## Singularities, Supersymmetry and Combinatorial Reciprocity

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## (Article begins on next page)

# Singularities, Supersymmetry and Combinatorial Reciprocity 

## A Dissertation Presented

B Y

Roberto E. Martinez II

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# Singularities, Supersymmetry and Combinatorial Reciprocity 


#### Abstract

This work illustrates a method to investigate certain smooth, orientable, codimension-two, real submanifolds of spheres of arbitrary odd dimension (with complements that fiber over the circle) using a novel supersymmetric quantum invariant. Algebraic (fibered) links in $S^{2 n+1}$ [310], including Brieskorn-Pham manifolds and homology spheres with exotic differentiable structure [63], [64], [374], are examples of said manifolds with a relative diffeomorphism-type that is determined by the corresponding (multivariate) Alexander polynomial [480], [257].

The twist-regularized Wess-Zumino model is a two-dimensional, interacting, (partially-broken) supersymmetric, topological (constructive) quantum field theory on a spacetime torus [222], [223], [225]. Given a suitable complex analytic superpotential $f$, the supersymmetric partition function or elliptic genus, $\mathfrak{Z}^{f}=\operatorname{Tr} \Gamma e^{-\beta H-i \sigma P-i \theta J}$ admits an explicit representation involving a ratio of Jacobi theta functions depending only the weights of $f$ and spacetime-twist parameters (op. cit.). Said genus is a weak Jacobi form and enjoys a translationalunimodular $\mathbb{Z}^{2} \ltimes \mathrm{SL}_{2}(\mathbb{Z})$-symmetry despite the model possessing no a priori conformal structure.

I propose that the elliptic genus $\mathfrak{Z}^{f}$ of the twist-regularized Wess-Zumino model with superpotential $f$ encodes the reduced Alexander polynomial $\Delta_{K_{f}}$


of the algebraic link $K_{f}$. That is, by specializing to the Steenbrink series of the mixed Hodge structure (of a corresponding fiber) [436], [437], [438], [439] from a $q$-expansion of $\mathfrak{Z}^{f}$, one may isolate the singularity spectrum, determine the eigenvalues of the Picard-Lefschetz monodromy (acting on said fiber) and compute the corresponding characteristic polynomial and the reduced Alexander polynomial as a factor [310], [315]. Moreover, a $\mathbb{Z}_{2}$-symmetry of the elliptic genus descends to classical functional equations satisfied by the Steenbrink series, Hilbert-Poincaré series of the local algebra [363], the Lefschetz zeta function of an infinite cyclic covering of (the complement of the interior of a tubular neighborhood of) $K_{f}$ [309], [352], and the reduced Alexander polynomial of $K_{f}$, all of which imply a reciprocity law for the singularity spectrum. Furthermore, the number of quantum mechanical grounds states is the zero-twist limit of the elliptic genus [225], the Fredholm index of a supercharge [474], and coincides with the rank of the middle homology group of the Milnor fiber and the dimension of the local algebra. Finally, since the isotopy-type of algebraic knots in $S^{3}$ are classified by their (univariate) Alexander polynomials [257], the corresponding moduli space of twist-regularized Wess-Zumino models admits a similar classification by the corresponding elliptic genera.

Although comparably different and quite general, the proposed method complements the observation that the Jones polynomial of links in $S^{3}$ may be interpreted as arising from Chern-Simons (gauge) theory [475].

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## Preface

Mathematics, rightly viewed, possesses not only truth, but supreme beauty

- a beauty cold and austere, without appeal to any part of our weaker nature, without the gorgeous trappings of painting or music, yet sublimely pure, and capable of a stern perfection such as only the greatest art can show. The true spirit of delight, the exaltation, the sense of being more than man, which is the touchstone of the highest excellence, is to be found in mathematics as surely as in poetry. - Bertrand Russell

The key to maintaining an unreasonable effectiveness of physics in mathematics and vice versa is through the continued investigation of topics where deep and inspiring connections bridge the two, often estranged, fields. Yearning for increased cross-fertilization, Supersymmetry (SUSY) is a purported symmetry of nature that manifests as an involution interchanging particles of integral and half-integral spin and as a rather general algebraic structure at the helm of representation theory, whose elements act on operator-valued tempered distributions related to said particles.

From a physical perspective, SUSY is believed to be spontaneously broken near or above the electroweak scale ( $\sim 246 \mathrm{GeV}$ ), which may account for the dearth of supersymmetric partner particles, or sparticles. Such scarcity of physical evidence, despite the plethora of suggestive theories and computations, forms the impetus to craft more powerful and robust particle accelerators to
delve into energetic terra incognita. From a mathematical perspective, however, SUSY is defined through a $\mathbb{Z}_{2}$-graded Lie superalgebra of operators and the curio* underlying large-scale cancellations and simplifications in certain computations. More recently, mathematicians have taken an interest in SUSY, not only for the myriad of novel intrinsic symmetries but also for its overwhelmingly predictive power for yielding new insight-even full-fledged solutions-to problems which have hitherto been completely intractable. It is therefore reasonable, if not paramount, to desire a thorough understanding of SUSY from complementary physical and mathematical perspectives. Such a compelling principle lies at the foundation of this modest work.
R. E. M. II

Cambridge, MA

[^0]
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## Litterarum radices amarae, fructus dulces.*

- Cato

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For their constant and unwavering faith in my abilities to succeed, I mention my dear family. With it, I ask their forgiveness for having effectively lost me to studies. Alas, few have been born and grown into adolescence, some have aged and passed on and many have been forsaken by the wrath of natural disasters, yet their resolve remains strong and their arms remain open ready to accept me upon my eventual return-and I shall return! And if asked why my studies have seemed to matter above all else, I shall say to them:

Although you, my dear family, have given me life, it was mathematics that has given me a reason to live....

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$$


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Had I the heavens' embroidered cloths,
Enwrought with golden and silver light,
The blue and the dim and the dark cloths
Of night and light and the half-light,
I would spread the cloths under your feet:
But I, being poor, have only my dreams;
I have spread my dreams under your feet;
Tread softly because you tread on my dreams.

- William Butler Yeats
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## Chapter o

Prologue

# Mathematics knows no races or geographic boundaries; for mathematics, the cultural world is one country. - David Hilbert 

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## o.1. Hilbert's Sixth Problem

For his address of the Le Congrès International de Mathématiciens at the Sorbonne in Paris, France, on 8 August 1900, Hilbert prepared a list of 24 unsolved problems, publishing 23 and announcing only 10, which covered virtually all
active areas of mathematics including physics. Perhaps Hilbert's most ambitious request is the sixth problem, which requires the axiomatization of physics, not unlike his work in formalizing Euclidean geometry. At that time, quantum theory had not yet been realized, as Planck's derivation of Wien's Law of blackbody radiation based on thermodynamic entropy, ideal oscillators and quantized energy was proposed on 19 October 1900, and published the following year. Relativity, too, would have to wait until 1905, when Einstein published his first treatise contextualizing the work of Maxwell, Hertz, Doppler and Lorentz. In retrospect, we understand Hilbert's sixth problem as one of classical physics without relativity. Despite this often ignored chronology, many have sought the axiomatization of quantum theory with relativity - and rightly so. The great predictive power of the union stands alone as the pinnacle of human investigation.

Beginning with the Dirac-von Neumann Axioms of quantum systems and measurements [465], mathematicians began to develop axiomatic schemes to distill and isolate the essential criteria that every reasonable quantum theory should satisfy. This was followed by Jost-Hall-Wightman Axioms of quantum field theory [444], Lehmann-Symanzik-Zimmermann and Haag-Ruelle Axioms for scattering [174], Haag-Kastler Axioms of local quantum physics [173, 174], Nelson Axioms of Euclidean Markov fields [338, 339] and, more recently, Osterwalder-Schrader Axioms of Euclidean field theory [365, 366]. For years and to the dismay of many, the only completely tractable models that satisfied these axiomatic approaches were non-interacting or free field theories and concrete composites thereof. It was not until the work of Glimm and Jaffe
that mathematical methods could be used to unify quantum theory with relativity in a mathematically consistent and highly non-trivial fashion, when the essential self-adjointness of the Hamiltonian in a non-linear, relativistic field theory in two-dimensional Minkowski spacetime was realized [145, 146, 147, 148]. The collective work of Glimm and Jaffe [149, 150], Nelson, Simon, Spencer and Symanzik, among others, in understanding the construction, existence of mass gaps and phase transitions of free and interacting quantum field theories in two and more dimensions, is the study of Constructive Quantum Field Theory. For a historical and personal account of the development of the field, consult [224].

## o.2. The Standard Model

According to the Standard Model (SM) with gauge group $\operatorname{SU}(3) \times \operatorname{SU}(2) \times$ $\mathrm{U}(1)$, eight massless, spin-0 bosons (the gluons, $\{g\}$ ), three massive, spin- 1 vector bosons (the two charged and neutral weak bosons, $W^{+}, W^{-}$and $Z$ ) and a massless, spin- 1 scalar boson (the photon, $\gamma$ ) mediate three fundamental interactions in the observable universe*: the strong force, the electroweak force and the electromagnetic force, respectively. Predicted in 1968 by the Glashow-WeinbergSalam Model (GWSM) with gauge group $\mathrm{SU}(2) \times \mathrm{U}(1)$ and observed indirectly in 1973 (Gargamelle Bubble Chamber) and directly in 1983 (UA1/2) at CERN [207], the mediators of the short-range weak nuclear force, the intermediate vector

[^2]bosons, $W^{ \pm}\left(m_{W} \approx 80.385 \mathrm{GeV}\right)^{*}$ and $\mathrm{Z}\left(m_{Z} \approx 91.1876 \mathrm{GeV}\right)$, are responsible for the stability of all interacting matter via nuclear transmutation by beta decay and electron capture. While the gluons and photon are massless by local gauge invariance, an unbroken $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge symmetry requires massless vector bosons. The $\mathrm{GWSM}^{\dagger}$ solves this mass discrepancy and preserves renormalizability by invoking the Higgs mechanism [126, 190, 191], which purports the existence of a precursor quantum field, the elusive Higgs field, that spontaneously breaks $S U(2) \times U(1)$ and manifests as a massless scalar, the photon, three asymmetrically massive vector bosons, $W^{ \pm}$and $Z$, and a massive scalar, the Higgs boson. The resulting bosons are then available to couple with elementary fermionic particles or leptons, such as the electron $e^{-}$, muon $\mu^{-}$, tauon $\tau^{-}$ and their corresponding neutrinos $v_{e^{-}}, v_{\mu^{-}}$and $v_{\tau^{-}}$.

The GWSM alone is insufficient to predict a light Higgs boson mass, in contrast to that of the photon and the ratio of those of the weak bosons. Instead the GWSM implies that it be directly proportional to an unconstrained variable, the Higgs boson self-coupling parameter, $\lambda$, by the relation $m_{h}=\sqrt{2 \lambda} v_{h}$, where $v_{h}$

[^3]is the vacuum expectation value of the Higgs boson. Precise muon lifetime experiments incorporating two-loop, Quantum Electrodynamic (QED) corrections yield a Fermi coupling $G_{F}=1.166364(5) \times 10^{-5} \mathrm{GeV}^{-2}$ (CODATA 2010), from which one infers the value $v_{h}=\frac{1}{\sqrt[4]{2} \sqrt{G_{F}}} \approx 246.221 \mathrm{GeV}$. By imposing (renormalization group-improved) unitarity bounds on the corresponding elastic scattering amplitudes, one derives the upper bound $m_{h} \leqslant 2 \sqrt[4]{2} \sqrt{\frac{\pi}{3 G_{F}}} \approx 712.664 \mathrm{GeV}$ [260, 283]. Enhancing further the GWSM with a Yang Mills SU(3)-gauge theory, Quantum Chromodynamics (QCD), yields the SM with an additional six massive, color-charged spin- $\frac{1}{2}$ fermions or quarks (up $u$, down $d$, strange $s$, charm $c$, top $t$ and bottom $b$ ) and, with their antiparticles, conspire in pairs to form the meson families (e.g., $\pi, \eta, K, D$ and $B$ ) and in triplets* to form the baryon families (e.g., nucleons, $\Lambda, \Delta, \Sigma, \Xi$, and $\Omega$ ) through the strong interaction. However, isolated quarks or anti-quarks are believed to be essentially unobservable due to their low-energy confinement $[164,384]$ and high-energy asymptotic freedom [163] which allows only a rather weak coupling with gluons. In total, there are eighteen parameters ${ }^{\dagger}$ which determine the SM: three gauge coupling parameters, three charged lepton masses, six quark masses, three flavor mixing angles, one charge-parity (CP)-violating phase, the Higgs boson mass and vacuum

[^4]expectation value (determined by the masses of the $W^{ \pm}$and $Z$ vector bosons) [396].

While the literature is rich with theoretical proposals that engage electroweak symmetry-breaking and the Higgs mechanism* in more appealing ways, the SM is most likely the simplest and definitely the most wellunderstood. According to the SM, the three neutrinos $\left(v_{e}, v_{\mu}\right.$ and $\left.v_{\tau}\right)$ and their antiparticles are massless spin- $\frac{1}{2}$ fermions. However, experimental evidence suggest neutrino oscillations between flavor types, which a priori require massive neutrinos [102]. Coupling parameter unification (e.g., grand unification), baryon asymmetry, hierarchy problem, dark matter, naturalness, etc., are additional issues which are not addressed by SM, per se. Therefore, if one is to properly model the universe (sans gravity), the SM must be modified, extended and/or subsumed accordingly.

### 0.3. Supersymmetry

Supersymmetry (SUSY) is a conjectured symmetry of nature between integer-spin particles, the mediators of the fundamental forces, and half-integerspin particles, the constituents of matter. In dimensions three and greater, a given Lagrangian represents a supersymmetric quantum theory if and only if there

[^5]

Figure 0.1. The Minimally Supersymmetry Standard Model Particles
exists an infinitesimal field transformation interchanging the integer and halfinteger spin fields and admitting an equivalent representation as a graded Lie algebra of field operators. ${ }^{*}$ Given a boson $b \in \mathfrak{B}$ and fermion $f \in \mathfrak{F}$, where $\mathfrak{B}$ and $\mathfrak{F}$ are suitable Fock spaces, the images $\hat{b}=Q b$ and $\hat{f}=Q f$, where $Q$ is a supersymmetric charge operator, are the corresponding super-partners-the former a super-fermion, the latter a super-boson. In theories with unbroken supersymmetry, the mass of super-partners is identical to their partners, while in those with broken supersymmetry, the mass of super-partners is comparatively larger, and may explain why no super-partners have yet been observed.
0.3.1. Supersymmetry and the Standard Model. In a supersymmetric extension of the SM, namely, the Minimally Supersymmetric Standard Model (MSSM), a type III THDM proposed by Dimopoulos and Georgi [112], the

[^6]squared-mass of the light, CP even, scalar component of the Higgs field, the Higgs boson, is independently quadratically and logarithmically divergent in a sharp momentum cut-off [113]. However, certain quark-squark* interactions provide perturbative counter-terms that dramatically suppress such divergences, which is one of the many appealing features of SUSY. In particular, the MSSM with soft SUSY-breaking (near the electroweak scale) postulates two Higgs doublets leading to five potentially observable Higgs particles: two vector bosons, $H^{+}$and $H^{-}$, two CP even scalars, $h$ and $H$, and a CP odd scalar, $A$, satisfying the following mass inequalities at tree level ${ }^{\dagger}$ : $m_{W^{ \pm}} \leqslant m_{H^{ \pm}} \leqslant m_{H}, m_{h} \leqslant m_{Z} \leqslant m_{H}$ and $m_{h} \leqslant m_{A} \leqslant m_{H^{ \pm}}$, respectively [170]. At one-loop level, the MSSM predicts an explicit upper bound on the light Higgs boson mass $m_{h}$ within the decoupling limit ${ }^{\ddagger}$ through the quartic coupling contributions from the aforementioned vector bosons and (broken supersymmetric) radiative corrections from the top-stop quark sector with mixing parameter $\alpha$,
$$
m_{h}^{2} \leqslant m_{Z}^{2}+\frac{3 G_{F}}{\sqrt{2} \pi^{2}}\left(m_{t, 1}^{4} \log \frac{m_{\tilde{t}}}{m_{t, 1}}+m_{t, 2}^{4} \alpha^{2}\left(6-3 \alpha^{2}\right)\right),
$$

[^7]where (in natural units) the pole top quark mass $m_{t} \approx 172.9 \mathrm{GeV}$ and is given at two different energy scales, $m_{t, 1} \approx 157 \mathrm{GeV}$ and $m_{t, 2} \approx 150$. GeV [115]. Assuming nearly maximal mixing ( $\alpha \approx 1$ ) and conjecturing $m_{\tilde{t}} \approx 1 \mathrm{TeV}$, one computes $m_{h} \leqslant 132 \mathrm{GeV}$ [172]. However, neglecting stop mixing, one computes the upper bound $m_{h} \leqslant 110 \mathrm{GeV}$ [115], which violates the LEP exclusion $m_{h}>114.4 \mathrm{GeV}$ [265].
0.3.2. Recent Discovery of a New Boson. By early 2010, groups at the Tevatron at Fermilab and the Large Hadron Collider (LHC) working independently observed curious activity in $p p$-collisions in the range $115-130 \mathrm{GeV}$. As of 2011, the CMS and ATLAS experiments at CERN improved known bounds for a light Higgs boson by exclusion to the interval $114 \mathrm{GeV} \leqslant m_{h} \leqslant 157 \mathrm{GeV}$ (at 90-95\% confidence), consistent with a TeV-scale stop mass, maximal mixing in the decoupling limit and the MSSM upper bound. By mid 2012, CERN announced the observation of a new boson with a mass of approximately $125 \cdot 3^{*} \mathrm{GeV}$ and decay channels consistent with those of a light Higgs boson predicted by the SM [24]. While many anticipate a full resolution of the experimental search for a Higgs boson in the very near future, a complete physical model predicting precisely its mass remains hitherto undiscovered.
o.3.3. Spacetime Symmetry. Beyond the axiomatic approaches, early mathematical results in quantum field theory involved spacetime symmetries. One

[^8]notable example is a curious and wide-sweeping result of Coleman and Mandula, which proved every local spacetime symmetry (under certain reasonable assumptions) must submit to a restricted form.

Proposition o. 1 (Coleman, Mandula, [88]). Let G be an arcwise-connected symmetry group of the S-matrix (in the weak operator topology), where

$$
\begin{equation*}
S=1-i(2 \pi)^{4} \delta\left(P_{\mu}-P_{\mu}^{\prime}\right) T \tag{0.1}
\end{equation*}
$$

such that the following conditions hold:

1. The group $G$ contains a subgroup locally isomorphic to the Poincaré group;
2. All particle types correspond to positive-energy representations of the Poincaré group. For any positive real $M$, there are finitely many particle types of mass less than $M$;
3. Elastic-scattering amplitudes are analytic functions of the center-of-mass energy s and invariant momentum transfer $t$ in some neighborhood of the physical region, except at normal thresholds;
4. Let $|p\rangle$ and $\left|p^{\prime}\right\rangle$ be any two one-particle momentum eigenstates, and let $\left|p, p^{\prime}\right\rangle$ be the two-particle state created from these. Then $T\left|p, p^{\prime}\right\rangle$ does not vanish except perhaps for certain isolated values of s; and,
5. The generators of $G$, written as integral operators in momentum space, have distributions for their kernels.

Then, $G$ is necessarily locally isomorphic to the direct product of an internal symmetry group and the Poincaré group.

Remark 0.3.1. Proposition 0.1 involves Lorentz invariance (1.), particle finiteness (2.), weak elastic analyticity (3.), the occurrence of scattering (4.) and, according to Coleman and Mandula, an ugly technical assumption (5.).

Although SUSY may have first been anticipated in the mathematical work of Frölicher and Nijenhuis $[137,138]$ and perhaps rediscovered by Miyazawa [318, 319], it is generally believed to have been introduced independently in the physics literature by Golfand and Likhtman [152], Volkov and Akulov [464], and Wess and Zumino [470]. In particular, Wess and Zumino introduced a renormalizable four-dimensional supersymmetric quantum field theory with cubic interaction.

Haag, Lopuszanski and Sohnius [175] generalized the Coleman-Mandula Theorem to formally include SUSY as a spacetime symmetry by considering Lie super-algebras containing both commuting (even degree) and anti-commuting (odd degree) generators. As a direct consequence, certain quantum field theories whose operators form a (possibly broken) Lie super-algebra circumvent the restriction of the Coleman-Mandula Theorem and exhibit larger spacetime symmetry than once believed possible. Although it is plausible that an unbounded cascade of ever-increasing spacetime symmetry groups might arise from more complicated Lie (super-)algebraic structures underlying the set of quantum operators, the imposition of reasonable phenomenological constraints suggest that SUSY is the most general (local) symmetry allowed in four-dimensional Minkowski spacetime.

## o.4. The $\mathbf{W Z}_{\theta, \phi}$ Model

In 1987, Jaffe, Lesniewski and Lewenstein [219] studied the vacuum structure of Wess-Zumino Quantum Mechanics, that is, a supersymmetric model of holomorphic quantum mechanics with a bosonic, polynomial superpotential $V$. They calculated the Fredholm or Witten index* of the supercharge $Q_{V}^{+}$ (satisfying $\left.\left(Q_{V}^{+}\right)^{2}=H+P\right)$, viz.,

$$
\begin{align*}
\operatorname{ind}\left(Q_{V}^{+}\right) & =\lim _{\beta \rightarrow \infty} \operatorname{Tr}_{\mathscr{H} b^{b}} \Gamma e^{-\beta H}  \tag{0.2a}\\
& =n_{+}-n_{-}  \tag{0.2b}\\
& =\operatorname{deg} \partial V \tag{0.2c}
\end{align*}
$$

where $n_{+}=\operatorname{ker} Q_{V}^{+}$is the number of bosonic ground states and $n_{-}=\operatorname{ker}\left(Q_{V}^{+}\right)^{*}$ is the number of fermionic ground states. In particular, they proved a Vanishing Theorem of the absence of fermionic ground states, namely, $n_{-}=0$, i.e., $Q_{V}^{+}$is injective. This fact proves that the Witten index for this model is non-negative.

In 1999, Jaffe [222] studied a twist-regularized, interacting, bosonic quantum field theory with a twisted, bosonic partition function

$$
\begin{equation*}
\mathfrak{Z}_{g}^{b}(\beta)=\operatorname{Tr}_{\mathscr{H}^{b}} U\left(g^{-1}\right) e^{-\beta H} \tag{0.3}
\end{equation*}
$$

[^9]where $U(g)$ is a unitary representation of a group $G$ and $H$ is a $U(g)$-invariant, self-adjoint Hamiltonian $H$ on a bosonic Hilbert space $\mathscr{H}^{b}$. Taking $U=$ $e^{-i \sigma P-i \theta J}$ and $G$ equal to corresponding group of transformations, he proved twist positivity, namely,
\[

$$
\begin{align*}
\mathfrak{Z}_{\gamma}^{b}(\theta, \sigma, \beta) & =\operatorname{Tr}_{\mathscr{H} e^{b}} e^{-\beta H-i \sigma P-i \theta J}  \tag{0.4a}\\
& =\prod_{i=1}^{n} \prod_{k \in \hat{\mathbb{T}}}\left|1-\gamma_{i}(k)\right|^{-2}>0, \tag{o.4b}
\end{align*}
$$
\]

which holds for fixed $\theta, \sigma, \beta>0$ and any $g \in G$, thus implying the existence of a twisted Feynman-Kac representation of the interacting Hamiltonian, $H=H_{0}+V$.

In 2000, Jaffe [223] studied a particular generalization of the aforementioned bosonic field theory, as a twist-regularized, supersymmetric, generalized WessZumino model $\left(\mathrm{WZ}_{\theta, \phi}\right)$ on a (1,1)-spacetime torus $\mathbb{T}=S^{1} \times S^{1}$ of circumference $\ell$. Within the confines of mathematical approaches to quantum field theory, the $\mathrm{WZ}_{\theta, \phi}$ model remains to date the only interacting supersymmetric quantum field theory that satisfies a weaker, finite-volume version of the OsterwalderSchrader Axioms and wherein the ground-state structure is somewhat understood. Given a weighted homogeneous potential $V$ of the bosonic fields with weights $\left\{\omega_{1}, \ldots, \omega_{n}\right\} \subset \mathbb{Q} \cap\left(0, \frac{1}{2}\right]$, which satisfies the elliptic bounds (a technical estimate which precludes flat-directions and ensures a trace-class heat kernel), Jaffe computed the twist, boson-fermion elliptic genus* (or $\mathbb{Z}_{2}$-graded partition

[^10]function) $\mathfrak{Z}^{V}: \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ of complex twist $z=\frac{1}{2 \pi}(\theta-\phi \tau)$ and spacetime $\tau=\frac{1}{\ell}(\sigma+i \beta)$ parameters,
\[

$$
\begin{align*}
\mathfrak{Z}^{V}(z, \tau) & =\operatorname{Tr}_{\mathscr{F}} \Gamma e^{-\beta H-i \sigma P-i \theta J}  \tag{0.5a}\\
& =e^{i \theta \hat{c} / 2} \prod_{i=1}^{n} \prod_{k \geqslant 0} \frac{\left(1-y^{-\left(1-\omega_{i}\right)} q^{k}\right)\left(1-y^{\left(1-\omega_{j}\right)} q^{k+1}\right)}{\left(1-y^{-\omega_{i}} q^{k}\right)\left(1-y^{\omega_{i}} q^{k+1}\right)}  \tag{0.5b}\\
& =y^{-\hat{c} / 2} \prod_{i=1}^{n} \frac{\vartheta_{1}\left(\left(1-\omega_{i}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{i} z, \tau\right)}, \tag{0.5c}
\end{align*}
$$
\]

where $\hat{c}=n-2 \sum_{i=1}^{n} \omega_{i}, y=e^{2 \pi i z}$ and $q=e^{2 \pi i \tau}$. This is possible since the elliptic genus $\mathfrak{Z}^{\lambda V}$ is constant in $\lambda \in[0,1]$, and $\mathfrak{Z}^{\lambda V}$ is evaluated in the limit $\lambda \rightarrow 0$. As a result of the representation as a ratio Jacobi theta functions, the elliptic genus $\mathfrak{Z}^{V}$ is a weak Jacobi form and satisfies the following $\mathbb{Z}^{2} \ltimes \mathrm{SL}_{2}(\mathbb{Z})$ symmetry: For $\gamma_{\delta}=\left((m, n),\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)\right) \in \mathbb{Z}^{2} \ltimes \mathrm{SL}_{2}(\mathbb{Z})$ and $(z, \tau) \in \mathbb{C} \times \mathbb{H}$, one has the transformation law

$$
\begin{equation*}
\left.\mathfrak{J}^{V}\right|_{\gamma_{\delta}}(z, \tau)=y^{\hat{c} / 2} \mathbf{e}_{c, d}^{\hat{\imath}\left[c z^{2}-(2 m+1) z-a^{\prime} \tau-b^{\prime}\right]} \mathfrak{J}^{V}(z, \tau), \tag{0.6}
\end{equation*}
$$

where $a^{\prime}=m a+n c$ and $b^{\prime}=m b+n d$ and $\mathbf{e}_{c, d}^{z}=e^{\pi i z /(c \tau+d)}$.

### 0.5. Algebraic Links

A link is a closed, oriented 1-manifold smoothly embedded in $S^{3}$ or $\mathbb{R}^{3}$; a knot is a link consisting of a single, connected component. In particular, a knot may be viewed as a homeomorphism from $S^{1}$ to $\mathbb{R}^{3}$.


Figure 0.2. The Trefoil $\left(3_{1}\right)$ and Cinquefoil (51) [389]
It is prudent to consider a simple and illustrative, yet quite general, family of algebraic (fibered) links. Given two integers $p, q>1$, the polynomial $f=x^{p}+$ $y^{q}$ over $\mathbb{C}^{2}$ is a complex analytic map with an isolated critical point at the origin. Therefore, the complex algebraic variety or algebraic hypersurface $V_{f, 0}=f^{-1}(0)$ is singular only at $\mathbf{0}$. In [61], Brauner proves that the intersection of $V_{f, 0}$ with a sufficiently small 3-sphere, which we shall denote by $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{3}$, is a $(p, q)$ torus link ${ }^{*} T_{p, q}$ with GCD $(p, q)$ components. The torus knots $T_{2,3}$ and $T_{2,5}$ are shown in Figure o.2. The reduced Alexander polynomial of the torus link is given by

$$
\begin{equation*}
\Delta_{T_{p, q}}(t)=\frac{\left(t^{\mathrm{LCM}(p, q)}-1\right)^{\operatorname{GCD}(p, q)}(t-1)^{\delta_{r, 1}}}{\left(t^{p}-1\right)\left(t^{q}-1\right)} \tag{0.7}
\end{equation*}
$$

where $r=\operatorname{GCD}(p, q)$. Various prime links, including torus links, and their link data can be found in Appendix A.
0.5.1. Milnor Fibration. Pham and Brieskorn studied algebraic hypersurfaces and complete intersections of complex analytic polynomials of the form

[^11]$f=\sum_{i=0}^{n} z_{i}^{a_{i}}$ with $a_{i} \geqslant 1$, generalizing the torus links to higher dimensions [63], [64], [374]. Pham, in particular, proved that $V_{f, 1}=f^{-1}(1)$ is a deformation retraction of, hence homotopy equivalent to, the join $C_{a_{0}} \star \cdots \star C_{a_{n}}$, where $C_{n} \cong \mathbb{Z}_{n}$ is the multiplicative group of the $n^{\text {th }}$ roots-of-unity (or simply a cyclic group of order $n$ ) viewed as a pointed, discrete topological space $\left(C_{n}, 1\right)$ with the identity element 1 identified as the base-point. Since one has the homotopy equivalences $C_{n} \simeq \bigvee^{n-1} S^{0}$ and $S^{n} \star S^{m} \simeq S^{n+m+1}$ for $n, m \geqslant 0$, it follows that in fact $V_{f, 1}$ has the homotopy-type of a wedge sum of $n$-spheres $\bigvee^{\mu} S^{n}$, where $\mu=\prod_{i=0}^{n}\left(a_{i}-1\right)$.

Milnor studied the map $\phi_{f}=\frac{f}{\|f\|}: S_{\varepsilon}^{2 n+1} \backslash V_{f} \rightarrow S^{1}$, where $f$ is a given complex analytic function of $n+1$ variables with an isolated critical point at the origin. He proved that $\phi_{f}$ is a fibration over $S^{1}$ with fibers $F_{f, \theta}=\phi_{f}^{-1}\left(e^{i \theta}\right)$ and described various topological features of the intersection $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{2 n+1}$, most notably proving that it forms a link [310]. In particular, he computed the homotopy type of the fiber $F_{f, 0}$ as that of a wedge sum of spheres $\bigvee^{\mu} S^{n}$, where $\mu=\operatorname{rank} \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right)$, and the corresponding algebraic link $K_{f}$ is $(n-2)$ connected, possibly knotted and with (possibly reduced) Alexander polynomial equal to the characteristic polynomial of the monodromy homomorphism $h_{*}(f): \tilde{H}_{n}\left(F_{f, 0}, \mathbb{C}\right) \rightarrow \tilde{H}_{n}\left(F_{f, 2 \pi}, \mathbb{C}\right)$. For torus links,

$$
\Delta_{h_{*}}(t)=\frac{\left(t^{\mathrm{LCM}(p, q)}-1\right)^{\operatorname{GCD}(p, q)}(t-1)}{\left(t^{p}-1\right)\left(t^{q}-1\right)}
$$

For the case that $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous singularities with isolated critical points at the origin and with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$, Milnor proved the diffeomorphism $F_{f, 0} \cong{ }_{d} V_{f, 1}$, thereby generalizing the relevant collective work of Brauner, Brieskorn and Pham. Milnor and Orlik gave a method to compute the characteristic polynomial $\Delta_{h_{*}}(t)=\operatorname{det}\left(t \mid-h_{*}\right)$ explicitly in terms of the weights and also defined the algebra

$$
\begin{equation*}
\mathcal{A}_{f}=\mathbb{C}\left\{z_{0}, \ldots, z_{n}\right\} /\left\langle\partial_{0} f, \ldots, \partial f_{n}\right\rangle, \tag{o.8}
\end{equation*}
$$

where

$$
\begin{equation*}
\mu=\operatorname{dim}_{\mathbb{C}} \mathcal{A}_{f}=\prod_{i=0}^{n}\left(\frac{1}{\omega_{i}}-1\right) \tag{0.9}
\end{equation*}
$$

providing an algebraic interpretation to the rank of the middle homology group of the fiber $F_{f, 0}$.
0.5.2. Milnor Conjecture. In the same monograph, Milnor proved the relation $\mu=2 \delta-r+1$ satisfied by a square-free complex algebraic plane curve $f$, where $\delta$ is the delta invariant, that is, the number of double points of $V_{f, 0}$, and $r$ is the number of branches of $V_{f, 0}$ passing through the origin. Milnor conjectured that $\delta$ coincides with the unknotting number $u\left(K_{f}\right)$ of the corresponding link and is completely determined by the (singular) homology of fibers $F_{f, \theta}$. For knots of the torus-type $T_{p, q}$ corresponding to $f=x^{p}+y^{q}$, where $p$ and $q$ are coprime, then $r=1$ and $\delta$ is one-half the rank of the middle reduced homology group $\tilde{H}_{1}\left(F_{f, 0} ; \mathbb{Z}\right)$ or, equivalently, the genus $g\left(F_{f, 0}\right)=\frac{\mu}{2}=\frac{1}{2}(p-1)(q-1)$. In
general, the integers $\delta$ and $r$ count the positive lattice points (all interior and some boundary) and the number of interior lattice points of the hypotenuse (only boundary), respectively, of the lattice right triangle $\operatorname{conv}\left\{\mathbf{0}, p \mathbf{e}_{1}, q \mathbf{e}_{2}\right\}$.
Thus, the unknotting number of said knots is a combinatorial and homological invariant, for $u\left(T_{p, q}\right)=\frac{1}{2}(\mu+r-1)$.

Using Donaldson invariants, in 1992, Kronheimer and Mrowka proved that complex curves in K 3 surfaces satisfy the genus minimizing property [247, 248]. Among the many consequences of this important work is a proof of the Milnor Conjecture on the delta invariant and unknotting number. Using SeibergWitten invariants*, in 1994, Kronheimer and Mrowka succeeded in proving the Thom Conjecture, that a (connected) complex projective algebraic curve $\mathcal{C}_{d}$ of degree $d \geqslant 1$ in $\mathbb{C P}^{2}$ minimizes the genus in its homology class [247, 248]. That is, if $\mathcal{C}$ is an oriented two-dimensional manifold smoothly embedded in $\mathbb{C P}^{2}$ with homology class $\left[\mathcal{C}_{d}\right]$ for some (complex projective algebraic) curve $\mathcal{C}_{d}$, then its genus is bounded from below, $g(\mathcal{C}) \geqslant g\left(\mathcal{C}_{d}\right)=\frac{1}{2}(d-1)(d-2)$.
0.5.3. Faltings Theorem. Topological invariants such as the genus have also proved to be indispensable in the study of rational points on elliptic curves. In 1983, Faltings proved a generalization of the Mordell Conjecture which states that a non-singular algebraic curve of genus $g>1$ over a number field, that

[^12]is, a finite degree extension of the field of rationals, $Q$, has finite many rational points. The Fermat curve $\mathcal{W}_{d}: \mathbb{Q}^{2} \rightarrow \mathbb{R}$ of degree $d$, defined by the locus $\left\{(x, y) \in \mathbb{Q}^{2} \mid x^{d}+y^{d}=1\right\}$, has genus $g\left(\mathcal{W}_{d}\right)=\binom{d-1}{2}$. Falting's Theorem, therefore, implies that $\mathcal{W}_{d}$ contains finitely many rational points for $d>2$, proving a weak form of the Fermat Conjecture* (or Fermat's Last Theorem). The extent to which Milnor's construction of algebraic links and their invariants can shed further light on properties of elliptic curves is an active area of research.
0.5.4. Signature of a Manifold. Let $M^{4 k}$ denote a closed,oriented, $4 k$-manifold. Consider the self-cup product map $B: H^{2 k}\left(M^{4 k} ; \mathbb{Z}\right) / T \rightarrow$ $H^{4 k}\left(M^{4 k} ; \mathbb{Z}\right) \cong \mathbb{Z}$, where $B: x \mapsto x \smile x$, which is a quadratic form of type $(p, q)$ [314]. The Thom signature $\sigma\left(M^{4 k}\right)$ is defined as the signature of the quadratic form $B$, that is, $p-q$, and is a homomorphism $\sigma: \Omega_{4 k} \rightarrow \mathbb{Z}$ given by a Q-linear combination of the Pontryagin numbers [203]. The signature $\sigma\left(M^{4 k}\right)$ is therefore a cobordism invariant.
o.5.4.1. Hirzebruch Signature Theorem. Hirzebruch related the Thom signature to the L-genus [203]. In particular, the Hirzebruch-Thom signature $\sigma\left(M^{n}\right)$ is defined for any compact, smooth, oriented differential manifold $M^{n}$ of positive dimension, and is the value of the pairing of the $L$-genus with the fundamental

[^13]homology class [ $M^{n}$ ],
\[

\sigma\left(M^{n}\right)=\left\{$$
\begin{array}{cc}
0 & n \neq 0 \bmod 4  \tag{0.10}\\
\left\langle L_{k},\left[M^{n}\right]\right\rangle & n=4 k
\end{array}
$$\right.
\]

where $L_{k}=L_{k}\left(p_{1}, \ldots, p_{k}\right)$ is a Q-polynomial of degree at most $k$ over oriented cobordism invariants, namely, the Pontryagin classes $p_{k}=p_{k}\left(T M^{n}\right) \in$ $H^{4 k}\left(M^{n} ; \mathbb{Z}\right)$. In general, $L_{k}$ is given in terms of the complexified tangent bundle of $M^{n}, L_{k}=\prod_{i=1}^{2 k} \frac{x_{i}}{\tanh x_{i}}$, where $x_{i}=c_{i}\left(M^{n}\right)$ are the Chern roots of $M^{n}$. The fact that $\sigma\left(M^{n}\right)$ is an integer imposes strict divisibility criteria on the Pontryagin classes of $M^{n}$. For 4, 8 and 12-manifolds, the signature relations are

$$
\begin{align*}
\sigma\left(M^{4}\right) & =\frac{1}{3}\left\langle p_{1},\left[M^{4}\right]\right\rangle  \tag{0.11}\\
\sigma\left(M^{8}\right) & =\frac{1}{3^{2} \cdot 5}\left\langle 7 p_{2}-p_{1}^{2},\left[M^{8}\right]\right\rangle  \tag{0.12}\\
\sigma\left(M^{12}\right) & =\frac{1}{3^{3} \cdot 5 \cdot 7}\left\langle 2 p_{1}^{3}+(2 \cdot 31) p_{3}-13 p_{1} p_{2},\left[M^{12}\right]\right\rangle \tag{0.13}
\end{align*}
$$

respectively. The signature often has curious divisibility properties. According to Hirzebruch [202], if $b_{4}\left(M^{12}\right)=0$ (the fourth betti number), then

$$
\begin{equation*}
\left\langle 2 p_{1}^{3}-13 p_{1} p_{2},\left[M^{12}\right]\right\rangle=0 \tag{0.14}
\end{equation*}
$$

so the corresponding signature satisfies

$$
\begin{equation*}
945 \sigma\left(M^{12}\right)=62\left\langle p_{3},\left[M^{12}\right]\right\rangle \tag{0.15}
\end{equation*}
$$

and is therefore divisible by 62 as $\left\langle p_{3},\left[M^{12}\right]\right\rangle \in \mathbb{Z}$ and $\operatorname{GCD}(945,62)=1$.

### 0.5.5. Exotic Spheres.

Definition o.2. A homology $n$-sphere is an $n$-manifold possessing the same homology groups as those of an $n$-sphere, i.e., $H_{i}(X ; \mathbb{Z}) \cong\{0\}$ for $1 \leqslant i \leqslant n-1$ and $H_{0}(X ; \mathbb{Z}) \cong H_{n}(X ; \mathbb{Z}) \cong \mathbb{Z}$. A homotopy $n$-sphere is an $n$-manifold homotopy equivalent to $S^{n}$. A topological $n$-sphere is an $n$-manifold homeomorphic to $S^{n}$. An exotic $n$-sphere is a topological $n$-sphere not diffeomorphic to $S^{n}$.

Remark o.5.1. Every homotopy $n$-sphere is a homology $n$-sphere. Every topological $n$-sphere is a homotopy $n$-sphere.

Proposition o. 3 (Poincaré Conjecture). For $n \geqslant 2$, every homotopy $n$-sphere is homeomorphic to an $n$-sphere, i.e., a topological $n$-sphere.

Proof. The case $n=2$ is classical. In 1961, Smale proved the cases $n \geqslant 5$. In 1982, Freedman proved the case $n=4$. In 2003, Perelman proved the case $n=3$.

Conjecture o.4 (Smooth Poincaré Conjecture). For $n \geqslant 2$, every homotopy $n$-sphere is diffeomorphic to $S^{n}$.

Remark o.5.2. The case $n=2$ is classical. In 1956, Milnor gave a counterexample for $n=7$. Milnor and Kervaire produced counter-examples for $n \geqslant 7$. The conjecture is known to hold for $n=5,6$. Perelman proved the case $n=3$. The case $n=4$ is open.
o.5.6. Milnor 7-Sphere. While investigating $S^{3}$-bundles over $S^{4}$ with rotation and structural group $\operatorname{SO}(4)$, in 1956, Milnor discovered that the 7 -sphere has several differentiable structures [305]. In particular, Milnor constructs a Thom space $T$ with boundary $M$ and signature $\sigma(T)=1$ and $\left\langle p_{1}^{2},[T]\right\rangle=k^{2}$ for some integer $k$ congruent to 2 modulo 4 . However, by equation (4.114),

$$
\begin{equation*}
\left\langle p_{2},[T]\right\rangle=\frac{1}{7}\left(45 \sigma(T)+\left\langle p_{1}^{2},[T]\right\rangle\right)=\frac{1}{7}\left(45+k^{2}\right), \tag{0.16}
\end{equation*}
$$

which is not an integer if $k$ is not congruent to $\pm 2$ modulo 7 . Therefore, $M$ is not diffeomorphic to $S^{7}$ in the excluded cases.

Proposition 0.5 (Reeb). Given a compact n-manifold $M$ and a Morse function $f: M \rightarrow \mathbb{R}$ with exactly two critical points, then $M$ is homeomorphic to $S^{n}$.

Milnor proves that $M$ is a compact, oriented smooth 7-dimensional manifold satisfying the assumptions of Reeb's Sphere Theorem, so $M$ is homeomorphic to $S^{7}$. For a detailed discussion of this intriguing topic, see [311], Chapter 20 in [316] and Chapter 4 in [306].
0.5.7. Homotopy Spheres. Let $\Sigma^{n},\left[\Sigma^{n}\right]$ and $\Theta_{n}=\left\{\left[\Sigma^{n}\right] \mid \Sigma^{n} \simeq S^{n}\right\}$ denote a homotopy $n$-sphere, an equivalence class of $n$-spheres up to oriented $h$-cobordism, and the additive abelian group of such classes under the operation of connected sum, with inverse given by reversing orientation [237, 311]. By the work of Smale, Freedman, Perelman and others, the $h$-Cobordism Theorem implies that the elements of $\Theta_{n}$ are in fact oriented diffeomorphism classes. In
particular, every homotopy $n$-sphere is a topological $n$-sphere for $n \geqslant 0$. There is a cyclic subgroup $b P_{n+1}<\Theta_{n}$ consisting of the homotopy spheres which bound ( $n+1$ )-dimensional parallelizable (smooth) manifolds. The groups $\Theta_{n}$ and $b P_{n+1}$ are the Milnor-Kervaire groups. For $2 \leqslant n \leqslant 6, \Theta_{n}$ and $b P_{n+1}$ are trivial. For $m \geqslant 2$,

$$
\begin{equation*}
\left|b P_{4 m}\right|=2^{2 m-2}\left(2^{2 m-1}-1\right) \operatorname{num}\left(\frac{4\left|B_{2 m}\right|}{m}\right), \tag{0.17}
\end{equation*}
$$

where $B_{m}$ is the $m^{\text {th }}$-Bernoulli number. Milnor and Kervaire prove that $b P_{2 m+1}$ is trivial for $m \geqslant 1$. Recent work by Hill,Hopkins and Ravenel showed $b P_{2^{l}-2} \cong \mathbb{Z}_{2}$ for $l \geqslant 8$. Essential to the complete understanding of $b P_{4 m+2}$ is the computation of the Kervaire Invariant. Based on the work of Kervaire, et al., the current state of knowledge of the order of these groups is the following:

$$
\left|b P_{4 m+2}\right|=\left\{\begin{array}{cl}
1 & m \in\{1,3,7,15\}  \tag{0.18}\\
1 \text { or } 2 & m=31 \\
2 & \text { otherwise }
\end{array}\right.
$$

where the group $b P_{126}$ is hitherto not known. The number of exotic spheres in dimension $n$ is inferred from a careful study of the group $\Theta_{n}$, and its order $\left|\Theta_{n}\right|$, that is, the number of $h$-cobordism classes of smooth homotopy $n$-spheres as a function of $n \geqslant 1$ [237] (A001676). The Milnor-Kervaire numbers $\left|\Theta_{n}\right|$ and $\left|b P_{n+1}\right|$ for $7 \leqslant n \leqslant 20$ are given in Table o.1. For example, there are 27 exotic spheres in dimension 7.

Table 0.1. Milnor-Kervaire Groups $\Theta_{n}$ and $b P_{n+1}(7 \leqslant n \leqslant 20)$

| $n$ | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $\left\|\Theta_{n}\right\|$ | 28 | 2 | 8 | 6 | 992 | 1 | 3 | 2 | 16256 | 2 | 16 | 16 | 523264 | 24 |
| $\left\|b P_{n+1}\right\|$ | 28 | 1 | 2 | 1 | 992 | 1 | 1 | 1 | 8128 | 1 | 2 | 1 | 261632 | 1 |

0.5.8. Brieskorn-Pham Manifolds. Consider the polynomial

$$
\begin{equation*}
f=z_{0}^{5}+z_{1}^{3}+z_{2}^{2}+z_{3}^{2}+z_{4}^{2}+z_{5}^{2}+z_{6}^{2} \tag{0.19}
\end{equation*}
$$

over $\mathbb{C}^{7}$, and define the 1-parameter family of complex hypersurfaces $V_{f, \kappa}=$ $f^{-1}(\kappa)$ with $\kappa \in \mathbb{C}$ a regular value of $f$ and sufficiently close to the origin [63]. By the ADE classification of simple singularities, the singularity $f$ is a 4-stabilization of the $E_{8}$ surface singularity $\left(x^{2}=y^{3}+z^{5}\right.$ over $\left.\mathbb{C}^{3}\right)$ and corresponds to a Milnor fiber $F_{\Sigma^{4} E_{8}} \cong V_{f, 1} \simeq \bigvee^{8} S^{6}$ with Milnor number $\mu\left(\Sigma^{4} E_{8}\right)=\mu\left(E_{8}\right)=8$. The intersection $V_{f, \kappa} \cap B_{\varepsilon}^{14}$ with a 14-ball of sufficiently small radius $\varepsilon>\kappa$ is a 12 -manifold with boundary. The boundary $K_{\Sigma^{4} E_{8}}^{11}=\partial\left(V_{f, \kappa} \cap B_{\varepsilon}^{14}\right)=V_{f, \kappa} \cap S_{\varepsilon}^{13}$, the 4-iterated stabilization of the 5-iterated cyclic branched covering of the trefoil knot, has reduced Alexander polynomial

$$
\begin{align*}
\Delta_{f}(t) & =\frac{\left(t^{15}-1\right)(t-1)}{\left(t^{5}-1\right)\left(t^{3}-1\right)}  \tag{0.20a}\\
& =1-t+t^{3}-t^{4}+t^{5}-t^{7}+t^{8}  \tag{o.20b}\\
& =\Phi_{15}(t) \tag{0.20c}
\end{align*}
$$

where $\Phi_{n}(t)$ is the $n^{\text {th }}$-cyclotomic polynomial. According to Milnor, since $\Delta_{f}(1)=\Phi_{15}(1)=1$, then $K_{\Sigma^{4} E_{8}}^{11}$ is a topological sphere. The quotient space
$M_{\Sigma^{4} E_{8}}^{12}=V_{f, \kappa} \cap B_{\varepsilon}^{14} / K_{\Sigma^{4} E_{8}}^{11}$ is a 5-connected 12-manifold (without boundary) with $b_{i}\left(M_{f}^{12}\right)=0$ for $1 \leqslant i \leqslant 5$ and signature $\sigma\left(M_{\Sigma^{4} E_{8}}^{12}\right)=-8$. As the signature is not divisible by 62, it follows that $M_{\Sigma^{4} E_{8}}^{12}$ is not a differentiable manifold. Although $K_{\Sigma^{4} E_{8}}^{11}$ is homeomorphic to $S^{11}$, it is not diffeomorphic to it. Hence, $K_{\Sigma^{4} E_{8}}^{11}$ is an exotic 11-sphere [63]. This example represents one of 992 (oriented diffeomorphism classes of) differentiable structures on $S^{11}$ - all representable by the 1-parameter family of polynomials

$$
\begin{equation*}
f=z_{0}^{6 k-1}+z_{1}^{3}+z_{2}^{2}+z_{3}^{2}+z_{4}^{2}+z_{5}^{2}+z_{6}^{2} \quad 1 \leqslant k \leqslant 992 . \tag{0.21}
\end{equation*}
$$

In fact, up to diffeomorphism, all exotic spheres in dimensions $4 m-1$ admit a similar realization.

Proposition o. 6 (Brieskorn [63]). Let $\Sigma_{k}^{4 m-1}$ be the link of the BrieskornPham singularity $f=z_{0}^{6 k-1}+z_{1}^{3}+\sum_{i=2}^{2 m} z_{i}^{2}$. Then $\Sigma_{k}^{4 m-1}$ is a homotopy sphere with signature $\sigma\left(\Sigma_{k}^{4 m-1}\right)=(-1)^{m} 8 k$ and represents $\frac{\sigma_{m}}{8}$ differential structures in $b P_{4 m}$, where

$$
\begin{equation*}
\sigma_{m}=2^{2 m+1}\left(2^{2 m-1}-1\right) \operatorname{num}\left(\frac{4\left|B_{2 m}\right|}{m}\right) \tag{0.22}
\end{equation*}
$$

that is, $\Sigma_{k}^{4 m-1} \in b P_{4 m}$ for $1 \leqslant k \leqslant \frac{\sigma_{m}}{8}$.

These techniques were generalized and employed by Milnor to disprove the Smooth Poincare Conjecture in large dimensions. That is, given a homotopy $n$-sphere $\Sigma^{n}$, then it is not necessarily true that $M^{n}$ is diffeomorphic to $S^{n}$.

## o.6. Atiyah-Singer Index Theorem

Atiyah and Singer proved that the signature $\sigma(M)$ is the Fredholm index of an elliptic operator $D=d^{*}+d$, the signature operator of a compact manifold $M$, where $d$ is the exterior derivative and $D^{2}=d^{*} d+d d^{*}=\Delta$ is the Laplacian restricted to the +1-eigenspace of even forms in the complex bundle $\Omega^{*}(M)=$ $\Omega^{+}(M) \oplus \Omega^{-}(M)$ under a specific $\mathbb{Z}_{2}$-action involving the Hodge star $*$ modulo a normalizing power of $i$ [192]. That is, the index of $D$ is given by

$$
\begin{align*}
\operatorname{ind}(D) & =(-1)^{\ell}\left\langle\operatorname{ch}\left(\Lambda^{+} T_{\mathrm{C}}^{*} M-\Lambda^{-} T_{\mathrm{C}}^{*} M\right) \frac{\operatorname{td}\left(T_{\mathrm{C}} M\right)}{\mathrm{e}(T M)},[M]\right\rangle  \tag{0.23a}\\
& =\left\langle\prod_{i=1}^{2 k} \frac{x_{i}}{\tanh x_{i}},[M]\right\rangle \tag{o.23b}
\end{align*}
$$

which is precisely $\left\langle L_{k}(M),[M]\right\rangle$, the signature of $M$. Compare these formulas to the Euler characteristic $\chi(M)=\langle\mathrm{e}(T M),[M]\rangle$ and the index of the Dolbeault operator and $\operatorname{ind}(\bar{\partial})=\left\langle\operatorname{td}\left(T_{C} M\right),[M]\right\rangle$, where e, ch and td denote the Euler, Chern and Todd classes, respectively. If $M$ is a compact, oriented 4-manifold with a virtual vector bundle $E$, the Atiyah-Singer Index Theorem states that there is a Dirac operator $D_{A}^{+}$corresponding the $\hat{A}$-genus with coefficients in $E$ (Chapter 2, [323]),

$$
\begin{align*}
\operatorname{ind}\left(D_{A}^{+}\right) & =\langle\hat{A}(T M) \operatorname{ch}(E),[M]\rangle  \tag{0.24}\\
& =-\frac{\operatorname{dim} E}{24}\left\langle p_{1}(T M),[M]\right\rangle+\frac{1}{2}\left\langle c_{1}(E)^{2},[M]\right\rangle \tag{0.25}
\end{align*}
$$

Combined with the Hirzebruch Signature Theorem, $\sigma(M)=\frac{1}{3}\left\langle p_{1}(T M),[M]\right\rangle$,

$$
\begin{equation*}
\operatorname{ind}\left(D_{A}^{+}\right)=-\frac{1}{8} \sigma(M)+\frac{1}{2}\left\langle c_{1}(E)^{2}-c_{2}(E),[M]\right\rangle \tag{0.26}
\end{equation*}
$$

In particular, if $M$ is smooth spin 4-manifold, then the index of $D_{A}^{+}$is even and the term involving the Chern number is zero.

Proposition o. 7 (Rokhlin). If $M$ is a closed, oriented, smooth spin 4-manifold, then $\sigma(M)$ is divisible by 16.

### 0.7. Durfee Conjecture

Recall that a complex analytic germ is a non-degenerate if and only if it possesses and isolated critical point at the origin. Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, $V_{f, \kappa}=f^{-1}(\kappa)$ and $\bar{F}_{f, 0} \cong{ }_{d} V_{f, \kappa} \cap \bar{B}_{\varepsilon}$ for sufficiently small $0<\varepsilon<\kappa$ denote a non-degenerate, complex analytic germ, the corresponding hypersurface and (closed) Milnor fiber, respectively. Denote the signature of $F_{f, 0}$ by $\sigma(f)$ and by $p_{g}(f)=\operatorname{dim}_{\mathbb{C}} H^{n-1}\left(\tilde{V}_{f, 0}, \mathcal{O}_{\tilde{V}_{f, 0}}\right)$ the geometric genus (or first plurigenus) of any minimal resolution given by a proper analytic map $\pi:\left(\tilde{V}_{f, 0}, E\right) \rightarrow\left(V_{f, 0}, 0\right)$ with exceptional locus $E=\pi^{-1}(0)$ such that $\tilde{V}_{f, 0} \backslash E \rightarrow V_{f, 0} \backslash\{0\}$ is an analytic isomorphism and $\pi^{-1}\left(V_{f, 0} \backslash\{0\}\right)$ is dense in $\tilde{V}_{f, 0}$ [478]. Laufer [256] proves the identity

$$
\begin{equation*}
12 p_{g}(f)=1+\mu(f)-\chi(E)-K^{2} \tag{0.27}
\end{equation*}
$$

where $K^{2}=K \cdot K$ is the self-intersection number of the canonical divisor on $\tilde{V}_{f, 0}$, and $\chi(E)$ is the Euler characteristic of $E$. Although the Euler characteristic $\chi(E)$ and self-intersection number $K^{2}$ are topological invariants, the geometric genus $p_{g}$ is not as there are homeomorphic singularities with distinct Milnor numbers, c.f. equation (0.27). In [118], Durfee conjectured that non-degenerate, weighted homogeneous, surface singularities $(n=2)$ satisfy $\sigma(f) \leqslant 0$ and $6 p_{g}(f) \leqslant \mu(f)$ with equality of the latter inequality only in the case $\mu(f)=0$. In the case that $f$ is non-degenerate, strict positivity of $\mu$ is known for $n \geqslant 1$. In 1993, Xu and Yau sharpen and prove the Durfee Conjecture for surface singularities.

Proposition o. $8(\mathrm{Xu}, \mathrm{Yau},[478])$. Let $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a nondegenerate, weighted homogeneous germ, and $V_{f, 0}=f^{-1}(0)$ its corresponding hypersurface. Let $\mu(f), p_{g}(f), v(f)$ and $\sigma(f)$ denote the Milnor number, geometric genus, multiplicity and signature of $f$, respectively. Then

$$
\begin{equation*}
6 p_{g}(f) \leqslant \mu(f)-v(f)+1 \tag{o.28}
\end{equation*}
$$

with equality if and only if $V_{f, 0}$ is defined by a homogeneous polynomial. Moreover, if $\sigma$ denotes the signature of the Milnor fiber of $f$, then

$$
\begin{equation*}
\sigma(f) \leqslant-\frac{1}{3} \mu(f)-\frac{2}{3}(v(f)-1) \tag{0.29}
\end{equation*}
$$

Conjecture 0.9 (Yau). Given a non-degenerate, weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, let $\mu(f), p_{g}(f)$ and $v(f)$ denote the Milnor number,
geometric genus and multiplicity of $f$, respectively. The following inequality holds:

$$
\begin{equation*}
(n+1)!p_{g}(f) \leqslant \mu(f)-(v(f)-1)^{n+1}+v(f)(v(f)-1) \cdots(v(f)-n) \tag{0.30}
\end{equation*}
$$

with equality if and only if $f$ is homogeneous.

Conjecture 0.9 is true for $n=3$ [478] and $n=4$ [269]. Sȩkalski [423] proves that the multiplicity of a weighted homogeneous polynomial depends only on its weights, $v(f)=\min \left\{k \in \mathbb{N} \left\lvert\, k \geqslant \min \left\{\frac{1}{\omega_{i}}\right\}\right.\right\}$.
0.7.1. The Milnor Number as a Fredholm Index. By the Durfee equality relating the signature $\sigma$ and the Milnor number $\mu$ for $n=2$, it is clear that $\mu$ is the Fredholm index of a Dirac operator and the pairing of an elliptic genus on the fundamental homology class of a corresponding manifold (both up to sign). It is reasonable to suggest that these interpretations continue to hold for $n>2$, that there is a Dirac operator and genus whose bilinear pairing yields a Fredholm index equal to the Milnor number $\mu$.

Proposition 0.10 (Durfee, [118]). The signature $\sigma=-\frac{1}{3}\left(2 \mu+K^{2}+s+2 h\right)$.

For $n=1, \sigma=-\mu$, so negative definite. This proves that $\mu$ is the index of a Dirac operator for $n=1$. In the next section, we extend this result to $n>1$.

The Milnor number of a weighted homogeneous polynomial and corresponding algebraic variety is but one of a myriad of manifold invariants with interpretations ranging from the differential, analytic, geometric, topological,

K-theoretic, algebraic and the combinatorial. In this work, we add an additional interpretation: supersymmetric quantum physical.

## o.8. Quantum Field Theory and the Alexander Polynomial

Definition o.11. A topological quantum field theory is a quantum field theory that computes topological invariants.

The $\mathrm{WZ}_{\theta, \phi}$ model is an interacting topological quantum field theory defined within the framework of constructive quantum field theory.

Definition o.12. A complex analytic function $g: \mathbb{C}^{m} \rightarrow \mathbb{C}$ satisfies the elliptic bounds if and only if there are positive constants $\varepsilon, M, \rho<\infty$ such that for any non-negative multi-index $\alpha$ and for all $z=\left(z_{1}, \ldots, z_{m}\right)$ satisfying $\|z\|>\rho$, one has $\left\|\partial^{\alpha} g\right\| \leqslant \varepsilon\|\partial g\|^{2}+M$ and $\|z\|^{2}+\|g\| \leqslant M\left(\|\partial g\|^{2}+1\right)$, where $\partial g=$ $\left(\partial_{1} g, \ldots, \partial_{m} g\right)$.

Remark o.8.1. For non-degenerate, weighted homogeneous polynomials, the latter bound is redundant.

Proposition 0.13. Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a non-degenerate, weighted homogeneous polynomial satisfying the elliptic bounds. The elliptic genus $\mathfrak{Z}^{f}$ determines the reduced Alexander polynomial of the algebraic link $K_{f}$, which is a complete (cobordism and isotopy) invariant if $K_{f} \subset S^{3}$ is a knot.

Proof. Define the spacetime-twist parameters $\tau=\frac{\sigma+i \beta}{\ell} \in \mathbb{H}, z=\frac{\theta-\tau \phi}{2 \pi} \in$ C and the associated nomes $q=e^{2 \pi i \tau}$ and $y=e^{2 \pi i z}$. Denote the weights of
$f$ by $\omega$, and define $\hat{c}=\sum_{i=0}^{n} 1-2 \omega_{i}$, the central charge. Since $f$ satisfies the standard hypotheses, the corresponding elliptic genus $\mathfrak{Z}^{f}: \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ exists and, assuming $\phi=0$, admits the following exact representation [223],

$$
\begin{align*}
\mathfrak{Z}^{f}(z, \tau) & =y^{\hat{c} / 2} \prod_{i=0}^{n} \prod_{k \geqslant 0} \frac{\left(1-y^{-\left(1-\omega_{i}\right)} q^{k}\right)\left(1-y^{\left(1-\omega_{j}\right)} q^{k+1}\right)}{\left(1-y^{-\omega_{i}} q^{k}\right)\left(1-y^{\omega_{i}} q^{k+1}\right)}  \tag{0.31}\\
& =y^{-(n+1) / 2} \operatorname{Sp}(f ; y)+O(q), \tag{0.32}
\end{align*}
$$

where the Steenbrink series

$$
\begin{align*}
\operatorname{Sp}(f ; y) & =\prod_{i=0}^{n} \frac{y^{1-\omega_{i}}-1}{1-y^{-\omega_{i}}}  \tag{0.33}\\
& =\sum_{j=1}^{\mu} y^{\gamma_{j}} \tag{0.34}
\end{align*}
$$

and $\mu(f)=\operatorname{rank} H_{n}\left(F_{f, 0} ; \mathbb{Z}\right)$ [436]. The spectrum $\operatorname{Sp}(f)=\left\{\gamma_{j}\right\}_{1 \leqslant j \leqslant \mu}$ of the mixed Hodge structure of a generic fiber $F_{f, \vartheta}$ determines the characteristic polynomial $\Delta_{h_{*}}(t)=\operatorname{det}\left(t \mid-h_{*}\right)$ of the Picard-Lefschetz monodromy $h_{*}: H_{n}\left(F_{f, \vartheta} ; \mathbb{C}\right) \rightarrow H_{n}\left(F_{f, \vartheta+2 \pi} ; \mathbb{C}\right)($ op. cit. $)$, viz.,

$$
\begin{equation*}
\Delta_{h_{*}}(t)=\prod_{j=1}^{\mu}\left(t-e^{2 \pi i \gamma_{j}}\right), \tag{0.35}
\end{equation*}
$$

the reduced Alexander polynomial of $K_{f}$ [310], viz.,

$$
\begin{equation*}
\Delta_{h_{*}}(t) \doteq(t-1)^{1-\delta_{r, 1}} \Delta_{K_{f}}(t, \ldots, t) \tag{o.36}
\end{equation*}
$$

the Lefschetz zeta function [352],

$$
\begin{align*}
\zeta_{K_{f}}(t) & =\exp \sum_{k \geqslant 0} \Lambda\left(h^{\circ k}\right) \frac{t^{k}}{k}  \tag{0.37}\\
& =\prod_{l \geqslant 0} \operatorname{det}\left(I-t h_{*, l}\right)^{(-1)^{l+1}} \tag{0.38}
\end{align*}
$$

where $h: V_{f, 1} \rightarrow V_{f, 1}$ is the transformation $h(z)=\left(e^{2 \pi i \omega_{0}} z_{0}, \ldots, e^{2 \pi i \omega_{n}} z_{n}\right)$, and the Lefschetz number

$$
\begin{equation*}
\Lambda\left(h^{\circ k}\right)=\sum_{l \geqslant 0}(-1)^{l} \operatorname{Tr}\left(h_{*, l}^{k}: H_{l}\left(V_{f, 1} ; \mathbb{Q}\right) \rightarrow H_{l}\left(V_{f, 1} ; \mathbb{Q}\right)\right) \tag{0.39}
\end{equation*}
$$

equals the Euler characteristic $\chi_{k}=\left\{z \in V_{f, 1} \mid h^{\circ k}(z)=z\right\}$ [310], viz.,

$$
\begin{equation*}
\zeta_{K_{f}}(t)=(-1)^{\mu n}(1-t)^{-1} \Delta_{h_{*}}(t) \tag{0.40}
\end{equation*}
$$

If $n=r=1$, the diffeomorphism-type of the relative pair $\left(S^{3}, K_{f}\right)$ [257]. We have therefore proven the following claim.

REmARK o.8.2. The symmetry $\mathfrak{Z}^{f}(-z, \tau)=\mathfrak{Z}^{f}(z, \tau)$ implies the reflexivity

$$
\begin{equation*}
\operatorname{Sp}(f ; y)=y^{n+1} \operatorname{Sp}\left(f ; \frac{1}{y}\right) \tag{0.41}
\end{equation*}
$$

the reciprocity $\gamma_{\mu+1-j}=n+1-\gamma_{j}$ for $1 \leqslant j \leqslant \mu$, and the functional equation

$$
\begin{equation*}
\Delta_{h_{*}}(t)=(-1)^{\mu n} t^{\mu} \Delta_{h_{*}}\left(\frac{1}{t}\right) . \tag{0.42}
\end{equation*}
$$

The iterated limit of zero twist angles, $\theta, \phi \rightarrow 0$, of the elliptic genus $\mathfrak{Z}^{f}$ computes the Fredholm index of the corresponding supercharge operator $Q_{f}^{+}$, where $\left(Q_{f}^{+}\right)^{2}=H+f$,

$$
\begin{align*}
\operatorname{ind}\left(Q_{f}^{+}\right) & =\lim _{\theta, \phi \rightarrow 0} \mathfrak{J}^{f}(z, \tau)  \tag{o.43a}\\
& =\prod_{i=1}^{n} \frac{\operatorname{deg} \partial_{i} f}{\operatorname{deg} \varphi_{i}}  \tag{o.43b}\\
& =\prod_{i=1}^{n}\left(\frac{1}{\omega_{i}}-1\right) \tag{o.43c}
\end{align*}
$$

which is the multiplicity of the quantum mechanical ground state of the $\mathrm{WZ}_{\theta, \phi}$ model. Comparing this limit to equation (o.9), the relation of the Milnor number of $f$, namely, $\mu(f)=\operatorname{dim}$ ker $Q_{f}^{+}$. Such a relation was anticipated by the Vanishing Theorem of Klimek and Lesniewski [239], wherein the space ker $Q_{f}^{+}$is related to the Koszul cohomology of $f$, and it was shown that $\operatorname{dim} \operatorname{ker}\left(Q_{f}^{+}\right)^{*}=0$. See [158], [220], [221] and [223] for further details.

Remark o.8.3. Given $p, q \in \mathbb{N}_{>1}$, the polynomial $f=x^{p}+y^{q}$ has weights $\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and corresponds to the torus link $T_{p, q}=V_{f, 0} \cap S^{3}$ with $\operatorname{GCD}(p, q)$ components and unknotting number $u\left(T_{p, q}\right)=\frac{1}{2}(p q-p-q+\operatorname{GCD}(p, q))$ [247]. Since $f$ satisfies the standard hypotheses,

$$
\begin{equation*}
\mathfrak{Z}^{f}(z, \tau)=y^{\frac{1}{p}+\frac{1}{q}-1} \sum_{k=0}^{p-2} \sum_{l=0}^{q-2} y^{k / p+l / q}+O\left(e^{2 \pi i \tau}\right) \tag{0.44}
\end{equation*}
$$

Therefore, $\operatorname{Sp}(f)=\left\{\frac{k}{p}+\frac{l}{q}\right\}_{1 \leqslant k \leqslant p-1,1 \leqslant l \leqslant q-1}=\left\{2-\frac{k}{p}+\frac{l}{q}\right\}_{1 \leqslant k \leqslant p-1,1 \leqslant l \leqslant q-1}$. Setting $\zeta_{n}=e^{2 \pi i / n}$ for $n \in \mathbb{N}$,

$$
\begin{align*}
\Delta_{h_{*}}(t) & =\prod_{k=1}^{p-1} \prod_{l=1}^{q-1}\left(t-\zeta_{p}^{k} \zeta_{q}^{l}\right)  \tag{0.45}\\
& =\frac{\left(t^{\mathrm{LCM}(p, q)}-1\right)^{\operatorname{GCD}(p, q)}(t-1)}{\left(t^{p}-1\right)\left(t^{q}-1\right)}  \tag{0.46}\\
& =(-t)^{\mu} \Delta_{h_{*}}\left(\frac{1}{t}\right) \tag{0.47}
\end{align*}
$$

where $\mu(f)=\lim _{\theta \rightarrow 0} \mathfrak{Z}^{f}(z, \tau)=(p-1)(q-1)$ is the Fredholm index of the supercharge $Q_{f}^{+}[223]$, as above. In particular, if $p$ and $q$ are coprime, then $T_{p, q}$ is a knot, and the index $\mu(f)=2 u\left(T_{p, q}\right)=2 g\left(F_{f, \vartheta}\right)$, twice the genus of a corresponding generic fiber [310, 247].

### 0.9. Geometric Genera of Weighted Homogeneous Surface Singularities

Other original contributions of this work include the exact geometric genus and signature of an arbitrary weighted homogeneous surface singularity, q.v., Propositions 5.56 and 5.44 , as well as an identity relating the geometric genus and Milnor number of the $t$-dilate of a weighted homogeneous polynomial in
$\mathbb{C}^{3}$, from which one can give a proof of the Durfee Conjecture for asymptotically small weights and compute the exact error term of the Durfee-Yau Theorem,

$$
\begin{align*}
p_{g}\left(f_{t}\right)= & \frac{1}{6}
\end{align*} \mu_{\mathrm{alg}}\left(f_{t}\right)+\frac{1}{6}\left(\frac{1}{\omega_{1} \omega_{2}}-\frac{1}{2 \omega_{2} \omega_{3}}-\frac{1}{2 \omega_{3} \omega_{1}}\right) t^{2} .
$$

### 0.10. Relating the Milnor Number, Signature and Geometric Genus

In Proposition 5.55, we relate the geometric genus, Milnor number and signature for an arbitrary non-degenerate, weighted homogeneous surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, namely,

$$
\begin{equation*}
4 p_{g}(f)=\sigma\left(F_{f, 0}\right)+\mu_{\mathrm{a} \mid g}(f)+\varsigma_{0} \tag{0.49}
\end{equation*}
$$

where $\sigma_{0}$ is the number of zero eigenvalues of the intersection form of the mixed Hodge structure of the corresponding fiber.

### 0.11. Signature of Weighted Homogeneous Surface Singularities

In Proposition 5.56, we compute the signature $\sigma\left(F_{f, 0}\right)$ of the Milnor fiber $F_{f, 0}$ of a non-degenerate, weighted homogeneous surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with integral weights $\left\{q_{1}, q_{2}, q_{3}\right\}$ and weighted degree $d$, where $\omega_{i}=\frac{q_{i}}{d}$, namely,

$$
\begin{align*}
\sigma\left(F_{f, 0}\right)= & 1-\varsigma_{0}-\frac{d^{3}}{3 q_{1} q_{2} q_{3}}+\frac{d^{2}}{q_{1} q_{2}}+\left(\frac{q_{1}}{3 q_{2} q_{3}}-\frac{1}{q_{1}}-\frac{1}{q_{2}}\right) d \\
& -\frac{q_{1}}{q_{3}}\left(1+\frac{q_{1}}{3 q_{2}}\right)\left\{\frac{d}{q_{1}}\right\} \\
& +\frac{q_{1}}{q_{3}}\left(1+\frac{q_{1}}{q_{2}}\right)\left\{\frac{d}{q_{1}}\right\}^{2}-\frac{2 q_{1}^{2}}{3 q_{2} q_{3}}\left\{\frac{d}{q_{1}}\right\}^{3} \\
& +\frac{2 q_{2}}{q_{3}} \sum_{i=1}^{\left\lfloor d / q_{1}\right\rfloor}\left\{\frac{d-i q_{1}}{q_{2}}\right\}\left(1-\left\{\frac{d-i q_{1}}{q_{2}}\right\}\right) \\
& -4 \sum_{i=1}^{\left\lfloor d / q_{1}\right\rfloor\left\lfloor\left(d-i q_{1}\right) / q_{2}\right\rfloor} \sum_{j=1}^{\left\{\frac{d-i q_{1}-j q_{2}}{q_{3}}\right\},} \tag{0.50}
\end{align*}
$$

where

$$
\begin{equation*}
\varsigma_{0}=\frac{d^{2}}{q_{1} q_{2} q_{3}}-\sum_{1 \leqslant i<j \leqslant 3} \frac{d}{\operatorname{LCM}\left(q_{i}, q_{j}\right)}+\sum_{1 \leqslant i \leqslant 3} \frac{\operatorname{GCD}\left(d, q_{i}\right)}{q_{i}}-1 . \tag{0.51}
\end{equation*}
$$

### 0.12. Signature of Torus Links

We compute the signature of a torus link, q.v., Proposition 6.61,

$$
\begin{align*}
\sigma\left(T_{p, q}\right)= & \frac{2 p q}{3 \mathrm{LCM}(p, q, 2)^{2}}-\frac{p q}{2}+\frac{2 q}{3 p}+\frac{2 p}{3 q}-1 \\
& -4\left(p^{\prime} \mathfrak{s}\left(\frac{2 q \tau}{d}, \frac{p p^{\prime} \tau}{d}\right)+q^{\prime} \mathfrak{s}\left(\frac{2 p \tau}{d}, \frac{q q^{\prime} \tau}{d}\right)+r \mathfrak{s}\left(\frac{p q \tau}{d}, \frac{2 r \tau}{d}\right)\right) \tag{0.52}
\end{align*}
$$

where $p^{\prime}=\operatorname{GCD}(2, q), q^{\prime}=\operatorname{GCD}(p, 2), r=\operatorname{GCD}(p, q)$ and

$$
\begin{equation*}
\frac{d}{\tau}=\frac{\operatorname{GCD}(p, 2) \operatorname{GCD}(p, q) \operatorname{GCD}(2, q)}{\operatorname{GCD}(p, q, 2)} . \tag{0.53}
\end{equation*}
$$

### 0.13. Résumé of Volume 1

This work illustrates a method to investigate certain smooth, codimensiontwo, real submanifolds of spheres of arbitrary odd dimension (with complements that fiber over the circle) using a novel supersymmetric quantum invariant. Algebraic (fibered) links in $S^{2 n+1}$ [310], including Brieskorn-Pham homology spheres with exotic differentiable structure, are examples of said manifolds with a relative diffeomorphism-type that is determined by the corresponding (multivariate) Alexander polynomial [480, 257].

The twist-regularized Wess-Zumino $\left(\mathrm{WZ}_{\theta, \phi}\right)$ model on a spacetime torus defined within the framework of Constructive Quantum Field Theory [150] and studied from a mathematical perspective by Jaffe et al. [222, 221, 223] is a two-dimensional, interacting, (partially broken) supersymmetric, topological (constructive) quantum field theory on a spacetime torus which exhibits stunning mathematical properties including a hidden translational-unimodular $\mathbb{Z}^{2} \ltimes \mathrm{SL}_{2}(\mathbb{Z})$-symmetry despite having no a priori conformal structure. Given a suitable complex analytic superpotential $f$, the supersymmetric partition function or elliptic genus, $\mathfrak{Z}^{f}=\operatorname{Tr} \Gamma e^{-\beta H-i \sigma P-i \theta J}$ admits an explicit representation as a ratio of Jacobi theta functions involving only the weights of $f$ and
spacetime-twist parameters (op. cit.). Said genus is a weak Jacobi form and enjoys a translational-unimodular $\mathbb{Z}^{2} \ltimes \operatorname{SL}(2 ; \mathbb{Z})$-symmetry despite having no $a$ priori conformal structure.

In contrast to the closely related Landau-Ginzburg (LG) model studied by Kawai, Vafa, Warner and Ceccotti et al., the moduli space of the $\mathrm{WZ}_{\theta, \phi}$ model lacks rigorous classification. The purpose of this work is to illustrate how the topological, algebraic, analytic, geometric, combinatorial and arithmetic facets of singularity theory of complex hypersurfaces [310] can produce a unifying structure for, and generate further insights into, the moduli space of said model.

I propose that the elliptic genus $\mathfrak{Z}^{f}$ of the twist-regularized Wess-Zumino model with superpotential $f$ encodes the reduced Alexander polynomial of the algebraic link $K_{f}$. That is, by specializing to the Steenbrink series (of the mixed Hodge structure of a corresponding fiber) in a certain expansion of $\mathfrak{Z}^{f}$, one may isolate the eigenvalues of the Picard-Lefschetz monodromy (acting on said fiber) - the singularity spectrum - and thereby compute the corresponding characteristic polynomial of which the Alexander polynomial is a factor.

Moreover, a $\mathbb{Z}_{2}$-symmetry of the elliptic genus descends to classical functional equations satisfied by the Steenbrink series, Hilbert-Poincaré series of the local algebra [310], the Lefschetz zeta function of the infinite cyclic covering $M_{K_{f}, \infty}$ of (the complement of the interior of a tubular neighborhood $T\left(K_{f}\right)$ of) $K_{f}$ [352], and the reduced Alexander polynomial of $K_{f}$, all of which imply
a reciprocity law for the singularity spectrum [436]. Although comparably different, the proposed method complements the observation that the Jones polynomial of links in $S^{3}$ may be interpreted as arising from Chern-Simons (gauge) theory [471].

We discuss also new functional symmetries and features of the elliptic genus $\mathfrak{Z}^{f}$ such as $q$-invariance and generalized twist positivity. These identifications are suggestive of the relevance and essential nature of algebraic topology and singularity theory in two-dimensional supersymmetric quantum field theories.

Finally, since algebraic knots in $S^{3}$ are classified by their (univariate) Alexander polynomials [257], the corresponding moduli space of twist-regularized Wess-Zumino models admits a similar classification of said algebraic knots by their corresponding elliptic genera. We propose a more general classification based on the corresponding algebraic link. Our classification scheme is not only by simple topological type of the corresponding singularity (as done for the LG model) but rather by subtle (co-)homological and combinatorial data codified by the local algebraic structure (inner modality, multiplicity, geometric genus, etc.), monodromy (characteristic polynomial), associated fiber (homotopy-type, genus, etc.) and corresponding algebraic link invariants (Alexander polynomial, linking and unknotting numbers, signature, etc.). These data provide a finer taxonomic hierarchy than that given by the LG/ADE correspondence and are generalizable to non-isolated singularities, complete intersections and polar weighted homogeneous functions.

This work represents the first step toward understanding this beautiful tapestry of ideas.

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## Part 1

# Singularities of Complex Algebraic 

## Hypersurfaces



## Chapter 1

# Topological Structure of Isolated Singularities 

Poetry should surprise by a fine excess and not by a singularity, it should strike the reader as a wording of his own highest thoughts, and appear almost a remembrance. - John Keats

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1.11. Topological Morphisms on the Milnor Fiber ..... 74
1.12. Complex Topological K-Theory of the Milnor Fiber ..... 87
Following a brief review of the classical Brouwer degree theory of continuous maps between real manifolds, we discuss fibrations and the Ehresmann Fibration Theorem. Generalizing to hypersurfaces of complex analytic germs with a critical locus in a neighborhood of the origin [310], we then review the Milnor Fibration Theorem and the open-book decomposition of the diffeomorphism class of corresponding fibers and (fibered) boundary link.

With increasing specialization, our discussion begins with the most generic complex analytic case where said critical locus is of arbitrary dimension and unspecified topological density, proceeding then to those complex analytic germs with an isolated critical point at the origin, specializing further still to weighted homogeneous polynomials and finally concluding the introductory discussion with singularities of the Brieskon-Pham type. We continue to the construction of a wedge sum of spheres via CW-quotients, various morphisms acting on Milnor fibers and finally the complex topological K-theory of wedge sums, cartesian products and smash products of spheres. We study these analytic objects from a topological setting from the points-of-view of homology*, homotopy and complex topological K-theory, including a thorough discussion of related numerical invariants. Correspondingly, we define the differential, topological and K-theoretic indices. In Chapter 3, we discuss the related notion of local multiplicities of holomorphic maps between more general complex domains and Grothendieck residues.

Generalizations of the Milnor Fibration Theorem to complete intersections, hypersurface singularities with non-trivial critical loci and real analytic maps, including some recent work on the Sebastiani-Thom Equivalence, are also briefly mentioned in the sequel.

[^14]
### 1.1. Brouwer Degree

The degree of a continuous map $\phi: X \rightarrow Y$ between oriented differential manifolds (without boundary and of the same dimension), where $X$ is compact and $Y$ is connected, is a classical numerical invariant of differential topology [307, 169, 198]. The theory of degrees has its origin in the nascent days of homology theory, and it's influence in modern mathematics is quite profound and still present [67]. Degree theory is without doubt an indispensable tool for computing useful invariants to distinguish certain topological spaces. In fact, the degree formalism has also been instrumental in the development of many classical and modern areas of mathematical research, namely, homotopy theory, fixed point theory, index theory and topological/algebraic K-theory. The mathematical province of these topics is immeasurable and their applications are discussed at length in the vast literature of Differential Topology [307].

Although there are various definitions of the degree of a map, we focus on two primary and complementary formulations: cohomological and homological. While the former is suitable for definition, the latter is more amenable to generalization, which is necessary for our needs.

Consider a pair of oriented differential manifolds $(X, Y)$ and a map $\phi: X \rightarrow$ $Y$, as described above. After selecting a local coordinate atlas at a generic point $x \in X$, there is a value $y=\phi(x) \in Y$ such that the map $\phi$ induces a local pullback (differential) homomorphism $\varphi_{x}^{*}=(d \phi)_{x}: T_{x} X \rightarrow T_{y} Y$ between corresponding local tangent manifolds. Let $U_{x}$ be an open neighborhood of $x \in X$.

As a generalization of the Inverse Function Theorem, if $(d \phi)_{x}: T_{x} X \rightarrow T_{\phi(x)} Y$ is a linear isomorphism, then $\left.\phi\right|_{U_{x}}: U_{x} \rightarrow \phi\left(U_{x}\right)$ is a diffeomorphism. Conversely, if for each $x \in X$, the map $\left.\phi\right|_{U_{x}}: U_{x} \rightarrow \phi\left(U_{x}\right)$ is a diffeomorphism, then $(d \phi)_{x}: T_{x} X \rightarrow T_{\phi(x)} Y$ is a linear isomorphism. In particular, if $y$ is a regular value, then $(d \phi)_{x}$ is an isomorphism and the preimage $\phi^{-1}(y)$ consists of a finite set of points. By patching local charts, we construct a differential $\varphi^{*}=d \phi: T X \rightarrow T Y$ between tangent bundles, as illustrated by the following commutative diagram,

where $\pi_{X}$ and $\pi_{Y}$ are canonical projections to $X$ and $Y$, respectively.
Given a (normalized) top-form $\omega$ as an element of the exterior algebra $\bigwedge^{\operatorname{dim} X} T X$, whose existence is guaranteed by construction, define the Brouwer degree of the map $\phi$ as the pairing $\langle\phi(X), \omega\rangle$, or equivalently $\left\langle X, \phi^{*}(\omega)\right\rangle$, that is,

$$
\begin{align*}
\operatorname{deg}_{B} \phi & =\int_{\phi(X)} \omega  \tag{1.2a}\\
& =\int_{X} \phi^{*}(\omega)  \tag{1.2b}\\
& =\sum_{x \in \phi^{-1}(y)} \operatorname{sgn} \phi^{*}(x), \tag{1.2C}
\end{align*}
$$

which is independent of the regular value $y \in Y$ so chosen arbitrarily. The integral equality is a consequence of the natural duality between (smooth) singular
homology of chains and the de Rham cohomology of differential forms on $X$, viz., $\left\langle\phi_{*}[\gamma],[\eta]\right\rangle=\left\langle[\gamma], \phi^{*}[\eta]\right\rangle$, where $\gamma$ is a representative cycle of the homology class $[\gamma]$ and $\eta$ is a representative closed differential form of the de Rham cohomology class $[\eta]$. The integral-sum equality follows from the assumption that the target space is both compact and that the set $\phi^{-1}(y)$ consists of regular points, so finite. The value of $\operatorname{sgn} \phi^{*}(x)$ is 1 if and only if $\phi^{*}$ is orientationpreserving and -1 otherwise, so the integrality of the Brouwer degree of $\phi$ is clear.

As illustrated by the following commutative diagram, the map $\phi$ induces a push-forward homomorphism $\phi_{*}$ between corresponding reduced-homology groups,

$$
\begin{array}{rll}
\tilde{H}_{i}(X ; \mathbb{Z}) & \stackrel{\phi_{*}}{ }> & \tilde{H}_{i}(Y ; \mathbb{Z})  \tag{1.3}\\
\left.\tilde{H}_{i}(-; \mathbb{Z})\right|_{X} & & \\
X & & \\
& & \tilde{H}_{i}(-; \mathbb{Z}) \\
Y
\end{array}
$$

where $\tilde{H}_{i}(-; \mathbb{Z})$ denotes the $i^{\text {th }}$-reduced homology functor (with $\mathbb{Z}$ coefficients).
If the manifolds $X$ and $Y$ are homology $n$-spheres, then the $n^{\text {th }}$-reduced homology group is infinite-cyclic, i.e., $\tilde{H}_{n}\left(S^{n} ; \mathbb{Z}\right) \cong \mathbb{Z}$, and the induced map $\phi_{*}$ is therefore an endomorphism of $\mathbb{Z}$. Consequently, the Brouwer degree of the map $\phi$ is simply the image of the generator $1 \in \mathbb{Z}$ under said endomorphism,
that is, $\operatorname{deg}_{B}(\phi)=\phi_{*}(1)$, as in the commutative diagram,


In fact, this homological approach to the degree of a map $\phi: X \rightarrow Y$ extends more generally to any pair of $n$-dimensional CW-spaces each possessing a corresponding reduced-homology group that is infinite-cyclic, namely, $\tilde{H}_{i}(X ; \mathbb{Z}) \cong \tilde{H}_{i}(Y ; \mathbb{Z}) \cong \mathbb{Z}$ for some common index $0<i \leqslant n$, though not necessarily $n$.

Definition 1.1. Two continuous maps $\phi_{0}, \phi_{1}: X \rightarrow Y$ are smoothly homotopic, i.e., there exists a smooth $\Phi: X \times[0,1] \rightarrow Y$ such that $\Phi(x, 0)=\phi_{0}(x)$ and $\Phi(x, 1)=\phi_{1}(x)$, if and only if $\operatorname{deg}_{B} \phi_{0}=\operatorname{deg}_{B} \phi_{1}$.

Definition 1.2. A map is null-homotopic if it has vanishing degree.

As shown by Hopf, the degree is a complete homotopy invariant*.

Proposition 1.3. Let $X$ and $Y$ be CW complexes. The following is true:

[^15]1. If the identity automorphism $\boldsymbol{i d}_{X}: X \rightarrow X$ is null-homotopic, then $X$ is contractible (to a point) implying the triviality of both the associated homology and homotopy;
2. Given two continuous maps $\phi: X \rightarrow Y$ and $\psi: Y \rightarrow X$ as above, if $\phi \circ \psi$ and $\psi \circ \phi$ are smoothly homotopic to the identities $i \boldsymbol{d}_{X}: X \rightarrow X$ and $\boldsymbol{i d}_{Y}: Y \rightarrow Y$, respectively, then the spaces $X$ and $Y$ have equivalent homotopy-type; and,
3. In particular, if there is a inclusion map $\phi: X \hookrightarrow Y$, then there is a deformation retraction from $Y$ onto $X$.

