseudo Euclidean space $\mathbb{S}^d$.

Proof. For the first statement, we can pick d vertices, together with the first vertex at the origin, to form a subset of size $d+1$, that has a linear span of dimension d in p. We denote by p_0 the configuration p restricted to this subset. From Lemma 26, the principal submatrix of $\mathbf{G}(p)$ associated with this basis must have rank d. The result then follows from Lemma 28.

If $v \leq d+1$ and the points are in general position, then the result follows directly from Lemma 27.

Acknowledgements We would like to thank Robert Connelly, Bill Jackson, John Owen, Louis Theran, and for helpful conversations and suggestions. We would especially like to thank Walter Whiteley for sharing with us his explanation of the Pogorelov map.

References

1. Basu, S., Pollack, R., and Roy, M. *Algorithms in real algebraic geometry*, vol. 10. Springer-Verlag New York Inc, 2006.
2. Bochnak, J., Coste, M., and Roy, M. F. *Real Algebraic Geometry*. Springer, 1998.
3. Connelly, R. On generic global rigidity. *DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 4* (1991), 147–155.
4. Connelly, R. Generic global rigidity. *Discrete Comput. Geom 33*, 4 (2005), 549–563.
5. Connelly, R., and Whiteley, W. Global rigidity: the effect of coning. *Discrete & Computational Geometry 43*, 4 (2010), 717–735.
6. Frenkel, P. *New constructions in classical invariant theory*. PhD thesis, Budapest University of Technology and Economics, 2007.
7. Gortler, S., Healy, A., and Thurston, D. Characterizing generic global rigidity. *American Journal of Mathematics 132*, 4 (2010), 897–939.
8. Jackson, B., and Owen, J. The number of equivalent realisations of a rigid graph. *Arxiv preprint arXiv:1204.1228* (2012).
9. Parshin, A., and Shafarevich, I. *Algebraic geometry IV: linear algebraic groups, invariant theory*, vol. 55. Springer, 1994.
10. Saliola, F., and Whiteley, W. Some notes on the equivalence of first-order rigidity in various geometries. *Arxiv preprint arXiv:0709.3354* (2007).
11. Whiteley, W. personal communication, 2012.
12. Whitney, H. Elementary structure of real algebraic varieties. *Ann. of Math. (2) 66* (1957), 545–556.