For a discussion of these and related results, consult [198] and [307].

### 1.2. Ehresmann Fibration Theorem

Definition 1.4. Given a quadruple $(F, E, B, \phi)$ consisting of a fiber $F$, a total space $E$ and a connected, pointed base space $(B, b)$ with a countable open cover $\mathscr{U}$, a (projection of a) fibration is a continuous surjection $\phi: E \rightarrow B$ such that $\phi^{-1}(b)=F$ and, for each point $x \in B$, there is an open neighborhood $U_{x} \subset \mathscr{U}$ such that there is a (fiber-preserving and trivializing) homeomorphism $h: \phi^{-1}\left(U_{x}\right) \rightarrow U_{x} \times F$ and a projection $\pi: U_{x} \times F \rightarrow U_{x}$ yielding the commutative diagram


Remark 1.2.1. A fibration may be defined universally as a surjection satisfying the homotopy lifting property (Chapter 7, [296]).

Remark 1.2.2. A fibration $\phi$ is often implied, and a fibration quadruple $(F, E, B, \phi)$ is written simply as a sequence of maps $F \hookrightarrow E \rightarrow B$.

Proposition 1.5 (Hopf, [208]). There is a fibration $S^{1} \hookrightarrow S^{3} \rightarrow S^{2}$.

A cross-section of the Hopf fibration is given in Figure 1.1.

Remark 1.2.3. Although the Hopf fibration is locally trivial, it is not globally trivial as $S^{3}$ does not factor as the Cartesian product $S^{2} \times S^{1}$.


Figure 1.1. The Hopf Fibration (Adapted from [229])

Definition 1.6. Given two differentiable manifolds $X$ and $Y$ with tangent bundles $T X=\bigcup_{x \in X} T_{x} X$ and $T Y=\bigcup_{y \in Y} T_{y} Y$, respectively, a local submersion at a point $x \in X$ is a smooth, differentiable map $\phi: X \rightarrow Y$ with a surjective, linear,
local differential $(d \phi)_{x}: T_{x} X \rightarrow T_{\phi(x)} Y$. If $\phi$ is a local submersion everywhere on $X$, then it is a submersion on $X$, and there is a commutative diagram,


Definition 1.7. A continuous map between topological spaces is proper if any pre-image of a compact subset is compact subset.

Proposition 1.8 (Ehresmann, [121]). A smooth, proper, surjective submersion $\phi: X \rightarrow Y$ between smooth, oriented, differentiable manifolds, where $X$ is compact and without boundary, is a locally trivial fibration, i.e., the projection of a locally trivial fiber bundle.

Proof. See Theorem 3.1 in [420].

### 1.3. Milnor Fibration Theorem

Let $U_{x}$ be a neighborhood of the point $x \in \mathbb{C}^{n+1}$ and write $U=U_{0}$, where $\mathbf{0}$ denotes the origin. Let $B_{\varepsilon}^{2 n+2}=\left\{\left(z_{0}, \ldots, z_{n}\right) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} \bar{z}_{i} z_{i} \leqslant \varepsilon \in \mathbb{R}_{>0}\right\}$ denote the complex $(n+1)$-dimensional ball with radius $\varepsilon$ and centered on the origin with boundary (sphere) $S_{\varepsilon}^{2 n+1}$.

Given a complex analytic germ ${ }^{*} f:(U, \mathbf{0}) \rightarrow(\mathbb{C}, 0)$ and a representative function (denoted by the same symbol) $f$, define the parametrized family of

[^16]corresponding complex hypersurfaces of $f-\kappa$, namely, $\left\{V_{f, \kappa}=f^{-1}(\kappa)\right\}_{\kappa \in \mathbb{C}}$. Let $\Sigma\left(V_{f, \kappa}\right)$ denote the set of singular points of $V_{f, \kappa}$, which corresponds to the critical locus of $f$. The following classical result of Milnor is a landmark which partially generalizes the Ehresmann Fibration Theorem to complex analytic maps.

Proposition 1.9 (Milnor, [310]). Let $f:(U, \mathbf{0}) \rightarrow(\mathbb{C}, 0)$ be a complex analytic germ. Consider the map $\phi_{f}=\frac{f}{\|f\|}: S_{\varepsilon}^{2 n+1} \backslash V_{f, 0} \rightarrow S^{1}$ with the image identified with $\partial \bar{\Delta}$, the standard unit circle in $\mathbb{C}$. There is an $\varepsilon_{0}>0$ such that for all $\varepsilon$ satisfying $0<\varepsilon<\varepsilon_{0}$, the map $\phi_{f}$ is the projection of a locally trivial fibration over $S^{1}$.

Proof. See $\S 1-\$ 4$ in [310], Chapter 1 in [420] and [230].
The fibration map $\phi_{f}$ induces an open-book decomposition $\left(K_{f}, \phi_{f}\right)$ of $S_{\varepsilon}^{2 n+1}$, where the 1-parameter diffeomorphism class of fibers $\left\{F_{f, \theta}=\phi_{f}^{-1}\left(e^{i \theta}\right)\right\}_{\theta \in S^{1}}$ constitute the pages and the hypersurface intersection $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{2 n+1}$ constitutes the binding (Figure 1.2).

Additionally, Milnor proves the following facts:
M1. Each page $F_{f, \theta}=V_{f, \kappa} \cap B_{\varepsilon}^{2 n+2}$ is a non-compact, smooth, parallelizable manifold of (real) dimension $2 n$ with the homotopy-type of an $n$-dimensional, finite CW-complex;
M2. The closure $\bar{F}_{f, \theta}$ is a compact manifold with boundary $V_{f, \kappa} \cap S_{\varepsilon}^{2 n+1}$ and, for sufficiently small but non-zero $\kappa$ and $\varepsilon, \bar{F}_{f, \theta}$ is diffeomorphic to the intersection $V_{f, \kappa} \cap B_{\varepsilon}^{2 n+2}$ independent of any minute variation of $\varepsilon$ and $\kappa$; and,


Figure 1.2. Open book $\left(K_{f}, \phi_{f}\right)$ with pages $\left\{F_{f, \theta}\right\}_{\theta \in S^{1}}$ and binding $K_{f}$ (Adapted from [310])

M3. The relative pair $\left(B_{\varepsilon}^{2 n+2}, V_{f, 0} \cap B_{\varepsilon}^{2 n+2}\right)$ is locally homeomorphic to the cone over the relative pair $\left(S_{\varepsilon}^{2 n+1}, K_{f}\right)=\left(S_{\varepsilon}^{2 n+1}, V_{f, 0} \cap S_{\varepsilon}^{2 n+1}\right)$.

Suppose the origin is an isolated critical point ${ }^{*}$ of $f$ (i.e., an isolated root of the system $\left.\partial f\right|_{U}=0$ ); therefore, it is an isolated singularity of $V_{f, 0}$ (implying $V_{f, 0}^{\times}=V_{f, 0} \backslash \Sigma\left(V_{f, 0}\right)$ is a non-singular, complex manifold of dimension $n$ - in fact, a Stein manifold, so holomorphically convex and separable) and M1-3 can be sharpened substantially to the following:
$\mathbf{M 1}^{\prime}$. Each page $F_{f, \theta}$ is $(n-1)$-connected, has trivial reduced-homology except in (middle) dimension $n$ and, therefore, has the homotopy-type of a wedge sum or bouquet of $n$-spheres, $\bigvee^{\mu} S^{n}$;
$\mathbf{M 2}^{\prime}$. For $n \neq 2$, the compact manifold with boundary $\bar{F}_{f, \theta}$ is diffeomorphic to a handlebody obtained by adjoining a number of $n$-handles to $B^{2 n}$; and,

[^17]$\mathbf{M}_{3}{ }^{\prime}$. The binding $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{2 n+1}$ (as a transversal intersection) is a smooth ( $n-2$ )-connected, codimension-two (real) submanifold of $S_{\varepsilon}^{2 n+1}$ manifested as a possibly linked, disjoint union $\coprod S^{2 n-1}$, a fibered link.

Remark 1.3.1. Until recently, the $h$-Cobordism Theorem was unproven for $n=2$, hence the restriction $n \neq 2$ in $\mathbf{M 2}^{\prime}$ [259]. Since the Poincaré Conjecture ${ }^{*}$ is true in $\mathbb{R}^{3}$, the $h$-Cobordism Theorem follows.


Figure 1.3. The Milnor Fibration and Corresponding Boundary Link

### 1.4. Milnor Fiber

As previously noted, no particular choice of page need be made as each fiber is identical up to diffeomorphism. Henceforth, let $F_{f, 0}=\phi_{f}^{-1}(1)$ denote the

[^18]Milnor fiber of $f$. Correspondingly, we refer to a representative fiber $F_{f, \theta}$ of the 1-parameter diffeomorphism class $\left\{F_{f, \theta}\right\}_{\theta \in S^{1}}$ as a generic Milnor fiber of $f$.

Proposition 1.10 (Andreotti, Frankel, [13]). A Stein $k$-manifold $X \subset \mathbb{C}^{n}$ has the homotopy type of a CW-complex of real dimension at most $k$. In particular,

$$
\begin{equation*}
H^{i}(X ; \mathbb{Z}) \cong H_{i}(X ; \mathbb{Z}) \cong\{0\} \quad i>k \tag{1.6}
\end{equation*}
$$

Proof. See Theorem 7.1 in $\S 7$ in [306] for a proof of a special case.
According to Milnor, a generic fiber $F_{f, \theta}$ has a trivial tangent bundle and no compact components, so it is a parallelizable manifold. Regardless of the topological density of the critical points of $f$ in $U$, since $V_{f, 0}^{\times}$is a Stein manifold, Proposition 1.10 then implies that said fiber has the homotopy-type of (at most) an $n$ dimensional, finite CW-complex (Theorem 5.1, [310]). In particular, if the origin is either a regular or isolated singular point of $V_{f, 0}$ and and the hypersurface $V_{f, 0}$ intersects $S_{\varepsilon}^{2 n+1}$ transversally, then $F_{f, \theta}$ is an $(n-1)$-connected manifold (Lemma 6.4, op. cit.) with trivial homology in all dimensions less than $n$ (Corollary 6.3,op. cit.). Consequently, there are no cohomology classes in dimension $n+1$, so the middle (reduced) homology group $\tilde{H}_{n}\left(F_{f, \theta} ; \mathbb{Z}\right)$ is torsion-free and free abelian of finite rank by the Hurewicz Theorem.

Proposition 1.11 (Serre). Let X be a finite, ( $m-1$ )-connected CW complex with $m \geqslant 2$. The Hurewicz homomorphism $\pi_{k}(X) \rightarrow H_{k}(X ; \mathbb{Z})$ is a $\mathscr{C}$-isomorphism for $0 \leqslant k<2 m-1$.

Proof. See Theorem 18.3 in [316].
In addition to Whitehead's Theorem, this uniquely specifies the homotopytype of a generic fiber $F_{f, \theta}$, as in $\mathbf{M r}^{\prime}$ (Figure 1.4).

Proposition 1.12 (Milnor, [310]). Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a complex analytic germ with an isolated critical point at the origin. The homotopy-type of a generic fiber $F_{f, \theta}$ is that of a wedge sum of $n$-spheres, $\bigvee^{\mu} S^{n}$.

Proof. See in Theorem 6.5 in [310].
Such a homotopy type is rather pervasive and may be constructed from a quotient of a general CW-structure, q.v., §1.9.1 and §1.9.2. Since the (reduced) homology and homotopy of a bouquet is concentrated in the dimension of the constituent spheres, the number of such spheres, the rank of the non-vanishing homology group, is a topological invariant of the corresponding singularity.


Figure 1.4. A Bouquet of (Pointed) Spheres

### 1.5. Algebraic Links

The local topological nature of the critical locus in the vicinity of the origin leads to dramatically different isotopy and diffeomorphism types of the corresponding fibered link. In particular, the topology of the critical locus is key in determining the topological type of the ambient fibers with two fundamentally different cases to consider. On the one hand, if the origin is a regular or simple point (i.e., a non-singular point) $f$, then $V_{f, 0}$ is a smooth manifold, $F_{f, \theta}$ is contractible and the binding $K_{f}$ is a smooth differentiable $(2 n-1)$-manifold diffeomorphic to the standard, unknotted codimension-two ( $2 n-1$ )-sphere embedded in $S_{\varepsilon}^{2 n+1}$. On the other hand, if the origin is an isolated singular point of $V_{f, 0}$, i.e., an isolated root of the system $\left.\partial f\right|_{U}=\mathbf{0}$, then a generic fiber $F_{f, \theta}$ is not contractible and the binding $K_{f}$ is knotted (Corollary 7.3,op. cit.) with possibly many linked components, as in $\mathbf{M} 3^{\prime}$, e.g., the torus link $T_{p, q}$. Illustrated in Figure 1.3, the transverse intersection ${ }^{*} K_{f}=V_{f, 0} \cap S_{\varepsilon}^{3}$ is a pair of linked unknots, $T_{2,2}$, the Hopf link. Two fibers ( $F_{f, 0}$ and $F_{f, \pi}$ ) bounding the Hopf link are shown in Figure 1.5.

REmARK 1.5.1. With regard to the Hopf fibration $S^{1} \hookrightarrow S^{3} \rightarrow S^{2}$, fibering over each pair of distinct points in $S^{2}$, there is a Hopf link in $S^{3}$.
1.5.1. Monodromy. That $S_{\varepsilon}^{2 n+1} \backslash V_{f, 0} \cong S_{\varepsilon}^{2 n+1} \backslash K_{f}$ is fibered over $S^{1}$ implies the existence of a 1-parameter family of homomorphisms $h_{\theta^{\prime}}: \bar{F}_{f, \theta} \rightarrow \bar{F}_{f, \theta+\theta^{\prime}}$.

[^19]

Figure 1.5. Milnor Fibers of the Hopf Link (Adapted from [114])
Corresponding to a full rotation, the map $h=h_{2 \pi}$ induces a non-trivial pushforward endomorphism $h_{*}: \tilde{H}_{n}\left(\bar{F}_{f, 0} ; \mathbb{C}\right) \rightarrow \tilde{H}_{n}\left(\bar{F}_{f, 0}, \mathbb{C}\right)$, as illustrated by the commutative diagram

$$
\begin{array}{rc}
\tilde{H}_{n}\left(\bar{F}_{f, 0} ; \mathbb{C}\right) & \stackrel{h_{*}}{ }>\tilde{H}_{n}\left(\bar{F}_{f, 0} ; \mathbb{C}\right) \\
\tilde{H}_{n}(-; \mathrm{C}) \mid & \tilde{H}_{n}(-; \mathrm{C}) \\
\bar{F}_{f, 0} \xrightarrow{h} & \\
& \bar{F}_{f, 0}
\end{array}
$$

where $\tilde{H}_{n}(-; \mathbb{C})$ denotes the reduced homology functor (with complex coefficients) and an associated Wang sequence [176],

$$
\{0\} \longrightarrow H_{n}\left(F_{f, 0} ; \mathbb{Z}\right) \xrightarrow{1-h_{*}} H^{n}\left(F_{f, 0} ; \mathbb{Z}\right) \longrightarrow \tilde{H}_{n-1}\left(K_{f} ; \mathbb{Z}\right) \longrightarrow\{0\}
$$

When the context is clear, we refer to both the fiber map $h=h(f)$ and its induced morphism $h_{*}=h_{*}(f)$ as the (Picard-Lefschetz) monodromy of $f$.

There is a distinguished basis of the homology group ${ }^{*} H_{n}\left(F_{f, 0} ; \mathbb{Z}\right) \cong \mathbb{Z}^{\mu}$ given by the vanishing cycles $\left\{\Delta_{1}, \ldots, \Delta_{\mu}\right\}$, which allows one to compute explicitly the corresponding intersection matrix $S=\left(\Delta_{i} \circ \Delta_{j}\right)$ and monodromy
*To simplify notation, we write $\mathbb{Z}^{r}$ in place of the direct summation $\oplus_{i=1}^{r} \mathbb{Z}$.
in terms of intersection numbers of said cycles, $h_{*}=\prod_{i=0}^{n+1} h_{*, i}$, where $h_{*, i}=$ $\alpha+(-1)^{n(n+1) / 2}\left(\alpha \circ \Delta_{i}\right) \Delta_{i}$ and $\alpha \in H_{n}\left(F_{f, 0} ; \mathbb{Z}\right)[22]$.

Proposition 1.13. Given a non-degenerate, isolated singularity
$f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, the eigenvalues of the monodromy $h_{*}: \tilde{H}_{n}\left(\bar{F}_{f, 0} ; \mathbb{C}\right) \rightarrow$ $\tilde{H}_{n}\left(\bar{F}_{f, 0}, \mathbb{C}\right)$ are roots of unity.

Proof. See Theorem 3.1 in [161] and §5.C in [276].

Corollary 1.14. The characteristic polynomial $\Delta_{h_{*}}(t)=\operatorname{det}\left(t I-h_{*}\right)$ of the monodromy $h_{*}: \tilde{H}_{n}\left(\bar{F}_{f, 0} ; \mathbb{C}\right) \rightarrow \tilde{H}_{n}\left(\bar{F}_{f, 0}, \mathbb{C}\right)$ is monic, reflexive and has constant coefficient equal to $\pm 1$, that is,

$$
\begin{equation*}
\Delta_{h_{*}}(t)=t^{\mu}+b_{\mu-1} t^{\mu-1}+\cdots+b_{1} t^{1}+(-1)^{\mu n} \quad b_{1}, \ldots, b_{\mu} \in \mathbb{Z} \tag{1.7}
\end{equation*}
$$

where $b_{k}=b_{\mu-k}$ for $1 \leqslant k \leqslant \mu-1$.
By carefully studying the monodromy map and associated Wang sequence, Milnor proved the following result.

Proposition 1.15 (Milnor, [310]). Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a complex analytic germ with an isolated critical point at the origin. For $n \neq 2$, the algebraic link $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{2 n+1}$ with $r$ components is a topological $(2 n-1)$-sphere if and only if the characteristic polynomial $\Delta_{h_{*}}(t)=\operatorname{det}\left(t \mid-h_{*}\right)$ of the associated monodromy map $h_{*}: \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{C}\right) \rightarrow \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{C}\right)$ coincides with the reduced Alexander polynomial

$$
\begin{equation*}
\Delta_{K_{f}}(t)=(t-1)^{1-\delta_{r, 0}} \Delta_{K_{f}}\left(t_{1}, \ldots, t_{r}\right) \tag{1.8}
\end{equation*}
$$

and satisfies $\Delta_{h_{*}}(1)= \pm 1$. The degree of $\Delta_{h_{*}}$ is the number of spheres in the homotopy type of the fiber, $F_{f, 0} \simeq \bigvee^{\mu} S^{n}$.

Proof. See Lemma 8.2 and Theorem 8.5 in [310].

Remark 1.5.2. There are counter-examples to Proposition 1.15 for $n=2$. In this case, however, replacing topological 3-sphere with homology 3-sphere reinstates its validity.

We return to the discussion of algebraic links and their classification in §4.6. As will be evident in the sequel, the Milnor fiber $F_{f, 0}$ does not uniquely determine the topological type of the corresponding link $K_{f}$. However, there are numerical invariants of $F_{f, 0}$ (and therefore of $f$ ) which describe various salient features of its boundary and vice versa. Such invariants will occupy our attention for the majority of this volume.

### 1.6. Weighted Homogeneous Singularities

If the complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial, i.e. $\tilde{f}=f\left(z_{0}^{q_{0}}, \ldots, z_{n}^{q_{n}}\right)$ is a homogeneous polynomial of degree $d$ for some set of positive integers $\left\{q_{0}, \ldots, q_{n}, d\right\}$, then the hypersurface $V_{f, \kappa}$, where $\kappa$ is a regular value of $f$ sufficiently close to the origin, completely determines the topological type of a generic fiber $F_{f, \theta}$. In terms of the reduced weights $\left\{\omega_{i}=\frac{q_{i}}{d}\right\}$, the computation of the associated numerical invariants is greatly simplified. Focusing M1-3' to the weighted homogeneous case,
$\mathbf{M} \mathbf{2}^{\prime \prime}$. If $f$ is a non-degenerate, weighted homogeneous polynomial, then $F_{f, \theta}$ diffeomorphic to $V_{f, 1}=f^{-1}(1)$ as a deformation retraction. In this case, one writes $F_{f, \theta} \cong_{d} V_{f, 1}$; and,
$\mathbf{M} 3^{\prime \prime}$. If $f$ has reduced weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\} \subset \mathbb{Q} \cap(0,1)$, then the monodromy automorphism $h: \bar{F}_{f, 0} \rightarrow \bar{F}_{f, 0} \cong{ }_{d} V_{f, 1}$ is explicitly given by the $\mathbb{C}^{\times}$-action

$$
\begin{equation*}
h:\left(z_{0}, \ldots, z_{n}\right) \mapsto\left(e^{2 \pi i \omega_{0}} z_{0}, \ldots, e^{2 \pi i \omega_{n}} z_{n}\right) . \tag{1.9}
\end{equation*}
$$

The $k$-orbit $h^{k}: V_{f, 1} \rightarrow V_{f, 1}$ has Lefschetz number equal to the Euler characteristic $\chi_{k}=\sum_{1<d \mid k} d r_{d}$ of the fixed-point manifold

$$
\begin{equation*}
\left\{\left(z_{0}, \ldots, z_{n}\right) \in V_{f, 1} \mid h^{k}\left(z_{0}, \ldots, z_{n}\right)=\left(z_{0}, \ldots, z_{n}\right)\right\} \tag{1.10}
\end{equation*}
$$

where the integers $\left\{r_{d}\right\}$ are encoded in the characteristic polynomial

$$
\begin{equation*}
\Delta_{h_{*}}(t)=(t-1)^{(-1)^{n+1}} \prod_{1<d \mid N}\left(t^{d}-1\right)^{(-1)^{n} r_{d}} \tag{1.11}
\end{equation*}
$$

where $N$ denotes the period of the monodromy $h, v i z ., h^{N}=1$, and $r_{d}$ is non-zero only when $d$ divides $N$. In particular, the degree of the characteristic polynomial may be recovered from the exponents, namely,

$$
\begin{equation*}
\mu=(-1)^{n+1}+(-1)^{n} \sum_{1<d \mid N} d r_{d}=(-1)^{n+1}\left(1-\chi_{N}\right) \tag{1.12}
\end{equation*}
$$

Denote the $n^{\text {th }}$-root-of-unity by $\zeta_{n}=e^{2 \pi i / n}$. Writing the rational weights in the reduced form $\left\{\omega_{i}=\frac{r_{i}}{s_{i}}\right\}$, Milnor and Orlik (Theorem 4, [315]) prove that the characteristic polynomial of a weighted homogeneous singularity $f$ is determined by the divisor

$$
\begin{equation*}
\operatorname{div} \Delta_{h_{*}}(t)=\prod_{i=0}^{n}\left(\frac{1}{r_{i}} \Lambda_{s_{i}}-\Lambda_{1}\right) \tag{1.13}
\end{equation*}
$$

where $\Lambda_{s_{i}}=\sum_{k=0}^{n-1}\left\langle\zeta_{n}^{k}\right\rangle$ is the divisor of $t^{s_{i}}-1$ in the group algebra $\mathbb{Z} \mathbb{C}^{\times}$with product $\Lambda_{a} \Lambda_{b}=\operatorname{GCD}(a, b) \Lambda_{\mathrm{LCM}(a, b)}$ for $a, b \in \mathbb{N}$ and identity $\Lambda_{1}$. Here, a divisor $c \Lambda_{d}$ contributes $\left(t^{d}-1\right)^{c}$ to the numerator if $\operatorname{sign}(c)=1$ or the denominator if $\operatorname{sign}(c)=-1$, provided that $d \geqslant 1$. In principle, the coefficients $\left\{c_{k}\right\}$ may computed in terms of the weights.

Proposition 1.16 (Milnor, Orlik, [315]). Given $\operatorname{div} \Delta_{h_{*}}(t)=\sum_{k \geqslant 1} c_{k} \Lambda_{k}$, then $\kappa=\sum_{k \geqslant 1} c_{k}$ and $\rho=\prod_{k \geqslant 2} k^{c_{k}}$ are non-negative integers, where $\kappa$ is the greatest power of the linear factor $t-1$ dividing $\Delta_{h_{*}}(t)$ and $\Delta_{h_{*}}(1)=\rho \delta_{0, \kappa}$.

### 1.7. Numerical Invariants of the Milnor Fiber

In the sections, chapters, parts and volumes to follow, we endeavor to deepen our understanding of the underlying differential, topological, algebraic, analytic, geometric, combinatorial, arithmetic, categorical, real and supersymmetric structure respecting the diffeomorphism class of generic Milnor fibers and their boundary (algebraic) links. A central theme of this volume is how the

Table 1.1. Invariant Indices of Isolated Singularities

| Index | Symbol | Value | Brief Description |
| :--- | :--- | :--- | :--- |
| Differential | $\mu_{\text {diff }}$ | $\operatorname{deg}_{B}\left(\phi_{\left.\partial f\right\|_{U}}\right)$ | Brouwer Degree of the Map $\left.\phi_{\partial f}\right\|_{U}$ |
| Topological | $\mu_{\text {top }}$ | $\operatorname{rank} \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right)$ | Rank of the $n^{\text {th }}$-Reduced Homology Group of $F_{f, 0}$ |
| $K$-Theoretic | $\mu_{K}$ | $\operatorname{rank} \tilde{K}^{n}\left(F_{f, 0}\right)$ | Rank of the $n^{\text {th }}$-Reduced Grothendieck Group of $F_{f, 0}$ |
| Analytic | $\mu_{\text {anal }}$ | $\operatorname{Res} \omega\left(\left.\partial f\right\|_{U}\right)$ | Grothendieck Residue of the Logarithmic Form $\omega\left(\left.\partial f\right\|_{u}\right)$ |
| Geometric | $\mu_{\text {geom }}$ | $\operatorname{mult}(f)$ | Local Geometric Multiplicity of $f$ on $U$ |
| Algebraic | $\mu_{\text {alg }}$ | $\operatorname{dim}_{C} \mathcal{A}_{f}$ | Complex Dimension of the Local Algebra of $f$ |
| Cohomological | $\mu_{\text {co }}$ | $b_{1}\left(M_{K, \infty}\right)$ | First Betti Number of Infinite Cyclic Covering |
| Combinatorial | $\mu_{\text {comb }}$ | $\operatorname{MV} \mathcal{K}(f)$ | Mixed Volume of the Polytope $\mathcal{K}(f)$ |
| Quantum | $\mu_{\mathrm{qm}}$ | $\operatorname{ind}\left(Q_{\partial f}^{+}\right)$ | Multiplicity of the Ground State in the $\mathrm{WZ}_{\theta, \phi}$ Model |
| Arithmetic | $\mu_{\mathrm{nt}}$ | $\operatorname{ord} \mathcal{O}(f)$ | Number of Integral Solutions of $0<\omega_{i} x_{i}<1$ |

associated numerical invariants are interrelated and to what extent they elucidate the subtle and salient features of algebraic links (Table 1.1). We summarize these points in the following claim.

Proposition 1.17. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a complex analytic germ with an isolated critical point at the origin, then the following statements are true:

1. The first five indices given in Table 1.1 are well-defined, coincident positive integers equal to the rank of the of the $n^{\text {th }}$-reduced Grothendieck group of the corresponding Milnor fiber;
2. If $f$ is weighted homogeneous, then the first eight indices in said table are coincident positive integers equal to the dimension of the local algebra of $f$;
3. If $f$ is weighted homogeneous and elliptic, then the first nine indices in said table are coincident positive integers equal to the Fredholm index of a supercharge of a two-dimensional, supersymmetric, quantum field theory with interaction $f$; and,
4. If $f$ is Brieskorn-Pham (or more generally, quasi-Brieskorn-Pham and elliptic), then all indices in said table are coincident positive integers equal to the number of non-negative integral solutions of a corresponding system of Diophantine inequalities.

Chapter 11 culminates in the proof of Proposition 1.17. The experienced or eager reader may choose to forgo the next few chapters and proceed directly to Part 2, which concerns supersymmetry and the twist-regularized Wess-Zumino model.

### 1.8. Local Geometric Multiplicity

Of the vast number of invariants that arise in the context of isolated singularities of complex hypersurfaces, we begin our investigation with the two most easily described, the local geometric multiplicity and Brouwer degree.

A critical point $x$ of a function $f$ is non-degenerate or Morse if the Hessian (matrix) $H(f)=(\partial f)$ evaluated at $x$ is non-singular [306]. A function is Morse if it only has Morse critical points. The local geometric multiplicity mult ${ }_{x}(f)$ of $f$ at a degenerate critical point $x$ is the number of Morse points into which $x$ splits as a result of a perturbation $f+\varepsilon g$ in $U_{x}$ (Figure 1.6), where $g$ is a Morse function and $\varepsilon>0$. The function $f_{\varepsilon}=f+\varepsilon g$ is the (complex) morsification of $f$.

If the real parameter $\varepsilon$ is sufficiently small, the non-negative integer $\operatorname{mult}_{x}(f)$ does not depend on the Morse perturbation $g$. If $x$ is the origin, we write instead mult $(f)$.


Figure 1.6. A Morse Perturbation and Local Geometric Multiplicity equal to 3
1.8.1. Geometric Index. Define the geometric index of a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ as the local geometric multiplicity in a neighborhood of the origin,

$$
\begin{equation*}
\mu_{\text {geom }}(f):=\operatorname{mult}(f)=\left|V_{f, \kappa} \cap B_{\varepsilon}^{2 n+2}\right|, \tag{1.14}
\end{equation*}
$$

where $\kappa \in \mathbb{C}^{\times}$is a regular value of $f$ sufficiently close to the origin and $\varepsilon>0$ is sufficiently small.
1.8.2. Differential Index. Given a complex analytic germ $f:(U, 0) \rightarrow(\mathbb{C}, 0)$, where $U \subset \mathbb{C}^{n+1}$ is a neighborhood of the origin, one defines a fibration $\phi_{f}=$ $\frac{f}{\|f\|}: S_{\varepsilon}^{2 n+1} \backslash V_{f, 0} \rightarrow S^{1}$, which is a map between spheres of different dimensions. The gradient map $\left.\partial f\right|_{U}=\left(\partial_{0} f, \ldots, \partial_{n} f\right):\left(U^{n+1}, \mathbf{0}\right) \rightarrow\left(\mathbb{C}^{n+1}, \mathbf{0}\right)$ furnishes a map between spheres of the same dimension, namely, $\phi_{\left.\partial f\right|_{u}}=\frac{\left.\partial f\right|_{u}}{\left\|\left.\partial f\right|_{u}\right\|}: S_{\varepsilon}^{2 n+1} \rightarrow S_{\varepsilon}^{2 n+1}$.

As mentioned in the preceding discussion, the case of an isolated point at the origin $\{\mathbf{0}\}$ of the system $\left.\partial f\right|_{U}=\mathbf{0}$, i.e., $\Sigma\left(V_{f, 0}\right)=\{\mathbf{0}\}$, is especially curious and will retain our attention for majority of this work. Assume, for now, that $f$
is an isolated singularity, and define the differential index of $f$ as the PoincaréHopf index ind ${ }_{P H} \partial f$ of the vector field $\partial f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow\left(\mathbb{C}^{n+1}, \mathbf{0}\right)$ or, equivalently, the degree of the map $\phi_{\partial f \mid u}$ at the origin*,

$$
\begin{equation*}
\mu_{\mathrm{diff}}(f):=\operatorname{ind}_{\mathrm{PH}} \partial f=\operatorname{deg}_{B}\left(\phi_{\partial f \mid u}\right) . \tag{1.15}
\end{equation*}
$$

In the forthcoming sections we motivate and define additional numerical invariants associated with singular points. We now shift our focus to related topological aspects of the fiber $F_{f, 0}$ including the associated singular homology and homotopy-type.

### 1.9. Homology/Homotopy Class of the Milnor Fiber

In this section we discuss a family of topological spaces central to the Milnor construction, a wedge sum of spheres. We begin with the simplest, non-trivial example involving quotient graphs and subsequently generalize to more complicated quotient space constructions.
1.9.1. Spanning Tree Quotients and Roses. Let $G$ be a non-trivial simple graph ${ }^{\dagger}$ with $e$ edges, $v$ vertices and containing a spanning tree, say $\Gamma_{G}$, which is guaranteed to exist by virtue of the assumption of connectedness. Recall that the Euler characteristic of $G$ is the difference between the number of its vertices and edges, $\chi(G)=v-e \leqslant 0$. By definition, $\Gamma_{G}$ has $v$ vertices and $v-1$ edges

[^20](hence spanning), so the quotient graph $G / \Gamma_{G}$ —the resulting graph from contracting $\Gamma_{G}$ to any one of its vertices -has $\mu(G)=e-(v-1)=1-\chi(G)$ edges, as loops, connected by the aforementioned (root) vertex. The quotient graph $G / \Gamma_{G}$ is then homotopy equivalent to a wedge sum of $\mu=\mu(G)$ circles, $\bigvee^{\mu} S^{1}$, or rose with petals (Figure 1.7).


Figure 1.7. Spanning Tree Quotient and Rose with Petals

Remark 1.9.1. It is a nice exercise to prove that both the spanning tree $\Gamma_{G}$ and the vertex of contraction can be chosen arbitrarily, and that the number of petals $\mu$ in the resulting quotient graph $G / \Gamma_{G}$ coincides with the first Betti number and the dimension of the cycle space of G, i.e., the number of fundamental cycles of $G$.

The homotopy groups of a rose with petals are especially simple to describe. Since $G / \Gamma_{G}$ is pathwise connected, $\pi_{0}\left(\bigvee^{\mu} S^{1}\right)$ is trivial. Since $G / \Gamma_{G}$ consists of $b_{1}$ loops held together by a basepoint, then by the van Kampen Theorem and that $\pi_{1}\left(S^{1}\right) \cong \mathbb{Z}$, it follows that $\pi_{1}\left(\bigvee^{\mu} S^{1}\right) \cong \mathbb{Z} * \cdots * \mathbb{Z} \cong F_{\mu}$, the free group on
$\mu$ generators. Thus, the quotient graph $G / \Gamma_{G} \simeq \bigvee^{\mu} S^{1}$ is an Eilenberg-Maclane space, $K\left(F_{\mu}, 1\right)$.

Remark 1.9.2. The above construction can be generalized to simplicial complexes with corresponding quotients involving (contractible) spanning subcomplexes.
1.9.2. Skeletal Quotients and Bouquets. Since any connected graph is a one-dimensional CW-complex, we generalize accordingly [177]. Let $X$ be any non-trivial $n$-dimensional CW-complex with cellular (or skeletal) decomposition $X_{0} \subset X_{1} \subset \cdots \subset X_{n}=X$, where $X_{k}$ denotes the inductively defined $k$-skeleton of $X$ by appropriately attaching a number of $k$-cells to $X_{k-1}$. If $X_{n}$ contains $\mu$ $n$-cells, then there is a homotopy equivalence to a wedge sum of $\mu n$-spheres, $X / X_{n-1} \simeq \bigvee^{\mu} S^{n}$. The number of spheres in the bouquet, $\mu$, can be identified with an alternating sum involving Betti numbers of $X$. In the next subsection, we prove that $\mu=(-1)^{n} \tilde{\chi}(X)$, where the reduced Euler characteristic $\tilde{\chi}(X)=$ $\chi(X)-1$ is determined by the reduced homology $\tilde{H}_{*}(X ; \mathbb{Z})$.


Figure 1.8. Equatorial Sphere Contraction and Mixed Wedge Sum of Spheres

A slightly more general mixed wedge sum can be constructed by choosing an appropriate equatorial CW structure, namely, $S^{n} / S^{k} \simeq S^{n} \vee S^{k+1}$ for $0 \leqslant$ $k<n$ (op. cit.; Figure 1.8). One may then inductively define the wedge sum of $n$-spheres by $S^{n} / \bigvee^{\mu-1} S^{n-1} \simeq \bigvee^{\mu} S^{n}$, which we call the slice construction. For instance, $\bigvee^{3} S^{2} \simeq S^{2} / S^{1} \vee S^{1} \simeq S^{2} /\left(S^{1} / S^{0}\right)$, as shown in Figure 1.9.

In contrast to $K\left(\mathrm{~F}_{\mu}, 1\right)$, the bouquet $\bigvee^{\mu} S^{n}$ is not - without some prior surgery - an Eilenberg-Maclane space $K\left(\mathrm{~F}_{\mu}, n\right)$. However, by simply attaching a (countable) number of $m$-cells with $m>n$, we can inductively construct a $K\left(F_{\mu}, n\right)$ space from $\bigvee^{\mu} S^{n}$ by ensuring the vanishing of higher homotopy groups. Thus, $\pi_{i}\left(\bigvee^{\mu} S^{n}\right)$ is trivial for $0 \leqslant i<n$ and $\pi_{n}\left(\bigvee^{\mu} S^{n}\right) \cong \mathrm{F}_{\mu}$ [184]. The Milnor fiber $F_{f, 0} \simeq \bigvee^{\mu} S^{n}$ is an Eilenberg-Maclane space $K\left(F_{\mu}, n\right)$. Moreover, there is an isomorphism involving the relative homotopy class of based maps $\left[X, K\left(F_{\mu}, n\right)\right] \cong H^{n}\left(X ; F_{\mu}\right)$, the $n^{\text {th }}$-singular cohomology group of $X$ with coefficients in $\mathrm{F}_{\mu}$. In general, if a connected CW-complex $X$ is a $K\left(\pi_{n}(X), n\right)$ space, then the loop space $\Omega X$ is a $K\left(\pi_{n}(X), n-1\right)$ space [184].

This concludes our discussion of a few standard constructions of the main family of topological spaces at the heart of the Milnor construction, a wedge sum of spheres, including a discussion of its known homotopy structure. We turn our attention now to the associated monodromy and related topological invariants. In particular, the homology of the Milnor fiber is vastly intriguing and will be our primary focus for the remainder of the section.


Figure 1.9. Slice Construction of a Wedge Sum of Spheres
1.9.3. Euler Characteristic of the Milnor Fiber. Recall that the Euler characteristic $\chi(X)$ of a topological space $X$ with a CW-structure $\left\{X_{i}\right\}$ is a homotopy invariant and depends only on its singular homology by the following identity

$$
\begin{align*}
\chi(X) & =\sum_{i=0}^{n}(-1)^{i}\left|X_{i}\right|  \tag{1.16a}\\
& =\sum_{i=0}^{n}(-1)^{i} b_{i}(X), \tag{1.16b}
\end{align*}
$$

where $b_{i}(X)=\operatorname{rank} H_{i}(X ; \mathbb{Z})$ is the $i^{\text {th }}$-Betti number of $Y$ [177]. The homology of the bouquet $\bigvee^{\mu} S^{n}$ is all but vanishing save $H_{0}\left(\bigvee^{\mu} S^{n} ; \mathbb{Z}\right) \cong \mathbb{Z}$, since it is path-connected, and

$$
\begin{equation*}
\tilde{H}_{n}\left(\bigvee^{\mu} S^{n} ; \mathbb{Z}\right) \cong \bigoplus^{\mu} \tilde{H}_{n}\left(S^{n} ; \mathbb{Z}\right) \cong \mathbb{Z}^{\mu} \tag{1.17}
\end{equation*}
$$

by the Hurewicz Theorem for wedge sum of spheres [97], which states the Hurewicz homomorphism $\psi_{i}: \pi_{i}\left(\bigvee^{\mu} S^{n}\right) \rightarrow \tilde{H}_{i}\left(\bigvee^{\mu} S^{n} ; \mathbb{Z}\right) \cong \mathbb{Z}^{m}$ is an isomorphism for $i>0$ and each $\mu>0$ [184, 197]. Thus, by considering ranks we have derived the fundamental relation $\chi\left(F_{f, 0}\right)=1+(-1)^{n} \mu$, where $\mu$ is nonnegative and counts the number of spheres in the bouquet. Equivalently, since $F_{f, 0} \simeq \bigvee^{\mu} S^{n}$, then the identity $\chi(X \vee Y)=\chi(X)+\chi(Y)-1$ for CW complexes $X$ and $Y$ implies

$$
\begin{equation*}
\chi\left(F_{f, 0}\right)=\mu \chi\left(S^{n}\right)-(\mu-1)=(-1)^{n} \mu+1 \tag{1.18}
\end{equation*}
$$

where $\chi\left(S^{n}\right)=1+(-1)^{n}$.
1.9.4. Topological Index. Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with isolated critical point at the origin, the topological index of $f$ is the number of spheres in the homotopy type of the corresponding Milnor fiber,

$$
\begin{equation*}
\mu_{\mathrm{top}}(f):=\operatorname{rank} \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right) \tag{1.19a}
\end{equation*}
$$

which is, of course, equal to $(-1)^{n} \tilde{\chi}\left(F_{f, 0}\right)=(-1)^{n+1}\left(1-\chi\left(F_{f, 0}\right)\right)$.
1.9.4.1. Poincaré Series of the Milnor Fiber. Yet a third (equivalent) method for computing the Euler characteristic $\chi\left(F_{f, 0}\right)$ of the Milnor fiber $F_{f, 0}$ involves computing the Poincaré series of $F_{f, 0}$. Recall that the generating function of the sequence of Betti numbers $\left\{b_{0}, b_{1}, \ldots\right\}$ of a given topological space $X$ (with finitely generated $\mathbb{Z}$-homology of finite type) is the Poincaré series, $P_{X}(t)=\sum_{i \geqslant 0} b_{i} t^{i}$. Since $P_{X \vee Y}(t)=P_{X}(t)+P_{Y}(t)-1$ for $C W$ complexes $X$ and $Y$, and $P_{S^{n}}(t)=$
$t^{n}+1$, it follows by iteration that $P_{\bigvee^{\mu} S^{n}}(t)=\mu P_{S^{n}}(t)-(\mu-1)=\mu t^{n}+1$. Therefore, the value $P_{F_{f, 0}}(-1)$ is precisely the desired Euler characteristic.

### 1.10. Invariance under Topological Morphisms

Given a diffeomorphism $F_{f, 0} \cong{ }_{d} F_{g, 0}$ or homotopy equivalence $F_{f, 0} \simeq F_{g, 0}$, there is an induced isomorphism between corresponding reduced-homology groups, hence $\mu_{\text {top }}(f)=\mu_{\text {top }}(g)$. The converse of these implications, however, is patently false. To see this for the latter morphism, choose $n, m>1, n \neq m$ with $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ and $g:\left(\mathbb{C}^{m+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ as analytic germs with isolated critical points at the respective origins, and assume $\mu_{\text {top }}(f)=\mu_{\text {top }}(g)=\mu$. By $\mathbf{M} \mathbf{2}^{\prime}, F_{f, 0} \simeq \bigvee^{\mu} S^{n}$ and $F_{g, 0} \simeq \bigvee^{\mu} S^{m}$. Since the reduced homology differs in dimensions $n$ and $m$, the fibers are neither homeomorphic nor weakly homotopy equivalent, and therefore cannot be diffeomorphic.

However, there is a context in which a stronger equivalence is preserved under continuous deformation.

Proposition 1.18 (Oka, $[355,357])$. Let $f_{t}:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a smooth 1-parameter family of analytic functions with an isolated critical point at the origin for $0 \leqslant t \leqslant 1$. If there is an $\varepsilon^{\prime}>0$ such that for each $\varepsilon>0$ satisfying $\varepsilon<\varepsilon^{\prime}$ the intersection $V_{f_{t}}=f_{t}^{-1}(0) \cap S_{\varepsilon}^{2 n+1}$ is transversal for $0 \leqslant t \leqslant 1$, then $F_{f_{t}, 0}$ is a parametrized family of fibers within the same isomorphism class.

It is often the case that one would like to compare fibers and/or their complex analytic germs through fiber morphisms or certain algebraic deformations.

To begin to understand how such objects may be approached, we first consider a weaker equivalence, like that of homotopy, and similar structure-preserving operations and fiber transformations.

It is clear that the homotopy-type of the hypersurface at the origin is a local topological invariant. Thus, any parametrized deformation of the hypersurface preserving the local topological-type also preserves the topological index. We close this section with a result of Lê and Ramanujam on the converse of this invariance.

Proposition 1.19 (Lê, Ramanujam, [259]). Let $\Delta$ denote any open disk about the origin, $0 \in \mathbb{C}$. Let $f_{t}:\left(\Delta \times \mathbb{C}^{n+1}, \Delta \times \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a 1-parameter family of analytic functions such that $\operatorname{dim} \Sigma\left(V_{f_{t}, 0}\right)=0$ at the origin. For sufficiently small $t$, the following statements are true:

1. The fiber-homotopy type of the Milnor fibrations of $f_{t}$ at the origin is constant; and,
2. If $n \neq 2$, the diffeomorphism-type of the Milnor fibrations of $f_{t}$ is constant, as well as the local topological-type of the hypersurfaces $V_{f_{t}}$.

Proof. See [290] for a discussion of this result with relevant citations and a generalization to non-isolated singularities.

In the next section we consider special cases of the Milnor fiber up to homotopy-type (in the class of wedge sums of spheres) and some relations satisfied by their corresponding numerical invariants.

### 1.11. Topological Morphisms on the Milnor Fiber

1.11.1. Inclusion-Exclusion Property. In direct analogy with the disjoint union $\sqcup$ and direct sum $\oplus$ in the categories of sets Set and abelian groups $\mathbf{A b}$, respectively, the wedge sum $v$ is the coproduct in the category of pointed topological spaces Top. Since any singleton space or point $\{\bullet\}$ is a zero object in Top., a contractible Milnor fiber $F_{f, 0}$ with the homotopy-type of a point $\bigvee^{0} S^{n} \simeq\{\bullet\}$ and with Euler characteristic equal to 1, corresponds to the value $\mu_{\text {top }}(f)=0$, consistent with equation (1.19a).

If the context is clear, we shall on occasion write $\mu_{\text {top }}\left(F_{f, 0}\right)$ instead of $\mu_{\text {top }}(f)$ so as to illustrate a dependence on the fiber $F_{f, 0}$ of the complex analytic germ $f$.

Suppose $X$ is a finite, compact CW-complex. For any closed subcomplex $Y \subset X$, the Euler characteristic satisfies the Excision Property ${ }^{*}, \chi(X)=\chi(Y)+$ $\chi(X \backslash Y)$, which if $X=Y_{1} \cup Y_{2}$ (subcomplex union) and the short sequence

$$
\{0\} \longrightarrow \tilde{H}_{i}(X ; \mathbb{Z}) \longrightarrow \tilde{H}_{i}\left(Y_{1} ; \mathbb{Z}\right) \oplus \tilde{H}_{i}\left(Y_{2} ; \mathbb{Z}\right) \longrightarrow \tilde{H}_{i}\left(Y_{1} \cap Y_{2} ; \mathbb{Z}\right) \longrightarrow\{0\}
$$

is exact for $i \geqslant 0$, implies the Inclusion-Exclusion Property: $\chi(X)=\chi\left(Y_{1}\right)+$ $\chi\left(Y_{2}\right)-\chi\left(Y_{1} \cap Y_{2}\right)$. Together with equation (1.19a), the topological index satisfies the inequality,

$$
\begin{align*}
\mu_{\mathrm{top}}\left(F_{f, 0} \cup F_{g, 0}\right) & =\mu_{\mathrm{top}}\left(F_{f, 0}\right)+\mu_{\mathrm{top}}\left(F_{g, 0}\right)-\mu_{\mathrm{top}}\left(F_{f, 0} \cap F_{g, 0}\right)  \tag{1.20a}\\
& \leqslant \mu_{\mathrm{top}}\left(F_{f, 0}\right)+\mu_{\mathrm{top}}\left(F_{g, 0}\right), \tag{1.20b}
\end{align*}
$$

[^21]provided that the union and intersection of fibers can be realized through Milnor's construction. Since bouquets are closed under wedge sums, it follows that $\bigvee^{\mu} S^{n} \vee \bigvee^{\mu^{\prime}} S^{n} \simeq \bigvee^{\mu^{\prime \prime}} S^{n}$ if and only if $\mu+\mu^{\prime}=\mu^{\prime \prime}$. Thus, one way to achieve a strictly additive relation in equation (1.2ob) is for $F_{f, 0} \cup F_{g, 0} \simeq \bigvee^{\mu+\mu^{\prime}} S^{n}$ and $F_{f, 0} \cap F_{g, 0} \simeq\{\bullet\}$ (since the topological index of a point is zero) with $f, g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ both non-degenerate. That is, if a composite fiber has the homotopy-type of the wedge sum of $F_{f, 0} \vee F_{g, 0}$, then
\[

$$
\begin{equation*}
\mu_{\mathrm{top}}\left(F_{f, 0} \vee F_{g, 0}\right)=\mu_{\mathrm{top}}\left(F_{f, 0}\right)+\mu_{\mathrm{top}}\left(F_{g, 0}\right) \tag{1.21}
\end{equation*}
$$

\]

Equation (1.21) also follows from the additivity over wedge sums of CWcomplexes satisfied by the reduced Euler characteristic, viz.,

$$
\begin{equation*}
\tilde{\chi}(X \vee Z)=\tilde{\chi}(X)+\tilde{\chi}(Z) \tag{1.22}
\end{equation*}
$$

where $X$ and $Z$ are arbitrary CW-complexes. From the perspective of reduced homology, we have the following decomposition by exactness:

$$
\begin{equation*}
\tilde{H}_{i}\left(\bigvee^{\mu+\mu^{\prime}} S^{n} ; \mathbb{Z}\right) \cong \tilde{H}_{i}\left(\bigvee^{\mu} S^{n} ; \mathbb{Z}\right) \oplus \tilde{H}_{i}\left(\bigvee^{\mu^{\prime}} S^{n} ; \mathbb{Z}\right) \quad i \geqslant 0 \tag{1.23}
\end{equation*}
$$

By considering the corresponding ranks, one concludes equation (1.21).

The construction of a wedge sum of fibers on the level of their corresponding complex analytic germs is not immediately obvious* for $n>1$, so we shall endeavor to seek a more suitable structure.
1.11.2. The Join of Pham. Let $C_{n}$ denote the $n^{\text {th }}$-roots of unity (or cyclic group of order $n$ ) viewed as the disjoint union of $n$ trivial pointed spaces with a distinguished point as a base-point, or as an edgeless graph $\bar{K}_{n}$ - the graph complement ${ }^{\dagger}$ of the complete graph $K_{n}$ — with a vertex as a distinguished basepoint. If $f=\sum_{i=0}^{n} z_{i}^{a_{i}}$, then Pham [374] demonstrated the homotopy equivalence $V_{f, 1} \simeq \mathrm{C}_{a_{0}} \star \cdots \star \mathrm{C}_{a_{n}}$, where $\star$ denotes the join operation (Figure 1.10). Such a polynomial is referred to as Brieskorn-Pham, and the corresponding manifold $V_{f, 1}$ is known as the Join of Pham.

Recall that the topological join of two CW complexes is defined as

$$
\begin{equation*}
X \star Y \cong C X \times Y \sqcup_{X \times Y} C Y \times X, \tag{1.24}
\end{equation*}
$$

where $C X=(X \times I) /(X \times\{0\})$ denotes the cone over $X$, and $I$ denotes the unit interval (Figure 1.10). That $X$ is a deformation retraction of $C X$ implies

[^22]constancy of both homology and homotopy when taking the cone. In fact, the cone map $C$ is a endofunctor of the category Top.


Figure 1.10. Join Space $X \star Y$ of Pointed CW-complexes $(X, x)$ and $(Y, y)$

Since $\mathrm{C}_{a_{i}} \simeq \bigvee^{a_{i}-1} S^{0}$ and $S^{n} \star S^{m} \simeq S^{n+m+1}$ for $n, m \geqslant 0$, it follows that $F_{f, 0} \cong V_{f, 1}$ has the following homotopy-type of a wedge sum of spheres,

$$
\begin{align*}
\mathrm{C}_{a_{0}} \star \cdots \star \mathrm{C}_{a_{n}} & \simeq \bigvee^{a_{0}-1} S^{0} \star \cdots \star \bigvee^{a_{n}-1} S^{0}  \tag{1.25a}\\
& \simeq \bigvee^{a_{0}-1} \cdots \bigvee^{a_{n}-1} S^{0} \star \cdots \star S^{0}  \tag{1.25b}\\
& \simeq \bigvee^{\mu} S^{n}, \tag{1.25c}
\end{align*}
$$

where $\mu=\mu_{\text {top }}(f)=\prod_{i=0}^{n}\left(a_{i}-1\right)$.
In particular, the fiber of $f=z^{2}$ has the homotopy-type consisting of two isolated points, since $F_{z^{2}, 0} \simeq C_{2} \simeq S^{0}$, the 0-sphere. Like the trivial fiber $F_{z, 0}$ discussed the previous subsection, the fiber of the square $F_{z^{2}, 0}$ has special significance in the algebraic structure of the homotopy class of fibers.

The homotopy type of $V_{f, 1}$ can be seen directly by considering the complement of a tubular neighborhood of the graph $\mathrm{C}_{a_{0}} \star \cdots \star \mathrm{C}_{a_{n}}$ within a ball. Consider the example, $f=x^{2}+y^{2}$ and $g=x^{3}+y^{2}$ are shown in Figure 1.11. Here,
$F_{f, 0} \simeq S^{1}$ and $F_{g, 0} \simeq S^{1} \vee S^{1}$. Alternatively, the homotopy type can be inferred from the spanning-tree contraction of §1.9.1.


Figure 1.11. Two Joins of Pham: $f=x^{2}+y^{2}$ and $g=x^{3}+y^{2}$

The importance of singularities of the Brieskorn-Pham type cannot be overstated, and the follow classical result of Brieskorn attests to this fact.

Proposition 1.20 (Brieskorn, [63]). For $n \geqslant 4$, any homotopy $(2 n-1)$-sphere that bounds a parallelizable manifold is diffeomorphic to an algebraic link $K_{f}$ of the polynomial $f=\sum_{i=0}^{n} z_{i}^{a_{i}}$ for some set of positive integers $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}_{>1}$.

Proof. See Korollar 2 (Corollary 2) in [63] and Chapter 1 in [420].
Milnor generalized Brieskorn and Pham's work to the context of nondegenerate complex analytic germ and proved that the corresponding fibers have the homotopy-type of a wedge sum of $n$-spheres, the number of which is effectively computable. The fiber-join decomposition proposed by Pham was further generalized by Sebastiani and Thom [421] and recently by Massey [289]. In summary, we have the following classical result.

Proposition 1.21 (Milnor,[310]). Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with an isolated critical point at the origin, the fiber $F_{f, 0}=$ $V_{f, \kappa} \cap B_{\varepsilon}^{2 n}$ is a parallelizable manifold with homotopy type of a wedge sum of spheres $\bigvee^{\mu} S^{n}$, where $\mu$ is a non-negative integer and zero only in the case that the origin is a regular point of $f$.

Proof. See Theorem 7.2 in [420].
In the next section we broaden our view of the structure of the space of Milnor fibers up to homotopy and begin our transition toward the algebraic aspects of the Milnor construction.
1.11.3. Sebastiani-Thom Equivalence. Now that some basic features of the topology and construction of wedge sum of spheres have been established, we discuss a fundamental equivalence which allows the construction of a product Milnor fiber from constituent fibers and gives a simple means of identifying the monodromy, homology and homotopy-type of the resulting object.

Let $f_{\alpha}: U_{\alpha} \rightarrow \mathbb{C}$ be a complex analytic germ with domain $U_{\alpha} \subset \mathbb{C}^{n_{\alpha}}$. Define the projection $\pi_{\alpha^{\prime}}: \prod_{\alpha} U_{\alpha} \rightarrow U_{\alpha^{\prime}}$ and the Sebastiani-Thom sum $\boxplus_{\alpha} f_{\alpha}=\sum_{\alpha} f_{\alpha} \circ$ $\pi_{\alpha}$ with product domain $\prod_{\alpha} U_{\alpha} \subset \mathbb{C}^{\sum_{\alpha} n_{\alpha}}$ such that $\pi_{\alpha^{\prime}}\left(\square_{\alpha} f_{\alpha}\right)=f_{\alpha^{\prime}}$. The map $f \mapsto f \boxplus g$ shall hereafter be referred to as an augmentation of $f$ by $g$.

REMARK 1.11.1. If $f=x^{a} y^{b}$ and $g=x^{c} y^{d}+x^{a^{\prime}} y^{b^{\prime}}+x^{c^{\prime}} y^{d^{\prime}}$, then

$$
\begin{align*}
& f \boxplus g=x^{a} y^{b}+z^{c} w^{d}+z^{a^{\prime}} w^{b^{\prime}}+z^{c^{\prime}} w^{d^{\prime}}  \tag{1.26a}\\
& g \boxplus f=x^{c} y^{d}+x^{a^{\prime}} y^{b^{\prime}}+x^{c^{\prime}} y^{d^{\prime}}+z^{a} w^{b} \tag{1.26b}
\end{align*}
$$

which are equivalent up to an action of $S_{4}$ on the variables.

Proposition 1.22 (Sebastiani, Thom [421]). Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a neighborhood of the origin. Assume that the complex analytic germ $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ has an isolated critical point at the origin. The monodromy of $f=\square_{\alpha=1}^{s} f_{\alpha}:\left(X_{\alpha=1}^{s} U_{\alpha}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ factors as the tensor product $h_{*}(f)=\bigotimes_{\alpha=1}^{s} h_{*}\left(f_{\alpha}\right)$, and the fiber $F_{f, 0}$ has the homotopy-type of the iterated join space $F_{f_{1}, 0} \star \cdots \star F_{f_{s}, 0}$.

Proof. See [421] and [356].
1.11.4. Cones and Suspensions. Recall that given any pointed space $X=$ $\left(X, x_{0}\right)$, the free suspension $S X:=S^{0} \star X \cong C X \sqcup_{X} C X$, where $C X$ denotes the cone (space) of $X$ (Figures 1.12 and 1.13). Define also the reduced suspension $\Sigma X$, which is $S X$ with the line connecting basepoints of $S^{0}$ and $X$ contracted to a point. Indeed, there is a homeomorphism $\Sigma(X \star Y) \cong \Sigma X \star \Sigma Y$ and homotopy equivalence $\Sigma X \simeq S X$.

The following short exact sequence of spaces $X \hookrightarrow C X \rightarrow S X$ induces a corresponding long exact sequence in reduced homology,

$$
\cdots \longrightarrow \tilde{H}_{i}(X ; \mathbb{Z}) \longrightarrow \tilde{H}_{i}(C X ; \mathbb{Z}) \longrightarrow \tilde{H}_{i}(S X ; \mathbb{Z}) \longrightarrow
$$



Figure 1.12. Cone $C X$ of a pointed CW-complex $(X, x)$
Since $C X$ is contractible, i.e., $C X \simeq\{\bullet\}$, then $\tilde{H}_{k}(C X ; \mathbb{Z}) \cong \tilde{H}_{i}(\{\bullet\} ; \mathbb{Z}) \cong\{0\}$, and the long exact sequence splits into short exact sequences of the form,

$$
\{0\} \longrightarrow \tilde{H}_{i}(S X ; \mathbb{Z}) \longrightarrow \tilde{H}_{i-1}(X ; \mathbb{Z}) \longrightarrow\{0\}
$$

which imply $\tilde{H}_{i}(S X ; \mathbb{Z}) \cong \tilde{H}_{i-1}(X ; \mathbb{Z})$ for $i \geqslant 1$.


Figure 1.13. Free Suspension $S X$ of a Pointed CW-complex $(X, x)$

To avoid cumbersome notation we shall, for the moment, write $F_{0}(f)$ in place of $F_{f, 0}$ for the Milnor fiber of $f$. Consider the Sebastiani-Thom sum $\Sigma f:=f \boxplus z^{2}$ where $f$ is a non-degenerate complex analytic germ, as above. By Proposition 1.22, $F_{0}(\Sigma f) \simeq F_{0}(f) \star F_{0}\left(z^{2}\right)$, but $F_{0}\left(z^{2}\right)$ is diffeomorphic to
$S^{0} \cong\left\{z \in \mathbb{C} \mid z^{2}=1\right\}=\{ \pm 1\}$. Thus, the fiber of the Sebastiani-Thom sum $\Sigma f$ has the homotopy-type of the free suspension of the Milnor fiber of $f$, that is, $S F_{f, 0} \cong S^{0} \star F_{0}(f) \simeq S\left(\bigvee^{\mu} S^{n}\right) \simeq \bigvee^{\mu} S^{n+1}$. Correspondingly, one observes in homology only a shift of indices and an invariance of the topological index. By considering a corresponding Mayer-Vietoris sequence or simply a slight modification of the long exact sequence above, one is led to the commutative diagram

for all $i \geqslant 0$. The suspension functor $S$ induces a group isomorphism between the middle reduced-homology groups, $S_{*}: \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right) \rightarrow \tilde{H}_{n+1}\left(S F_{f, 0} ; \mathbb{Z}\right)$, and therefore

$$
\begin{equation*}
\mu_{\mathrm{top}}(\Sigma f)=\mu_{\mathrm{top}}(f) \tag{1.27}
\end{equation*}
$$

The suspension operation, as applied to wedge sum of spheres, can be inferred and understood by the two key examples, $S S^{1} \simeq S^{2}$ and $S\left(S^{1} \vee S^{1}\right) \simeq S^{2} \vee S^{2}$, as illustrated in Figure 1.14.


Figure 1.14. Reduced Suspensions of Wedge Sums of Spheres
1.11.4.1. Suspensions of Cartesian Products. For pointed spaces $\left\{X_{1}, \ldots, X_{n}\right\}$, one has the following reduced suspension [131],

$$
\begin{equation*}
\Sigma\left(X_{i=1}^{n} X_{i}\right) \simeq \bigvee_{k=1}^{n} \bigvee_{1 \leqslant i_{1}<\cdots<i_{k} \leqslant n}\left(\Sigma X_{i_{1}}\right) \wedge X_{i_{2}} \wedge \cdots \wedge X_{i_{k}} \tag{1.28}
\end{equation*}
$$

For CW complexes $\left\{X_{1}, \ldots, X_{n}\right\}$, the reduced suspension is homotopy equivalent to the unreduced suspension, so

$$
\begin{equation*}
S\left(X_{i=1}^{n} X_{i}\right) \simeq \bigvee_{k=1}^{n} \bigvee_{1 \leqslant i_{1}<\cdots<i_{k} \leqslant n}\left(S X_{i_{1}}\right) \wedge X_{i_{2}} \wedge \cdots \wedge X_{i_{k}} \tag{1.29}
\end{equation*}
$$

The iterated suspension of a mixed product of spheres is a mixed wedge sum of spheres, namely, for $N>0$,

$$
\begin{equation*}
S^{N}\left(X_{i=1}^{n} S^{n_{i}}\right) \simeq \bigvee_{k=1}^{n} \bigvee_{1 \leqslant i_{1}<\cdots<i_{k} \leqslant n} S^{n_{i_{1}}+\cdots+n_{i_{k}}+N} \tag{1.30}
\end{equation*}
$$

since $S^{l} \wedge S^{m} \simeq S^{l+m}$ for $l, m \geqslant 0$. In particular, if the dimensions of spheres in the product are the same, as for the case of generalized tori, namely, $\mathbb{T}_{m}^{n} \simeq$ $\chi^{n} S^{m}$, then one has for $N>0$,

$$
\begin{align*}
S^{N}\left(\mathbb{T}_{m}^{n}\right) & \simeq \bigvee_{k=1}^{n} \bigvee_{1 \leqslant i_{1}<\cdots<i_{k} \leqslant n} S^{k m+N}  \tag{1.31}\\
& =\bigvee_{k=1}^{n} \bigvee^{\binom{n}{k}} S^{k m+N} . \tag{1.32}
\end{align*}
$$

1.11.5. Joins. With regard to the wedge sum of spheres, we have the following homotopy equivalence as a result of two homeomorphisms,

$$
\begin{align*}
\bigvee_{i=1}^{r} S^{n_{i}} \star \bigvee_{j=1}^{s} S^{m_{j}} & \cong \bigvee_{i=1}^{r} S^{n_{i}} \star_{r} \bigvee_{j=1}^{s} S^{m_{j}}  \tag{1.33a}\\
& \cong \bigvee_{i=1}^{r} \bigvee_{j=1}^{s} S^{n_{i} \star_{r}} S^{m_{j}}  \tag{1.33b}\\
& \simeq \bigvee_{i=1}^{r} \bigvee_{j=1}^{s} S^{n_{i}+m_{j}+1} \tag{1.33c}
\end{align*}
$$

where ${ }_{~_{r}}$ denotes reduced join. If $n_{i}=n$ and $m_{j}=m$, as in the join of two (not necessarily distinct) Milnor fibers, then

$$
\begin{equation*}
\bigvee^{r} S^{n} \star \bigvee^{s} S^{m} \simeq \bigvee^{r s} S^{n+m+1} \tag{1.34}
\end{equation*}
$$

and the product identity of topological indices follows as a consequence.*
By Proposition 1.22, consider the fiber of a Sebastiani-Thom summation singularity $F_{f \boxplus g, 0} \simeq F_{f, 0} \star F_{g, 0}$. As a direct consequence of the Künneth formula and the fact that $\tilde{H}_{*}\left(F_{f, 0} ; \mathbb{Z}\right)$ is without torsion, the reduced-homology groups

[^23]factor over fiber joins [131],
\[

$$
\begin{align*}
\tilde{H}_{k+1}\left(F_{f \boxplus g, 0} ; \mathbb{Z}\right) & \cong \tilde{H}_{k+1}\left(F_{f, 0} \star F_{g, 0} ; \mathbb{Z}\right) \quad k \geqslant 0  \tag{1.35}\\
& \cong \bigoplus_{i+j=k} \tilde{H}_{i}\left(F_{f, 0} ; \mathbb{Z}\right) \otimes \tilde{H}_{j}\left(F_{g, 0} ; \mathbb{Z}\right) . \tag{1.36}
\end{align*}
$$
\]

Since the homology is concentrated in the middle dimensions, $n$ and $m$, respectively, the only non-trivial group is

$$
\begin{equation*}
\tilde{H}_{n+m+1}\left(F_{f \boxplus g, 0} ; \mathbb{Z}\right) \cong \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right) \otimes \tilde{H}_{m}\left(F_{g, 0} ; \mathbb{Z}\right) \tag{1.37}
\end{equation*}
$$

which is free abelian of rank $\mu_{\text {top }}(f) \mu_{\text {top }}(g)$. Thus, the topological indices satisfy the multiplicative identity,

$$
\begin{equation*}
\mu_{\text {top }}(f \boxplus g)=\mu_{\text {top }}(f) \mu_{\text {top }}(g) \tag{1.38}
\end{equation*}
$$

1.11.6. Iterated Suspensions and Stabilization. Define the $N$-stabilization of $f$ by the recurrence $\Sigma^{N} f=\left(\Sigma^{N-1} f\right) \boxplus z^{2}$, where $\Sigma^{0} f=f$ and $\Sigma^{1} f=f \boxplus z^{2}$ denoted $\Sigma f$. Since the suspension functor $S$ preserves homotopy groups (by modifying only indices), then the corresponding fibers satisfy $F_{\Sigma^{N} f, 0} \simeq S^{N} F_{f, 0}$ and $\mu_{\text {top }}\left(\Sigma^{N} f\right)=\mu_{\text {top }}(f)$ by a sequential application of the Freudenthal Suspension and Hurewicz Theorems. This particular invariance of the topological index under $N$-stabilization is consistent with equation (1.38) (choosing $a_{i}=2$ for $1 \leqslant i \leqslant N$ ) and is a special case of the Sebastiani-Thom equivalence.
1.11.7. Iterated Cone and Suppression. Define the $N$-suppression of $f$ as the recurrence $C^{N} f=\left(C^{N-1} f\right) \boxplus z$, where $C^{0} f=f$ and $C^{1} f=f \boxplus z$ denoted by $C f$. There is a homeomorphism $C S^{n} \simeq_{h} B^{n+1}$, which is contractible and, hence, one has the homotopy equivalence $C S^{n} \simeq\{\bullet\}$. Since the cone factors through the wedge sum, we have $F_{C f, 0} \simeq C F_{f, 0} \simeq\{\bullet\}$, including $F_{z, 0} \simeq\{\bullet\}$. It follows that

$$
\begin{equation*}
\mu_{\mathrm{top}}\left(C^{N} f\right)=\delta_{N, 0} \mu_{\mathrm{top}}(f), \tag{1.39}
\end{equation*}
$$

where the Kronecker delta function $\delta_{N, 0}$ is 1 if $N=0$ and 0 otherwise.
1.11.8. Iterated Cyclic and Free Suspensions. A special case of Proposition 1.22 implies the fiber $F_{f \boxplus z^{k}, 0}$ has the homotopy-type of a wedge sum of $k-1$ identical copies of the free suspension of the Milnor fiber $F_{f, 0}$, that is, $\bigvee^{k-1} S F_{f, 0}$. We call this operation the $k$-iterated cyclic suspension or simply $k$-cyclic suspension of $F_{f, 0}$ (and/or, when the context is clear, of $f$ ). Denote the $k$-iterated free suspension of $F$ by $S^{k} F_{f, 0}=S^{k-1}\left(S F_{f, 0}\right)$ for $k \geqslant 0$. By considering the corresponding homology groups directly or using equation (1.38), we calculate

$$
\begin{align*}
\mu_{\mathrm{top}}\left(F_{f \boxplus z^{k}, 0}\right) & =\sum_{i=1}^{k-1} \mu_{\mathrm{top}}\left(S F_{f, 0}\right)  \tag{1.40a}\\
& =(k-1) \mu_{\mathrm{top}}\left(F_{f, 0}\right) \tag{1.4ob}
\end{align*}
$$

since the suspension functor preserves the topological index,

$$
\begin{equation*}
\mu_{\mathrm{top}}\left(S F_{f, 0}\right)=\mu_{\mathrm{top}}\left(F_{f, 0}\right) \tag{1.41}
\end{equation*}
$$

The case $k=1$ is well-defined and consistent with $F_{f \boxplus z, 0} \simeq C F_{f, 0} \simeq\{\bullet\}$.
By the Sebastiani-Thom equivalence and an iteration of equation (1.40a), the topological index of the Brieskorn-Pham polynomial $f=\sum_{i=0}^{n} z_{i}^{a_{i}}$ with $a_{i}>0$ is the product $\prod_{i=0}^{n}\left(a_{i}-1\right)$, consistent with Pham's construction. For any $f$ and $g=\sum_{i=1}^{N} z_{i}^{k_{i}}$ with $k_{i}>0$, one has $F_{f \boxplus g, 0} \simeq \bigvee^{k_{1}-1} \cdots \bigvee^{k_{N}-1} S^{N} F_{f, 0}$ and

$$
\begin{align*}
\mu_{\text {top }}\left(F_{f \boxplus g, 0}\right) & =\sum_{i=1}^{k_{1}-1} \cdots \sum_{i=1}^{k_{N}-1} \mu_{\mathrm{top}}\left(S^{N} F_{f, 0}\right)  \tag{1.42a}\\
& =\left(\prod_{i=1}^{N}\left(k_{i}-1\right)\right) \mu_{\mathrm{top}}\left(F_{f, 0}\right), \tag{1.42b}
\end{align*}
$$

consistent with the Sebastiani-Thom equivalence and $\mu_{\mathrm{top}}(g)=\prod_{i=1}^{N}\left(k_{i}-1\right)$. If, however, $f=0$ then, $S^{N}\{\bullet\} \simeq S^{N}$, so one concludes that $F_{g, 0}$ is homotopy equivalent to a $\mu(g)$ wedge sum of $N$-spheres, consistent with $\mathbf{M 2}{ }^{\prime \prime}$.

### 1.12. Complex Topological K-Theory of the Milnor Fiber

In this section we compute the Grothendieck groups of the Milnor fiber of an isolated singularity and discuss its relation to the corresponding homology introduced earlier in the chapter. In particular, we elucidate the role of Bott periodicity in certain transformations of the corresponding monodromy.

Based on Grothendieck's ground-breaking work on studying algebraic varieties from a topological and categorical point-of-view, Hirzebruch and Atiyah
developed topological K-theory of vector bundles and their (vector bundle) automorphisms [27]. Swan's Theorem, as generalized by Serre, then proves a categorical equivalence between vector bundles over a topological space $X$ and the finitely-generated projective modules over the ring $C(X)$ of continuous $\mathbb{R}$ valued functions on $X$. Projections in matrix algebras are then related to free modules through their direct summands, projective modules. These are the preferred objects of study in algebraic K-theory. The interested reader should consult [27] and [431] for complementary classical and modern approaches to the major results of algebraic K-theory. For the reader particularly interested in the relationship between $K$-theory and (Fredholm) index theory, consult [192].
1.12.1. Higher Homotopy Groups of Spheres. Hitherto we have considered only $\pi_{i}\left(S^{n}\right)$, or more precisely, $\pi_{i}\left(\bigvee^{\mu} S^{n}\right)$ for $0 \leqslant i \leqslant n$. What of the case $i>n$ ? In contrast to the homology groups $H_{i}\left(S^{n} ; \mathbb{Z}\right)$ which are trivial for $i \neq n$, the higher homotopy groups $\pi_{i}\left(S^{n}\right)$, although known to be abelian and finitelygenerated, are not completely understood for $i>n$.

Hopf met this challenge with partial success by proving that $\pi_{2 n-1}\left(S^{n}\right)$ is non-trivial for $n \geqslant 1$ and constructing the fibration $S^{1} \hookrightarrow S^{3} \rightarrow S^{2}$ to prove $\pi_{3}\left(S^{2}\right) \cong \pi_{3}\left(S^{3}\right) \oplus \pi_{2}\left(S^{1}\right) \cong \mathbb{Z}[208,209]$. In a landmark work, Serre proved that the homotopy group $\pi_{i}\left(S^{n}\right)$ is finite for $i>n$ save $\pi_{4 k-1}\left(S^{2 k}\right) \cong \mathbb{Z} \oplus G$, where $G$ is a finite group. Consider, for example, the first 12 non-trivial homotopy groups of $S^{4}$; these are $\pi_{4}\left(S^{4}\right) \cong \mathbb{Z}, \pi_{7}\left(S^{4}\right) \cong \mathbb{Z} \oplus \mathbb{Z}_{12}, \pi_{10}\left(S^{4}\right) \cong \mathbb{Z}_{24} \oplus \mathbb{Z}_{3}$, $\pi_{11}\left(S^{4}\right) \cong \mathbb{Z}_{15}, \pi_{14}\left(S^{4}\right) \cong \mathbb{Z}_{120} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_{2}$ and $\pi_{15}\left(S^{4}\right) \cong \mathbb{Z}_{84} \oplus \mathbb{Z}_{2}^{5}$,
while $\pi_{i}\left(S^{4}\right) \cong \mathbb{Z}_{2}^{r}$, where $r=1$ for $i \in\{5,6,12\}, r=2$ for $i \in\{8,9\}$ and $r=3$ for $i=13$. In general, the fibration $S^{3} \hookrightarrow S^{7} \rightarrow S^{4}$ implies $\pi_{i}\left(S^{4}\right) \cong \pi_{i}\left(S^{7}\right) \oplus \pi_{i-1}\left(S^{3}\right)$ for $i>1$. This example illustrates the potentially complicated nature of higher homotopy groups of spheres. Forming the basis of an exciting and active research program, computing $\pi_{i}\left(S^{n}\right)$ for $i>n$ is difficult. Aside from a few isolated cases, the relevance of higher homotopy groups of spheres in physics remains virtually unstudied.

Although one might contend that such abstract constructions are limited only to pure mathematics, (complex topological) Grothendieck groups of spheres are indeed relevant to modern theoretical physics-two applications immediately come to mind. For instance, since $\tilde{K}^{0}\left(S^{n}\right) \cong \pi_{n-1}(\mathrm{U}(k))$ for $n \geqslant 1$ and $k>\left\lfloor\frac{n+1}{2}\right\rfloor$, where $\mathrm{U}(k)$ denotes the Lie group of $k \times k$ unitary (complex) matrices, then $\tilde{K}^{0}\left(S^{2}\right) \cong \pi_{1}(\mathrm{U}(k)) \cong \mathbb{Z}$, which corresponds to the charge of a Dirac monopole, and $\tilde{K}^{0}\left(S^{4}\right) \cong \pi_{3}(\mathrm{U}(k)) \cong \mathbb{Z}$, which corresponds to the instanton number.
1.12.2. $K$-Theoretic Index. In this section, we discuss the relevance of the groups $\tilde{K}^{p}\left(\bigvee_{i=1}^{m} S^{n_{i}}\right)$ and $\tilde{K}^{p}\left(S^{N} \times \times_{i=1}^{m} S^{n_{i}}\right)$ in quantum field theory. In particular, we calculate the complex topological $K$-theory ${ }^{*}$ of the Milnor fiber $F_{f, 0}$. Since $F_{f, 0}$ has the homotopy of a wedge sum of spheres, it suffices to compute

[^24]$\tilde{K}^{p}\left(\bigvee^{\mu} S^{n}\right)$ for $\mu, n \in \mathbb{Z}_{\geqslant 0}$ and $p \in \mathbb{Z}$. This construction then allows for the elucidation of how Bott periodicity and suspension/stabilization are related through the Sebastiani-Thom isomorphism.

Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with isolated critical point at the origin, the K-theoretic index is the rank of the $n^{\text {th }}$-Grothendieck group of the corresponding Milnor fiber,

$$
\begin{equation*}
\mu_{K}(f):=\operatorname{rank} \tilde{K}^{n}\left(F_{f, 0}\right) \tag{1.43}
\end{equation*}
$$

1.12.3. Grothendieck Groups. We follow Sections 2.5-2.6 of [369] closely and make the appropriate generalizations necessary for the main result of this section. Recall the following basic facts.

Definition 1.23. A locally compact Hausdorff space is a Hausdorff (topological) space with a compact neighborhood about each point.

Remark 1.12.1. A proper map between locally compact Hausdorff spaces is continuous at infinity.

Proposition 1.24 (Park, [369]). Let $p \in\{-1,0\}$. Let $Y$ be a closed subspace of a locally compact Hausdorff space $X$. Let $\tau$ be the inclusion map $Y \hookrightarrow X$. Suppose there is a map $\pi: X \rightarrow Y$, continuous at infinity, such that $\left.\pi \circ \imath\right|_{Y}$ acts as the identity on $Y$. Then there is a split exact sequence

$$
\begin{equation*}
\{0\} \longrightarrow K^{p}(X \backslash Y) \longrightarrow K^{p}(X) \frac{i^{*}}{\pi^{*}} K^{p}(Y) \longrightarrow\{0\} \tag{1.44}
\end{equation*}
$$

and, therefore, an isomorphism $K^{p}(X) \cong K^{p}(Y) \oplus K^{p}(X \backslash Y)$.

Proof. See Theorem 2.6.11 in [369].
The following classical Grothendieck groups are indispensable for our calculation of $K^{n}\left(F_{f, 0}\right)$,

$$
K^{p}\left(\mathbb{R}^{n}\right) \cong\left\{\begin{array}{cl}
\mathbb{Z} & p+n \text { even }  \tag{1.45}\\
\{0\} & p+n \text { odd }
\end{array}\right.
$$

for $p \in \mathbb{Z}$ and $n \in \mathbb{N}$ and, in particular,

$$
K^{0}\left(S^{n}\right) \cong\left\{\begin{array} { c l } 
{ \mathbb { Z } ^ { 2 } } & { n \text { even } }  \tag{1.46}\\
{ \mathbb { Z } } & { n \text { odd } }
\end{array} \quad \text { and } \quad K ^ { - 1 } ( S ^ { n } ) \cong \left\{\begin{array}{cc}
\{0\} & n \text { even } \\
\mathbb{Z} & n \text { odd } .
\end{array}\right.\right.
$$

More generally, for any locally compact Hausdorff spaces $X$ and $Y$,

$$
\begin{align*}
K^{0}(X \times Y) & \cong\left(K^{0}(X) \otimes K^{0}(Y)\right) \oplus\left(K^{-1}(X) \otimes K^{-1}(Y)\right)  \tag{1.47}\\
K^{-1}(X \times Y) & \cong\left(K^{0}(X) \otimes K^{-1}(Y)\right) \oplus\left(K^{0}(X) \otimes K^{-1}(Y)\right)  \tag{1.48}\\
\tilde{K}^{0}(X \star Y) & \cong\left(\tilde{K}^{0}(X) \otimes \tilde{K}^{-1}(Y)\right) \oplus\left(\tilde{K}^{0}(X) \otimes \tilde{K}^{-1}(Y)\right)  \tag{1.49}\\
\tilde{K}^{-1}(X \star Y) & \cong\left(\tilde{K}^{0}(X) \otimes \tilde{K}^{0}(Y)\right) \oplus\left(\tilde{K}^{-1}(X) \otimes \tilde{K}^{-1}(Y)\right) . \tag{1.50}
\end{align*}
$$

In particular, one has

$$
\begin{equation*}
K^{0}\left(X \times S^{1}\right) \cong K^{0}(X) \oplus K^{-1}(X) \cong K^{-1}\left(X \times S^{1}\right) \tag{1.51}
\end{equation*}
$$

By induction on $n$, it follows that the Grothendieck groups of the $n$-torus $\mathbb{T}^{n} \cong$ $\mathbb{T}^{n-1} \times S^{1}$ for $n \in \mathbb{N}$, where $\mathbb{T}^{0} \cong\{\bullet\}$, are given by

$$
K^{0}\left(\mathbb{T}^{n}\right) \cong\left\{\begin{array} { c c } 
{ \mathbb { Z } ^ { 2 ^ { n - 1 } } } & { n \geqslant 1 }  \tag{1.52}\\
{ \mathbb { Z } } & { n = 0 }
\end{array} \quad \text { and } \quad K ^ { - 1 } ( \mathbb { T } ^ { n } ) \cong \left\{\begin{array}{cc}
\mathbb{Z}^{2^{n-1}} & n \geqslant 1 \\
\{0\} & n=0
\end{array}\right.\right.
$$

If $(X,\{\bullet\})$ is a pointed space, then define the reduced group $\tilde{K}^{p}(X):=$ $\tilde{K}^{p}(X,\{\bullet\})$ as the kernel of the map $\tilde{K}^{p}(X) \rightarrow \tilde{K}^{p}(\{\bullet\})$. It follows that $K^{p}(X) \cong$ $\tilde{K}^{p}(X) \oplus K^{p}(\{\bullet\})$. Thus, for instance, the Grothendieck group of a point is computable in terms of that of the 0 -sphere $S^{0}, K^{p}(\{\bullet\}) \cong \tilde{K}^{p}\left(S^{0}\right)$. In general, one computes

$$
\tilde{K}^{p}\left(S^{n}\right) \cong\left\{\begin{array}{rl}
\mathbb{Z} & p+n \text { even }  \tag{1.53}\\
\{0\} & p+n \text { odd }
\end{array}\right.
$$

In general, for a pair $(X, Y)$ define the relative group $K^{p}(X, Y)=\tilde{K}^{p}(X / Y)$. To extend the definition beyond $p \in\{-1,0\}$, define $K^{p}(X, Y)=\tilde{K}^{0}\left(\Sigma^{|p|}(X / Y)\right)$ for $p \in \mathbb{N}$, where $\Sigma^{r} Z=S^{r} \wedge Z$ denotes the $r$-iterated reduced suspension of $Z$ and $\wedge$ denotes the smash product. In particular, $K^{p}(X)=\tilde{K}^{0}\left(\Sigma^{|p|} X\right)$.

Proposition 1.25. For pointed, locally compact Hausdorff spaces $X$ and $Y$,

$$
\begin{equation*}
\tilde{K}^{p}(X \times Y) \cong \tilde{K}^{p}(X \wedge Y) \oplus \tilde{K}^{p}(X) \oplus \tilde{K}^{p}(Y) \quad p \in \mathbb{Z} \tag{1.54}
\end{equation*}
$$

To extend the values of $p$ to $\mathbb{Z}$, we invoke Bott periodicity, which is a manifestation of the periodicity of the (stable) homotopy groups of the infinite
unitary group $\mathrm{U}=\lim _{k} \mathrm{U}(k)$ and infinite (complex) general linear group $\mathrm{GL}=\lim _{k} \mathrm{GL}(k ; \mathbb{C})$, namely, $\pi_{n}(\mathrm{U}) \cong \pi_{n}(\mathrm{GL}(\mathbb{C})) \cong \mathbb{Z}$ for odd $n$ and trivial otherwise. Equivalently, the double loop space $\Omega^{2} B U$ of the classifying space $B U$ is homotopy equivalent to $B U \times \mathbb{Z}$ [45].

Proposition 1.26 (Bott). Let X be a pointed, locally compact Hausdorff space. The double suspension induces the following 2-periodic isomorphism of Grothendieck groups,

$$
\begin{equation*}
\tilde{K}^{p+2}(X) \cong \tilde{K}^{p}(X) . \tag{1.55}
\end{equation*}
$$

### 1.12.4. Grothendieck Groups of a Bouquet of Spheres.

Proposition 1.27. The reduced Grothendieck groups of a wedge sum of $m n-$ spheres are

$$
\tilde{K}^{p}\left(\bigvee^{m} S^{n}\right) \cong\left\{\begin{array}{cc}
\mathbb{Z}^{m} & p+n \text { even }  \tag{1.56}\\
\{0\} & p+n \text { odd }
\end{array}\right.
$$

Proof. Let $\imath_{m}$ denote the inclusion map $\bigvee^{m-1} S^{n} \hookrightarrow \bigvee^{m} S^{n}$. By identifying two $n$-spheres, define the surjection $\pi_{m}: \bigvee^{m} S^{n} \rightarrow \bigvee^{m-1} S^{n}$. Note that $\pi_{m}$ is proper, so continuous at infinity. The composite map $(\pi \circ \imath)_{m}=\pi_{m} \circ \imath_{m}$ acts as the identity when restricted to the compact subspace $\bigvee^{m-1} S^{n}$.

The complement of $\bigvee^{m-1} S^{n}$ in $\bigvee^{m} S^{n}$ is homeomorphic to $\mathbb{R}^{n}$, so Proposition 1.24 implies the isomorphism $K^{p}\left(\bigvee^{m} S^{n}\right) \cong K^{p}\left(\bigvee^{m-1} S^{n}\right) \oplus K^{p}\left(\mathbb{R}^{n}\right)$ for $p \in\{-1,0\}$ and $m, n \geqslant 0$. By iteration of the above argument for $m \geqslant 0$, we have
the isomorphism

$$
\begin{equation*}
K^{p}\left(\bigvee^{m} S^{n}\right) \cong K^{p}\left(S^{n}\right) \oplus \bigoplus_{i=1}^{m-1} K^{p}\left(\mathbb{R}^{n}\right) \tag{1.57}
\end{equation*}
$$

Thus, by the isomorphisms of equations (1.45) and (1.46), we have

$$
\begin{align*}
K^{0}\left(\bigvee^{m} S^{n}\right) & \cong K^{0}\left(S^{n}\right) \oplus \bigoplus_{i=1}^{m-1} K^{0}\left(\mathbb{R}^{n}\right)  \tag{1.58a}\\
& \cong\left\{\begin{array}{cc}
\mathbb{Z}^{m+1} & n \text { even } \\
\mathbb{Z} & n \text { odd }
\end{array}\right. \tag{1.58b}
\end{align*}
$$

and

$$
\begin{align*}
K^{-1}\left(\bigvee^{m} S^{n}\right) & \cong K^{-1}\left(S^{n}\right) \oplus \bigoplus_{i=1}^{m-1} K^{-1}\left(\mathbb{R}^{n}\right)  \tag{1.59a}\\
& \cong \begin{cases}\{0\} & n \text { even } \\
\mathbb{Z}^{m} & n \text { odd }\end{cases} \tag{1.59b}
\end{align*}
$$

To conclude the proof, recall the isomorphism $K^{p}\left(\bigvee^{m} S^{n}\right) \cong \tilde{K}^{p}\left(\bigvee^{m} S^{n}\right) \oplus$ $K^{p}(\{\bullet\})$ and $\tilde{K}^{p+2}\left(\bigvee^{m} S^{n}\right) \cong \tilde{K}^{p}\left(\bigvee^{m} S^{n}\right)$ (Bott Periodicity), which allows one to extend the values of $p$ to $\mathbb{Z}$.

Remark 1.12.2. In general, for pointed, locally compact Hausdorff spaces $\left\{X_{i}\right\}$, one has the reduced Grothendieck group isomorphism

$$
\begin{equation*}
\tilde{K}^{p}\left(\bigvee_{i=1}^{m} X_{i}\right) \cong \bigoplus_{i=1}^{m} \tilde{K}^{p}\left(X_{i}\right) \quad p \in \mathbb{Z} \tag{1.60}
\end{equation*}
$$

Using this identity, one proves

$$
\begin{equation*}
\tilde{K}^{p}\left(\bigvee_{i=1}^{m} S^{n_{i}}\right) \cong \mathbb{Z}^{r} \quad r=\left|\left\{p+n_{1}, \ldots, p+n_{m}\right\} \cap 2 \mathbb{Z}\right| \tag{1.61}
\end{equation*}
$$

so if $p$ is even (resp., odd), then $r$ counts the even (resp., odd) spheres in the wedge sum. Thus, if $n$ is even (resp., odd), then the rank of $\tilde{K}^{0}\left(F_{f, 0}\right)$ (resp., $\left.\tilde{K}^{-1}\left(F_{f, 0}\right)\right)$ counts the number of even (resp. odd) spheres, q.v. Corollary 9.21. $\quad \Delta$

By Proposition 1.12, the Milnor fiber arising from a complex analytic map $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with isolated critical point at the origin has the homotopytype of a wedge of spheres. We have therefore computed the Grothendieck groups of a Milnor fiber. The next result relates these groups to those of the corresponding homology.

Proposition 1.28. Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a complex analytic germ with an isolated critical point at the origin and denote by $F_{f, 0}=\phi_{f}^{-1}(1)$ the corresponding Milnor fiber, where $\phi_{f}=\frac{f}{\|f\|}: S^{2 n+1} \backslash V_{f, 0} \rightarrow S^{1}$. There is a group isomorphism

$$
\tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right) \cong \tilde{K}^{n}\left(F_{f, 0}\right) \cong\left\{\begin{array}{cc}
\tilde{K}^{-1}\left(F_{f, 0}\right) & n \text { odd }  \tag{1.62}\\
\tilde{K}^{0}\left(F_{f, 0}\right) & n \text { even }
\end{array}\right.
$$

Proof. The fiber $F_{f, 0}$ has the homotopy type of a wedge sum of spheres, $\bigvee^{\mu} S^{n}$. Thus, $\tilde{K}^{n}\left(F_{f, 0}\right)$ is free abelian of rank $\mu=\mu_{\text {top }}(f)$.

Corollary 1.29. Given two complex analytic germs $f$ and $g$,

$$
\begin{align*}
\tilde{K}^{0}\left(F_{f \boxplus g}\right) & \cong\left(\tilde{K}^{0}\left(F_{f}\right) \otimes \tilde{K}^{-1}\left(F_{g}\right)\right) \oplus\left(\tilde{K}^{0}\left(F_{f}\right) \otimes \tilde{K}^{-1}\left(F_{g}\right)\right)  \tag{1.63}\\
\tilde{K}^{-1}\left(F_{f \boxplus g}\right) & \cong\left(\tilde{K}^{0}\left(F_{f}\right) \otimes \tilde{K}^{0}\left(F_{g}\right)\right) \oplus\left(\tilde{K}^{-1}\left(F_{f}\right) \otimes \tilde{K}^{-1}\left(F_{g}\right)\right) \tag{1.64}
\end{align*}
$$

Proof 1. Combine Proposition 1.28 and Bott periodicity with Künneth formula, $\tilde{H}_{n+m+1}\left(F_{f \boxplus g, 0} ; \mathbb{Z}\right) \cong \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right) \otimes \tilde{H}_{m}\left(F_{g, 0} ; \mathbb{Z}\right)$.

Proof 2. Combine equations (1.49) and (1.50) with the Sebastiani-Thom equivalence $F_{f \boxplus g} \simeq F_{f} \star F_{g}$.

Remark 1.12.3. Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with $n$ even, the stabilization map $f \mapsto \Sigma f$, Proposition 1.28 (as applied only to the fundamental Grothendieck groups without recourse to the Bott isomorphism), Freudenthal suspension theorem and the Sebastiani-Thom equivalence imply the following Bott-like isomorphism, $\tilde{K}^{0}\left(F_{\Sigma^{2} f, 0}\right) \cong \tilde{K}^{0}\left(F_{f, 0}\right)$, since

$$
\begin{array}{cc}
\tilde{K}^{-1}\left(F_{\Sigma f, 0}\right) \cong \tilde{K}^{-1}\left(S F_{f, 0}\right) & \text { Sebastiani-Thom equivalence } \\
\tilde{K}^{-1}\left(S F_{f, 0}\right) \cong \tilde{H}_{n+1}\left(S F_{f, 0} ; \mathbb{Z}\right) & \text { Proposition } 1.28  \tag{1.65}\\
\tilde{H}_{n+1}\left(S F_{f, 0} ; \mathbb{Z}\right) \cong \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right) & \text { Freudenthal suspension theorem } \\
\tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right) \cong \tilde{K}^{0}\left(F_{f, 0}\right) & \text { Proposition 1.28. }
\end{array}
$$

Contradistinctively, Proposition 1.28, the Bott isomorphism and the SebastianiThom equivalence imply the following Freudenthal suspension-like isomorphism, $\tilde{H}_{n+2}\left(S^{2} F_{f, 0} ; \mathbb{Z}\right) \cong \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right)$, since

$$
\begin{align*}
\tilde{H}_{n+2}\left(S^{2} F_{f, 0} ; \mathbb{Z}\right) & \cong \tilde{K}^{n+2}\left(S^{2} F_{f, 0}\right) & & \text { Proposition } 1.28 \\
\tilde{K}^{n+2}\left(S^{2} F_{f, 0}\right) & \cong \tilde{K}^{n+4}\left(F_{f, 0}\right) & & \text { Definition }  \tag{1.66}\\
\tilde{K}^{n+4}\left(F_{f, 0}\right) & \cong \tilde{K}^{n}\left(F_{f, 0}\right) & & \text { Bott isomorphism } \\
\tilde{K}^{n}\left(F_{f, 0}\right) & \cong \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right) & & \text { Proposition 1.28. }
\end{align*}
$$

Thus, there is a rather direct relationship between Bott periodicity and the Sebastiani-Thom equivalence.

Remark 1.12.4. On the level of weighted homogeneous singularities, Bott periodicity manifests as a 2-periodic map between the characteristic polynomials of $\Sigma^{2 N-1} f$ and $\Sigma^{2 N} f$ for $N \geqslant 1$, q.v., Corollary 2.57.

Let $X$ be a (possibly suspended) Milnor fiber. Define the maps $\bar{S}_{*}: \tilde{K}^{p}(X) \rightarrow$ $\tilde{K}^{p+1}(S X)$ and $\psi: \tilde{K}^{p}(X) \rightarrow \tilde{K}^{p-1}(S X)$. Note that $\bar{S}_{*}$ is an induced functor by the homology functor $S_{*}$. Let $B$ denote the Bott bijection $B: \tilde{K}^{p}(X) \rightarrow \tilde{K}^{p+2}(X)$. Since $X$ has the homotopy-type of a wedge sum of spheres, there is an isomorphism $M_{*}$ from the middle reduced homology group to the corresponding reduced Grothendieck group, namely, $M_{*}: \tilde{H}_{*}(X ; \mathbb{Z}) \rightarrow \tilde{K}^{*}(X)$. Proposition 1.27 asserts that $\psi$ is an isomorphism, so it follows that $\bar{S}_{*}=\psi \circ B$ and $B^{-1} \circ \bar{S}_{*} \circ \bar{S}_{*} \circ B^{-1}$ are isomorphisms, too. The previous remarks above can then
be summarized in the following commutative diagram

where, in particular, the Freudenthal suspension-like isomorphism holds:


Remark 1.12.5. Recall the topological index $\mu_{\text {top }}\left(F_{f, 0}\right)=(-1)^{n} \tilde{\chi}\left(F_{f, 0}\right)$, where $\tilde{\chi}\left(F_{f, 0}\right)=\chi\left(F_{f, 0}\right)-1$. According to Proposition 1.27, if $n$ is odd, then in terms of the (unreduced) Grothendieck groups, rank $K^{0}\left(F_{f, 0}\right)=1$ and $\operatorname{rank} K^{-1}\left(F_{f, 0}\right)=1-\chi\left(F_{f, 0}\right)=\mu_{\text {top }}\left(F_{f, 0}\right)$. Correspondingly, if $n$ is even, then $\operatorname{rank} K^{-1}\left(F_{f, 0}\right)=0$ and $\operatorname{rank} K^{0}\left(F_{f, 0}\right)=1+\mu_{\mathrm{top}}\left(F_{f, 0}\right)=\chi\left(F_{f, 0}\right)$. These imply the identity, true for any locally Hausdorff space,

$$
\begin{equation*}
\tilde{\chi}\left(F_{f, 0}\right)=\operatorname{rank} \tilde{K}^{0}\left(F_{f, 0}\right)-\operatorname{rank} \tilde{K}^{-1}\left(F_{f, 0}\right) \tag{1.67}
\end{equation*}
$$

and suggests a generalization of the topological index as the difference of ranks of Grothendieck groups of the corresponding Milnor fiber. We explore this point more fully in §9.2.2.

This concludes our development of the formal topological aspects of the Milnor fibration. We now turn our attention to some related algebraic objects, structures and operations naturally associated with complex analytic singularities.

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## Chapter 2

## Algebraic Structure of Isolated Singularities


#### Abstract

Two kinds of symbol must surely be distinguished. The algebraic symbol comes naked into the world of mathematics and is clothed with value by its masters. A poetic symbol-like the Rose, for Love, in Guillaume de Lorriscomes trailing clouds of glory from the real world, clouds whose shape and colour largely determine and explain its poetic use. In an equation, $x$ and $y$ will do as well as a and b; but the Romance of the Rose could not, without loss, be re-written as the Romance of the Onion, and if a man did not see why, we could only send him back to the real world to study roses, onions, and love, all of them still untouched by poetry, still raw. - C. S. Lewis*


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In this chapter we develop and study the algebraic structure of complex analytic germs with and without isolated critical point at the origin. In particular, we focus our attention on a class of complex analytic polynomials with explicit
*The Personal Heresy: A Controversy (1936)
local algebras, Hilbert-Poincaré series, monodromies, corresponding characteristic polynomials and singularity spectra. We refer the reader to [26] for basic definitions and classical results of commutative algebra.

### 2.1. Local Algebras

Recall the space of germs of analytic functions about the origin $\mathcal{O}_{0, n}$ or, equivalently, the ring of convergent power series $\mathbb{C}\left\{z_{0}, \ldots, z_{n}\right\}$, equipped with the compact-open topology is a unique factorization domain. The ideal $\mathfrak{m}=$ $\mathfrak{m}_{n} \subset \mathcal{O}_{0, n}$ of analytic functions which vanish at the origin is maximal in $\mathcal{O}_{0, n}$. Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a complex analytic germ, equivalently, $f \in \mathfrak{m}$, and define the Jacobi ideal $J_{\partial f}:=\left\langle\partial_{0} f, \ldots, \partial_{n} f\right\rangle \subset \mathcal{O}_{0, n}$, where $\partial_{i} f=\frac{\partial f}{\partial z_{i}}$ denotes the $i^{\text {th }}$-directional derivative of $f$.

Definition 2.1. The local algebra $\mathcal{A}_{f}$ of $f$ is the Artinian ring $\mathcal{O}_{0, n} / J_{\partial f}$.
2.1.1. Algebraic Index. Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with an isolated critical point at the origin, the algebraic index of $f$ is the (complex) dimension of the corresponding local algebra,

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f):=\operatorname{dim}_{\mathbb{C}} \mathcal{A}_{f}=\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{0, n} / J_{\partial f} . \tag{2.1}
\end{equation*}
$$

Following Milnor, if $f$ has an isolated critical point at the origin, then $\mathcal{A}_{f}$ is finite dimensional and $\mu_{\mathrm{alg}}(f)$ is well-defined.

## 2.2. $\mu$-Constant Deformation

To facilitate the transition from the topological structure to the algebraic structure of isolated singularities, we quote the following algebro-topological result. Lê proves that the algebraic index is a topological invariant with respect to deformations of the corresponding complex hypersurface [258].

Proposition 2.2 (Lê, [258]). If $f, g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ are complex analytic germs with isolated critical points at the origin and $V_{f, 0}$ and $V_{g, 0}$ are topologically equivalent, then $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{alg}}(g)$.

A partial converse to Proposition 2.2 also holds.

Proposition 2.3 (Lê, Ramanujam, [259]). Any $\mu$-constant deformation of the hypersurface $V_{f, 0}$ of a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, provided that $n \neq 2$, is topologically constant.
2.2.1. Non-Degeneracy and Łojasiewicz Inequality. By Hilbert's Nullstellensatz, the local algebra $\mathcal{A}_{f}$ is finite dimensional if and only if there exists an integer $k>0$ such that $\mathfrak{m}^{k} \subseteq J_{\partial f}$, or equivalently, that $\mathfrak{m}=\operatorname{Rad}\left(J_{\partial f}\right)$, the radical of the ideal $J_{\partial f}$. It suffices to prove that the origin is an isolated critical point of $f$, which is implied by the existence of constants $\varepsilon, \ell>0$ such that $|\partial f| \geqslant \varepsilon|z|{ }^{\ell}$ in an open neighborhood of the origin, where $|\cdot|$ denotes any norm on $\mathbb{C}^{n+1}$. This inequality is known in the literature as the (complex) Łojasiewicz inequality, and $\ell_{0}(f)=\inf \{\ell\}$ is the Łojasiewicz exponent of $f$ (at the origin), as introduced by

Teissier [454]. In the same article, Teissier conjectures that $\ell_{0}(f)$ is a topological invariant, q.v., §2.7.5.

Definition 2.4. A complex analytic singularity is non-degenerate if and only if it satisfies a Łojasiewicz inequality, that is, if and only if it has an isolated critical point at the origin.

A closely related invariant, also studied by Teissier [454], is the following.

Definition 2.5. The degree of topological determinacy of a complex analytic singularity is the integer $\left\lfloor\ell_{0}(f)\right\rfloor+1$.
2.2.2. Biholomorphisms. Given two complex analytic germs $f, g \in \mathfrak{m} \subset$ $\mathcal{O}_{0, n}$, a local holomorphic change of variables yields the ideal isomorphism $J_{\partial f} \cong J_{\partial g}$ and algebra isomorphism $\mathcal{A}_{f} \cong \mathcal{A}_{g}$ (where the latter does not imply the former, but the former implies the latter), including the identity $\mu_{\mathrm{alg}}(f)=$ $\mu_{\mathrm{alg}}(g)$.

More generally, suppose that $f$ and $g$ are analytic functions of $n+1$ and $m+1$ complex variables, respectively. If there are isomorphisms $\psi: J_{\partial f} \rightarrow J_{\partial g}$ and $\psi^{\prime}: \mathcal{A}_{f} \rightarrow \mathcal{A}_{g}$, one can consider the following commutative diagram of exact sequences,


By the Five Lemma, there is an isomorphism $\psi^{\prime}: \mathcal{O}_{0, n} \rightarrow \mathcal{O}_{0, m}$, implying $n=m$. Such isomorphisms have been classified as biholomorphisms. However, although rather subtle, it is possible to compare singularities with domains of differing dimension.
2.2.3. Sebastian-Thom Equivalence. A preliminary algebraic characterization of any mathematical structure involves determining its prime or irreducible components; complex analytic germs are no exception. Let $f_{\alpha}: U_{\alpha} \rightarrow \mathbb{C}$ be a complex analytic function with domain $U_{\alpha} \subset \mathbb{C}^{n_{\alpha}}$ in a neighborhood of the origin. Define the projection $\pi_{\alpha^{\prime}}: \prod_{\alpha} U_{\alpha} \rightarrow U_{\alpha^{\prime}}$. Recall the Sebastiani-Thom summation $\square_{\alpha} f_{\alpha}=\sum_{\alpha} f_{\alpha} \circ \pi_{\alpha}$ with product domain $\times_{\alpha} U_{\alpha} \subset \mathbb{C}^{\sum_{\alpha} n_{\alpha}}$ such that $\pi_{\alpha^{\prime}}\left(\boxplus_{\alpha} f_{\alpha}\right)=f_{\alpha^{\prime}}$. Define the $N$-stabilization $\Sigma^{N} f=\Sigma^{N-1}\left(f \boxplus z^{2}\right)$, where $\Sigma f=f \boxplus z^{2}$. Since $\mathcal{A}_{z^{2}}=\mathbb{C}\{z\} /\langle 2 z\rangle \cong \mathbb{C}$, it follows that $\mathcal{A}_{\Sigma f} \cong \mathcal{A}_{f}$ and the identity

$$
\begin{equation*}
\mu_{\mathrm{alg}}(\Sigma f)=\mu_{\mathrm{alg}}(f) \tag{2.2}
\end{equation*}
$$

REMARK 2.2.1. Compare equations (1.27) and (2.2).

Proposition 2.6. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a neighborhood of the origin. Assume that the complex analytic map $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is non-degenerate. The local algebra $\mathcal{A}_{f}$ of $f=\boxplus_{\alpha} f_{\alpha}:\left(\times_{\alpha} U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ factors as a tensor product $\mathcal{A}_{f_{1}} \otimes \cdots \otimes \mathcal{A}_{f_{s}}$.

In particular,

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\prod_{\alpha=1}^{s} \mu_{\mathrm{alg}}\left(f_{\alpha}\right) \tag{2.3}
\end{equation*}
$$

Proof. It suffices to consider two weighted homogeneous polynomials $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ and $g:\left(\mathbb{C}^{m+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$. The tensor product factorization of the monodromy over Sebastiani-Thom summations, namely, $h_{*}(f \boxplus g) \cong h_{*}(f) \otimes h_{*}(g), q . v$. Proposition 1.22, induces an algebra isomorphism $\mathcal{A}_{f \boxplus g} \cong \mathcal{A}_{f} \otimes \mathcal{A}_{g}$ by the following decomposition,

$$
\begin{align*}
\mathcal{A}_{f \boxplus g} & =\mathcal{O}_{0,(n+1)(m+1)} / J_{\partial(f \boxplus g)}  \tag{2.4a}\\
& \cong \mathcal{O}_{0, n} / J_{\partial f} \otimes \mathcal{O}_{0, m} / J_{\partial g}  \tag{2.4b}\\
& =\mathcal{A}_{f} \otimes \mathcal{A}_{g} \tag{2.4c}
\end{align*}
$$

as summarized in the following commutative diagram


The projections $\pi_{1}^{*}: \mathcal{A}_{f \boxplus g} \rightarrow \mathcal{A}_{f}$ and $\pi_{2}^{*}: \mathcal{A}_{f \boxplus g} \rightarrow \mathcal{A}_{g}$ are defined through the projections $\pi_{1}: \mathrm{A}_{f} \oplus \mathrm{~A}_{g} \rightarrow \mathrm{~A}_{f}$ and $\pi_{2}: \mathrm{A}_{f} \oplus \mathrm{~A}_{g} \rightarrow \mathrm{~A}_{g}$, respectively. By construction, there is an isomorphism $\mathcal{A}_{f \boxplus g} \cong \mathcal{A}_{f} \otimes \mathcal{A}_{g}$ and, therefore,

$$
\begin{align*}
\mu_{\mathrm{alg}}(f \boxplus g) & =\operatorname{dim}_{\mathrm{C}} \mathcal{A}_{f} \otimes \mathcal{A}_{g}  \tag{2.5}\\
& =\left(\operatorname{dim}_{\mathrm{C}} \mathcal{A}_{f}\right)\left(\operatorname{dim}_{\mathrm{C}} \mathcal{A}_{g}\right)  \tag{2.6}\\
& =\mu_{\mathrm{alg}}(f) \mu_{\mathrm{alg}}(g) . \tag{2.7}
\end{align*}
$$

Remark 2.2.2. As the Hilbert-Poincaré series factors over tensor products,

$$
\begin{align*}
P_{\mathcal{A}_{f \boxplus g}}(t) & =P_{\mathcal{A}_{f} \otimes \mathcal{A}_{g}}(t)  \tag{2.8a}\\
& =\sum_{k \geqslant 0} \operatorname{dim}_{\mathbb{C}}\left(\mathcal{A}_{f} \otimes \mathcal{A}_{g}\right)_{k} t^{k}  \tag{2.8b}\\
& =\sum_{k \geqslant 0} \operatorname{dim}_{\mathbb{C}}\left(\bigoplus_{k^{\prime}+k^{\prime \prime \prime}=k} \mathcal{A}_{f, k^{\prime}} \otimes \mathcal{A}_{g, k^{\prime \prime}}\right) t^{k}  \tag{2.8c}\\
& =\sum_{k^{\prime} \geqslant 0} \sum_{k^{\prime \prime} \geqslant 0} \operatorname{dim}_{\mathbb{C}} \mathcal{A}_{f, k^{\prime}} \operatorname{dim}_{\mathcal{C}} \mathcal{A}_{g, k^{\prime \prime}} t^{k^{\prime}+k^{\prime \prime}}  \tag{2.8d}\\
& =P_{\mathcal{A}_{f}}(t) P_{\mathcal{A}_{g}}(t), \tag{2.8e}
\end{align*}
$$

which yields a second proof of the identity $\mu_{\mathrm{alg}}(f \boxplus g)=\mu_{\mathrm{alg}}(f) \mu_{\mathrm{alg}}(g)$.

### 2.3. Right, Right-Left, Contact and Stable Equivalence

Recall that $\mathcal{O}_{0, n} \cong \mathbb{C}\left\{z_{0}, \ldots, z_{n}\right\}$ denotes the polynomial ring of convergent power series about the origin, and $\mathcal{O}_{0, n}^{\times}$denote the subset of invertible elements of $\mathcal{O}_{0, n}$, the group of units.

Definition 2.7. Two complex analytic germs $f, g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ are said to be right-equivalent, denoted by $f \sim_{r} g$, if and only if there is a biholomorphism $\Phi \in \mathcal{O}_{0, n} \rightarrow \mathcal{O}_{0, n}$ and $f=g \circ \Phi$.

Definition 2.8. Two complex analytic germs $f, g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ are said to be right-left-equivalent, denoted by $f \sim_{r, l} g$, if and only if there are two biholomorphisms $\Phi: \mathcal{O}_{0, n} \rightarrow \mathcal{O}_{0, n}$ and $\Psi: \mathcal{O}_{0,0} \rightarrow \mathcal{O}_{0,0}$ and $f=\Psi \circ g \circ \Phi$.

Definition 2.9. Two complex analytic germs $f, g$, as above, are said to be contact-equivalent if and only if there is an automorphism $\varphi \in \operatorname{Aut}\left(\mathcal{O}_{0, n}\right)$ and unit $u \in \mathcal{O}_{0, n}^{\times}$such that $f=u \cdot \varphi(g)$. In this case, we write $f \sim_{c} g$.

Proposition 2.10. Let $f, g \in \mathcal{O}_{0, n}$. The following statements are true:

1. There is an automorphism $\psi \in \operatorname{Aut}\left(\mathcal{O}_{0, n}\right)$ such that $\psi(f)=g$ if and only if $f \sim_{r} g ;$
2. There is an algebra isomorphism $\mathcal{O}_{0, n} /\langle f\rangle \cong \mathcal{O}_{0, n} /\langle g\rangle$ if and only if $f \sim_{c} g$;
3. There is a space isomorphism $\left(V_{f, 0}, \mathbf{0}\right) \cong\left(V_{g, 0}, \mathbf{0}\right)$ if and only if $f \sim_{c} g$; and,
4. Right equivalence implies right-left equivalence.
5. Right equivalence implies contact equivalence.

Proof. See Definition 2.9, Remark 2.9.1 and Exercise 2.1.3 in [168].
Therefore, a right equivalence class corresponds to an equivalence class of corresponding complex analytic germs up to a change of variables. Similarly, a contact equivalence class corresponds to an isomorphism class of Milnor fibers up to a change of variables.

Remark 2.3.1. The coefficients of polynomial singularities may be ignored effectively in many instances, but not all. Consider two related complex analytic germs, $f$ and $\tilde{f}$, where the latter is the former with all coefficients replaced by 1. By a rescaling of the coordinates, if there is a coordinate diffeomorphism $\Phi$ such that $\tilde{f}=f \circ \Phi$, then $f \sim_{r} \tilde{f}$.

REmARK 2.3.2. The polynomials $f=x^{n}+y^{2}-z^{2}$ and $g=x^{n}+y z$ over $\mathbb{C}^{3}$ are right-equivalent for $n>1$. Consider the automorphism $\psi$ of $\mathcal{O}_{0,2}$ given by $\psi:(x, y, z) \mapsto(x, y-z, y+z)$, which yields $\psi(g)=x^{n}+y^{2}-z^{2}=f$.

Contact equivalence does not imply right equivalence.

Remark 2.3.3. Consider $f_{\lambda}=x^{a}+y^{b}+z^{c}+\lambda x y z$ with $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1$. For $\lambda, \lambda^{\prime}>0, f_{\lambda} \sim_{c} f_{\lambda^{\prime}}$, but $f_{\lambda} \sim_{r} f_{\lambda^{\prime}}$ if and only if $\lambda=\lambda^{\prime}$.

We refer the reader to Appendix B for tables of weighted homogeneous singularities with inner modality less than or equal to six (up to right and stable equivalence). Consult [19], [20] for various lists of families of singularities by type.

Remark 2.3.4. Consider $f_{r, s}=x^{r}+x y^{s}$ over $\mathbb{C}^{2}$, where $r, s \in \mathbb{N}$, with weights $\left\{\frac{1}{r}, \frac{r-1}{r s}\right\}$. The following statements are true:

1. For $s \in \mathbb{N}, f_{1, s} \sim_{r} x+y^{2}\left(A_{0}\right.$-singularity $)$;
2. For $r \in \mathbb{N}_{>1}, f_{r, 1} \sim_{r} x^{2}+y^{2}$ ( $A_{1}$-singularity);
3. For $s \in \mathbb{N}, f_{2, s} \sim_{r} x^{2}+y^{2 s}\left(A_{2 s-1}\right.$-singularity $)$;
4. For $r \in \mathbb{N}_{>2}, f_{r, 2} \sim_{r} x^{2} y+y^{r}$ ( $D_{r+1}$-singularity);
5. $f_{3,3} \sim_{r} x^{3}+x y^{3}+y^{5}$ (E7-singularity);
6. $f_{3,4} \sim_{r} x^{3}+x^{2} y^{2}+x y^{5}+y^{6}\left(J_{2,0}\right.$-singularity $)$;
7. $f_{4,3} \sim_{r} x^{4}+y^{4}+a x^{2} y^{2}$, where $a^{2}-4 \neq 0$ ( $X_{9}$-singularity); and,
8. $f_{4,4} \sim_{r} x^{4}+x^{2} y^{3}+x y^{4}+y^{6}\left(W_{13}\right.$-singularity $)$.

The local algebra $\mathcal{A}_{f}$ carries a $\mathbb{C}\{t\}$-algebra structure with pointwise multiplication (.) defined by $t \cdot h=f h$ for all $h \in \mathcal{A}_{f}$ [293]. The next two important results characterize analytic germs by their local algebras up to isomorphism.

Proposition 2.11 (Mather-Yau, [293, 294]). Non-degenerate, complex analytic germs $f, g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ are right-equivalent if and only if their corresponding local algebras, $\mathcal{A}_{f}$ and $\mathcal{A}_{g}$, respectively, are isomorphic as $\mathbb{C}\{t\}$-algebras.

Recall that we have defined a $N$-stabilization of a singularity by $N$ iterations of a Sebastiani-Thom summation with a square, viz., $\Sigma^{N_{g}}=\Sigma^{N-1}\left(g \boxplus z^{2}\right)$.

Definition 2.12. Two non-degenerate, complex analytic germs $f, g$, as above, are said to be stably-equivalent if and only if there is a non-negative integer $N$ such that $f \sim_{r} \Sigma^{N} g$.

Right equivalence implies stable equivalence, but the converse is not true.

Proposition 2.13 (Mather). Two non-degenerate, complex analytic germs
$f, g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ are stably-equivalent if and only if their corresponding local algebras are isomorphic as $\mathbb{C}$-algebras.

Let $\simeq$ denote an isomorphism between $\mathbb{C}\{t\}$-algebras. We illustrate the above with the following commutative diagram,


REMARK 2.3.5. The polynomials $f=x^{n}$ and $g=x^{n}+y z$ over $\mathbb{C}$ and $\mathbb{C}^{3}$, respectively, are stably-equivalent for $n>1$, as their corresponding local algebras are isomorphic, viz.,

$$
\begin{equation*}
\mathcal{A}_{g}=\mathbb{C}\{x, y, z\} /\left\langle x^{n-1}, y, z\right\rangle \cong \mathbb{C}\{x\} /\left\langle x^{n-1}\right\rangle=\mathcal{A}_{f} \tag{2.9}
\end{equation*}
$$

REMARK 2.3.6. If $f=\sum_{i=0}^{n} z_{i}^{a}$ and $g=\sum_{i=0}^{m} z_{i}^{b}$ and $f \sim_{r} g$, then $a=b=2$ or $a=b$ and $n=m$.

REMARK 2.3.7. If $f \sim_{r} \Sigma^{N} g$ for some $N>0$, then $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{alg}}(g)$, but the converse is not true. Let $n(f)$ denote the dimension of the domain of $f$. Varchenko proves that stable equivalence implies right equivalence in the sense that if $\Sigma^{N} f \sim_{r} \Sigma^{N} g$ and $n(f)=n(g)$, then $f \sim_{r} g$.
2.3.1. Mather-Yau Algebra and Tjurina Number. Define the Tjurina ideal $T_{f}:=\langle f\rangle+J_{\partial f}=\left\langle f, \partial_{0} f, \ldots, \partial_{n} f\right\rangle$. The moduli algebra, Tjurina algebra or Mather-Yau algebra $\mathcal{M}_{f}$ of $f$ is the quotient ring $\mathcal{O}_{0, n} / T_{f}$. The complex dimension $\tau(f)=\operatorname{dim}_{\mathbb{C}} \mathcal{M}_{f}$ is the Tjurina number of $f$. By the Nullstellensatz, $\mathcal{M}_{f}$ is finite dimensional if and only if there exists an integer $k>0$ such that $\mathfrak{m}^{k} \subseteq T_{f}$, or equivalently, $\mathfrak{m}=\operatorname{Rad}\left(T_{f}\right)$. In general, if $f$ is non-degenerate [276], then

$$
\begin{equation*}
\tau(f) \leqslant \mu_{\mathrm{alg}}(f) \tag{2.10}
\end{equation*}
$$

Proposition 2.14. Let $f, g \in \mathcal{O}_{0, n}$. The following statements are true:

1. If $f \sim_{r} g$, then there is an algebra isomorphism $\mathcal{A}_{f} \cong \mathcal{A}_{g}$;
2. If $f \sim_{r} g$ or $f \sim_{c} g$, then there is an algebra isomorphism $\mathcal{M}_{f} \cong \mathcal{M}_{g}$; and,
3. If $f \sim_{c} g$, then $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{alg}}(g)$.

Proof. See Lemma 2.10 in [168].

Thus, if $f \sim_{r} g$, then $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{alg}}(g)$, but the converse is not true. Moreover, if $f \sim_{c} g$, then $\tau(f)=\tau(g)$, but the converse is not true. Mather and Yau prove the following partial converse.

Proposition 2.15 (Mather, Yau). Let $f, g \in \mathcal{O}_{0, n}$. Then $f \sim_{c} g$ if and only if there is an algebra isomorphism $\mathcal{M}_{f} \cong \mathcal{M}_{g}$.

Proof. See Theorem 2.26 in [168].

REmARK 2.3.8. The polynomials $f=x^{n}+y^{2}+z^{2}$ and $g=x^{n}+y z$ over $\mathbb{C}^{3}$ are contact-equivalent for $n \geqslant 2$, as their corresponding Mather-Yau algebras are isomorphic, viz.,

$$
\begin{align*}
\mathcal{M}_{g} & =\mathbb{C}\{x, y, z\} /\left\langle x^{n}+y z, x^{n-1}, y, z\right\rangle  \tag{2.11}\\
& =\mathbb{C}\{x, y, z\} /\left\langle x^{n-1}, y, z\right\rangle  \tag{2.12}\\
& \cong \mathbb{C}\{x, y, z\} /\left\langle x^{n}+y^{2}+z^{2}, x^{n-1}, y, z\right\rangle  \tag{2.13}\\
& =\mathcal{M}_{f} . \tag{2.14}
\end{align*}
$$

2.3.2. Relationships among Singularity Equivalences. Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, define the Yau algebras $\mathcal{Y}_{f}=\mathcal{O}_{0, n} /(\langle f\rangle+$ $\left.\mathfrak{m} J_{\partial f}\right)$ and $\mathcal{U}_{f}=\mathcal{O}_{0, n} / \mathfrak{m} J_{\partial f}$. Define the following sets:

$$
\begin{align*}
\mathscr{Q}(f) & =\left\{g \in \mathfrak{m} \mid \mathcal{A}_{g} \cong \mathcal{A}_{f}\right\}  \tag{2.15}\\
\mathscr{R}(f) & =\left\{g \in \mathfrak{m} \mid g \sim_{r} f\right\}  \tag{2.16}\\
\mathscr{R} \mathscr{L}(f) & =\left\{g \in \mathfrak{m} \mid g \sim_{r, l} f\right\}  \tag{2.17}\\
\mathscr{K}(f) & =\left\{g \in \mathfrak{m} \mid g \sim_{c} f\right\}  \tag{2.18}\\
\mathscr{A}(f) & =\left\{g \in \mathfrak{m} \mid \mathcal{M}_{g} \cong \mathcal{M}_{f}\right\}  \tag{2.19}\\
\mathscr{B}(f) & =\left\{g \in \mathfrak{m} \mid \mathcal{Y}_{g} \cong \mathcal{Y}_{f}\right\} \tag{2.20}
\end{align*}
$$

Yau [485] and Benson and Yau [49] studied various relationships among the sets above, namely, $\mathscr{Q}$-equivalence (isomorphic Milnor algebras), $\mathscr{R}$-equivalence (right equivalence) $\mathscr{R} \mathscr{L}$-equivalence (right-left equivalence), $\mathscr{K}$-equivalence (contact equivalence), $\mathscr{A}$-equivalence (isomorphic Mather-Yau algebras) and $\mathscr{B}$-equivalence (isomorphic Yau algebras) as specified by the following diagram:


Additionally, let $a(f)=\left\{g \in \mathfrak{m} \mid J_{\partial g} \subseteq J_{\partial f}\right\}$ and

$$
\begin{equation*}
f^{-1} \mathfrak{m}_{1}=\{F \in \mathbb{C}[f] \mid F \in \mathbb{C}\{f\}, f(\mathbf{0})=0\} \tag{2.21}
\end{equation*}
$$

that is, the module of convergent power series of the form $\sum_{k \geqslant 1} c_{k} f^{k}$ which vanish at the origin. Define a third Yau algebra, $\mathcal{Z}_{f}=\mathcal{O}_{0, n} / f^{-1} \mathfrak{m}_{1}+\mathfrak{m} J_{\partial f}$.

Proposition 2.16 (Mather [292]; Benson, Yau [49]). Given $f \in \mathcal{O}_{0, n}$, let $V_{f, 0}=f^{-1}(0)$. The following statements are equivalent:

1. The complex algebraic variety $V_{f, 0}$ is singular only at the origin;
2. The Milnor algebra $\mathcal{A}_{f}$ is finite-dimensional;
3. The Mather-Yau algebra $\mathcal{M}_{f}$ is finite-dimensional;
4. The Yau algebra $\mathcal{U}_{f}$ is finite dimensional;
5. The Yau algebra $\mathcal{Y}_{f}$ is finite-dimensional; and,
6. The Yau algebra $\mathcal{Z}_{f}$ is finite-dimensional.

Proof. See Theorem 3.2 in [49].

Proposition 2.17 (Shoshitaishvili [425]; Benson, Yau [49]). If a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is non-degenerate, then $\mathscr{Q}(f) \cong \mathscr{R} \mathscr{L}(f)$ if and only if $f^{-1} \mathfrak{m}_{1}+\mathfrak{m} J_{\partial f}=a(f)+\mathfrak{m} J_{\partial f}$.

Proof. See Theorem 3 in [485] and Theorem 5.7 in [49].

Proposition 2.18 (Mather, Yau [293], [294]; Yau [485]). If a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is non-degenerate, then $\mathscr{K}(f) \cong \mathscr{A}(f) \cong \mathscr{B}(f)$, and the following statements are equivalent:

1. $\mathfrak{m}(f) \subseteq \mathfrak{m} J_{\partial f}$;
2. $\mathscr{K}(f) \cong \mathscr{R} \mathscr{L}(f)$;
3. $f^{-1} \mathfrak{m}_{1}+\mathfrak{m} J_{\partial f}=\langle f\rangle+\mathfrak{m} J_{\partial f}$.

Proof. See Theorems 2 and 7 in [485].

### 2.4. Weighted Homogeneous Polynomials

Originally studied by Milnor and Orlik, we now consider a noteworthy class of complex analytic polynomials that allow for the explicit evaluation of their local algebras, corresponding Hilbert-Poincaré series and characteristic polynomials of their monodromy.

Definition 2.19. A complex analytic polynomial $f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ is weighted homogeneous if and only if there is set $\left\{\omega_{0}, \ldots, \omega_{n}\right\} \subset \mathbb{R}$ such that

$$
\begin{equation*}
\lambda f=f\left(\lambda^{\omega_{0}} z_{0}, \ldots, \lambda^{\omega_{n}} z_{n}\right) \quad \lambda \in \mathbb{C}^{\times}, \tag{2.22}
\end{equation*}
$$

where $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ are (reduced) weights.
Remark 2.4.1. If a polynomial is weighted homogeneous, then it necessarily vanishes at the origin.

Remark 2.4.2. Consider $f=x^{a}+x y^{b}$ over $\mathbb{C}$, where $a, b \in \mathbb{N}$. Then $f$ is a weighted homogeneous polynomial as it satisfies $\lambda^{a b} f=f\left(\lambda^{b} x, \lambda^{a-1} y\right)$ for $\lambda \in \mathbb{C}^{\times}$. The weights $\left\{\omega_{1}, \omega_{2}\right\}$ are solutions of the following system of linear equations, $a \omega_{1}=1$ and $\omega_{1}+b \omega_{2}=1$. Indeed, this system can be solved by simple substitution, and we find $\omega_{1}=\frac{1}{a} \in \mathbb{Q}$ and $\omega_{2}=\frac{1-\omega_{1}}{b}=\frac{a-1}{a b} \in \mathbb{Q}$.

Proposition 2.20. A complex analytic, weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\} \subset \mathbb{R}$ satisfies the weighted Euler equation

$$
\begin{equation*}
f=\sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i} f \tag{2.23}
\end{equation*}
$$

Proof. We defer the proof for the general case in Proposition 3.2.
Remark 2.4.3. The converse of Proposition 2.20 is true for weighted homogeneous polynomials (Theorem 3, §4.4, [65]). However, with a more general notion of weighted homogeneity, converse of Proposition 2.20, or more generally Proposition 3.2, is false, q.v., §3.1.

Proposition 2.21. If the weights of a weighted homogeneous polynomial are unique, then they are rational.

Proof. Suppose $f$ is a weighted homogeneous polynomial with unique weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$. The exponent vector, say $\left(a_{0}, \ldots, a_{n}\right)$, of a monomial of $f$ provides the coefficients of a linear Diophantine equation, namely, $\sum_{i=0}^{n} a_{i} x_{i}=1$, of which the weights are a unique solution. Since the exponent vectors are integral and all monomials satisfy a similar requirement, it follows that the weights are rationals.

Definition 2.22. Given rational weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$, let $d$ be the smallest positive integer such that $d \omega_{i}=q_{i} \in \mathbb{Z}$ for $0 \leqslant i \leqslant n$. The integers $\left\{q_{0}, \ldots, q_{n}\right\}$ and $d$ are the integral weights and weighted degree, respectively.

Remark 2.4.4. While it is false that every polynomial is weighted homogeneous, it is true that every polynomial is a linear combination of weighted homogeneous polynomials (of possibly different weighted degrees). If, however, the monomials of a polynomial have the same integral weights and weighted degree, then said polynomial is weighted homogeneous with those integral weights and weighted degree.

REMARK 2.4.5. Equivalently, a complex analytic polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ is weighted homogeneous if and only if there is a set of integers $\left\{q_{0}, \ldots, q_{n}, d\right\}$ such that $\tilde{f}=f\left(z^{q_{0}}, \ldots, z^{q_{n}}\right)$ is a homogeneous polynomial of degree $d$. In this case, $\tilde{f}$ satisfies the Euler equation $d \tilde{f}=\sum_{i=0}^{n} z_{i} \partial_{i} \tilde{f}$.

REmARK 2.4.6. A majority of authors in the physics community prefer the term quasi-homogeneous (polynomial) to weighted homogeneous (polynomial). In this work, we differentiate these terms for consistency with the mathematics literature. The adjective quasi-homogeneous shall be reserved either for polynomials that are right equivalent to weighted homogeneous polynomials (Definition 2.32) or for complex analytic varieties with isolated singularity at the origin and with a $\mathbb{C}^{\times}$-action containing the origin in the closure of each orbit (Definition 1.1, §III.1, [420]).

Proposition 2.23. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial with weights $\left\{q_{0}, \ldots, q_{n}\right\}$ and weighted degree $d$, then any derivative of $f$
satisfies a weighted Euler equation, namely, for $k \geqslant 0$,

$$
\begin{align*}
\partial_{i_{1}, \ldots, i_{k}} f & =\left(d-\sum_{l=1}^{k} q_{i_{l}}\right)^{-1} \sum_{i=0}^{n} q_{i} z_{i} \partial_{i, i_{1}, \ldots, i_{k}} f \quad 0 \leqslant i_{1}, \ldots, i_{k} \leqslant n  \tag{2.24}\\
& =\left(1-\sum_{l=1}^{k} \omega_{i_{l}}\right)^{-1} \sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i, i_{1}, \ldots, i_{k}} f \tag{2.25}
\end{align*}
$$

provided that $d-\sum_{l=1}^{k} q_{i_{l}}$ is not zero.

Remark 2.4.7. Therefore, by the converse of Proposition 3.2, $f_{i_{1}, \ldots, i_{k}}=$ $\partial_{i_{1}, \ldots, i_{k}} f$ is weighted homogeneous and satisfies

$$
\begin{equation*}
f_{i_{1}, \ldots, i_{k}}=\lambda^{-1} f_{i_{1}, \ldots, i_{k}}\left(\lambda^{v_{0}} z_{0}, \ldots, \lambda^{v_{n}} z_{n}\right) \quad \lambda \in \mathbb{C}^{\times} \tag{2.26}
\end{equation*}
$$

where $v_{i}=\frac{\omega_{i}}{1-\sum_{l=1}^{k} \omega_{i_{l}}}$ for $0 \leqslant i \leqslant n$.
Corollary 2.24. If a weighted homogeneous polynomial has a non-unique or zero weight, then it is degenerate, i.e., the origin is not an isolated critical point of said polynomial.

Proposition 2.25 (Saito, [409]). A complex analytic germ $f$ defines a nondegenerate, weighted homogeneous polynomial (hypersurface) singularity up to a change of variables if and only if $f$ defines a non-degenerate (hypersurface) singularity and $f \in J_{\partial f}$.

Proof. See §1.4 in [276].

Proposition 2.26 (Shoshitaishvili [425]; Mather, Yau [293], [294], [485]). If a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is non-degenerate, i.e., $f$ has an isolated critical point at the origin, then the following statements are equivalent:

1. $f$ is right-equivalent to a weighted homogeneous polynomial;
2. $f \in \mathfrak{m} J_{\partial f}$;
3. $a(f) \subseteq \mathfrak{m} J_{\partial f}$;

4a. $f^{-1} \mathfrak{m}_{1} \subseteq \mathfrak{m} J_{\partial f}$;
4b. $f^{-1} \mathfrak{m}_{1} \cong \mathfrak{m} J_{\partial f}$;
5. $\mathscr{Q}(f) \cong \mathscr{R}(f)$;
6. $\mathfrak{m} J_{\partial f} \cong a(f)+\mathfrak{m} J_{\partial f} ;$
7. $\mathfrak{m} J_{\partial f} \cong\langle f\rangle+\mathfrak{m} J_{\partial f}$
8. $\mathscr{K}(f) \cong \mathscr{R}(f)$; and,
9. $\mathscr{R}(f) \cong \mathscr{R} \mathscr{L}(f)$.

Proof. See Theorems 1,3 and 6 and Propositions 4 and 5 in [485] and Theorems 4.2, 4.14 and 4.15 in [49].

Remark 2.4.8. Yau claims that $1 . \Longrightarrow 5 . \Longrightarrow 3$. for degenerate singularities (Theorem 6,op cit.), so the fact that $J_{\partial g} \cong J_{\partial f}$ implies $g \in \mathfrak{m} J_{\partial g}$, where $f$ is weighted homogeneous (Lemma, op cit.), implies the equivalence $1 . \Longleftrightarrow 4 . \Longleftrightarrow$ 5. $\Longleftrightarrow 9$. for degenerate singularities.

The following result is a partial converse of Proposition 2.14.

Corollary 2.27 (Shoshitaishvili, [425]). Let $f$ and $g$ be a non-degenerate, weighted homogeneous polynomials with the same number of variables. Then $f \sim_{r} g$ if and only if there is an algebra isomorphism $\mathcal{A}_{f} \cong \mathcal{A}_{g}$.

### 2.4.1. Quasi-Brieskorn Pham Singularities.

Definition 2.28. A weighted homogeneous polynomial is Brieskorn-Pham if and only if it is a summation of powers of disjoint variables, e.g., $f=\sum_{i=0}^{n} z_{i}^{a_{i}}$, where $a_{0}, \ldots, a_{n} \in \mathbb{N}$.

REmARK 2.4.9. A weighted homogeneous function $f$ of a single complex variable satisfies the first-order ordinary differential equation $f=\omega z f^{\prime}$, and is, therefore, of the form $f=c z^{1 / \omega}$ with $c \in \mathbb{C}^{\times}$. If $f$ is a polynomial, it follows that $\omega$ is an inverse integer, and $f$ is necessarily Brieskorn-Pham. In general, however, the class of Brieskorn-Pham singularities is meager in the space of non-degenerate, weighted homogeneous singularities, which is itself meager in the space of complex analytic singularities.

Definition 2.29. A weighted homogeneous polynomial is quasi-BrieskornPham if and only if its reduced weights are inverse positive integers.

It is clear that every Brieskorn-Pham polynomial is quasi-Brieskorn-Pham, but the converse is not true.

Remark 2.4.10. Consider $f=x^{a}+x y^{d}+z^{c}$ over $\mathbb{C}^{3}$ with $a, c, d \in \mathbb{N}$. Then $f$ has weights $\left\{\frac{1}{a}, \frac{a-1}{a d}, \frac{1}{c}\right\}$. If there is an integer $b$ such that $\frac{1}{b}+\frac{1}{a d}=\frac{1}{d}$, that is, $b=$
$d+\frac{b}{a}$, so $a$ necessarily divides $b$, then $f$ is quasi-Brieskorn-Pham with weights $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$. Determining whether this family of singularities is quasi-BrieskornPham is therefore equivalent to computing Egyptian fractions. Additionally, if $\min \left\{a, \frac{a d}{a-1}, c\right\} \in \mathbb{N}$, then $f$ is weakly quasi-Brieskorn-Pham.

Definition 2.30. A weighted homogeneous polynomial is weakly quasi-Brieskorn-Pham if and only if the maximum of its reduced weights is an inverse positive integer.
2.4.2. Semi-weighted Homogeneity. Let $f_{0}, g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be weighted homogeneous polynomials, where $f_{0}$ has weights $\left\{\omega_{i}\right\}$ and $g$ has weighted degree strictly greater than that of $f_{0}$. The complex analytic map $f=f_{0}+\lambda$, where $\lambda \in \mathbb{C}^{\times}$, is called semi-weighted homogeneous with weighted homogeneous principal part $f_{0}$. In [461], Varchenko proved the invariance

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{alg}}\left(f_{0}\right) \tag{2.27}
\end{equation*}
$$

See Definition 2.17 and Corollary 2.18 in [168]. Equation 2.27 is a special case of the following result.

Proposition 2.31 (Varchenko, [461]). Let $f_{t}(z)=f_{0}(z)+\sum_{\alpha} \delta_{\alpha}(t) g_{\alpha}(z)$ be a deformation of a non-degenerate, weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$, where the functions $\delta_{\alpha}:(\mathbb{C}, 0) \rightarrow(\mathbb{C}, 0)$ are non-zero and $g_{\alpha}:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathrm{C}, 0)$ are holomorphic. For sufficiently small values of $t$, the family $\left\{f_{t}\right\}$ has constant
algebraic index $\mu_{\mathrm{alg}}\left(f_{t}\right)=\mu_{\mathrm{alg}}\left(f_{0}\right)$ if and only if each monomial of $g_{\alpha}$ has weighted degree higher than or equal to that of the principal part $f_{0}$.
2.4.3. Quasi-homogeneity and Almost Quasi-homogeneity. In [485], Yau proves a simple necessary and sufficient condition to ensure that a complex analytic germ is right equivalent to a weighted homogeneous polynomial.

Definition 2.32. A complex analytic germ $f \in \mathcal{O}_{0, n}$ is quasi-homogeneous if and only if $f \in \mathfrak{m} J_{\partial f}$ and almost quasi-homogeneous if and only if $\mathfrak{m}(f) \subseteq \mathfrak{m} J_{\partial f}$.

Proposition 2.33 (Yau,[485]). A complex analytic germ $f$ is quasihomogeneous if and only if it is right-equivalent to a weighted homogeneous polynomial.

REMARK 2.4.11. While the quasi-Brieskorn-Pham polynomial $f=x^{5}+y^{5}+$ $x^{3} y^{2}$ over $\mathbb{C}^{2}$ with weights $\left\{\frac{1}{5}, \frac{1}{5}\right\}$ is non-degenerate, the polynomial $f=x^{5}+$ $y^{5}+x^{3} y^{3}$ over $\mathbb{C}^{2}$ is non-degenerate and almost quasi-homogeneous, but not quasi-homogeneous (Example 5.13, [49]). Moreover, both $f \boxplus f$ over $\mathbb{C}^{4}$ over $\mathbb{C}^{2}$ is non-degenerate and not almost quasi-homogeneous (Examples 6.17, op. cit.). The polynomial $g=\left(x^{4}+y\right)\left(x^{9}+y^{2}\right)$ over $\mathbb{C}^{2}$ is non-degenerate and not almost quasi-homogeneous (Example 5.15,op. cit.). Moreover, $g \sim_{r} x^{3}+x^{2} y^{4}+y^{13}$ ( $J_{4,1}$-singularity) and $\mu_{\mathrm{alg}}(g)=23$.

### 2.5. Hilbert-Poincaré Series of the Local Algebra

Computing the invariants of the local algebra of weighted homogeneous polynomials is especially straight-forward and has led to a plethora of classification schemes [21], [23], [494], [493], [415] and [449]. The algebraic index $\mu_{\mathrm{alg}}(f)$ can be calculated explicitly in terms of the reduced weights $\boldsymbol{\omega}=\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ of $f$. Equip the local algebra $\mathcal{A}_{f}$ with the follow positive grading of the indeterminates, $\operatorname{deg}_{w} z_{i}=q_{i}$ for $0 \leqslant i \leqslant n$. Define the weighted degree of a monomial $z_{0}^{a_{i 0}} \cdots z_{n}^{a_{i n}}$ with said gradation to be the integer $\sum_{l=0}^{n} a_{i l} q_{l}$. If $f$ is weighted homogeneous with integral weights ${ }^{*}\left\{q_{0}, \ldots, q_{n}\right\}$, then each monomial of $f$ has equal weighted degree $d=\sum_{l=0}^{n} a_{i l} q_{l}$ equal to the weighted degree of $f$. It follows that $f=d^{-1} \sum_{i} q_{i} z_{i} \partial_{i} f$ and $\partial_{l} f=\left(d-q_{l}\right)^{-1} \sum_{i} q_{i} z_{i} \partial_{i l} f$ for $0 \leqslant l \leqslant n$. We write $\operatorname{deg}_{w} f=d$ and $\operatorname{deg}_{w} \partial_{i} f=d-q_{i}$ for $0 \leqslant i \leqslant n$.
2.5.1. Hilbert-Poincaré Series. We refer the reader to Chapter 4,6 and 10 in [87] for basic background material in rings and modules.

Let $R$ denote a Noetherian local ring with a maximal ideal $\mathfrak{m}$. Consider a finitely-generated $R$-module $M_{R}$ with Krull dimension $r$. The Hilbert-Poincaré series of $M_{R}$ is the generating function [71],

$$
\begin{equation*}
P_{M_{R}}(t)=\sum_{k \geqslant 0}\left(\operatorname{dim}_{R / \mathfrak{m}} \mathfrak{m}^{k} M_{R} / \mathfrak{m}^{k+1} M_{R}\right) t^{k} \tag{2.28}
\end{equation*}
$$

[^25]Proposition 2.34 (Hilbert). Let $R$ denote a Noetherian local ring. The HilbertPoincaré series $P_{M_{R}}(t)$ of a finitely-generated $R$-module $M_{R}$ with Krull dimension $r$ is a rational function of the form $(1-t)^{-r} H_{M_{R}}(t)$, where $H_{M_{R}}$ is a $\mathbb{Z}$-polynomial.

Proposition 2.35. Let $R$ denote a Noetherian local ring. Suppose $M_{R}$ is a finitely-generated $R$-module. If $M_{R}$ is Cohen-Macaulay, then the numerator of the Hilbert-Poincaré series, namely, $H_{M_{R}}(t)$, has non-negative integral coefficients.

Proof. See Chapter 4 in [71].

REMARK 2.5.1. For a field $\mathbb{F}$, the polynomial ring $M_{R}=\mathbb{F}\left[x_{1}, \ldots, x_{n}\right]$, where $x_{i}$ is an indeterminant of degree 1 , has Krull dimension $n$. The corresponding Hilbert-Poincaré series is $P_{M_{R}}(t)=(1-t)^{-n}$.

The following generalization holds.

Proposition 2.36 (Hilbert, Serre). Let $R_{*}=\oplus_{d \geqslant 0} R_{d}$ be a positively-graded, commutative Noetherian ring, (finitely) generated as a $R_{0}$-algebra by homogeneous indeterminates of positive degrees $r_{1}, \ldots, r_{n}$, respectively. If $M_{*}=\bigoplus_{d \geqslant 0} M_{d}$ is a positively graded, finitely generated $R_{*}$-module, then each homogeneous component $M_{d}$ is a finitely generated $R_{0}$-module. Moreover, the Hilbert-Poincaré series $P_{M_{*}}$ is a rational function,

$$
\begin{equation*}
P_{M_{*}}(t)=\frac{H_{M_{*}}(t)}{\prod_{j=1}^{n}\left(1-t^{r_{j}}\right)} \quad H_{M_{*}} \in \mathbb{Z}[t] . \tag{2.29}
\end{equation*}
$$

Proof. See Theorem 6.3.2 in [87].
2.5.2. Hilbert-Poincaré Series of Weighted Homogeneous Singularities. Suppose $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial. As a consequence of the filtration $\mathcal{A}_{f}=\oplus_{k \geqslant 0} \mathcal{A}_{f, k}$, where $\mathcal{A}_{f, k}$ is the component generated by weighted degree $k$ basis monomials, one can compute the HilbertPoincaré series of the local algebra $\mathcal{A}_{f}$ with relative ease [23]. By the additivity of the Hilbert-Poincaré series over direct summations, one has

$$
\begin{equation*}
P_{\mathcal{A}_{f}}(t)=\sum_{k \geqslant 0} \operatorname{dim}_{\mathbb{C}} \mathcal{A}_{f, k} t^{k} \tag{2.30}
\end{equation*}
$$

However, by eliminating those polynomials generated by the ideal $J_{\partial f}$ and by the multiplicativity of the series over tensor products, one has

$$
\begin{align*}
P_{\mathcal{A}_{f}}(t) & =\left(1-t^{\operatorname{deg}_{w} \partial_{n} f}\right) \cdots\left(1-t^{\operatorname{deg}_{w} \partial_{0} f}\right) P_{\mathcal{O}_{n, 0}}(t)  \tag{2.31a}\\
& =\prod_{i=0}^{n} \frac{1-t^{d-q_{i}}}{1-t^{q_{i}}} \tag{2.31b}
\end{align*}
$$

Under certain conditions, including when $f$ is non-degenerate (q.v., §2.5.3 and Remark 2.5.7), the Hilbert-Poincaré series $P_{\mathcal{A}_{f}}(t)$ is, in fact, a reflexive polynomial of non-negative degree $D=\sum_{i=0}^{n} d-2 q_{i}$ and satisfies the functional identity, $P_{\mathcal{A}_{f}}(t)=t^{D} P_{\mathcal{A}_{f}}\left(\frac{1}{t}\right)$. The Arnol'd-Saito singularity index is the rational $\beta(f)=\sum_{i=0}^{n}\left(\frac{1}{2}-\omega_{i}\right)=\frac{D}{2 d}$. It is clear that if $f$ is a non-degenerate, then $0 \leqslant 2 \beta(f)<n+1$ [409].

Writing $P_{\mathcal{A}_{f}}(t)=\sum_{l \geqslant 0} \mu_{l} t^{l}$, where $\mu_{l}=\operatorname{dim}_{\mathbb{C}} \mathcal{A}_{f, l}$ is the number of basis monomials with weighted degree equal to $l$, one infers the coefficient identity $\mu_{l}=\mu_{D-l}$ for $0 \leqslant l \leqslant D$. Therefore, since $\mu_{0}=1$, there is a unique highest degree $D$ basis monomial. By symmetry, the middle coefficient(s) occurs when $l=\left\lfloor\frac{D}{2}\right\rfloor$ (and $l=\left\lceil\frac{D}{2}\right\rceil$ ). In such case, the dimension of corresponding local algebra $\mathcal{A}_{f}$ admits an exceedingly simple representation in terms of the reduced weights,

$$
\begin{align*}
\mu_{\mathrm{alg}}(f) & =\sum_{l \geqslant 0} \mu_{l}  \tag{2.32a}\\
& =\lim _{t \rightarrow 1} P_{\mathcal{A}_{f}}(t)  \tag{2.32b}\\
& =\prod_{i=0}^{n}\left(\frac{1}{\omega_{i}}-1\right) . \tag{2.32c}
\end{align*}
$$

Assuming that $f$ is non-degenerate, then $\mu_{\mathrm{alg}}(f)$ is a positive integer, which is equivalent to the radical ideal condition $\mathfrak{m}=\operatorname{Rad}\left(J_{\partial f}\right)$. It is important to note that although the reduced weights are rational numbers, the product above is not a priori a non-negative integer. The fact that $\mu_{\mathrm{alg}}(f)$ is the (complex) dimension of a C-algebra implies both its non-negativity and integrality. We shall return to similar structure-preserving dualities in the context of the combinatorics of polytopes in Volume 2.
2.5.3. Non-Degeneracy, Revisited. The algebraic index does not determine the weights, the dimension of the corresponding critical points in a neighborhood of the origin, the structure of the local algebra or the form of the HilbertPoincaré series. The following five examples illustrate the special character of the weights of non-degenerate weighted homogeneous polynomials.

REMARK 2.5.2. Consider $f=x^{a}+x y^{b}+z_{2}^{2}+\cdots+z_{n}^{2}$ over $\mathbb{C}^{n+1}$ with $a, b \in \mathbb{N}$. For $a>1$ and $b \geqslant 1, f$ is a non-degenerate, weighted homogeneous with weights $\left\{\frac{1}{a}, \frac{a-1}{a b}, \frac{1}{2}, \ldots, \frac{1}{2}\right\}$ and local algebra

$$
\begin{equation*}
\mathcal{A}_{f} \cong \mathbb{C}\{x, y\} /\left\langle a x^{a-1}+y^{b}, b x y^{b-1}\right\rangle \tag{2.33}
\end{equation*}
$$

with algebraic index $\mu_{\mathrm{alg}}(f)=(a-1)\left(\frac{a b}{a-1}-1\right)=a b-a+1$.
REmARK 2.5.3. Consider $f=x y+y^{k}+z_{2}^{2}+\cdots+z_{n}^{2}$ over $\mathbb{C}^{n+1}$ for $k>1$ with weights $\left\{\frac{k-1}{k}, \frac{1}{k}, \frac{1}{2}, \ldots, \frac{1}{2}\right\}$. Moreover, $\partial f=\left(y, x+k y^{k-1}, z_{2}, \ldots, z_{n}\right)$, so $f$ has an isolated critical point at the origin for $k>1$ and $n \geqslant 1$. Furthermore,

$$
\begin{equation*}
\mathcal{A}_{f} \cong \mathbb{C}\left\{x, y, z_{1}, \ldots, z_{n}\right\} /\left\langle x, y, z_{1}, \ldots, z_{n}\right\rangle \cong \mathbb{C}, \tag{2.34}
\end{equation*}
$$

so $\mu_{\mathrm{alg}}(f)=1$ for $k>1$ and $n \geqslant 1$.
REMARK 2.5.4. Consider $f=x^{5} y^{6}+x^{4} y^{9}, g=x^{7} y^{3}+x^{6} y^{5}$ and $h=x^{7}+y^{21}$ over $\mathbb{C}^{2}$. Although $h$ is non-degenerate, $f$ and $g$ are degenerate, as

$$
\begin{equation*}
\partial f=\left(5 x^{4} y^{6}+4 x^{3} y^{9}, 6 x^{5} y^{5}+9 x^{4} y^{8}\right) \tag{2.35}
\end{equation*}
$$

and

$$
\begin{equation*}
\partial g=\left(7 x^{6} y^{3}+6(x y)^{5}, 3 x^{7} y^{2}+5 x^{6} y^{4}\right) \tag{2.36}
\end{equation*}
$$

which have continua of critical points on

$$
\begin{equation*}
\left\{(x, 0) \in \mathbb{C}^{2} \mid x \in \mathbb{C}\right\} \cup\left\{(0, y) \in \mathbb{C}^{2} \mid y \in \mathbb{C}\right\} \tag{2.37}
\end{equation*}
$$

The weights of $f$ and $h$ are $\left\{\frac{1}{7}, \frac{1}{21}\right\}$, while those of $g$ are $\left\{\frac{2}{17}, \frac{1}{17}\right\}$. By equation (2.32c), the algebraic indices of $f, g$ and $h$ are identical and equal to 120. Although $P_{\mathcal{A}_{f}}(t)=P_{\mathcal{A}_{h}}(t)$, the corresponding Hilbert-Poincaré series of $f$ and $g$ differ as $\operatorname{deg} P_{\mathcal{A}_{f}}(t)=34$ and $\operatorname{deg} P_{\mathcal{A}_{g}}(t)=28$.

REMARK 2.5.5. Consider $f=x^{2} y^{6}+x^{5} y$ over $\mathbb{C}^{2}$ and $h=f ⿴ z^{4}$ over $\mathbb{C}^{3}$. Here, the weights of $f$ are $\left\{\frac{5}{28}, \frac{3}{28}\right\}$, so $\mu_{\mathrm{alg}}(f)=\frac{115}{3} \notin \mathbb{N}$, which clearly cannot coincide with the dimension of $\mathcal{A}_{f}$ (infinite dimensional, in this case). As

$$
\begin{equation*}
\partial f=\left(x y\left(2 y^{5}+5 x^{3}\right), x^{2}\left(6 y^{5}+x^{3}\right)\right), \tag{2.38}
\end{equation*}
$$

there is a continuum of critical points on $\left\{(0, y) \in \mathbb{C}^{2} \mid y \in \mathbb{C}\right\}$, so $f$ is degenerate. Similarly,

$$
\begin{equation*}
\partial h=\left(5 x^{4} y+2 x y^{6}, x^{5}+6 x^{2} y^{5}, 4 z^{3}\right), \tag{2.39}
\end{equation*}
$$

then $h$ has a continuum of critical points on $\left\{(0, y, 0) \in \mathbb{C}^{3} \mid y \in \mathbb{C}\right\}$, so $h$ is also degenerate. However, the Hilbert-Poincaré series $P_{\mathcal{A}_{f}}(t)$ is not a reflexive

$\mathbb{Z}_{\geqslant 0}$-polynomial of degree 54 . By the multiplicativity of the algebraic index over Sebastiani-Thom summation, $\mu_{\mathrm{alg}}\left(f \boxplus z^{4}\right)=\left(\frac{115}{3}\right)(3)=115$, which coincides with the limit $\lim _{t \rightarrow 1} P_{\mathcal{A}_{h}}(t)$.

Remark 2.5.6. Consider a putative weighted homogeneous polynomial $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with integral weights $q_{0}=2, q_{1}=3$ and $q_{2}=4$ and weighted degree $d=11$. The corresponding Hilbert-Poincaré series is

$$
\begin{align*}
P_{\mathcal{A}_{f}}(t)= & \frac{\left(1-t^{7}\right)\left(1-t^{8}\right)\left(1-t^{9}\right)}{\left(1-t^{2}\right)\left(1-t^{3}\right)\left(1-t^{4}\right)}  \tag{2.40a}\\
= & 1+t^{2}+t^{3}+2 t^{4}+t^{5}+3 t^{6}+t^{7}+3 t^{8}+t^{9} \\
& \quad+3 t^{10}+3 t^{12}-t^{13}+2 t^{14}-t^{15} \ldots \tag{2.4Ob}
\end{align*}
$$

where $\mu_{\mathrm{agg}}(f)=\lim _{t \rightarrow 1} P_{\mathcal{A}_{f}}(t)=21$. However, as $\mu_{13}=\mu_{15}=-1$, no such weighted homogeneous polynomial $f$ can exist with these weights, otherwise $\mathcal{A}_{f, 13}$ and $\mathcal{A}_{f, 15}$ would be ill-defined.

Proposition 2.37. The following statements are true:

1. For each $n \geqslant 1$, there is a non-degenerate, non-Brieskorn-Pham, quasi-Brieskorn-Pham singularities over $\mathbb{C}^{n+1}$;
2. For each $n \geqslant 1$, there is a non-degenerate, weighted homogeneous polynomial over $\mathbb{C}^{n+1}$ with weights arbitrarily close to both 0 and 1 ;
3. Neither the algebraic index nor the dimension of the corresponding critical locus uniquely specifies the corresponding Hilbert-Poincaré series;
4. Given a weighted homogeneous polynomial $f$, if a weight $\omega_{i}$ is identically zero, then $f$ has a non-compact continuum of zeros in the direction of $z_{i}$ through the origin and, therefore, the local algebra $\mathcal{A}_{f}$ is infinite-dimensional. However, the converse is not true. That is, a weighted homogeneous polynomial with positive weights less than $\frac{1}{2}$ and a Hilbert-Poincaré series that is a reflexive $\mathbb{Z}_{\geqslant 0}$-polynomial may possess a non-compact continuum of critical points and an infinite-dimensional local algebra; and,
5. A list of positive integers $\left\{q_{0}, \ldots, q_{n}, d\right\}$ does not a priori correspond to the integral weights and weighted degree of a non-degenerate, weighted homogeneous polynomial, even in the case that the corresponding reduced weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\} \subset \mathbb{Q} \cap\left(0, \frac{1}{2}\right)$ and the product $\prod_{i=0}^{n}\left(\frac{1}{\omega_{i}}-1\right)$ is a positive integer.

Proof. See Remarks 2.5.2, 2.5.3, 2.5.4, 2.5.5 and 2.5.6.

Remark 2.5.7. The aforementioned remarks serve to illustrate the unique role a non-degenerate, weighted homogeneous singularity takes among the space of all weighted homogeneous polynomials. To recapitulate, a weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is non-degenerate if and only if the following data is true:

1. The origin is an isolated critical point of $f$;
2. The polynomial $f$ satisfies the Łojasiewicz inequality;
3. The radical of the Jacobi ideal $\operatorname{Rad}\left(J_{\partial f}\right)$ is the maximal ideal $\mathfrak{m}$;
4. The local algebra $\mathcal{A}_{f}=\mathcal{O}_{0, n} / J_{\partial f}$ is finite and positive-dimensional;
5. The Hilbert-Poincaré series $P_{\mathcal{A}_{f}}$ is a monic, reflexive $\mathbb{Z}_{\geqslant 0}$-polynomial;
6. The limit $\lim _{t \rightarrow 1} P_{\mathcal{A}_{f}}(t)$ exists and is a positive integer;
7. The product $\prod_{i=0}^{n}\left(\frac{1}{\omega_{i}}-1\right)$ is a positive integer;
8. There is a set $\left\{\omega_{0}, \ldots, \omega_{n}\right\} \subset \mathbb{Q} \cap(0,1)$ such that

$$
\begin{equation*}
f=\lambda^{-1} f\left(\lambda^{\omega_{0}} z_{0}, \ldots, \lambda^{\omega_{n}} z_{n}\right) \quad \lambda \in \mathbb{C}^{\times} ; \tag{2.41}
\end{equation*}
$$

and,
9. There is a set $\left\{q_{0}, \ldots, q_{n}, d\right\} \subset \mathbb{N}$ such that

$$
\begin{equation*}
d f=\sum_{i=0}^{n} q_{i} z_{i} \partial_{i} f \tag{2.42}
\end{equation*}
$$

where 1. $\Longleftrightarrow 2 . \Longleftrightarrow 3 . \Longleftrightarrow 4 . \Longrightarrow 5 . \Longrightarrow 6 . \Longrightarrow 7$. and $1 . \Longrightarrow 8 . \Longleftrightarrow 9 .$, and the first four and last statements are independent of weighted homogeneity.

### 2.6. Characteristic Polynomial of the Monodromy

According to Milnor, for $n \in \mathbb{N} \backslash\{2\}$, it is fruitful to consider the short exact (Wang) sequence [420],

$$
\{0\} \longrightarrow H_{n}\left(K_{f}\right) \longrightarrow H_{n}\left(F_{f, 0}\right) \xrightarrow{I-h_{*}} H_{n}\left(F_{f, 0}\right) \longrightarrow H_{n}\left(S_{\varepsilon}^{2 n+1} \backslash K_{f}\right) \longrightarrow\{0\}
$$

and duality isomorphisms $H_{n}\left(S_{\varepsilon}^{2 n+1} \backslash K_{f}\right) \cong H^{n}\left(K_{f}\right) \cong H_{n-1}\left(K_{f}\right)$. The map I- $h_{*}$ is an isomorphism if and only if $H_{n}\left(K_{f}\right) \cong H_{n-1}\left(K_{f}\right)$ is trivial if and only if $\Delta_{h_{*}}(1)=\operatorname{det}\left(I-h_{*}\right)= \pm 1$. In this case, the corresponding algebraic link $K_{f}$ is a homotopy $(2 n-1)$-sphere, therefore a topological $(2 n-1)$-sphere (Theorem 8.5, [310]).
2.6.o.1. Characteristic Polynomial of Weighted Homogeneous Polynomials. Given a non-degenerate, weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, the hypersurface $V_{f, 1}=f^{-1}(1)$ admits the following explicit representation in space [420],

$$
\begin{equation*}
V_{f, 1}=\left\{\left(\lambda_{0} e^{2 \pi i \omega_{0}}, \ldots, \lambda_{n} e^{2 \pi i \omega_{n}}\right) \in \mathbb{C}^{n+1} \mid \lambda_{i} \geqslant 0 \wedge \sum_{i=0}^{n} \lambda_{i}=1\right\} \tag{2.43}
\end{equation*}
$$

As the monodromy $h:\left(z_{0}, \ldots, z_{n}\right) \rightarrow\left(e^{2 \pi i \omega_{0}} z_{0}, \ldots, e^{2 \pi i \omega_{n}} z_{n}\right)$ induces a homeomorphism $h: V_{f, 1} \rightarrow V_{f, 1}$ and a diffeomorphism $F_{f, 0} \cong_{d} V_{f, 1}$ as a deformation retraction $[310,420]$, Milnor computes the zeta function and the characteristic polynomial of the monodromy in the weighted homogeneous case [310].
2.6.o.2. Companion Matrices of Monic Polynomials.

Definition 2.38. Let $\mathbb{F}$ be a field. The companion matrix $\mathrm{C}_{f}$ of a monic polynomial $f=\sum_{i=0}^{n} b_{k} x^{k} \in \mathbb{F}[x]$ is the matrix

$$
\mathrm{C}_{f}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0  \tag{2.44}\\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-b_{0} & -b_{1} & -b_{2} & \cdots & -b_{n-1}
\end{array}\right) \in \mathbb{F}^{n \times n}
$$

Proposition 2.39. The companion matrix $\mathrm{C}_{f}$ of a polynomial $f \in \mathbb{F}[x]$ satisfies the following:

1. $f(x)=\operatorname{det}\left(x \mid-C_{f}\right)$;
2. $f\left(C_{f}\right)=0$; and,
3. The eigenvalues of $\mathrm{C}_{f}$ are the roots of $f$.

Proof. The first statement follows from direct computation, while the second statement is a consequence of the Cayley-Hamilton Theorem. The third statement is a consequence of these.

An important result regarding companion matrices is the following decomposition.

Proposition 2.40. Let $\mathbb{F}$ be a field. Suppose $f \in \mathbb{F}[x]$ is a monic polynomial that factors into irreducible powers, say, $f=f_{1}^{r_{1}} \ldots f_{n}^{r_{n}}$, where $f_{1}, \ldots, f_{n} \in \mathbb{F}[x]$ and $r_{1}, \ldots, r_{n} \in \mathbb{N}$. The companion matrix $\mathrm{C}_{f}$ is similar to the direct summation of companion matrices of said irreducible powers, namely, $\mathrm{C}_{f} \sim \bigoplus_{i=1}^{n} \mathrm{C}_{f_{i}^{r_{i}}}$.

Proof. See Corollary 4.7 in Chapter 7, [215].
2.6.o.3. Monodromy of Brieskorn-Pham Singularities. The monodromy can be computed explicitly for Brieskorn-Pham singularities [361]. The monodromy matrix $h_{*}$ of the Brieskorn-Pham singularity $f=\sum_{k=0}^{n} f_{k}$, where $f_{k}=z^{a_{k}}$, is the Kronecker (tensor) product of companion matrices of the characteristic polynomial of $f_{k}$, namely $\Delta_{z^{a_{k}}}(t)=\frac{t^{a_{k}-1}}{t-1}$. In particular, $h_{*}=\bigotimes_{k=0}^{n} h_{a_{k}}$, where $h_{a_{k}}$
is the $\left(a_{k}-1\right) \times\left(a_{k}-1\right)$-matrix $(0, \pm 1)$-matrix

$$
h_{a_{k}}=\left(\begin{array}{ccccc}
0 & 1 & 0 & \cdots & 0  \tag{2.45}\\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 \\
-1 & -1 & -1 & \cdots & -1
\end{array}\right)
$$

which can be computed directly by considering the intersection matrix of the basis of vanishing cycles on the corresponding fiber. Let $\zeta_{n}$ denote the $n^{\text {th }}$-root of unity $e^{2 \pi i / n}$. It is relatively straightforward to show that $h_{a_{k}}$ may be diagonalized to the form

$$
h_{a_{k}} \sim\left(\begin{array}{ccccc}
\zeta_{a_{k}} & 0 & 0 & \cdots & 0  \tag{2.46}\\
0 & \zeta_{a_{k}}^{2} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \zeta_{a_{k}}^{a_{k}-1}
\end{array}\right)
$$

Defining $\mu=\prod_{i=0}^{n}\left(a_{i}-1\right)$, it follows that $h_{*}$ may also be diagonalized into the $\mu \times \mu$-matrix

$$
h_{*} \sim\left(\begin{array}{ccccc}
\zeta_{a_{0}} \cdots \zeta_{a_{n}} & 0 & 0 & \cdots & 0  \tag{2.47}\\
0 & \zeta_{a_{0}}^{2} \cdots \zeta_{a_{n}} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & \zeta_{a_{0}}^{a_{0}-1} \cdots \zeta_{a_{n}}^{a_{n}-1}
\end{array}\right)
$$

where the entries are all possible products of the form $\zeta_{0}^{k_{0}} \cdots \zeta_{n}^{k_{n}}$, where $1 \leqslant k_{i} \leqslant$ $a_{i}-1$ for $0 \leqslant i \leqslant n$. It follows that

$$
\begin{equation*}
\Delta_{h_{*}}(t)=\prod_{k_{0}=1}^{a_{0}-1} \cdots \prod_{k_{n}=1}^{a_{n}-1}\left(t-\zeta_{a_{0}}^{k_{0}} \cdots \zeta_{a_{n}}^{k_{n}}\right) \tag{2.48}
\end{equation*}
$$

2.6.1. Characteristic Polynomial from the Hilbert-Poincaré Series. Possessing the Hilbert-Poincaré series allows one to explicitly calculate the characteristic polynomial of the corresponding monodromy. For a weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with weights $\left\{\omega_{i}\right\}=\left\{\frac{q_{i}}{d}\right\} \subset \mathbb{Q} \cap(0,1)$, recall that the Hilbert-Poincaré series is the product

$$
\begin{equation*}
P_{\mathcal{A}_{f}}(t)=\prod_{i=0}^{n}\left(\frac{1-t^{d-q_{i}}}{1-t^{q_{i}}}\right) . \tag{2.49}
\end{equation*}
$$

Under the map $t \mapsto t^{1 / d}$, define the reduced Hilbert-Poincaré series,

$$
\begin{equation*}
\bar{P}_{\mathcal{A}_{f}}(t)=P_{\mathcal{A}_{f}}\left(t^{1 / d}\right)=\prod_{i=0}^{n}\left(\frac{1-t^{1-\omega_{i}}}{1-t^{\omega_{i}}}\right) . \tag{2.50}
\end{equation*}
$$

If $f$ is non-degenerate, then $P_{\mathcal{A}_{f}}(t)$ is a monic and reflexive $\mathbb{Z}_{\geqslant 0}$-polynomial, and $\bar{P}_{\mathcal{A}_{f}}(t)$ is a Puiseux series satisfying $\bar{P}_{\mathcal{A}_{f}}(t)=t^{\hat{c}} \bar{P}_{\mathcal{A}_{f}}\left(\frac{1}{t}\right)$, where $\hat{c}=2 \beta(f)$, twice the Arnol'd-Saito index.

Proposition 2.41. Given two non-degenerate, weighted homogeneous polynomials $f, g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, if $P_{\mathcal{A}_{f}}(t)=P_{\mathcal{A}_{g}}(t)$, then $f$ and $g$ have identical weights up to permutation.

Write $\mu=\mu_{\mathrm{alg}}(f)$ and suppose $\bar{P}_{\mathcal{A}_{f}}(t)=\sum_{j=1}^{\mu} t^{\alpha_{j}}$. By Proposition 1.13, the corresponding monodromy $h_{*}=h_{*}(f)$ has only roots of unity as eigenvalues, say, $\left\{e^{2 \pi i \gamma_{1}}, \ldots, e^{2 \pi i \gamma_{\mu}}\right\}$, where spectrum $\operatorname{Sp}(f)=\left\{\gamma_{i}\right\} \subset \mathbb{Q}$.

Proposition 2.42. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ and reduced Hilbert-Poincaré series $\bar{P}_{\mathcal{A}_{f}}(t)=$ $\sum_{j=1}^{\mu} t^{\alpha_{j}}$, then the elements of its spectrum satisfy

$$
\begin{equation*}
\gamma_{j}=\alpha_{j}+\sum_{i=0}^{n} \omega_{i} \quad 1 \leqslant j \leqslant \mu \tag{2.51}
\end{equation*}
$$

Proof. Omitted.
Thus, the characteristic polynomial $\Delta_{f}(t)=\Delta_{h_{*}}(t)=\operatorname{det}\left(t \mid-h_{*}\right)$ is monic with degree $\mu$ and constant term $\pm 1$, namely,

$$
\begin{align*}
\Delta_{f}(t) & =\prod_{j=1}^{\mu}\left(t-e^{2 \pi i \gamma_{j}}\right)  \tag{2.52a}\\
& =\prod_{j=1}^{\mu}\left(t+(-1)^{n} e^{2 \pi i\left(\alpha_{j}-\beta\right)}\right) \tag{2.52b}
\end{align*}
$$

where $\beta=\beta(f)$ is the Arnol'd-Saito index of $f$.

Proposition 2.43. The local algebra $\mathcal{A}_{f}$ determines the corresponding characteristic polynomial $\Delta_{f}$.

Proof. The local algebra determines a Hilbert-Poincaré series, which determines the Arnol'd-Saito index by its reflexivity. Apply Proposition 2.42.

Corollary 2.44. The local algebra $\mathcal{A}_{f}$ determines the corresponding (possibly reduced) Alexander polynomial $\Delta_{K_{f}}(t, \ldots, t)$ of the corresponding algebraic link $K_{f}$.

REMARK 2.6.1. If the corresponding algebraic link is a knot, then the characteristic polynomial coincides with the Alexander polynomial.
2.6.2. Characteristic Polynomial from the Lefschetz Zeta Function. According to Milnor, the fixed-point manifold of the $k$-orbit $h^{k}: \bar{F}_{f, 0} \rightarrow \bar{F}_{f, 0}$, where $F_{f, 0} \cong_{d} V_{f, 1}$, has Euler characteristic $\chi_{k}=\sum_{1<d \mid k} d r_{d}$ in terms of the exponent $r_{d}$ equal to the Lefschetz number $\Lambda\left(h^{\circ k}\right)$, where $h^{\circ k}\left(z_{0}, \ldots, z_{n}\right)=$ $\left(\zeta_{d}^{k q_{0}} z_{0}, \ldots, \zeta_{d}^{k q_{n}} z_{n}\right)$. If $\zeta_{d}^{k q_{i}}$ is not unity for all $0 \leqslant i \leqslant n$, then $\chi_{k}=0$. Otherwise, for some $k$ if there is an $0 \leqslant i \leqslant n$ such that $u^{k q_{i}}=1$ or $k \omega_{i} \in \mathbb{Z}$, then the corresponding fixed point manifold of $h^{k}$ is non-trivial and of the form $\bar{F}_{f, 0} \cap L$, where $L$ is a hyperplane defined by $z_{i_{1}}=\cdots=z_{i_{j}}=0$ for some $j \in \mathbb{N}$. In these distinguished cases, the corresponding Euler characteristic is (putatively) nontrivial and computable. From these data, Milnor computes the zeta function in terms of the integers $r_{d}$ and notes that the characteristic polynomial is a factor of the rational form [310], thereby proving

$$
\begin{equation*}
\Delta_{f}(t)=(t-1)^{(-1)^{n+1}} \prod_{1<d \mid N}\left(t^{d}-1\right)^{(-1)^{n} r_{d}} \tag{2.53}
\end{equation*}
$$

where $\mu=(-1)^{n+1}+(-1)^{n} \sum_{1<d \mid N} d r_{d}$ and $N$ is the period of the fiber map $h$.

Remark 2.6.2. If the period $N$ is a prime $p$, then

$$
\begin{equation*}
\Delta_{f}(t)=(t-1)^{(-1)^{n+1}}\left(t^{p}-1\right)^{(-1)^{n}} \chi_{p} / p \tag{2.54}
\end{equation*}
$$

where $\mu=(-1)^{n+1}\left(1-\chi_{p}\right)$. Such a case occurs if, for example, the weighted degree of $f$ is a prime $p$. Consider $f=x^{3}+x y^{2}$ with $q_{0}=q_{1}=1$ and $d=3$. Thus, $\chi_{3}=-3$, and the characteristic polynomial is

$$
\begin{equation*}
\Delta_{f}(t)=(t-1)\left(t^{3}-1\right)=t^{4}-t^{3}-t+1 \tag{2.55}
\end{equation*}
$$

Proposition 2.45. The exponents $\left\{r_{d}\right\}$ of the characteristic polynomial satisfy

$$
\begin{equation*}
r_{k}=\frac{1}{k} \sum_{d \mid k} \chi_{d} \mu\left(\frac{k}{d}\right) \tag{2.56}
\end{equation*}
$$

where $\mu$ is the Möbius (arithmetic) function.

Proof. The Euler characteristic $\chi_{k}$ and factor $k r_{k}$ are multiplicative arithmetic functions of $k$. The (Möbius) inverse of the Dirichlet convolution $\chi_{k}=$ $k r_{k} * 1$ is $k r_{k}=\chi_{k} * \mu$.

Remark 2.6.3. One has

$$
\begin{align*}
& r_{1}=\chi_{1}  \tag{2.57}\\
& r_{2}=\frac{1}{2}\left(\chi_{2}-\chi_{1}\right)  \tag{2.58}\\
& r_{3}=\frac{1}{3}\left(\chi_{3}-\chi_{1}\right)  \tag{2.59}\\
& r_{4}=\frac{1}{4}\left(\chi_{4}-\chi_{2}\right)  \tag{2.60}\\
& r_{5}=\frac{1}{5}\left(\chi_{5}-\chi_{1}\right)  \tag{2.61}\\
& r_{6}=\frac{1}{6}\left(\chi_{6}-\chi_{3}-\chi_{2}+\chi_{1}\right), \text { etc. } \tag{2.62}
\end{align*}
$$

Proposition 2.46. The characteristic polynomial satisfies the identity

$$
\begin{equation*}
\Delta_{f}(t)=(-1)^{\mu n} t^{\mu} \Delta_{f}\left(\frac{1}{t}\right) \tag{2.63}
\end{equation*}
$$

Proof. By the reflexivity of the reduced Hilbert-Poincaré series,

$$
\begin{align*}
\bar{P}_{\mathcal{A}_{f}}^{\prime}(t) & =\hat{c} t^{\hat{c}-1} \bar{P}_{\mathcal{A}_{f}}\left(\frac{1}{t}\right)+t^{\hat{c}}\left(\bar{P}_{\mathcal{A}_{f}}\left(\frac{1}{t}\right)\right)^{\prime}  \tag{2.64}\\
& =\hat{c} t^{\hat{c}-1} \bar{P}_{\mathcal{A}_{f}}\left(\frac{1}{t}\right)-t^{\hat{c}-2} \bar{P}_{\mathcal{A}_{f}}\left(\frac{1}{t}\right), \tag{2.65}
\end{align*}
$$

where $\bar{P}_{\mathcal{A}_{f}}^{\prime}(t)=\sum_{j=1}^{\mu} \alpha_{j} t^{\alpha_{j}-1}$. Hence, $\bar{P}_{\mathcal{A}_{f}}^{\prime}(1)=\hat{c} \bar{P}_{\mathcal{A}_{f}}(1)-\bar{P}_{\mathcal{A}_{f}}^{\prime}(1)$, and, consequently, $\sum_{j=1}^{\mu} \alpha_{j}=\frac{\mu}{2} \sum_{i=0}^{n} 1-2 \omega_{i}=\frac{\mu}{2}(n+1)-\mu \sum_{i=0}^{n} \omega_{i}$. Thus, since $\gamma_{j}=\alpha_{j}+\sum_{i=0}^{n} \omega_{i}, \sum_{j=1}^{\mu} \gamma_{j}=\sum_{j=1}^{\mu} \alpha_{j}+\mu \sum_{i=0}^{n} \omega_{i}=\frac{\mu}{2}(n+1)$. By equation
(2.52a),

$$
\begin{align*}
\Delta_{f}\left(\frac{1}{t}\right) & =t^{-\mu} \prod_{j=1}^{\mu}\left(1-t e^{2 \pi i \gamma_{j}}\right)  \tag{2.66}\\
& =(-1)^{\mu}\left(\prod_{j=1}^{\mu} e^{2 \pi i \gamma_{j}}\right) t^{-\mu} \Delta_{f}(t)  \tag{2.67}\\
& =(-1)^{\mu} e^{\pi i \mu(n+1)} t^{-\mu} \Delta_{f}(t) \tag{2.68}
\end{align*}
$$

which is $(-1)^{\mu n} t^{-\mu} \Delta_{f}(t)$ and completes the proof.

Corollary 2.47. If $\Delta_{f}(t)=\sum_{i=0}^{\mu} b_{k} t^{k}$, then

$$
\begin{align*}
b_{k} & =(-1)^{\mu n} b_{\mu-k} \quad 0 \leqslant k \leqslant \mu  \tag{2.69}\\
& =(-1)^{\mu-k} \sum_{1 \leqslant i_{1}<\cdots<i_{\mu-k} \leqslant \mu} \cos \left(2 \pi\left(\gamma_{i_{1}}+\cdots+\gamma_{i_{\mu-k}}\right)\right) \tag{2.70}
\end{align*}
$$

In particular, $b_{0}=(-1)^{\mu n}$ and $b_{\mu}=1$.

Proof. Observe

$$
\begin{align*}
\sum_{k=0}^{\mu}(-1)^{\mu n} b_{\mu-k} t^{k} & =\sum_{k=0}^{\mu}(-1)^{\mu n} b_{k} t^{\mu-k}  \tag{2.71}\\
& =(-1)^{\mu n} t^{\mu} \Delta_{f}\left(\frac{1}{t}\right)  \tag{2.72}\\
& =\Delta_{f}(t)  \tag{2.73}\\
& =\sum_{i=0}^{\mu} b_{k} t^{k} \tag{2.74}
\end{align*}
$$

Thus, $b_{k}=(-1)^{\mu n} b_{\mu-k}$,

$$
\begin{equation*}
b_{0}=\Delta_{f}(0)=\prod_{j=1}^{\mu}\left(-e^{2 \pi i \gamma_{j}}\right)=(-1)^{\mu} e^{\pi i \mu(n+1)}=(-1)^{\mu n} \tag{2.75}
\end{equation*}
$$

and $b_{\mu}=(-1)^{\mu n} b_{0}=1$. Finally,

$$
\begin{equation*}
\prod_{j=1}^{\mu}\left(t-e^{2 \pi i \gamma_{j}}\right)=\sum_{k=0}^{\mu}(-1)^{\mu-k} e_{\mu-k}\left(e^{2 \pi i \gamma_{1}}, \ldots, e^{2 \pi i \gamma_{\mu}}\right) t^{k} \tag{2.76}
\end{equation*}
$$

where $e_{k}$ is the $k^{\text {th }}$-elementary symmetric polynomial,

$$
\begin{equation*}
e_{k}\left(x_{1}, \ldots, x_{n}\right)=\sum_{1 \leqslant i_{1}<\cdots<i_{k} \leqslant n} x_{i_{1}} \cdots x_{i_{k}} \tag{2.77}
\end{equation*}
$$

It follows that

$$
\begin{equation*}
b_{k}=(-1)^{\mu-k} \sum_{1 \leqslant i_{1}<\cdots<i_{\mu-k} \leqslant \mu} e^{2 \pi i\left(\gamma_{i_{1}}+\cdots+\gamma_{i_{\mu-k}}\right)} \tag{2.78}
\end{equation*}
$$

However, since $b_{k}$ is a priori real, then only the the real part of the complex exponentials contribute to the summation.

Proposition 2.48. Given a non-degenerate, weighted homogeneous singularity $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, for $m \geqslant 0$, the characteristic polynomial $\Delta_{f}(t)$ satisfies

$$
\begin{align*}
\left((-1)^{m+\mu n}-1\right) \Delta_{f}^{(m)}(1) & =\sum_{k=0}^{m-1}(k-\mu)_{m-k}\binom{m}{k} \Delta_{f}^{(k)}(1)  \tag{2.79}\\
\left((-1)^{\mu(n+1)}-(-1)^{m}\right) \Delta_{f}^{(m)}(-1) & =\sum_{k=0}^{m-1}(-1)^{k}(k-\mu)_{m-k}\binom{m}{k} \Delta_{f}^{(k)}(-1) \tag{2.80}
\end{align*}
$$

where $(x)_{k}=x(x+1) \cdots(x+k-1)$. In particular,

$$
\begin{equation*}
\sum_{k=0}^{\mu}( \pm 1)^{k}(k-\mu)_{\mu-k+1}\binom{\mu+1}{k} \Delta_{f}^{(k)}( \pm 1)=0 \tag{2.81}
\end{equation*}
$$

Proof. By Proposition 2.46,

$$
\begin{equation*}
(-1)^{\mu n} \Delta_{f}^{\prime}(t)=\mu t^{\mu-1} \Delta_{f}\left(\frac{1}{t}\right)-t^{\mu-2} \Delta_{f}^{\prime}\left(\frac{1}{t}\right) \tag{2.82}
\end{equation*}
$$

By repeatedly differentiating say $m$ times,

$$
\begin{equation*}
(-1)^{\mu n} \Delta_{f}^{(m)}(t)=(-1)^{m} \sum_{k=0}^{m}(k-\mu)_{m-k}\binom{m}{k} t^{\mu-m-k} \Delta_{f}^{(k)}\left(\frac{1}{t}\right) . \tag{2.83}
\end{equation*}
$$

Since the degree of $\Delta_{f}(t)$ is $\mu$, take $m=\mu+1$. Hence,

$$
\begin{equation*}
\sum_{k=0}^{\mu}(k-\mu)_{\mu+1-k}\binom{\mu+1}{k} t^{-1-k} \Delta_{f}^{(k)}\left(\frac{1}{t}\right)=0 \tag{2.84}
\end{equation*}
$$

Corollary 2.49. The characteristic polynomial of a non-degenerate, weighted homogeneous germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ satisfies the identities:

$$
\begin{align*}
\mu \Delta_{f}(1) & =\left(1+(-1)^{\mu n}\right) \Delta_{f}^{\prime}(1)  \tag{2.85}\\
\mu \Delta_{f}(-1) & =-\left(1+(-1)^{\mu(n+1)}\right) \Delta_{f}^{\prime}(-1) \tag{2.86}
\end{align*}
$$

In particular, if $\mu$ and $n$ are odd, then $K_{f}$ is not a topological sphere. If $\mu$ or $n$ is even and $\mu>0$, then the corresponding algebraic link $K_{f}$ is a topological sphere if and only if $\Delta_{f}^{\prime}(1)= \pm \frac{\mu}{2}$.

Remark 2.6.4. Assume $\mu>0$. In particular, if $\mu$ and $n$ are odd, then $\Delta_{f}(1)=0$ and $\Delta_{f}(-1)=-\frac{2}{\mu} \Delta_{f}^{\prime}(-1)$; if $\mu$ is even, $\Delta_{f}( \pm 1)= \pm \frac{2}{\mu} \Delta_{f}^{\prime}( \pm 1)$; and, if $\mu$ and $n$ are even, then $\Delta_{f}(1)=\frac{2}{\mu} \Delta_{f}^{\prime}(1)$ and $\Delta_{f}(-1)=0$. Additionally, if $\Delta_{f}^{\prime}(1)=0$, then $\Delta_{f}(1)=0$. If either $\mu$ or $n$ are even, then $\Delta_{f}(1)=\frac{2}{\mu} \Delta_{f}^{\prime}(1)$. In particular, if $n$ is odd, $\Delta_{f}(-1)=-\frac{2}{\mu} \Delta_{f}^{\prime}(-1)$.

Corollary 2.50. For a non-degenerate, weighted homogeneous singularity $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, the exponents of the characteristic polynomial of the monodromy satisfy

$$
\begin{equation*}
\sum_{d \mid N} r_{d}=\mu n \bmod 2 \tag{2.87}
\end{equation*}
$$

Proof. By equation (2.53),

$$
\begin{align*}
\Delta_{f}\left(\frac{1}{t}\right) & =\left(\frac{1}{t}-1\right)^{(-1)^{n+1}} \prod_{1<d \mid N}\left(t^{-d}-1\right)^{(-1)^{n} r_{d}}  \tag{2.88}\\
& =t^{(-1)^{n}}(1-t)^{(-1)^{n+1}} \prod_{1<d \mid N} t^{(-1)^{n+1} d r_{d}}\left(1-t^{d}\right)^{(-1)^{n} r_{d}}  \tag{2.89}\\
& =t^{(-1)^{n}+(-1)^{n+1} \sum_{1<d \mid N} d r_{d}}(-1)^{(-1)^{n+1}+(-1)^{n} \sum_{1<d \mid N} r_{d}} \Delta_{f}(t) \tag{2.90}
\end{align*}
$$

Since

$$
\begin{equation*}
\mu=(-1)^{n+1}\left(1-\chi\left(F_{f, 0}\right)\right)=(-1)^{n+1}+(-1)^{n} \sum_{1<d \mid N} d r_{d} \tag{2.91}
\end{equation*}
$$

then by Proposition 2.46, the factor

$$
\begin{equation*}
(-1)^{n+1}+(-1)^{n} \sum_{1<d \mid N} r_{d} \tag{2.92}
\end{equation*}
$$

has the same parity as $\mu n$.

Proposition 2.51. Given two non-degenerate, weighted homogeneous polynomials $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ and $g:\left(\mathbb{C}^{m+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ and $\left\{v_{0}, \ldots, v_{m}\right\}$, respectively, and corresponding reduced Hilbert-Poincaré series,

$$
\begin{equation*}
\bar{P}_{\mathcal{A}_{f}}(t)=\sum_{j=1}^{\mu(f)} t^{\alpha_{j}} \quad \text { and } \quad \bar{P}_{\mathcal{A}_{g}}(t)=\sum_{j=1}^{\mu(g)} t^{\alpha_{j}^{\prime}}, \tag{2.93}
\end{equation*}
$$

if $\mu(f)=\mu(g)$ and $\left\{\alpha_{j}+\sum_{i=0}^{n} \omega_{i}\right\} \equiv\left\{\alpha_{j}^{\prime}+\sum_{i=0}^{m} v_{i}\right\} \bmod 1$, then $\Delta_{f}(t)=\Delta_{g}(t)$.

Proof. Equation (2.52a), the definition of the sequence $\left\{\gamma_{j}\right\}$ and the $2 \pi$ periodicity of complex exponentials implies the claim.

### 2.6.3. Milnor-Orlik Invariants for Weight Homogeneous Singularities.

Recall that if $n \neq 2$, then the algebraic link $K_{f}$ of an isolated singularity is a topological sphere if and only if $\Delta_{f}(1)= \pm 1$. Milnor and Orlik prove the following result.

Proposition 2.52 (Milnor, Orlik, [315]). Given $\operatorname{div} \Delta_{f}(t)=\sum_{k \geqslant 1} c_{k} \Lambda_{k}$ then $\kappa=\sum_{k \geqslant 1} c_{k}$ and $\rho=\prod_{k \geqslant 2} k^{c_{k}}$ are non-negative integers, where $\kappa$ is the greatest power of the linear factor $t-1$ dividing $\Delta_{f}(t)$ and $\Delta_{f}(1)=\rho \delta_{0, \kappa}$.

Proof. Consider the identities $\Delta_{f}(t)=(t-1)^{\kappa} \prod_{k \geqslant 1}\left(1+t+\cdots+t^{k-1}\right)^{c_{k}}$,

$$
\begin{align*}
(t-1)^{\kappa} & =\sum_{k=0}^{\kappa}\binom{\kappa}{k}(-1)^{\kappa-k} t^{k}  \tag{2.94}\\
\left(1+t+\cdots+t^{k-1}\right)^{c_{k}} & =\sum_{i_{0}+\cdots+i_{k-1}=c_{k}}\binom{c_{k}}{i_{0}, \cdots, i_{k-1}} 1^{i_{0}} t^{i_{1}} \cdots t^{(k-1) i_{k-1}} . \tag{2.95}
\end{align*}
$$

Remark 2.6.5. Hereafter, the integers $\kappa$ and $\rho$ shall be referred to as the Milnor-Orlik invariants.

For a non-degenerate, weighted homogeneous singularity $f$ with weights $\omega_{0}, \ldots, \omega_{n}$,

$$
\begin{align*}
& \operatorname{div} \Delta_{f}(t)=(-1)^{n+1} \Lambda_{1}+\sum_{k=1}^{n+1}(-1)^{n-k+1} \\
& \cdot \sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} c_{i_{1}, \ldots, i_{k}} \Lambda_{\mathrm{LCM}\left(s_{i_{1}}, \ldots, s_{i_{k}}\right)} \tag{2.96a}
\end{align*}
$$

where $\omega_{i}=\frac{r_{i}}{s_{i}}$ and $c_{i_{1}, \ldots, i_{k}}=\left(\operatorname{LCM}\left(s_{i_{1}}, \ldots, s_{i_{k}}\right) \omega_{i_{1}} \cdots \omega_{i_{k}}\right)^{-1}$. The corresponding Milnor-Orlik algebraic link invariants are given by

$$
\begin{equation*}
\kappa=(-1)^{n+1}+(-1)^{n}\left(r_{0}+\cdots+r_{n}\right)+\sum_{k=2}^{n+1}(-1)^{n-k+1} \sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} c_{i_{1}, \ldots, i_{k}} \tag{2.97}
\end{equation*}
$$

and

$$
\begin{equation*}
\rho=\left(r_{0} \cdots r_{n}\right)^{(-1)^{n}} \prod_{k=2}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \operatorname{LCM}\left(s_{i_{1}}, \ldots, s_{i_{k}}\right)^{(-1)^{n-k+1} 1 / c_{i_{1}}, \ldots, i_{k}} . \tag{2.98}
\end{equation*}
$$

2.6.4. Polynomial Tensor Products. Generalizing earlier work of Brawley and Carlitz, Glasby [143] studied the tensor product decomposition of arbitrary polynomials over arbitrary (commutative) polynomial rings, elucidating the relationship between factorizations into irreducibles and tensor products decomposition of related fields.

Definition 2.53. Let $f=\sum_{i=1}^{m} a_{k} x^{k}$ and $g=\sum_{k=1}^{n} b_{k} x^{k}$ be degree $m$ and $n$ polynomials over the ring $\mathbb{Z}_{m, n}=\mathbb{Z}\left[a_{1}, \ldots, a_{m}, b_{1}, \ldots, b_{n}\right]$, respectively. Let $f=a_{m} \prod_{i=1}^{m}\left(x-\alpha_{i}\right)$ and $g(x)=b_{n} \prod_{j=1}^{n}\left(x-\beta_{j}\right)$ be complete factorizations of $f$ and $g$ in the splitting fields of $f$ and $g$ over the field of fractions of $\mathbb{Z}_{m, n}$. The polynomial tensor product of $f$ and $g$ is the polynomial of degree $m n$,

$$
\begin{equation*}
(f \otimes g)(x)=a_{m}^{n} b_{n}^{m} \prod_{i=1}^{m} \prod_{j=1}^{n}\left(x-\alpha_{i} \beta_{j}\right) \tag{2.99}
\end{equation*}
$$

Recall that an integer partition of $k$ of length $m$ is a list $\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ such that $k=\lambda_{1}+\cdots+\lambda_{m}$. Let $P_{k, m, n}$ denote the set of integer partitions $\lambda=$ $\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ of $k$ with $n \geqslant \lambda_{1} \geqslant \cdots \geqslant \lambda_{m} \geqslant 0$. Given $\lambda \in P_{k, m, n}$, define the dual partition $\lambda^{\prime}$ by $\lambda_{i}^{\prime}=\left|\left\{k \mid \lambda_{k} \geqslant i\right\}\right|$ for $1 \leqslant i \leqslant n$.

Proposition 2.54 (Glasby, [143]). Given $f, g \in \mathbb{Z}_{m, n}$, where $f=\sum_{i=1}^{m} a_{k} x^{k}$ and $g=\sum_{k=1}^{n} b_{k} x^{k}$, then $(f \otimes g)(x)=\sum_{k=0}^{m n} c_{k} x^{k} \in \mathbb{Z}_{m, n}$, where $c_{k}=$
$\sum_{\lambda \in P_{k, m, n}} \sum_{l \geqslant \lambda} \gamma_{\lambda, l}^{k} a_{\lambda^{\prime}} b_{l}$, for some integers $\gamma_{\lambda, l}^{k}$ such that $\gamma_{\lambda, \lambda}^{k}=(-1)^{m n-k}$ for all $\lambda \in P_{k, m, n}$.

Proof. See Theorem 2.1 in [143].

Proposition 2.55 (Glasby, [143]). Let $R$ be an integral domain. The set $R[x]^{\times}$ of non-zero polynomials forms a commutative semi-ring (with unity) with standard multiplication as addition and tensor product as multiplication.

### 2.6.5. Monodromy and Sebastiani-Thom Summation.

PROPOSITION 2.56. The characteristic polynomial of the monodromy of a Sebastiani-Thom summation of singularities factors as the (polynomial) tensor product of the respective characteristic polynomials of the constituent factors, viz.,

$$
\begin{equation*}
\Delta_{f_{1} \boxplus \cdots \boxplus f_{s}}(t)=\left(\Delta_{f_{1}} \otimes \cdots \otimes \Delta_{f_{s}}\right)(t) \tag{2.100}
\end{equation*}
$$

Proof 1. With regard to Sebastiani-Thom summation, recall the corresponding Hilbert-Poincaré series is multiplicative over tensor products. Writing $P_{\mathcal{A}_{f}}(t)=\sum_{i=1}^{\mu(f)} t^{\alpha_{i}}$ and $P_{\mathcal{A}_{g}}(t)=\sum_{i=1}^{\mu(f)} t^{\alpha_{i}^{\prime}}$, one has

$$
\begin{equation*}
P_{\mathcal{A}_{f \boxplus g}}(t)=\sum_{i=1}^{\mu(f)} \sum_{j=1}^{\mu(g)} t^{\alpha_{i}+\alpha_{j}^{\prime}} . \tag{2.101}
\end{equation*}
$$

Define $\gamma_{k}=\alpha_{k}+\sum_{k=1}^{n} \omega_{k}$ and $\gamma_{k^{\prime}}^{\prime}=\alpha_{k^{\prime}}^{\prime}+\sum_{k^{\prime}=1}^{m} v_{k^{\prime}}$. The characteristic polynomial factors as a (polynomial) tensor product,

$$
\begin{align*}
\Delta_{f \boxplus g}(t) & =\prod_{i=1}^{\mu(f)} \prod_{j=1}^{\mu(g)}\left(t-e^{2 \pi i\left(\gamma_{i}+\gamma_{j}^{\prime}\right)}\right)  \tag{2.102}\\
& =\prod_{i=1}^{\mu(f)}\left(t-e^{2 \pi i \gamma_{i}}\right) \otimes \prod_{j=1}^{\mu(g)}\left(t-e^{2 \pi i \gamma_{j}^{\prime}}\right)  \tag{2.103}\\
& =\left(\Delta_{f} \otimes \Delta_{g}\right)(t) \tag{2.104}
\end{align*}
$$

as claimed.
Proof 2. The Kronecker product decomposition of the monodromy matrix of the Sebastiani-Thom summation and the factorization of the corresponding characteristic polynomials [143] implies

$$
\begin{align*}
\Delta_{f \boxplus g}(t) & =\operatorname{det}\left(t \mathbb{1}-h_{*}(f \boxplus g)\right)  \tag{2.105a}\\
& =\operatorname{det}\left(t \mathbb{1}-h_{*}(f) \otimes h_{*}(g)\right)  \tag{2.105b}\\
& =\operatorname{det}\left(t \mathbb{1}-h_{*}(f)\right) \otimes \operatorname{det}\left(t \mathbb{1}-h_{*}(g)\right)  \tag{2.105c}\\
& =\Delta_{f}(t) \otimes \Delta_{g}(t), \tag{2.105d}
\end{align*}
$$

where equations (2.105c) and (2.105d) are polynomial tensor products.

Corollary 2.57. Let $f$ be a non-degenerate, weighted homogeneous polynomial, and let $\Sigma^{N} f$ denote the $N$-stabilization of $f$. The characteristic polynomial $\Delta_{f}$ satisfies
the following:

$$
\Delta_{\Sigma^{N} f}(t)=\left\{\begin{array}{cc}
\Delta_{f}(t) & \text { Neven }  \tag{2.106}\\
(-1)^{\mu(f)} \Delta_{f}(-t) & \text { N odd }
\end{array}\right.
$$

Proof. The suspension map $f \mapsto \Sigma f$ is involutive on the characteristic polynomial,

$$
\begin{align*}
\Delta_{f \boxplus z^{2}}(t) & =\prod_{i=1}^{\mu(f)}\left(t-e^{2 \pi i\left(\gamma_{i}+\frac{1}{2}\right)}\right)  \tag{2.107}\\
& =\prod_{i=1}^{\mu(f)}\left(t+e^{2 \pi i \gamma_{i}}\right)  \tag{2.108}\\
& =(-1)^{\mu(f)} \Delta_{f}(-t) \tag{2.109}
\end{align*}
$$

Thus, $\Delta_{\Sigma^{2 N} f}(t)=\Delta_{f}(t)$ and $\Delta_{\Sigma^{2 N-1} f}=(-1)^{\mu(f)} \Delta_{f}(-t)$ for $N \in \mathbb{N}$.
REMARK 2.6.6. Consider the local algebra of a monomial $f=z^{a}$ with gradation $q=1$ and degree $d=a$, namely,

$$
\begin{align*}
\mathcal{A}_{z^{a}} & =\mathbb{C}\{z\} /\left\langle z^{a-1}\right\rangle  \tag{2.110}\\
& \cong\left\{c_{0}+c_{1} z+\cdots+c_{a-2} z^{a-2} \mid\left(c_{0}, \ldots, c_{a-2}\right) \in \mathbb{C}^{a-1}\right\}  \tag{2.111}\\
& \cong\langle 1\rangle \oplus\langle z\rangle \oplus \cdots \oplus\left\langle z^{a-2}\right\rangle \tag{2.112}
\end{align*}
$$

so $\mathcal{A}_{z^{a}, k}=\left\langle z^{k}\right\rangle$ and $\operatorname{dim}_{\mathbb{C}} \mathcal{A}_{z^{a}, k}=1$ for $0 \leqslant k \leqslant a-2$. Thus, $\operatorname{dim}_{\mathbb{C}} \mathcal{A}_{z^{a}}=a-1$. The corresponding Hilbert-Poincaré series is simply $P_{\mathcal{A}_{z^{a}}}(t)=\sum_{k=0}^{a-2} t^{k}$ and, therefore, $P_{\mathcal{A}_{z^{a}}}\left(t^{1 / d}\right)=\sum_{k=1}^{a-1} t^{(k-1) / a}$, so $\alpha_{k}=\frac{k-1}{a}$ and $\gamma_{k}=\alpha_{k}+\frac{1}{a}=\frac{k}{a}$ for
$1 \leqslant k \leqslant a-1$. Thus,

$$
\begin{equation*}
\Delta_{z^{a}}(t)=\prod_{k=1}^{a-1}\left(t-\zeta_{a}^{k}\right)=\sum_{k=0}^{a-1} t^{k}=\frac{t^{a}-1}{t-1} \tag{2.113}
\end{equation*}
$$

where $\zeta_{n}=e^{2 \pi i / n}$. Thus, $\Delta_{z^{a}}(1)=a$ and $\Delta_{z^{a}}(-1)=\frac{1}{2}\left(1-(-1)^{a}\right)$.
Remark 2.6.7. Consider $f=\sum_{i=0}^{n} f_{i}$, where $f_{i}=z_{i}^{a_{i}}$. By Proposition 2.6, the local algebra is the tensor product,

$$
\begin{equation*}
\mathcal{A}_{f} \cong \bigotimes_{k=0}^{n} \mathcal{A}_{f_{i}} \cong \bigotimes_{i=0}^{n} \bigoplus_{k=0}^{a_{i}-2}\left\langle z^{k}\right\rangle \tag{2.114}
\end{equation*}
$$

and the corresponding characteristic polynomial is the polynomial tensor product, by Proposition 2.56,

$$
\begin{align*}
\Delta_{f}(t) & =\bigotimes_{i=0}^{n} \Delta_{f_{i}}(t)  \tag{2.115}\\
& =\bigotimes_{i=0}^{n} \frac{t^{a_{i}}-1}{t-1}  \tag{2.116}\\
& =\prod_{k_{0}=1}^{a_{1}-1} \cdots \prod_{k_{n}=1}^{a_{n}-1}\left(t-\zeta_{a_{0}}^{k_{0}} \cdots \zeta_{a_{n}}^{k_{n}}\right) \tag{2.117}
\end{align*}
$$

which serves as another proof of Lemma 4 in [63].

Remark 2.6.8. According to Milnor, Grothendieck proved that equation (2.117) is necessarily a product of cyclotomic polynomials [310]. We give a proof of this fact in Proposition 6.78.

REMARK 2.6.9. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a homogeneous polynomial of degree $d$, then

$$
\begin{equation*}
\Delta_{f}(t)=\prod_{k_{0}=1}^{d-1} \prod_{k_{1}=1}^{d-1}\left(t-\zeta_{d}^{k_{0}+k_{1}}\right)=(t-1)\left(t^{d}-1\right)^{d-2} \tag{2.118}
\end{equation*}
$$

REMARK 2.6.10. If $f$ and $g$ are quasi-Brieskorn-Pham singularities with weights $\left\{\frac{1}{2}, \frac{1}{3}\right\}$ and $\left\{\frac{1}{3}, \frac{1}{3}\right\}$, respectively, then the corresponding characteristic polynomials are simply $\Delta_{f}(t)=\Phi_{6}(t)$ and $\Delta_{g}(t)=\Phi_{1}(t)^{2} \Phi_{3}(t)$, where $\Phi_{n}$ is the $n^{\text {th }}$-cyclotomic polynomial defined as the product over primitive roots-ofunity,

$$
\begin{equation*}
\Phi_{n}(t)=\prod_{\substack{1 \leqslant k \leqslant n \\ \operatorname{GCD}(k, n)=1}}\left(t-\zeta_{n}^{k}\right) \tag{2.119}
\end{equation*}
$$

Such cases are indicative of a more general identity, q.v., §6.7.

### 2.7. Algebraic Morphisms of the Singularity

2.7.1. Quasi-Homogeneity, Revisited. Recall $V_{f, 0}=f^{-1}(0)$ denotes the complex codimension-one (algebraic) hypersurface of an analytic complex germ $f$. Since $f$ is weighted homogeneous with weights $\omega$, the hypersurface $V_{f, 0}$ is an invariant set under the $\mathbb{C}^{\times}$-action of $\lambda^{\omega} \cdot\left(z_{0}, \ldots, z_{n}\right)=\left(\lambda^{\omega}{ }_{0} z_{0}, \ldots, \lambda^{\omega_{n}} z_{n}\right)$.

Given a complex analytic hypersurface $V$, let $\mathcal{O}_{V}$ denote the sheaf of (germs of) holomorphic functions on $V$. Let $\mathcal{O}_{0, V}$ denote the stalk of (germs of) holomorphic functions of $\mathcal{O}_{V}$ at the origin.

Definition 2.58. A hypersurface $V$ is quasi-homogeneous if and only if there is a weighted homogeneous polynomial $f$ and an algebra isomorphism $\mathcal{O}_{0, V} \cong \mathcal{A}_{f}$.

Saito [409] has shown that a complex analytic germ $f$ is right equivalent to a weighted homogeneous germ if and only if the algebraic index and Tjurina number coincide. In fact, more is true.

Proposition 2.59 (Saito, [409]). Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a nondegenerate, complex analytic polynomial. Then the following statements are equivalent:

1. The polynomial $f$ is right-equivalent to a weighted homogeneous polynomial;
2. The polynomial $f$ is contact-equivalent to a weighted homogeneous polynomial;
3. The algebraic index and Tjurina number coincide, i.e., $\mu_{\mathrm{alg}}(f)=\tau(f)$;
4. The polynomial $f$ is an element of the Jacobi ideal $J_{\partial f}$;
5. The hypersurface $V_{f, 0}$ is quasi-homogeneous for a suitable choice of variables; and,
6. The Poincaré complex of the hypersurface $V_{f, 0}$ is exact,

$$
0 \stackrel{1}{\longrightarrow} \mathbb{C} \xrightarrow{1} \mathcal{O}_{V} \xrightarrow{d} \Omega_{V}^{1} \xrightarrow{d} \cdots \xrightarrow{d} \Omega_{V}^{n} \xrightarrow{0} 0 .
$$

Proof. For the first two statements, see Lemma 2.13 of [168].

Remark 2.7.1. The third statement of Proposition 2.59 furnishes a coordinate-free characterization of weighted homogeneity.

Remark 2.7.2. Boubakri, Greuel and Markwig have recently generalized the third statement to germs over polynomial rings of algebraically closed fields of positive characteristic (Proposition 2.3, [58]).

Proposition 2.60 (Saito, [409]). Let $f \in \mathcal{O}_{0, n}$ be a non-degenerate, complex analytic germ. The following statements are true:

1. If $f$ is weighted homogeneous with weights $0<\omega_{0} \leqslant \cdots \leqslant \omega_{n}<1$ and $f \in \mathfrak{m}^{3}$, then $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ is unique and $0<\omega_{i}<\frac{1}{2}$;
2. If $f \in J_{\partial f}$, then $f$ is stably equivalent to a weighted homogeneous polynomial $g$, that is, $f \simeq_{r} \Sigma^{n-k_{g}}$. In particular, $0<\omega_{0} \leqslant \cdots \leqslant \omega_{k}<\omega_{k+1}=\cdots=\omega_{n}=$ $\frac{1}{2}$; and,
3. If $f, g \in \mathcal{O}_{0, n}$ are right equivalent and weighted homogeneous with weights $0<\omega_{0} \leqslant \cdots \leqslant \omega_{n} \leqslant \frac{1}{2}$ and $0<\nu_{0} \leqslant \cdots \leqslant v_{n} \leqslant \frac{1}{2}$, then $\omega_{i}=v_{i}$.
2.7.2. Orlik-Saito Isomorphisms. In 1970, Orlik proved the topological invariance of the weights of a non-degenerate, weighted homogeneous polynomial in $\mathbb{C}^{3}$ [361]. In particular, he considered relative homeomorphisms between non-degenerate weighted homogeneous hypersurfaces. In 1971, Saito [409] proved the weights are local analytic invariants for any non-degenerate weighted homogeneous polynomial $f$ and are determined uniquely by the analytic isomorphism class of the corresponding hypersurface $V_{f, 0}$.

In 1979, Yoshinaga and Suzuki [492] proved a similar claim for $\mathbb{C}^{2}$, and, in 1986, Nishimura [350] provided a substantially simplified proof of the same result. Specializing to singularities of the Brieskorn-Pham type, in 1978, Yoshinaga and Suzuki proved the following uniqueness result.

Proposition 2.61 (Yoshinaga, Suzuki, [491]). Given two Brieskorn-Pham polynomials $f, g:\left(\mathbb{C}^{n+1}\right) \rightarrow(\mathbb{C}, 0)$ with ordered exponents $2 \leqslant a_{0} \leqslant \cdots \leqslant a_{n}$ and $2 \leqslant b_{0} \leqslant \cdots \leqslant b_{n}$, respectively. If the hypersurfaces $V_{f, 0}$ and $V_{g, 0}$ have identical topological type at the origin, then $a_{i}=b_{i}$ for $0 \leqslant i \leqslant n$.

Remark 2.7.3. For Brieskorn-Pham singularities, it follows that an equivalence of topological-type of hypersurfaces implies an isomorphism of local algebras, equal algebraic indices, equivalent monodromies, equal characteristic polynomials and diffeomorphic (and isotopic) algebraic links. In this case, one has the following commutative diagram


In 1983, Yoshinaga generalized Proposition 2.61 to the following.

Proposition 2.62 (Yoshinaga, [489]). Let $f, g:\left(\mathbb{C}^{n+1}\right) \rightarrow(\mathbb{C}, 0)$ be Brieskorn-Pham polynomials with ordered exponents $2 \leqslant a_{0} \leqslant \cdots \leqslant a_{n}$ and $2 \leqslant b_{0} \leqslant \cdots \leqslant b_{n}$, respectively. Then the following statements are equivalent:

1. The hypersurfaces $V_{f, 0}$ and $V_{g, 0}$ have identical topological type at the origin;
2. The ordered exponents coincide, that is, $a_{i}=b_{i}$ for $0 \leqslant i \leqslant n$; and,
3. The characteristic polynomials $\Delta_{f}(t)$ and $\Delta_{g}(t)$ coincide.

Remark 2.7.4. Lê proved (1.) $\Longrightarrow$ (3.) in [258]. Oka proved the local topological type of a weighted homogeneous singularity is determined by its weights, the implication (2.) $\Longrightarrow$ (1.) in [355]. Yoshinaga proved (3.) $\Longrightarrow$ (2.) in [489].

As a consequence of a theorem of Lê [258], if $V_{f, 0}$ and $V_{g, 0}$ have identical topological type (at the origin), then the characteristic polynomials $\Delta_{f}(t)$ and $\Delta_{g}(t)$ are identical and, hence, the algebraic indices coincide, $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{alg}}(g)$. One concludes that the characteristic polynomial of the monodromy of an isolated singularity is a topological invariant. However, this does not imply that the corresponding algebraic links are isotopic, as there are many examples of nonisotopic links with equal reduced Alexander polynomials (but distinct Alexander polynomials), $q \cdot v .$, Remark 4.13.1. In fact, Proposition 2.62 is an immediate corollary of the following substantially stronger result which holds for weighted homogeneous polynomials.

Proposition 2.63 (Yoshinaga, [489]). Let $f, g:\left(\mathrm{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathrm{C}, 0)$ be two non-degenerate, weighted homogeneous polynomials with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ and $\left\{v_{0}, \ldots, v_{n}\right\}$. Write $\omega_{i}=\frac{r_{i}}{s_{i}}$ and $v_{i}=\frac{t_{i}}{u_{i}}$ (in reduced rational form). The characteristic polynomials $\Delta_{f}(t)$ and $\Delta_{g}(t)$ coincide if and only if the following is true:

1. The sets $\left\{2, s_{0}, \ldots, s_{n}\right\}$ and $\left\{2, u_{0}, \ldots, u_{n}\right\}$ are equal; and,
2. For any $s \in\left\{2, s_{0}, \ldots, s_{n}\right\}, \prod_{s_{i}=s}\left(1-\frac{1}{\omega_{i}}\right)=\prod_{u_{i}=s}\left(1-\frac{1}{v_{i}}\right)$,
where an empty product is indicative of a value equal 1.
2.7.3. Local Homeomorphisms. Given two complex analytic germs $f$ and $g$, the corresponding hypersurfaces $V_{f, 0}$ and $V_{g, 0}$ have identical topological type at the origin if and only if there exists two neighborhoods (of the origin) $U_{f}$ and $U_{g}$ and a homeomorphism $\eta: U_{f} \rightarrow U_{g}$ such that $\eta(0)=0$ and $\eta\left(V_{f, 0} \cap U_{f}\right)=$ $V_{g, 0} \cap U_{g}$ (Figure 2.1).


Figure 2.1. A Local Homeomorphism between Complex Hypersurfaces
In 1988, Saeki proved a certain invariance of singularities over $\mathbf{C}^{2}$ or $\mathbb{C}^{3}$ depends only on the local topological type of the corresponding hypersurfaces in the neighborhood of the origin.

Proposition 2.64 (Saeki, [407]; Yoshinaga, [491]; Nishimura [350]). Let $n \in\{1,2\}$. Given two non-degenerate, weighted homogeneous polynomials
$f, g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with weights $\boldsymbol{\omega}$ and $\boldsymbol{v}$, respectively, if $\left(\mathbb{C}^{n+1}, V_{f, 0}\right)$ and $\left(\mathbb{C}^{n+1}, V_{g, 0}\right)$ are locally homeomorphic, then the weights $\omega$ and $v$ are identical up to permutation.

In this case, one has the following commutative diagram,


### 2.7.4. Multiplicity.

Definition 2.65. The multiplicity of a complex analytic singularity is the minimum degree of the constituent monomials in its series expansion.

Conjecture 2.66 (Zariski). If $f$ and $g$ are complex analytic singularities and there is a local homeomorphism $\left(B_{\varepsilon}^{2 n}, V_{f, 0} \cap B_{\varepsilon}^{2 n}\right) \cong\left(B_{\varepsilon}^{2 n}, V_{g, 0} \cap B_{\varepsilon}^{2 n}\right)$, then $f$ and $g$ are equimultiple, i.e., multiplicities of $f$ and $g$ coincide.

The conjecture is true for weighted homogeneous polynomials [167], [353], [487]. Sękalski [423] proved the multiplicity of a weighted homogeneous polynomial depends only on its weights, viz.,

$$
\begin{align*}
v(f) & =\min \left\{k \in \mathbb{N} \left\lvert\, k \geqslant \min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}\right.\right\}  \tag{2.120}\\
& =\left\lceil\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}\right\rceil . \tag{2.121}
\end{align*}
$$

Therefore, the multiplicity is a topological invariant (Lemma 6, [407]).

Proposition 2.67 (Saeki, [407]). Given two non-degenerate, weighted homogeneous polynomials, $f, g:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, if $\left(\mathbb{C}^{2}, V_{f, 0}\right)$ and $\left(\mathbb{C}^{2}, V_{g, 0}\right)$ are locally homeomorphic, then the multiplicities of $f$ and $g$ coincide.
2.7.5. Łojasiewicz Exponent. For the cases of weighted homogeneous polynomials over $\mathbb{C}^{2}$ and $\mathbb{C}^{3}$, Krasiński, Oleksik and Płoski give an explicit formula for the Łojasiewicz exponent $\ell_{0}(f)$ in terms of the weights [246], thus proving a topological invariance of $\ell_{0}(f)$ for $n=1,2$.

REmARK 2.7.5. Suppose $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial with weights $\left\{\omega_{0}, \omega_{1}, \omega_{2}\right\}$. Then

$$
\begin{equation*}
\ell_{0}(f)=\max \left\{\frac{1}{\omega_{0}}-1, \frac{1}{\omega_{1}}-1, \frac{1}{\omega_{2}}-1\right\} . \tag{2.122}
\end{equation*}
$$

Let $\frac{1}{\omega}-1=\max _{0 \leqslant i \leqslant 2}\left\{\frac{1}{\omega_{i}}-1\right\}$. If $\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}$ does not necessarily lie in the half-closed interval $\left(0, \frac{1}{2}\right]$, then $\ell_{0}(f)=\min \left\{\frac{1}{\omega}-1, \mu_{\text {alg }}(f)\right\}$ (Theorem 3,op. cit.). Sękalski shows that the latter formula does not hold for $n>3$.

Proposition 2.68 (Krasiński, Oleksik and Płoski, [246]). If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial, then

$$
\begin{equation*}
\ell_{0}(f) \leqslant \max _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}-1\right\} \leqslant \mu_{\mathrm{alg}}(f) \tag{2.123}
\end{equation*}
$$

Proof. See Proposition 1 in [246].

Remark 2.7.6. The Łojasiewicz exponent and the corresponding algebraic index are equal if and only if the rank of the local Hessian $\left(\partial_{i j} f\right)$ at the origin equals or exceeds $n$ (Lemma 2, [246]).

In 2010, Tan, Yau and Zuo [452] establish the identity

$$
\begin{equation*}
\ell_{0}(f)=\max _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}-1\right\}, \tag{2.124}
\end{equation*}
$$

thus demonstrating the topological nature of the Łojasiewicz exponent of a nondegenerate, weighted homogeneous polynomial over $\mathbb{C}^{n+1}$ for $n>1$, thereby proving the Teissier Conjecture [454].

Proposition 2.69. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a neighborhood of the origin. Assume that the complex analytic map $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial with weights $\left\{\omega_{i_{\alpha}}\right\}$. The multiplicity and Łojasiewicz exponent of the Sebastian-Thom summation $f=\square_{\alpha} f_{\alpha}:\left(\times_{\alpha} U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ satisfy

$$
\begin{align*}
v(f) & =\min _{1 \leqslant \alpha \leqslant s}\left\{v\left(f_{\alpha}\right)\right\}  \tag{2.125}\\
\ell_{0}(f) & =\max _{1 \leqslant \alpha \leqslant s}\left\{\ell_{0}\left(f_{\alpha}\right)\right\} . \tag{2.126}
\end{align*}
$$

In particular, $v\left(\Sigma^{N} f\right)=\min \{2, v(f)\}$ and $\ell_{0}\left(\Sigma^{N} f\right)=\ell_{0}(f)$ for $N \geqslant 1$.

Proof. The claimed identities follow from the classical identities

$$
\begin{align*}
\max \left\{\max \left\{x_{1}, \ldots, x_{n-1}\right\}, x_{n}\right\} & =\max \left\{x_{1}, \ldots, x_{n}\right\}  \tag{2.127}\\
\min \left\{\min \left\{x_{1}, \ldots, x_{n-1}\right\}, x_{n}\right\} & =\min \left\{x_{1}, \ldots, x_{n}\right\} \tag{2.128}
\end{align*}
$$

for $\left\{x_{1}, \ldots, x_{n}\right\} \subset \mathbb{R}_{\geqslant 0}$. Moreover, the multiplicity of $g=\sum_{i=0}^{n} z_{i}^{2}$ is 2 .

Proposition 2.70. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial, then the multiplicity and Łojasiewicz exponent of $f$ satisfy

$$
v(f) \leqslant \begin{cases}\left\lfloor\ell_{0}(f)\right\rfloor+1 & \text { if } \min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\} \in \mathbb{N}  \tag{2.129}\\ \left\lfloor\ell_{0}(f)\right\rfloor+2 & \text { if } \min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\} \notin \mathbb{N}\end{cases}
$$

In particular,

$$
\begin{equation*}
v(f) \leqslant\left\lceil\sqrt[n+1]{\mu_{\mathrm{alg}}(f)}\right\rceil+1 \leqslant\left\lceil\ell_{0}(f)\right\rceil+1 \tag{2.130}
\end{equation*}
$$

Proof. By the identity $\lceil x\rceil=x-\{x\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}(x)$ on $\mathbb{R}_{\geqslant 0}$, one has

$$
\begin{align*}
v(f) & =\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}-\left\{\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}\right)  \tag{2.131}\\
& \leqslant \max _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}-1\right\}+1-\left\{\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}\right)  \tag{2.132}\\
& \leqslant \ell_{0}(f)+1-\left\{\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}\right) . \tag{2.133}
\end{align*}
$$

Since $v(f)$ is an integer,

$$
v(f) \leqslant \begin{cases}\left\lfloor\ell_{0}(f)\right\rfloor+1 & \text { if } \min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\} \in \mathbb{N}  \tag{2.134}\\ \left\lfloor\ell_{0}(f)+\left\{\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}\right\}\right\rfloor+1 & \text { if } \min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\} \notin \mathbb{N}\end{cases}
$$

and the first inequality follows. For $\left\{x_{1}, \ldots, x_{n}\right\} \subset \mathbb{R}_{\geqslant 0}$,

$$
\begin{equation*}
\min \left\{x_{1}, \ldots, x_{n}\right\} \leqslant \prod_{i=1}^{n} \sqrt[n]{x_{i}} \leqslant \max \left\{x_{1}, \ldots, x_{n}\right\} \tag{2.135}
\end{equation*}
$$

implies

$$
\begin{equation*}
\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}-1\right\} \leqslant \prod_{i=1}^{n}\left(\frac{1}{\omega_{i}}-1\right)^{1 / n} \leqslant \max _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}-1\right\} \tag{2.136}
\end{equation*}
$$

from which the final series of inequalities follow.
Certain invariants uniquely determine the weights of singularities in few complex dimensions.

Proposition 2.71. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial, then the multiplicity and Łojasiewicz exponent of $f$ satisfy the sharp inequalities

$$
\begin{align*}
\frac{1}{\omega_{1}}+\frac{1}{\omega_{2}} & \leqslant v(f)+\ell_{0}(f)+1  \tag{2.137}\\
\frac{1}{\omega_{1} \omega_{2}} & \leqslant v(f)+\ell_{0}(f)+\mu_{\mathrm{alg}}(f) \tag{2.138}
\end{align*}
$$

with equality if and only if $f$ is weakly quasi-Brieskorn-Pham.

Proof. The identity $\max \{x, y\}=x+y-\min \{x, y\}$ implies

$$
\begin{align*}
\ell_{0}(f) & =\max \left\{\frac{1}{\omega_{1}}, \frac{1}{\omega_{2}}\right\}-1  \tag{2.139}\\
& =\frac{1}{\omega_{1}}+\frac{1}{\omega_{2}}-\min \left\{\frac{1}{\omega_{1}}, \frac{1}{\omega_{2}}\right\}-1  \tag{2.140}\\
& \geqslant \frac{1}{\omega_{1}}+\frac{1}{\omega_{2}}-v(f)-1  \tag{2.141}\\
& =\frac{1}{\omega_{1} \omega_{2}}-\mu_{\mathrm{alg}}(f)-v(f) . \tag{2.142}
\end{align*}
$$

Proposition 2.72. Let $f, g:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be two non-degenerate, weighted homogeneous polynomials. If the Łojasiewicz exponents coincide, viz., $\ell_{0}(f)=\ell_{0}(g)$, and the algebraic indices coincide, viz., $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{alg}}(g)$, then $f$ and $g$ have identical weights up to permutation. In particular, the multiplicities coincide, viz., $v(f)=v(g)$.

Proof. The proof is immediate once one recalls the identities

$$
\begin{align*}
v(f) & =\left[\min \left\{\frac{1}{\omega_{1}}, \frac{1}{\omega_{2}}\right\}\right]  \tag{2.143}\\
\ell_{0}(f) & =\max \left\{\frac{1}{\omega_{1}}-1, \frac{1}{\omega_{2}}-1\right\}  \tag{2.144}\\
\mu_{\mathrm{alg}}(f) & =\left(\frac{1}{\omega_{1}}-1\right)\left(\frac{1}{\omega_{2}}-1\right) . \tag{2.145}
\end{align*}
$$

The latter two equations may be combined to unique solve for the weights.

REMARK 2.7.7. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is weakly quasi-Brieskorn-Pham, then the weights $\left\{\frac{1}{\omega_{1}}, \frac{1}{\omega_{2}}\right\}$ are uniquely determined by the quantities $v(f)+\ell_{0}(f)+1$
and $\mu_{\mathrm{alg}}(f)$ by solving the quadratic equation,

$$
\begin{equation*}
x^{2}-\left(v(f)+\ell_{0}(f)+1\right) x+v(f)+\ell_{0}(f)+\mu_{\mathrm{alg}}(f)=0 . \tag{2.146}
\end{equation*}
$$

Therefore, in this case, the rational $v(f)+\ell_{0}(f)$ is a topological invariant, which is not obvious despite the fact that both $v(f)$ and $\ell_{0}(f)$ are topological invariants.

Remark 2.7.8. Proposition 2.72 does not hold in $\mathbb{C}^{n+1}$ for $n>1$. Consider $f=x^{9}+\sum_{i=1}^{n} z_{i}^{2}$ and $g=x^{5}+y^{3}+\sum_{i=2}^{n} z_{i}^{2}$ over $\mathbb{C}^{n+1}$. For $n>1, \mu_{\mathrm{alg}}(f)=$ $\mu_{\mathrm{alg}}(g)=8, v(f)=v(g)=2$, while $\ell_{0}(f)=8$ and $\ell_{0}(g)=4$.

Remark 2.7.9. If, instead, the multiplicities and Łojasiewicz exponents coincide, it does not follow that the algebraic indices necessarily coincide. Take $f=x^{2} y+y^{4}$ with weights $\left\{\frac{3}{8}, \frac{1}{4}\right\}$ and $g=x^{3}+y^{4}$ with weights $\left\{\frac{1}{3}, \frac{1}{4}\right\}$. Clearly, the indices differ, namely, $\mu_{\mathrm{alg}}(f)=5$ and $\mu_{\mathrm{alg}}(g)=6$.

Proposition 2.73. Let $f, g:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be two non-degenerate, quasi-Brieskorn-Pham polynomials. If $\ell_{0}(f)=\ell_{0}(g), v(f)=v(g)$ and $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{alg}}(g)$, then $f$ and $g$ have identical weights up to permutation.

Proof. Let $\left\{a_{1}, a_{2}, a_{3}\right\}$ and $\left\{b_{1}, b_{2}, b_{3}\right\}$ denote the inverse weights of $f$ and $g$, respectively. Without loss of generality, we may assume the orderings $2 \leqslant a_{1} \leqslant$ $a_{2} \leqslant a_{3}$ and $2 \leqslant b_{1} \leqslant b_{2} \leqslant b_{3}$. By assumption, $a_{3}=\ell_{0}(f)+1=\ell_{g}(g)+1=b_{3}$ and $a_{1}=v(f)=v(g)=b_{1}$. Finally, since $\left(a_{1}-1\right)\left(a_{2}-1\right)\left(a_{3}-1\right)=\mu_{\mathrm{alg}}(f)=$
$\mu_{\mathrm{alg}}(g)=\left(b_{1}-1\right)\left(b_{2}-1\right)\left(b_{3}-1\right)$, it follows that $a_{2}=b_{2}$, which completes the proof.

### 2.8. Exponent Matrices

Definition 2.74. Given a polynomial $f=\sum_{i=1}^{m} c_{i} z_{1}^{a_{i 1}} \cdots z_{n}^{a_{i n}} \in \mathfrak{m}$, where $c_{i} \in \mathbb{C}^{\times}$, the $m \times n$ non-negative integral matrix $\mathrm{A}_{f}=\left(a_{i j}\right)$ is the exponent matrix of $f$ defined up to an action of $S_{n} \times S_{m}$ on the rows and columns.

If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial with $m$ monomials and weights* $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$, then said weight multiset $\omega$ solves the matrix equation $\mathrm{A}_{f} \boldsymbol{\omega}=\mathbf{1}_{m}$, and is also defined up to an action of $S_{m}$ inherited from the $S_{m} \times S_{n}$-action on $\mathrm{A}_{f}$. If $\mathrm{A}=\mathrm{A}_{f}$ is square and non-singular, then the weights are uniquely determined by Cramer's Rule, $\omega_{i}=\left(\mathrm{A}^{-1} \mathbf{1}_{n+1}\right)_{i}=$ $\operatorname{det} \mathrm{A}^{-1} \mathrm{~A}_{i}$ where, in general, $\mathrm{A}_{i}$ denotes the matrix A (of size $m \times(n+1)$ ) with the $i^{\text {th }}$-column replaced with the vector $\mathbf{1}_{m}$. In §9.2.1, we discuss the uniqueness of the weights when $A_{f}$ is not square.

Definition 2.75. A weighted homogeneous polynomial is a square if and only if it possesses an equal number of monomials as variables, that is, its exponent matrix is a square matrix.

[^26]REMARK 2.8.1. A square, weighted homogeneous polynomial $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ has the form (modulo coefficients),

$$
\begin{equation*}
f=x^{a_{11}} y^{a_{12}}+x^{a_{21}} y^{a_{22}} \quad\left\{a_{i j}\right\} \subset \mathbb{Z}_{\geqslant 0} \tag{2.147}
\end{equation*}
$$

and, if $\mathrm{A}_{f}=\left(a_{i j}\right)$ is non-singular, has the following weights,

$$
\begin{equation*}
\omega_{1}=\frac{a_{22}-a_{12}}{a_{11} a_{22}-a_{12} a_{21}} \quad \text { and } \quad \omega_{2}=\frac{a_{11}-a_{21}}{a_{11} a_{22}-a_{12} a_{21}} . \tag{2.148}
\end{equation*}
$$

Similarly, a square, weighted homogeneous polynomial $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ has the form (modulo coefficients),

$$
\begin{equation*}
f=x^{a_{11}} y^{a_{12}} z^{a_{13}}+x^{a_{21}} y^{a_{22}} z^{a_{23}}+x^{a_{31}} y^{a_{32}} z^{a_{33}} \quad\left\{a_{i j}\right\} \subset \mathbb{Z}_{\geqslant 0} \tag{2.149}
\end{equation*}
$$

and, if $\mathrm{A}_{f}=\left(a_{i j}\right)$ is non-singular, has the following weights,

$$
\begin{align*}
& \omega_{1}=\frac{a_{13} a_{22}-a_{12} a_{23}-a_{13} a_{32}+a_{23} a_{32}+a_{12} a_{33}-a_{22} a_{33}}{a_{13} a_{22} a_{31}-a_{12} a_{23} a_{31}-a_{13} a_{21} a_{32}+a_{11} a_{23} a_{32}+a_{12} a_{21} a_{33}-a_{11} a_{22} a_{33}}  \tag{2.150}\\
& \omega_{2}=\frac{a_{21} a_{33}-a_{11} a_{33}-a_{23} a_{31}+a_{13} a_{31}+a_{11} a_{23}-a_{13} a_{21}}{a_{13} a_{22} a_{31}-a_{12} a_{23} a_{31}-a_{13} a_{21} a_{32}+a_{11} a_{23} a_{32}+a_{12} a_{21} a_{33}-a_{11} a_{22} a_{33}}  \tag{2.151}\\
& \omega_{3}=\frac{a_{12} a_{21}-a_{11} a_{22}-a_{12} a_{31}+a_{22} a_{31}+a_{11} a_{32}-a_{21} a_{32}}{a_{13} a_{22} a_{31}-a_{12} a_{23} a_{31}-a_{13} a_{21} a_{32}+a_{11} a_{23} a_{32}+a_{12} a_{21} a_{33}-a_{11} a_{22} a_{33}} . \tag{2.152}
\end{align*}
$$

In both cases, the algebraic index has a representation involving only the exponents of the corresponding singularity, albeit unwieldy.

Remark 2.8.2. Consider $f=x z^{a}+x^{b} y+y^{c} z$ over $\mathbb{C}^{3}$, where $a, b, c \in \mathbb{N}$. The corresponding matrix of exponents is the $3 \times 3$ matrix,

$$
A_{f}=\left(\begin{array}{lll}
1 & 0 & a  \tag{2.153}\\
b & 1 & 0 \\
0 & c & 1
\end{array}\right)
$$

and since $\mathrm{A}_{f}$ is non-singular, $f$ has the following weights,

$$
\begin{equation*}
\omega_{1}=\frac{1-a+a c}{1+a b c}, \quad \omega_{2}=\frac{1-b+a b}{1+a b c} \quad \text { and } \quad \omega_{3}=\frac{1-c+b c}{1+a b c} \tag{2.154}
\end{equation*}
$$

The corresponding algebraic index is simply $\mu_{\mathrm{alg}}(f)=a b c$.
Proposition 2.76. Let $\mathrm{A}_{f}$ be the exponent matrix of a non-degenerate, square, weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$. If the local algebra $\mathcal{A}_{f}$ is positive dimensional, then $\mathrm{A}_{f}$ has full rank and satisfies

$$
\begin{equation*}
0<\left|\operatorname{det} \mathrm{A}_{f, i}\right|<\left|\operatorname{det} \mathrm{A}_{f}\right| \quad 1 \leqslant i \leqslant n . \tag{2.155}
\end{equation*}
$$

Proof. The inequality is a necessary consequence of the bounds $0<\omega<$ 1. The positive dimensionality of the local algebra $\mathcal{A}_{f}$ follows from equation (2.32C).

Remark 2.8.3. As in Remark 2.5.5, the polynomial $f=x^{2} y^{6}+x^{5} y$ implies that the converse of the Proposition 2.76 does not hold for degenerate singularities. That is, although $\mathrm{A}_{f}$ has rank 2, the determinantal inequality is insufficient to ensure positive dimensionality of the corresponding local algebra as, in this
case, $\operatorname{det} \mathrm{A}_{f}=-28, \operatorname{det} \mathrm{~A}_{f, 1}=-5$ and $\operatorname{det} \mathrm{A}_{f, 2}=-3$, which violate the converse.

Definition 2.77. A weighted homogeneous polynomial is reduced if and only if its integral weights and weighted degree equal the numerators and denominators of the corresponding reduced weights, respectively.

REMARK 2.8.4. A quasi-Brieskorn-Pham singularity is reduced if and only if it is a homogeneous polynomial.
2.8.1. Kobayashi Duality. There is a natural involution on the class of weighted homogeneous germs, namely, the transpose map which takes an exponent matrix to its transpose. Such a map then descends to a map between weighted homogeneous polynomials. Consider

$$
\left(\mathrm{A}_{f}\right)^{\top}=\left(\begin{array}{ccccccc}
1 & a_{2} & 0 & 0 & 0 & \cdots & 0  \tag{2.156}\\
0 & 1 & a_{3} & 0 & 0 & \cdots & 0 \\
0 & 0 & 1 & a_{4} & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \ddots & a_{n} \\
a_{1} & 0 & 0 & 0 & \cdots & 0 & 1
\end{array}\right)=\mathrm{A}_{\tilde{f}},
$$

where $\tilde{f}$ is the corresponding weighted homogeneous polynomial

$$
\begin{equation*}
\tilde{f}=z_{1}^{a_{1}} z_{n}+\sum_{i=1}^{n-1} z_{i} z_{i+1}^{a_{i+1}} \tag{2.157}
\end{equation*}
$$

which we shall call the Kobayashi dual of

$$
\begin{equation*}
f=z_{1} z_{n}^{a_{1}}+\sum_{i=1}^{n-1} z_{i}^{a_{i+1}} z_{i+1} \tag{2.158}
\end{equation*}
$$

Proposition 2.78. Let $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a square, weighted homogeneous singularity with non-singular exponent matrix $\mathrm{A}_{f}=\left(a_{i j}\right)$. Define the vector $\left(\mathbf{a}_{i}\right)_{j}=$ $\left(a_{i j}\right)$. The weights of the Kobayashi dual $\tilde{f}:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ are

$$
\begin{align*}
& \omega_{1}=\frac{\left(\mathbf{a}_{2} \times \mathbf{a}_{3}\right) \cdot \mathbf{1}_{3}}{\operatorname{det} \mathrm{~A}_{f}}  \tag{2.159}\\
& \omega_{2}=\frac{\left(\mathbf{a}_{3} \times \mathbf{a}_{1}\right) \cdot \mathbf{1}_{3}}{\operatorname{det} \mathrm{~A}_{f}}  \tag{2.160}\\
& \omega_{3}=\frac{\left(\mathbf{a}_{1} \times \mathbf{a}_{2}\right) \cdot \mathbf{1}_{3}}{\operatorname{det} \mathrm{~A}_{f}} \tag{2.161}
\end{align*}
$$

Proof. This is an exercise is elementary linear algebra. See [120].
We mention briefly that the duality map is integral to establishing a curious manifestation of Mirror Symmetry with Arnol'd's Strange Duality relating the Dolgachev and Gabrielov numbers of certain non-degenerate, weighted homogeneous singularities [23]. For details, consult [240], [119] and [120].

### 2.8.2. Weight Preserving Maps.

Definition 2.79. Two matrices $A$ and $B$ are permutation equivalent if and only if there are permutation matrices $\mathrm{P}_{r}$ and $\mathrm{P}_{c}$, acting on the rows and columns, respectively, such that $A=P_{r} \mathrm{BP}_{c}$.

If two matrices $A$ and $B$ are permutation equivalent, write $A \simeq B$. We define a topologically trivial morphism on the space of singularities.

Definition 2.80. Two singularities are permutation equivalent if and only if the corresponding exponent matrices are permutation equivalent.

Write $f \cong g$ if and only if $\mathrm{A}_{f} \simeq \mathrm{~A}_{g}$. The following proposition is obvious.

Proposition 2.81. Given two weighted homogeneous singularities $f$ and $g$, if $f \cong g$, then the weights of $f$ and $g$ coincide up to permutation.

Proof. If $A_{f} \simeq A_{g}$, then there are permutation matrices $\mathrm{P}_{r}$ and $\mathrm{P}_{c}$ such that $\mathrm{A}_{f}=\mathrm{P}_{r} \mathrm{~A}_{g} \mathrm{P}_{c}$. Thus, $\mathbf{1}=\mathrm{A}_{f} \boldsymbol{\omega}=\left(\mathrm{P}_{r} \mathrm{~A}_{g} \mathrm{P}_{c}\right) \boldsymbol{\omega}=\left(\mathrm{P}_{r} \mathrm{~A}_{g}\right)\left(\mathrm{P}_{c} \boldsymbol{\omega}\right)$. Since $\mathrm{P}_{r} \mathrm{~A}_{g}$ represents the exponent matrix of $g$ (with permuted monomials but not permuted variables), then it follows that $\boldsymbol{\omega}$ is the weight multiset of $g$ (up to permutation).

Corollary 2.82. Permutation equivalence implies right equivalence.

Proof. The required biholomorphism is simply a permutation of variables.

The converse of Corollary 2.82 is not true, however. We shall return to this topic in §2.9.2. In the meantime, consider the following non-linear map.

Proposition 2.83. Given a square singularity $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with nonsingular exponent matrix $\mathrm{A}_{f}=\left(a_{i j}\right)$, if there is a non-trivial, integral solution $(\alpha, \beta)$
satisfying the system of Diophantine inequalities,

$$
\begin{array}{r}
\min \left\{a_{11}, a_{21}\right\}+\alpha \geqslant 0 \\
\min \left\{a_{12}, a_{22}\right\}+\beta \geqslant 0 \\
\left(a_{11}-a_{21}\right) \beta+\left(a_{22}-a_{12}\right) \alpha=0 \tag{2.164}
\end{array}
$$

then there is a square singularity $g:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with non-singular exponent matrix $\mathrm{A}_{g}=\left(b_{i j}\right)$ and a translation map $T_{\alpha, \beta}$ preserving the weights,

$$
T_{\alpha, \beta}\left(\begin{array}{ll}
a_{11} & a_{12}  \tag{2.165}\\
a_{21} & a_{22}
\end{array}\right) \mapsto\left(\begin{array}{ll}
b_{11} & b_{12} \\
b_{21} & b_{22}
\end{array}\right)=\left(\begin{array}{ll}
a_{11}+\alpha & a_{12}+\beta \\
a_{21}+\alpha & a_{22}+\beta
\end{array}\right) .
$$

Proof. The first two inequalities ensure that the image of $T_{\alpha, \beta}$ is an exponent matrix. The last equality holds if and only if the $T_{\alpha, \beta}$ preserves the determinant. Finally, under the same map, the following equalities hold, $a_{11}-a_{21}=b_{11}-b_{21}$ and $a_{22}-a_{12}=b_{22}-b_{12}$, so the weights remain unchanged by Cramer's Rule.

Consider the line $\ell$ defined by the locus

$$
\begin{equation*}
\left\{(\alpha, \beta) \in \mathbb{R}^{2} \mid\left(a_{11}-a_{21}\right) \beta+\left(a_{22}-a_{12}\right) \alpha=0\right\} \tag{2.166}
\end{equation*}
$$

which has slope $\frac{a_{12}-a_{22}}{a_{11}-a_{21}}$ in the $(\alpha, \beta)$-plane. The constraints $\alpha \geqslant-\min \left\{a_{11}, a_{21}\right\}$ and $\beta \geqslant-\min \left\{a_{12}, a_{22}\right\}$ define a region in the plane. For finitely many solutions, the line must intersect said region in at least one point. The intersection is defined by the extremal points $\alpha_{\max }=\frac{a_{21}-a_{11}}{a_{12}-a_{22}} \min \left\{a_{12}, a_{22}\right\}$ and
$\beta_{\max }=\frac{a_{22}-a_{12}}{a_{11}-a_{21}} \min \left\{a_{11}, a_{21}\right\}$. In particular, if $\min \left\{a_{12}, a_{22}\right\}=\min \left\{a_{12}, a_{22}\right\}=0$, $a_{11} \neq a_{21}$ and $a_{12} \neq a_{22}$, the latter two yielding a finite and non-zero slope, then there is only the trivial solution at the origin. There are positively many solutions if and only if the slope of $\ell$ is negative and intersects the region nontrivially. There are infinitely many solutions if and only if the slope of $\ell$ is zero, positive or infinite and intersects the region non-trivially.

This technique does not distinguish between degenerate and non-degenerate singularities. It conserves only the weights, as the following two remarks illustrate.

REMARK 2.8.5. Consider the quasi-Brieskorn-Pham singularity $f=x^{3}+x y^{4}$ over $C^{2}$, which is non-degenerate and has weights $\left\{\frac{1}{3}, \frac{1}{6}\right\}$. There are two integral points on the line $2 \beta+4 \alpha=0$ in the planar region $\left\{(\alpha, \beta) \in \mathbb{R}^{2} \mid \alpha \geqslant-1 \wedge \beta \geqslant 0\right\}$, namely, the origin and the point $(-1,2)$. It follows that $g=x^{2} y^{2}+y^{6}$ has the same weights as those of $f$. However, $g$ is degenerate, as $\partial g=\left(2 x y^{2}, 2 x^{2} y+\right.$ $\left.6 y^{5}\right)$, and there is a continuum of critical points on $\left\{(x, 0) \in \mathbb{C}^{2} \mid x \in \mathbb{C}\right\}$. Applying the same technique to $g$ yields $f$.

REmARK 2.8.6. Consider $f=x^{10} y+x^{5} y^{7}$ over $\mathbb{C}^{2}$. There are two integral points on the line $5 \beta+6 \alpha=0$ in the planar region $\left\{(\alpha, \beta) \in \mathbb{R}^{2} \mid \alpha \geqslant-5 \wedge \beta \geqslant\right.$ $-1\}$, namely, the origin and the point $(-5,6)$. It follows that $g=x^{5} y^{7}+y^{13}$ has the same weights as those of $f$, namely, $\left\{\frac{6}{65}, \frac{1}{13}\right\}$. However, both $f$ and $g$ are degenerate. Applying the same technique to $g$ yields $f$.

Proposition 2.83 generalizes to three dimensions, where one determines whether a plane intersects a certain solid region. The solutions of the Diophantine system of inequalities correspond to lattice points of this intersection, which is a 2-simplex. The details are similar to those of the two-dimensional case, so we omit further discussion.

Proposition 2.84. Given a square singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with nonsingular exponent matrix $\mathrm{A}_{f}=\left(a_{i j}\right)$, if there is a non-trivial, integral solution $(\alpha, \beta, \gamma)$ satisfying the system of Diophantine inequalities,

$$
\begin{array}{r}
\min \left\{a_{11}, a_{21}, a_{31}\right\}+\alpha \geqslant 0 \\
\min \left\{a_{12}, a_{22}, a_{32}\right\}+\beta \geqslant 0 \\
\min \left\{a_{13}, a_{23}, a_{33}\right\}+\gamma \geqslant 0 \\
A \alpha+B \beta+C \gamma=0 \tag{2.170}
\end{array}
$$

where

$$
\begin{align*}
& A=a_{13} a_{22}-a_{12} a_{23}-a_{13} a_{32}+a_{23} a_{32}+a_{12} a_{33}-a_{22} a_{33}  \tag{2.171}\\
& B=a_{21} a_{33}-a_{11} a_{33}-a_{23} a_{31}+a_{13} a_{31}+a_{11} a_{23}-a_{13} a_{21}  \tag{2.172}\\
& C=a_{12} a_{21}-a_{11} a_{22}-a_{12} a_{31}+a_{22} a_{31}+a_{11} a_{32}-a_{21} a_{32} \tag{2.173}
\end{align*}
$$

then there is a square singularity $g:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with non-singular exponent matrix $\mathrm{A}_{g}=T_{\alpha, \beta, \gamma} \mathrm{A}_{f}$ and a translation map $T_{\alpha, \beta, \gamma}$ preserving the weights,

$$
T_{\alpha, \beta, \gamma}\left(\begin{array}{lll}
a_{11} & a_{12} & a_{13}  \tag{2.174}\\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{array}\right) \mapsto\left(\begin{array}{lll}
a_{11}+\alpha & a_{12}+\beta & a_{13}+\gamma \\
a_{21}+\alpha & a_{22}+\beta & a_{23}+\gamma \\
a_{31}+\alpha & a_{32}+\beta & a_{33}+\gamma
\end{array}\right) .
$$

Proof. We defer the proof for the general case in Proposition 2.86.
There is no doubt that the astute reader has recognized the coefficients of the Diophantine system of inequalities (cf., equations (2.150) through (2.152)). To proceed to the general case, we require an auxiliary result.

Proposition 2.85. Given $N n \times n$ matrices $A^{1}, \ldots, A^{N}$,

$$
\begin{equation*}
\operatorname{det}\left(\sum_{i=1}^{N} \mathrm{~A}^{i}\right)=\sum_{\pi \in[N]^{[n]}} \operatorname{det} \mathrm{A}^{\pi} \tag{2.175}
\end{equation*}
$$

where $[N]^{[n]}=\{\pi:\{1, \ldots, n\} \mapsto\{1, \ldots, N\}\}$ and $\left(\mathrm{A}^{\pi}\right)_{i j}=\mathrm{A}_{i j}^{\pi(i)}$.
Proof. The identity follows from repeatedly expanding by minors.

Proposition 2.86. Given a square singularity $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with non-singular exponent matrix $\mathrm{A}_{f}=\left(a_{i j}\right)$, if there is a non-trivial, integral solution
$\left(\alpha_{0}, \ldots, \alpha_{n}\right)$ satisfying the system of Diophantine inequalities,

$$
\begin{align*}
\min \left\{a_{0 j}, \ldots, a_{n j}\right\}+\alpha_{j} & \geqslant 0 \quad 0 \leqslant j \leqslant n  \tag{2.176}\\
\sum_{i=0}^{n}\left(\operatorname{det} \mathrm{~A}_{f, i}\right) \alpha_{i} & =0 \tag{2.177}
\end{align*}
$$

then there is a square singularity $g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with non-singular exponent matrix $\mathrm{A}_{g}$ and a translation map $T_{\boldsymbol{\alpha}}$ preserving the weights of $f$ such that

$$
\begin{equation*}
T_{\boldsymbol{\alpha}} \mathrm{A}_{f}=\mathrm{A}_{g}=\mathrm{A}_{f}+J_{n+1} \boldsymbol{\alpha} \tag{2.178}
\end{equation*}
$$

where $J_{n}$ denotes the $n \times n$ matrix of 1 s and $\boldsymbol{\alpha}=\left(\alpha_{0}, \ldots, \alpha_{n}\right)^{\top}$.

Proof. Cramer's rule implies

$$
\begin{align*}
\frac{\operatorname{det} \mathrm{A}_{g, i}}{\operatorname{det} \mathrm{~A}_{g}} & =\frac{\operatorname{det}\left(\mathrm{A}_{f}+J_{n} \boldsymbol{\alpha}\right)_{i}}{\operatorname{det}\left(\mathrm{~A}_{f}+J_{n} \boldsymbol{\alpha}\right)}  \tag{2.179}\\
& =\frac{\operatorname{det}\left(\mathrm{A}_{f, i}+\left(J_{n} \boldsymbol{\alpha}\right)^{(i)}\right)}{\operatorname{det} \mathrm{A}_{f}+\sum_{i=0}^{n}\left(\operatorname{det} \mathrm{~A}_{f, i}\right) \alpha_{i}}  \tag{2.180}\\
& =\frac{\operatorname{det} \mathrm{A}_{f, i}}{\operatorname{det} \mathrm{~A}_{f}} \tag{2.181}
\end{align*}
$$

by two applications of equation (2.175), where $\left(J_{n} \alpha\right)^{(i)}$ is the matrix $J_{n} \alpha$ with 0 s replacing the $i^{\text {th }}$-column.

Conjecture 2.87. A square singularity over $\mathbb{C}^{n+1}$ has at least one weight outside of the interval $\left(0, \frac{1}{2}\right]$ if and only if the corresponding Diophantine system of inequalities specified in Proposition 2.86 has infinitely many solutions. Otherwise, all
weights are in said interval if and only if the number of solutions is finite, including the trivial solution.

Remark 2.8.7. The conjecture makes no claim about degeneracy or the integrality of the corresponding algebraic index.

Remark 2.8.8. There are other transformations that preserve the weights. For example, given a weighted homogeneous polynomial $f$, if there is a weighted homogeneous polynomial $g$ and matrix $T$ such that $\mathrm{A}_{g}=\mathrm{TA}_{f}$, where $\mathrm{T} \mathbf{1}=\mathbf{1}$, then $\mathrm{A}_{g} \boldsymbol{\omega}=\mathrm{T}\left(\mathrm{A}_{f} \boldsymbol{\omega}\right)=\mathbf{1}$, and $f$ and $g$ share the same weights.

REMARK 2.8.9. In §9.2, we generalize some of these results to weighted homogeneous polynomials with non-square exponent matrices.

### 2.9. Algebraic Morphisms on Exponent Matrices

2.9.1. Sebastiani-Thom Equivalence, Revisited. Let $f_{\alpha}: U_{\alpha} \rightarrow \mathbb{C}$ be a complex analytic function with domain $U_{\alpha} \subset \mathbb{C}^{n_{\alpha}}$ in a neighborhood of the origin. Define the projection $\pi_{\alpha^{\prime}}: \prod_{\alpha} U_{\alpha} \rightarrow U_{\alpha^{\prime}}$. Recall the Sebastiani-Thom summation $\square_{\alpha} f_{\alpha}=\sum_{\alpha} f_{\alpha} \circ \pi_{\alpha}$ with product domain $\prod_{\alpha} U_{\alpha} \subset \mathbb{C}^{\sum_{\alpha} n_{\alpha}}$ such that $\pi_{\alpha^{\prime}}\left(\boxplus_{\alpha} f_{\alpha}\right)=f_{\alpha^{\prime}}$.

If $f \in \mathcal{O}_{0, n}$ and $g \in \mathcal{O}_{0, m}$ are weighted homogeneous with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ and $\left\{v_{0}, \ldots, v_{m}\right\}$, respectively, then $f \boxplus g \in \mathcal{O}_{0, n+m+1}$ is weighted homogeneous with weights $\left\{\omega_{0}, \ldots, \omega_{n}, v_{0}, \ldots, v_{m}\right\}$, which is compatible with equation (2.3). With regard to the corresponding exponent matrices, one has the
following direct sum decomposition,

$$
\begin{equation*}
\mathrm{A}_{f \boxplus g}=\mathrm{A}_{f} \oplus \mathrm{~A}_{g} . \tag{2.182}
\end{equation*}
$$

If $\mathrm{A}_{f}$ and $\mathrm{A}_{g}$ are non-singular, square matrices, then $\mathrm{A}_{f \boxplus g}$ is also nonsingular by the determinant identity $\operatorname{det}(A \oplus B)=(\operatorname{det} A)(\operatorname{det} B)$. The converse, however, is not true. Moreover, if $f$ and $g$ are non-degenerate, then $f \boxplus g$ is nondegenerate. The converse, however, is not true.

REMARK 2.9.1. Consider $f=x^{2} y^{6}+x^{5} y$ over $\mathbb{C}^{2}$, which is degenerate, q.v., Remark 2.5.5, and $h=f \boxplus z^{4}$ over $\mathbb{C}^{3}$ with exponent matrix

$$
A_{h}=\left(\begin{array}{lll}
2 & 6 & 0  \tag{2.183}\\
5 & 1 & 0 \\
0 & 0 & 4
\end{array}\right)=A_{f} \oplus(4)
$$

Clearly, $z^{4}$ is non-degenerate. However, as $\partial h=\left(5 x^{4} y+2 x y^{6}, x^{5}+6 x^{2} y^{5}, 4 z^{3}\right)$, then $h$ has a continuum of critical points on $\left\{(0, y, 0) \in \mathbb{C}^{3} \mid y \in \mathbb{C}\right\}$. Hence, $h$ is also degenerate. Note that the Hilbert-Poincaré series $P_{\mathcal{A}_{f}}(t)$ is not a reflexive $\mathbb{Z}_{\geqslant 0}$-polynomial (or even a polynomial at all), while that of $\mathcal{A}_{h}$ is a reflexive $\mathbb{Z}_{\geqslant 0}$-polynomial of degree 54 . By the multiplicativity of the algebraic index over Sebastiani-Thom summation, $\mu_{\mathrm{alg}}\left(f \boxplus z^{4}\right)=\left(\frac{115}{3}\right)(3)=115$, which is an integer that coincides with the limit $\lim _{t \rightarrow 1} P_{\mathcal{A}_{h}}(t)$.
2.9.2. Kronecker Products. In this section we discuss a tensor product acting on the vector space $\operatorname{Mat}(\mathbb{F})$ of matrices over a field $\mathbb{F}$ of arbitrary characteristic. We focus specifically on the (simple, associative) matrix algebra $\operatorname{Mat}(n, \mathbb{F})$ of square matrices of size $n \in \mathbb{N}$. We define complementary operations on the space of weighted homogeneous polynomials and discuss certain invariance properties thereof. We refer the reader to [212,213] and [68] for details and relevant proofs.

Definition 2.88. The Kronecker product of two matrices $A \in \mathbb{F}^{m, n}$ and $\mathrm{B} \in \mathbb{F}^{r, s}$ is the matrix

$$
\mathrm{A} \otimes \mathrm{~B}=\left(\begin{array}{ccc}
a_{11} \mathrm{~B} & \ldots & a_{1 n} \mathrm{~B}  \tag{2.184}\\
a_{21} \mathrm{~B} & \ldots & a_{2 n} \mathrm{~B} \\
\vdots & \ddots & \vdots \\
a_{m 1} \mathrm{~B} & \ldots & a_{m n} \mathrm{~B}
\end{array}\right) \in \mathbb{F}^{m r, n s}
$$

The Kronecker product satisfies the following properties [213]:

1. (Scalar Multiplication) For each $\lambda \in \mathbb{F}, \lambda(\mathrm{A} \otimes \mathrm{B})=\lambda \mathrm{A} \otimes B=\mathrm{A} \otimes \lambda B$;
2. (Distributivity over Transpose) $(A \otimes B)^{\top}=A^{\top} \otimes B^{\top}$;

3a. (Left Distributivity over Sum) $A \otimes(B+C)=A \otimes B+A \otimes C$;
3b. (Right Distributivity over Sum) $(A+B) \otimes C=A \otimes C+B \otimes C$; and,
4. (Associativity) $(A \otimes B) \otimes C=A \otimes(B \otimes C)$,
all of which are rather simple to prove by direct calculation. If $A$ and $B$ are invertible matrices, then so is $A \otimes B$ with inverse $(A \otimes B)^{-1}=A^{-1} \otimes B^{-1}$. In
particular, if A and B are non-singular matrices of size $n \times n$ and $m \times m$, respectively, then $A \otimes B$ is non-singular by the $\operatorname{determinant}$ identity $\operatorname{det}(A \otimes B)=$ $(\operatorname{det} \mathrm{A})^{m}(\operatorname{det} \mathrm{~B})^{n}$. The converse, however, is not true.

For matrices $A, B, C$ and $D$ such that the matrix products $A C$ and $B D$ are welldefined, the equality $(A \otimes B)(C \otimes D)=(A C) \otimes(B D)$ holds and is known as the mixed product identity. According to Knuth [245], the second and last two identities were shown by Hurwitz [216].

Let $I_{n}$ denote the (multiplicative) identity matrix of size $n$ and $Z_{r, s}$ denote the $r \times s$ matrix of zeros. A few special cases include: $\mathrm{I}_{n} \otimes \mathrm{I}_{m}=\mathrm{I}_{n m}, \mathrm{I}_{m} \otimes \mathrm{~A}=\oplus_{i=1}^{m} \mathrm{~A}$ with $1 \otimes A=A \otimes 1=A$ and $A \otimes Z_{r, s}=Z_{r, s} \otimes A=Z_{m r, n s}$ for any $m \times n$ matrix $A$.

In general, $A \otimes B \neq B \otimes A$ for arbitrary $A, B$, so the Kronecker product is not a commutative operation on matrices. However, the following equivalence holds.

Proposition 2.89. For matrices $\mathrm{A}, \mathrm{B}$, there are two permutation matrices $\mathrm{P}_{r}$ and $\mathrm{P}_{c}$ such that $\mathrm{A} \otimes \mathrm{B}=\mathrm{P}_{r}(\mathrm{~B} \otimes \mathrm{~A}) \mathrm{P}_{c}$, and the pair $(\mathrm{A} \otimes \mathrm{B}, \mathrm{B} \otimes \mathrm{A})$ is permutation equivalent. In particular, if A and B have the same size, then $\mathrm{P}_{c}=\mathrm{P}_{r}^{\top}$, and the pair $(A \otimes B, B \otimes A)$ is permutation-similar.

Proof. The claim follows from direct computation.
Equipped with the Kronecker product, putatively more complicated weighted homogeneous polynomials can be generated from simpler ones.

Definition 2.90. The Kronecker product $f$ Q $g$ of weighted homogeneous polynomials $f$ and $g$ is the weighted homogeneous polynomial with unit coefficients and exponent matrix $\mathrm{A}_{f} \otimes \mathrm{~A}_{g}$.

REMARK 2.9.2. If $f=x^{a} y^{b}+x^{c} y^{d}$ and $g=x^{a^{\prime}} y^{b^{\prime}}+x^{c^{\prime}} y^{d^{\prime}}$, then

$$
\begin{align*}
& f \boxtimes g=x^{a a^{\prime}} y^{a b^{\prime}} z^{b a^{\prime}} w^{b b^{\prime}}+x^{a c^{\prime}} y^{a d^{\prime}} z^{b c^{\prime}} w^{b d^{\prime}} \\
& \quad+x^{c a^{\prime}} y^{c b^{\prime}} z^{d a^{\prime}} w^{d b^{\prime}}+x^{c c^{\prime}} y^{c d^{\prime}} z^{d c^{\prime}} w^{d d^{\prime}}  \tag{2.185a}\\
& g \boxtimes f=x^{a a^{\prime}} y^{b a^{\prime}} z^{a b^{\prime}} w^{b b^{\prime}}+x^{c a^{\prime}} y^{d a^{\prime}} z^{c b^{\prime}} w^{d b^{\prime}} \\
& \quad+x^{a c^{\prime}} y^{b c^{\prime}} z^{a d^{\prime}} w^{b d^{\prime}}+x^{c c^{\prime}} y^{d c^{\prime}} z^{c d^{\prime}} w^{d d^{\prime}} \tag{2.185b}
\end{align*}
$$

and $f \boxtimes g \cong g \boxtimes f$. In particular, if $f=x^{a}+y^{d}$ and $g=x^{a^{\prime}}+y^{d^{\prime}}$, then

$$
\begin{align*}
& f \boxtimes g=x^{a a^{\prime}}+y^{a d^{\prime}}+z^{d a^{\prime}}+w^{d d^{\prime}}  \tag{2.186}\\
& g \boxtimes f=x^{a a^{\prime}}+y^{d a^{\prime}}+z^{a d^{\prime}}+w^{d d^{\prime}} \tag{2.187}
\end{align*}
$$

Moreover, for powers, in particular, the Kronecker product coincides with ordinary exponentiation, viz., $z^{a} \boxtimes z^{b}=z^{a b}$.

Remark 2.9.3. A Kronecker product need not be square.

For an $m \times n$ matrix $A$, let $\hat{A}$ denote any $m \times n$ matrix PA, where P is a permutation matrix. Clearly, $A \simeq \hat{A}$.

Proposition 2.91. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a neighborhood of the origin. Assume that the complex analytic map $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted
homogeneous singularity. Then $f=\mathbb{X}_{\alpha=1}^{s} f_{\alpha}:\left(\otimes_{\alpha=1}^{s} U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is weighted homogeneous over $\prod_{\alpha=1}^{S} n_{\alpha}$ complex variables with weights $\left\{\omega_{i_{1}} \cdots \omega_{i_{s}}\right\}$, where $1 \leqslant$ $i_{\alpha} \leqslant n_{\alpha}$ and $1 \leqslant \alpha \leqslant s$.

Proof. Consider two weighted homogeneous polynomials, $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ and $g:\left(\mathbb{C}^{m+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, with $n^{\prime}$ and $m^{\prime}$ monomials, respectively. Recall $\mathbf{1}_{n}$ denotes the vector consisting of $n 1 \mathrm{~s}$. Since $\mathrm{A}_{f} \boldsymbol{\omega}=\mathbf{1}_{n^{\prime}}$ and $\mathrm{A}_{g} \boldsymbol{v}=\mathbf{1}_{m^{\prime}}$, then

$$
\begin{align*}
\mathbf{1}_{n^{\prime} m^{\prime}} & =\mathbf{1}_{n^{\prime}} \otimes \mathbf{1}_{m^{\prime}}  \tag{2.188a}\\
& =\left(\mathrm{A}_{f} \boldsymbol{\omega}\right) \otimes\left(\mathrm{A}_{g} \boldsymbol{v}\right)  \tag{2.188b}\\
& =\left(\mathrm{A}_{f} \otimes \mathrm{~A}_{g}\right)(\boldsymbol{\omega} \otimes \boldsymbol{v})  \tag{2.188c}\\
& =\mathrm{A}_{f \boxtimes g}(\boldsymbol{\omega} \otimes \boldsymbol{v}) \tag{2.188d}
\end{align*}
$$

by the mixed product identity. It follows that $f$ $\mathcal{g}$ is a weighted homogeneous polynomial with exponent matrix $\mathrm{A}_{f} \otimes \mathrm{~A}_{g}$ and weight $\boldsymbol{\omega} \otimes \boldsymbol{v}$. In this case, $f \boxtimes g$ satisfies the identity

$$
\begin{equation*}
(f \boxtimes g)\left(z_{0}, \ldots, z_{n m+n+m}\right)=\lambda^{-1}(f \boxtimes g)\left(\lambda^{\omega_{0} v_{0}} z_{0}, \ldots, \lambda^{\omega_{n} v_{m}} z_{n m+n+m}\right) \tag{2.189}
\end{equation*}
$$

where $\lambda \in \mathbb{C}^{\times}$. Observe that there are two permutation matrices $\mathrm{P}_{r}$ and $\mathrm{P}_{c}$ such that

$$
\begin{align*}
\mathrm{A}_{f \boxtimes g}(\boldsymbol{\omega} \otimes \boldsymbol{v}) & =\left(\mathrm{P}_{r}\left(\mathrm{~A}_{g} \otimes \mathrm{~A}_{f}\right) \mathrm{P}_{c}\right)(\boldsymbol{\omega} \otimes \boldsymbol{v})  \tag{2.190a}\\
& =\left(\mathrm{P}_{r}\left(\mathrm{~A}_{g} \otimes \mathrm{~A}_{f}\right)\right)\left(\mathrm{P}_{c}(\boldsymbol{\omega} \otimes \boldsymbol{v})\right)  \tag{2.19ob}\\
& =\widehat{\mathrm{A}_{g \boxtimes f}}(\widehat{\boldsymbol{\omega} \otimes \boldsymbol{v}}) . \tag{2.190c}
\end{align*}
$$

Since the matrix $\mathrm{A}_{g \boxtimes f}$ is defined up to an action of $S_{n^{\prime} m^{\prime}} \times S_{(n+1)(m+1)}$ (by permuting the monomials and/or relabeling of the variables), then $\widehat{\mathrm{A}_{g \boxtimes f}} \simeq \mathrm{~A}_{g \boxtimes f}$. Similarly, since the weight $\boldsymbol{\omega} \otimes \boldsymbol{v}$ is a multiset, then $\widehat{\boldsymbol{\omega} \otimes \boldsymbol{v}} \simeq \boldsymbol{\omega} \otimes \boldsymbol{v}$. Thus, both $\mathrm{A}_{f \boxtimes g}$ and $\mathrm{A}_{g \boxtimes f}$ define the same weighted homogeneous polynomial up to an action of $S_{n^{\prime} m^{\prime}} \times S_{(n+1)(m+1)}$ and therefore $g \boxtimes f \cong f \boxtimes g$. This completes the proof.

Remark 2.9.4. Proposition 2.91 makes no claim about the critical locus of the Kronecker product singularity $f \boxtimes g$.

Since the set of diagonal matrices is closed under the operation of Kronecker product, it follows that the set of Brieskorn-Pham polynomials is closed under $\boxtimes$. More generally, since the Kronecker product of weight multisets preserves inverse integrality, the set of quasi-Brieskorn-Pham polynomials is closed under囚.

Definition 2.92. Given a weighted homogeneous polynomial $f$, the $t$ dilate of $f$ is $f_{t}=f\left(z_{0}^{t}, \ldots, z_{n}^{t}\right)$, where $t \in \mathbb{N}$.

REMARK 2.9.5. It follows that $f_{t}=f \boxtimes z^{t}$. In fact, the numerical invariants of $f_{t}$ are related to those of $f$ and many have combinatorial interpretations in terms of $t$-dilates of the Newton and weight polytopes of $f$, q.v., Chapter $5 . \Delta$

Given two weighted homogeneous polynomials $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ and $g:\left(\mathbb{C}^{m+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with local algebras $\mathcal{A}_{f}$ and $\mathcal{A}_{g}$, respectively, define the Kronecker product $\mathcal{A}_{f} \boxtimes \mathcal{A}_{g}$ of the corresponding local algebras as

$$
\begin{align*}
\mathcal{A}_{f} \boxtimes \mathcal{A}_{g} & =\mathcal{O}_{0, n} / J_{\partial f} \boxtimes \mathcal{O}_{0, n} / J_{\partial g}  \tag{2.191}\\
& =\mathcal{O}_{0, n m+n+m} / J_{\partial(f \boxtimes g)} \tag{2.192}
\end{align*}
$$

with the corresponding commutative diagram


The projections $\pi_{1}^{*}: \mathcal{A}_{f \boxtimes g} \rightarrow \mathcal{A}_{f}$ and $\pi_{2}^{*}: \mathcal{A}_{f \boxtimes g} \rightarrow \mathcal{A}_{g}$ are defined through the projections $\pi_{1}: \mathrm{A}_{f} \otimes \mathrm{~A}_{g} \rightarrow \mathrm{~A}_{f}$ and $\pi_{2}: \mathrm{A}_{f} \otimes \mathrm{~A}_{g} \rightarrow \mathrm{~A}_{g}$, respectively. By
construction, there is an isomorphism $\mathcal{A}_{f \boxtimes g} \cong \mathcal{A}_{f} \boxtimes \mathcal{A}_{g}$ and, therefore,

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f \boxtimes g)=\operatorname{dim}_{\mathrm{C}} \mathcal{A}_{f} \boxtimes \mathcal{A}_{g} . \tag{2.193}
\end{equation*}
$$

We proceed to the computation of the algebraic index of a Kronecker product of weighted homogeneous polynomials.

Proposition 2.93. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a neighborhood of the origin. Assume that the complex analytic map $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, square singularity. The algebraic index of the Kronecker product $f=\searrow_{\alpha=1}^{s} f_{\alpha}:\left(\otimes_{\alpha=1}^{s} U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is the product

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\prod_{i_{1}=1}^{n_{1}} \cdots \prod_{i_{s}=1}^{n_{s}}\left(\frac{1}{\omega_{i_{1}} \cdots \omega_{i_{s}}}-1\right) \tag{2.194}
\end{equation*}
$$

REMARK 2.9.6. Define $\imath_{k}=\sum_{i=1}^{k} z_{i}$ for $k \geqslant 1$. The exponent matrix of $\imath_{k}$ is $\mathrm{A}_{l_{k}}=\mathrm{I}_{k}$. The identities $\mathrm{I}_{k} \otimes \mathrm{~A}=\oplus_{i=1}^{k} \mathrm{~A}$ and $f \boxtimes \imath_{k} \simeq \imath_{k} \boxtimes f$ and equation (2.3) imply

$$
\begin{align*}
\mu_{\mathrm{alg}}\left(f \boxtimes \iota_{k}\right) & =\mu_{\mathrm{alg}}\left(\sum_{f}^{k-1} f\right)  \tag{2.195a}\\
& =\mu_{\mathrm{ag}}(f \boxplus \cdots \boxplus f)  \tag{2.195b}\\
& =\mu_{\mathrm{ag}}(f)^{k} \tag{2.195c}
\end{align*}
$$

$\Delta$

Corollary 2.94. If $f$ and $g$ are two non-degenerate, weighted homogeneous polynomials, then the following identity holds:

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f \boxtimes g)=\mu_{\mathrm{alg}}(g \boxtimes f) \tag{2.196}
\end{equation*}
$$

Proof. The weight multisets of $f \boxtimes g$ and $g \boxtimes f$ are identical up to permutation.

Remark 2.9.7. If $\mu_{\mathrm{alg}}(f)$ and $\mu_{\mathrm{alg}}(g)$ are non-negative integers, it does not necessarily imply that $\mu_{\mathrm{alg}}(f \boxtimes g)$ is a non-negative integer. Consider $f=x^{2} y^{10}+x^{3} y^{5}$ over $\mathbb{C}^{2}$ with weights $\left\{\frac{1}{4}, \frac{1}{20}\right\}, g=x^{6} y^{10}+x^{8} y$ over $\mathbb{C}^{2}$ with weights $\left\{\frac{9}{74}, \frac{1}{37}\right\}$ and

$$
\begin{equation*}
f \boxtimes g=x^{12} y^{20} z^{60} w^{100}+x^{16} y^{2} z^{80} w^{10}+x^{18}(y z)^{30} w^{50}+x^{24} y^{3} z^{40} w^{5} \tag{2.197}
\end{equation*}
$$

over $\mathbb{C}^{4}$ with weights $\left\{\frac{9}{296}, \frac{1}{148}, \frac{9}{1480}, \frac{1}{740}\right\}$. Then $\mu_{\mathrm{alg}}(f)=57, \mu_{\mathrm{alg}}(g)=260$ and $\mu_{\mathrm{alg}}(f \boxtimes g)=\frac{15287451347}{27}$. Note that $f, g$ and $f \boxtimes g$ are degenerate.

Remark 2.9.8. If $\mu_{\mathrm{alg}}(f \boxtimes g)$ is a non-negative integer, then it does not necessarily imply that $\mu_{\mathrm{alg}}(f)$ and $\mu_{\mathrm{alg}}(g)$ are each non-negative integers. Consider $f=x^{6} y^{9}+x^{2} y^{8}$ over $\mathbb{C}^{2}$ with weights $\left\{-\frac{1}{30}, \frac{2}{15}\right\}, g=x^{10} y^{4}+x^{6} y^{8}$ over $\mathbb{C}^{2}$ with weights $\left\{\frac{1}{14}, \frac{1}{14}\right\}$ and

$$
\begin{equation*}
f \boxtimes g=x^{60} y^{24} z^{90} w^{36}+x^{36} y^{48} z^{54} w^{72}+x^{20} y^{8} z^{80} w^{32}+x^{12} y^{16} z^{48} w^{64} \tag{2.198}
\end{equation*}
$$

over $\mathbb{C}^{4}$ with weights $\left\{-\frac{1}{420},-\frac{1}{420}, \frac{1}{105}, \frac{1}{105}\right\}$. Then $\mu_{\mathrm{alg}}(f)=-\frac{403}{2}, \mu_{\mathrm{alg}}(g)=169$ and $\mu_{\mathrm{alg}}(f \boxtimes g)=1917038656$. Note that $f, g$ and $f \boxtimes g$ are degenerate.

Remark 2.9.9. If either $\mu_{\mathrm{alg}}(f)$ or $\mu_{\mathrm{alg}}(g)$ is zero, it does not necessarily imply that $\mu_{\mathrm{alg}}(f \boxtimes g)$ is zero. Consider $f=x^{9}+y$ over $\mathbb{C}^{2}$ with weights $\left\{\frac{1}{9}, 1\right\}$, $g=x^{3} y^{3}+y^{9}$ over $\mathbb{C}^{2}$ with weights $\left\{\frac{2}{9}, \frac{1}{9}\right\}$ and $f \boxtimes g=(x y)^{27}+y^{81}+(z w)^{3}+w^{9}$ over $\mathbb{C}^{4}$ with weights $\left\{\frac{2}{81}, \frac{1}{81}, \frac{2}{9}, \frac{1}{9}\right\}$. Then $\mu_{\mathrm{alg}}(f)=0, \mu_{\mathrm{alg}}(g)=28$ and $\mu_{\mathrm{alg}}(f \boxtimes$ $g)=88480$. Note that $f$ has no critical points at the origin, and $g$ and $f \boxtimes g$ are degenerate.

The Kronecker product factors over direct summations from the right but not from the left. The proof is simple yet instructive. Observe

$$
\begin{align*}
(A \otimes C) \oplus(B \otimes C) & =\left(\begin{array}{cc}
A \otimes C & 0 \\
0 & B \otimes C
\end{array}\right)  \tag{2.199a}\\
& =\left(\begin{array}{cc}
A \otimes C & 0 \otimes C \\
0 \otimes C & B \otimes C
\end{array}\right)=(A \oplus B) \otimes C \tag{2.199b}
\end{align*}
$$

which corrects a typographical error and generalizes Proposition 28 in [68]. In general, however,

$$
\mathrm{A} \otimes(\mathrm{~B} \oplus \mathrm{C})=\left(\begin{array}{ccc}
a_{11} \mathrm{~B} \oplus \mathrm{C} & \ldots & a_{1 n} \mathrm{~B} \oplus \mathrm{C}  \tag{2.200}\\
a_{21} \mathrm{~B} \oplus \mathrm{C} & \ldots & a_{2 n} \mathrm{~B} \oplus \mathrm{C} \\
\vdots & \ddots & \vdots \\
a_{m 1} \mathrm{~B} \oplus \mathrm{C} & \ldots & a_{m n} \mathrm{~B} \oplus \mathrm{C}
\end{array}\right) \neq\left(\begin{array}{cc}
\mathrm{A} \otimes \mathrm{~B} & \mathbf{0} \\
\mathbf{0} & \mathrm{~A} \otimes \mathrm{C}
\end{array}\right)
$$

the right side being $(A \otimes B) \oplus(A \otimes C)$. However, $A \otimes(B \oplus C) \simeq(B \oplus C) \otimes A$.

Proposition 2.95. Let $f, g$ and $h$ be weighted homogeneous polynomials. Then the following statements are true:

1. $(g \boxplus h) \boxtimes f=(g \boxtimes f) \boxplus(h \boxtimes f)$;
2. $f \boxtimes(g \boxplus h) \neq(f \boxtimes g) \boxplus(f \boxtimes h)$; however,
3. $f \boxtimes(g \boxplus h) \cong(f \boxtimes g) \boxplus(f \boxtimes h)$.

Therefore, $f \boxtimes(g \boxplus h) \cong(g \boxplus h) \boxtimes f \cong(h \boxplus g) \boxtimes f$.

Proof. The claim follows from the distributive identity of Kronecker products over Kronecker summations from the right, $\left(\mathrm{A}_{f} \oplus \mathrm{~A}_{g}\right) \otimes \mathrm{A}_{h}=\left(\mathrm{A}_{f} \otimes \mathrm{~A}_{h}\right) \oplus$ $\left(\mathrm{A}_{g} \otimes \mathrm{~A}_{h}\right)$ and the preceding discussion.

REMARK 2.9.10. Consider $f=x^{a}+y^{b}$ and $g=x^{a^{\prime}}+y^{b^{\prime}}$ over $\mathbb{C}^{2}$. Over $\mathbb{C}^{4}$,

$$
\begin{align*}
f \boxtimes g & =\left(x^{a} \boxplus x^{b}\right) \boxtimes\left(x^{a^{\prime}} \boxplus x^{b^{\prime}}\right)  \tag{2.201a}\\
& =\left(x^{a} \boxtimes\left(x^{a^{\prime}} \boxplus x^{b^{\prime}}\right)\right) \boxplus\left(x^{b} \boxtimes\left(x^{a^{\prime}} \boxplus x^{b^{\prime}}\right)\right)  \tag{2.201b}\\
& =\left(\left(x^{a} \boxtimes x^{a^{\prime}}\right) \boxplus\left(x^{a} \boxtimes x^{b^{\prime}}\right)\right) \boxplus\left(\left(x^{b} \boxtimes x^{a^{\prime}}\right) \boxplus\left(x^{b} \boxtimes x^{b^{\prime}}\right)\right)  \tag{2.201c}\\
& =x^{a a^{\prime}} \boxplus x^{a b^{\prime}} \boxplus x^{b a^{\prime}} \boxplus x^{b b^{\prime}}  \tag{2.201d}\\
& =x^{a a^{\prime}}+y^{a b^{\prime}}+z^{b a^{\prime}}+w^{b b^{\prime}} . \tag{2.201e}
\end{align*}
$$

Corollary 2.96. If $f, g$ and $h$ are non-degenerate, square singularities, then the corresponding algebraic indices satisfy the following identity,

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f \boxtimes(g \boxplus h))=\mu_{\mathrm{alg}}(f \boxtimes g) \mu_{\mathrm{alg}}(f \boxtimes h) \tag{2.202}
\end{equation*}
$$

Proof. The claim is a consequence of Proposition 2.95 and the multiplicativity of the algebraic index over Sebastiani-Thom summations, q.v., equation (2.3).

Corollary 2.97. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a neighborhood of the origin. Assume that the complex analytic map $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is quasi-Brieskorn-Pham polynomials with inverse weights $\left\{a_{i_{\alpha}}\right\}$. The algebraic index of Kronecker product $f=\boxtimes_{\alpha=1}^{s} f_{\alpha}:\left(\otimes_{\alpha=1}^{s} U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is the product

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\prod_{i_{1}=1}^{n_{1}} \cdots \prod_{i_{s}=1}^{n_{s}}\left(a_{i_{1}} \cdots a_{i_{s}}-1\right) \tag{2.203}
\end{equation*}
$$

which is zero if and only if $\mu_{\mathrm{alg}}\left(f_{\alpha}\right)=0$ for $1 \leqslant \alpha \leqslant s$.
REMARK 2.9.11. Consider the power $f^{N \boxtimes}=\bigotimes_{i=1}^{N} f$ and denote the set of consecutive positive integers $\{1, \ldots, n\}$ by the symbol $[n]$. It follows that

$$
\begin{align*}
\mu_{\mathrm{alg}}\left(f^{N \boxtimes}\right) & =\prod_{i_{1}=1}^{n} \cdots \prod_{i_{N}=1}^{n}\left(\frac{1}{\omega_{i_{1}} \cdots \omega_{i_{N}}}-1\right)  \tag{2.204a}\\
& =\prod_{i_{1}, \ldots, i_{N} \in[n]}\left(\frac{1}{\omega_{i_{1}} \cdots \omega_{i_{N}}}-1\right) . \tag{2.204b}
\end{align*}
$$

In particular, if $f=\sum_{i=1}^{n} z_{i}^{a_{i}}$, then

$$
\begin{equation*}
\mu_{\mathrm{alg}}\left(f^{N \boxtimes}\right)=\prod_{i_{1}, \ldots, i_{N} \in[n]}\left(a_{i_{1}} \cdots a_{i_{N}}-1\right) \tag{2.205}
\end{equation*}
$$

which is a priori a non-negative integer.
2.9.3. Kronecker Summation. The Kronecker sum* of two square matrices $\mathrm{A} \in \mathbb{C}^{n, n}$ and $\mathrm{B} \in \mathbb{C}^{m, m}$ is the square matrix $\mathrm{A} \odot \mathrm{B}=\mathrm{A} \otimes \mathrm{I}_{m}+\mathrm{I}_{n} \otimes \mathrm{~B} \in \mathbb{C}^{n m, n m}$, where $\otimes$ denotes the Kronecker product. In general, the Kronecker summation is not commutative.

Definition 2.98. The Kronecker summation $f \boxtimes g$ of square, weighted homogeneous polynomials $f$ and $g$ is the square, weighted homogeneous polynomial with unit coefficients and exponent matrix $\mathrm{A}_{f} \odot \mathrm{~A}_{g}$.

Remark 2.9.12. A Kronecker summation is necessarily square.

Equipped with the Kronecker sum, potentially more complicated weighted homogeneous polynomials can be generated from simpler ones.

REMARK 2.9.13. Consider $f=x^{a} y^{b}+x^{c} y^{d}$ and $g=x^{a^{\prime}} y^{b^{\prime}}+x^{c^{\prime}} y^{d^{\prime}}$, then

$$
\begin{align*}
& f \boxtimes g=x^{a+a^{\prime}} y^{b^{\prime}} z^{b}+x^{c^{\prime}} y^{a+d^{\prime}} w^{b}+x^{c} z^{d+a^{\prime}} w^{b^{\prime}}+y^{c} z^{c^{\prime}} w^{d+d^{\prime}}  \tag{2.206a}\\
& g \boxtimes f=x^{a+a^{\prime}} y^{b} z^{b^{\prime}}+x^{c} y^{d+a^{\prime}} w^{b^{\prime}}+x^{c^{\prime}} z^{a+d^{\prime}} w^{b}+y^{c^{\prime}} z^{c} w^{d+d^{\prime}} \tag{2.206b}
\end{align*}
$$

[^27]where $f \boxtimes g \cong g \boxtimes f$. If $f=x^{a}+y^{d}$ and $g=x^{a^{\prime}}+y^{d^{\prime}}$, then
\[

$$
\begin{align*}
& f \boxtimes g=x^{a+a^{\prime}}+y^{a+d^{\prime}}+z^{d+a^{\prime}}+w^{d+d^{\prime}}  \tag{2.207}\\
& g \boxtimes f=x^{a+a^{\prime}}+y^{d+a^{\prime}}+z^{a+d^{\prime}}+w^{d+d^{\prime}} \tag{2.208}
\end{align*}
$$
\]

Moreover, for powers, in particular, the direct summation coincides with ordinary multiplication, viz., $z^{a} \bullet z^{b}=z^{a+b}$.

Since the set of diagonal matrices is closed under the operation of Kronecker summation, it follows that the set of Brieskorn-Pham polynomials is closed under $\square$. However, the set of quasi-Brieskorn-Pham polynomials is not closed under the operation Kronecker summation, nor is the set of non-degenerate square singularities.

REMARK 2.9.14. Consider $f=x^{2}+x y^{4}$ over $\mathbb{C}^{2}$, which is non-degenerate, as $\partial f=\left(2 x+y^{4}, 3 x y^{3}\right)$, and has weights $\left\{\frac{1}{2}, \frac{1}{8}\right\}$. Consider the Kronecker summation of $f$ with itself, namely,

$$
\begin{equation*}
f \backsim f=x^{4}+x y^{6}+x z^{6}+y z w^{8} \tag{2.209}
\end{equation*}
$$

which is degenerate, as

$$
\begin{equation*}
\partial(f \boxminus f)=\left(4 x^{3}+y^{6}+z^{6}, 6 x y^{5}+z w w^{8}, 6 x z^{5}+w^{8} y, 8 y z w^{7}\right) \tag{2.210}
\end{equation*}
$$

and $f \boxtimes f$ has a continuum of critical points on $\left\{(0,0,0, w) \in \mathbb{C}^{4} \mid w \in \mathbb{C}\right\}$. Moreover, $f \boxtimes f$ has weights $\left\{\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{3}{32}\right\}$, so it is not quasi-Brieskorn-Pham.

REMARK 2.9.15. If $f$ and $g$ are square singularities with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ and $\left\{v_{0}, \ldots, v_{n}\right\}$, respectively, then it is not necessarily true that

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f \boxtimes g)=\prod_{i=1}^{n} \prod_{j=1}^{m}\left(\frac{1}{\omega_{i}}+\frac{1}{v_{j}}-1\right) \tag{2.211}
\end{equation*}
$$

despite an analogous identity for Kronecker products, i.e., equation (2.194). From initial studies, disentangling $\mu_{\mathrm{alg}}(f \boxtimes g)$ as a function of the weights of $f$ and $g$ seems rather unlikely, perhaps even impossible, except for a few isolated families of singularities.

Proposition 2.99. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a neighborhood of the origin. Assume that the complex analytic map $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a Brieskorn-Pham polynomial with exponents $\left\{a_{i_{\alpha}}\right\} \subset \mathbb{N}$. The algebraic index of Kronecker summation $f=\bullet_{\alpha=1}^{s} f_{\alpha}:\left(\otimes_{\alpha=1}^{s} U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is the product

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\prod_{i_{1}=1}^{n_{1}} \cdots \prod_{i_{s}=1}^{n_{s}}\left(a_{i_{1}}+\cdots+a_{i_{s}}-1\right) \tag{2.212}
\end{equation*}
$$

REMARK 2.9.16. If $f$ is Brieskorn-Pham with exponents $\left\{a_{1}, \ldots, a_{n}\right\} \subset \mathbb{N}$, then for any positive integer $m$,

$$
\begin{equation*}
\mu_{\mathrm{alg}}\left(f \boxminus \bullet_{i=1}^{m} z\right)=\prod_{i=1}^{n}\left(a_{i}+m-1\right) \tag{2.213}
\end{equation*}
$$

which is necessarily a positive integer.

REMARK 2.9.17. Consider the additive power $f^{N \boxminus}=\square_{i=1}^{N} f$. By Proposition 2.99, if $f=\sum_{i=1}^{n} z_{i}^{a_{i}}$, then

$$
\begin{equation*}
\mu_{\mathrm{alg}}\left(f^{N \boxminus}\right)=\prod_{i_{1}, \ldots, i_{N} \in[n]}\left(a_{i_{1}}+\cdots+a_{i_{N}}-1\right) \tag{2.214}
\end{equation*}
$$

which is necessarily a positive integer.

Define the Kronecker summation $\mathcal{A}_{f} \boxtimes \mathcal{A}_{g}$ of the corresponding local algebras of two weighted homogeneous polynomials $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ and $g:\left(\mathbb{C}^{m+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$,

$$
\begin{align*}
\mathcal{A}_{f} \boxtimes \mathcal{A}_{g} & =\mathcal{O}_{0, n} / J_{\partial f} \boxtimes \mathcal{O}_{0, n} / J_{\partial g}  \tag{2.215}\\
& =\mathcal{O}_{0, n m+n+m} / J_{\partial(f \boxminus g)} \tag{2.216}
\end{align*}
$$

with the corresponding commutative diagram


The projections $\pi_{1}^{*}: \mathcal{A}_{f \boxminus g} \rightarrow \mathcal{A}_{f}$ and $\pi_{2}^{*}: \mathcal{A}_{f \boxminus g} \rightarrow \mathcal{A}_{g}$ are defined through the projections $\pi_{1}: \mathrm{A}_{f} \boxminus \mathrm{~A}_{g} \rightarrow \mathrm{~A}_{f}$ and $\pi_{2}: \mathrm{A}_{f} \boxtimes \mathrm{~A}_{g} \rightarrow \mathrm{~A}_{g}$, respectively. By construction, there is an isomorphism $\mathcal{A}_{f \boxminus g} \cong \mathcal{A}_{f} \square \mathcal{A}_{g}$ and, therefore,

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f \boxminus g)=\operatorname{dim}_{\mathrm{C}} \mathcal{A}_{f} \sqsubset \mathcal{A}_{g} . \tag{2.217}
\end{equation*}
$$

### 2.9.4. Representative Graphs.

Definition 2.100. Let $G=(V, E)$ be a weighted, directed graph with vertex set $V$, edge set $E \subset V \times V$ and edge weights $\left\{e_{i j}\right\} \subset \mathbb{Z}_{\geqslant 0}$. The adjacency matrix $\mathrm{A}(G)=\left(a_{i j}\right)$ of $G$ is defined as follows:

$$
a_{i j}=\left\{\begin{array}{ccc}
e_{i j} & \text { if } & (i, j) \in E  \tag{2.218}\\
0 & \text { if } & (i, j) \notin E .
\end{array}\right.
$$

Definition 2.101. Two graphs are graph isomorphic if and only if there is a permutation matrix $P$ such that the corresponding adjacency matrices are $P$ similar, viz., $\mathrm{A}^{\prime}=\mathrm{PAP}^{-1}$. Two graphs are permutation equivalent if and only there are two permutation matrices P and Q such that the corresponding adjacency matrices are ( $\mathrm{P}, \mathrm{Q}$ )-similar, viz., $\mathrm{A}^{\prime}=\mathrm{PAQ}^{-1}$.

Remark 2.9.18. Isomorphic graphs are permutation equivalent, but the converse is not true.

Recall that two $n \times n$ matrices are permutation equivalent if and only if they equal up to an action of $S_{n} \times S_{n}$. Let [A] denote the equivalence class of graphs up to permutation equivalence with representative adjacency matrix $A$. Since
adjacency matrices are necessarily square, we shall allow the addition of rows to ensure squareness.

Definition 2.102. The representative graph $G_{f}$ of a square singularity $f$ is the most connected graph corresponding to the equivalence class of adjacency matrices $\left[\mathrm{A}_{f}\right]$, where $\mathrm{A}_{f}$ is the exponent matrix of $f$ (up to the addition of rows).

Remark 2.9.19. As permutation matrices have unit determinant and the determinant is multiplicative over matrix products, the determinant is an invariant on permutation equivalence classes of exponent matrices.

We discuss now a selection of typical representative graphs (Figure 2.2).


Figure 2.2. Five Connected Graphs
2.9.4.1. Path Graph. The representative graph of the weighted homogeneous polynomial $f=z_{2}+\sum_{i=1}^{n-2} z_{i} z_{i+2}+z_{n-1}$ for $n>1$ is the path graph $P_{n}$ with
(tri-diagonal) adjacency matrix

$$
\mathrm{A}\left(P_{n}\right)=\left(\begin{array}{ccccccc}
0 & 1 & 0 & 0 & \cdots & 0 & 0  \tag{2.219}\\
1 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 & 1 & 0
\end{array}\right) .
$$

In this case the polynomial $f$ is degenerate for $n>1$, singular for $n \equiv 1 \bmod 4$, or has at least one zero weight for $n>3$ and $n \not \equiv 1 \bmod 4$. The first few polynomials are

$$
\begin{array}{ll}
P_{2}: & f=z_{1}+z_{2} \\
P_{3}: & f=z_{2}+z_{1} z_{3}+z_{2} \\
P_{4}: & f=z_{2}+z_{1} z_{3}+z_{2} z_{4}+z_{3} \tag{2.222}
\end{array}
$$

where the red text denotes a graph that cannot be realized from a square adjacency matrix due to a repetition of monomials.
2.9.4.2. Cycle Graph. The representative graph of the weighted homogeneous polynomial $f=z_{2} z_{n}+z_{1} z_{3}+\cdots+z_{n-2} z_{n}+z_{1} z_{n-1}$ is a cycle graph $C_{n}$ with
adjacency matrix

$$
\mathrm{A}\left(C_{n}\right)=\left(\begin{array}{ccccccc}
0 & 1 & 0 & 0 & \cdots & 0 & 1  \tag{2.223}\\
1 & 0 & 1 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 1 & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & 1 \\
1 & 0 & \cdots & 0 & 0 & 1 & 0
\end{array}\right) .
$$

In this case the $n$ weights of $f$ are $\left\{\frac{1}{2}, \ldots, \frac{1}{2}\right\}$ if $n \not \equiv 0 \bmod 4$. See for example, [91]. The first few polynomials are

$$
\begin{array}{ll}
C_{2}: & f=z_{1}^{2}+z_{2}^{2} \\
C_{3}: & f=z_{2} z_{3}+z_{1} z_{3}+z_{1} z_{2} \\
C_{4}: & f=z_{2} z_{4}+z_{1} z_{3}+z_{2} z_{4}+z_{1} z_{3} . \tag{2.226}
\end{array}
$$

Remark 2.9.20. These quasi-Brieskorn-Pham singularities are right equivalent to simple $A_{1}$-singularities (e.g., $f \sim_{r} \sum_{i=1}^{n} z_{i}^{2}$ ) and have trivial HilbertPoincaré series, i.e., $P_{\mathcal{A}_{f}}(t)=1$.
2.9.4.3. Star Graph. The representative graph of the weighted homogeneous polynomial $f=z_{2} \cdots z_{n}+(n-1) z_{1}=z_{2} \cdots z_{n}+z_{1}+\cdots+z_{1}$ is the star graph
$S_{n+1}$ with adjacency matrix

$$
\mathrm{A}\left(S_{n+1}\right)=\left(\begin{array}{cc}
0 & \mathbf{1}_{n-1}^{\top}  \tag{2.227}\\
\mathbf{1}_{n-1} & \mathrm{Z}_{n-1}
\end{array}\right)
$$

where $\mathbf{1}_{n}$ denotes a column vector of $n 1 \mathrm{~s}$, and $Z_{n}$ denotes the $n \times n$ matrix of 0 s. In this case the polynomial $f$ is degenerate for $n>1$. By the repetition of monomials, the star graphs cannot be realized by a square, weighted homogeneous singularity for $n>2$.
2.9.4.4. Complete Graph. The representative graph of the weighted homogeneous polynomial $f=\sum_{i=1}^{n} z_{1} \cdots \hat{z}_{i} \cdots z_{n}$, where the circumflex denotes omission, is the complete graph $K_{n}$ with adjacency matrix

$$
\mathrm{A}\left(K_{n}\right)=\left(\begin{array}{ccccccc}
0 & 1 & 1 & 1 & \cdots & 1 & 1  \tag{2.228}\\
1 & 0 & 1 & 1 & \cdots & 1 & 1 \\
1 & 1 & 0 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 0 & 1 & 1 \\
1 & 1 & \cdots & 1 & 1 & 0 & 1 \\
1 & 1 & \cdots & 1 & 1 & 1 & 0
\end{array}\right)
$$

that is, $\mathrm{J}_{n}-\mathrm{I}_{n}$, where $\mathrm{J}_{n}$ denote the $n \times n$ matrix of 1 s . In this case, the $n$ weights of $f$ are $\left\{\frac{1}{n-1}, \cdots \frac{1}{n-1}\right\}$ for $n>2$. For $n=2, f=z_{1}+z_{2} \sim_{r} z_{1}+z_{2}^{2}\left(A_{0}-\right.$ singularity). For $n>2, f$ is degenerate.
2.9.4.5. Weighted and Directed Edges. By allowing weighted and directed edges, one may enlarge the families of permissible representative graphs to include the adjacency matrices/exponent matrices of a myriad of square, weighted homogeneous polynomials.

Remark 2.9.21. Consider $f=\sum_{i=1}^{n}\left(z_{1} \cdots \hat{z}_{i} \cdots z_{n}\right)^{m}$ with $n>1$ and $m \geqslant 1$. Then $f$ is degenerate for $n>3$. Moreover, $f$ has weights $\left\{\frac{1}{m(n-1)}, \ldots, \frac{1}{m(n-1)}\right\}$, and $G_{f}$ is the complete weighted graph $m K_{n}$ (that is, $K_{n}$ with all edge weights equal to $m$ ).

To what extent does a representative graph determine the invariants of a singularity is an intriguing question. Recall that the characteristic polynomial of a graph is that of its adjacency matrix, $p_{G}(\lambda)=\operatorname{det}\left(\lambda I-A_{G}\right)$. The following is one such example.

Proposition 2.103. If $f$ is a Brieskorn-Pham polynomial with representative graph $G_{f}$, then

$$
\begin{equation*}
p_{G_{f}}(1)=\mu_{\mathrm{alg}}(f) \tag{2.229}
\end{equation*}
$$

Proof. The characteristic polynomial of an $n \times n$ diagonal matrix $D=\left(d_{i i}\right)$ evaluated at $\lambda=1$ is the product $\prod_{i=1}^{n}\left(\lambda_{i}-1\right)$, where $\lambda_{i}=d_{i i}$.
2.9.4.6. Graph Morphisms. Using the exponent matrix approach and its graphical interpretation, it is now possible to create new singularities from old in a compatible, consistent and visual manner. For example, it is relatively

Table 2.1. Various Closed Operations on Singularities

| Singularity | Exponent Matrix | Local Algebra | Representative Graph |
| :---: | :---: | :---: | :---: |
| $f \boxplus g$ | $\mathrm{~A}_{f} \oplus \mathrm{~A}_{g}$ | $\mathcal{A}_{f} \otimes \mathcal{A}_{g}$ | $G_{f} \sqcup \mathrm{G}_{g}$ |
| $f \boxtimes g$ | $\mathrm{~A}_{f} \otimes \mathrm{~A}_{g}$ | $\mathcal{A}_{f} \boxtimes \mathcal{A}_{g}$ | $G_{f} \square G_{g}$ |
| $f \boxtimes g$ | $\mathrm{~A}_{f} \odot \mathrm{~A}_{g}$ | $\mathcal{A}_{f} \boxtimes \mathcal{A}_{g}$ | $G_{f} \times \mathrm{G}_{g}$ |

straightforward to demonstrate that the Sebastiani-Thom summation of two weighted homogeneous polynomials corresponds to the graph disjoint union of the respective representative graphs. Similar structures arise in an equally simple way.

Proposition 2.104. Let $\sqcup, \square$ and $\times$ denote the graph (disjoint) union, graph cartesian product and graph tensor product, respectively. The following identities hold:

$$
\begin{align*}
& G_{f \boxplus g}=G_{f} \sqcup G_{g}  \tag{2.230}\\
& G_{f \boxtimes g}=G_{f} \square G_{g}  \tag{2.231}\\
& G_{f \boxtimes g}=G_{f} \times G_{g} . \tag{2.232}
\end{align*}
$$

Proof. The graph identities follow immediately from the following adjacency matrix identities $\mathrm{A}_{f \boxplus g}=\mathrm{A}_{f} \oplus \mathrm{~A}_{g}, \mathrm{~A}_{f \boxminus g}=\mathrm{A}_{f} \odot \mathrm{~A}_{g}$ and $\mathrm{A}_{f \boxtimes g}=\mathrm{A}_{f} \otimes \mathrm{~A}_{g}$, respectively, where $\boxplus$ denotes Sebastiani-Thom summation, $\oplus$ denotes (matrix) direct summation, $\odot$ denotes (matrix) Kronecker summation and $\otimes$ denotes (matrix) Kronecker product.

Remark 2.9.22. Consider $f=\sum_{i=1}^{n} z_{i}^{a_{i}}$. Then $G_{f}=G_{z^{a_{1}}} \sqcup \cdots \sqcup G_{z^{a_{n}}}$, the disjoint union of loops with edge weights $\left\{a_{i}\right\}$.

REMARK 2.9.23. Consider $f^{\square n}=\sum_{i=1}^{2^{n+1}} z_{i}^{n+1}$. Then $G_{f} \cong P_{2} \cong K_{2}$, the edge graph, and $G_{f \boxminus n}=G_{f} \square \cdots \square G_{f} \cong Q_{n+1}$, the $(n+1)$-cube graph.

We summarize the three operations on singularities, exponent matrices, local algebras and representative graphs discussed in this section in Table 2.1.

This concludes our analysis of some of the algebraic structure that lies at the heart of complex analytic singularities. We turn our attention to more analytical aspects.

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## Chapter 3

## Analytic Structure of Isolated Singularities

> A commodity appears at first sight an extremely obvious, trivial thing. But its analysis brings out that it is a very strange thing, abounding in metaphysical subtleties and theological niceties. - Karl Marx

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In this chapter we generalize the notion of weighted homogeneity to meromorphic functions and derive a new analytic criterion for weighted homogeneous polynomials. We discuss also the Grothendieck residue and define the analytic index of a singularity. Finally, we make some remarks on the mixed Hodge structure of weighted homogeneous singularities.

### 3.1. Generalized Weighted Homogeneity

Definition 3.1. Let $U \subset \mathbb{C}^{n+1}$ be a domain. A meromorphic function $f: U \rightarrow \mathbb{C}$ is weighted homogeneous on $U$ if and only if there is a real $m$ and a set
$\left\{\omega_{0}, \ldots, \omega_{n}\right\} \subset \mathbb{R}$ such that

$$
\begin{equation*}
\lambda^{m} f=f\left(\lambda^{\omega_{0}} z_{0}, \ldots, \lambda^{\omega_{n}} z_{n}\right) \quad \lambda \in \mathbb{C}^{\times},\left(z_{0}, \ldots, z_{n}\right) \in U \tag{3.1}
\end{equation*}
$$

where the $m$ is the co-degree and $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ are (reduced) weights.

Remark 3.1.1. Although it may seem prudent to absorb the co-degree into the weights, it is not always possible to do so.

Remark 3.1.2. A function with rational weights need not be a polynomial. Consider two complex analytic, homogeneous polynomials $f$ and $g$ of degree $d$ and $d^{\prime}$, respectively, then the rational function $\frac{f}{g}$ is weighted homogeneous over $\mathbb{C}^{n+1} \backslash V_{g, 0}$ with co-degree $d-d^{\prime}$ and reduced weights $\{1, \ldots, 1\}$.

Remark 3.1.3. A weighted homogeneous function with non-zero co-degree $m$ and (possibly non-unique) weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ is a weighted homogeneous function with co-degree 1 and weights $\left\{\frac{\omega_{0}}{m}, \ldots, \frac{\omega_{n}}{m}\right\}$.

Proposition 3.2. Let $U \subset \mathbb{C}^{n+1}$ be a domain. A meromorphic, weighted homogeneous function $f: U \rightarrow \mathbb{C}$ with co-degree $m \in \mathbb{R}$ and weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\} \subset$ $\mathbb{R}$ satisfies the weighted Euler equation

$$
\begin{equation*}
m f=\sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i} f \quad\left(z_{0}, \ldots, z_{n}\right) \in U \tag{3.2}
\end{equation*}
$$

Proof. To show the claimed implication, let

$$
\begin{equation*}
f^{\lambda}=\lambda^{\omega} \cdot f:=f\left(\lambda^{\omega_{0}} z_{0}, \ldots, \lambda^{\omega_{n}} z_{n}\right) \tag{3.3}
\end{equation*}
$$

and consider the total differential $\mathrm{d} f^{\lambda}$,

$$
\begin{align*}
\mathrm{d} f^{\lambda} & =\sum_{i=0}^{n} \frac{\partial f^{\lambda}}{\partial\left(\lambda^{\omega_{i} z_{i}}\right)} \mathrm{d}\left(\lambda^{\omega_{i}} z_{i}\right)  \tag{3.4a}\\
& =\sum_{i=0}^{n} \omega_{i} \lambda^{\omega_{i}-1} z_{i} \frac{\partial f^{\lambda}}{\partial\left(\lambda^{\omega_{i}} z_{i}\right)} \mathrm{d} \lambda  \tag{3.4b}\\
& =\lambda^{-1} \sum_{i=0}^{n} \omega_{i} z_{i} \frac{\partial f^{\lambda}}{\partial z_{i}} \mathrm{~d} \lambda \tag{3.4c}
\end{align*}
$$

By definition, $f^{\lambda}=\lambda f$ for $\lambda \in \mathbb{C}^{\times}$, since $f$ is weighted homogeneous. It follows that $\mathrm{d} f^{\lambda}=m \lambda^{m-1} f \mathrm{~d} \lambda$ and, therefore, $m f=\sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i} f$, as claimed.

Remark 3.1.4. The finite Puiseux series $f=\sum_{i=0}^{n} c_{i} z_{i}^{1 / a_{i}}$, where $c_{i} \in \mathbb{C}^{\times}$and $a_{i} \in \mathbb{Q}_{>0}^{\times}$, is weighted homogeneous with co-degree 1 and weights $\left\{a_{0}, \ldots, a_{n}\right\} \subset$ $\mathbb{N}$ and satisfies the weighted Euler equation $f=\sum_{i=0}^{n} a_{i} z_{i} \partial_{i} f$.

Corollary 3.3. Let $U \subset \mathbb{C}^{n+1}$ be a domain. If a meromorphic, weighted homogeneous function $f: U \rightarrow \mathbb{C}$ with non-zero co-degree and non-zero weights has any critical points on $U$, then it also vanishes at those points.

Corollary 3.4. Let $U \subset \mathbb{C}^{n+1}$ be a domain. If a meromorphic, weighted homogeneous function $f: U \rightarrow \mathbb{C}$ with non-zero co-degree has any critical points on $U$ and any zero weights in some directions, then it has a continuum of critical points in the corresponding directions.

Denote by $\mathcal{W H}(\boldsymbol{\omega})$ the equivalence class of weighted homogeneous functions with weight $\omega$ and domain extended to the generalized Riemann torus
$\hat{\mathbb{C}}^{n+1}$. Denote by $\mathcal{W} \mathcal{H}(\boldsymbol{\omega} ; m) \subset \mathcal{W} \mathcal{H}(\boldsymbol{\omega})$ the subclass of those functions with co-degree $m$. Define 0 and 1 to be the unique constant weighted homogeneous functions of arbitrary co-degree and the zero co-degree, respectively.

Proposition 3.5. For any $\boldsymbol{\omega} \in \mathbb{R}^{n+1}$, the equivalence class $\mathcal{W H}(\boldsymbol{\omega})$ is a group. For $m \in \mathbb{R}$, the subclass $\mathcal{W H}(\boldsymbol{\omega} ; m)$ is a vector space over $\mathbb{C}$. In particular, the subclass $\mathcal{W H}(\boldsymbol{\omega} ; 0)$ is a unital associative algebra over $\mathbb{C}$.

Proof. Given $f \in \mathcal{W} \mathcal{H}(\boldsymbol{\omega} ; m)$ and $g \in \mathcal{W} \mathcal{H}\left(\boldsymbol{\omega} ; m^{\prime}\right)$, the product $f g$ is also weighted homogeneous with the same weights and satisfies

$$
\begin{align*}
\sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i}(f g) & =g \sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i} f+f \sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i} g  \tag{3.5}\\
& =\left(m+m^{\prime}\right) f g, \tag{3.6}
\end{align*}
$$

illustrating the fact that the $f g$ has co-degree $m+m^{\prime}$, so $f g \in \mathcal{W} \mathcal{H}\left(\boldsymbol{\omega} ; m+m^{\prime}\right)$. Similarly, the ratio $\frac{f}{g}$ is weighted homogeneous with the same weights and satisfies

$$
\begin{align*}
g^{2} \sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i}\left(\frac{f}{g}\right) & =g \sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i} f-f \sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i} g  \tag{3.7}\\
& =\left(m-m^{\prime}\right) f g \tag{3.8}
\end{align*}
$$

illustrating the fact that $\frac{f}{g}$ has co-degree $m-m^{\prime}$, so $\frac{f}{g} \in \mathcal{W} \mathcal{H}\left(\boldsymbol{\omega} ; m-m^{\prime}\right)$. Note that the pointwise product is closed, commutative and associative on $\mathcal{W H}(\boldsymbol{\omega})$.

Since inverses exist and there is a multiplicative identity, namely, 1 , it follows that $\mathcal{W H}(\boldsymbol{\omega})$ is a group.

Given $f, g \in \mathcal{W} \mathcal{H}(\boldsymbol{\omega} ; m)$, consider the linear combination $\alpha f+\beta g$, where $\alpha, \beta \in \mathbb{C}$. For any $\lambda \in \mathbb{C}^{\times}$,

$$
\begin{align*}
\lambda^{m}(\alpha f+\beta g) & =\alpha \lambda^{m} f+\beta \lambda^{m} g  \tag{3.9}\\
& =\alpha f\left(\lambda^{\omega} \cdot z\right)+\beta g\left(\lambda^{\omega} \cdot z\right)  \tag{3.10}\\
& =(\alpha f+\beta g)\left(\lambda^{\omega} \cdot z\right) \tag{3.11}
\end{align*}
$$

so $\alpha f+\beta g \in \mathcal{W} \mathcal{H}(\boldsymbol{\omega} ; m)$. Given $f \in \mathcal{W} \mathcal{H}(\boldsymbol{\omega} ; m)$ with $m \in \mathbb{R}$, for any $g \in$ $\mathcal{W H}(\boldsymbol{\omega} ; 0)$, one has $f g, \frac{f}{g} \in \mathcal{W} \mathcal{H}(\boldsymbol{\omega}, m)$. Thus, the summation and product are closed, commutative and associative operations on $\mathcal{W H}(\boldsymbol{\omega} ; 0)$ with identities 0 and 1, respectively. Inverses also exist for all non-zero elements.

Let $m(f)$ denote the co-degree of $f$.
Corollary 3.6. For $s \in \mathbb{N}, r_{1}, \ldots, r_{s} \in \mathbb{Z}$ and $f_{1}, \ldots, f_{s} \in \mathcal{W H}(\omega)$,

$$
\begin{equation*}
m\left(f_{1}^{r_{1}} \cdots f_{s}^{r_{s}}\right)=\sum_{i=1}^{s} r_{i} m\left(f_{i}\right) \tag{3.12}
\end{equation*}
$$

Proof. By Proposition 3.5, one has $m(f g)=m(f)+m(g)$ and $m\left(\frac{f}{g}\right)=$ $m(f)-m(g)$. Therefore, for any integer $a$, one has $m\left(f^{a}\right)=a m(f)$. The claim now follows.

Definition 3.7. If an entire function is weighted homogeneous on its domain with non-zero co-degree, then it is simply weighted homogeneous.

Remark 3.1.5. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a complex analytic function (so differentiable), weighted homogeneous with co-degree 1 and weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$, then for $k \geqslant 0$,

$$
\begin{equation*}
f=\sum_{0 \leqslant i_{1}, \ldots, i_{k} \leqslant n} \omega_{i_{1}} \cdots \omega_{i_{k}} D_{i_{1}} \cdots D_{i_{k}} f, \tag{3.13}
\end{equation*}
$$

where $D_{i}=z_{i} \partial_{i}$. Let $\left[\lambda_{1}, \cdots, \lambda_{m}\right]$ denote the number of subset partitions of the sequence $\left[\lambda_{1}+\cdots+\lambda_{m}\right]=\left\{1,2, \ldots, \lambda_{1}+\cdots+\lambda_{m}\right\}$ into $\left\{\lambda_{1}, \ldots, \lambda_{m}\right\}$-parts with canonical ordering (increasing in each part) (A036040). For example, $[2,2]=3$ and $[2,1,1]=6$, since $\{1,2,3,4\}$ may be partitioned into $\{2,2\}$-parts as $12 \mid 34$, $13 \mid 24$ and $14 \mid 23$, and into $\{2,1,1\}$-parts as follows $12|3| 4,13|2| 4,14|2| 3,23|1| 4$, $24|1| 3$ and $34|1| 2$, respectively. By expanding equation (3.13),

$$
\begin{equation*}
f=\sum_{\left(\lambda_{1}, \ldots, \lambda_{m}\right) \vdash k}\left[\lambda_{1}, \ldots, \lambda_{m}\right] \sum_{0 \leqslant i_{1}, \ldots, i_{m} \leqslant n} \omega_{i_{1}}^{\lambda_{1}} \cdots \omega_{i_{m}}^{\lambda_{m}} z_{i_{1}} \cdots z_{i_{k}} \partial_{i_{1}, \ldots, i_{m}} f . \tag{3.14}
\end{equation*}
$$

For instance, if $k=2$, then

$$
\begin{align*}
f & =\sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i}\left(\sum_{j=0}^{n} \omega_{j} z_{j} \partial_{j} f\right)  \tag{3.15}\\
& =\sum_{i, j=0}^{n} \omega_{i} \omega_{j} z_{i} \partial_{i}\left(z_{j} \partial_{j} f\right)  \tag{3.16}\\
& =\sum_{i=0}^{n} \omega_{i}^{2} z_{i} \partial_{i} f+\sum_{i, j=0}^{n} \omega_{i} \omega_{j} z_{i} z_{j} \partial_{i j} f \tag{3.17}
\end{align*}
$$

from which one may deduce the identity

$$
\begin{equation*}
\sum_{i, j=0}^{n} \omega_{i} \omega_{j} z_{i} z_{j} \partial_{i j} f=\sum_{i=0}^{n} \omega_{i}\left(1-\omega_{i}\right) z_{i} \partial_{i} f \tag{3.18}
\end{equation*}
$$

while if $k=3$ or $k=4$, one has

$$
\begin{equation*}
f=\sum_{i=0}^{n} \omega_{i}^{3} z_{i} \partial_{i} f+3 \sum_{i, j=0}^{n} \omega_{i}^{2} \omega_{j} z_{i} z_{j} \partial_{i j} f+\sum_{i, j, k=0}^{n} \omega_{i} \omega_{j} \omega_{k} z_{i} z_{j} z_{k} \partial_{i j k} f \tag{3.19}
\end{equation*}
$$

and

$$
\begin{align*}
f= & \sum_{i=0}^{n} \omega_{i}^{4} z_{i} \partial_{i} f+4 \sum_{i, j=0}^{n} \omega_{i}^{3} \omega_{j} z_{i} z_{j} \partial_{i j} f+3 \sum_{i, j=0}^{n} \omega_{i}^{2} \omega_{j}^{2} z_{i} z_{j} \partial_{i j} f \\
& +6 \sum_{i, j, k=0}^{n} \omega_{i}^{2} \omega_{j} \omega_{k} z_{i} z_{j} z_{k} \partial_{i j k} f+\sum_{i, j, k, l=0}^{n} \omega_{i} \omega_{j} \omega_{k} \omega_{l} z_{i} z_{j} z_{k} z_{l} \partial_{i j k l} f, \tag{3.20}
\end{align*}
$$

respectively.

Definition 3.8. A weighted homogeneous polynomial is a complex analytic polynomial that is weighted homogeneous on its domain with co-degree 1.

Let $U \subset \mathbb{R}^{n}$ denote a convex neighborhood of the origin. The following result is indispensable in Morse Theory [306].

Proposition 3.9 (Milnor, [306]). If $f:(U, \mathbf{0}) \rightarrow(\mathbb{R}, 0)$ is a smooth function, then there are smooth functions $\left\{g_{1}, \ldots, g_{n}\right\}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ defined on $U$ such that

$$
\begin{equation*}
f=\sum_{i=1}^{n} x_{i} g_{i} \quad \text { and } \quad g_{i}(\mathbf{0})=\partial_{i} f(\mathbf{0}) \tag{3.21}
\end{equation*}
$$

Proof. Define the $\mathbb{R}$-action $t \cdot\left(x_{1}, \ldots, x_{n}\right)=\left(t x_{1}, \ldots, t x_{n}\right)$. Let $f^{\prime}\left(t x_{1}, \ldots, t x_{n}\right)$ denote the derivative of the composite function

$$
\begin{equation*}
(f \circ t)\left(x_{1}, \ldots, x_{n}\right)=f\left(t x_{1}, \ldots, t x_{n}\right) \tag{3.22}
\end{equation*}
$$

Similarly, let

$$
\begin{equation*}
\left(\partial_{i} f \circ t\right)\left(x_{1}, \ldots, x_{n}\right)=\left(\partial_{i} f\right)\left(t x_{1}, \ldots, t x_{n}\right) \tag{3.23}
\end{equation*}
$$

denote the $i^{\text {th }}$-directional derivative of $f\left(x_{1}, \ldots, x_{n}\right)$ subsequently evaluated at the point $\left(t x_{1}, \ldots, t x_{n}\right)$. Observe that

$$
\begin{align*}
f\left(x_{1}, \ldots, x_{n}\right) & =\int_{0}^{1} f^{\prime}\left(t x_{1}, \ldots, t x_{n}\right) d t  \tag{3.24}\\
& =\int_{0}^{1} \sum_{i=1}^{n} x_{i}\left(\partial_{i} f\right)\left(t x_{1}, \ldots, t x_{n}\right) d t  \tag{3.25}\\
& =\sum_{i=1}^{n} x_{i} g_{i}\left(x_{1}, \ldots, x_{n}\right) \tag{3.26}
\end{align*}
$$

where

$$
\begin{equation*}
g_{i}\left(x_{1}, \ldots, x_{n}\right)=\int_{0}^{1}\left(\partial_{i} f\right)\left(t x_{1}, \ldots, t x_{n}\right) d t \tag{3.27}
\end{equation*}
$$

Since $f$ is smooth and vanishes at the origin, it follows that $g_{i}$ is smooth and vanishes at the origin, as claimed.

REMARK 3.1.6. Given a set of functions $\left\{h_{1}(t, \mathbf{x}), \ldots, h_{n}(t, \mathbf{x})\right\}: \mathbb{R}^{n+1} \rightarrow$ $\mathbb{R}^{2}$, differentiable in both variables, the function $\left(\partial_{i} f\right)\left(h_{1}(t, \mathbf{x}), \ldots, h_{n}(t, \mathbf{x})\right)$
denotes the directional derivative of $f\left(x_{1}, \ldots, x_{n}\right)$ subsequently evaluated at the point $\left(h_{0}(t, \mathbf{x}), \ldots, h_{n}(t, \mathbf{x})\right)$. Contradistinctively, the function $\partial_{i} f\left(h_{1}(t, \mathbf{x}), \ldots, h_{n}(t, \mathbf{x})\right)$ denotes the directional derivative of the composite function $f\left(h_{1}(t, \mathbf{x}), \ldots, h_{n}(t, \mathbf{x})\right)$.

### 3.2. Weighted Homogeneous Polynomials, Revisited

Proposition 3.10. A complex analytic polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is weighted homogeneous if and only if there are unique, positive reals $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ such that

$$
\begin{equation*}
f\left(z_{0}, \ldots, z_{n}\right)=\int_{0}^{1} f\left(t^{\omega_{0}} z_{0}, \ldots, t^{\omega_{n}} z_{n}\right) d^{\times} t \tag{3.28}
\end{equation*}
$$

In particular, $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ are the corresponding weights.

Proof. For any complex analytic polynomial $f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ (and therefore any polynomial over $\mathbb{R}^{n+1}$ ) and any $\left\{\omega_{0}, \ldots, \omega_{n}\right\} \subset \mathbb{R}_{>0}$, the Fundamental Theorem of Calculus [346] implies the identity

$$
\begin{align*}
f\left(z_{0}, \ldots, z_{n}\right)-f(\mathbf{0}) & =\int_{0}^{1} f^{\prime}\left(t^{\omega_{0}} z_{0}, \ldots, t^{\omega_{n}} z_{n}\right) d t  \tag{3.29}\\
& =\sum_{i=0}^{n} \omega_{i} z_{i} \int_{0}^{1} \partial_{i} f\left(t^{\omega_{0}} z_{0}, \ldots, t^{\omega_{n}} z_{n}\right) d^{\times} t \tag{3.30}
\end{align*}
$$

since

$$
\begin{align*}
f^{\prime}\left(t^{\omega_{0}} z_{0}, \ldots, t^{\omega_{n}} z_{n}\right) & =\sum_{i=0}^{n}\left(\frac{d\left(t^{\omega_{i}} z_{i}\right)}{d t}\right)^{-1} \frac{\partial f}{\partial\left(t^{\omega_{i}} z_{i}\right)}  \tag{3.31}\\
& =t^{-1} \sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i} f\left(t^{\omega_{0}} z_{0}, \ldots, t^{\omega_{n}} z_{n}\right) . \tag{3.32}
\end{align*}
$$

See Lemma 4.1 in [49]. Thus, if $f$ vanishes at the origin and there are unique, positive reals $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ such that

$$
\begin{equation*}
f\left(z_{0}, \ldots, z_{n}\right)=\int_{0}^{1} f\left(t^{\omega_{0}} z_{0}, \ldots, t^{\omega_{n}} z_{n}\right) d^{\times} t \tag{3.33}
\end{equation*}
$$

then each directional derivative of $f$ satisfies a similar identity, namely,

$$
\begin{align*}
\partial_{i} f\left(z_{0}, \ldots, z_{n}\right) & =\partial_{i} \int_{0}^{1} f\left(t^{\omega_{0}} z_{0}, \ldots, t^{\omega_{n}} z_{n}\right) d^{\times} t  \tag{3.34}\\
& =\int_{0}^{1} \partial_{i} f\left(t^{\omega_{0}} z_{0}, \ldots, t^{\omega_{n}} z_{n}\right) d^{\times} t \tag{3.35}
\end{align*}
$$

which when combined with equation (3.30) yields the following weighted Euler equation,

$$
\begin{equation*}
f=\sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i} f \tag{3.36}
\end{equation*}
$$

Thus, by the converse of Proposition 3.2, $f$ is weighted homogeneous with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$. Conversely, suppose $f$ is weighted homogeneous with
said weights. The identity $\lambda f\left(z_{0}, \ldots, z_{n}\right)=f\left(\lambda^{\omega_{0}} z_{0}, \ldots, \lambda^{\omega_{n}} z_{n}\right)$ implies

$$
\begin{align*}
\int_{0}^{1} f\left(t^{\omega_{0}} z_{0}, \ldots, t^{\omega_{n}} z_{n}\right) d^{\times} t & =f\left(z_{0}, \ldots, z_{n}\right) \int_{0}^{1} t d^{\times} t  \tag{3.37}\\
& =f\left(z_{0}, \ldots, z_{n}\right) \tag{3.38}
\end{align*}
$$

which completes the proof of the equivalence.

Remark 3.2.1. More generally, for any point $\mathbf{a}=\left(a_{0}, \ldots, a_{n}\right) \in \mathbb{C}$,

$$
\begin{align*}
f\left(z_{0}+a_{0}, \ldots, z_{n}+a_{n}\right)-f(\mathbf{a}) & =\int_{0}^{1} f^{\prime}\left(t^{\omega_{0}} z_{0}+a_{0}, \ldots, t^{\omega_{n}} z_{n}+a_{n}\right) d t  \tag{3.39}\\
& =\sum_{i=0}^{n} \omega_{i} z_{i} \int_{0}^{1} \partial_{i} f\left(t^{\omega_{0}} z_{0}+a_{0}, \ldots, t^{\omega_{n}} z_{n}+a_{n}\right) d^{\times} t .
\end{align*}
$$

REMARK 3.2.2. If $\omega_{i}=1$ for $0 \leqslant i \leqslant n$, then

$$
\begin{equation*}
\partial_{i} f\left(t z_{0}, \ldots, t z_{n}\right)=t\left(\partial_{i} f\right)\left(t z_{0}, \ldots, t z_{n}\right) \quad 0 \leqslant i \leqslant n, \tag{3.40}
\end{equation*}
$$

so equation (3.25) is a special case of equation (3.30).
REMARK 3.2.3. The integral of equation (3.30) is not well-defined if the weights are non-positive or if $f$ is a non-polynomial, rational function.

Corollary 3.11. If a complex analytic polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ satisfies the system of integro-differential equalities,

$$
\begin{equation*}
\omega_{i} \partial_{i} f=\int_{0}^{1} \partial_{i} f\left(t z_{0}, \ldots, t z_{n}\right) d^{\times} t \quad 0 \leqslant i \leqslant n \tag{3.41}
\end{equation*}
$$

for some set of rationals $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$, then $f$ is weighted homogeneous.
Proof. By the Fundamental Theorem of Calculus, any complex analytic polynomial $f: \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ satisfies the identity

$$
\begin{equation*}
f\left(z_{0}, \ldots, z_{n}\right)-f(\mathbf{0})=\sum_{i=0}^{n} z_{i} \int_{0}^{1} \partial_{i} f\left(t z_{0}, \ldots, t z_{n}\right) d^{\times} t \tag{3.42}
\end{equation*}
$$

If $f$ vanishes at the origin and satisfies equation 3.41 , then equation 3.42 implies that $f$ satisfies a weighted Euler equation. Thus, by a similar argument used in the proof of Proposition 3.10, $f$ is a weighted homogeneous polynomial with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$.

Remark 3.2.4. The converse of Corollary 3.11 holds for Brieskorn-Pham polynomials.

Remark 3.2.5. The converse of Corollary 3.11 is not true in general. Consider the non-degerate, weighted homogeneous polynomial $f=x^{2}+x y^{2}$ over $\mathbb{C}^{2}$, which has weights $\left\{\frac{1}{2}, \frac{1}{4}\right\}$. While $\partial_{x} f=2 x+y^{2}$ and $\partial_{y} f=2 x y$,

$$
\begin{equation*}
\int_{0}^{1} \partial_{x} f(t x, t y) d^{\times} t=x+\frac{1}{3} y^{3} \quad \text { and } \quad \int_{0}^{1} \partial_{x} f(t x, t y) d^{\times} t=\frac{2}{3} x y . \tag{3.43}
\end{equation*}
$$

Corollary 3.12. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial, then the putative weights can be determined by the equations

$$
\begin{equation*}
\int_{0}^{1} \frac{\partial_{i} f\left(t z_{0}, \ldots, t z_{n}\right)}{\partial_{i} f\left(z_{0}, \ldots, z_{n}\right)} d^{\times} t=\omega_{i} \quad 0 \leqslant i \leqslant n . \tag{3.44}
\end{equation*}
$$

REMARK 3.2.6. If there are functions $h(\lambda)$ and $\left\{g_{0}(\lambda), \ldots, g_{n}(\lambda)\right\}$ with logarithmic derivatives on $\mathbb{C}^{\times}$such that $f$ satisfies the transformation law

$$
\begin{equation*}
h(\lambda) f=f\left(g_{0}(\lambda) z_{0}, \ldots, g_{n}(\lambda) z_{n}\right) \quad \lambda \in \mathbb{C}^{\times}, \tag{3.45}
\end{equation*}
$$

then $f$ satisfies the weighted Euler equation

$$
\begin{equation*}
(\log h(\lambda))^{\prime} f=\sum_{i=0}^{n}\left(\log g_{i}(\lambda)\right)^{\prime} z_{i} \partial_{i} f \tag{3.46}
\end{equation*}
$$

Taking $g_{i}(\lambda)=\lambda^{\omega_{i}}$ and $h(\lambda)=\lambda^{m}$ yields the case in the statement of Proposition 3.2. Alternatively, the choice $g_{i}(\lambda)=\omega_{i} \lambda^{\omega_{i}}$ and $h(\lambda)=\lambda$ yields the transformation law $\lambda f=f\left(\omega_{0} \lambda^{\omega_{0}} z_{0}, \ldots, \omega_{n} \lambda^{\omega_{n}} z_{n}\right)$ for $\lambda \in \mathbb{C}^{\times}$with the weighted Euler equation $f=\sum_{i=0}^{n} \omega_{i} z_{i} \partial_{i} f$.

Using Proposition 2.60 among other related results, Hertling and Kurbel have recently begun the classification of non-degenerate, weighted homogeneous singularities [206], complete up to 4 variables and with algebraic index less than or equal to 2000 . The number of equivalence classes $N_{n}$ of nondegenerate, weighted homogeneous polynomials over $\mathbb{C}^{n+1}$ is given in Table 3.1.

Remark 3.2.7. For $n \in\{1,2,3,4\}$, the number of equivalence classes $N_{n}$ of non-degenerate, weighted homogeneous polynomials over $\mathbb{C}^{n+1}$ coincides with the number of endofunctions of $n+1$ points (A001372).

Their classification includes the following result.

Table 3.1. Number of Equivalence Classes of Non-degenerate Weighted Homogeneous Singularities over $\mathbb{C}^{n+1}$ for $1 \leqslant n \leqslant 5$

| $n$ | 1 | 2 | 3 | 4 | 5 |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $N_{n}$ | 3 | 7 | 19 | 47 | 128 |

Proposition 3.13 (Hertling, Kurbel, [206]). Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a non-degenerate weighted homogeneous polynomial with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ and weighted degree $d$. Let $p_{i}$ denote the $(i+1)^{\text {th }}$ prime. Then

$$
d \leqslant \mu_{\mathrm{alg}}(f) \cdot \begin{cases}\prod_{i=0}^{n} \frac{p_{i}}{p_{i}-1} & 0<\omega_{i} \leqslant \frac{1}{2}  \tag{3.47}\\ \prod_{i=0}^{n-1} \frac{p_{i}}{p_{i}-1} & 0<\omega_{i}<\frac{1}{2}, n>1\end{cases}
$$

Remark 3.2.8. Merten's Third Theorem (Theorem 429, [178]) is the limit

$$
\begin{equation*}
\lim _{n \rightarrow \infty}(\log n) \prod_{p \leqslant n}\left(1-\frac{1}{p}\right)=e^{-\gamma} \tag{3.48}
\end{equation*}
$$

which follows from the effective bound

$$
\begin{equation*}
\prod_{p \leqslant n} \frac{p}{p-1}=e^{\gamma}(\log n)(1+o(1)) \tag{3.49}
\end{equation*}
$$

Consequently, by Proposition 3.13, one has

$$
d=\mu_{\mathrm{alg}}(f) e^{\gamma}(1+o(1)) \begin{cases}\log (n+1) & 0<\omega_{i} \leqslant \frac{1}{2}  \tag{3.50}\\ \log n & 0<\omega_{i}<\frac{1}{2}, n>1\end{cases}
$$

### 3.3. Flat Directions and Elliptic Bounds

As will become apparent in the sequel, it is of interest to determine whether a given complex analytic function, say $f$, grows sufficiently quickly (at infinity). Said function has no flat directions if and only if there exists two fixed, positive constants $\varepsilon, M<\infty$ such that for any non-negative multi-index $\alpha$ and for all $|z|>r$, where $r>0$ is sufficiently large,

$$
\begin{equation*}
\left|\partial^{\alpha} f\right| \leqslant \varepsilon|\partial f|^{2}+M \quad \text { and } \quad|z|^{2}+|f| \leqslant M\left(|\partial f|^{2}+1\right) \tag{3.51}
\end{equation*}
$$

where $z=\left(z_{0}, \ldots, z_{n}\right)$ and $|\partial f|^{2}=\sum_{j=0}^{n}\left|\partial_{i} f\right|^{2}$, the squared-magnitude of the gradient of the $\partial f$. Although a function satisfying the elliptic bounds is sometimes referred to as elliptic, we shall not have occasion for such terminology as it conflicts with conventional mathematical use of the adjective.

If a singularity has an isolated critical point at the origin, then it cannot have flat directions (in a neighborhood of the origin), otherwise there would be a continuum of zeros in all neighborhoods of the origin. Hence, non-degenerate, complex analytic functions are de facto elliptic, at least near the origin. We discuss now the Łojasiewicz inequality as it relates to the elliptic bounds for a suitable class of complex-valued functions.

Proposition 3.14. If a weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ satisfies a weighted Euler equation with unique weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$, has a Łojasiewicz exponent $\ell_{0}(f)$ greater than or equal to unity, and for any non-negative
multi-index $\alpha$ there are non-negative constants $A, B, C<\infty$ not all zero such that

$$
\begin{equation*}
\left|\partial^{\alpha} f\right| \leqslant A|\partial f|^{2}+B|f|+C|z|^{2} \quad|z|>r \tag{3.52}
\end{equation*}
$$

where $r>0$ is sufficiently large, then $f$ satisfies the elliptic bounds.

Proof. By assumption, $f$ satisfies a weighted Euler equation, so

$$
\begin{align*}
|f| & \leqslant\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right)\left|\sum_{i=0}^{n}\right| z_{i}| | \partial_{i} f| |  \tag{3.53}\\
& \leqslant\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right)\left(\sum_{i=0}^{n}\left|z_{i}\right|^{2}\right)^{1 / 2}\left(\sum_{i=0}^{n}\left|\partial_{i} f\right|^{2}\right)^{1 / 2}  \tag{3.54}\\
& =\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right)|z||\partial f| \tag{3.55}
\end{align*}
$$

by the Cauchy-Schwartz Inequality. By assumption, $f$ satisfies the Łojasiewicz inequality, $|\partial f| \geqslant \varepsilon|z|^{\ell}$, where $\varepsilon>0$ and $\ell=\ell_{0}(f) \geqslant 1$. Thus,

$$
\begin{align*}
|z|^{2}+|f| & \leqslant|z|^{2}+\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right)|z||\partial f|  \tag{3.56}\\
& \leqslant\left(\frac{|\partial f|}{\varepsilon}\right)^{2 / \ell}+\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right)\left(\frac{|\partial f|}{\varepsilon}\right)^{1 / \ell}|\partial f|  \tag{3.57}\\
& \leqslant\left(\left(\frac{1}{\varepsilon}\right)^{2 / \ell}+\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right)\left(\frac{1}{\varepsilon}\right)^{1 / \ell}\right)|\partial f|^{2}  \tag{3.58}\\
& \leqslant M_{f}\left(|\partial f|^{2}+1\right), \tag{3.59}
\end{align*}
$$

where $M_{f}=\varepsilon^{-2 / \ell}+\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right) \varepsilon^{-1 / \ell}>0$. By assumption, for each non-negative multi-index $\alpha$, there are non-negative constants $A, B, C<\infty$ not all zero such that

$$
\begin{align*}
\left|\partial^{\alpha} f\right| & \leqslant A|\partial f|^{2}+B|f|+C|z|^{2}  \tag{3.60}\\
& \leqslant A|\partial f|^{2}+B\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right)|z||\partial f|+C\left(\frac{|\partial f|}{\varepsilon}\right)^{2 / \ell}  \tag{3.61}\\
& \leqslant A|\partial f|^{2}+B\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right)\left(\frac{|\partial f|}{\varepsilon}\right)^{1 / \ell}|\partial f|+C\left(\frac{|\partial f|}{\varepsilon}\right)^{2 / \ell}  \tag{3.62}\\
& \leqslant\left(A+B\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right)\left(\frac{1}{\varepsilon}\right)^{1 / \ell}+C\left(\frac{1}{\varepsilon}\right)^{2 / \ell}\right)|\partial f|^{2}  \tag{3.63}\\
& \leqslant \varepsilon^{\prime}|\partial f|^{2}+M_{f} \tag{3.64}
\end{align*}
$$

by equation (3.55), where $\varepsilon^{\prime}=A+B\left(\max _{0 \leqslant i \leqslant n}\left|\omega_{i}\right|\right) \varepsilon^{-1 / \ell}+C \varepsilon^{-2 / \ell}>0$.

Proposition 3.15. If a non-degenerate, weighted homogeneous polynomial has unique weights in $\left(0, \frac{1}{2}\right]$, then it satisfies the latter of the elliptic bounds.

Proof. By assumption, since $f$ is non-degenerate, it satisfies a Łojasiewicz inequality with exponent $\ell_{0}(f)$, and its weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\} \subset \mathbb{Q}$ are unique and confined to the interval $\left(0, \frac{1}{2}\right]$. Moreover, $\ell_{0}(f)=\max _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}-1\right\}$, and it follows that $\ell_{0}(f) \geqslant 1$. The bound $|f| \leqslant\left(\max _{0 \leqslant i \leqslant n} \omega_{i}\right)|z||\partial f|$ also follows, as does the existence of a fixed positive $M_{f}<\infty$ such that $|f| \leqslant M_{f}\left(|\partial f|^{2}+1\right)$.

Proposition 3.16. Any Brieskorn-Pham polynomial with exponents not less than 2 satisfies the elliptic bounds. Any polynomial of the form $f=x^{a}+x y^{b}+$ $\sum_{i=2}^{n} z_{i}^{c_{i}}$, where $\left\{a, b, c_{2}, \ldots, c_{n}\right\} \subset \mathbb{N}_{>1}$, satisfies the elliptic bounds.

### 3.4. Grothendieck Residue

The Brouwer degree is a proper generalization of the winding number $w(\gamma)$ of a closed Jordan curve $\gamma$, viewed as an endomorphism of $\mathbb{C}^{\times} \simeq S^{1}$. A generalization of the degree to complex maps between higher dimensional complex domains is achieved by the Grothendieck residue [428], which we now introduce.

Recall that $U_{x}$ is a neighborhood of a point $x \in \mathbb{C}^{n+1}$, and $\mathcal{O}_{x, n}$ is the space of complex analytic germs over $U_{x}$. Consider a complex analytic map $\left.f\right|_{U_{x}}=\left(f_{0}, \ldots, f_{n}\right):\left(U_{x}, x\right) \rightarrow\left(\mathbb{C}^{n+1}, \mathbf{0}\right)$, where $x$ is an isolated root of the system $\left.f\right|_{u_{x}}=\mathbf{0}$. Given any $h \in \mathcal{O}_{x, n}$, define the logarithmic meromorphic form $\omega_{h}(f)=h \bigwedge_{i=0}^{n} \frac{d z_{i}}{f_{i}}$, and choose the cycle $\gamma_{f, x}=\left\{z \in U_{x} \mid\left\|f_{i}\right\|=\varepsilon\right\}$ centered at $x$ and oriented by the non-negativity of $\bigwedge_{i=1}^{n} d \arg f_{i}$ for sufficiently small but positive $\varepsilon$ [368], [162]. The cycle $\gamma_{f, x}$ is an oriented compact submanifold of $U_{x} \backslash V_{f}$, where $V_{f}=\left.\bigcap_{i=1}^{n} f_{i}\right|_{\mathcal{U}_{x}} ^{-1}(0)$ is the complex algebraic variety defined by the common locus of zeros of the system $\left.f\right|_{U_{x}}=\mathbf{0}$. The Grothendieck residue of the meromorphic form $\omega_{h}=\omega_{h}(f)$ at the point $x$ is defined as the pairing

$$
\operatorname{Res}_{x}\left[\begin{array}{c}
\omega_{h}  \tag{3.65}\\
f
\end{array}\right]:=\frac{1}{(2 \pi i)^{n+1}} \int_{\gamma_{f, x}} \omega_{h}(f) \in \mathbb{Z}
$$

equivalent to $\left\langle\left[\gamma_{f, x}\right],\left[\omega_{h}(f)\right]\right\rangle$ up to a multiplicative constant. The latter implies that the Grothendieck residue depends only on the homology/cohomology class of $\gamma_{f, x}$ and $\omega_{h}(f)$, respectively. When $h$ coincides with the Jacobian matrix of $f, \frac{\partial f}{\partial z}=\frac{\partial\left(f_{0}, \ldots, f_{n}\right)}{\partial\left(z_{0}, \ldots, z_{n}\right)}$, we write $\omega(f)=\bigwedge_{i=0}^{n} \frac{d f_{i}}{f_{i}}$ for the logarithmic meromorphic form $\omega_{\partial f}(f)$ and $\operatorname{Res} \omega(f)$ for the Grothendieck residue of $\omega(f)$ at the origin.
3.4.1. Analytic Index. Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, define the related logarithmic meromorphic form $\omega\left(\left.\partial f\right|_{U}\right)=\bigwedge_{i=0}^{n} \frac{d f_{i}}{f_{i}}$, where $f_{i}=\partial_{i} f$. Define the cycle $\gamma_{f}=\left\{\left(z_{0}, \ldots, z_{n}\right) \in U| | f_{i}\left(z_{0}, \ldots, z_{n}\right) \mid=\varepsilon\right\}$ centered at the origin with orientation induced by the argument of the local complex exponential parametrization of $f_{i}$. Define the analytic index of the complex analytic germ $f$ at the origin as the residue of the logarithmic meromorphic form $\omega\left(\left.\partial f\right|_{U}\right)$,

$$
\begin{equation*}
\mu_{\mathrm{anal}}(f):=\operatorname{Res} \omega\left(\left.\partial f\right|_{U}\right) \tag{3.66}
\end{equation*}
$$

### 3.5. Mixed Hodge Structure

In this section, we discuss some facets of the mixed Hodge structure of the Milnor fiber of a non-degenerate weighted homogeneous polynomial [373]. We refer the reader to [436], [438] and [439] for basic definitions and notation.

For a non-degenerate, weighted homogeneous singularity $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$, let $I$ denote the exponent vectors of the monomials comprising a monomial basis $\left\{z^{a}=z_{0}^{a_{0}} \cdots z_{n}^{a_{n}} \mid a=\left(a_{0}, \ldots, a_{n}\right) \in I \subset \mathbb{Z}_{\geqslant 0}^{n+1}\right\}$ of the local alge$\operatorname{bra} \mathcal{A}_{f}$. For each $a=\left(a_{0}, \ldots, a_{n}\right) \in I$, define $l(a)=\sum_{i=0}^{n}\left(a_{i}+1\right) \omega_{i}$, where
$\left\{\omega_{0}, \ldots, \omega_{n}\right\} \subset \mathbb{Q} \cap(0,1)$ are the weights of $f$. The rationals $\{l(a)\}_{a \in I}$ are the eigenvalues of the Gauss-Manin connection, $\nabla z^{a}=l(a) z^{a}$ [250].

Proposition 3.17 (Arnol'd, [19]; Steenbrink, [436]). For even $n$, the diagonalized intersection form $S$ on $H_{n}\left(F_{f, 0} ; \mathbb{R}\right)$ has $\varsigma_{+}$positive, $\varsigma_{0}$ zero and $\varsigma_{-}$negative eigenvalues, respectively, where

$$
\begin{align*}
\varsigma_{+} & =\mid\{a \in I \mid l(a) \notin \mathbb{Z} \wedge[l(a)\rfloor \text { is even }\} \mid  \tag{3.67}\\
\varsigma_{0} & =|\{a \in I \mid l(a) \in \mathbb{Z}\}|  \tag{3.68}\\
\varsigma_{-} & =\mid\{a \in I \mid l(a) \notin \mathbb{Z} \wedge\lfloor l(a)\rfloor \text { is odd }\} \mid . \tag{3.69}
\end{align*}
$$

Define the Arnol'd-Steenbrink series [437],

$$
\begin{align*}
\operatorname{Sp}(f ; t) & =t^{\sum_{i=0}^{n} \omega_{i}} \bar{P}_{\mathcal{A}_{f}}(t)  \tag{3.70}\\
& =\prod_{i=0}^{n} \frac{t^{\omega_{i}}-t}{1-t^{\omega_{i}}}  \tag{3.71}\\
& =\sum_{j=1}^{\mu} t^{\gamma_{j}} \tag{3.72}
\end{align*}
$$

and denote the spectrum of $f$ by $\operatorname{Sp}(f)=\left\{\gamma_{j}\right\}$.
Proposition 3.18 (Steenbrink, [437]). If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a nondegenerate, weighted homogeneous polynomial, then the local algebra $\mathcal{A}_{f}$ determines the spectrum of $f$, namely, $\operatorname{Sp}(f)=\{l(a)\}_{a \in I}$.

Proposition 3.19. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a neighborhood of the origin. Assume that the complex analytic map $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is non-degenerate. Then

$$
\begin{equation*}
\operatorname{Sp}(f ; t)=\prod_{i=1}^{s} \operatorname{Sp}\left(f_{i} ; t\right) \tag{3.73}
\end{equation*}
$$

and, therefore,

$$
\begin{equation*}
\operatorname{Sp}(f)=\bigoplus_{i=1}^{s} \operatorname{Sp}\left(f_{i}\right)=\left\{\gamma_{1, i_{1}}+\cdots+\gamma_{s, i_{s}}\right\}_{1 \leqslant i_{\alpha} \leqslant \mu\left(f_{\alpha}\right)} \tag{3.74}
\end{equation*}
$$

Proof. Suppose $f$ and $g$ are non-degenerate weighted homogeneous polynomials with weights $\omega_{1}, \ldots, \omega_{n}$ and $v_{1}, \ldots, v_{m}$, respectively. Then $f \boxplus g$ has weights $\left\{\omega_{1}, \ldots, \omega_{n}, v_{1}, \ldots, v_{m}\right\}$. By Proposition 2.6, the local algebra $\mathcal{A}_{f \boxplus g} \cong$ $\mathcal{A}_{f} \otimes \mathcal{A}_{g}$, so $P_{\mathcal{A}_{f \boxplus g}}(t)=P_{\mathcal{A}_{f}}(t) P_{\mathcal{A}_{g}}(t)$. Thus,

$$
\begin{align*}
\operatorname{Sp}(f \boxplus g ; t) & =t^{\sum_{i=1}^{n} \omega_{i}+\sum_{j=1}^{n} v_{j}} P_{\mathcal{A}_{f} \boxplus g}(t)  \tag{3.75}\\
& =\left(t^{\sum_{i=1}^{n} \omega_{i}} P_{\mathcal{A}_{f}}(t)\right)\left(t^{\sum_{j=1}^{n} v_{j}} P_{\mathcal{A}_{g}}(t)\right)  \tag{3.76}\\
& =\operatorname{Sp}(f ; t) \operatorname{Sp}(g ; t)  \tag{3.77}\\
& =\sum_{i=1}^{\mu(f)} \sum_{j=1}^{\mu(g)} t^{\gamma_{1, i}+\gamma_{2, j}}, \tag{3.78}
\end{align*}
$$

which implies the claimed spectrum decomposition.

Corollary 3.20. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighed homogeneous polynomial, then $\operatorname{Sp}\left(\Sigma^{N} f ; t\right)=t^{N / 2} \operatorname{Sp}(f ; t)$ and, therefore,

$$
\begin{equation*}
\operatorname{Sp}\left(\Sigma^{N} f\right)=\operatorname{Sp}(f) \oplus\left\{\frac{N}{2}\right\}=\left\{\gamma_{i}+\frac{N}{2}\right\} \tag{3.79}
\end{equation*}
$$

According to Steenbrink, it is useful to consider the weight filtration $\operatorname{Gr}_{k+1}^{W} H^{k}\left(F_{f, 0}\right)$ of the cohomology group $H^{k}\left(F_{f, 0}\right)$ (with compact support $\left.H_{c}^{k}\left(F_{f, 0}\right)\right)$, where

$$
\begin{equation*}
\operatorname{Gr}_{n}^{W} H_{c}^{n}\left(F_{f, 0} ; \mathbb{C}\right)=\bigoplus_{p+q=n} H^{p, q}\left(F_{f, 0}\right) \tag{3.80}
\end{equation*}
$$

Define the corresponding intersection form $S$ by $S(\alpha, \beta)=\langle\alpha, j(\beta)\rangle, \alpha$ and $\beta$ are cycles and $j: H_{c}^{k}\left(F_{f, 0}\right) \rightarrow H^{k}\left(F_{f, 0}\right)$ according to the commutative diagram


The number of eigenvalues of $S$ according to sign can be counted by the mixed Hodge structure and rational cohomology of the fiber $F_{f, 0}$.

Proposition 3.21 (Steenbrink [436], Arnol'd [19]). Let $f:\left(\mathbb{C}^{n+1}, 0\right) \rightarrow$ $(\mathbb{C}, 0)$ be a non-degenerate, weighted homogeneous polynomial. Assume $n$ is even, and
the intersection form $S$ is diagonal on some basis of $H_{c}^{k}\left(F_{f, 0} ; \mathbb{Q}\right)$. Then $S$ has $\varsigma_{+}$positive, $\varsigma_{0}$ zero and $\varsigma_{-}$negative eigenvalues, respectively, where

$$
\begin{align*}
\zeta_{+} & =\sum_{\substack{p+q=k \\
q \text { even }}} h^{p, q}  \tag{3.81}\\
\zeta_{0} & =\operatorname{dim} G r_{n+1}^{W} H^{k}\left(F_{f, 0} ; \mathbb{Q}\right)  \tag{3.82}\\
\zeta_{-}= & \sum_{\substack{p+q=k \\
q \text { odd }}} h^{p, q}, \tag{3.83}
\end{align*}
$$

where $h^{p, q}$ denotes the $(p, q)$-Hodge number of $F_{f, 0}$. Moreover, these integers depend only the spectrum of $f$ through the following identities:

$$
\begin{align*}
\varsigma_{+} & =|\{\gamma \in \operatorname{Sp}(f) \mid \sin (\pi \gamma)>0\}|  \tag{3.84}\\
\varsigma_{0} & =|\{\gamma \in \operatorname{Sp}(f) \mid \sin (\pi \gamma)=0\}|  \tag{3.85}\\
\varsigma_{-} & =|\{\gamma \in \operatorname{Sp}(f) \mid \sin (\pi \gamma)<0\}| \tag{3.86}
\end{align*}
$$

Proposition 3.22. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial with weights $\left\{\omega_{i}\right\}$ and spectrum $\operatorname{Sp}(f)=\left\{\gamma_{j}\right\}$, then the following identities hold:

1. A spectral reciprocity relation, $\gamma_{\mu+1-j}=n+1-\gamma_{j}$, for $1 \leqslant j \leqslant \mu$;
2. $\sum_{j=1}^{\mu} \gamma_{j}=\frac{\mu}{2}(n+1)$; and,
3. $\sum_{\gamma \in \operatorname{Sp}(f)} e^{\pi i \gamma}=e^{\pi i(n+1) / 2} \prod_{i=0}^{n} \cot \left(\frac{\pi \omega_{i}}{2}\right)$.

Proof. By virtue of the identity,

$$
\begin{align*}
t^{n+1} \operatorname{Sp}\left(f ; \frac{1}{t}\right) & =t^{n+1} \prod_{i=0}^{n} \frac{t^{-\omega_{i}}-t^{-1}}{1-t^{-\omega_{i}}}  \tag{3.87}\\
& =\prod_{i=0}^{n} \frac{t^{1-\omega_{i}}-1}{1-t^{-\omega_{i}}}  \tag{3.88}\\
& =\operatorname{Sp}(f ; t) \tag{3.89}
\end{align*}
$$

the first identity holds. Consequently, $\sum_{j=1}^{\mu} \gamma_{j}=\mu(n+1)-\sum_{j=1}^{\mu} \gamma_{j}$, which implies the second identity. Finally,

$$
\begin{align*}
\operatorname{Sp}\left(f ; e^{\pi i}\right) & =\prod_{i=0}^{n} \frac{e^{\pi i \omega_{i}}-e^{\pi i}}{1-e^{\pi i \omega_{i}}}  \tag{3.90}\\
& =\prod_{i=0}^{n}-\frac{e^{\pi i \omega_{i}}+1}{e^{\pi i \omega_{i}}-1}  \tag{3.91}\\
& =\prod_{i=0}^{n} i e^{\pi i / 2} \frac{e^{\pi i \omega_{i}}+1}{e^{\pi i \omega_{i}}-1}  \tag{3.92}\\
& =e^{\pi i(n+1) / 2} \prod_{i=0}^{n} \cot \left(\frac{\pi \omega_{i}}{2}\right) . \tag{3.93}
\end{align*}
$$

Corollary 3.23. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial with weights $\left\{\omega_{i}\right\}$ and spectrum $\operatorname{Sp}(f)=\left\{\gamma_{j}\right\}$, then the
following identities hold:

$$
\sum_{\gamma \in \operatorname{Sp}(f)} \cos (\pi \gamma)=\left\{\begin{array}{cc}
-\prod_{i=0}^{n} \cot \left(\frac{\pi \omega_{i}}{2}\right) & n \equiv 1 \bmod 4  \tag{3.94}\\
\prod_{i=0}^{n} \cot \left(\frac{\pi \omega_{i}}{2}\right) & n \equiv 3 \bmod 4 \\
0 & n \text { is even }
\end{array}\right.
$$

and

$$
\sum_{\gamma \in \operatorname{Sp}(f)} \sin (\pi \gamma)=\left\{\begin{array}{cc}
\prod_{i=0}^{n} \cot \left(\frac{\pi \omega_{i}}{2}\right) & n \equiv 1 \bmod 4  \tag{3.95}\\
-\prod_{i=0}^{n} \cot \left(\frac{\pi \omega_{i}}{2}\right) & n \equiv 3 \bmod 4 \\
0 & n \text { is odd }
\end{array}\right.
$$

In particular, if $n$ is even,

$$
\begin{equation*}
\sum_{\substack{\gamma \in \operatorname{Sp}(f) \\ \cos (\pi \gamma)>0}} \cos (\pi \gamma)=\sum_{\substack{\gamma \in \operatorname{Sp}(f) \\ \cos (\pi \gamma)<0}} \cos (\pi \gamma) \tag{3.96}
\end{equation*}
$$

while if $n$ is odd,

$$
\begin{equation*}
\sum_{\substack{\gamma \in \operatorname{Sp}(f) \\ \sin (\pi \gamma)>0}} \sin (\pi \gamma)=\sum_{\substack{\gamma \in \operatorname{Sp}(f) \\ \sin (\pi \gamma)<0}} \sin (\pi \gamma) \tag{3.97}
\end{equation*}
$$

### 3.5.1. Signature of the Milnor Fiber.

Definition 3.24. The signature of the Milnor fiber $F_{f, 0}$ is the signature of the intersection form $S$, namely,

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=\varsigma_{+}-\varsigma_{-} \tag{3.98}
\end{equation*}
$$

Corollary 3.25. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial, then the signature of the Milnor fiber $F_{\Sigma^{N} f, 0}$ satisfies

$$
\sigma\left(F_{\Sigma^{N} f, 0}\right)=\left\{\begin{array}{cc}
(-1)^{N / 2} \sigma\left(F_{f, 0}\right) & n+N \text { is even }  \tag{3.99}\\
0 & n+N \text { is odd }
\end{array}\right.
$$

Proof. If $N \equiv 3 \bmod 4$, then

$$
\begin{align*}
\varsigma_{+}\left(\Sigma^{N} f\right) & =\left|\left\{\gamma \in \operatorname{Sp}(f) \mid \operatorname{Im} e^{\pi i \gamma+\pi i N / 2}>0\right\}\right|  \tag{3.100}\\
& =\left|\left\{\gamma \in \operatorname{Sp}(f) \mid \operatorname{Im} e^{\pi i \gamma}<0\right\}\right|  \tag{3.101}\\
& =\zeta_{-}(f)  \tag{3.102}\\
\varsigma_{-}\left(\Sigma^{N} f\right) & =\left|\left\{\gamma \in \operatorname{Sp}(f) \mid \operatorname{Im} e^{\pi i \gamma+\pi i N / 2}<0\right\}\right|  \tag{3.103}\\
& =\left|\left\{\gamma \in \operatorname{Sp}(f) \mid \operatorname{Im} e^{\pi i \gamma}>0\right\}\right|  \tag{3.104}\\
& =\zeta_{+}(f) . \tag{3.105}
\end{align*}
$$

Alternatively, if $N \equiv 1 \bmod 4$ or even, there is no overall sign change to the imaginary parts of the exponentials, so $\varsigma_{ \pm}\left(\Sigma^{N} f\right)=\varsigma_{ \pm}(f)$. However, if $n$ is odd, then $\varsigma_{+}(f)=\varsigma_{-}(f)$, so $\sigma\left(F_{\Sigma^{N} f, 0}\right)=0$. Thus, if $n$ is odd and $N$ is even, $\sigma\left(F_{\Sigma^{N} f, 0}\right)=0$. Since $\mu_{\mathrm{alg}}\left(\Sigma^{N} f\right)=\mu_{\mathrm{alg}}(f)$, it follows that $\varsigma_{0}\left(\Sigma^{N} f\right)=\varsigma_{0}(\Sigma f)$ if $N$ is odd and $\zeta_{0}(f)$ otherwise. The other cases are handled similarly.

3•5.1.1. Dual Steenbrink Numbers. Define the following integers,

$$
\begin{align*}
& \omega_{+}=|\{\gamma \in \operatorname{Sp}(f) \mid \cos (\pi \gamma)>0\}|  \tag{3.106}\\
& \omega_{0}=|\{\gamma \in \operatorname{Sp}(f) \mid \cos (\pi \gamma)=0\}|  \tag{3.107}\\
& \omega_{-}=|\{\gamma \in \operatorname{Sp}(f) \mid \cos (\pi \gamma)<0\}| . \tag{3.108}
\end{align*}
$$

REmARK 3.5.1. As the total number of eigenvalues equals the size of $S$,

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\varsigma_{+}+\varsigma_{0}+\varsigma_{-}=\omega_{+}+\omega_{0}+\omega_{-} \tag{3.109}
\end{equation*}
$$

REmARK 3.5.2. The shift map $\gamma \rightarrow \gamma+\frac{1}{2}$ lifts to a involution between $\left\{\varphi_{+}, \varphi_{0}, \varphi_{-}\right\}$and $\left\{\omega_{+}, \omega_{0}, \omega_{-}\right\}$.

Define $\operatorname{sgn}^{ \pm}(x)$ to 1 if and only if $\pm x>0$ and 0 otherwise. Define $\operatorname{sgn}^{0}(x)=$ $\delta_{x, 0}$. Observe $\operatorname{sgn}(x)=\operatorname{sgn}^{+}(x)-\operatorname{sgn}^{-}(x)$.

Conjecture 3.26. For $a_{0}, \ldots, a_{n} \in \mathbb{N}$,

$$
\begin{align*}
\zeta_{+} & =\sum_{k_{0}=1}^{a_{0}-1} \cdots \sum_{k_{n}=1}^{a_{n}-1} \operatorname{sgn}^{+} \sin \left(\pi \sum_{i=0}^{n} \frac{k_{i}}{a_{i}}\right)  \tag{3.110}\\
\zeta_{0} & =\sum_{k_{0}=1}^{a_{0}-1} \cdots \sum_{k_{n}=1}^{a_{n}-1} \operatorname{sgn}^{0} \sin \left(\pi \sum_{i=0}^{n} \frac{k_{i}}{a_{i}}\right)  \tag{3.111}\\
\zeta_{-} & =\sum_{k_{0}=1}^{a_{0}-1} \cdots \sum_{k_{n}=1}^{a_{n}-1} \operatorname{sgn}^{-} \sin \left(\pi \sum_{i=0}^{n} \frac{k_{i}}{a_{i}}\right) . \tag{3.112}
\end{align*}
$$

Conjecture 3.27. For $a_{0}, \ldots, a_{n} \in \mathbb{N}$,

$$
\begin{align*}
& \omega_{+}=\sum_{k_{0}=1}^{a_{0}-1} \cdots \sum_{k_{n}=1}^{a_{n}-1} \operatorname{sgn}^{+} \cos \left(\pi \sum_{i=0}^{n} \frac{k_{i}}{a_{i}}\right)  \tag{3.113}\\
& \omega_{0}=\sum_{k_{0}=1}^{a_{0}-1} \cdots \sum_{k_{n}=1}^{a_{n}-1} \operatorname{sgn}^{0} \cos \left(\pi \sum_{i=0}^{n} \frac{k_{i}}{a_{i}}\right)  \tag{3.114}\\
& \omega_{-}=\sum_{k_{0}=1}^{a_{0}-1} \cdots \sum_{k_{n}=1}^{a_{n}-1} \operatorname{sgn}^{-} \cos \left(\pi \sum_{i=0}^{n} \frac{k_{i}}{a_{i}}\right) . \tag{3.115}
\end{align*}
$$

Conjecture 3.28. Given a non-degenerate, weighted homogeneous germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, the integers $\left(\varsigma_{+}, \varsigma_{0}, \varsigma_{-}\right)$and $\left(\omega_{+}, \omega_{0}, \omega_{-}\right)$can be computed from counting certain lattice points of the corresponding weight polytope $\mathcal{W}(f)$.

This concludes our remarks on the analytic structure of complex analytic singularities. We proceed now with a discussion of more geometric aspects.

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## Chapter 4

# Geometric Structure of Isolated Singularities 

Children and lunatics cut the Gordian knot which the poet spends his life patiently trying to untie. - Jean Cocteau

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In this chapter we review elementary features of links including some wellknown link invariants. The algebraic link of an isolated singularity will take an central role. We discuss a number of important geometric features of the algebraic links and pay particular attention to those of Brieskorn-Pham singularities. While many topics are covered in this chapter, they are done so expeditiously.

We refer the reader to [403], [335] and [274] for beautiful and elucidating introductions to the mathematical theory of links.

### 4.1. Links

Recall that a $\operatorname{link} k^{*}$ is a compact, oriented 1-submanifold without boundary, possibly with multiple components and smoothly embedded in $S^{3}\left(\right.$ or $\left.\mathbb{R}^{3}\right)$. A $k n o t$ is a link of one (connected) component. Higher-dimensional links are defined analogously as compact, oriented codimension-two submanifolds without boundary, possibly with multiple components and smoothly embedded in $S^{n}$ (or $\mathbb{R}^{n}$ ). A link projection or diagram is a static representation of a link. One often conflates a link with its projections.

Two links $L=\coprod K_{i} \subset S^{n}$ and $L^{\prime}=\coprod K_{i}^{\prime} \subset S^{n}$ are link equivalent if and only if there is an orientation-preserving homeomorphism $\varphi: S^{n} \rightarrow S^{n}$ which descends to an orientation-preserving homeomorphism between the connected components $\tilde{\varphi}: K_{i} \rightarrow K_{i}^{\prime}$, that is,


Two links $L=\coprod K_{i} \subset \mathbb{R}^{n}$ and $L^{\prime}=\coprod K_{i}^{\prime} \subset \mathbb{R}^{n}$ are ambient isotopic if and only if there is an orientation-preserving homeomorphism $\varphi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$
*We consider only tame links.


Figure 4.1. A Medley of Links [418]
which descends to an orientation-preserving homeomorphism between the connected components $\tilde{\varphi}: L_{i} \rightarrow L_{i}^{\prime}$. In particular, two links $L=\coprod K_{i} \subset S^{3}$ and $L^{\prime}=\coprod K_{i}^{\prime} \subset S^{3}$ are planar isotopic if and only if there is a finite sequence of Reidemeister moves applied to one yielding the other. The Reidemeister moves (types I, II and III) are shown in Figure 4.2. The latter two notions coincide if
$L, L^{\prime} \subset S^{3}$. For this section, unless explicitly stated otherwise, we consider only links in $S^{3}$.


Figure 4.2. Reidemeister Moves (Types I, II and III) [418]

Eight inequivalent links with two and three components are shown in Figures 4.3 and 4.22.


Figure 4.3. Four Inequivalent Links with Two Components $\left(2_{1}^{2}, 4_{1}^{2}, 5_{1}^{2}\right.$ and $6_{1}^{2}$ ) [418]
4.1.1. Link Orientation and Chirality. As each knot has exactly two canonical orientations, namely, the two orientations in any of its projections $K$ and $-K$. Each link of $r$ components has $2^{r}$ orientations resulting from those of its constituents. Chirality, in the context of knots, refers to the asymmetry of orientation between a knot $K$ and its reflected image or mirror $K^{*}$ through a transversal plane. Including the asymmetric case, the following symmetries exhaust the possible knot types with respect to orientation and chirality [93],
2. (Reversible) $K=-K$;
3. ((+)-Amphichiral) $K=K^{*}$;
4. ((-)-Amphichiral) $K=-K^{*}$; and,
5. (Strongly Amphichiral) $K=-K=K^{*}=-K^{*}$.

Remark 4.1.1. Figure 4.4 depicts two enantiomorphs of the trefoil knot. In fact, all torus knots are chiral. The figure-eight knot is equivalent to its reverse and mirror image, so it is strongly amphichiral.


Figure 4.4. Enantiomorphs of the Trefoil $\operatorname{Knot}\left(3_{1}\right.$ and $\left.3_{1}^{*}\right)$ [389]
4.1.2. Prime Knots. The connected sum of two n-manifolds involves removing the interiors of two $B^{n}$ from each and gluing the resulting two $S^{n-1}$. In particular, the connected sum of knots in $S^{3}$ ( or $\mathbb{R}^{3}$ ) involves removing two open intervals $\left(B^{1}\right)$ and gluing the resulting two endpoints $\left(S^{0}\right)$.

Table 4.1. Number of Prime Knots by Crossing Number

| $c(K)$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $N(K)$ | 1 | 1 | 2 | 3 | 7 | 21 | 49 | 165 | 552 | 2176 | 9988 | 46972 | 253293 | 1388705 |

The initial orientations of link connected-summands is relevant for the orientation of the connected summation. For example, the granny knot $3_{1} \# 3_{1}$ is the
connected sum of two trefoil knots with the same orientation, while the square knot $3_{1} \# 3_{1}^{*}$ is the connected sum of two trefoil knots with its mirror image (Figure 4.5 ).

Definition 4.1. A knot is prime if it cannot be decomposed into a connected sum of knots. A knot that is not prime is composite.


Figure 4.5. Two Connected Sums of Two Trefoil Knots ( $3_{1} \# 3_{1}$ and $3_{1} \# 3_{1}^{*}$ ) [389]

## Proposition 4.2. The following statements are true:

1. The connected sum of any knot with the unknot is ambient isotopic to the (former) knot; and,
2. No non-trivial connected sum is ambient isotopic to the unknot.

Proof. See §G, Chapter 2 in [403].

Proposition 4.3 (Schubert, 1949). The set of oriented knots equipped with the operation of connected sum is a monoid with unique prime factorization (up to a permutation of knot summands).


Figure 4.6. Prime Links Ordered by Increasing Crossing Number [418]
4.1.3. Link Crossing Number. The crossing number of a link $L$ is the minimum number of crossings in any link diagram of $L$. The crossing number is a link invariant. For example, there are only three inequivalent prime knots with
six crossings, namely, $6_{1}, 6_{2}$ and $6_{3}$ (Figure 4.7). The number $N(K)$ of prime knots with crossing number $c(K) \leqslant 16$ is known (Table 4.1).


Figure 4.7. Three Prime Knots with Six Crossings $\left(6_{1}, 6_{2}\right.$ and $\left.6_{3}\right)$ [389]

The crossing number of the torus link $T_{p, q}$ and twist $\operatorname{knot} T_{n}$, where $n$ denotes the number of half-twists, is known, namely, $c\left(T_{p, q}\right)=\min \{p(q-1), q(p-$ $1)\}$ and $c\left(T_{n}\right)=n+2$, respectively. Conversely, the number of torus links $C_{n}$ with crossing number $n$ is given by the formula [333] (A051764),

$$
\begin{equation*}
C_{n}=\sum_{\substack{\sqrt{n}<k \leqslant n \\ \operatorname{GCD}\left(k, 1+\frac{n}{k}\right)=1}} 1 . \tag{4.1}
\end{equation*}
$$

4.1.4. Link Unknotting Number. Let $K=K_{0}$ be a finite, tame knot. By considering a sequence of stepwise crossing changes, for any $K$ there is a finite unknotting sequence $\mathcal{K}=\left\{K_{0}, K_{1}, \ldots, K_{n}\right\}$, where $K_{i+1}$ and $K_{i}$ differ by the sign of a single crossing, and $K_{n}$ is isotopic to the unknot.

Definition 4.4. The unknotting number $u(K)=\inf _{\mathcal{K}}\{|\mathcal{K}|\}$ is the fewest number of crossing changes to unknot $K$.


Figure 4.8. Six Twist Knots $\left(3_{1}, 4_{1}, 5_{2}, 6_{1}, 7_{2}\right.$ and $\left.8_{1}\right)$ [389]
For links, an analogous unknotting number is defined as the fewest number of crossings across all projections to completely disentangle a given link into a disjoint union of unknots (Figure 4.11).

Proposition 4.5 (Scharlemann, Thompson, [419]). Every knot with unknotting number equal to 1 is prime.

Proof. See [419] and [496].
The converse of Proposition 4.5 , however, is not true; not all prime knots have unknotting number equal to 1 . For instance, a twist $\operatorname{knot} T_{n}$ has unknotting number 1 for $n \geqslant 1$, since it is equivalent to the unknot with $n$ half-twists and single-linked ends (Figure 4.8). Since each prime knot has unknotting at least 1 , it follows that every composite knot has unknotting number at least 2.
4.1.5. Linking Number. Consider an oriented link $L$ with at least two disjoint, oriented components $K$ and $K^{\prime}$ with Seifert surfaces $F_{K}$ and $F_{K^{\prime}}$, respectively. Without loss of generality, one may assume that $K$ intersects $F_{K^{\prime}}$ transversally with a finite number of points $K \cap F_{K^{\prime}}=\left\{p_{1}, \ldots, p_{n}\right\}$ by appropriately perturbing $F_{K^{\prime}}$. By considering the tangent vector of a point of intersection with respect to the direction $K$ through $F_{K^{\prime}}$, one may assign a value $\varepsilon(p)$ equal 1 or -1 with the right-hand-rule.

Definition 4.6. The linking number $\operatorname{lk}\left(K, K^{\prime}\right)$ of two disjoint, oriented knots $K$ and $K^{\prime}$ is the integer

$$
\begin{equation*}
\mathrm{lk}\left(K, K^{\prime}\right)=\sum_{p \in K \cap F_{K^{\prime}}} \varepsilon(p) . \tag{4.2}
\end{equation*}
$$

If the sum is empty, then the corresponding linking number is 0 .
Remark 4.1.2. Milnor invariants generalize the linking number to crossings involving more than two components [304] (also Chapter 8 in [324]).

Proposition 4.7. Let $K$ and $K^{\prime}$ be two disjoint, oriented knots. Let $c_{+}\left(K, K^{\prime}\right)$ and $c_{-}\left(K, K^{\prime}\right)$ denote the total number of positive and negative crossings between $K$ and $K^{\prime}$. The linking number $\operatorname{lk}\left(K, K^{\prime}\right)$ admits the equivalent representation the summation

$$
\operatorname{lk}\left(K, K^{\prime}\right)=\frac{1}{2} \sum_{K \neg K^{\prime}} c_{+}\left(K, K^{\prime}\right)-c_{-}\left(K, K^{\prime}\right) .
$$



Figure 4.9. Linking Number at Each Crossing Type
Proof. This is an exercise in combinatorics, so we omit further details.

Remark 4.1.3. Figure 4.9 illustrates the four crossing types.

Remark 4.1.4. Proposition 4.7 implies $\operatorname{lk}\left(K, K^{\prime}\right)=\operatorname{lk}\left(K^{\prime}, K\right)$.

Definition 4.8. The total linking number $\operatorname{lk}(L)$ of an oriented $\operatorname{link} L$ is one-half of the sum of the linking numbers over all crossings in any projection, viz.,

$$
\begin{equation*}
\operatorname{lk}(L)=\sum_{1 \leqslant i<j \leqslant r} \operatorname{lk}\left(K_{i}, K_{j}\right) . \tag{4.4}
\end{equation*}
$$

The total linking number of the torus link is known, namely, $\operatorname{lk}\left(T_{p, q}\right)=p q$. Figure 4.10 depicts four torus knots. A closely related invariant of the linking number is the writhe.


Figure 4.10. Four Torus Knots $\left(0_{1}, 3_{1}, 5_{1}\right.$ and $\left.7_{1}\right)$ [389]
Definition 4.9. The writhe $\mathrm{w}(L)$ of an oriented link $L$ is the sum of its linking numbers over all crossings in any projection, viz.,

$$
\begin{equation*}
\mathrm{w}(L)=\sum_{K \sqcap K^{\prime} \subset L} \operatorname{lk}\left(K, K^{\prime}\right) . \tag{4.5}
\end{equation*}
$$

REmARK 4.1.5. The writhe of the Hopf link (with similarly oriented components) is 2 , an oriented trefoil is 3 and Borromean rings (with similarly oriented components) is 0 .
4.1.6. Knot and Link Groups. A knot group $\pi(K)$ of a knot $K \subset S^{3}$ is the fundamental group of the knot complement, $\pi_{1}\left(S^{3} \backslash K\right)$. In any given projection, there is a finite number of crossings which divides said knot into $p$ arcs labeled $\left\{\gamma_{1}, \ldots, \gamma_{p}\right\}$. Using any orientation of $K$, order the labels of the arcs such that arcs with consecutive labels meet a crossing (assuming, of course, $\gamma_{p}$ meets $\gamma_{1}$ ). Underneath each crossing $\gamma_{i} \rightarrow \gamma_{j}$ (incoming) and $\gamma_{i+1} \rightarrow \gamma_{j+1}$ (outgoing), place a sufficiently small oriented loop in the form of a square with sides labeled (clockwise from the top) $x_{j}, x_{i+1}, x_{j}$ and $x_{i}$ (coinciding with the closest arc) with a side orientation that yields a positive crossing at each crossing of the
square sides and arcs. The square represents concatenated loops surrounding each arc with positive orientation that extend to infinity, from which one computes directly the fundamental group of the complement. This analysis can be summarized by the Wirtinger presentation,

$$
\begin{equation*}
\pi(K) \cong\left\langle x_{1}, \ldots, x_{p} \mid r_{1}, \ldots, r_{p}\right\rangle \tag{4.6}
\end{equation*}
$$

with relations only of the form $x_{i} x_{j} x_{i+1}^{-1} x_{j}^{-1}$ (positive crossing) or $x_{i} x_{j}^{-1} x_{i+1}^{-1} x_{j}$ (negative crossing). We refer the reader to Chapter 2 in [324] for a detailed construction of this brief discussion.

REMARK 4.1.6. The knot group is a knot invariant. Equivalent knots have isomorphic knot groups, but the converse is not true.

The link group $\pi(L)$ is a distinguished quotient group of $\pi_{1}\left(S^{3} \backslash L\right)$, where components are considered up to link homotopy (each component is allowed to pass through itself but not any other component) [304]. Let $L_{r}$ be an $r$ component link in $S^{3}$. The fundamental group of the complement $\pi_{1}\left(S^{3} \backslash L_{r}\right)$ is an Eilenberg-Maclane space, $K(\pi, 1)$ only for $r=1$. However, for $r>1$, the homology groups are known,

$$
H_{k}\left(S^{3} \backslash L_{r}\right) \cong \begin{cases}\mathbb{Z} & k=0  \tag{4.7}\\ \mathbb{Z}^{r} & k=1 \\ \mathbb{Z}^{r-1} & k=2 \\ \{0\} & k \geqslant 3\end{cases}
$$

and, in particular, $H_{1}\left(S^{3} \backslash L_{r}\right) \cong \pi_{1}\left(S^{3} \backslash L_{r}\right)^{\text {ab }}$, the abelianization of the fundamental group (Chapter 2, [111]).

Remark 4.1.7. Equivalent links have isomorphic links groups, but the converse is not true.

Remark 4.1.8. Since the fundamental group of the complement of a disjoint union of unknots is the free product of those of each unknot, namely, $\mathbb{Z}$, it follows that for the trivial link with $r$ components, $\pi\left(0_{1}^{r}\right) \cong \mathrm{F}_{r}$, the free group on $r$ generators.


Figure 4.11. Unlink with Five Components [418]

Remark 4.1.9. Define the Braid Group

$$
\begin{equation*}
B_{n}=\left\langle\sigma_{1}, \ldots, \sigma_{n-1} \mid \sigma_{i} \sigma_{i+1} \sigma_{i}=\sigma_{i+1} \sigma_{i} \sigma_{i+1}, \sigma_{i} \sigma_{j}=\sigma_{j} \sigma_{i}\right\rangle_{1 \leqslant i \leqslant n-2,|i-j| \geqslant 2} \tag{4.8}
\end{equation*}
$$

The Wirtinger presentation of the knot group of the trefoil is precisely

$$
\begin{align*}
\pi\left(T_{2,3}\right) & \cong\left\langle x, y, z \mid y x z^{-1} x^{-1}, z y x^{-1} y^{-1}, x z y^{-1} z^{-1}\right\rangle  \tag{4.9}\\
& \cong\langle x, y \mid x y x=y x y\rangle \tag{4.10}
\end{align*}
$$

which is the braid group $B_{3}$, since $z=y x y^{-1}$, and the various relations can be simplified accordingly. A similar, but substantially more involved analysis proves that for any coprime, positive integers $p$ and $q$,

$$
\begin{equation*}
\pi\left(T_{p, q}\right) \cong\left\langle x, y \mid x^{p}=y^{q}\right\rangle \tag{4.11}
\end{equation*}
$$

Picantin computed the link group of the torus link.

Proposition 4.10 (Picantin, [375]). The link group of the torus link $T_{p, q}$ is a group with $p$ generators $\left\{x_{1}, \ldots, x_{p}\right\}$ satisfying the relations

$$
x_{i}\left(x_{1} \cdots x_{p}\right)^{\left\lfloor\frac{q-i}{p}\right\rfloor+1}=\left(x_{1} \cdots x_{p}\right)^{\left\lfloor\frac{q-i}{p}\right\rfloor+1} x_{p\left(\left\lfloor\frac{q-i}{p}\right\rfloor+1\right)-q+i} \quad 1 \leqslant i \leqslant p
$$

Proof. See Lemma 2.1 in [375].

Proposition 4.11 (Neuwirth, Burde, Zeischang). A knot is the unknot or a torus knot if and only if its knot group admits a non-trivial center.

### 4.1.7. Seifert Surfaces.

Definition 4.12. A Seifert surface $F$ of an oriented $\operatorname{link} L \subset S^{3}$ is a compact, connected, orientable surface embedded in $S^{3}$ with boundary $\partial F=L$.

Proposition 4.13 (Frankl, Pontrjagin; Seifert). Each link has a Seifert surface, as well as an algorithm to construct it.


Figure 4.12. A Seifert Surface of the Trefoil Knot (Adapted from [114])
Proof. See [133] and [422].

Remark 4.1.10. Figure 4.12 illustrates a Seifert surface of the trefoil with three levels of transparency.
4.1.8. Alexander Polynomials. For this section, we refer the reader unfamiliar with the theory of knots and links to the fine book [403].

Given an oriented knot $K \subset S^{3}$ with a tubular neighborhood $T(K)$, define $M_{K}$ as the complement $S^{3} \backslash T(K)^{\circ}$. Let $M_{K, r}$ and $\Sigma_{K, r}$ denote the $r$-cyclic covering and $r$-cyclic branched covering of $M_{K}$, respectively, where the former can be constructed by gluing $r$ copies of a Seifert surface of $K$. The homology group $H_{1}\left(M_{K, r} ; R\right)$, where the ring $R$ being infinite cyclic or rational, is a finitelygenerated $R\left[t, \frac{1}{t}\right]$-module, where $t$ is the generator of the cyclic group $\mathbb{Z}_{r}$ acting on $H_{1}\left(M_{K, r} ; \mathbb{Z}\right)$. In particular, $H_{1}\left(M_{K, r} ; \mathbb{Z}\right) \cong H_{1}\left(\Sigma_{K, r} ; \mathbb{Z}\right) \oplus \mathbb{Z}$ (§D, [403]). The infinite-cyclic covering $M_{K, \infty}$, and consider the homology group $H_{1}\left(M_{K, \infty} ; R\right)$ with coefficients $R$ analogously.

Definition 4.14. The $k^{\text {th }}$-Alexander ideal $\Lambda_{k}$ of $K$ is the $k^{\text {th }}$-elementary ideal of $H_{1}\left(M_{K, \infty} ; \mathbb{Z}\right)$.

Definition 4.15. The $k^{\text {th }}$-Alexander polynomial $\Delta_{K, k}(t)$ of $K$ is a generator of the minimal principal ideal of $\mathbb{Z}\left[t, \frac{1}{t}\right]$ containing $\Lambda_{k}$. The Alexander polynomial of a knot is $\Delta_{K}(t)=\Delta_{K, 0}(t)$, where in particular $\Delta_{0_{1}}(t)=1$.

Definition 4.16. The $k^{\text {th }}$-Alexander matrix of $K$ is any presentation matrix of $H_{1}\left(M_{K, \infty} ; \mathbb{Z}\right)$ corresponding to the $k^{\text {th }}$-Alexander ideal.

Proposition 4.17. Given a knot $K$ with (zeroth) Alexander polynomial $\Delta_{K}(t)$,

$$
\begin{equation*}
H_{1}\left(M_{K, \infty} ; \mathbb{Z}\right) \cong \mathbb{Z}\left[t, \frac{1}{t}\right] / \Delta_{K}(t) \tag{4.12}
\end{equation*}
$$

Proof. See Chapters 7 and 8 in [403].

Remark 4.1.11. The Alexander ideals, polynomials and matrices are knot invariants.

Remark 4.1.12. All Alexander polynomials of $K$ are defined only up to products by units in the corresponding ideal $\mathbb{Z}\left[t, \frac{1}{t}\right]$, namely $t,-t, \frac{1}{t}$ and $-\frac{1}{t}$. $\quad \Delta$

Proposition 4.18. Let $S_{K, k}$ denote the Seifert matrix corresponding to the $k^{\text {th_ }}$ Alexander module of a knot $K$. The matrix $t S_{K, k}-S_{K, k}^{\top}$ is an Alexander matrix for (the $k^{\text {th }}$-Alexander module of) $K$. The following statements are true:

1. The Alexander polynomial $\Delta_{K, k}(t)$ is the determinant $\operatorname{det}\left(t S_{K, k}-S_{K, k}^{\top}\right)$ up to a unit in $\mathbb{Z}\left[t, \frac{1}{t}\right], \Delta_{K, k}(t) \doteq \operatorname{det}\left(t S_{K, k}-S_{K, k}^{\top}\right)$;
2. $\operatorname{deg} \Delta_{K, k}(t)$ is even;
3. $\Delta_{K, k}(t) \doteq \Delta_{K, k}\left(\frac{1}{t}\right)$;
4. $\Delta_{K, k}(1)= \pm 1$;
5. $\Delta_{K, k}(t)$ is factor of $\Delta_{K, k-1}(t)$;

In particular, if $K$ is a connected sum $K^{\prime} \# K^{\prime \prime}$, then $\Delta_{K \# K^{\prime}, k}(t)=\Delta_{K^{\prime}, k}(t) \Delta_{K^{\prime \prime}, k}(t)$.

Proof. The proof of the third statement follows from the fact that there is an integer $m$ such that $\operatorname{det}\left(t S_{K, k}-S_{K, k}^{\top}\right)= \pm t^{m} \operatorname{det}\left(\frac{1}{t} S_{K, k}-S_{K, k}^{\top}\right)$. The fourth statement follows from the specific form of the Seifert matrix. The last statement follows from the fact that $S_{K, k}=S_{K^{\prime}, k} \oplus S_{K^{\prime \prime}, k}$ and the determinant identity

$$
\begin{align*}
\operatorname{det}\left(t S_{K, k}-S_{K, k}^{\top}\right) & =\operatorname{det}\left(t S_{K^{\prime}, k} \oplus S_{K^{\prime \prime}, k}-\left(S_{K^{\prime}, k} \oplus S_{K^{\prime \prime}, k}\right)^{\top}\right)  \tag{4.13}\\
& \doteq \operatorname{det}\left(\left(t S_{K^{\prime}, k}-S_{K^{\prime}, k}^{\top}\right) \oplus\left(t S_{K^{\prime \prime}, k}-S_{K^{\prime \prime}, k}^{\top}\right)\right)  \tag{4.14}\\
& \doteq \operatorname{det}\left(t S_{K^{\prime}, k}-S_{K^{\prime}, k}^{\top}\right) \operatorname{det}\left(t S_{K^{\prime \prime}, k}-S_{K^{\prime \prime}, k}^{\top}\right) \tag{4.15}
\end{align*}
$$

For proofs of the remaining statements, see Propositions 8.11, 8.12 and 8.14 in [74] and also Theorem 3, §C, Chapter 8 in [403].

Remark 4.1.13. Principal Seifert matrices for the prime knots $3_{1}, 4_{1}$ and $5_{1}$ are the following:

$$
S_{3_{1}}=\left(\begin{array}{cc}
-1 & -1  \tag{4.16}\\
0 & -1
\end{array}\right), \quad S_{4_{1}}=\left(\begin{array}{cc}
1 & 0 \\
-1 & -1
\end{array}\right)
$$

and

$$
S_{5_{1}}=\left(\begin{array}{cccc}
-1 & -1 & 0 & -1  \tag{4.17}\\
0 & -1 & 0 & 0 \\
-1 & -1 & -1 & -1 \\
0 & -1 & 0 & -1
\end{array}\right)
$$

from which one computes the following Alexander polynomials,

$$
\begin{align*}
& \Delta_{3_{1}}(t) \doteq \operatorname{det}\left(t S_{3_{1}}-S_{3_{1}}^{\top}\right)=1-t+t^{2}  \tag{4.18}\\
& \Delta_{4_{1}}(t) \doteq \operatorname{det}\left(t S_{4_{1}}-S_{4_{1}}^{\top}\right)=-\left(1-3 t+t^{2}\right)  \tag{4.19}\\
& \Delta_{5_{1}}(t) \doteq \operatorname{det}\left(t S_{5_{1}}-S_{5_{1}}^{\top}\right)=1-t+t^{2}-t^{3}+t^{4} \tag{4.20}
\end{align*}
$$

For the convenience of the reader, the (zeroth) Alexander polynomials of all prime knots up to seven crossings is given in the Appendix.

Definition 4.19. The knot determinant det $K$ is the value $\left|\Delta_{K}(-1)\right|$.

Remark 4.1.14. The knot determinant is a knot invariant.
Remark 4.1.15. One has $\operatorname{det} 3_{1}=3$ and $\operatorname{det} 4_{1}=\operatorname{det} 5_{1}=5$.
Proposition 4.20. Let $\Sigma_{K, r}$ denote an r-cyclic branched covering of a knot $K \subset \mathrm{~S}^{3}$. The order of the homology group $H_{1}\left(\Sigma_{K, 2}\right)$ is equal to det $K$, which is an odd integer.

Proof. See Theorem 1, Corollary 3 and Exercise 2 in §D, Chapter 8 in [403].

REMARK 4.1.16. The isomorphism $H_{1}\left(M_{K, r}\right) \cong H_{1}\left(\Sigma_{K, r}\right) \oplus \mathbb{Z}$ implies the evenness of $\left|H_{1}\left(M_{K, 2}\right)\right|$.

Proposition 4.21 (Fox, [132]; Weber [469]). If $M_{K, r}$ is an $r$-fold cyclic covering over $K \subset S^{3}$, then the order of the first integral homology group of $M_{K, r}$ is the following product

$$
\begin{equation*}
\left|H_{1}\left(M_{K, r} ; \mathbb{Z}\right)\right|=\prod_{k=0}^{r-1} \Delta_{K}\left(\zeta_{r}^{k}\right) \tag{4.21}
\end{equation*}
$$

where a zero product signifies an infinite order.

Remark 4.1.17. Consider the trefoil $3_{1}$ with Alexander polynomial $\Delta_{3_{1}}(t)=t^{2}-t+1$. By Proposition 4.21,

$$
\begin{align*}
\left|H_{1}\left(M_{3_{1}, r} ; \mathbb{Z}\right)\right| & =\prod_{k=0}^{r-1}\left(\zeta_{r}^{2 k}-\zeta_{r}^{k}+1\right)  \tag{4.22}\\
& =2-2 \cos \left(\frac{\pi r}{3}\right)  \tag{4.23}\\
& =\left\{\begin{array}{rl}
0 & r \equiv 0 \bmod 6 \\
1 & r \equiv\{1,5\} \bmod 6 \\
3 & r \equiv\{2,4\} \bmod 6 \\
4 & r \equiv 3 \bmod 6
\end{array}\right. \tag{4.24}
\end{align*}
$$

$\Delta$

Proposition 4.22 (Jensen). If $f: \mathbb{C} \rightarrow \mathbb{C}$ is analytic in a region that contains the circle $\partial \Delta_{\varepsilon}$ centered at the origin, $f(0) \neq 0$ and has (possibly some) zeros
$\left\{\alpha_{1}, \ldots, \alpha_{n}\right\} \subset \Delta_{\varepsilon}^{\circ}$, then

$$
\begin{equation*}
\log |f(0)|=\sum_{i=0}^{n} \log \frac{\left|\alpha_{i}\right|}{\varepsilon}+\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left|f\left(r e^{i \theta}\right)\right| d \theta \tag{4.25}
\end{equation*}
$$

Proof. See Theorem 9.1.2 in [166].

Proposition 4.23 (González-Acuña, Short, [155]). If the Alexander polynomial $\Delta_{K}(t)$ of a knot $K$ does not vanish on the circle, then the first homology group of an $r$-cyclic covering $M_{K, r}$ satisfies the following limit,

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \frac{1}{r} \log \left|H_{1}\left(M_{K, r} ; \mathbb{Z}\right)\right|=\log m\left(\Delta_{K}\right) \tag{4.26}
\end{equation*}
$$

where $m\left(\Delta_{K}\right)$ denotes the Mahler measure of $\Delta_{K}$.

Proof. We sketch briefly Theorem 11.4 in [324]. The Mahler measure of a monic polynomial $f(z)=\prod_{i=1}^{n}\left(z-\alpha_{i}\right)$ is defined as

$$
\begin{align*}
m(f) & =\exp \left(\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left|f\left(e^{i \theta}\right)\right| d \theta\right)  \tag{4.27}\\
& =\prod_{i=1}^{n} \max \left\{\left|\alpha_{i}\right|, 1\right\} \geqslant 1 \tag{4.28}
\end{align*}
$$

By Proposition 4.22, it follows that

$$
\begin{align*}
\log m\left(\Delta_{K}\right) & =\int_{0}^{2 \pi} \log \left|\Delta_{K}\left(e^{i \theta}\right)\right| d \theta  \tag{4.29}\\
& =\int_{0}^{1} \log \left|\Delta_{K}\left(e^{2 \pi i \tau}\right)\right| d \tau  \tag{4.30}\\
& =\lim _{r \rightarrow \infty} \frac{1}{r} \prod_{k=0}^{r-1}\left|\Delta_{K}\left(e^{2 \pi i k / r}\right)\right| . \tag{4.31}
\end{align*}
$$

Finally, apply Proposition 4.21.
Remark 4.1.18. Consider the figure-eight ( $4_{1}$ ) and Stevedore ( $6_{1}$ ) knots with Alexander polynomials $\Delta_{4_{1}}(t)=1-3 t+t^{2}$, which has roots $\left\{\frac{1}{2}(3-\right.$ $\left.\sqrt{5}), \frac{1}{2}(3+\sqrt{5})\right\}$, and $\Delta_{6_{1}}(t)=2-5 t+2 t^{2}$, which has roots $\left\{\frac{1}{2}, 2\right\}$, respectively. By Proposition 4.23,

$$
\begin{align*}
& \lim _{r \rightarrow \infty} \frac{1}{r} \log \left|H_{1}\left(M_{4_{1}, r} ; \mathbb{Z}\right)\right|=\log \left(\frac{1}{2}(3+\sqrt{5})\right)  \tag{4.32}\\
& \lim _{r \rightarrow \infty} \frac{1}{r} \log \left|H_{1}\left(M_{6_{1}, r} ; \mathbb{Z}\right)\right|=\log 2 \tag{4.33}
\end{align*}
$$

For a discussion of the Mahler measure of the Alexander polynomial and its relation to homology of the corresponding knot within the context of $p$-adic number theory and dynamical systems, see [351].
4.1.9. Zeta Function of a Knot. The zeta function of a knot provides an explicit link between topology, geometry, analysis and number theory. Recall there is a map $h: M_{K, \infty} \rightarrow M_{K, \infty}$ and its push-forward $h_{*}: H_{1}\left(M_{K, \infty} ; \mathbb{Q}\right) \rightarrow$
$H_{1}\left(M_{K, \infty} ; \mathbb{Q}\right)$, where $\langle h\rangle \cong \mathbb{Z}$ acts on $H_{1}\left(M_{K, \infty} ; \mathbb{Z}\right)$. The Lefschetz zeta function of a knot $K \subset S^{3}$ is that of $h$, equal to the product and (finite) rational function

$$
\begin{align*}
\zeta_{K}(t) & =\exp \left(\sum_{n \geqslant 1} \Lambda\left(h^{n}\right) \frac{t^{n}}{n}\right)  \tag{4.34}\\
& =\prod_{l=0}^{\operatorname{dim}_{K, \infty}} \operatorname{det}\left(\mathrm{I}-t h_{*, i}\right)^{(-1)^{l+1}}, \tag{4.35}
\end{align*}
$$

where $\Lambda(h)=\sum_{l=0}^{\operatorname{dim}^{M} M_{K, \infty}}(-1)^{l} \operatorname{Tr}\left(h_{*, l}: H_{i}\left(M_{K, \infty} ; \mathbb{Q}\right) \rightarrow H_{i}\left(M_{K, \infty} ; \mathbb{Q}\right)\right)$ is the Lefschetz number of $h$. In particular, for knots in $S^{3}, H_{i}\left(M_{K, \infty} ; \mathbb{Z}\right) \cong\{0\}$ for $i>1$ (Proposition 8.9, [74]). Thus,

$$
\begin{align*}
\Lambda(h)= & \operatorname{Tr}\left(h_{*, 0}: H_{0}\left(M_{K, \infty} ; \mathbb{Q}\right) \rightarrow H_{0}\left(M_{K, \infty} ; \mathbb{Q}\right)\right) \\
& -\operatorname{Tr}\left(h_{*, 1}: H_{1}\left(M_{K, \infty} ; \mathbb{Q}\right) \rightarrow H_{1}\left(M_{K, \infty} ; \mathbb{Q}\right)\right) . \tag{4.36}
\end{align*}
$$

Proposition 4.24 (Noguchi, [352]). The Lefschetz zeta function $\zeta_{K}$ of a knot $K \subset S^{3}$ is determined by the corresponding (suitably normalized) Alexander polynomial $\Delta_{K}(t)$ and satisfies

$$
\begin{equation*}
\zeta_{K}(t)=\frac{\Delta_{K}(t)}{\Delta_{K}(0)(1-t)} \tag{4.37}
\end{equation*}
$$

Remark 4.1.19. According to Noguchi, a functional equation satisfied by the zeta function (cf., Weil Conjectures) implies a similar functional equation for the Alexander polynomial, viz., $\Delta_{K}\left(\frac{1}{t}\right)=t^{b_{1}} \Delta_{K}(t)$, where $b_{1}$ is the first betti number $M_{K, \infty}$.

Remark 4.1.20. In [310], Milnor computes the zeta function of an algebraic link.
4.1.9.1. Multivariate Alexander Polynomials of Links. Each link $L$ of $r$ components has a polynomial invariant which partially encodes the homology of its complement known as the $k^{\text {th }}$-Alexander polynomial $\Delta_{L, k}\left(t_{1}, \ldots, t_{r}\right)$, with one variable corresponding to each connected component. These are defined in a similar fashion as its one-component counterpart, so we leave the discussion for standard treatments of the subject. Define the reduced Alexander polynomial of a link $L$ as $(t-1)^{1-\delta_{r, 1}} \Delta_{L}(t, \ldots, t)$.

REmARK 4.1.21. It is conventional to normalize all Alexander polynomials to have the term with lowest degree be the constant term.

REmARK 4.1.22. For a torus links $T_{p, q}$, the reduced Alexander polynomial admits the representation

$$
\Delta_{T_{p, q}}(t, \ldots, t) \doteq \frac{\left(t^{\operatorname{lcm}(p, q)}-1\right)^{\operatorname{gcd}(p, q)}(t-1)}{\left(t^{p}-1\right)\left(t^{q}-1\right)}
$$

where $r=\operatorname{GCD}(p, q)$. Thus, if $\{p, q\} \neq\left\{p^{\prime}, q^{\prime}\right\}$, then $T_{p, q} \not \not T_{p^{\prime}, q^{\prime}}$.

### 4.1.10. Link Signature.

Definition 4.25. The signature $\sigma(L)$ of a link $L$ is the signature of the Seifert matrix $S_{L}+S_{L}^{\top}$, that is, the difference of the number of positive and negative eigenvalues.

Proposition 4.26. Let $L$ be a link. The signature $\sigma(L)$ satisfies the following:

1. The signature is additive over connected sums of links, i.e., if $L=L_{1} \# L_{2}$, then

$$
\begin{equation*}
\sigma(L)=\sigma\left(L_{1}\right)+\sigma\left(L_{2}\right) \tag{4.38}
\end{equation*}
$$

2. If $L^{*}$ and $-L$ denote the mirror and reverse-orientations of $L$, respectively, then

$$
\begin{equation*}
\sigma\left(L^{*}\right)=-\sigma(L)=-\sigma(-L) \tag{4.39}
\end{equation*}
$$

In particular, if $L$ is amphichiral, then $\sigma(L)=0$;
3a. If $u(L)$ denotes the unknotting number of $L$, then $|\sigma(L)| \leqslant 2 u(L)$;
3b. If $L$ is a link with non-zero reduced Alexander polynomial $\Delta_{L}(t)$, then

$$
\begin{equation*}
|\sigma(L)| \leqslant \operatorname{deg} \Delta_{L}(t) ; \tag{4.40}
\end{equation*}
$$

4a. If $L$ is a knot, then $\sigma(L)$ is even;
4b. If $L$ is a link with $r$ components such that $\Delta_{L}(-1) \neq 0$, then $\sigma(L)$ has the opposite parity of $r$;
5. If $L$ is a knot and $\Delta_{L}(t)$ denotes its Alexander polynomial, then

$$
\operatorname{sign} \Delta_{L}(-1)=(-1)^{\sigma(L) / 2} \text { and }\left|\Delta_{L}(-1)\right| \equiv(-1)^{\sigma(L) / 2} \bmod 4
$$

6. If $L$ is an oriented link with $r$ components $K_{1}, \ldots, K_{r}$, and $L^{\prime}$ is obtained from $L$ by reversing the orientation of one component, then

$$
\begin{equation*}
\sigma(L)+\operatorname{lk}(L)=\sigma\left(L^{\prime}\right)+\operatorname{lk}\left(L^{\prime}\right) \tag{4.41}
\end{equation*}
$$

Proof. For Statements 1., 2., 3a., 4a., 5. and 6. consult Chapters 5 and 6 in [335]. Statements 3b. and 4b. can be found in [424].

Remark 4.1.23. The signature is a link invariant.

### 4.1.11. Link Genera.

Definition 4.27. The link genus $g(L)$ of a link $L$ as the infimum genus of the Seifert surfaces of $L$.

Remark 4.1.24. The link genus is a link invariant.

Proposition 4.28. The genus of a connected sum of links $L \# L^{\prime}$ satisfies

$$
g\left(L \# L^{\prime}\right)=g(L)+g\left(L^{\prime}\right)
$$

Proof. The connected sum of two Riemann surfaces with genera $g$ and $g^{\prime}$, respectively, is a Riemann surface of genus $g+g^{\prime}$. While for links, one works with Seifert surfaces, the proof of the claim is entirely analogous.

Proposition 4.29. A knot $K$ is equivalent to the unknot if and only if the knot group $\pi(K) \cong \mathbb{Z}$ if and only if the knot genus $g(K)=0$. A link $L$ is trivial (a disjoint union of unknots) if and only if the link group $\pi(L)$ is a free group.

Proof. See [335] or [74].

Proposition 4.30 (Crowell, [94]). The genus of a non-split link $L$ with $r$ components satisfies the following inequality,

$$
\operatorname{deg} \Delta_{L}(t) \geqslant 2 g(L)+r-1
$$

If $L$ is an alternating link, then the inequality is an equality.

Proof. See Theorem 3.5 in [94].

Remark 4.1.25. By definition, twist knots are alternating.

Remark 4.1.26. The only alternating torus links are the elementary torus links, e.g., $T_{p, 2}$ and $T_{2, q}$, respectively. For torus knots (with a single component) [310], the Seifert surface is a singly-punctured Riemann surface with genus

$$
g\left(T_{p, q}\right)=\frac{1}{2} \operatorname{deg} \Delta_{T_{p, q}}(t)=\frac{1}{2}(p-1)(q-1) .
$$

Definition 4.31. The slice genus (or 4-ball genus) $g^{*}(L)$ of a link $L$ as the infimum of the genera of all orientable surfaces $G$ which admit a smooth, proper embedding in $B^{4}$ taking $\partial G$ to $L$.

Remark 4.1.27. One has $g(L) \geqslant g^{*}(L)$ [142].

Proposition 4.32 (Murasugi,[330]). The slice genus $g^{*}\left(L_{r}\right)$ of an $r$ component link $L_{r}$ with signature $\sigma\left(L_{r}\right)$ and unknotting number $u\left(L_{r}\right)$ satisfies the
bound

$$
\begin{equation*}
\frac{1}{2}\left(\left|\sigma\left(L_{r}\right)\right|-r+1\right) \leqslant g^{*}\left(L_{r}\right) \leqslant u\left(L_{r}\right) . \tag{4.42}
\end{equation*}
$$

Now that we've established a number of link invariants, we turn to a few important families of links.

### 4.2. Fourier Links

A knot is Fourier- $\left(n_{x}, n_{y}, n_{z}\right)$ if and only if admits the following parametrization $(x(t), y(t), z(t)) \subset \mathbb{R}^{3}$ [234],

$$
\begin{align*}
& x(t)=\sum_{i=1}^{n_{x}} A_{x, i} \cos \left(n_{x, i} \theta+\varphi_{x, i}\right)  \tag{4.43}\\
& y(t)=\sum_{i=1}^{n_{y}} A_{y, i} \cos \left(n_{y, i} \theta+\varphi_{y, i}\right)  \tag{4.44}\\
& z(t)=\sum_{i=1}^{n_{z}} A_{z, i} \cos \left(n_{z, i} \theta+\varphi_{z, i}\right), \tag{4.45}
\end{align*}
$$

where $\theta \in[0,2 \pi),\left\{n_{x}, n_{y}, n_{z}\right\} \subset \mathbb{N},\left\{n_{x, i}, n_{y, i}, n_{z, i}\right\} \subset \mathbb{Z}$, and

$$
\left\{A_{x, i}, A_{y, i}, A_{z, i}, \varphi_{x, i}, \varphi_{y, i}, \varphi_{z, i}\right\} \subset \mathbb{R}
$$

A link is Fourier if and only if each of its components is a Fourier knot. By elementary Fourier analysis, it follows that every knot in 3-space is a Fourier$(1,1, n)$ knot for some positive integer $n$ [254]. Hence, every link is Fourier. We refer the reader to $\S_{1} .6$, Chapter 1 in [93] for more details.


Figure 4.13. Four Lissajous $\operatorname{Knots}\left(5_{2}, 6_{1}, 8_{2}\right.$ and $\left.3_{1} \# 3_{1}^{*}\right)$ [389]
4.2.1. Lissajous Knots. The Lissajous knots are Fourier-( $1,1,1$ ) knots (Figure 4.13). To avoid self-intersection, one requires certain conditions on the coefficients, namely, $n_{x, 1}, n_{y, 1}, n_{z, 1}$ be mutually coprime and $n_{j, 1} \phi_{j}-n_{i, 1} \phi_{i} \notin \pi \mathbb{Z}$ for $i, j \in\{x, y, z\}$. Depending on the parity of the integral frequencies $n_{x, 1}, n_{y, 1}$ and $n_{z, 1}$, Lissajous knots are invariant under certain Euclidean symmetries such as a reflection or rotation in the coordinate axes. The former symmetry implies that said knot is strongly amphichiral, while the latter symmetry implies that said knot is 2 -periodic. Such symmetries imply further that the corresponding Alexander polynomial is a perfect square modulo 2 and has Arf-Kervaire invariant zero [183], [332], which excludes most knots from being Lissajous.

### 4.3. Torus Links

A torus link $T_{p, q}$ is a link composed of closed orbits on a torus, where each orbit wraps about the meridian $p$ times and about the longitude $q$ times. In general, $T_{p, q} \simeq T_{q, p}, T_{p, q} \simeq T_{-p,-q} \simeq-T_{p, q}, T_{p,-q} \simeq T_{p, q}^{*}$, so torus links are invertible. Figure 4.14 shows the first sixty-four of the family of torus links arranged by increasing crossing number.


Figure 4.14. Torus Links Ordered by Increasing Crossing Number [418]
If $\operatorname{GCD}(p, q)=1$, then the torus link $T_{p, q}$ has only one component and is, therefore, a torus knot. Four torus links are shown in Figure 4.10.

As shown in [214], the torus knot $T_{p, q}$ admits the following parametrization $(x(t), y(t), z(t)) \subset \mathbb{R}^{3}$,

$$
\begin{align*}
& x(t)=\cos (p \theta)  \tag{4.46}\\
& y(t)=\cos \left(q \theta+\frac{\pi}{2 p}\right)  \tag{4.47}\\
& z(t)=\cos \left(p \theta+\frac{\pi}{2}+\cos \left((q-p) \theta+\frac{\pi}{2 p}-\frac{\pi}{4 q}\right)\right. \tag{4.48}
\end{align*}
$$

where $\theta \in[0,2 \pi)$. Thus torus knots are Fourier- $(1,1,2)$ knots. Although all torus knots are Fourier knots, there are some torus knots that are not Lissajous knots [254].


Figure 4.15. Eight Trivial Torus Knots ( $T_{1, q}$ and $T_{p, 1}$ for $2 \leqslant p, q \leqslant 5$ ) [418]

Torus knots of the form $T_{1, q}$ or $T_{p, 1}$ are isotopic to the unknot for $p, q \geqslant 1$ and consist of a single curve wrapping $q$-times about the meridian or $p$-times about the longitude along a torus, respective, as shown in Figure 4.15. Similarly, torus links of the form $T_{p, p}$ consist of $p$ interlinked unknots, as shown in Figure 4.18. Notable examples of proper torus links are the Hopf Link $T_{2,2}$, Solomon's
$\operatorname{Knot}^{*} T_{2,4}$ and the Hopf Link $T_{3,3} \simeq 6_{3}^{3}$. Some torus links of the type $T_{2, q}$ and $T_{p, 2}$ are shown in Figure 4.16, while those of type $T_{3, q}$ and $T_{p, 3}$ are shown in Figure 4.17.


Figure 4.16. Six Torus Links ( $T_{2, q}$ and $T_{p, 2}$ for $p, q \in\{3,4,5\}$ ) [418]


Figure 4.17. Six Torus Links ( $T_{3, q}$ and $T_{p, 3}$ for $p, q \in\{4,5,6\}$ ) [418]
*Solomon's Knot is technically a link.

### 4.4. Hopf Links

Hopf links are isotopic to torus links of the form $T_{p, p}$. Figure 4.18 illustrates a few examples of Hopf links.


Figure 4.18. Six Hopf Links ( $T_{p, p}$ for $2 \leqslant p \leqslant 6$ ) [418]

### 4.5. Fibered Links

Definition 4.33. A link $L \subset \mathbb{R}^{3}$ is fibered ${ }^{*}$ if and only if there is a $S^{1}$ parametrized family of Seifert surfaces $S_{\theta}$ such that $L=\left\{S_{\theta} \cap S_{\theta^{\prime}} \mid \theta \neq \theta^{\prime} \in S^{1}\right\}$.

REMARK 4.5.1. Torus links, including the Hopf links, are fibered.

Neuwirth and Stallings proved the following result.

Proposition 4.34 (Neuwirth 1962; Stallings 1963). For a knot $K \subset S^{3}$, the following are equivalent:
*A fibered knot is also called Neuwirth-Stallings knot.

1. The complement $S^{3} \backslash K$ is the total space of a fiber bundle over the circle, the fiber $F$ being a connected surface (In particular, $K$ is fibered);
2. The commutator subgroup $\pi(K)^{a b}$ is free; and,
3. The commutator subgroup $\pi(K)^{a b}$ is finitely-generated.

If a knot $K$ satisfies any one of the previous conditions, then the following is also true:

1. The Alexander polynomial $\Delta_{K}(t)$ is monic (up to sign) and

$$
\begin{equation*}
\operatorname{rank} \pi(K)^{a b}=\frac{1}{2} \operatorname{deg} \Delta_{K}(t) ; \tag{4.49}
\end{equation*}
$$

2. The fiber $F$ is diffeomorphic to a compact, orientable surface with one point removed and with genus $\frac{1}{2} \operatorname{deg} \Delta_{K}(t)$; and,
3. The knot genus equals $g(K)=\frac{1}{2} \operatorname{deg} \Delta_{K}(t)$.

As a corollary, one has the following result.

Corollary 4.35 (Saveliev, [416]). If $K$ is a fibered knot with Seifert surface F, then the closure $\bar{F}$ is a Seifert surface of $K$ with (minimal) genus $g(K)$.

Proof. See Corollary 8.3 in [416].

Remark 4.5.2. A twist knot $T_{n}$ with more than two half-twists $n$ is not fibered, as

$$
\Delta_{T_{n}}(t) \doteq \begin{cases}\frac{n+1}{2} t^{2}-n t+\frac{n+1}{2} & n \text { odd } \\ \frac{n}{2} t^{2}-(n+1) t+\frac{n}{2} & n \text { even } .\end{cases}
$$

In particular, as $\Delta_{T_{4}}(t) \doteq 2 t^{2}-5 t+2$ (also shared by $9_{46}$ ), the Stevedore knot $T_{4} \simeq 6_{1}$ is not fibered.


Figure 4.19. Fibered Knots of Genus $1\left(3_{1}\right.$ and $\left.4_{1}\right)$ [389]

Proposition 4.36 (Stallings, Rolfsen, [403]). If a link $L \subset S^{3}$ is fibered, then the commutator subgroup $\pi(L)^{\mathrm{ab}}$ is finitely generated, in which case said subgroup is free with rank equal to $2 g(L)$, the link genus of $L$. If $L$ is a knot, then the converse is true.

Proof. See Chapter 10, §H, Proposition 3 in [403].

Proposition 4.37 (Burde, Zieschang, [74]). The only fibered knots of genus one are the trefoil knot $3_{1}$ and the figure-eight knot $4_{1}$.

Remark 4.5.3. Although it is spanned by a Seifert surface with minimal genus, the 3-twist knot $5_{2}$ is not fibered [74].

### 4.6. Algebraic Links

Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with an isolated critical point at the origin, Milnor defines the map $\phi_{f}=\frac{f}{\|f\|}: S_{\varepsilon}^{2 n+1} \backslash V_{f, 0} \rightarrow S^{1}$ with
$\varepsilon \in \mathbb{R}_{>0}$. He proves that there is an $\varepsilon_{0}>0$ such that for all $\varepsilon$ satisfying $0<\varepsilon<\varepsilon_{0}$, the map $\phi_{f}$ is the projection of a locally smooth, trivial fibration over $S^{1}$. In fact, the $\operatorname{map} \phi_{f}$ induces an open-book decomposition $\left(K_{f}, \phi_{f}\right)$ of $S_{\varepsilon}^{2 n+1}$, where the intersection $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{2 n+1}$.

Definition 4.38. A link $L \subset \mathbb{R}^{2 n-1}$ is algebraic if and only if there is a complex analytic map $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with isolated critical point at the origin and whose generic Milnor fiber $F_{f, \theta}$ bounds $L$.

The boundary link $K_{f}$ of the Milnor fiber $F_{f, 0}$ is algebraic since it is the boundary of an open-book decomposition arising from (the isolated singularity of) a complex analytic map $f$ by construction [310]. Moreover, $K_{f}$ is fibered, as the generic Milnor fibers $F_{f, \theta}$ form the $S^{1}$-parametrized family of Seifert surfaces of $K_{f}$. It follows that every algebraic link is fibered. The complement $S_{\varepsilon}^{2 n+1} \backslash K_{f}$ fibers over $S^{1}$ with fiber $F_{f, 0}$, as in the diagram below,


Two fibers of the trefoil $\operatorname{knot}\left(F_{f, 0}\right.$ and $\left.F_{f, \pi}\right)$ are shown in Figure 4.20.
However, not all links are algebraic and, in particular, not all fibered links are algebraic. Durfee proves a bijection between the equivalence classes of fibered links up to isotopy and integral unimodular bilinear forms, the Seifert
forms of said links [116], [232]. Such bilinear forms are necessarily upper triangular, a condition that excludes many fibered knots from being algebraic. For example, the figure-eight knot is prime, alternating, hyperbolic and fibered but not algebraic, and the Stevedore knot (or double figure-eight) is neither fibered nor algebraic (Figure 4.19) but is prime, alternating, hyperbolic, ribbon and slice [403]. For a review of knots and links, see [335].


Figure 4.20. Milnor Fibers of the Trefoil Knot (Adapted from [114])
In general, the class of algebraic links forms a subclass of iterated torus links, which are iterative cables a torus link. In particular, much is known about algebraic links in $S^{3}$ (i.e., $n=1$ ), including their complete classification [257, 258]. According to Lê, the isotopy class of an algebraic link is determined by its Puiseux pairs [257]. When $n=2$, certain surface singularities can be realized as well-known homology spheres [125]. If $n \geqslant 3$, then $K_{f}$ is simply connected (only connected for $n=2$ ) and may have exotic differential structure [63].

Milnor's work allows one to determine a significant amount of the topology of an algebraic link from studying certain maps of corresponding fiber.

Proposition 4.39 (Milnor, [310]). Let $f:\left(C^{n+1}, \mathbf{0}\right) \rightarrow(\mathrm{C}, 0)$ be a complex analytic germ with an isolated critical point at the origin. For $n \neq 2$, the algebraic link $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{2 n+1}$ with $r$ components is a topological $(2 n-1)$-sphere if and only if the characteristic polynomial $\Delta_{h_{*}}(t)=\operatorname{det}\left(t l-h_{*}\right)$ of the associated monodromy map $h_{*}: \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{C}\right) \rightarrow \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{C}\right)$ coincides with the reduced Alexander polynomial
 the number of spheres in the homotopy type of the fiber, $F_{f, 0} \simeq \bigvee^{\mu} S^{n}$.

Proof. See Lemma 8.2, Theorem 8.5 and Lemma 10.1 in [310].
Remark 4.6.1. There are counter-examples to Proposition 4.39 for $n=2$. In this case, however, replacing topological 3-sphere with homology 3-sphere reinstates its validity.

Remark 4.6.2. Consider $f=x^{p}+y^{p r}$. Since $\operatorname{GCd}(p, p r)=p$ and $\operatorname{LCM}(p, p r)=p r$, the characteristic polynomial of $T_{p, q}$ can be used to compute that of $T_{p, p r}$,

$$
\begin{equation*}
\Delta_{f}(t)=\frac{\left(t^{p r}-1\right)^{p-1}(t-1)}{\left(t^{p}-1\right)} . \tag{4.50}
\end{equation*}
$$

The Alexander multivariate polynomial of $T_{p, p r}$ is given by

$$
\begin{equation*}
\Delta_{T_{p, p r}}\left(t_{1}, \ldots, t_{p}\right)=\frac{\left(\left(t_{1} \cdots t_{p}\right)^{r}-1\right)^{p-1}}{t_{1} \cdots t_{p}-1} \tag{4.51}
\end{equation*}
$$

It is clear that $\Delta_{f}(t)=(t-1) \Delta_{T_{p, p r}}(t, \ldots, t)$.

Remark 4.6.3. In general, the Hosokawa polynomial [235] of $T_{p, q}$ is

$$
\begin{align*}
\hat{\Delta}_{T_{p, q}}(t) & =(t-1)^{2-\operatorname{GCD}(p, q)} \frac{\left(t^{\mathrm{LCM}(p, q)}-1\right)^{\mathrm{GCD}(p, q)}}{\left(t^{p}-1\right)\left(t^{q}-1\right)}  \tag{4.52}\\
& =(t-1)^{2-\operatorname{GCD}(p, q)} \Delta_{T_{p, q}}(t, \ldots, t) \tag{4.53}
\end{align*}
$$

It follows that $\Delta_{f}(t)=(t-1)^{1-\delta_{r, 1}} \Delta_{T_{p, q}}(t, \ldots, t)$.


Figure 4.21. Non-Algebraic Knots $\left(4_{1}\right.$ and $\left.6_{1}\right)$ [389]

Remark 4.6.4. The Alexander (and characteristic) polynomials of the figure-eight and Stevedore knots are $\Delta_{4_{1}}(t)=t^{2}-3 t+1$ and $\Delta_{6_{1}}(t)=$ $2 t^{2}-5 t+2$, respectively. While the Alexander polynomial of the former is monic, its roots are $\frac{1}{2}(3 \pm \sqrt{5})$. Similarly, $\Delta_{6_{1}}(t)$ is neither monic nor has constant coefficient $\pm 1$. These facts violate Proposition 1.14 and Corollary 1.13. Consequently, neither the figure-eight knot nor the Stevedore knot (and $9_{46}$ ) can be algebraic.

Remark 4.6.5. Perron [371] proved that the figure-eight knot does arise as the boundary link of the Milnor fiber of the real polynomial map $f:\left(\mathbb{R}^{4}, \mathbf{0}\right) \rightarrow$
$\left(\mathbb{R}^{2}, 0\right)$ given by $f(x, y, z, w)=\left(f_{1}\left(x, y, z^{2}-w^{2}, 2 z w\right), f_{2}\left(x, y, z^{2}-w^{2}, 2 z w\right)\right)$, where

$$
\begin{align*}
& f_{1}(x, y, z, w)=2 x f_{3}(x, y, z, w)+z f_{3}(\sqrt{3} x, i y, i z, i w)  \tag{4.54}\\
& f_{2}(x, y, z, w)=2 y f_{3}(x, y, z, w)+\sqrt{2} x w  \tag{4.55}\\
& f_{3}(x, y, z, w)=3 x^{2}-y^{2}-z^{2}-w^{2} \tag{4.56}
\end{align*}
$$

Remark 4.6.6. It is not known if all fibered links arise from the open-book decomposition of Milnor fibers of real ${ }^{*}$ polynomial maps. A result of Akbulut and King [9] states that each link in $S^{3}$ arises from a polynomial map with a weakly isolated singularity at the origin, defining the notion of a weakly algebraic link.

Corollary 4.40. If, in particular, $n=1$ and $V_{f, 0}$ has a single analytic branch through the origin, then $\pi\left(K_{f}\right)$ is free with rank equal to $\operatorname{deg} \Delta_{f}$, and the genera of $K_{f}$ and $\bar{F}_{f, \theta}$ for any $\theta \in S^{1}$ are equal to $\frac{1}{2} \operatorname{deg} \Delta_{f}$.

Proof. See Corollary 10.2 in [310].
4.6.1. Cohomological Index. Given a complex analytic germ
$f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, define the cohomology index to be the first betti number of the infinite cyclic covering $M_{K_{f}, \infty}$ of the corresponding algebraic link $K_{f}$,

[^28]namely,
\[

$$
\begin{equation*}
\mu_{\mathrm{co}}(f)=b_{1}\left(M_{K_{f}, \infty}\right) \tag{4.57}
\end{equation*}
$$

\]

### 4.7. Torus Links, Revisited

A complex analytic germ $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ defines a complex algebraic plane curve $V_{f, 0}=f^{-1}(0)$, and the corresponding algebraic link $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{3}$ is a (possibly-knotted) disjoint union $\coprod S^{1}$ with components depending on the local analytic and topological structure of $V_{f, 0}$ at the origin. One notable example of such algebraic links is that of the torus type, $T_{p, q}$, as mentioned in the Prologue. For the convenience of the reader, we revisit the aforementioned discussion now.

Given two integers $p, q>1$ consider the complex plane curve $f=x^{p}+y^{q}$ over $\mathbb{C}^{2}$. The origin is an isolated critical point of $f$ and, therefore, the complex algebraic hypersurface $V_{f, 0}$ is singular only at the origin. In [61], Brauner proved that the intersection of $V_{f, 0}$ with a sufficiently small 3-sphere, namely, $K_{f}=V_{f} \cap S_{\varepsilon}^{3}$, is a $(p, q)$-torus link $T_{p, q}$ consisting of $\operatorname{GCD}(p, q)$ connected components explicitly defined by the system $\left\{x=r e^{i \theta}, y=\rho e^{i \phi}, r^{p} e^{i p \theta}=\right.$ $\left.\rho^{q} e^{i(q \phi+\pi)}, r^{2}+\rho^{2}=\varepsilon^{2}\right\}$ with fixed and positive $\varepsilon$ and $p \theta \equiv q \varphi+\pi \bmod 2 \pi$. That is, $T_{p, q}=\left\{\left(r e^{i q \varphi}, \rho e^{i(p \varphi+\pi / q)}\right) \mid \varphi \in[0,2 \pi)\right\}$ (Proposition 2.18, [74]) or, in an equivalent but normalized form, as the locus $\left\{(x, y) \in \mathbb{C} \mid x^{p}=-y^{q}, x=\right.$ $\left.e^{i m \varphi / p}, y=e^{i(m+n) \varphi / q+\pi i / q}, \varphi \in[0,2 \pi)\right\}$ intersecting a torus of radius $\sqrt{2}$, where $1 \leqslant n \leqslant \operatorname{GCD}(p, q)$ and $0 \leqslant m \leqslant \operatorname{LCM}(p, q)$ [312].
4.7.1. Torus Links with Core. Under Milnor's construction, the link of the singularity $f=x^{2}-x y^{2}$ is isotopic to $T_{2,4}$, while that of the singularity $f=x^{3}-$ $x y^{2}$ is isotopic to the (triple) Hopf link $T_{3,3}$. In general, the link of the singularity $f=x^{2}-x y^{q}$, where $q \geqslant 1$, is isotopic to the torus link $T_{2,2 q}$. More generally, still, the singularity $f=x^{p}-x y^{q}=x\left(x^{p-1}-y^{q}\right)$, where $p>1$ and $q \geqslant 1$, corresponds to the torus link $T_{p-1, q}$ linked with an unknot (as the core of a torus on which the torus link wraps) in an alternating fashion, which we denote by $O T_{p, q}$. The two sets of algebraic links $\left\{T_{p, q}\right\}$ and $\left\{O T_{r, s}\right\}$ are not isomorphic up to isotopy. Although $O T_{r, 1} \simeq T_{2,2}$ for $r \geqslant 2, T_{2,3} \nsucceq O T_{r, s}$ for any $(r, s) \in \mathbb{N}^{2}$. Only in the case that $p$ divides $q$ (up to sign) does there exist an isotopy $T_{p, q} \simeq O T_{r, s}$ for $r=p$ and $s=\frac{q}{p}(p-1)$. That is, if $r s$ does not divide $r-1$, then $O T_{r, s}$ will not be isotopic to any torus link.
4.7.2. Multilinks. If a complex analytic map $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ factors* into a product of irreducible, analytic maps $f=\prod_{i=1}^{r} f_{i}^{r_{i}}$, where $r_{i} \in \mathbb{N}$, then the algebraic link $K_{f}$ is a multilink $K_{f}=\bigcup_{i=1}^{r} r_{i} K_{f_{i}}$, where $K_{f_{i}}$ is the link of $f_{i}$ [420]. According to Hillman (§5, [196]), given two irreducible, coprime polynomials $f$ and $g$ with algebraic links $L_{f}$ and $L_{g}$, respectively, the linking number of these components within the algebraic link $L_{f g}$ is the algebraic invariant

$$
\begin{equation*}
\operatorname{lk}\left(L_{f}, L_{g}\right)=\operatorname{dim}_{\mathbb{C}} \mathcal{O}_{0,1} /\langle f, g\rangle \tag{4.58}
\end{equation*}
$$

[^29]Definition 4.41. The center and core of a torus link are the center and core of the enveloping torus, respectively.

Definition 4.42. A torus link is said to be centrally linked by a second torus link if and only if the core of the latter link passes through the center of the former link with no further linking.

Remark 4.7.1. The Hopf link consists of an unknot centrally linked by an unknot.

REMARK 4.7.2. Consider $f=\left(x+y^{2}\right)\left(x^{2}+y^{5}\right)$ over $\mathbb{C}^{2}$, which is rightequivalent to a $J_{2,1}$-singularity $x^{3}+x^{2} y^{2}+y^{7}$ [20]. The algebraic link $K_{f}$ consists of $T_{2,1}$ centrally linking (the core of) $T_{2,5}$ [117].

REMARK 4.7.3. Consider $f=\prod_{i=1}^{r}\left(a_{i} x^{p_{i}}+b_{i} y^{q_{i}}\right)$ over $\mathbb{C}^{2}$ with $a_{i}, b_{i} \in \mathbb{C}^{\times}$. The algebraic link $K_{f}$ consists of $r$ linked torus links $\bigcup_{i=1}^{r} T_{p_{i}, q_{i}}$, each centrally linking the other $r-1$ torus links.

Remark 4.7.4. Consider $f=x^{p}-y^{q}=\prod_{k=0}^{r-1}\left(x^{p / r}-\zeta_{p}^{k} y^{q / r}\right)$ over $\mathbb{C}^{2}$ with $p, q \geqslant 1$ and $r=\operatorname{GCD}(p, q)$. The link $K_{f}$ consists of $r$ linked torus links $\bigcup_{k=1}^{r} T_{p / r, q / r}$, each centrally linking the other $r-1$ torus links. If $p$ and $q$ are coprime, then $f=x^{p}-y^{q}$ is irreducible over $\mathbb{C}[x, y]$, so $K_{f} \simeq T_{p, q}$. If $p=q$, the algebraic link $K_{f}$ consists of $p$ linked unknots, each centrally linking the other $p-1$ unknots.

### 4.7.3. Alexander Polynomial of the Torus Link.

4.7.3.1. Method I. The Brieskorn-Pham singularity ${ }^{*} f=x^{p}+y^{q}$ is a nondegenerate, weighted homogeneous polynomial with weights $\left\{\frac{1}{p}, \frac{1}{q}\right\}$, and the corresponding algebraic link $K_{f}=V_{f} \cap S_{\varepsilon}^{3}$ is the torus link $T_{p, q}$ with GCD $(p, q)$ linked components. The corresponding divisor is computed by

$$
\begin{align*}
\operatorname{div} \Delta_{f}(t) & =\operatorname{div} \Delta_{z^{p}}(t) \cdot \operatorname{div} \Delta_{z^{q}}(t)  \tag{4.59a}\\
& =\left(\Lambda_{p}-\Lambda_{1}\right)\left(\Lambda_{q}-\Lambda_{1}\right)  \tag{4.59b}\\
& =\operatorname{GCD}(p, q) \Lambda_{\mathrm{LCM}(p, q)}-\Lambda_{p}-\Lambda_{q}+\Lambda_{1} \tag{4.59c}
\end{align*}
$$

Since a divisor $a \Lambda_{b}$ contributes $\left(t^{b}-1\right)^{a}$ to the numerator if $\operatorname{sign}(a)=1$ or the denominator if $\operatorname{sign}(a)=-1$, provided that $b \geqslant 1$, the corresponding characteristic polynomial is the rational function

$$
\begin{equation*}
\Delta_{f}(t)=\frac{\left(t^{\mathrm{LCM}(p, q)}-1\right)^{\mathrm{GCD}(p, q)}(t-1)}{\left(t^{p}-1\right)\left(t^{q}-1\right)} \tag{4.60}
\end{equation*}
$$

which is precisely the Alexander polynomial $\Delta_{T_{p, q}}$ of the torus link $T_{p, q}$ (up to a multiplicative power of $t$ ) when $p$ and $q$ are coprime, in which case,

$$
\begin{equation*}
\Delta_{T_{p, q}}(t)=\operatorname{GCD}\left(\frac{t^{p q}-1}{t^{p}-1}, \frac{t^{p q}-1}{t^{q}-1}\right) . \tag{4.61}
\end{equation*}
$$

[^30]This identity follows from computing (the Jacobian of) the knot group $\pi\left(T_{p, q}\right)$, a crucial step in proving cyclicity of the Alexander module (Proposition 9.14 and Example 9.15 in [74]).

Remark 4.7.5. Consider $f=x^{p}-x y^{q}$ with weights $\left\{\frac{1}{p}, \frac{p-1}{p q}\right\}$. Then $K_{f}=$ $O T_{p, q}$ and

$$
\begin{align*}
\operatorname{div} \Delta_{f}(t) & =\left(\Lambda_{p}-\Lambda_{1}\right)\left(\frac{1}{p-1} \Lambda_{p q}-\Lambda_{1}\right)  \tag{4.62}\\
& =\frac{\operatorname{GCD}(p, p q)}{p-1} \Lambda_{\mathrm{LCM}(p, p q)}-\Lambda_{p}-\frac{1}{p-1} \Lambda_{p q}+\Lambda_{1}  \tag{4.63}\\
& =\Lambda_{p q}-\Lambda_{p}+\Lambda_{1} \tag{4.64}
\end{align*}
$$

since $\operatorname{GCD}(p, p q)=p$ and $\operatorname{LCM}(p, p q)=p q$. Thus, the corresponding characteristic polynomial is the rational function

$$
\begin{equation*}
\Delta_{f}(t)=\frac{\left(t^{p q}-1\right)(t-1)}{\left(t^{p}-1\right)} \tag{4.65}
\end{equation*}
$$

4.7.3.2. Method II. A second way to compute the same reduced Alexander polynomial is by first computing the reduced Hilbert-Poincaré series of $f$, then
using equation (2.52a). Here, one has

$$
\begin{align*}
\bar{P}_{\mathcal{A}_{f}}(t) & =\left(\frac{1-t^{1-1 / p}}{1-t^{1 / p}}\right)\left(\frac{1-t^{1-1 / q}}{1-t^{1 / q}}\right)  \tag{4.66a}\\
& =\left(\sum_{k=0}^{p-2} t^{k / p}\right)\left(\sum_{l=0}^{q-2} t^{l / q}\right)  \tag{4.66b}\\
& =\sum_{k=0}^{p-2} \sum_{l=0}^{q-2} t^{k / p+l / q} \tag{4.66c}
\end{align*}
$$

Writing $\bar{P}_{\mathcal{A}_{f}}(t)=\sum_{j=1}^{(p-1)(q-1)} t^{\alpha_{j}}$, where $\left\{\alpha_{j}\right\}_{j}=\left\{\frac{k}{p}+\frac{l}{q}\right\}_{(l, k) \in\{(0,0), \ldots,(p-2, q-2)\}}$, define the shifted exponents $\gamma_{j}=\alpha_{j}+\frac{1}{p}+\frac{1}{q}$. One then computes the characteristic polynomial

$$
\begin{align*}
\Delta_{f}(t) & =\prod_{j=1}^{(p-1)(q-1)}\left(t-e^{2 \pi i \gamma_{j}}\right)  \tag{4.67a}\\
& =\prod_{k=1}^{p-1} \prod_{l=1}^{q-1}\left(t-\zeta_{p}^{k} \zeta_{q}^{l}\right)  \tag{4.67b}\\
& =\frac{\left(t^{\mathrm{LCM}(p, q)}-1\right)^{\mathrm{GCD}(p, q)}(t-1)}{\left(t^{p}-1\right)\left(t^{q}-1\right)} \tag{4.67c}
\end{align*}
$$

where we have used the following identities: $\prod_{k=0}^{n-1}\left(t-\zeta_{n}^{k}\right)=t^{n}-1$ for $n \in \mathbb{N}$ and

$$
\begin{align*}
\prod_{k=0}^{p-1}\left(t-\zeta_{p}^{k}\right) \otimes \prod_{l=0}^{q-1}\left(t-\zeta_{q}^{l}\right) & =\prod_{k=0}^{p-1} \prod_{l=0}^{q-1}\left(t-\zeta_{p}^{k} \zeta_{q}^{l}\right)  \tag{4.68a}\\
& =\left(t^{\mathrm{LCM}(p, q)}-1\right)^{\operatorname{GCD}(p, q)} \tag{4.68b}
\end{align*}
$$

We generalize these methods in $\S 4.8$.
4.7.3.3. Method III. A third way to compute the reduced Alexander polynomial for the torus link $T_{p, q}$ is to compute the exponents $\left\{r_{d}\right\}_{d \mid N}$ and the euler characteristics $\chi_{k}=\sum_{d \mid k} d r_{d}$ of the fixed point manifolds under the iterated orbits, $h^{k}$. Here, $N=\operatorname{LCm}(p, q)$. We compute

$$
\begin{equation*}
\left(r_{d}, d\right) \in\{(1,1),(-1, p),(-1, q),(\operatorname{GCD}(p, q), \operatorname{LCM}(p, q))\} \tag{4.69}
\end{equation*}
$$

Thus,

$$
\begin{align*}
\Delta_{f}(t) & =(t-1) \prod_{1<d \mid N}\left(t^{d}-1\right)^{-r_{d}}  \tag{4.70}\\
& =\frac{\left(t^{\mathrm{LCM}(p, q)}-1\right)^{\mathrm{GCD}(p, q)}(t-1)}{\left(t^{p}-1\right)\left(t^{q}-1\right)}, \tag{4.71}
\end{align*}
$$

The degree of the characteristic polynomial is the algebraic index,

$$
\begin{align*}
\mu & =1-\sum_{1<d \mid N} d r_{d}  \tag{4.72}\\
& =1-p-q+\operatorname{GCD}(p, q) \operatorname{LCM}(p, q)  \tag{4.73}\\
& =(p-1)(q-1), \tag{4.74}
\end{align*}
$$

since $\operatorname{GCD}(p, q) \operatorname{LCm}(p, q)=p q$. In light of Proposition 4.39, one has the following result.

Remark 4.7.6. The associated characteristic polynomials of a few torus links are as follows,

$$
\begin{aligned}
& \Delta_{T_{2,2}}(t)=t-1=\Phi_{1}(t) \\
& \Delta_{T_{2,3}}(t)=t^{2}-t+1=\Phi_{6}(t) \\
& \Delta_{T_{2,4}}(t)=t^{3}-t^{2}+t-1=\Phi_{1}(t) \Phi_{4}(t) \\
& \Delta_{T_{3,3}}(t)=t^{4}-t^{3}-t+1=\Phi_{1}(t)^{2} \Phi_{3}(t) \\
& \Delta_{T_{3,4}}(t)=t^{6}-t^{5}+t^{3}-t+1=\Phi_{6}(t) \Phi_{12}(t) \\
& \Delta_{T_{3,5}}(t)=t^{8}-t^{7}+t^{5}-t^{4}+t^{3}-t+1=\Phi_{15}(t) \\
& \Delta_{T_{4,4}}(t)=t^{9}-t^{8}-2 t^{5}+2 t^{4}+t-1=\Phi_{1}(t)^{3} \Phi_{2}(t)^{2} \Phi_{4}(t)^{2}
\end{aligned}
$$

where the cyclotomic polynomial $\Phi_{n}(t)$ is defined as the product,

$$
\Phi_{n}(t)=\prod_{\substack{1 \leqslant k \leqslant n \\ \operatorname{gcd}(k, n)=1}}\left(t-\zeta_{n}^{k}\right)
$$

For the convenience of the reader, the characteristic polynomial of the torus link $T_{p, q}$ for $2 \leqslant p, q \leqslant 10$ (up to isotopy) is given in the Appendix.

Proposition 4.43. A torus link is a homotopy sphere if and only if it is a knot.

Proof 1. Writing $\operatorname{div} \Delta_{f}(t)=\sum_{k \geqslant 1} \varepsilon_{k} \Lambda_{k}$, from equation (4.59c) one infers the only non-zero coefficients, $\varepsilon_{1}=1, \varepsilon_{p}=\varepsilon_{q}=-1$ and $\varepsilon_{\mathrm{LCM}(p, q)}=\operatorname{GCD}(p, q)$. Thus, the Milnor-Orlik algebraic link invariants are $\kappa=\operatorname{GCD}(p, q)-1$ and
$\rho=\frac{1}{p q} \operatorname{LCM}(p, q)^{\operatorname{GCD}(p, q)}$. By Proposition 2.52,

$$
\begin{equation*}
\Delta_{f}(1)=\frac{1}{p q} \mathrm{LCM}(p, q)^{\operatorname{GCD}(p, q)} \delta_{1, \mathrm{GCD}(p, q)}=\delta_{1, \mathrm{GCD}(p, q)} \tag{4.75}
\end{equation*}
$$

as $\operatorname{LCM}(p, q)=p q$ if and only if $\operatorname{GCD}(p, q)=1$.
Proof 2. By applying the Residue Theorem to equation (4.67c), one proves

$$
\begin{equation*}
\lim _{t \rightarrow 1} \Delta_{f}(t)=\left.\operatorname{Res}\left(\frac{\left(t^{\mathrm{LCM}(p, q)}-1\right)^{\operatorname{GCD}(p, q)}}{\left(t^{p}-1\right)\left(t^{q}-1\right)}\right)\right|_{t=1}=\delta_{1, \operatorname{GCD}(p, q)} \tag{4.76}
\end{equation*}
$$

REmARK 4.7.7. The choice $\{p, q\}=\{2,3\}$ yields the trefoil $\operatorname{knot} T_{2,3} \simeq$ $T_{3,2}$ and the corresponding characteristic and reduced Alexander polynomials coincide, namely,

$$
\begin{equation*}
\Delta_{f}(t)=\Delta_{T_{2,3}}(t)=1-t+t^{2} \tag{4.77}
\end{equation*}
$$

As $\Delta_{T_{2,3}}(1)=1$, it follows that $T_{2,3}$ is a topological sphere.
Remark 4.7.8. The choice $p=q=3$ yields the (triple) Hopf link $T_{3,3}$ and the corresponding characteristic polynomial

$$
\begin{equation*}
\Delta_{f}(t)=(t-1)\left(t^{3}-1\right)=1-t-t^{3}+t^{4} \tag{4.78}
\end{equation*}
$$

while the Alexander (multivariate) polynomial of $T_{p, p r}$ is

$$
\begin{equation*}
\Delta_{T_{p, p r}}\left(t_{1}, \ldots, t_{p}\right)=\frac{\left(\left(t_{1} \cdots t_{p}\right)^{r}-1\right)^{p-1}}{t_{1} \cdots t_{p}-1} \quad p \geqslant 2, r \geqslant 1 \tag{4.79}
\end{equation*}
$$

As $\Delta_{T_{3,3}}(1)=0$, it follows that $T_{3,3}$ is not a topological sphere. Note that for $p=3$ and $r=1, \Delta_{T_{3,3}}(t, t, t)=t^{3}-1=(t-1)^{-1} \Delta_{f}(t)$.

Corollary 4.44. The following identities hold:

$$
\Delta_{T_{p, q}}(0)=\left\{\begin{array}{cc}
1 & \text { p or } q \text { is odd }  \tag{4.80}\\
-1 & \text { otherwise }
\end{array} \quad \Delta_{T_{p, q}(1)}=\left\{\begin{array}{cc}
1 & \operatorname{GCD}(p, q)>1 \\
0 & \text { otherwise } .
\end{array}\right.\right.
$$

Remark 4.7 .9 . Figure 4.22 shows the striking resemblance of the Borromean rings $6_{2}^{3}$ (left) and the Triple Hopf link $T_{3,3} \simeq 6_{3}^{3}$. The multivariate Alexander polynomial of the Borromean rings is

$$
\begin{equation*}
\Delta_{6_{2}^{3}}\left(t_{1}, t_{2}, t_{3}\right)=\left(t_{1}-1\right)\left(t_{2}-1\right)\left(t_{3}-1\right) . \tag{4.81}
\end{equation*}
$$

However, there is no integer $k$ such that $(t-1)\left(t^{3}-1\right)=t^{k}(t-1) \Delta_{6_{2}^{3}}(t, t, t)$, so one concludes that $T_{3,3}$ is not isotopy equivalent to the Borromean rings, $q \cdot v$. , Proposition 4.39.


Figure 4.22. Four Prime Links with Three Components $\left(6_{1}^{3}, 6_{2}^{3}, 6_{3}^{3}\right.$ and $\left.7_{1}^{3}\right)$ [418]

Corollary 4.45. The link $O T_{p, q}$ is not a homotopy sphere for $p, q \geqslant 1$.
Proof. The value of equation (4.65) at unity is 0 for $p, q \geqslant 1$.

### 4.8. Triangle Groups and Brieskorn-Pham 3-Manifolds



Figure 4.23. Five Constructible Regular $n$-gons ( $n=3,4,5,6,8$ ).
4.8.1. Regular Polyhedra. Denote the regular $n$-gon by the Schläfli symbol $\{n\}$. A regular polyhedron $\{p, q\}$ is a connected subspace of $\mathbb{R}^{3}$ with $f\{p\}$-faces and $q$ meeting at each of the $v$ (regular) vertices (i.e., $\{q\}$ vertex figure), and $e$ edges joining exactly two vertices and exactly two faces, where

$$
\begin{align*}
& v=\frac{1}{\frac{q}{2 p}-\frac{q}{4}+\frac{1}{2}}  \tag{4.82}\\
& e=\frac{1}{\frac{1}{p}+\frac{1}{q}-\frac{1}{2}}  \tag{4.83}\\
& f=\frac{1}{\frac{p}{2 q}-\frac{p}{4}+\frac{1}{2}} . \tag{4.84}
\end{align*}
$$

For every regular polyhedron $\{p, q\}$ with $v$ vertices, $e$ edges and $f$ faces, there is a (regular) dual polyhedron $\{q, p\}$ with $f$ vertices, $e$ edges and $v$ faces. In particular, both $\{p, q\}$ and $\{q, p\}$ have Euler characteristic $v-e+f=2$.

Definition 4.46. A polyhedron is finite and/or convex if and only if it is not an infinite and/or convex subspace of $\mathbb{R}^{3}$.

Table 4.2. Face Data of the Five Platonic Solids

| $\{p, q\}$ | $v$ | $e$ | $f$ |
| :---: | :---: | :---: | :---: |
| $\{4,3\}$ | 8 | 12 | 6 |
| $\{3,4\}$ | 6 | 12 | 8 |
| $\{3,3\}$ | 4 | 6 | 4 |
| $\{3,5\}$ | 12 | 30 | 20 |
| $\{5,3\}$ | 20 | 30 | 12 |

Proposition 4.47. There are exactly five closed and convex polyhedra, namely, the Platonic solids: hexahedron (cube) $\{4,3\}$, octahedron $\{3,4\}$, tetrahedron $\{3,3\}$, icosahedron $\{3,5\}$ and dodecahedron $\{5,3\}$.


Figure 4.24. The Platonic Solids and their Boundary Complexes: The Hexahedron (Cube), Octahedron, Tetrahedron, Icosahedron and Dodecahedron [73]

Let $S_{n}$ denote the symmetric group on $n$ letters of order $n!, A_{n}<S_{n}$ denote the alternating group on $n$ letters of order $\frac{n!}{2}, D_{n}$ denote the dihedral group of order $2 n$, and $\mathbb{Z}_{n}<D_{n}$ denote the cyclic group of order $n$.

For the next few sections, we refer the reader to [420] and [312].
4.8.2. Triangle Groups. Consider a generalized triangle $\triangle_{a, b, c}$ (a triangle on the sphere, plane or pseudo-sphere) with internal angles $\frac{\pi}{a}, \frac{\pi}{b}$ and $\frac{\pi}{c}$. Consider all Euclidean transformations generated by reflections through the edges of $\triangle_{a, b, c}$. Any reflection is involutive, and any composition of two reflections through adjacent edges is equivalent to a rotation by the angle $\frac{2 \pi}{a}, \frac{2 \pi}{b}$ or $\frac{2 \pi}{c}$, respectively. For fixed $(a, b, c) \in \mathbb{N}_{>1}^{3}$, these Euclidean operations generate a Triangle Group $\Delta(a, b, c)$ defined by the presentation,

$$
\begin{equation*}
\Delta(a, b, c)=\left\langle x, y, z \mid x^{2}=y^{2}=z^{2}=(x y)^{a}=(y z)^{b}=(z x)^{c}=e\right\rangle \tag{4.85}
\end{equation*}
$$

where $e$ denotes the identity element. For any $\pi \in S_{3}$,

$$
\begin{equation*}
\Delta(a, b, c) \cong \Delta(\pi(a), \pi(b), \pi(c)) \tag{4.86}
\end{equation*}
$$

4.8.3. von Dyck Groups. The unique normal subgroup of index 2 of the triangle group $\Delta(a, b, c)$ is the von Dyck group $D(a, b, c)$ defined by the presentation,

$$
\begin{equation*}
D(a, b, c)=\left\langle r, s, t \mid r^{a}=s^{b}=t^{c}=r s t=e\right\rangle, \tag{4.87}
\end{equation*}
$$

where $r=x y, s=y z$ and $t=z x$. The order of a von Dyck group $D(a, b, c)$ is as follows:

$$
|D(a, b, c)|=\left\{\begin{array}{cc}
\frac{2}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1} & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1  \tag{4.88}\\
\infty & \text { otherwise }
\end{array}\right.
$$

Note that $D(a, b, c) \cong D(\pi(a), \pi(b), \pi(c))$ for any $\pi \in S_{3}$. The von Dyck groups are partitioned into three families depending on the value of the parameter

$$
\begin{equation*}
\kappa=\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \tag{4.89}
\end{equation*}
$$

namely, the spherical $(\kappa>1)$, euclidean $(\kappa=1)$ and hyperbolic $(\kappa<1)$.

Proposition 4.48. The von Dyck groups are partitioned as follows:

1. (Spherical) $D(2,2, n), D(2,3,3), D(2,3,4)$ and $D(2,3,5)$;
2. (Euclidean) $D(2,3,6), D(2,4,4)$ and $D(3,3,3)$; and,
3. (Hyperbolic) Infinitely many groups $\{D(a, b, c)\}$ such that $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}<1$.

Proof. Simply determine those integers $(a, b, c)$ which satisfy the three cases $\kappa>1, \kappa=1$ and $\kappa<1$, respectively.

For certain parameter values [420], the group $D(a, b, c)$ and the natural geometric object on which it acts is well known, for it pertains to a finite subgroups of $\mathrm{SO}(3)$. For example, for $n \geqslant 1$,

$$
\begin{array}{lc}
D(1, n, n) \cong \mathbb{Z}_{n}<\operatorname{SO}(2) & (\text { acting on }\{n\}) \\
D(2,2, n) \cong D_{n}<\mathrm{O}(2) & (\text { acting on }\{n\}) \\
D(2,3,3) \cong A_{4}<\operatorname{SO}(3) & (\text { acting on }\{3,3\})  \tag{4.90}\\
D(2,3,4) \cong S_{4}<\operatorname{SO}(3) & (\text { acting on }\{3,4\} \text { or }\{4,3\}) \\
D(2,3,5) \cong A_{5}<\operatorname{SO}(3) & (\text { acting on }\{3,5\} \text { or }\{5,3\}) .
\end{array}
$$

In general, the abelianization of the von Dyck group is the cartesian product of cyclic groups, that is, $D(a, b, c)^{\mathrm{ab}} \cong \mathbb{Z}_{m} \times \mathbb{Z}_{n}$, where $m=\operatorname{GCD}(a, b, c)$ and $n=\operatorname{LCM}(\operatorname{GCD}(a, b), \operatorname{GCD}(b, c), \operatorname{GCD}(c, a))$ and $m n=\frac{a b c}{\operatorname{LCM}(a, b, c)}$.
4.8.4. Binary von Dyck Groups. The binary von Dyck group or centrally extended triangle group is defined by the presentation,

$$
\begin{equation*}
\Gamma(a, b, c)=\left\langle r, s, t \mid r^{a}=s^{b}=t^{c}=r s t\right\rangle \tag{4.91}
\end{equation*}
$$

and satisfies the quotient $D(a, b, c) \cong \Gamma(a, b, c) /\langle r s t\rangle$, where $\langle r s t\rangle$ has order 2 . The order of the binary von Dyck group $\Gamma(a, b, c)$ is as follows:

$$
|\Gamma(a, b, c)|=\left\{\begin{array}{cc}
\frac{4}{a} \frac{1}{b}+\frac{1}{c}-1 & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1  \tag{4.92}\\
\infty & \text { otherwise }
\end{array}\right.
$$

with group exponent $2 \operatorname{LCM}(a, b, c)$ and number of conjugacy classes $a+b+c-$ 1. The commutator subgroup $\Pi(a, b, c)=[\Gamma(a, b, c), \Gamma(a, b, c)]$ has order

$$
|\Pi(a, b, c)|=\left\{\begin{array}{cc}
=\frac{4}{a b c\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right)^{2}} & \frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1  \tag{4.93}\\
\infty & \text { otherwise }
\end{array}\right.
$$

Thus, the abelianization $\Gamma(a, b, c)^{\mathrm{ab}} \cong \Gamma(a, b, c) / \Pi(a, b, c)$ has order $\mid a b+b c+$ $c a-a b c \mid$ (Corollary 3.2, [312]).

Proposition 4.49 (Milnor, [312]). Let P denote either the Euclidean, spherical or hyperbolic planes. Let $\bar{G}$ denote the connected Lie group of orientation preserving
isometries of $P$. The fundamental group of the coset space $\bar{G} / D(a, b, c)$ is isomorphic to $\Gamma(a, b, c)$.

REMARK 4.8.1. Consider the following binary von Dyck groups:

1. Let $Q=\left\langle i, j, k \mid i^{2}=j^{2}=k^{2}=i j k\right\rangle$ be the quaternion group;
2. Let $\tilde{T}=\left\langle r, s, t \mid r^{2}=s^{3}=t^{3}=r s t\right\rangle$ be binary tetrahedral group (with $r=s t, s=\frac{1}{2}(1+i+j+k)$ and $\left.t=\frac{1}{2}(1+i+j-k)\right) ;$
3. Let $\tilde{O}=\left\langle r, s, t \mid r^{2}=s^{3}=t^{4}=r s t\right\rangle$ be the binary octahedral group (with $r=s t, s=-\frac{1}{2}(1+i+j+k)$ and $t=\frac{1}{\sqrt{2}}(1+i) ;$ and,
4. Let $\tilde{I}=\left\langle r, s, t \mid r^{2}=s^{3}=t^{5}=r s t\right\rangle$ be the binary icosahedral group (with $r=s t, s=\frac{1}{2}(1+i+j+k), t=\frac{1}{2}\left(\phi+\phi^{-1} i+j\right)$ and $\left.\phi=\frac{1}{2}(1+\sqrt{5})\right)$.

Observe $\tilde{T} \cong \operatorname{SL}\left(2, \mathbb{Z}_{3}\right) \cong Q \rtimes \mathbb{Z}_{3}, \tilde{O} \cong 2 \cdot S_{4}^{-}$, the Schur cover of $S_{4}$ of (-)-type, and $\tilde{I} \cong \mathrm{SL}\left(2, \mathbb{Z}_{5}\right) \cong 2 \cdot A_{5}$, the Schur cover of $A_{5}$ [165].

Proposition 4.50. The following group isomorphisms hold:

1. $\Pi(2,2, r) \cong \mathbb{Z}_{r}$ for $r \in \mathbb{N}_{>1}$;
2. $\Pi(2,3,3) \cong \Gamma(2,2,2)$;
3. $\Pi(2,3,4) \cong \Gamma(2,3,3)$; and,
4. $\Pi(2,3,5) \cong \Gamma(2,3,5)$.

Proof. See Chapter 2 in [420].

REMARK 4.8.2. A group is perfect if and only if it is isomorphic to its commutator subgroup. For instance, the alternating group $A_{5}$, the binary von Dyck
group $\Gamma(2,3,5)$, any non-abelian simple group and any quasi-simple group are perfect.
4.8.5. Lens Spaces. For this section, we refer the reader to [308]. Given $d, q_{0}, \ldots, q_{n} \in \mathbb{Z}$, such that $\operatorname{GCD}\left(d, q_{i}\right)=1$ for $0 \leqslant i \leqslant n$, the orbit space of the (free) $\mathbb{Z}_{d}$-action $\left(z_{0}, \ldots, z_{n}\right) \mapsto\left(\tau_{d}^{q_{0}} z_{0}, \ldots, \zeta_{d}^{q_{n}} z_{n}\right)$ on the unit sphere $S^{2 n+1} \subset \mathbb{C}^{n+1}$ is the Lens space $L\left(d ; q_{0}, \ldots, q_{n}\right)$, which is a compact, connected, orientable $(2 n+1)$-manifold. The quotient map $S^{2 n+1} \rightarrow L\left(d ; q_{0}, \ldots, q_{n}\right)$ is a $d$-fold covering. Note that $L\left(d, q_{0} \ldots, q_{n}\right) \cong L\left(d ; q_{\pi(0)}, \ldots, q_{\pi(n)}\right)$ for any permutation $\pi \in S_{n+1}$. Lens spaces enjoy the following homotopy ${ }^{*}$ groups

$$
\pi_{i}\left(L\left(d ; q_{0}, \ldots, q_{n}\right)\right) \cong \begin{cases}\mathbb{Z}_{d} & i=1  \tag{4.94}\\ \pi_{i}\left(S^{2 d-1}\right) & i \geqslant 2\end{cases}
$$

and homology groups

$$
H_{i}\left(L\left(d ; q_{0}, \ldots, q_{n}\right)\right) \cong \begin{cases}\mathbb{Z} & i \in\{0,2 n-1\}  \tag{4.95}\\ \mathbb{Z}_{d} & i=2 k-1, k \in\{1, \ldots, n-1\} \\ \{0\} & \text { else }\end{cases}
$$

Proposition 4.51. Given two Lens spaces $L\left(d ; q_{0}, \ldots, q_{n}\right)$ and $L\left(d^{\prime} ; q_{0}^{\prime}, \ldots, q_{n}^{\prime}\right)$, the following statements are true:

[^31]1. (Homotopy Equivalence) $L\left(d ; q_{0}, \ldots, q_{n}\right) \simeq L\left(d^{\prime} ; q_{0}^{\prime}, \ldots, q_{n}^{\prime}\right)$ if and only if $q_{0} \cdots q_{n} \equiv \pm k^{n} q_{0}^{\prime} \cdots q_{n}^{\prime} \bmod d$ for some $k \in \mathbb{Z}_{d} ;$
2. (Topological Equivalence) $L\left(d ; q_{0}, \ldots, q_{n}\right) \cong L\left(d^{\prime} ; q_{0}^{\prime}, \ldots, q_{n}^{\prime}\right)$ if and only if there is a permutation $\pi \in S_{n+1}$ and $k \in \mathbb{Z}_{d}$ such that $q_{i} \equiv \pm k q_{\pi(i)}^{\prime} \bmod d$ for $0 \leqslant i \leqslant n$; and,
3. (h-Cobordism Equivalence) $L\left(d ; q_{0}, \ldots, q_{n}\right) \cong_{h-c} L\left(d^{\prime} ; q_{0}^{\prime}, \ldots, q_{n}^{\prime}\right)$ if and only if $L\left(d ; q_{0}, \ldots, q_{n}\right) \cong L\left(d^{\prime} ; q_{0}^{\prime}, \ldots, q_{n}^{\prime}\right)$.

Proof. See [25], [66], [308] and [359].

### 4.8.6. Brieskorn-Pham 3-Manifolds.

Definition 4.52. A spherical 3-manifold is a 3-manifold isomorphic to a quotient $S^{3} / \Gamma$, where $\Gamma$ is a finite subgroup of $S O(4)$ which acts freely by rotations.

Remark 4.8.3. The Poincaré homology sphere or dodecahedral space is a spherical 3-manifold and homology 3-sphere isomorphic to $S^{3} / \tilde{I}$, where $\tilde{I}<\operatorname{SU}(2)$ is the binary icosahedral group, and $S^{3}$ is realized as the 3-dimensional quaternionic Lie group $\operatorname{Spin}(3) \cong \operatorname{Sp}(1) \cong S U(2)$, the double cover of $S O$ (3). However, it is not a topological 3-sphere as its fundamental group is $\tilde{I}$, and it is the only non-trivial homology 3-sphere (excluding $S^{3}$ ) with a finite fundamental group.

Proposition 4.53 (Milnor, [312]). The Brieskorn-Pham 3-manifold $\Sigma(p, q, r)$ is homeomorphic to the r-fold cyclic branched covering of $S^{3}$, branched along the torus $\operatorname{link} T_{p, q}$.

The classification of the triangle groups implies the following result.

Proposition 4.54 (Milnor, [312]). Let $\mathrm{N}<\operatorname{SL}(3, \mathbb{R})$ denote the Heisenberg (matrix) group of upper triangular $3 \times 3$-matrices of the form

$$
\left(\begin{array}{lll}
1 & a & c  \tag{4.96}\\
0 & 1 & b \\
0 & 0 & 1
\end{array}\right) \quad a, b, c \in \mathbb{R}
$$

and $\mathrm{SL} \widetilde{(2, \mathbb{R})}$ denotes the universal cover of $\mathrm{SL}(2, \mathbb{R})$. The following is true:

1. If $\kappa>0$, then $\Sigma(a, b, c)$ is diffeomorphic to $\mathrm{SU}(2) / \Pi$, where $\Pi$ is a finite subgroup;
2. If $\kappa=0$, then $\Sigma(a, b, c)$ is diffeomorphic to $N / \Pi$, where $\Pi$ is a discrete and uniform subgroup; and,
3. If $\kappa<0$, then $\Sigma(a, b, c)$ is diffeomorphic to $\overline{\mathrm{SL}(2, \mathbb{R})} / \Pi$, where $\Pi$ is a cocompact subgroup.

Corollary 4.55 (Milnor, [312]). Given three positive integers $a, b$ and $c$ satisfying $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1$, the Brieskorn-Pham 3-manifold $\Sigma(a, b, c)$ is diffeomorphic to the quotient manifold $S^{3} / \Pi(a, b, c)$, where $\Pi(a, b, c)$ is the commutator subgroup of the binary von Dyck group $\Gamma(a, b, c)$.

Proof. See Theorem 4.5 in [312].

Remark 4.8.4. The Brieskorn-Pham 3-manifold $\Sigma(a, b, c)$ with relatively coprime integers $\{a, b, c\} \subset \mathbb{N}$ is a Seifert 3-manifold, where in particular, $\Sigma(a, b, 1), \Sigma(a, 1, c)$ and $\Sigma(1, b, c)$ are topological 3-spheres, $\Sigma(2,2, n)$ is diffeomorphic to the Lens space $L(n ; n-1), \Sigma(2,3,3), \Sigma(2,3,4)$ and $\Sigma(2,3,5)$ are diffeomorphic to the spherical 3-manifolds $S^{3} / Q, S^{3} / \tilde{O}$ and $S^{3} / \tilde{I}$, respectively. $\quad \triangle$
4.8.7. Seifert Invariants. For this section, we refer the reader to [342]. Let $\mathrm{A}=\left(\alpha_{i j}\right)$ be an $(n-1) \times(n+1)$ matrix such that no $(n-1) \times(n-1)$ subdeterminant vanishes. Define $n$ Brieskorn-Pham polynomials $\mathbb{C}^{n+1}$ with coefficients from the rows of A , say, $f_{i}=\alpha_{i 0} z_{0}^{a_{0}}+\cdots+\alpha_{i n} z_{n}^{a_{n}}$, for $0 \leqslant i \leqslant n-1$, where $a_{0}, \ldots, a_{n} \in \mathbb{N}$. Consider the complete intersection

$$
\begin{equation*}
V_{A}\left(a_{0}, \ldots, a_{n}\right)=\bigcap_{i=0}^{n-1} f_{i}^{-1}(0) \tag{4.97}
\end{equation*}
$$

which admits a $\mathbb{C}^{\times}$-action $\lambda \cdot\left(z_{0}, \ldots, z_{n}\right)=\left(\lambda^{q_{0}} z_{0}, \ldots, \lambda^{q_{n}} z_{n}\right)$, where $q_{i}=\frac{N}{a_{i}}$ and $N=\operatorname{LCM}\left(a_{0}, \ldots, a_{n}\right)$, that restricts to a fixed-point free $\mathbb{C}^{\times}$-action on the Brieskorn-Pham 3-manifold $\Sigma\left(a_{0}, \ldots, a_{n}\right)=V_{A}\left(a_{0}, \ldots, a_{n}\right) \cap S^{2 n-1}$. For any permutation $\pi \in S_{n+1}, \Sigma\left(a_{0}, \ldots, a_{n}\right) \cong \Sigma\left(a_{\pi(0)}, \ldots, a_{\pi(n)}\right)$. Said manifold is therefore a Seifert 3-manifold, which is characterized up to (equivariant) diffeomorphism by
its Seifert invariants $\left\{g ; d_{0}\left(c_{0}, \beta_{0}\right), \ldots, d_{n}\left(c_{n}, \beta_{n}\right)\right\}$ (Theorem 2.1, [342]), where

$$
\begin{align*}
c_{i} & =\frac{N}{\operatorname{LCM}\left(a_{0}, \ldots, \hat{a}_{i}, \ldots, a_{n}\right)} \geqslant 1  \tag{4.98}\\
d_{i} & =\frac{a_{0} \cdots \hat{a}_{i} \cdots a_{n}}{\operatorname{LCM}\left(a_{0}, \ldots, \hat{a}_{i}, \ldots, a_{n}\right)} \geqslant 1  \tag{4.99}\\
g & =1+\frac{1}{2}(n-1) \frac{a_{0} \cdots a_{n}}{N}-\frac{1}{2} \sum_{i=0}^{n} d_{i} \geqslant 0  \tag{4.100}\\
\chi & =-\frac{a_{0} \cdots a_{n}}{N^{2}}, \tag{4.101}
\end{align*}
$$

which satisfy $\operatorname{GCD}\left(c_{i}, \beta_{i}\right)=1$ and $\sum_{i=0}^{n} \beta_{i} \frac{d_{i}}{c_{i}}=-\chi$ or equivalently $\sum_{i=0}^{n} \beta_{i} q_{i}=1$. The integer $g=g\left(\Sigma\left(a_{0}, \ldots, a_{n}\right) / S^{1}\right)$ is the $g e n u s$ of the Seifert surface (or base orbifold) $\Sigma\left(a_{0}, \ldots, a_{n}\right) / S^{1}$ and $\chi=\chi\left(\Sigma\left(a_{0}, \ldots, a_{n}\right)\right)$ is the Euler number of the fiber, respectively. In particular, for a Brieskorn-Pham 3-manifold $\Sigma=\Sigma(a, b, c)$, one has

$$
\begin{align*}
g\left(\Sigma / S^{1}\right) & =\frac{a b c}{2 \operatorname{LCM}(a, b, c)}-\frac{1}{2}(\operatorname{GCD}(a, b)+\operatorname{GCD}(b, c)+\operatorname{GCD}(a, c))+1  \tag{4.102}\\
\chi(\Sigma) & =-\frac{a b c}{\operatorname{LCM}(a, b, c)^{2}} . \tag{4.103}
\end{align*}
$$

Proposition 4.56 (Milnor, [312]). For positive integers $a, b$ and $c$, if

$$
\begin{equation*}
m=\operatorname{LCM}(a, b)=\operatorname{LCM}(a, c)=\operatorname{LCM}(b, c) \tag{4.104}
\end{equation*}
$$

then the Brieskorn-Pham 3-manifold $\Sigma(a, b, c)$ fibers smoothly as a smooth $S^{1}$-bundle with Chern number $-\frac{a b c}{m^{2}}=-\operatorname{GCD}(a, b, c)$ over a Riemann surface of Euler characteristic $\frac{a b c}{m} \kappa$, where $\kappa=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1$, and genus $1-\frac{a b c}{2 m} \kappa$.

Proof. See Lemmata 7.1 and 7.2 and Theorem $7 \cdot 3$ in [312].

### 4.8.8. Homeomorphic Brieskorn-Pham 3-Manifolds.

Proposition 4.57. Given $a, b, c, a^{\prime}, b^{\prime}, c^{\prime} \in \mathbb{N}$, there is a homemorphism $\Sigma(a, b, c) \cong \Sigma\left(a^{\prime}, b^{\prime}, c^{\prime}\right)$ if

$$
\begin{align*}
& \operatorname{GCD}(a, b, c)=\operatorname{GCD}\left(a^{\prime}, b^{\prime}, c^{\prime}\right)  \tag{4.105}\\
& m=\operatorname{LCM}(a, b)=\operatorname{LCM}(b, c)=\operatorname{LCM}(a, c)  \tag{4.106}\\
& m^{\prime}=\operatorname{LCM}\left(a^{\prime}, b^{\prime}\right)=\operatorname{LCM}\left(b^{\prime}, c^{\prime}\right)=\operatorname{LCM}\left(a^{\prime}, c^{\prime}\right)  \tag{4.107}\\
& \frac{a b c}{m}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1\right)=\frac{a^{\prime} b^{\prime} c^{\prime}}{m^{\prime}}\left(\frac{1}{a^{\prime}}+\frac{1}{b^{\prime}}+\frac{1}{c^{\prime}}-1\right), \tag{4.108}
\end{align*}
$$

the latter equality being equivalent to $g\left(\Sigma(a, b, c) / S^{1}\right)=g\left(\Sigma\left(a^{\prime}, b^{\prime}, c^{\prime}\right) / S^{1}\right)$.

Proof. For positive integers $a, b$ and $c$, if $m=\operatorname{LCm}(a, b)=\operatorname{Lcm}(a, c)=$ $\operatorname{LCM}(b, c)$, then according to Proposition $4 \cdot 56$, the Chern number $-\frac{a b c}{m^{2}}$ and and Euler characteristic $\frac{a b c}{m} \kappa$, where $\kappa=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1$, are diffeomorphism invariants. That is, if $\Sigma(a, b, c)$ and $\Sigma\left(a^{\prime}, b^{\prime}, c^{\prime}\right)$ fiber as $S^{1}$-bundles with equal Chern numbers over surfaces of equal (base-orbifold) genera, then there is a homeomorhism $\Sigma(a, b, c) \cong \Sigma\left(a^{\prime}, b^{\prime}, c^{\prime}\right)$ as well as a diffeomorphism $\Sigma(a, b, c) \cong_{d}$ $\Sigma\left(a^{\prime}, b^{\prime}, c^{\prime}\right)$, since the notion of homeomorphism and diffeomorphism coincide
for smooth 3-manifolds. The assumption on $m$ implies

$$
\begin{align*}
\frac{a b c}{m^{2}} & =\tau=\operatorname{GCD}(a, b, c)  \tag{4.109}\\
\frac{a b c}{m} \kappa & =l^{\prime}-\frac{d}{\tau}=2-2 g\left(\Sigma(a, b, c) / S^{1}\right) \tag{4.110}
\end{align*}
$$

where $l^{\prime}=\operatorname{GCD}(a, b)+\operatorname{GCD}(b, c)+\operatorname{GCD}(a, c)$ and $\frac{d}{\tau}=\frac{\operatorname{GCD}(a, b) \operatorname{GCD}(b, c) \operatorname{GCD}(a, c)}{\operatorname{GCD}(a, b, c)}$.

### 4.9. Brieskorn-Pham Manifolds as Homotopy Spheres

The fibered link $K_{f}$ consists of a linked, disjoint union $\coprod S^{2 n-1}$ embedded in $S_{\varepsilon}^{2 n+1}$ and, for $n \neq 2$, is a homotopy sphere if and only if the characteristic polynomial $\Delta_{h_{*}}(t)=\operatorname{det}\left(t I-h_{*}\right)$ of the associated monodromy map $h_{*}$ satisfies $\Delta_{h_{*}}(1)= \pm 1$. The degree of $\Delta_{h_{*}}$ is precisely the number of spheres in the bouquet, $\bigvee^{\mu} S^{n}$.

Let $f=\sum_{i=0}^{n} z_{i}^{a_{i}}$ and define the hypersurface $V_{f, 0}=f^{-1}(0)$. The algebraic link of the Brieskorn-Pham singularity $f$ is the Brieskorn-Pham $(2 n-1)$ manifold $\Sigma\left(a_{0}, \ldots, a_{n}\right)=V_{f, 0} \cap S^{2 n+1}$. The following is a simple set of graphical criteria which imply when the manifold $\Sigma\left(a_{0}, \ldots, a_{n}\right)$ is a homotopy sphere.

Proposition 4.58 (Milnor, Brieskorn, [63]). Let $f=\sum_{i=0}^{n} z_{i}^{a_{i}}$ and $\Delta_{f}=\Delta_{h_{*}}$ denote the associated characteristic polynomial of the monodromy $h_{*}$ of $f$. Define the following graph $\Gamma_{\mathbf{a}}$ on $n+1$ vertices labelled by the exponents $\left\{a_{i}\right\}$ and containing the edge $e_{i j}=\left\langle a_{i}, a_{j}\right\rangle$ if and only if $\operatorname{GCD}\left(a_{i}, a_{j}\right)>1$ (with loops ignored). The manifold
$K_{f}=\Sigma\left(a_{0}, \ldots, a_{n}\right)$ is a homotopy sphere if and only if $\Delta_{f}(1)= \pm 1$ if and only if the following is true:

1. The graph $\Gamma_{\mathbf{a}}$ contains at least two isolated vertices; or,
2. The graph $\Gamma_{\mathbf{a}}$ contains at least one isolated vertex and a connected subgraph on an odd set of vertices such that $\operatorname{GCD}\left(a_{i}, a_{j}\right)=2$ for any two distinct vertices therein.

Proof. See Bedingung B (Condition B) and Satz 1 (Theorem 1) in [63].


Figure 4.25. The Brieskorn Graph $\Gamma_{(2,3,4,5,6)}$

Remark 4.9.1. The Brieskorn-Pham manifold $\Sigma(2,3,4,5,6)$ is not a homotopy sphere, as Brieskorn graph $\Gamma_{(2,3,4,5,6)}$ (Figure 4.25) does not satisfy the criteria of Proposition $4 \cdot 58$. However, this fact can be verified directly by computing
the characteristic polynomial, namely,

$$
\begin{aligned}
\Delta_{h_{*}}(t) & =\Phi_{5}(t)^{2} \Phi_{10}(t)^{2} \Phi_{15}(t) \Phi_{20}(t)^{4} \Phi_{30}(t)^{2} \Phi_{60}(t)^{3} \\
& =\left(1+t+t^{2}-t^{5}+t^{8}-t^{11}+t^{14}+t^{15}+t^{16}\right)^{2}\left(1-t^{2}+t^{4}-t^{6}+t^{8}\right)^{4} \\
& \cdot\left(1+t^{2}-t^{6}-t^{8}-t^{10}+t^{14}+t^{16}\right)^{3}\left(1-t+t^{3}-t^{4}+t^{5}-t^{7}+t^{8}\right),
\end{aligned}
$$

so $\Delta_{h_{*}}(1)=(5)^{2}(1)^{4}(1)^{3}(1)=25$.

### 4.10. Brieskorn-Pham Manifolds as Stiefel Manifolds

Let $V_{k}\left(\mathbb{R}^{d}\right)=\left\{\mathbf{v} \in \mathbb{R}^{d \times k} \mid \mathbf{v}^{\top} \mathbf{v}=1\right\}$ denote the Stiefel manifold of orthonormal $k$-frames in $\mathbb{R}^{d}$. Viewed as a homogeneous space, one has $V_{k}\left(\mathbb{R}^{d}\right) \cong$ $\mathrm{O}(d) / \mathrm{O}(d-k)$, hence $\operatorname{dim} V_{k}\left(\mathbb{R}^{d}\right)=k d-\binom{k+1}{2}$. In particular, $V_{2}\left(\mathbb{R}^{d}\right)$ can be identified with the unit tangent bundle to $S^{d-1}$, where each point on $S^{d-1}$ has $S^{d-2}$ as a fiber.

Proposition 4.59 (Durfee, [116]). The algebraic link $K_{f} \subset S^{2 n+1}$ of the Morse function $f=z_{0}^{2}+\cdots+z_{n}^{2}$ over $\mathbb{C}^{n+1}$ is diffeomorphic to the tangent $S^{n-1}$-bundle of $S^{n}$ with Seifert form $(-1)^{n(n+1) / 2}$.

Proof. See Proposition 2.2 in [116].
The next result generalizes Proposition 4.59.

Proposition 4.60 (Davis, [100, 101]). For $n \neq 2$,

1. If $n$ is even, the link $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{2 n+1}$ is homeomorphic to $V_{2}\left(\mathbb{R}^{n+1}\right)$ if and only if $\Delta_{h_{*}}(1)= \pm 2$; or,
2. If $n$ is odd, the link $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{2 n+1}$ is homeomorphic to $V_{2}\left(\mathbb{R}^{n+1}\right)$ or

$$
S^{n} \times S^{n-1} \text { if and only if } \Delta_{h_{*}}(1)=0 \text { and }(t-1)^{2} \nmid \Delta_{h_{*}}(t) \text {. }
$$

Remark 4.10.1. Observe that for $f=z_{0}^{2}+\cdots+z_{n}^{2}$ over $\mathbb{C}^{n+1}$, one has the characteristic polynomial $\Delta_{h_{*}}(t)=t+(-1)^{n}$. By Proposition $4.60, K_{f}=\Sigma\left(2_{n+1}\right)$ is homeomorphic to $V_{2}\left(\mathbb{R}^{n+1}\right)$ for $n>2$, which is the content of Proposition 4.59.

Remark 4.10.2. Davis [100] proves that any Brieskorn-Pham singularity over $\mathbb{C}^{n+1}$ with even $n$ and $\Delta_{h_{*}}(1)= \pm 2$ is necessary the example in Remark 4.10.1.

Remark 4.10.3. Observe that for $f=z_{0}^{4}+z_{1}^{2}+\cdots+z_{n}^{2}$ over $\mathbb{C}^{n+1}$, one has the characteristic polynomial $\Delta_{h_{*}}(t)=\left(t^{2}+1\right)\left(t+(-1)^{n}\right)$. By Proposition 4.60, $K_{f}=\Sigma\left(4,2_{\underline{n}}\right)$ is homeomorphic to either $V_{2}\left(\mathbb{R}^{n+1}\right)$ or $S^{n} \times S^{n-1}$ for odd $n>2$.

Remark 4.10.4. Observe that for $f=z_{0}^{4}+z_{1}^{4}+z_{2}^{2}+\cdots+z_{n}^{2}$ over $\mathbb{C}^{n+1}$, one has the characteristic polynomial

$$
\begin{equation*}
\Delta_{h_{*}}(t)=\left(t^{2}+1\right)^{2}(t+1)^{\frac{5}{2}+(-1)^{n} \frac{1}{2}}(t-1)^{\frac{5}{2}-(-1)^{n} \frac{1}{2}} . \tag{4.111}
\end{equation*}
$$

By Proposition $4.60, K_{f}=\Sigma\left(4,4,2_{n-1}\right)$ is not homeomorphic to $V_{2}\left(\mathbb{R}^{n+1}\right)$ for $n>2$.

### 4.11. Brieskorn-Pham Manifolds as Exotic Spheres

4.11.1. Hirzebruch Signature Theorem. Let $M^{4 k}$ denote a closed, oriented, $4 k$-manifold. Consider the self-cup product map $B: x \mapsto x \smile x$, which is a quadratic form of type $(p, q)$, where $x$ is an element of the middle cohomology group $H^{2 k}\left(M^{4 k} ; \mathbb{Z}\right) / T$, where $T$ is the corresponding torsion, mapping to an element in the top cohomology group $H^{4 k}\left(M^{4 k} ; \mathbb{Z}\right) \cong \mathbb{Z}$.

Definition 4.61. The Thom signature $\sigma\left(M^{4 k}\right)$ is defined as the signature of the quadratic form $B$, that is, $p-q$.

Thom proved that the signature $\sigma\left(M^{4 k}\right)$ is a homomorphism from the cobordism class $\Omega_{4 k}$ to the integers, therefore a cobordism invariant, and is a $\mathbb{Q}$-linear combination of the Pontryagin numbers [203].

Hirzebruch related the Thom signature to the L-genus [203]. In particular, the Hirzebruch-Thom signature $\sigma\left(M^{n}\right)$ is defined for any compact, smooth, oriented differential manifold $M^{n}$ of positive dimension, and is the value of the pairing of the $L$-genus with the fundamental homology class $\left[M^{n}\right]$,

$$
\sigma\left(M^{n}\right)=\left\{\begin{array}{cc}
0 & n \neq 0 \bmod 4  \tag{4.112}\\
\left\langle L_{k},\left[M^{n}\right]\right\rangle & n=4 k
\end{array}\right.
$$

where $L_{k}=L_{k}\left(p_{1}, \ldots, p_{k}\right)$ is a Q-polynomial of degree at most $k$ over oriented cobordism invariants, namely, the Pontryagin classes $p_{k}=p_{k}\left(T M^{n}\right) \in$ $H^{4 k}\left(M^{n} ; \mathbb{Z}\right)$. In general, $L_{k}$ is given in terms of the complexified tangent bundle of $M^{n}, L_{k}=\prod_{i=1}^{2 k} \frac{x_{i}}{\tanh x_{i}}$, where $x_{i}=c_{i}\left(M^{n}\right)$ are the Chern roots of $M^{n}$. The fact
that $\sigma\left(M^{n}\right)$ is an integer imposes strict divisibility criteria on the Pontryagin classes of $M^{n}$. For 4, 8 and 12-manifolds, the signature relations are

$$
\begin{align*}
\sigma\left(M^{4}\right) & =\frac{1}{3}\left\langle p_{1},\left[M^{4}\right]\right\rangle  \tag{4.113}\\
\sigma\left(M^{8}\right) & =\frac{1}{3^{2} \cdot 5}\left\langle 7 p_{2}-p_{1}^{2},\left[M^{8}\right]\right\rangle  \tag{4.114}\\
\sigma\left(M^{12}\right) & =\frac{1}{3^{3} \cdot 5 \cdot 7}\left\langle 2 p_{1}^{3}+(2 \cdot 31) p_{3}-13 p_{1} p_{2},\left[M^{12}\right]\right\rangle \tag{4.115}
\end{align*}
$$

respectively. The signature often has curious divisibility properties. According to Hirzebruch [202], if $b_{4}\left(M^{12}\right)=0$ (the fourth betti number), then

$$
\begin{equation*}
\left\langle 2 p_{1}^{3}-13 p_{1} p_{2},\left[M^{12}\right]\right\rangle=0, \tag{4.116}
\end{equation*}
$$

so the corresponding signature satisfies

$$
\begin{equation*}
945 \sigma\left(M^{12}\right)=62\left\langle p_{3},\left[M^{12}\right]\right\rangle \tag{4.117}
\end{equation*}
$$

and is therefore divisible by 62 as $\left\langle p_{3},\left[M^{12}\right]\right\rangle \in \mathbb{Z}$ and $\operatorname{GCD}(945,62)=1$.

### 4.11.2. Homology, Homotopy, Topological and Exotic Spheres.

Definition 4.62. A topological $n$-sphere is a smooth, closed, oriented $n$ manifold that is homeomorphic to the $n$-sphere. A homology $n$-sphere is a $n$ manifold that is homology equivalent to the $n$-sphere, sharing the same homology groups. A homotopy $n$-sphere is a $n$-manifold that is homotopy equivalent to the $n$-sphere, sharing the same homology and homotopy groups, so, in particular, a homology $n$-sphere.

Definition 4.63. A homology $n$-sphere is an $n$-manifold possessing the same homology groups as those of an $n$-sphere, i.e., $H_{i}(X ; \mathbb{Z}) \cong\{0\}$ for $1 \leqslant i \leqslant n-1$ and $H_{0}(X ; \mathbb{Z}) \cong H_{n}(X ; \mathbb{Z}) \cong \mathbb{Z}$. A homotopy $n$-sphere is an $n$ manifold homotopy equivalent to $S^{n}$. A topological $n$-sphere is an $n$-manifold homeomorphic to $S^{n}$. An exotic $n$-sphere is a topological $n$-sphere not diffeomorphic to $S^{n}$.

Remark 4.11.1. Every homotopy $n$-sphere is a homology $n$-sphere. Every topological $n$-sphere is a homotopy $n$-sphere.

According to the combined work of Smale $(n \geqslant 5)$, Freedman $(n=4)$, Perelman $(n=3)$ and Möbius, von Dyck, Dehn, Heegaard and Rado ( $n=1,2$ ), a smooth homotopy $n$-sphere is a topological $n$-sphere provided that $n \geqslant 1$. Their work is summarized in the following landmark result.

Proposition 4.64 (Poincaré Conjecture). For $n \geqslant 2$, every homotopy $n$-sphere is homeomorphic to an $n$-sphere, i.e., a topological n-sphere.

It is natural then to consider whether a differential analogue of the Poincaré Conjecture holds for spheres.

Conjecture 4.65 (Smooth Poincaré Conjecture). For $n \geqslant 2$, every homotopy $n$-sphere is diffeomorphic to $S^{n}$.

The case $n=2$ is classical. Perelman proved the case $n=3$. The case $n=4$ remains open. In 1956, Milnor gave a counter-example for $n=7$ [305]. Milnor
and Kervaire proved that it holds for $n \in\{5,6\}$ and produced counter-examples for $n>7$ [313].
4.11.3. Milnor 7-Sphere. While investigating $S^{3}$-bundles over $S^{4}$ with rotation and structural group SO(4), in 1956, Milnor discovered that the 7-sphere has several differentiable structures [305]. In particular, Milnor constructs a Thom space $T$ with boundary $M$ and signature $\sigma(T)=1$ and $\left\langle p_{1}^{2},[T]\right\rangle=k^{2}$ for some integer $k$ congruent to 2 modulo 4 . However, by equation (4.114),

$$
\begin{equation*}
\left\langle p_{2},[T]\right\rangle=\frac{1}{7}\left(45 \sigma(T)+\left\langle p_{1}^{2},[T]\right\rangle\right)=\frac{1}{7}\left(45+k^{2}\right), \tag{4.118}
\end{equation*}
$$

which is not an integer if $k$ is not congruent to $\pm 2$ modulo 7 . Therefore, $M$ is not diffeomorphic to $S^{7}$ in the excluded cases.

Proposition 4.66 (Reeb). Given a compact n-manifold $M$ and a Morse function $f: M \rightarrow \mathbb{R}$ with exactly two critical points, then $M$ is homeomorphic to $S^{n}$.

Proof. See Theorem 4.1 in [306].
Milnor proves that $M$ is a compact, oriented smooth 7-dimensional manifold satisfying the assumptions of Reeb's Sphere Theorem, so $M$ is homeomorphic to $S^{7}$. For a detailed discussion of this intriguing topic, see [306], [311] and Chapter 20 in [316].
4.11.4. Homotopy Spheres. Let $\Sigma^{n},\left[\Sigma^{n}\right]$ and $\Theta_{n}=\left\{\left[\Sigma^{n}\right] \mid \Sigma^{n} \simeq S^{n}\right\}$ denote a homotopy $n$-sphere, an equivalence class of $n$-spheres up to oriented $h$ cobordism, and the additive abelian group of such classes under the operation
of connected sum, with inverse given by reversing orientation [237, 311]. By the work of Smale, Freedman, Perelman and others, the $h$-Cobordism Theorem implies that the elements of $\Theta_{n}$ are in fact oriented diffeomorphism classes. In particular, every homotopy $n$-sphere is a topological $n$-sphere for $n \geqslant 0$. There is a cyclic subgroup $b P_{n+1}<\Theta_{n}$ consisting of the homotopy spheres which bound ( $n+1$ )-dimensional parallelizable (smooth) manifolds. The groups $\Theta_{n}$ and $b P_{n+1}$ are the Milnor-Kervaire groups. For $2 \leqslant n \leqslant 6, \Theta_{n}$ and $b P_{n+1}$ are trivial. For $m \geqslant 2$,

$$
\begin{equation*}
\left|b P_{4 m}\right|=2^{2 m-2}\left(2^{2 m-1}-1\right) \operatorname{num}\left(\frac{4\left|B_{2 m}\right|}{m}\right), \tag{4.119}
\end{equation*}
$$

where $B_{m}$ is the $m^{\text {th }}$-Bernoulli number. Milnor and Kervaire prove that $b P_{2 m+1}$ is trivial for $m \geqslant 1$. Recent work by Hill, Hopkins and Ravenel concerning the Kervaire Invariant One problem implies $b P_{2^{l-2}} \cong \mathbb{Z}_{2}$ for $l \geqslant 8$ [210]. Essential to the complete understanding of $b P_{4 m+2}$ is the computation of the Kervaire Invariant. Based on the work of Kervaire, et al., the current state of knowledge of the order of these groups is the following:

$$
\left|b P_{4 m+2}\right|=\left\{\begin{array}{cl}
1 & m \in\{1,3,7,15\}  \tag{4.120}\\
1 \text { or } 2 & m=31 \\
2 & \text { otherwise }
\end{array}\right.
$$

where the group $b P_{126}$ is hitherto still not completely understood. The number of exotic spheres in dimension $n$ is inferred from a careful study of the group
$\Theta_{n}$, and its order $\left|\Theta_{n}\right|$, that is, the number of $h$-cobordism classes of smooth homotopy $n$-spheres as a function of $n \geqslant 1$ [237] (A001676).

In higher dimensions, an algebraic link $K_{f}$ may have curious differential structure, quite different from that of the ordinary $(2 n-1)$-sphere. If $f=\sum_{i=0}^{n} z_{i}^{a_{i}}$ with $a_{i} \geqslant 1$, one often writes $\Sigma\left(a_{0}, \ldots, a_{n}\right)$ in place of $K_{f}$, not to be confused for the Brieskorn-Pham 3-manifold arising from a complete intersection in §4.8.7. Again, $\Sigma\left(a_{0}, \ldots, a_{n}\right) \cong \Sigma\left(a_{\pi(0)}, \ldots, a_{\pi(n)}\right)$ for any permutation $\pi \in S_{n+1}$. Let $a_{\underline{\underline{b}}}$ denote $a$ repeated $b$ times as in ' $a, \ldots, a^{\prime}$.

Proposition 4.67 (Brieskorn, Milnor [310]). If $a_{0}, \ldots, a_{n} \in \mathbb{N}$ are pairwise coprime, then the manifold $\Sigma\left(a_{0}, \ldots, a_{n}\right)$ is an integral homology $(2 n-1)$-sphere. If, in addition, $n>2$, then $\Sigma\left(a_{0}, \ldots, a_{n}\right)$ is a topological $(2 n-1)$-sphere.
4.11.5. Brieskorn-Pham Manifolds as Exotic Spheres. Consider the polynomial $f=z_{0}^{5}+z_{1}^{3}+\sum_{i=2}^{6} z_{i}^{2}$ over $\mathbb{C}^{7}$, and define the 1-parameter family of complex hypersurfaces $V_{f, \kappa}=f^{-1}(\kappa)$ with $\kappa \in \mathbb{C}$ sufficiently close to the origin [63]. By the ADE classification of simple singularities, the singularity $f$ is a 4 -stabilization of the $E_{8}$ surface singularity $\left(x^{2}=y^{3}+z^{5}\right.$ over $\mathbb{C}^{3}$ ) and corresponds to a Milnor fiber $F_{\Sigma^{4} E_{8}} \simeq \bigvee^{8} S^{6}$ with topological index $\mu_{\text {top }}\left(\Sigma^{4} f\right)=\mu_{\text {top }}(f)=8$. The intersection $V_{f, \kappa} \cap B_{\varepsilon}^{14}$, where $\kappa \in \mathbb{C}$ is a regular value of $f$ sufficiently close to the origin and $\epsilon>0$ is sufficiently small, is a 12-manifold with boundary

$$
\begin{equation*}
K_{\Sigma^{4} E_{8}}^{11}=\partial\left(V_{f, \kappa} \cap B_{\varepsilon}^{14}\right)=V_{f, \kappa} \cap S_{\varepsilon}^{13} . \tag{4.121}
\end{equation*}
$$

This algebraic link is the 4-iterated stabilization of the 5-iterated cyclic branched covering of the trefoil knot and has reduced Alexander polynomial

$$
\begin{align*}
\Delta_{f}(t) & =\frac{\left(t^{15}-1\right)(t-1)}{\left(t^{5}-1\right)\left(t^{3}-1\right)}  \tag{4.122a}\\
& =1-t+t^{3}-t^{4}+t^{5}-t^{7}+t^{8}  \tag{4.122b}\\
& =\Phi_{15}(t) \tag{4.122c}
\end{align*}
$$

where $\Phi_{n}(t)$ is the $n^{\text {th }}$-cyclotomic polynomial. According to Milnor, since $\Delta_{f}(1)=\Phi_{15}(1)=1$, then $K_{\Sigma^{4} E_{8}}^{11}$ is a topological sphere. The quotient space $M_{\Sigma^{4} E_{8}}^{12}=V_{f, \kappa} \cap B_{\varepsilon}^{14} / K_{\Sigma^{4} E_{8}}^{11}$ is a 5-connected 12-manifold (without boundary) with $b_{i}\left(M_{f}^{12}\right)=0$ for $1 \leqslant i \leqslant 5$ and signature $\sigma\left(M_{\Sigma^{4} E_{8}}^{12}\right)=-8$ [202]. As the signature is not divisible by 62 , it follows that $M_{\Sigma^{4} E_{8}}^{12}$ is not a differentiable manifold. In particular, although $K_{\Sigma^{4} E_{8}}^{11}$ is homeomorphic to $S^{11}$, it cannot be diffeomorphic to it. Hence, $K_{\Sigma^{4} E_{8}}^{11}$ is an exotic 11-sphere [63]. This example represents one of 992 (oriented diffeomorphism classes of) differentiable structures on $S^{11}$ - all representable by the 1-parameter family of polynomials $f=z_{0}^{6 k-1}+z_{1}^{3}+\sum_{i=2}^{6} z_{i}^{2}$ for $1 \leqslant k \leqslant 992$. In fact, up to diffeomorphism, all exotic spheres in dimensions $4 m-1$ admit a similar realization.

Proposition 4.68 (Hirzebruch, Brieskorn [63]). Let $\Sigma_{k}^{4 m-1}$ be the link of the Brieskorn-Pham singularity $f=z_{0}^{6 k-1}+z_{1}^{3}+\sum_{i=2}^{2 m} z_{i}^{2}$. Then $\Sigma_{k}^{4 m-1}$ is a homotopy sphere with signature $\sigma\left(\Sigma_{k}^{4 m-1}\right)=(-1)^{m} 8 k$ and represents $\frac{\sigma_{m}}{8}$ differential structures
in $b P_{4 m}$, where

$$
\begin{equation*}
\sigma_{m}=2^{2 m+1}\left(2^{2 m-1}-1\right) \operatorname{num}\left(\frac{4\left|B_{2 m}\right|}{m}\right), \tag{4.123}
\end{equation*}
$$

that is, $\Sigma_{k}^{4 m-1} \in b P_{4 m}$ for $1 \leqslant k \leqslant \frac{\sigma_{m}}{8}$.
4.11.5.1. Arf-Kervaire Invariant. The differentiable structure of an algebraic link $K_{f}$ is determined by certain values of the characteristic polynomial, namely, $\Delta_{f}( \pm 1)$, and certain invariants of the corresponding fiber, namely, the signature $\sigma\left(F_{f, 0}\right)$ (for even $n>2$ ) and Arf-Kervaire invariant $c\left(F_{f, 0}\right)$ (for odd $n$ ) [310], [420]. For certain values of $n$, it is determined completely by the characteristic polynomial.

Proposition 4.69 (Levine, [268]). If a fibered link $L$ is a topological ( $2 n-1$ )sphere for some odd $n$, then the Arf-Kervaire invariant of the fiber $F$ is determined by the characteristic polynomial,

$$
c(F)= \begin{cases}0 & \Delta_{L}(-1) \equiv \pm 1 \bmod 8 \\ 1 & \Delta_{L}(-1) \equiv \pm 3 \bmod 8\end{cases}
$$

The link $L$ is diffeomorphic to $S^{2 n-1}$ if and only if $c(F)=0$.

Proof. See also [331].
Thus, for odd $n$, if the algebraic link $K_{f}$ is a topological sphere, then its diffeomorphism structure is determined by the monodromy through the value $\Delta_{f}(-1)$. Using this result, Brieskorn proves the following.

Proposition 4.70 (Brieskorn, [63]). The manifold $\Sigma\left(d, 2_{\underline{n}}\right)$ is an exotic ( $2 n-$ $1)$-sphere for odd $d, n>2$ if and only if $d \equiv \pm 3 \bmod 8$.

### 4.12. Characteristic Polynomial of Brieskorn-Pham Manifolds

Proposition 4.71. For positive integers $a_{0}, \ldots, a_{n}$,

$$
\begin{equation*}
\bigotimes_{i=0}^{n}\left(t^{a_{i}}-1\right)=\left(t^{\mathrm{LCM}\left(a_{0}, \ldots, a_{n}\right)}-1\right)^{N\left(a_{0}, \ldots, a_{n}\right)}, \tag{4.124}
\end{equation*}
$$

where

$$
\begin{align*}
N\left(a_{0}, \ldots, a_{n}\right) & =\frac{a_{0} \cdots a_{n}}{\operatorname{LCM}\left(a_{0}, \ldots, a_{n}\right)}  \tag{4.125a}\\
& =\prod_{l=2}^{n} \prod_{\left\{i_{1}, \ldots, i_{l}\right\} \subseteq\left\{a_{0}, \ldots, a_{n}\right\}} \operatorname{GCD}\left(i_{1}, \ldots, i_{l}\right)^{(-1)^{l}} . \tag{4.125b}
\end{align*}
$$

Proof. We prove the identity by induction on $n$. The identity is true for $n=1$ by the classical result,

$$
\begin{equation*}
\left(t^{a}-1\right) \otimes\left(t^{b}-1\right)=\left(t^{\mathrm{LCM}(a, b)}-1\right)^{a b / \mathrm{LCм}(a, b)} \tag{4.126}
\end{equation*}
$$

Assume the claim holds for some auxiliary index $k>1$. Then it holds for $k+1$, as

$$
\begin{align*}
\bigotimes_{i=0}^{k+1}\left(t^{a_{i}}-1\right) & =\left(t^{\mathrm{LCM}\left(a_{0}, \ldots, a_{k}\right)}-1\right)^{N\left(a_{0}, \ldots, a_{k}\right)} \otimes\left(t^{a_{k+1}}-1\right)  \tag{4.127}\\
& =\left(\left(t^{\mathrm{LCM}\left(a_{0}, \ldots, a_{k}\right)}-1\right) \otimes\left(t^{a_{k+1}}-1\right)\right)^{N\left(a_{0}, \ldots, a_{k}\right)}  \tag{4.128}\\
& =\left(t^{\mathrm{LCM}\left(a_{0}, \ldots, a_{k+1}\right)}-1\right)^{N\left(a_{0}, \ldots, a_{k+1}\right)} \tag{4.129}
\end{align*}
$$

since

$$
\begin{align*}
N\left(a_{0}, \ldots, a_{k+1}\right) & =\frac{N\left(a_{0}, \ldots, a_{k}\right) \operatorname{LCM}\left(a_{0}, \ldots, a_{k}\right) a_{k+1}}{\operatorname{LCM}\left(\operatorname{LCM}\left(a_{0}, \ldots, a_{k}\right), a_{k+1}\right)}  \tag{4.130}\\
& =\prod_{l=2}^{k+1} \prod_{\left\{i_{1}, \ldots, i_{l}\right\} \subseteq\left\{a_{0}, \ldots, a_{k+1}\right\}} \operatorname{GCD}\left(i_{1}, \ldots, i_{l}\right)^{(-1)^{l}}, \tag{4.131}
\end{align*}
$$

which follows from the following arithmetic identity: For $1<l<k$,

$$
\begin{align*}
& \operatorname{LCM}\left(a_{1}, \ldots, a_{k}\right)=\operatorname{LCM}\left(a_{1}, \ldots, a_{k-l}, \operatorname{LCM}\left(a_{k-l+1}, \ldots, a_{k}\right)\right)  \tag{4.132}\\
& \operatorname{GCD}\left(a_{1}, \ldots, a_{k}\right)=\operatorname{GCD}\left(a_{1}, \ldots, a_{k-l}, \operatorname{GCD}\left(a_{k-l+1}, \ldots, a_{k}\right)\right) . \tag{4.133}
\end{align*}
$$

This proves the inductive hypothesis and completes the proof of the claim.

REMARK 4.12.1. The classical identity $t^{n}-1=\prod_{k=0}^{n-1}\left(t-\zeta_{n}^{k}\right)$, where $\zeta_{n}=$ $e^{2 \pi i / n}$, combined with Proposition 4.71 implies the factorization,

$$
\begin{equation*}
\left(t^{\mathrm{LCM}\left(a_{0}, \ldots, a_{n}\right)}-1\right)^{N\left(a_{0}, \ldots, a_{n}\right)}=\prod_{k_{0}=0}^{a_{0}-1} \cdots \prod_{k_{n}=0}^{a_{n}-1}\left(t-\zeta_{a_{0}}^{k_{0}} \cdots \zeta_{a_{n}}^{k_{n}}\right) . \tag{4.134}
\end{equation*}
$$

For a Brieskorn-Pham polynomial $f$ with exponents $\left\{a_{0}, \ldots, a_{n}\right\}$, the algebraic link $K_{f}$ is the Brieskorn-Pham manifold $\Sigma\left(a_{0}, \ldots, a_{n}\right)$. One way to determine if $\Sigma\left(a_{0}, \ldots, a_{n}\right)$ is an integral homology sphere ( $n=2$ ) or topological sphere $(n>2)$ is to evaluate the corresponding characteristic polynomial $\Delta_{f}(t)$ at unity.

Proposition 4.72. The characteristic polynomial of a non-degenerate, quasi-Brieskorn-Pham singularity with inverse weights $\left\{a_{0}, \ldots, a_{n}\right\}$ is the alternating product

$$
\begin{align*}
\Delta_{f}(t) & =\prod_{k_{0}=1}^{a_{0}-1} \cdots \prod_{k_{n}=1}^{a_{n}-1}\left(t-\zeta_{a_{0}}^{k_{0}} \cdots \zeta_{a_{n}}^{k_{n}}\right)  \tag{4.135}\\
& =(t-1)^{(-1)^{n+1}} \prod_{k=1}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n}\left(t^{\mathrm{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)}-1\right)^{(-1)^{n-k+1} N\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)},
\end{align*}
$$

where $N\left(a_{0}, \ldots, a_{n}\right)$ is defined in Proposition 4.71.

Proof 1. The reduced Poincaré series of the local algebra of $f$ is given by the iterated summation,

$$
\begin{align*}
\bar{P}_{\mathcal{A}}(t) & =\prod_{i=0}^{n} \frac{1-t^{1-1 / a_{i}}}{1-t^{1 / a_{i}}}  \tag{4.136a}\\
& =\sum_{k_{0}=0}^{a_{0}-2} \cdots \sum_{k_{n}=0}^{a_{n}-2} t^{k_{0} / a_{0}+\cdots+k_{n} / a_{n}} \tag{4.136b}
\end{align*}
$$

which is a specialization of the weighted Ehrhart function of the minimal $(n+1)$-orthotope $\square$ enclosing the simplicial $(n+1)$-polytope $\operatorname{conv}\left\{0,\left(a_{0}-\right.\right.$ 2) $\left.\mathbf{e}_{1}, \ldots,\left(a_{n}-2\right) \mathbf{e}_{n+1}\right\}$, namely,

$$
\begin{equation*}
\mathscr{L}_{\square}\left(1 ; x_{0}, \ldots, x_{n}\right)=\sum_{i_{0}=0}^{a_{0}-2} \cdots \sum_{i_{n}=0}^{a_{n}-2} x_{0}^{i_{0}} \cdots x_{n}^{i_{n}} \tag{4.137}
\end{equation*}
$$

where $x_{i}=t^{1 / a_{i}}, q . v .$, Volume 2. Define the set $I$ of intersection lattice points $\square \cap \mathbb{Z}^{n+1}=\left\{(0, \ldots, 0), \ldots,\left(a_{0}-2, \ldots, a_{n}-2\right)\right\}$. Writing $\bar{P}_{\mathcal{A}}(t)=\sum_{j=1}^{\mu} t^{\alpha_{j}}$, where
$\mu=\mu_{\mathrm{alg}}(f)=\prod_{i=0}^{n}\left(a_{i}-1\right)$ and $\left\{\alpha_{j}\right\}_{j=0}^{\mu}$ is the list $\left\{\sum_{i=0}^{n} \frac{k_{i}}{a_{i}}\right\}_{\left(k_{0}, \ldots, k_{n}\right) \in I}$ reordered, define the shifted exponents $\gamma_{j}=\alpha_{j}+\sum_{i=0}^{n} \frac{1}{a_{i}}$. As a consequence of Proposition 4.71, noting the indices of the product of linear terms involving roots-of-unity, one then computes the characteristic polynomial as a rational function in $\mathbb{Z}(t)$,

$$
\begin{align*}
\Delta_{f}(t) & =\prod_{j=0}^{\mu}\left(t-e^{2 \pi i \gamma_{j}}\right) \\
& =\prod_{k_{0}=1}^{a_{0}-1} \cdots \prod_{k_{n}=1}^{a_{n}-1}\left(t-\zeta_{a_{0}}^{k_{0}} \ldots \zeta_{a_{n}}^{k_{n}}\right)  \tag{4.138b}\\
& =(t-1)^{(-1)^{n+1}} \prod_{k=1}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n}\left(t^{\mathrm{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)}-1\right)^{(-1)^{n-k+1} N\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)}
\end{align*}
$$

with $\sum_{k=0}^{n+1}\binom{n+1}{k}=2^{n+1}$ terms.

Remark 4.12.2. Therefore, $\left\{\operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)\right\}_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n}$ forms the set of divisors of the period $N=\operatorname{LCM}\left(a_{0}, \ldots, a_{n}\right)$ with corresponding exponents of the form

$$
\begin{equation*}
r_{\mathrm{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)}=(-1)^{k} \frac{a_{i_{1}} \cdots a_{i_{k}}}{\operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)} \tag{4.139}
\end{equation*}
$$

and, hence, the degree of the characteristic polynomial $\Delta_{f}(t)$ is

$$
\begin{align*}
\mu_{\mathrm{alg}}(f) & =(-1)^{n+1}+\sum_{k=1}^{n+1}(-1)^{n+1-k} \sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} a_{i_{1}} \cdots a_{i_{k}}  \tag{4.140}\\
& =\sum_{k=0}^{n+1}(-1)^{n+1-k} e_{k}\left(a_{0}, \ldots, a_{n}\right)  \tag{4.141}\\
& =\prod_{i=0}^{n}\left(a_{i}-1\right) . \tag{4.142}
\end{align*}
$$

Proof 2. For a non-degenerate, quasi-Brieskorn-Pham singularity with inverse weights $\left\{a_{0}, \ldots, a_{n}\right\}$, the aforementioned representation of the characteristic polynomial follows directly from the divisor identity

$$
\begin{align*}
\operatorname{div} \Delta_{f}(t) & =\prod_{i=0}^{n}\left(\Lambda_{a_{i}}-\Lambda_{1}\right)  \tag{4.143a}\\
& =\sum_{k=0}^{n+1}(-1)^{n+1-k} e_{k}\left(\Lambda_{a_{0}}, \ldots, \Lambda_{a_{n}}\right) \tag{4.143b}
\end{align*}
$$

where $e_{k}$ is the $k^{\text {th }}$-elementary symmetric polynomial,

$$
\begin{align*}
e_{0}\left(\Lambda_{a_{0}}, \ldots, \Lambda_{a_{n}}\right) & =\Lambda_{1}  \tag{4.144a}\\
e_{1}\left(\Lambda_{a_{0}}, \ldots, \Lambda_{a_{n}}\right) & =\sum_{i=0}^{n} \Lambda_{a_{i}}  \tag{4.144b}\\
e_{k}\left(\Lambda_{a_{0}}, \ldots, \Lambda_{a_{n}}\right) & =\sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \Lambda_{a_{i_{1}}} \cdots \Lambda_{a_{i_{k}}} \quad k>1 \\
& =\sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \frac{a_{i_{1} \cdots a_{i_{k}}}^{\operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)} \Lambda_{\mathrm{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)}}{} \tag{4.144d}
\end{align*}
$$

using the following identity: For $1<l<k$,

$$
\begin{align*}
& \operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)=\operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k-l}}, \operatorname{LCM}\left(a_{i_{k-l+1}}, \ldots, a_{i_{k}}\right)\right)  \tag{4.145}\\
& \operatorname{GCD}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)=\operatorname{GCD}\left(a_{i_{1}}, \ldots, a_{i_{k-l}}, \operatorname{GCD}\left(a_{i_{k-l+1}}, \ldots, a_{i_{k}}\right)\right) \tag{4.146}
\end{align*}
$$

4.12.0.2. Milnor-Orlik Invariants. Recall that if $\operatorname{div} \Delta_{f}(t)=\sum_{k \geqslant 1} c_{k} \Lambda_{k}$, the Milnor-Orlik invariants are the non-negative integers

$$
\begin{equation*}
\kappa=\sum_{k \geqslant 1} c_{k} \quad \text { and } \quad \rho=\prod_{k \geqslant 2} k^{c_{k}} \tag{4.147}
\end{equation*}
$$

Milnor and Orlik [315] prove $\Delta_{f}(1)=\delta_{\kappa, 0} \rho$, where $\kappa$ is higher power of $t-1$ which divides $\Delta_{f}(t)$, q.v., Proposition 2.52.

Corollary 4.73. The Milnor-Orlik invariants for a non-degenerate, quasi-Brieskorn-Pham singularity with inverse weights $\left\{a_{0}, \ldots, a_{n}\right\}$ are

$$
\begin{equation*}
\kappa=(-1)^{n+1}+(-1)^{n}(n+1)+\sum_{k=2}^{n+1}(-1)^{n-k+1} \sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \frac{a_{i_{1} \cdots a_{i_{k}}}^{\operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)}}{} \tag{4.148}
\end{equation*}
$$

and

$$
\rho=\left(a_{0} \cdots a_{n}\right)^{(-1)^{n}} \prod_{k=2}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)^{(-1)^{n-k+1} a_{i_{1}} \cdots a_{i_{k}} / \operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right) .}
$$

Corollary 4.74. Given a non-degenerate, quasi-Brieskorn-Pham polynomial $f$ with pairwise coprime inverse weights $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}$, the corresponding characteristic polynomial is the product

$$
\begin{equation*}
\Delta_{f}(t)=(t-1)^{(-1)^{n+1}} \prod_{d \mid N}\left(t^{d}-1\right)^{m(d)} \tag{4.149}
\end{equation*}
$$

where $N=a_{0} \cdots a_{n}$ and $m(d)$ is 1 if $d$ is a product of $l+1$ elements from $\left\{a_{0}, \ldots, a_{n}\right\}$ and $l+n$ is even and -1 otherwise. Moreover,

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=(-1)^{n+1}+\sum_{d \mid N} d m(d) \tag{4.150}
\end{equation*}
$$

Proof. If $\left\{a_{1}, \ldots, a_{n}\right\}$ is pairwise coprime, then by Proposition 4.72 ,

$$
\begin{equation*}
\Delta_{f}(t)=(t-1)^{(-1)^{n+1}} \prod_{i=0}^{n}\left(t^{a_{0} \cdots a_{i}}-1\right)^{(-1)^{i+n}} \tag{4.151}
\end{equation*}
$$

REMARK 4.12.3. Define the $q$-integer $[n]_{q}=\frac{q^{n}-1}{q-1}$. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a homogeneous polynomial of degree $d$, then the characteristic polynomial is the product of two terms, namely,

$$
\begin{align*}
\Delta_{f}(t) & =(t-1)^{(-1)^{n+1}} \prod_{k=1}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n}\left(t^{d}-1\right)^{(-1)^{n-k+1} d^{k-1}}  \tag{4.152}\\
& =(t-1)^{(-1)^{n+1}} \prod_{k=1}^{n+1}\left(t^{d}-1\right)^{(-1)^{n-k+1} d^{k-1}\binom{n+1}{k}}  \tag{4.153}\\
& =(t-1)^{(-1)^{n+1}}\left(t^{d}-1\right)^{(-1)^{n}\left(1-(1-d)^{n+1}\right) / d}  \tag{4.154}\\
& =(t-1)^{(-1)^{n+1}}\left(t^{d}-1\right)^{(-1)^{n}[n+1]_{1-d}} \tag{4.155}
\end{align*}
$$

with degree

$$
\begin{align*}
\mu_{\mathrm{alg}}(f) & =(-1)^{n+1}+(-1)^{n} d[n+1]_{1-d}  \tag{4.156}\\
& =(-1)^{n+1}+(-1)^{n}\left(1-(1-d)^{n+1}\right)  \tag{4.157}\\
& =(d-1)^{n+1} \tag{4.158}
\end{align*}
$$

Moreover, the Milnor-Orlik invariants are simply

$$
\begin{align*}
\kappa & =(-1)^{n+1}+(-1)^{n}(n+1)+\sum_{k=2}^{n+1}(-1)^{n-k+1}\binom{n+1}{k} d^{k-1}  \tag{4.159}\\
& =(-1)^{n} \frac{1}{d}\left((1-d)^{n}-1\right)(d-1)  \tag{4.160}\\
& =(-1)^{n}[n]_{1-d}(1-d)  \tag{4.161}\\
& =(-1)^{n+1}+(-1)^{n}[n+1]_{1-d} \tag{4.162}
\end{align*}
$$

and

$$
\begin{aligned}
\rho & =d^{(-1)^{n}(n+1)+\sum_{k=2}^{n+1}(-1)^{n-k+1}\binom{n+1}{k} d^{k-1}} \\
& =d^{(-1)^{n} \frac{1}{d}\left(1-(1-d)^{n+1}\right)} \\
& =d^{(-1)^{n}[n+1]_{1-d} .}
\end{aligned}
$$

Furthermore, for $n \geqslant 0$ and $d \geqslant 1$,

$$
\Delta_{f}(1)= \begin{cases}1 & d=1  \tag{4.163}\\ 1+(-1)^{n} & d=2 \\ d & n=0 \\ 0 & d>2, n>0\end{cases}
$$

and

$$
\Delta_{f}(-1)= \begin{cases}1 & d=1 \text { or } n=0  \tag{4.164}\\ (-1)^{n}-1 & d=2 \\ 0 & \text { even } d \geqslant 4 \\ (-2)^{(-1)^{n+1}}\left((-1)^{d}-1\right)^{(-1)^{n}[n+1]_{1-d}} & \text { otherwise }\end{cases}
$$

Remark 4.12.4. In §6.8, we discuss the fundamental relationship between the characteristic polynomials of quasi-Brieskorn-Pham singularities and cyclotomic polynomials in the setting of abstract arithmetic and combinatorial number theory.

Remark 4.12.5. As in Remark 4.13.1, non-degenerate, quasi-BrieskornPham singularities with equal weights have equal Hilbert-Poincaré series, monodromies, corresponding characteristic polynomials and algebraic indices but may have different local algebras and not necessarily isotopic algebraic links. For example, the algebraic link of a quasi-Brieskorn-Pham singularity is not necessarily isotopic to a Brieskorn-Pham manifold.
4.12.1. Topological Spheres. We proceed now to the explicit computation a few families of Brieskorn-Pham manifolds which are topological spheres.

Proposition 4.75. For $n>2$, the following statements are true:

1. Given a set of pairwise coprime integers $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}_{>1}$, the BrieskornPham manifold $\Sigma\left(a_{0}, \ldots, a_{n}\right)$ is a topological sphere;
2. For $a>1$, the Brieskorn-Pham manifold $\Sigma(a, \ldots, a)$ is not a topological sphere; and,
3. For $a, b>1$ such that $\operatorname{GCD}(a, b)=1$, the Brieksorn-Pham manifold $\Sigma(a, \ldots, a, b)$ is a topological sphere only for odd $n$ and $a=2$.

Proof. If $\left\{a_{0}, \ldots, a_{n}\right\}$ is pairwise coprime, then $\operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)=$ $a_{i_{1}} \cdots a_{i_{k}}$ for $0 \leqslant i_{1}<\cdots<i_{k} \leqslant n$, and

$$
\begin{align*}
\sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \frac{a_{i_{1}} \cdots a_{i_{k}}}{\operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)} & =\sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} 1  \tag{4.165}\\
& =\binom{n+1}{k} \tag{4.166}
\end{align*}
$$

for $k, n \geqslant 1$. Thus, the corresponding Milnor-Orlik invariants are simply

$$
\begin{align*}
\kappa & =(-1)^{n+1}+\sum_{k=1}^{n+1}(-1)^{n-k+1}\binom{n+1}{k}  \tag{4.167}\\
& =(1-1)^{n+1}=0 \tag{4.168}
\end{align*}
$$

and

$$
\begin{equation*}
\Delta_{f}(1)=\delta_{0, \kappa} \prod_{k=1}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n}\left(a_{i_{1}} \cdots a_{i_{k}}\right)^{(-1)^{n-k+1}} \tag{4.169}
\end{equation*}
$$

which equals 1 if $n>0$ and $a_{0}$ if $n=0$, which proves the first statement. If $a_{i}=a$ for $0 \leqslant i \leqslant n$, then $\operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)=a$ for $0 \leqslant i_{1}<\cdots<i_{k} \leqslant n$, and

$$
\begin{equation*}
\sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \frac{a_{i_{1}} \cdots a_{i_{k}}}{\operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)}=a^{k-1}\binom{n+1}{k} \tag{4.170}
\end{equation*}
$$

for $k, n \geqslant 1$. Thus, the corresponding Milnor-Orlik invariants are simply

$$
\begin{align*}
\kappa & =(-1)^{n+1}+\sum_{k=1}^{n+1}(-1)^{n-k+1} a^{k-1}\binom{n+1}{k}  \tag{4.171}\\
& =(a-1)^{n+1} \tag{4.172}
\end{align*}
$$

and

$$
\begin{equation*}
\rho=a^{(n+1)\left(a^{n+1}+(-1)^{n}\right) /(a+1)} . \tag{4.173}
\end{equation*}
$$

Since $\kappa$ does not vanish unless $a=1$, it follows that $\Delta_{f}(1)=\delta_{0, \kappa}$ for any value of $\kappa$, which implies the second statement. Finally, given $a_{i}=a$ for $0 \leqslant i \leqslant n$ and $a_{n+1}=b$, one computes

$$
\begin{align*}
\kappa & =(-1)^{n} n+\sum_{k=2}^{n+1}(-1)^{n-k+1}\left[a^{k-1}\binom{n}{k}+a^{k-2}\left(\binom{n+1}{k}-\binom{n}{k}\right)\right]  \tag{4.174a}\\
& =(-1)^{n} n+(-1)^{n+1} n=0 \tag{4.174b}
\end{align*}
$$

and

$$
\begin{align*}
\rho & =\left(a^{n} b\right)^{(-1)^{n}} \prod_{k=2}^{n+1} a^{(-1)^{n-k+1} a^{k-1}\binom{n}{k}}(a b)^{(-1)^{n-k+1} a^{k-2}\left(\binom{n+1}{k}-\binom{n}{k}\right)}  \tag{4.175a}\\
& =b^{(-1)^{n}+\sum_{k=2}^{n+1}(-1)^{n-k+1} a^{k-2}\left(\binom{n+1}{k}-\binom{n}{k}\right)} . \tag{4.175b}
\end{align*}
$$

For positive integers $a$ and $n$,

$$
\begin{equation*}
\sum_{k=2}^{n+1}(-1)^{n-k+1} a^{k-2}\left(\binom{n+1}{k}-\binom{n}{k}\right)=(-1)^{n} \frac{(1-a)^{n}-1}{a} \tag{4.176}
\end{equation*}
$$

which follows from the binomial expansions,

$$
\begin{equation*}
\sum_{k=2}^{n+1}(-1)^{k} a^{k}\binom{n}{k}=n a+(1-a)^{n}-1 \tag{4.177}
\end{equation*}
$$

and

$$
\begin{equation*}
\sum_{k=2}^{n+1}(-1)^{k} a^{k}\binom{n+1}{k}=(n+1) a+(1-a)^{n+1}-1 \tag{4.178}
\end{equation*}
$$

Hence, there is a positive integer $m_{n, a}=(-1)^{n}\left(\frac{(1-a)^{n}-1}{a}+1\right)$ such that

$$
\Delta_{f}(1)= \begin{cases}1 & n=1 \text { or } n \text { is odd and } a=2  \tag{4.179}\\ b & n \text { is even and } a=2 \\ b^{m_{n, a}} & a>2\end{cases}
$$

This concludes the proof of the third statement.

Remark 4.12.6. The Brieskorn-Pham manifold $\Sigma\left(3,2_{\underline{n}}\right)$ generalizes the trefoil knot and has the corresponding characteristic polynomial

$$
\Delta_{\Sigma\left(3,2_{n}\right)}(t)= \begin{cases}t^{2}-t+1 & n \text { is odd }  \tag{4.180}\\ t^{2}+t+1 & n \text { is even. }\end{cases}
$$

Therefore, since $\Delta_{\Sigma(3,2 \underline{n})}(1)=1$ for odd $n$ and $\Delta_{\Sigma(3,2 \underline{n})}(1)=3$ for even $n$, it follows that $\Sigma\left(3,2_{\underline{n}}\right)$ is a topological $(2 n-1)$-sphere for odd $n$. This result is consistent with Proposition 4.75.

For $n=2$ and $p, q, r \in \mathbb{N}$,

$$
\begin{aligned}
\prod_{k=0}^{p-1} \prod_{l=0}^{q-1} \prod_{m=0}^{r-1}\left(t-\zeta_{p}^{k} \zeta_{q}^{l} \zeta_{r}^{m}\right) & =\prod_{k=0}^{p-1}\left(t-\zeta_{p}^{k}\right) \otimes \prod_{l=0}^{q-1}\left(t-\zeta_{q}^{l}\right) \otimes \prod_{m=0}^{r-1}\left(t-\zeta_{r}^{m}\right) \\
& =\left(t^{\mathrm{LCM}(p, q, r)}-1\right)^{p q r / \mathrm{LCM}(p, q, r)} \\
& =\left(t^{\mathrm{LCM}(p, q, r)}-1\right)^{\operatorname{GCD}(p, q) \operatorname{GCD}(q, r) \operatorname{GCD}(r, p) / \operatorname{GCD}(p, q, r)}
\end{aligned}
$$

which, when combined with previous computations, implies

$$
\begin{align*}
\Delta_{(p, q, r)}(t)= & \frac{\left(t^{\operatorname{LCM}(p, q, r)}-1\right)^{G(p, q, r)}}{\left(t^{\operatorname{LCM}(p, q)}-1\right)^{G(p, q)}\left(t^{\mathrm{LCM}(q, r)}-1\right)^{G(q, r)}} \\
& \cdot \frac{\left(t^{p}-1\right)\left(t^{q}-1\right)\left(t^{r}-1\right)}{\left(t^{\mathrm{LCM}(r, p)}-1\right)^{G(r, p)}(t-1)} \tag{4.182}
\end{align*}
$$

where $G(a, b)=\operatorname{GCD}(a, b)$ and

$$
\begin{equation*}
G(p, q, r)=\frac{G(p, q) G(q, r) G(r, p)}{\operatorname{GCD}(p, q, r)} \tag{4.183}
\end{equation*}
$$

Equivalently, for $p, q, r \in \mathbb{N}$, the divisor formula yields

$$
\begin{aligned}
\operatorname{div} \Delta_{(p, q, r)}(t)= & \left(\Lambda_{p}-\Lambda_{1}\right)\left(\Lambda_{q}-\Lambda_{1}\right)\left(\Lambda_{r}-\Lambda_{1}\right) \\
= & \operatorname{GCD}(p, q) \operatorname{GCD}(\operatorname{LCM}(p, q), r) \Lambda_{\operatorname{LCM}(\operatorname{LCM}(p, q), r)} \\
& -\operatorname{GCD}(p, q) \Lambda_{\mathrm{LCM}(p, q)}-\operatorname{GCD}(q, r) \Lambda_{\mathrm{LCM}(q, r)} \\
& -\operatorname{GCD}(r, p) \Lambda_{\mathrm{LCM}(r, p)}+\Lambda_{p}+\Lambda_{q}+\Lambda_{r}-\Lambda_{1}
\end{aligned}
$$

One then uses the arithmetic identities

$$
\begin{equation*}
\operatorname{LCM}(\operatorname{LCM}(p, q), r)=\operatorname{LCM}(p, q, r) \tag{4.185}
\end{equation*}
$$

and

$$
\begin{equation*}
\operatorname{GCD}(\operatorname{LCM}(p, q), r)=\frac{\operatorname{GCD}(q, r) \operatorname{GCD}(r, p)}{\operatorname{GCD}(p, q, r)} \tag{4.186}
\end{equation*}
$$

and derives equation (4.182). Thus, for pairwise coprime $p, q, r \in \mathbb{N}$,

$$
\begin{align*}
\Delta_{(p, q, r)}(t) & =\prod_{k=1}^{p-1} \prod_{l=1}^{q-1} \prod_{m=1}^{r-1}\left(t-\zeta_{p}^{k} \zeta_{q}^{l} \zeta_{r}^{m}\right)  \tag{4.187a}\\
& =\frac{\left(t^{p q r}-1\right)\left(t^{p}-1\right)\left(t^{q}-1\right)\left(t^{r}-1\right)}{\left(t^{p q}-1\right)\left(t^{q r}-1\right)\left(t^{r p}-1\right)(t-1)} \tag{4.187b}
\end{align*}
$$

Remark 4.12.7. Writing $\Delta_{(p, q, r)}$ in place of $\Delta_{f}$, where $f$ is quasi-BrieskornPham with inverse weights $\{p, q, r\} \subset \mathbb{N}$, we compute

$$
\begin{align*}
\Delta_{(2,3,5)}(t)= & 1+t-t^{3}-t^{4}-t^{5}+t^{7}+t^{8}  \tag{4.188}\\
\Delta_{(2,3,7)}(t)= & 1+t-t^{3}-t^{4}+t^{6}-t^{8}-t^{9}+t^{11}+t^{12}  \tag{4.189}\\
\Delta_{(2,5,7)}(t)= & 1+t-t^{5}-t^{6}-t^{7}-t^{8}+t^{10}+t^{11}+t^{12}+t^{13}+t^{14} \\
& \quad-t^{16}-t^{17}-t^{18}-t^{19}+t^{23}+t^{24} \tag{4.190}
\end{align*}
$$

which are the cyclotomic polynomials $\Phi_{30}(t), \Phi_{42}(t)$ and $\Phi_{70}(t)$, respectively. Observe the factorizations $30=2 \cdot 3 \cdot 5,42=2 \cdot 3 \cdot 7$ and $70=2 \cdot 5 \cdot 7$. Moreover,

$$
\begin{equation*}
\Delta_{(2,3,5)}( \pm 1)=\Delta_{(2,3,7)}( \pm 1)=\Delta_{(2,5,7)}( \pm 1)=1 \tag{4.191}
\end{equation*}
$$

so, in particular, the corresponding Brieskorn-Pham 3-manifolds are homology 3-spheres. Recall $\Sigma(2,3,5)$ is the Poincaré (integral) homology 3-sphere.

It is apparent that the complexity of the representation of the characteristic polynomial given in Proposition 4.72 exceeds the reasonable even for rather small values of $n$. In the next chapter, we give a simpler representation of the characteristic polynomial in terms of (polynomial tensor) products of cyclotomic polynomials.

### 4.13. Algebra Links by Topological Type

As before, let $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{2 n+1}$ denote the algebraic link of the singularity $f$. The topological type of a non-degenerate analytic map $f$ is the oriented homeomorphism class of $\left(B_{\varepsilon}^{n+1}, f^{-1}(0) \cap B_{\varepsilon}^{n+1}\right)$. In fact, the (oriented) diffeomorphism class of $\left(S_{\varepsilon}^{2 n+1}, K_{f}\right)$, where $K_{f}=f^{-1}(0) \cap S_{\varepsilon}^{2 n+1}$, determines the topological type of $f$ and vice-versa. In particular, the link $K_{f}$ is homeomorphic to the sphere $S^{2 n-1}$ if and only if $K$ has the homology of $S^{2 n-1}$. If $f$ is weighted homogeneous, more can be said. By a theorem of Oka [355, 357], the weights of a weighted homogeneous polynomial $f$ determine the topological type of the singularity of $V_{f}$ for $n \geqslant 0$. However, according to Orlik [361] only for $n=2$ does the topology of the corresponding algebraic link $K_{f}$ determine the weights of $f$; there are counter-examples for all other dimensions.

Remark 4.13.1. An equality of characteristic polynomials does not necessarily imply correspondingly isotopic algebraic links, while the converse may be true. Consider $f=x^{3}+y^{3}, g=x^{3}+x y^{2}$ and $h=x^{3}+y^{3}+z^{2}+w^{2}$, which are non-degenerate, quasi-Brieskorn-Pham polynomials with weights $\left\{\frac{1}{3}, \frac{1}{3}\right\}$, $\left\{\frac{1}{3}, \frac{1}{3}\right\}$ and $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right\}$, respectively. The corresponding Hilbert-Poincaré series are equal, namely, $\bar{P}_{\mathcal{A}_{f}}(t)=\bar{P}_{\mathcal{A}_{g}}(t)=\bar{P}_{\mathcal{A}_{h}}(t)=1+2 t^{1 / 3}+t^{2 / 3}$, and, therefore, so, too, are the characteristic series, $\Delta_{f}(t)=\Delta_{g}(t)=\Delta_{h}(t)=t^{4}-t^{3}-t+1$ by Proposition 2.51 and Corollary 2.57. However, $K_{f} \simeq T_{3,3} \simeq O T_{3,2} \simeq K_{g}$, but $K_{h} \simeq \Sigma(3,3,2,2) \nsucceq K_{g}$.

Definition 4.76. A map $h:(X, A) \rightarrow(Y, B)$ is a relative homeomorphism if and only if $h$ maps $X \backslash A$ homeomorphically onto $Y \backslash B$.

Proposition 4.77 (Saeki, [407]). Let $f, g:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be polynomials with isolated critical points at the origin. If $\left(\mathbb{C}^{2}, V_{f}\right)$ and $\left(\mathbb{C}^{2}, V_{g}\right)$ are locally homeomorphic, then $\left(S_{\varepsilon}^{3}, K_{f}\right)$ and $\left(S_{\varepsilon}^{3}, K_{g}\right)$ are relatively homeomorphic.

Proof. See Lemma 5 in [407].
Let $\pi\left(K_{f}\right)=\pi_{1}\left(S_{\varepsilon}^{2 n+1} \backslash K_{f}\right)$ denote the link group of $K_{f}$.

Proposition 4.78 (Saeki, [407]). Given non-degenerate weighted homogeneous polynomials $f, g:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, if $\pi\left(K_{f}\right) \cong \pi\left(K_{g}\right)$ and $\Delta_{f}(t)=\Delta_{g}(t)$, then $f$ and $g$ have identical weights up to permutation.

Proof. See Theorem 3 in [407].

Remark 4.13.2. Saeki remarks that Proposition 4.78 does not hold in general. Consider $f=z_{0}^{2} z_{1}+z_{0} z_{1}^{6}+z_{2}^{3}+z_{3}^{13}$ and $g=z_{0}^{3} z_{1}+z_{0} z_{1}^{4}+z_{2}^{3}+z_{3}^{13}$ over $\mathbb{C}^{4}$. Although the singularities $\Sigma^{n-3} f$ and $\Sigma^{n-3} g$ have different weights, namely, $\left\{\frac{5}{11}, \frac{1}{11}, \frac{1}{3}, \frac{1}{13}, \frac{1}{2}, \ldots, \frac{1}{2}\right\}$ and $\left\{\frac{3}{11}, \frac{2}{11}, \frac{1}{3}, \frac{1}{13}, \frac{1}{2}, \ldots, \frac{1}{2}\right\}$, respectively, their links are homeomorphic and corresponding characteristic polynomials are equal for $n \geqslant 3$. However, $\left(\mathbb{C}^{n+1}, V_{\Sigma^{n-3} f}\right)$ and $\left(\mathbb{C}^{n+1}, V_{\Sigma^{n-3} g}\right)$ are not locally homeomorphic at the origin for $n \geqslant 3$.

Proposition 4.79 (Saeki,[407]; Lê,[258]; Perron,[372]). Let $f, g:\left(\mathbb{C}^{n}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be polynomials with isolated critical points at the origin.

If $\left(\mathbb{C}^{n}, V_{f}\right)$ and $\left(\mathbb{C}^{n}, V_{g}\right)$ are locally homeomorphic, then $\pi\left(K_{f}\right) \cong \pi\left(K_{g}\right)$ and $\Delta_{f}(t)=\Delta_{g}(t)$.

Proof. See Lemma 2 in [407].
4.13.1. Lê's Classification. Let $C$ be a complex plane algebraic curve and $V_{C}$ denote the corresponding hypersurface. The problem of classifying algebraic knots of the form $K_{C}=V_{C} \cap S_{\varepsilon}^{3}$ was solved by Lê.

Proposition 4.8o (Lê, [257]). Let $K=K_{C}$ be the algebraic knot corresponding to a plane curve singularity $C$ with Puiseux pairs $P(C)=\left\{\left(m_{1}, n_{1}\right), \ldots,\left(m_{s}, n_{s}\right)\right\}$. Define $v_{i}=n_{i} \cdots n_{s}$ for $1 \leqslant i \leqslant s$ and $v_{s+1}=1, \lambda_{1}=m_{1}$ and $\lambda_{i}=$ $m_{i}-\left(m_{i-1} n_{i}+\lambda_{i-1} n_{i} n_{i-1}\right)$ for $2 \leqslant i \leqslant s$. The Alexander polynomial $\Delta_{K}(t)$ is the product $\prod_{i=1}^{S} P_{\lambda_{i}, n_{i}}\left(t^{v_{i+1}}\right)$, where

$$
\begin{equation*}
P_{\lambda, n}(t)=\frac{\left(t^{\lambda n}-1\right)(t-1)}{\left(t^{\lambda}-1\right)\left(t^{n}-1\right)} \tag{4.192}
\end{equation*}
$$

Proof. See [257].

Remark 4.13.3. The Puiseux pairs $\left\{\left(p_{i}, q_{i}\right)\right\}$ are coprime positive integers which satisfy the recurrence inequality $p_{i} q_{i} p_{i+1}<q_{i+1}$ for $1 \leqslant i<n$. An iterated torus knot of type $\left\{\left(p_{1}, q_{1}\right), \ldots,\left(p_{n}, q_{n}\right)\right\}$ is an inductively defined $\left(p_{i}, q_{i}\right)$ cabling beginning with a torus knot $T_{p_{1}, q_{1}}$. It is known that the connected components of any algebraic link are iterated torus knots. Recall, for example, that the plane curve $f=z_{0}^{p}+z_{1}^{q}$, where $p$ and $q$ are coprime, with hypersurface $C_{p, q}=f^{-1}(0)$ and Puiseux pairs $P\left(C_{p, q}\right)=\{(p, q)\}$ corresponds to the torus
$\operatorname{knot} T_{p, q}$. By Lê's formula, then

$$
\begin{equation*}
P_{p, q}(t)=\frac{\left(t^{p q}-1\right)(t-1)}{\left(t^{p}-1\right)\left(t^{q}-1\right)}=\Delta_{T_{p, q}}(t) \tag{4.193}
\end{equation*}
$$

Since Puiseux pairs are unique up to (permutation and) topological type, the classification of algebraic knots in $S^{3}$ is complete.

Corollary 4.81 (Lê, [257]). Two algebraic knots $K$ and $K^{\prime}$ are equivalent if and only if the corresponding Alexander polynomials are equal, i.e., $\Delta_{K}(t)=\Delta_{K^{\prime}}(t)$.

According to Lê, in the classical case $n=2$, the monodromy (hence the Alexander polynomial) determines the topological type of the corresponding knot. The case $n=3$ is open and $n \geqslant 4$, it is known that the Alexander polynomial does not determine the topological type of the corresponding knot. Lê proves that if two knots are cobordant, then their Alexander polynomials are in the same cobordism class, which means that there is a polynomial $p$ such that $\Delta_{K}(t) \Delta_{K^{\prime}}(t) \doteq p(t) p\left(t^{-1}\right)$ (up to a power of $\left.t\right)$. He proves also that the Alexander polynomial of an irreducible plane curve determines the cobordism type of the corresponding knot.
4.13.2. Yamamoto's Classification. In 1984, Yamamoto [480] proves that algebraic links in $S^{3}$ are classified (up to ambient isotopy) by their characteristic polynomials. He proves also for algebraic links in $S^{2 n+1}$ for any $n>1$, by way of explicitly constructing counter-examples, that there are distinct links (up to
isotopy) with identical characteristic polynomials. Therefore, the characteristic polynomial is insufficient to provide a complete classification of higher dimensional algebraic links up to ambient isotopy.

This concludes our discussion of some interesting geometric aspects of complex analytic singularities. We proceed now to some combinatorial structures.

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## Chapter 5

# Combinatorial Structure of Isolated Singularities 

Man is fond of counting his troubles, but he does not count his joys. If he counted them up as he ought to, he would see that every lot has enough happiness provided for it. - Fyodor Dostoevsky

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In this chapter we study some combinatorics related to isolated singularities, particularly the weighted-homogeneous type introduced in a previous chapter. We define the Newton and Weight polytopes and compute some of their geometric and combinatorial attributes. We relate these to the algebraic invariants of non-degenerate weighted homogeneous singularities-the algebraic, combinatorial, lattice and arithmetic indices among them—and pay special attention to the Brieskorn-Pham type or related singularities.

By enumerating lattice points in rational simplicial polytopes, we derive exact representations of the delta invariant and branch number of a weighted homogeneous plane curve in terms of the weights. By a similar technique, we compute the geometric genus of a weighted homogeneous singularity in $\mathbb{C}^{3}$. As a result of these novel representations, we derive new congruences and a threeterm reciprocity law for the Dedekind sum generalizing the corresponding classical results of Dedekind and Rademacher.

As a further consequence of our analysis of the geometric genus for $\mathbb{C}^{3}$, we prove a few special cases and develop sharper inequalities of the Durfee-YauZhang Theorem. We discuss a new method of proving the original Durfee Conjecture.

In volume 2 we discuss techniques for the enumeration of lattice points in polytopes. For the convenience of the reader, we reproduce some of these results and give a brief introduction to the necessary tools to understand this chapter. In particular, we include here some relevant passages from chapter 1 of volume 2.

For a detailed introduction to the foundation of these and related topics, see [43], [435] and [434].

### 5.1. Classical Ehrhart-Macdonald Theory

For $t \in \mathbb{R}_{>0}$, we define the solid $n$-polytope $\mathcal{P}=\boldsymbol{\operatorname { c o n v }}\left\{\mathbf{v}_{1}, \ldots, \mathbf{v}_{m}\right\}$ in two equivalent ways - as the closure of the convex hull $t \mathcal{P}=\boldsymbol{\operatorname { c o n v }}\left\{t \mathbf{v}_{1}, \ldots, t \mathbf{v}_{m}\right\}$ or the locus $\left\{\left(t x_{1}, \ldots, t x_{n}\right) \in \mathbb{R}^{n} \mid\left(x_{1}, \ldots, x_{n}\right) \in \mathcal{P}\right\}$. If $\mathcal{P}=\boldsymbol{\operatorname { c o n v }}\left\{\mathbf{0}, a_{1} \mathbf{e}_{1}, \ldots, a_{n} \mathbf{e}_{n}\right\}$ is
orthotopal simplicial, then the $t$-dilation*, admits an explicit algebraic description, $t \mathcal{P}=\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n} \left\lvert\, \sum_{i=1}^{n} \frac{x_{i}}{a_{i}} \leqslant t \wedge x_{i} \geqslant 0\right.\right\}$.

For a fixed polytope $\mathcal{P}$, define the enumerative function $\mathscr{L}_{\mathcal{P}}: \mathbb{R}_{\geqslant 0} \rightarrow \mathbb{Z}_{\geqslant 0}$ by $\mathscr{L}_{\mathcal{P}}(t)=\left|t \mathcal{P} \cap \mathbb{Z}^{n}\right|$, the number of $\mathbb{Z}^{n}$-lattice points intersecting $t \mathcal{P}$. Denote by $\ell_{\mathcal{P}}=\left.\mathscr{L}_{\mathcal{P}}\right|_{\mathbb{Z}}$ the restriction of $\mathscr{L}_{\mathcal{P}}$ on $\mathbb{Z}_{\geqslant 0}$. Define the order of $\mathcal{P}$, denoted by $\operatorname{ord}(\mathcal{P})$, as the integer $\ell_{\mathcal{P}}(1)$, the number of $\mathbb{Z}^{n}$-lattice points in $\mathcal{P}$.

It is useful to define the generating function $\mathscr{E}_{\mathcal{P}}(z)=\sum_{k \geqslant 1} \ell_{\mathcal{P}}(k) z^{k}$. We refer to $\mathscr{L}_{\mathcal{P}}, \mathscr{E}_{\mathcal{P}}$ and $\left\{\ell_{\mathcal{P}}(k)\right\}_{k \geqslant 1}$ as the Ehrhart function, Ehrhart series and Ehrhart sequence of the $n$-polytope $\mathcal{P}$, respectively. Similarly, define the enumerative functions for the interior ${ }^{\dagger}$ and boundary, $\mathscr{L}_{\mathcal{P}}(t)=\left|t \mathcal{P}^{\circ} \cap \mathbb{Z}^{n}\right|$, $\mathscr{E}_{\mathcal{P} \circ}(z)=\sum_{k \geqslant 1} \ell_{\mathcal{P} \circ}(k) z^{k}$ and $\mathscr{L}_{\partial \mathcal{P}}(t)=\left|t(\partial \mathcal{P}) \cap \mathbb{Z}^{n}\right|, \mathscr{E}_{\partial \mathcal{P}}(z)=\sum_{k \geqslant 1} \ell_{\partial \mathcal{P}}(k) z^{k}$, respectively.

Proposition 5.1 (Stanley, [434]). Let $f: \mathbb{Z}_{\geqslant 0} \rightarrow \mathbb{C}$ be an arithmetic function. The following statements are equivalent:

1. There are periodic functions $\left\{c_{1}, \ldots, c_{n}\right\}$ with least common period $d$ such that $f(k)=\sum_{l=1}^{n} c_{l}(k) k^{l}$.
2. There is an integer $d$ and polynomials $\left\{f_{0}, \ldots, f_{d-1}\right\}$ such that $f(k)=f_{l}(k)$ if $k \equiv l \bmod d$.
3. There are polynomials $\left\{g_{1}, \ldots, g_{m}\right\}$ such that $f(k)=\sum_{i=1}^{m} g_{i}(k) \zeta_{i}^{k}$, where $\zeta_{i}^{d}=1$, that is, $\zeta_{i}$ is a $d^{\text {th }}$-roots of unity.
*Of course, if $0<t<1$, then one uses " $t$-contraction" rather than " $t$-dilation".
${ }^{\dagger}$ The Ehrhart series $\mathscr{E} \mathcal{P}^{\circ}$ of the interior $\mathcal{P}^{\circ}$ is defined with lower summation index $k=1$.
4. The generating function $\sum_{k \geqslant 0} f(k) z^{k}$ is a rational function $\frac{P}{Q} \in \mathbb{C}(z)$, where $\operatorname{deg} P<\operatorname{deg} Q$ (in reduced form) and the roots of $Q$ are $d^{\text {th }}$-roots of unity.

Proof. See Theorem 1.1, Proposition 1.2 and Corollaries 1.3 and 1.5 in [434].

An arithmetic function $f: \mathbb{Z}_{\geqslant 0} \rightarrow \mathbb{C}$ satisfying any of the equivalent statements in Proposition 5.1 is called a quasi-polynomial of degree $n$ and period $N$. It is known that if $\mathcal{P}$ is a rational convex $n$-polytope (not necessarily simple, simplicial or orthotopal), then $\ell_{\mathcal{P}}$ is a quasi-polynomial of degree $\operatorname{dim} \mathcal{P}=n$ on $\mathbb{Z}_{\geqslant 0}[122,123]$. That is, there are $n+1$ periodic functions $c_{l}: \mathbb{Z}_{\geqslant 0} \rightarrow \mathbb{Q}$, each with finite period $\operatorname{per}\left(c_{l}\right) \geqslant 1$, such that $\ell_{\mathcal{P}}(t)=\sum_{l=0}^{n} c_{l}(t) t^{l}$ and $n!c_{l}(t) \in \mathbb{Z}[t]$ for $0 \leqslant l \leqslant n$. Moreover, the leading coefficient function $c_{n}(t)=c_{n}$ has period 1 and equals $\operatorname{vol}_{n}(\mathcal{P})$, the (continuous) $n$-content of $\mathcal{P}$ relative to the affine span $\operatorname{aff}(\mathcal{P}) \cap \mathbb{Z}^{n}$, and $c_{n-1}(0)=\frac{1}{2} \operatorname{vol}_{n-1}(\partial \mathcal{P})=\frac{1}{2} \sum_{F \subset \partial \mathcal{P}} \operatorname{vol}_{n-1}(F)$ (as a sum over facets, each normalized with respect to $\left.\operatorname{aff}(F) \cap \mathbb{Z}^{n}\right)$. Furthermore, the pe$\operatorname{riod} N=N(\mathcal{P})$ equal to $\operatorname{LCM}\left(\operatorname{per}\left(c_{0}\right), \ldots, \operatorname{per}\left(c_{n}\right)\right)$ divides the denominator $d=d(\mathcal{P})$.

Stanley proves that under certain conditions, $\mathscr{H}_{\mathcal{P}}(z)=(1-z)^{-n-1} \mathscr{E}_{\mathcal{P}}(z)$ is the Hilbert polynomial of a Cohen-Macaulay Ring corresponding to $\mathcal{P}$ and therefore has non-negative coefficients [434].

If, in particular, $\mathcal{P}$ is integral, then both the period and denominator of $\mathcal{P}$ are 1 (i.e., the coefficient functions $c_{l}$ are constant) and $\ell_{\mathcal{P}}(t)=\sum_{l=0} c_{l} t^{l} \in \mathbb{Q}[t]$
for $t \in \mathbb{Z}_{\geqslant 0}$ (See [299] and [435])*. In this case, three coefficients are known to carry (continuous) geometric/topological information ${ }^{\dagger}$, namely, $c_{n}=\operatorname{vol}_{n}(\mathcal{P})$, $c_{n-1}=\frac{1}{2} \operatorname{vol}_{n-1}(\partial \mathcal{P})$ and $c_{0}=\chi(\mathcal{P})$. The other coefficients, which had until recently lacked a similar satisfactory interpretation, also contain geometric and/or topological information. We shall refer to $\left\{c_{0}, \ldots, c_{n}\right\}$ as the (set of) Ehrhart coefficients of $\mathcal{P}$.

A polytope is pseudo-integral if and only if its Ehrhart function is a polynomial on the restricted domain $\mathbb{Z}_{\geqslant 0}$. All integral simplicial polytopes are pseudointegral, but the converse is not true.

Proposition 5.2. Let $\mathscr{L}_{\mathcal{P}}(t), \mathscr{L}_{\mathcal{P}}(t)$ and $\mathscr{L}_{\partial \mathcal{P}}(t)$ denote the Ehrhart functions of an orthopic simplicial n-polytope $\mathcal{P}=\boldsymbol{\operatorname { c o n v }}\left\{\mathbf{0}, a_{1} \mathbf{e}_{1}, \ldots, a_{n} \mathbf{e}_{n}\right\}$, its interior $\mathcal{P}^{\circ}$ and boundary $\partial \mathcal{P}$, respectively. Then $\mathscr{L}_{\partial \mathcal{P}}(t)=\mathscr{L}_{\mathcal{P}}(t)-\mathscr{L}_{\mathcal{P}} \circ(t)$, where

$$
\begin{align*}
& \mathscr{L}_{\mathcal{P}}(t)=\sum_{i_{1}=0}^{\left\lfloor a_{1} t\right\rfloor} \cdots \sum_{i_{n}=0}^{\left\lfloor a_{n}\left(t-\sum_{k=1}^{n-1} i_{k} / a_{k}\right)\right\rfloor} 1  \tag{5.1}\\
& \mathscr{L}_{\mathcal{P}^{\circ}}(t)=\sum_{i_{1}=1}^{\left\lceil a_{1} t\right\rceil-1} \cdots \sum_{i_{n}=1}^{\left\lceil a_{n}\left(t-\sum_{k=1}^{n-1} i_{k} / a_{k}\right)\right\rceil-1} 1 \quad t \in \mathbb{R}_{\geqslant 0} . \tag{5.2}
\end{align*}
$$

## Proof. See Volume 2.

${ }^{*}$ For an elementary proof for lattice polytopes, see Theorem 5.1 in [30].
${ }^{\dagger}$ This work was anticipated by Reeve as early as 1957 with his study of the volume of lattice 3 -polytopes. He proved an explicit formula to compute the corresponding volume depending only on $\ell_{\mathcal{P}}(1), \ell_{\mathcal{P} \circ}(1), \ell_{\mathcal{P}}(2)$ and $\ell_{\mathcal{P} \circ}(2)$, thereby generalizing Pick's Theorem (1899) of the lattice polygons.

Corollary 5.3. The number of positive lattice points intersecting the $t$-dilate of the orthotopal simplicial $n$-polytope $\mathcal{P}=\boldsymbol{\operatorname { c o n v }}\left\{\mathbf{0}, a_{1} \mathbf{e}_{1}, \ldots, a_{n} \mathbf{e}_{n}\right\}$ is given by

$$
\begin{equation*}
\mathscr{L}_{\mathcal{P}}(t ; \mathbb{N})=\sum_{i_{1}=1}^{\left\lfloor a_{1} t\right\rfloor} \cdots \sum_{i_{n}=1}^{\left\lfloor a_{n}\left(t-\sum_{k=1}^{n-1} i_{k} / a_{k}\right)\right\rfloor} 1 \tag{5.3}
\end{equation*}
$$

Proof. See Volume 2.
5.1.1. Enumerating Square Weighted Homogeneous Polynomials. Given a set of rational weights $\boldsymbol{\omega}=\left\{\omega_{1}, \ldots, \omega_{n}\right\}$, the number $N=N(\boldsymbol{\omega})$ of nonnegative integral solutions of the equation $\sum_{i=1}^{n} \omega_{i} a_{i}=1$, or of the equivalent Diophantine equation $\sum_{i=1}^{n} q_{i} a_{i}=d$, gives the number of weighted homogeneous monomials $z_{1}^{a_{1}} \cdots z_{n}^{a_{n}}$ with weighted degree $d$ and integral weights or gradation $\left\{q_{1}, \ldots, q_{n}\right\}$, where $q_{i}=\operatorname{deg} z_{i}$.

Proposition 5.4. The most general weighted homogeneous polynomial with weight $\boldsymbol{\omega}$ (modulo coefficients) is given by the difference

$$
\begin{align*}
\mathscr{L}_{\Delta(\mathcal{P})}(\mathbf{z} ; 1)= & \sum_{i_{1}=0}^{\left\lfloor 1 / \omega_{1}\right\rfloor} \cdots \sum_{i_{n}=0}^{\left\lfloor\left(1-\sum_{k=1}^{n-1} i_{k} \omega_{k}\right) / \omega_{n}\right\rfloor} z_{1}^{i_{1}} \cdots z_{n}^{i_{n}} \\
& -\sum_{i_{1}=0}^{\left\lceil 1 / \omega_{1}\right\rceil-1} \cdots \sum_{i_{n}=0}^{\left\lceil\left(1-\sum_{k=1}^{n-1} i_{k} \omega_{k}\right) / \omega_{n}\right\rceil-1} z_{1}^{i_{1}} \cdots z_{n}^{i_{n}} . \tag{5.4}
\end{align*}
$$

Proof. The integer $N$ also counts the non-negative lattice points intersecting the $(n-1)$-polytope $\triangle(\mathcal{P})=\operatorname{conv}\left\{\frac{\mathbf{e}_{1}}{\omega_{1}}, \ldots, \frac{\mathbf{e}_{n}}{\omega_{n}}\right\}$, which is a facet of the simplicial orthotopal $n$-polytope $\mathcal{P}=\boldsymbol{\operatorname { c o n v }}\left\{\mathbf{0}, \frac{\mathbf{e}_{1}}{\omega_{1}}, \ldots, \frac{\mathbf{e}_{n}}{\omega_{n}}\right\}$. By first enumerating the
non-negative lattice points intersecting $\mathcal{P}$, that is, the order $\mathscr{L}_{\mathcal{P}}(1)=\left|\mathcal{P} \cap \mathbb{Z}_{\geqslant 0}^{n}\right|$, and discounting those lattice points on the orthogonal boundaries as well as those in the interior $\mathcal{P}^{\circ}$, one computes the number of lattice points intersecting $\triangle(\mathcal{P})$. Add a monomial weight $z_{1}^{i_{1}} \cdots z_{i_{n}}^{i_{n}}$ to each lattice point $\left(i_{1}, \ldots, i_{n}\right) \in \mathbb{Z}_{\geqslant 0}^{n}$. By Proposition 5.2, the most general weighted homogeneous polynomial (modulo coefficients) in $n$ complex variables with weights $\boldsymbol{\omega}=\left\{\omega_{1}, \ldots, \omega_{n}\right\}$ is the difference of the corresponding generalized Ehrhart functions. This concludes the proof.

Let $W(\boldsymbol{\omega})$ denote the set of the monomials comprising the counting polynomial $\mathscr{L}_{\triangle(\mathcal{P})}(\mathbf{z} ; 1)$. Setting $z_{1}=\cdots=z_{n}=1$ gives the number $N$ of lattice points on $\triangle(\mathcal{P})$ which are correspondingly bijective with said monomials.

REMARK 5.1.1. To include all permutations of variables, then consider instead

$$
\begin{align*}
\mathscr{L}_{\triangle(\mathcal{P})}(\mathbf{z} ; 1)= & \sum_{\pi \in S_{n}} \sum_{i_{1}=0}^{\left\lfloor 1 / \omega_{1}\right\rfloor} \cdots \sum_{i_{n}=0}^{\left\lfloor\left(1-\sum_{k=1}^{n-1} i_{k} \omega_{k}\right) / \omega_{n}\right\rfloor} z_{\pi_{1}}^{i_{1}} \cdots z_{\pi_{n}}^{i_{n}} \\
& -\sum_{\pi \in S_{n}} \sum_{i_{1}=0}^{\left\lceil 1 / \omega_{1}\right\rceil-1} \cdots \sum_{i_{n}=0}^{\left\lceil\left(1-\sum_{k=1}^{n-1} i_{k} \omega_{k}\right) / \omega_{n}\right\rceil-1} z_{\pi_{1}}^{i_{1}} \cdots z_{\pi_{n}}^{i_{n}} . \tag{5.5}
\end{align*}
$$

Corollary 5.5. The number of weighted homogeneous monomials with weight $\boldsymbol{\omega}$ (modulo coefficients and permutations of variables) is the integer

$$
\begin{equation*}
|W(\boldsymbol{\omega})|=\mathscr{L}_{\triangle(\mathcal{P})}(\mathbf{1} ; 1) \tag{5.6}
\end{equation*}
$$

Remark 5.1.2. Neither Proposition 5.4 nor Corollary $5 \cdot 5$ differentiate nondegenerate from degenerate weighted homogeneous polynomials.

Recall that a weighted homogeneous polynomial is square if and only if the number of its variables equals that of its constituent monomials.

Corollary 5.6. The number of square weighted homogeneous polynomials with weight $\boldsymbol{\omega}=\left\{\omega_{1}, \ldots, \omega_{n}\right\}$ (modulo coefficients and permutations of variables) is the integer

$$
\begin{equation*}
\binom{|W(\boldsymbol{\omega})|}{|\boldsymbol{\omega}|}=\binom{N}{n} \tag{5.7}
\end{equation*}
$$

Proof. The number of weighted homogeneous polynomials with weight $\omega$ consisting of $k$ distinct monomials is the binomial coefficient $\binom{|W(\omega)|}{k}$.

### 5.2. Inner Modality and Restricted Integer Compositions

5.2.1. Inner Modality. Recall $D=\sum_{i=0}^{n} d-2 q_{i}$. Define the inner modality* $\bar{\mu}_{0}$ as the number of basis monomials in the local algebra $\mathcal{A}_{f}$ with weighted

[^32]degree greater than or equal to $d$. By the reflexive symmetry of the HilbertPoincaré series, one computes
\[

$$
\begin{align*}
\bar{\mu}_{0} & =\sum_{l \geqslant d} \mu_{l}  \tag{5.8a}\\
& =\sum_{D-l \geqslant d-D} \mu_{D-l}  \tag{5.8b}\\
& =\sum_{l \leqslant D-d} \mu_{l} \tag{5.8c}
\end{align*}
$$
\]

where $\mu_{l}$ is the number of basis monomials in $\mathcal{A}_{f}$ with weighted degree equal to $l$.

Proposition 5.7. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate weighted homogeneous polynomial with weights $\left\{q_{0}, \ldots, q_{n}\right\}$, weighted degree $d$ and local algebra $\mathcal{A}_{f}=\oplus_{l \geqslant 0} \mathcal{A}_{f, l}$, then

$$
\begin{equation*}
\operatorname{dim}_{\mathrm{C}} \mathcal{A}_{f, l}=(-1)^{l} \sum_{k=0}^{D}(-1)^{k}\left(\sum_{j=\max \{k, l\}}^{D} \frac{|s(j, l)|}{(j-k)!}\right) \frac{1}{k!} \prod_{i=0}^{n} \frac{k^{d-q_{i}}-1}{k^{q_{i}}-1} \tag{5.9}
\end{equation*}
$$

where $D=\sum_{i=0}^{n} d-2 q_{i}$ and $|s(j, l)|$ is the unsigned $(j, l)$-Stirling Number of the First Kind.

Proof. Based on the calculus of finite differences, a formula to compute any coefficient of a finite polynomial is given in Volume 2. As applied to the

Hilbert-Poincaré series of the corresponding weighted homogeneous polynomial, one computes

$$
\begin{equation*}
\mu_{l}=(-1)^{l} \sum_{k=0}^{D}(-1)^{k}\left(\sum_{j=\max \{k, l\}}^{D} \frac{|s(j, l)|}{(j-k)!}\right) \frac{P_{\mathcal{A}_{f}}(k)}{k!} . \tag{5.10}
\end{equation*}
$$

Remark 5.2.1. Consider the homogeneous polynomial $f=\sum_{i=0}^{n} z_{i}^{d}$. Thus, $D=(n+1)(d-2)$ and, by equation (5.8c), one computes

$$
\begin{equation*}
\mu_{l}=(-1)^{l} \sum_{k=0}^{(n+1)(d-2)}(-1)^{k}\left(\sum_{j=\max \{k, l\}}^{(n+1)(d-2)} \frac{|s(j, l)|}{(j-k)!}\right) \frac{1}{k!}\left(\sum_{r=0}^{d-2} k^{r}\right)^{n+1} \tag{5.11}
\end{equation*}
$$

If $n=1$, then the summation simplifies,

$$
\begin{equation*}
\mu_{l}=(d-|l-d+2|-1) \Theta(l) \Theta(2 d-l-4) \tag{5.12}
\end{equation*}
$$

and, in particular,

$$
\begin{equation*}
\mu_{\left\lfloor\frac{D}{2}\right\rfloor}=(d-1) \Theta(d-2) \tag{5.13}
\end{equation*}
$$

where $\Theta$ is the Heaviside function.

Corollary 5.8. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial with weights $\left\{q_{0}, \ldots, q_{n}\right\}$ and weighted degree $d$, the inner modality is given by

$$
\begin{equation*}
\bar{\mu}_{0}=\sum_{l \leqslant D-d}(-1)^{l} \sum_{k=0}^{D}(-1)^{k}\left(\sum_{j=\max \{k, l\}}^{D} \frac{|s(j, l)|}{(j-k)!}\right) \frac{1}{k!} \prod_{i=0}^{n} \frac{k^{d-q_{i}}-1}{k^{q_{i}}-1} \tag{5.14}
\end{equation*}
$$

where $D=\sum_{i=0}^{n} d-2 q_{i}$ and $|s(j, l)|$ is the unsigned $(j, l)$-Stirling Number of the First Kind.
5.2.2. Integer Compositions. Considering Remark 5.2.1, for $n=2, \mu_{\left\lfloor\frac{D}{2}\right\rfloor}$ is the number of compositions of $\left\lfloor\frac{3 d}{2}\right\rfloor$ into 3 (possibly repeated) positive integers no greater than $d-1$ (A077043). For $n=3$,

$$
\begin{equation*}
\mu_{\left\lfloor\frac{D}{2}\right\rfloor}=\frac{1}{3}(d-1)\left(2(d-1)^{2}+1\right) \Theta(d-2) \tag{5.15}
\end{equation*}
$$

which is the $(d-1)^{\text {th }}$-octahedral number (A005900), and is a reflexive sum of squares $1^{2}+\cdots+(d-1)^{2}+\ldots 1^{2}$. For $n=4, \mu_{\left\lfloor\frac{D}{2}\right\rfloor}$ is the number of compositions of $\left\lfloor\frac{5 d}{2}\right\rfloor$ into 5 (possibly repeated) positive integers no greater than $d-1$ (A077044). These examples illustrate a general phenomenon of counting integer compositions, which bear a similarity to integer partitions but differ in that the order of the summands is relevant.

Proposition 5.9. Suppose $f$ is a homogeneous polynomial of degree $d$ in $n+1$ complex indeterminates. For $d \geqslant 2$ and $n \geqslant 0$, the coefficient $\mu_{\left[\frac{D}{2}\right]}$ is the number of
positive compositions of $\left\lfloor\frac{d}{2}(n+1)\right\rfloor$ into $n+1$ (possibly repeated) positive integers no greater than $d-1$.

Proof. Let $f$ be a homogeneous polynomial of (homogeneous) degree $d$ in $n+1$ complex indeterminates. Let $c_{k}(N ;[a, b])$ denote the number of compositions of $N$ into $k$ parts, where each part is restricted to the interval $[a, b]$, e.g., $N=\sum_{i=1}^{k} s_{i}$ with $a \leqslant s_{i} \leqslant b$. The generating function of $c_{k}(N ;[a, b])$ is

$$
\begin{equation*}
G\left(c_{k}(N ;[a, b]) ; t\right)=t^{k a}\left(\frac{1-t^{b-a+1}}{1-t}\right)^{k} \tag{5.16}
\end{equation*}
$$

hence $c_{k}(N ;[a, b])=\left[t^{N}\right] G\left(c_{k}(N ;[a, b]) ; t\right)$, that is, the coefficient of $t^{N}$ of the generating function $G$ (as a series in $t$ ) is $c_{k}$. Setting $k=n+1, a=1$ and $b=d-1$, one finds

$$
\begin{align*}
G\left(c_{n+1}(N ;[1, d-1]) ; t\right) & =t^{n+1}\left(\frac{1-t^{d-1}}{1-t}\right)^{n+1}  \tag{5.17a}\\
& =t^{n+1} P_{\mathcal{A}_{f}}(t) \tag{5.17b}
\end{align*}
$$

where $P_{\mathcal{A}_{f}}$ is the Hilbert-Poincaré series of $f$. Since

$$
\begin{equation*}
\left\lfloor\frac{d}{2}(n+1)\right\rfloor+(n+1)=\left\lfloor\frac{1}{2}(n+1)(d-2)\right\rfloor \tag{5.18}
\end{equation*}
$$

one concludes that for $N=\left\lfloor\frac{d}{2}(n+1)\right\rfloor$,

$$
\begin{align*}
c_{n+1}\left(\left\lfloor\frac{1}{2}(n+1) d\right\rfloor ;[1, d-1]\right) & =\left[t^{\left\lfloor\frac{d}{2}(n+1)\right\rfloor}\right] t^{n+1} P_{\mathcal{A}_{f}}(t)  \tag{5.19a}\\
& =\left[t^{\left\lfloor\frac{1}{2}(n+1)(d-2)\right\rfloor}\right] P_{\mathcal{A}_{f}}(t)  \tag{5.19b}\\
& =\mu_{\left\lfloor\frac{D}{2}\right\rfloor^{\prime}} \tag{5.19c}
\end{align*}
$$

as claimed.

Remark 5.2.2. Suppose $d=4$ and $n=3$. The number of positive compositions of $\left\lfloor\frac{d}{2}(n+1)\right\rfloor=8$ into $n+1=4$ parts no greater than $d-1=3$ is 19 : That is, $2+2+2+2$, as well as the eighteen compositions:

$$
\begin{aligned}
& 1+1+3+3,1+3+1+3,1+3+3+1 \\
& 1+2+2+3,1+2+3+2,1+3+2+2 \\
& 2+1+2+3,2+1+3+2,2+2+1+3 \\
& 2+2+3+1,2+3+1+2,2+3+2+1 \\
& 3+1+2+2,3+2+1+2,3+2+2+1 \\
& 3+1+1+3,3+3+1+1,3+1+3+1
\end{aligned}
$$

The third octahedral number is $\frac{1}{3}(4-1)\left(2(4-1)^{2}+1\right)=19$, which coincides with the coefficient $\mu_{4}$ (as $D=8$ ) in the Hilbert-Poincaré series

$$
\begin{equation*}
P_{\mathcal{A}_{f}}(t)=1+4 t+10 t^{2}+16 t^{3}+19 t^{4}+16 t^{5}+10 t^{6}+4 t^{7}+t^{8} \tag{5.20}
\end{equation*}
$$

REMARK 5.2.3. By similar reasoning, for $0 \leqslant l \leqslant D$, by setting $k=n+1$, $a=0$ and $b=d-2$,

$$
\begin{align*}
\mu_{l} & =\left[t^{l}\right] P_{\mathcal{A}_{f}}(t)  \tag{5.21}\\
& =c_{n+1}(l ;[0, d-2]) \tag{5.22}
\end{align*}
$$

In particular, if $(n+1) a \leqslant l \leqslant D$ for some integer $a \geqslant 0$, then

$$
\begin{equation*}
\mu_{l}=c_{n+1}(l-(n-1) a ;[a, a+d-2]) . \tag{5.23}
\end{equation*}
$$

$\triangle$

Let $\left(i_{2}, \ldots, i_{b}\right)$ be the frequency vector of $N$, where $i_{j}$ is the frequency the integer $j$ occurs in each positive composition of $N$. For $2 \leqslant j \leqslant b$, define the following quantities

$$
\begin{align*}
& \alpha_{j}=N-k(j-1)-\sum_{\ell=j+1}^{b}(\ell-j+1) i_{\ell}  \tag{5.24}\\
& \beta_{j}=k-\sum_{\ell=j+1}^{b} i_{\ell}  \tag{5.25}\\
& \gamma_{j}=\left\lfloor\frac{1}{j-1}\left(N-k-\sum_{\ell=j+1}^{b}(\ell-1) i_{\ell}\right)\right\rfloor . \tag{5.26}
\end{align*}
$$

According to Theorem 2.1, [227],

$$
\begin{equation*}
c_{k}(N ;[a, b])=c_{k}(N-k(a-1) ;[1, b-(a-1)]) \tag{5.27}
\end{equation*}
$$

and

$$
\begin{equation*}
c_{k}(N ;[1, b])=\sum_{\substack{i_{2}=\alpha_{2}, i_{3}, \ldots, i_{b} \\ \max \left\{0, \alpha_{j}\right\} \leqslant i_{j} \leqslant \max \left\{\beta_{j}, \gamma_{j}\right\}}} \prod_{\ell=2}^{b}\binom{k-\sum_{j=2}^{\ell-1} i_{j}}{i_{\ell}} . \tag{5.28}
\end{equation*}
$$

In particular, if $\frac{k b-n}{k-1} \in \mathbb{N}$ and $\frac{k a+(b-a)-n}{k-1} \in \mathbb{Z}_{\geqslant 0}$, then

$$
\begin{equation*}
c_{k}(N ;[a, b])=\binom{N-k(a-1)-1}{k-1} . \tag{5.29}
\end{equation*}
$$

Corollary 5.10. Suppose $f$ is a homogeneous polynomial of (homogeneous)
degree $d$ in $n+1$ complex indeterminates. For $d \geqslant 2$ and $n \geqslant 0$,

$$
\begin{equation*}
\mu_{\left\lfloor\frac{D}{2}\right\rfloor}=\sum_{\substack{i_{2}=\alpha_{2}, i_{3}, \ldots, i_{d-1} \\ \max \left\{0, \alpha_{j}\right\} \leqslant i_{j} \leqslant \max \left\{\beta_{j}, \gamma_{j}\right\}}} \prod_{\ell=2}^{d-1}\binom{n+1-\sum_{j=2}^{\ell-1} i_{j}}{i_{\ell}}, \tag{5.30}
\end{equation*}
$$

where

$$
\begin{align*}
& \alpha_{j}=\left\lfloor\frac{d}{2}(n+1)\right\rfloor-(n+1)(j-1)-\sum_{\ell=j+1}^{d-1}(\ell-j+1) i_{\ell}  \tag{5.31a}\\
& \beta_{j}=n+1-\sum_{\ell=j+1}^{d-1} i_{\ell}  \tag{5.31b}\\
& \gamma_{j}=\left\lfloor\frac{1}{j-1}\left(\left\lfloor\frac{d}{2}(n+1)\right\rfloor-(n+1)-\sum_{\ell=j+1}^{d-1}(\ell-1) i_{\ell}\right)\right\rfloor . \tag{5.31c}
\end{align*}
$$

In particular, if $n=1$, then $\mu_{d-2}=d-1$.

Proof. The claimed representation follows immediately from previous discussion. The integers $(d, n)$ solving the integrality constraints $d-1+\frac{1}{n}(d-$ $\left.1-\left\lfloor\frac{d}{2}(n+1)\right\rfloor\right) \in \mathbb{N}$ and $1+\frac{1}{n}\left(d-1-\left\lfloor\frac{d}{2}(n+1)\right\rfloor\right) \in \mathbb{Z}_{\geqslant 0}$ pertain to the cases $n=1$ or $d=2$. Equation (5.29) yields $\mu_{d-2}=d-1$ if $n=1$ and $\mu_{0}=1$ if $d=2$.

Remark 5.2.4. Since $c_{n+1}(l ;[0, d-2])=c_{n+1}(l+n+1 ;[1, d-1])$, then all of the coefficients admit a similar interpretation. That is,

$$
\begin{align*}
& \mu_{l}=c_{n+1}(l+n+1 ;[1, d-1])  \tag{5.32}\\
&=\sum_{\substack{i_{2}=\alpha_{2}, i_{3}, \ldots, i_{d-1}}} \prod_{\max \left\{0, \alpha_{j}\right\} \leqslant i_{j} \leqslant \max \left\{\beta_{j}, \gamma_{j}\right\}}  \tag{5.33}\\
& \prod_{i}^{d-1}\binom{n+1-\sum_{j=2}^{\ell-1} i_{j}}{i_{\ell}},
\end{align*}
$$

where $\alpha_{j}=l-(n+1) j-\sum_{\ell=j+1}^{d-1}(\ell-j+1) i_{\ell}, \beta_{j}=n+1-\sum_{\ell=j+1}^{d-1} i_{\ell}$ and

$$
\begin{equation*}
\gamma_{j}=\left\lfloor\frac{1}{j-1}\left(l-\sum_{\ell=j+1}^{d-1}(\ell-1) i_{\ell}\right)\right\rfloor \tag{5.34}
\end{equation*}
$$

In particular, if $(d, n, l)$ satisfy the integrality constraints $d-2+\frac{d-l-2}{n} \in \mathbb{N}$ and $\frac{d-l-2}{n} \in \mathbb{Z}_{\geqslant 0}$, then

$$
\begin{equation*}
\mu_{l}=\binom{n+l}{n} \tag{5.35}
\end{equation*}
$$

5.2.3. Hilbert-Poincaré Series Coefficients and Lattice Points. Yoshinaga and Suzuki (Lemma 3.1, [493]) prove that the coefficient $\mu_{l}$ counts the number of non-negative integer solutions of the Diophantine equation $\sum_{i=0}^{n} q_{i} x_{i}=l$, provided $l \leqslant D-d,\left\{\omega_{i}\right\} \subset Q \cap\left(0, \frac{1}{2}\right)$ and $\sum_{i=0}^{n} \omega_{i} \geqslant \frac{2 n-1}{4}$. By our previous lattice point enumeration analysis, we give the following closed form expression.

Proposition 5.11. Suppose $f$ is a weighted homogeneous polynomial of weighted degree $d$ and integral weights $\left\{q_{0}, \ldots, q_{n}\right\}$, or equivalently reduced weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$. If $l \leqslant D-d, \omega_{i} \in\left(0, \frac{1}{2}\right)$ and $\sum_{i=0}^{n} \omega_{i} \geqslant \frac{2 n-1}{4}$, then $P_{\mathcal{A}_{f}}(t)=\sum_{l=0}^{D} \mu_{l} t^{l}$,
where $\mu_{l}=\mu_{D-l}$ and

$$
\begin{align*}
\mu_{l} & =\sum_{i_{0}=0}^{\left\lfloor l / q_{0}\right\rfloor} \cdots \sum_{i_{n}=0}^{\left\lfloor\left(l-\sum_{k=0}^{n-1} q_{k} i_{k}\right) / q_{n}\right\rfloor} 1-\sum_{i_{0}=0}^{\left\lceil l / q_{0}\right\rceil-1} \cdots \sum_{i_{n}=0}^{\left\lceil\left(l-\sum_{k=0}^{n-1} q_{k} i_{k}\right) / q_{n}\right\rceil-1} 1  \tag{5.36a}\\
& =\sum_{i_{0}=0}^{\left\lfloor d l / \omega_{1}\right\rfloor} \cdots \sum_{i_{n}=0}^{\left\lfloor\left(d l-\sum_{k=0}^{n-1} \omega_{k} i_{k}\right) / \omega_{n}\right\rfloor} 1-\sum_{i_{0}=0}^{\left\lceil d l / \omega_{1}\right\rceil-1} \cdots \sum_{i_{n}=0}^{\left\lceil\left(d l-\sum_{k=0}^{n-1} \omega_{k} i_{k}\right) / \omega_{n}\right\rceil-1} 1, \tag{5.36b}
\end{align*}
$$

which is the difference of two quasi-polynomials of degree $n$ in the variable dl.

Proof. Count the non-negative integral solutions of the Diophantine equation $\sum_{i=0} \frac{q_{i}}{l} x_{i}=1$ and use Proposition 5.2.

If $l \leqslant D-d$, then

$$
\begin{equation*}
\mu_{l}=\left|\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{N}^{n} \mid \sum_{i=1}^{n} q_{i} x_{i}=l\right\}\right| . \tag{5.37}
\end{equation*}
$$

In general, for $0 \leqslant l \leqslant\left\lfloor\frac{D}{2}\right\rfloor$,

$$
\begin{equation*}
\mu_{l} \leqslant\left|\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{Z}_{\geqslant 0}^{n} \mid \sum_{i=0}^{n} q_{i} x_{i}=l\right\}\right| \tag{5.38}
\end{equation*}
$$

In certain cases, the inner modality admits a Diophantine representation similar to that of the geometric genus. If $\sum_{i=1}^{n} \omega_{i} \geqslant \frac{3}{4}$, then

$$
\begin{align*}
\bar{\mu}_{0} & =\left|\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{Z}_{\geqslant 0}^{n} \mid \sum_{i=1}^{n} q_{i} x_{i} \leqslant 2 d-2 \sum_{i=1}^{n} q_{i}\right\}\right|  \tag{5.39a}\\
& =\left|\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{N}^{n} \mid \sum_{i=1}^{n} q_{i} x_{i} \leqslant 2 d-\sum_{i=1}^{n} q_{i}\right\}\right|  \tag{5.39b}\\
& =\left|\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{N}_{>1}^{n} \mid \sum_{i=1}^{n} q_{i} x_{i} \leqslant 2 d\right\}\right| . \tag{5.39c}
\end{align*}
$$

For weighted homogeneous polynomials over $\mathbb{C}^{2}$, we propose the following sharp upper bound for the coefficients of the corresponding Hilbert-Poincaré series.

Proposition 5.12. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial with integral weights $\left\{q_{1}, q_{2}\right\}$ and weighted degree $d$, then $P_{\mathcal{A}_{f}}(t)=\sum_{l=0}^{D} \mu_{l} t^{l}$, where $D=2\left(d-q_{1}-q_{2}\right)$,

$$
\begin{equation*}
\mu_{l} \leqslant \min \left\{\left\lfloor\frac{l}{q_{1}}\right\rfloor+1,\left\lfloor\frac{l}{q_{2}}\right\rfloor+1\right\} \quad 0 \leqslant l \leqslant D-d \tag{5.40}
\end{equation*}
$$

and

$$
\begin{equation*}
\mu_{l} \leqslant \min \left\{\left\lfloor\frac{D-l}{q_{1}}\right\rfloor+1,\left\lfloor\frac{D-l}{q_{2}}\right\rfloor+1\right\} \quad d \leqslant l \leqslant D \tag{5.41}
\end{equation*}
$$

Proof. In Volume 2, we compute the exact number of non-negative lattice points on the hypotenuse of the $t$-dilate of a rational 2-polytope, $\operatorname{conv}\left\{0, \frac{\mathbf{e}_{1}}{a_{1}}, \frac{\mathbf{e}_{2}}{a_{2}}\right\}$, namely,

$$
\begin{align*}
\mathscr{L}_{\left[\frac{\mathrm{te}_{1}}{a_{1}}, \frac{\mathrm{e}}{e_{2}}\right]}(1) & =1+\left\lfloor\frac{t}{a_{1}}\right\rfloor-\sum_{i=0}^{\left\lfloor t / a_{1}\right\rfloor} \chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{t-a_{1} i}{a_{2}}\right)  \tag{5.42}\\
& =1+\left\lfloor\frac{t}{a_{2}}\right\rfloor-\sum_{i=0}^{\left\lfloor t / a_{2}\right\rfloor} \chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{t-a_{2} i}{a_{1}}\right), \tag{5.43}
\end{align*}
$$

where $\chi_{X}^{+}$is the characteristic function of the intersection $X \cap \mathbb{R}_{\geqslant 0}$. Since $\mu_{l}$ is bounded from above by the number of non-negative lattice points on the hypotenuse of the $l$-dilate of a rational 2-polytope $\operatorname{conv}\left\{0, \frac{\mathbf{e}_{1}}{q_{1}}, \frac{\mathbf{e}_{2}}{q_{2}}\right\}$, the first upper
bound now follows. The reflexivity property, namely, $\mu_{l}=\mu_{D-l}$, implies the second upper bound.

Remark 5.2.5. Consider $f=x^{2}+y^{3}$ with $q_{1}=3, q_{2}=2$ and $d=6$. Thus, $D=2, \mu_{\mathrm{alg}}(f)=2$ and Proposition 5.12 implies $\mu_{l} \leqslant 1$ for $0 \leqslant l \leqslant 2$. Since $P_{\mathcal{A}_{f}}(t)$ is reflexive, these data uniquely specify $P_{\mathcal{A}_{f}}(t)=1+t^{2}$, without having to first compute the local algebra $\mathcal{A}_{f}$. This example illustrates the usefulness of the upper bounds.

### 5.3. Sebastiani-Thom Factorization

Even on the level of singularities, do we find elementary combinatorics.

Definition 5.13. The constituent singularities of a Sebastiani-Thom summation singularity are factors.

Definition 5.14. A non-degenerate, square singularity is SebastianiThom irreducible if and only if it is not permutation equivalent to a non-trivial Sebastiani-Thom summation of non-degenerate factors. Equivalently, the corresponding exponent matrix is not permutation equivalent to a direct summation of exponent matrices of non-degenerate singularities.

Definition 5.15. A non-degenerate singularity is Sebastiani-Thom reducible if and only if it is not Sebastiani-Thom irreducible.

A non-degenerate, Sebastiani-Thom reducible singularity $f \cong f_{1} \boxplus \cdots \boxplus f_{s}$ satisfies equation 2.3, which is a factorization of a positive integer into a product of divisors. In contrast to a standard non-degenerate Sebastiani-Thom summation, however, Remark 2.9.1 illustrates what might occur if one relaxes the condition of non-degeneracy of the factors, namely, the aforementioned product is a non-unique factorization into rationals.

Definition 5.16. A non-degenerate, Sebastiani-Thom reducible singularity is factored if and only if it is permutation equivalent to a Sebastiani-Thom summation of non-degenerate, Sebastiani-Thom irreducible factors.

Any Brieskorn-Pham singularity over $\mathbb{C}^{n+1}$ is Sebastiani-Thom reducible and factored for $n \geqslant 1$, as it is trivially and canonically permutation equivalent to a Sebastian-Thom summation of its constituent monomials, each of which is non-degenerate. Clearly, this factorization is unique up to a permutation of factors.

Problem 5.3.1. Determine whether or not the factorization of a nondegenerate, Sebastiani-Thom reducible singularity is unique up to a permutation of factors.

Problem 5.3.2. Determine whether or not Sebastiani-Thom factorization is well-defined in the more generalized context of right, contact or stable equivalence.

The number of possible factorizations of a square singularity is bounded from above by the number of integer compositions of the number of variables and the number of integer factorizations of the algebraic index into products of divisors, including order and those non-degenerate factors with algebraic index 1. In particular, the number of distinct, non-trivial Sebastiani-Thom sums forming a Brieskorn-Pham polynomial of $n+1$ variables is one less than the number of positive integer compositions of $n+1$ summed over $k$ possible non-zero parts, which is $\sum_{k=1}^{n+1}\binom{n}{k-1}-1=2^{n}-1$ (Figure 5.1).

| 0000 | 0000 |
| :---: | :---: |
| 0000 | 0000 |
| (0)0 0 | 0000 |
| 0000 | 00000 |
| 0000 | 00000 |
| 00000 | 0000 |
| 00000 | 00000 |
| 0000 | 00000 |

Figure 5.1. Sixteen Compositions of a Set with Five Elements

### 5.4. Newton and Weight Polytopes

5.4.1. Newton Polytope. Given a polynomial $f=\sum_{i=1}^{m} c_{i} z_{1}^{a_{i 1}} \cdots z_{n}^{a_{i n}}$, define the Newton Polytope $\mathcal{N}(f)$ as the convex hull $\operatorname{conv}\left\{\mathbf{a}_{1}, \ldots, \mathbf{a}_{n}\right\}$, where $\mathbf{a}_{i}=\left(a_{i 1}, \ldots, a_{i n}\right)^{\top}$. Define the $t$-dilate $f_{t}=f\left(z_{1}^{t}, \ldots, z_{n}^{t}\right)$. It follows that the Newton polytope of $f_{t}$ coincides with the $t$-dilate of that of $f$. Note, however, that it is not guaranteed that $f_{t}$ be non-degenerate for $t \geqslant 1$, or even that the corresponding algebraic index is integral for $t \geqslant 1$.

REmARK 5.4.1. Consider $g=x^{7} y^{3}+x^{6} y^{5}$ over $\mathbb{C}^{2}$, which is degenerate. Here, the weights of $g_{t}$ are $\left\{\frac{2}{17 t}, \frac{1}{17 t}\right\}$ and, therefore, $\mu_{\mathrm{alg}}\left(g_{t}\right)=\frac{289}{2} t^{2}-\frac{51}{2} t+1$. It is a simple matter to prove that the polynomial $p(t)=\frac{289}{2} t^{2}-\frac{51}{2} t+1$ takes positive integral values for $t \geqslant 1$ by induction, which follows from $p(1)=120$ and the identity $p(k+1)=p(k)+289 k^{2}+238 k+119$.

We invite the reader to consult [129] for a historical development of the subject of plane algebraic curves.
5.4.2. Combinatorial Index. Let $F_{k}^{\perp}$ denote a $k$-dimensional face of the Newton polytope $\mathcal{N}$ which intersects the coordinate axes. For any finite polytope $\mathcal{P}$, define the normalized $n$-content $\operatorname{vol}_{n}(\mathcal{P})=n!\operatorname{vol}_{n}(\mathcal{P})$. The 0 -face $F_{0}^{\perp}$ is the point at the origin and $\operatorname{vol}_{0}\left(F_{0}^{\perp}\right)=\operatorname{vol}_{0}^{\prime}\left(F_{0}^{\perp}\right)=1$. Given a complex analytic series $f \in \mathcal{O}_{0, n}$ with Newton Polytope $\mathcal{N}(f)$, Kushnirenko considers the polytope $\mathcal{K}(f)=\boldsymbol{\operatorname { c o n v }}\{\mathbf{0} \cup \mathcal{N}(f)\}$, the cone over $\mathcal{N}(f)$, and computes the following mixed volume (§10, [362]),

$$
\begin{align*}
\operatorname{MVK}(f) & =\sum_{k=0}^{n+1}(-1)^{k} \sum_{F_{n-k}^{\perp} \subset \mathcal{K}(f)} \operatorname{vol}_{n-k}^{\prime}\left(F_{n-k}^{\perp}\right)  \tag{5.44}\\
& =(n+1)!\operatorname{vol}_{n+1} F_{n+1}^{\perp}-n!\operatorname{vol}_{n} F_{n}^{\perp}+\cdots+(-1)^{n} F_{1}^{\perp}+(-1)^{n+1} \tag{5.45}
\end{align*}
$$

which we shall refer to as the Kushnirenko-Newton number (Proposition 2.16, [168]). Let $f:\left(\mathbb{C}^{n}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a semi-weighted homogeneous polynomial,
q.v., §2.4.2. Define the combinatorial index of $f$ as the Kushnirenko-Newton number of the cone $\mathcal{K}(f)$,

$$
\begin{equation*}
\mu_{\mathrm{comb}}(f):=\operatorname{MV} \mathcal{K}(f) \tag{5.46}
\end{equation*}
$$

Proposition 5.17 (Kushnirenko, [251]). Let $f$ be a weighted homogeneous polynomial. The following inequality holds,

$$
\begin{equation*}
\mu_{\text {comb }}(f) \leqslant \mu_{\mathrm{alg}}(f) \tag{5.47}
\end{equation*}
$$

with equality when the principal part of $f$ is non-degenerate.
5.4.3. Weight Polytope. Given a weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with reduced weights $\left\{\omega_{1}, \ldots, \omega_{n}\right\}$, define the Weight Polytope

$$
\begin{equation*}
\mathcal{W}(f)=\operatorname{conv}\left\{0, \frac{\mathbf{e}_{1}}{\omega_{1}}, \ldots, \frac{\mathbf{e}_{n}}{\omega_{n}}\right\} \tag{5.48}
\end{equation*}
$$

REMARK 5.4.2. It is clear that the weight polytope of a quasi-BrieskornPham polynomial is an integral polytope.

Proposition 5.18. If $f$ is quasi-Brieskorn-Pham, then $\mathcal{N}(f)=\mathcal{W}(f)$.
Proof. The identification follows from the equality of the convex hulls defining the respective polytopes.

REMARK 5.4.3. Consider the $r$-dilate $f \boxtimes z^{r}=x^{p r}-y^{q r}$ which equals the product $\prod_{k=0}^{r-1}\left(x^{p}-\zeta_{r}^{k} y^{q}\right)$ and shows that the map $f \mapsto f \boxtimes z^{r}$ induces a map
of algebraic links $T_{p, q} \mapsto T_{p r, q r} \cong \bigcup_{k=1}^{r} T_{p, q}$ and a map of polytopes $\mathcal{W}(f) \mapsto$ $r \mathcal{W}(f)$.


Figure 5.2. A Newton and Weight Polytope

Proposition 5.19. The algebraic index of the Kronecker sum $f \boxtimes z$, where $f$ is Brieskorn-Pham with exponents $\left\{a_{1}, \ldots, a_{n}\right\}$, is equal to the normalized $n$-content of $\mathcal{W}(f)$,

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f \boxminus z)=n!\operatorname{vol}_{n} \mathcal{W}(f) \tag{5.49}
\end{equation*}
$$

Proof. The proof is a consequence of equation (2.213).
5.4.4. Lattice Index. Let $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0)$ be a non-degenerate weighted homogeneous polynomial. Provide a weight of 2 to the interior points and a weight of 1 to the boundary points of the polytope $\mathcal{W}(f)$. Define the lattice
index of $f$ as the weighted lattice point summation

$$
\begin{align*}
\mu_{\text {lat }}(f) & =\sum_{p \in \mathcal{W}(f)^{\circ}} 2+\sum_{p \in \partial \mathcal{W}(f) \cap \mathbb{N}^{n}} 1  \tag{5.50a}\\
& =\sum_{p \in \mathcal{W}(f)} w(p) \tag{5.50b}
\end{align*}
$$

where the lattice point weight is defined succinctly as

$$
w(p)=\left\{\begin{array}{cc}
2 \quad \text { iff } p \in \mathcal{W}^{\circ}(f)  \tag{5.51}\\
1 & \text { iff } p \in \partial \mathcal{W}(f) \cap \mathbb{N}^{n} \\
0 & \text { iff } p \notin \mathcal{W}(f) \cap \mathbb{N}^{n}
\end{array}\right.
$$

Since the lattice index depends only on the weights of a weighted homogeneous singularity, it is a topological invariant of said singularity.
5.4.5. Arithmetic Index. Define the arithmetic index $\mu_{\mathrm{nt}}(f)$ of a weighted homogeneous polynomial $f$ with weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ as the number of positive integer solutions of the system of Diophantine inequalities, $0<\omega_{i} x_{i}<1$,

$$
\begin{equation*}
\mu_{\mathrm{nt}}(f):=\left|\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{N}^{n+1} \mid 0<\omega_{i} x_{i}<1\right\}\right| . \tag{5.52}
\end{equation*}
$$

Let $\mathcal{O}(f)$ denote the unique minimal $(n+1)$-orthotope which encloses the weight polytope of $f$. The Ehrhart function of the interior $\mathcal{O}^{\circ}$ of the unique minimal $(n+1)$-orthotope $\mathcal{O}$ enclosing the convex hull of the vertices $\left\{\mathbf{0}, b_{1} \mathbf{e}_{1}, \ldots, b_{n+1} \mathbf{e}_{n+1}\right\}$, where $\left\{b_{1}, \ldots, b_{n+1}\right\} \subset \mathbb{R}_{>0}$, is the product $\mathscr{L}_{\mathcal{O}^{\circ}}(t)=$ $\prod_{i=1}^{n+1}\left(\left\lceil b_{i} t\right\rceil-1\right)$. An inductive argument proves that this integer is equal to the
order of $\mathcal{O}^{\circ}(f)$, that is,

$$
\begin{equation*}
\mu_{\mathrm{nt}}(f)=\prod_{i=0}^{n}\left(\left\lceil\frac{1}{\omega_{i}}\right\rceil-1\right) \tag{5.53}
\end{equation*}
$$

Since the arithmetic Milnor depends only on the weights of a weighted homogeneous singularity, it is a topological invariant of said singularity. In light of equation (2.32c), the following inequality holds:

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f) \leqslant \mu_{\mathrm{nt}}(f) \tag{5.54}
\end{equation*}
$$

Proposition 5.20. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate weighted homogeneous polynomial, then one has the sharp upper bound

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f) \leqslant\left\lfloor\left(\sqrt[n+1]{\mu_{\mathrm{nt}}(f)}-\prod_{i=0}^{n} \sqrt[n+1]{\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{i}}\right)-\left\{\frac{1}{\omega_{i}}\right\}}\right)^{n+1}\right\rfloor \tag{5.55}
\end{equation*}
$$

with equality, $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{nt}}(f)$, in the case that $f$ is quasi-Brieskorn-Pham.

Proof. For any two non-negative sequences $\left\{x_{0}, \ldots, x_{n}\right\}$ and $\left\{y_{0}, \ldots, y_{n}\right\}$, a refinement of the Arithmetic-Geometric Inequality* is the following,

$$
\begin{equation*}
\prod_{i=0}^{n} \sqrt[n+1]{x_{i}}+\prod_{i=0}^{n} \sqrt[n+1]{y_{i}} \leqslant \prod_{i=0}^{n} \sqrt[n+1]{x_{i}+y_{i}} \leqslant \frac{1}{n+1} \sum_{i=0}^{n} x_{i}+\frac{1}{n+1} \sum_{i=0}^{n} y_{i} \tag{5.56}
\end{equation*}
$$

*See Chapter 2 of Volume 2 for a proof of this relation.

Write the Ehrhart function of $\mathcal{O}^{\circ}(f)$ as

$$
\begin{align*}
\mathscr{L}_{\mathcal{O}^{\circ}(f)}(t) & =\prod_{i=0}^{n}\left(\left\lceil a_{i} t\right\rceil-1\right)  \tag{5.57}\\
& =\prod_{i=0}^{n}\left(a_{i} t-\left\{a_{i} t\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(a_{i} t\right)-1\right) . \tag{5.58}
\end{align*}
$$

Thus, setting $x_{i}=\frac{1}{\omega_{i}}-1$ and $y_{i}=\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{i}}\right)-\left\{\frac{1}{\omega_{i}}\right\}$ and noting that both are non-negative ${ }^{*}$, the aforementioned inequality yields

$$
\begin{equation*}
\left(\sqrt[n+1]{\mu_{\mathrm{alg}}(f)}+\prod_{i=0}^{n} \sqrt[n+1]{\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{i}}\right)-\left\{\frac{1}{\omega_{i}}\right\}}\right)^{n+1} \leqslant \mathscr{L}_{\mathcal{O}^{\circ}(f)}(1) \tag{5.59}
\end{equation*}
$$

If one weight $\omega_{i}$ is an inverse integer, then $\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{i}}\right)=\left\{\frac{1}{\omega_{i}}\right\}=0$ and the product vanishes. In such case,

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f) \leqslant \mathscr{L}_{\mathcal{O}^{\circ}(f)}(1)=\mu_{\mathrm{nt}}(f) \tag{5.60}
\end{equation*}
$$

Otherwise, one has the refined bound

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f) \leqslant\left\lfloor\left(\sqrt[n+1]{\mu_{\mathrm{nt}}(f)}-\prod_{i=0}^{n} \sqrt[n+1]{1-\left\{\frac{1}{\omega_{i}}\right\}}\right)^{n+1}\right\rfloor . \tag{5.61}
\end{equation*}
$$

In the case that $f$ is a quasi-Brieskorn-Pham polynomial, the weights satisfy the identity $\left\lceil\frac{t}{\omega_{i}}\right\rceil=\frac{t}{\omega_{i}}$ for integral $t$, so equation (5.54) is an equality. This concludes the proof.

[^33]Remark 5.4.4. For any positive integer $t$, the inequality generalizes,

$$
\begin{equation*}
\mu_{\mathrm{alg}}\left(f_{t}\right) \leqslant \mu_{\mathrm{nt}}\left(f_{t}\right) \tag{5.62}
\end{equation*}
$$

where $f_{t}$ is the $t$-dilate of $f, q . v .$, Remark 2.9.5.

### 5.5. Milnor-Jung Formula

5.5.0.1. Dimension 1. To motivate the ensuing discussion we state the following trivial result for completeness. If $f:(\mathbb{C}, 0) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate weighted homogeneous polynomial with weight $\frac{1}{p}$ where $p \in \mathbb{N}$, then $\mu_{\mathrm{alg}}(f)=$ $\mu_{\mathrm{nt}}(f)=p-1$, where $p$ is the number of positive integer solutions $(x)$ of the Diophantine inequality $0<\frac{x}{p}<1$. Observe that the number of interior lattice points of the line $[0, p]$ on $\mathbb{Z}$ is the integer $p-1$. We generalize.

5•5.0.2. Dimension 2.

Proposition 5.21. If $f$ is quasi-Brieskorn-Pham with weights $\left\{\frac{1}{p}, \frac{1}{q}\right\}$, where $p, q \in \mathbb{N}$, then $\mu_{\mathrm{alg}}\left(f_{t}\right)=2 N_{t}-\operatorname{GCD}(p, q) t+1$, where $N_{t}$ is the number of positive integer solutions $(x, y)$ of the Diophantine inequality $\frac{x}{p}+\frac{y}{q} \leqslant t$.

Proof 1. In Volume 2, we prove that the number $N_{R}$ of non-negative integer solutions of the Diophantine equation $p x+q y \leqslant R$ is given by double summation

$$
\begin{equation*}
N_{R}=\sum_{i=0}^{\lfloor R / p\rfloor} \sum_{j=0}^{\lfloor(R-i p) / q\rfloor} 1 \tag{5.63}
\end{equation*}
$$

A classical result of Milnor and Orlik gives $\mu_{\mathrm{alg}}(f)=(p-1)(q-1)$. Let $R=$ $p q-p-q$. Thus, the result follows from the series of identities

$$
\begin{align*}
2 N_{R} & =2 \sum_{i=0}^{\lfloor q-1-q / p\rfloor} \sum_{j=0}^{\lfloor p-1-(i+1) p / q\rfloor} 1  \tag{5.64a}\\
& =2 \sum_{i=0}^{q-1}\left\lfloor\frac{i p}{q}\right\rfloor=(p-1)(q-1)+\operatorname{GCD}(p, q)-1 . \tag{5.64b}
\end{align*}
$$

Proof 2. If $f$ is quasi-Brieskorn-Pham, then $\mathscr{L}_{\mathcal{O}^{\circ}(f)}(t)=\mu_{\mathrm{alg}}\left(f_{t}\right)$. Since the number of interior lattice points along the hypotenuse of the $t$-dilate of $\mathcal{W}(f)$ is $\operatorname{GCD}(p, q) t-1$, by symmetry and comparison to $\mathcal{O}^{\circ}(f)$, this leaves only twice the positive lattice points of the $t$-dilate of $\mathcal{W}(f)$. This completes the proof of the identity.

Definition 5.22. Let $R$ be a polynomial ring over a field of characteristic zero. A polynomial $f \in R$ is square-free if and only if its factorization into irreducibles $f=\prod_{i=1}^{r} f_{i}^{r_{i}}$ implies that $r_{i}=1$ for $1 \leqslant i \leqslant r$.

A square-free, weighted homogeneous polynomial has no repeated factors vanishing at $\mathbf{0}$ if and only if the origin is an isolated critical point. Each analytically irreducible factor $f_{i}$ corresponds to an analytically irreducible component or branch of $V_{f, 0}$ at the origin. Milnor proves the following assertion.

Proposition 5.23 (Milnor, [310]). Let $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a square-free polynomial (i.e., corresponding to a reduced plane curve) with delta invariant $\delta(f)$ and
branch number $r(f)$. The algebraic index satisfies the identity

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=2 \delta(f)-r(f)+1 \tag{5.65}
\end{equation*}
$$

Proof. See Theorem 10.5 in [310] and Proposition 3.35 in [168].

REMARK 5.5.1. The branch number $r(f)$ counts the number of connected components of the corresponding link, $K_{f}$. In particular, if $r(f)=1$, then by a corollary (§10, Corollary 10.2 in [310]) of Milnor based on the work of Neuwirth and Stallings, the delta invariant $\delta$ coincides with the genus of the algebraic knot $K_{f}$, which is the minimum genus of all Seifert surfaces of $K_{f}, q . v .$, Proposition 4.39.

Proposition 5.24. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a square-free, non-degenerate weighted homogeneous polynomial, then

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{lat}}(f) \tag{5.66}
\end{equation*}
$$

Proof. Combine the Milnor-Jung formula with Proposition 5.26.

Definition 5.25. The generalized delta invariant $\delta(f)$ and generalized branch number $r(f)$ of a possibly degenerate, weighted homogeneous polynomial $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is the number of positive lattice points and one more than the number of interior lattice points of the hypotenuse of the corresponding
weight polytope $\mathcal{W}(f)$, respectively. That is,

$$
\begin{equation*}
\delta(f)=\left|\mathcal{W}(f) \cap \mathbb{N}^{2}\right| \quad \text { and } \quad r(f)=\left|\partial \mathcal{W}(f) \cap \mathbb{N}^{2}\right| \tag{5.67}
\end{equation*}
$$

Proposition 5.26. The generalized delta invariant $\delta(f)$ and generalized branch number $r(f)$ of a non-degenerate, square-free, weighted homogeneous polynomial $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ coincide with the delta invariant and branch number of $f$, respectively.

Proof. See Chapter 6 of [467].
Given a weighted homogeneous polynomial $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, we derive explicit formulas for $\delta(f)$ and $r(f)$ in terms of the weights.
5.5.1. Delta Invariant. Recall the delta invariant counts the number of double points of the plane curve $f$. When $f$ is non-degenerate and square-free, this integer coincides with the number of positive lattice points in the right triangle $\mathcal{W}(f)=\operatorname{conv}\left\{0, \frac{\mathbf{e}_{1}}{\omega_{1}}, \frac{\mathbf{e}_{2}}{\omega_{2}}\right\}$ and can be computed explicitly in terms of the weights. Accordingly, one has

$$
\begin{align*}
\delta\left(f_{t}\right) & =\sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor\left\lfloor\left(t-i \omega_{1}\right) / \omega_{2}\right\rfloor} \sum_{j=1} 1  \tag{5.68a}\\
& =\frac{1}{2 \omega_{2}}\left\lfloor\frac{t}{\omega_{1}}\right\rfloor\left(2 t-\omega_{1}-\omega_{1}\left\lfloor\frac{t}{\omega_{1}}\right\rfloor\right)-\sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor}\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}  \tag{5.68b}\\
& =\frac{t^{2}}{2 \omega_{1} \omega_{2}}-\frac{t}{2 \omega_{2}}+\frac{\omega_{1}}{2 \omega_{2}}\left(\left\{\frac{t}{\omega_{1}}\right\}-\left\{\frac{t}{\omega_{1}}\right\}^{2}\right)-\sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor}\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\} \tag{5.68c}
\end{align*}
$$

where $\{\cdot\}$ denotes the fractional part function. By interchanging the roles of the two weights, we have for $t=1$,

$$
\begin{align*}
\delta(f) & =\frac{1-\omega_{1}}{2 \omega_{1} \omega_{2}}+\frac{\omega_{1}}{2 \omega_{2}}\left(\left\{\frac{1}{\omega_{1}}\right\}-\left\{\frac{1}{\omega_{1}}\right\}^{2}\right)-\sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}  \tag{5.69a}\\
& =\frac{1-\omega_{2}}{2 \omega_{1} \omega_{2}}+\frac{\omega_{2}}{2 \omega_{1}}\left(\left\{\frac{1}{\omega_{2}}\right\}-\left\{\frac{1}{\omega_{2}}\right\}^{2}\right)-\sum_{i=1}^{\left\lfloor 1 / \omega_{2}\right\rfloor}\left\{\frac{1-i \omega_{2}}{\omega_{1}}\right\} . \tag{5.69b}
\end{align*}
$$

In symmetrized form,

$$
\begin{aligned}
& \delta(f)=\frac{2-\omega_{1}-\omega_{2}}{4 \omega_{1} \omega_{2}}-\frac{1}{2}\left(\frac{\omega_{1}}{\omega_{2}}\binom{\left\{\frac{1}{\omega_{1}}\right\}}{2}+\frac{\omega_{2}}{\omega_{1}}\binom{\left\{\frac{1}{\omega_{2}}\right\}}{2}\right. \\
&\left.+\sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}+\sum_{i=1}^{\left\lfloor 1 / \omega_{2}\right\rfloor}\left\{\frac{1-i \omega_{2}}{\omega_{1}}\right\}\right)
\end{aligned}
$$

where we use the notation $\binom{x}{2}=\frac{1}{2} x(x-1)$ for $x>0$.

Remark 5.5.2. Suppose $f$ is quasi-Brieskorn-Pham with weights $\omega_{1}=\frac{1}{a}$ and $\omega_{2}=\frac{1}{b}$, where $a, b \in \mathbb{N}$. One computes $\mu_{\mathrm{alg}}\left(f_{t}\right)=(a t-1)(b t-1)$ and

$$
\begin{align*}
\delta\left(f_{t}\right) & =\frac{a b}{2} t^{2}-\frac{b}{2} t-\sum_{i=1}^{a t}\left\{b\left(t-\frac{i}{a}\right)\right\}  \tag{5.70a}\\
& =\frac{a b}{2} t^{2}-\frac{1}{2}(a+b-\operatorname{GCD}(a, b)) t  \tag{5.70b}\\
& =\delta(f)\binom{t}{1}+a b\binom{t}{2} \tag{5.70c}
\end{align*}
$$

since

$$
\begin{equation*}
\sum_{i=1}^{a t}\left\{b\left(t-\frac{i}{a}\right)\right\}=\frac{1}{2}(a-\operatorname{GCD}(a, b)) t \quad a, b, t \in \mathbb{N} \tag{5.71}
\end{equation*}
$$

Correspondingly, the generalized branch number is easily computed $r\left(f_{t}\right)=$ $\operatorname{GCD}(a, b) t$, which establishes the Milnor-Jung formula for $t$-dilates of quasi-Brieskorn-Pham polynomials with weights $\left\{\frac{1}{a}, \frac{1}{b}\right\}$, namely,

$$
\begin{equation*}
\mu_{\mathrm{alg}}\left(f_{t}\right)=2 \delta\left(f_{t}\right)+r\left(f_{t}\right)-1 \tag{5.72}
\end{equation*}
$$

Remark 5.5.3. With $a=b=d \in \mathbb{N}_{>1}$, we recover the classical formula for the delta invariant of a homogeneous plane curve of degree $d$, namely,

$$
\begin{equation*}
\delta(f)=\frac{1}{2} d(d-1)=\binom{d}{2} \tag{5.73}
\end{equation*}
$$

Remark 5.5.4. Consider $f=x^{a}+x y^{b}$ over $\mathbb{C}^{n+1}$ with $a, b \in \mathbb{N}$, which is not quasi-Brieskorn-Pham. For $a>1$ and $b \geqslant 1, f$ is a non-degenerate, weighted homogeneous with weights $\left\{\frac{1}{a}, \frac{a-1}{a b}\right\}$. Since $\omega_{1}=\frac{1}{a}$ is an inverse integer, we use equation (5.68c) to compute $\delta\left(f_{t}\right)$, namely,

$$
\begin{equation*}
\delta\left(f_{t}\right)=\frac{a^{2} b}{2(a-1)} t^{2}-\frac{a b}{2(a-1)} t-\sum_{i=1}^{a t}\left\{\frac{a b}{a-1}\left(t-\frac{i}{a}\right)\right\} \tag{5.74}
\end{equation*}
$$

We establish elementary bounds for the delta invariant in terms of the weights.

Proposition 5.27. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial, then the generalized delta invariant $\delta(f)$ satisfies the elementary sharp bounds

$$
\begin{equation*}
0 \leqslant \delta(f) \leqslant \min \left\{\left\lfloor\frac{\left(2-\omega_{1}\right)^{2}}{8 \omega_{1} \omega_{2}}\right\rfloor,\left\lfloor\frac{\left(2-\omega_{2}\right)^{2}}{8 \omega_{1} \omega_{2}}\right\rfloor\right\} \tag{5.75}
\end{equation*}
$$

If $f$ is non-degenerate, then $\delta(f) \geqslant 1$.

Proof. Equations (5.69a) and (5.69b) and the fractional part difference bound, $0 \leqslant\{x\}-\{x\}^{2} \leqslant \frac{1}{4}$ for $0 \leqslant x \leqslant 1$, imply the upper bound. Thus,

$$
\begin{equation*}
\delta(f) \leqslant \min \left\{\frac{1-\omega_{1}}{2 \omega_{1} \omega_{2}}+\frac{\omega_{1}}{8 \omega_{2}}, \frac{1-\omega_{2}}{2 \omega_{1} \omega_{2}}+\frac{\omega_{2}}{8 \omega_{1}}\right\} \tag{5.76}
\end{equation*}
$$

which is the inequality. If $f$ is homogeneous of degree $d>1$,

$$
\begin{equation*}
\delta(f) \leqslant\left\lfloor\frac{1}{2} d(d-1)+\frac{1}{8}\right\rfloor=\frac{1}{2} d(d-1), \tag{5.77}
\end{equation*}
$$

proving the claimed sharpness of the bounds. If $f$ is non-degenerate, then each weight $\omega_{i}$ satisfies the bound $0<\omega_{i} \leqslant \frac{1}{2}$. Since $\triangle=\boldsymbol{\operatorname { c o n v }}\left\{\mathbf{0}, 2 \mathbf{e}_{1}, 2 \mathbf{e}_{2}\right\}$ has one positive lattice point at $(1,1)$ and $\triangle \subset \mathcal{W}(f)$, then $\delta(f) \geqslant 1$.
5.5.2. Branch Number. Recall the branch number $r(f)$ is the number of analytically irreducible branches of $V_{f, 0}$ passing through the origin. As a consequence of the Milnor-Jung formula and the fact that the delta invariant and algebraic index can be computed explicitly in terms of the corresponding weights,
the generalized branch number also admits such an explicit representation. We compute

$$
\begin{equation*}
r\left(f_{t}\right)=\frac{t}{\omega_{1}}+\frac{\omega_{1}}{\omega_{2}}\left(\left\{\frac{t}{\omega_{1}}\right\}-\left\{\frac{t}{\omega_{1}}\right\}^{2}\right)-2 \sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor}\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\} \tag{5.78}
\end{equation*}
$$

In particular, for $t=1$,

$$
\begin{align*}
r(f) & =\frac{1}{\omega_{1}}+\frac{\omega_{1}}{\omega_{2}}\left(\left\{\frac{1}{\omega_{1}}\right\}-\left\{\frac{1}{\omega_{1}}\right\}^{2}\right)-2 \sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}  \tag{5.79}\\
& =\frac{1}{\omega_{2}}+\frac{\omega_{2}}{\omega_{1}}\left(\left\{\frac{1}{\omega_{2}}\right\}-\left\{\frac{1}{\omega_{2}}\right\}^{2}\right)-2 \sum_{i=1}^{\left\lfloor 1 / \omega_{2}\right\rfloor}\left\{\frac{1-i \omega_{2}}{\omega_{1}}\right\} \tag{5.8o}
\end{align*}
$$

In symmetrized form,

$$
\begin{align*}
r(f)= & \frac{1}{2}\left(\frac{1}{\omega_{1}}+\frac{1}{\omega_{2}}\right)+\frac{\omega_{1}}{2 \omega_{2}}\left(\left\{\frac{1}{\omega_{1}}\right\}-\left\{\frac{1}{\omega_{1}}\right\}^{2}\right)+\frac{\omega_{2}}{2 \omega_{1}}\left(\left\{\frac{1}{\omega_{2}}\right\}-\left\{\frac{1}{\omega_{2}}\right\}^{2}\right) \\
& -\sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}-\sum_{i=1}^{\left\lfloor 1 / \omega_{2}\right\rfloor}\left\{\frac{1-i \omega_{2}}{\omega_{1}}\right\}  \tag{5.81a}\\
= & \frac{1}{2}\left(\frac{1}{\omega_{1}}+\frac{1}{\omega_{2}}\right)-\frac{\omega_{1}}{\omega_{2}}\binom{\left\{\frac{1}{\omega_{1}}\right\}}{2}-\frac{\omega_{2}}{\omega_{1}}\binom{\left\{\frac{1}{\omega_{2}}\right\}}{2}-\sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\} \\
& -\sum_{i=1}^{\left\lfloor 1 / \omega_{2}\right\rfloor}\left\{\frac{1-i \omega_{2}}{\omega_{1}}\right\} . \tag{5.81b}
\end{align*}
$$

If $f$ is a non-degenerate, weighted homogeneous polynomial and $\mu_{\mathrm{alg}}(f)$ is a non-negative integer, then $r(f)$ is a positive integer, which is not obvious from its representation involving only the weights.

Remark 5.5.5. Suppose $f$ is quasi-Brieskorn-Pham with inverse weights $a, b \in \mathbb{N}$. Then the generalized branch number of the corresponding $t$-dilate is given by

$$
\begin{align*}
r\left(f_{t}\right) & =\frac{1}{2}(a+b) t-\sum_{i=1}^{a t}\left\{b\left(t-\frac{i}{a}\right)\right\}-\sum_{i=1}^{b t}\left\{a\left(t-\frac{i}{b}\right)\right\}  \tag{5.82a}\\
& =\left(\frac{1}{2}(a+b)-\frac{1}{2}(a-\operatorname{GCD}(a, b))-\frac{1}{2}(b-\operatorname{GCD}(a, b))\right) t  \tag{5.82b}\\
& =\operatorname{GCD}(a, b) t \tag{5.82c}
\end{align*}
$$

REmARK 5.5.6. With $a=b=d \in \mathbb{N}_{>1}$, we recover the classical formula for the branch number of a homogeneous plane curve of degree $d$, namely, $r(f)=$ $d$, which is consistent with the values $\mu(f)=(d-1)^{2}$ and $\delta(f)=\binom{d}{2}$.

REMARK 5.5.7. Consider $f=x^{a}+x y^{b}$ over $\mathbb{C}^{n+1}$ with $a, b \in \mathbb{N}$, which is not quasi-Brieskorn-Pham. For $a>1$ and $b \geqslant 1, f$ is a non-degenerate, weighted homogeneous with weights $\left\{\frac{1}{a}, \frac{a-1}{a b}\right\}$. Since $\omega_{1}=\frac{1}{a}$ is an inverse integer, we use equation (5.78) to compute $r\left(f_{t}\right)$, namely,

$$
\begin{equation*}
r\left(f_{t}\right)=a t-2 \sum_{i=1}^{a t}\left\{\frac{a b}{a-1}\left(t-\frac{i}{a}\right)\right\} \tag{5.83}
\end{equation*}
$$

which may be verified directly using equation 5.68 c and the algebraic index,

$$
\begin{equation*}
\mu_{\mathrm{alg}}\left(f_{t}\right)=(a t-1)\left(\frac{a b t}{a-1}-1\right) \tag{5.84}
\end{equation*}
$$

As done previously for the delta invariant, we establish bounds for the branch number in terms of the weights.

Proposition 5.28. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial, then the generalized branch number $r(f)$ satisfies the elementary sharp bounds

$$
\begin{equation*}
0 \leqslant r(f) \leqslant \min \left\{\left\lfloor\frac{1}{\omega_{1}}+\frac{\omega_{1}}{4 \omega_{2}}\right\rfloor,\left\lfloor\frac{1}{\omega_{2}}+\frac{\omega_{2}}{4 \omega_{1}}\right\rfloor\right\} . \tag{5.85}
\end{equation*}
$$

If $f$ is non-degenerate, then $r(f) \geqslant 1$.

Proof. The proof of the bounds is similar to that given for Proposition 5.27, so the details are omitted. To establish sharpness, take $f$ to be a homogeneous polynomial of degree $d>1$. Then

$$
\begin{equation*}
d=r(f) \leqslant\left\lfloor d+\frac{1}{4}\right\rfloor=d \tag{5.86}
\end{equation*}
$$

as claimed.

Corollary 5.29. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial with $0<\omega_{1} \leqslant \omega_{2}$, then $\ell_{0}(f)=\frac{1}{\omega_{1}}-1$ and $1 \leqslant r(f) \leqslant\left\lfloor\frac{1}{\omega_{2}}+\frac{\omega_{2}}{4 \omega_{1}}\right\rfloor$.

Proof. If $\omega_{1} \leqslant \omega_{2}$, then $\frac{1}{\omega_{2}} \leqslant \frac{1}{\omega_{1}}$ and $4 \omega_{1}+\left(\omega_{1} \omega_{2}\right)^{2} \leqslant 4 \omega_{2}+\left(\omega_{1} \omega_{2}\right)^{2}$, which implies $\frac{1}{\omega_{2}}+\frac{\omega_{2}}{4 \omega_{1}} \leqslant \frac{1}{\omega_{1}}+\frac{\omega_{1}}{4 \omega_{2}}$ and the bound of Proposition 5.28 simplifies to the claimed upper bound for the branch number.

The next result relates the generalized branch number with the number of positive lattice points intersecting the hypotenuse of the corresponding weight polytope. Define the reflected triangle $\mathcal{W}^{\top}=\boldsymbol{\operatorname { c o n v }}\left\{\frac{\mathbf{e}_{1}}{\omega_{1}}, \frac{\mathbf{e}_{2}}{\omega_{2}}, \frac{\mathbf{e}_{1}}{\omega_{1}}+\frac{\mathbf{e}_{2}}{\omega_{2}}\right\}$.

Proposition 5.30. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate weighted homogeneous polynomial, then the generalized branch number satisfies the identity

$$
\begin{equation*}
r(f)=\operatorname{ord}\left(\mathcal{H}^{\circ}\right)+1+\Delta(\boldsymbol{\omega}) \tag{5.87}
\end{equation*}
$$

where $\mathcal{H}$ is the hypotenuse of $\mathcal{W}(f)$ and

$$
\begin{aligned}
\Delta(\boldsymbol{\omega})= & F(\boldsymbol{\omega})-F^{\boldsymbol{\top}}(\boldsymbol{\omega})+X(\boldsymbol{\omega})-X^{\top}(\boldsymbol{\omega}) \\
& +\left(\frac{1}{\omega_{1}}-1\right)\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{2}}\right)-\left\{\frac{1}{\omega_{2}}\right\}\right) \\
+ & \left(\frac{1}{\omega_{2}}-1\right)\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right)-\left\{\frac{1}{\omega_{1}}\right\}\right) \\
+ & \left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right)-\left\{\frac{1}{\omega_{1}}\right\}\right)\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{2}}\right)-\left\{\frac{1}{\omega_{2}}\right\}\right) .
\end{aligned}
$$

where

$$
\begin{align*}
F(\boldsymbol{\omega})= & -\left(\frac{\omega_{1}}{2 \omega_{2}}-1\right)\left\{\frac{1}{\omega_{1}}\right\}-\frac{\omega_{1}}{2 \omega_{2}}\left\{\frac{1}{\omega_{1}}\right\}^{2}-\sum_{i=1}^{\left\lceil 1 / \omega_{1}\right]-1}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}  \tag{5.88}\\
F^{\top}(\boldsymbol{\omega})= & \left\{\frac{1}{\omega_{1}}\right\}\left(1-\frac{1}{\omega_{2}}-\frac{3 \omega_{1}}{\omega_{2}}\right)+\left\{\frac{1}{\omega_{2}}\right\}\left(1-\frac{1}{\omega_{1}}\right)+\frac{\omega_{1}}{2 \omega_{2}}\left\{\frac{1}{\omega_{1}}\right\}^{2}+\left\{\frac{1}{\omega_{1}}\right\}\left\{\frac{1}{\omega_{2}}\right\} \\
& +2 \sum_{i=1}^{\left[1 / \omega_{1}\right]}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}-\sum_{i=1}^{\left\lceil 1 / \omega_{1}\right]-1}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\} \tag{5.89}
\end{align*}
$$

and

$$
\begin{align*}
X(\boldsymbol{\omega})= & -\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right)+\frac{\omega_{1}}{\omega_{2}} \chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right)\left\{\frac{1}{\omega_{1}}\right\}+\sum_{i=1}^{\left\lceil 1 / \omega_{1}\right]-1} \chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1-i \omega_{1}}{\omega_{2}}\right)  \tag{5.90}\\
X^{\top}(\boldsymbol{\omega})= & \chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right)\left(-1+\frac{1}{\omega_{2}}+\frac{\omega_{1}}{\omega_{2}}\left\{\frac{1}{\omega_{1}}\right\}-\left\{\frac{1}{\omega_{2}}\right\}\right) \\
& +\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{2}}\right)\left(-1+\frac{1}{\omega_{1}}-\left\{\frac{1}{\omega_{1}}\right\}\right)+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right) \chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{2}}\right) . \tag{5.91}
\end{align*}
$$

In particular, $\Delta(\boldsymbol{\omega})$ is small compared to $\operatorname{ord}\left(\mathcal{H}^{\circ}\right)$.

Proof. By splitting $\mathcal{O}$ into three sections, one has

$$
\begin{equation*}
\operatorname{ord} \mathcal{O}^{\circ}=\operatorname{ord}\left(\mathcal{W}^{\circ}\right)+\operatorname{ord}\left(\mathcal{H}^{\circ}\right)+\operatorname{ord}\left(\mathcal{W}^{\top \circ}\right) \tag{5.92}
\end{equation*}
$$

where

$$
\begin{align*}
\operatorname{ord}\left(\mathcal{O}^{\circ}\right) & =\sum_{i=1}^{\left\lceil 1 / \omega_{1}\right\rceil-1} \sum_{j=1}^{\left\lceil 1 / \omega_{2}\right\rceil-1} 1  \tag{5.93}\\
& =\left(\left\lceil\frac{1}{\omega_{1}}\right\rceil-1\right)\left(\left\lceil\frac{1}{\omega_{2}}\right\rceil-1\right) \tag{5.94}
\end{align*}
$$

and number of interior lattice points of $\mathcal{W}^{\circ}$ (see Volume 2),

$$
\begin{align*}
\operatorname{ord}\left(\mathcal{W}^{\circ}\right)= & \sum_{i=1}^{\left\lceil 1 / \omega_{1}\right]-1\left\lceil\left(1-i \omega_{1}\right) / \omega_{2}\right]-1} \sum_{j=1}^{1} 1  \tag{5.95}\\
= & \frac{\left(\omega_{1}-1\right)\left(2 \omega_{2}-1\right)}{2 \omega_{1} \omega_{2}}-\left(\frac{\omega_{1}}{2 \omega_{2}}-1-\frac{\omega_{1}}{\omega_{2}} \chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right)\right)\left\{\frac{1}{\omega_{1}}\right\}-\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right) \\
& -\frac{\omega_{1}}{2 \omega_{2}}\left\{\frac{1}{\omega_{1}}\right\}^{2}-\sum_{i=1}^{\left\lceil 1 / \omega_{1}\right\rceil-1}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}+\sum_{i=1}^{\left[1 / \omega_{1}\right]-1} \chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1-i \omega_{1}}{\omega_{2}}\right)  \tag{5.96}\\
= & \frac{\left(\omega_{1}-1\right)\left(2 \omega_{2}-1\right)}{2 \omega_{1} \omega_{2}}+F(\boldsymbol{\omega})+X(\boldsymbol{\omega}), \tag{5.97}
\end{align*}
$$

while that of the transpose is given by

$$
\begin{align*}
\operatorname{ord}\left(\mathcal{W}^{\boldsymbol{\top} \circ}\right) & =\sum_{i=1}^{\left\lceil 1 / \omega_{1}\right]-1} \sum_{j=\left\lfloor\left(1-i \omega_{1}\right) / \omega_{2}\right\rfloor+1}^{\left[1 / \omega_{2}\right]-1} 1  \tag{5.98}\\
& =\frac{\left(\omega_{1}-1\right)\left(2 \omega_{2}-1\right)}{2 \omega_{1} \omega_{2}}+F^{\boldsymbol{\top}}(\boldsymbol{\omega})+X^{\boldsymbol{\top}}(\boldsymbol{\omega}) \tag{5.99}
\end{align*}
$$

and that of the hypotenuse is given by

$$
\begin{equation*}
\operatorname{ord}\left(\mathcal{H}^{\circ}\right)=\left\lceil\frac{1}{\omega_{1}}\right\rceil-1-\sum_{i=1}^{\left\lceil 1 / \omega_{1}\right\rceil-1} \chi_{\mathbb{R} / \mathbb{Z}}^{+}\left(\frac{1-i \omega_{1}}{\omega_{2}}\right) \tag{5.100}
\end{equation*}
$$

Also, observe $\mu_{\mathrm{nt}}(f)=\operatorname{ord}\left(\mathcal{W}^{\circ}\right)+\operatorname{ord}\left(\mathcal{H}^{\circ}\right)+\operatorname{ord}\left(\mathcal{W}^{\top \circ}\right)$ and $\delta(f)=\operatorname{ord}\left(\mathcal{W}^{\circ}\right)+$ $\operatorname{ord}\left(\mathcal{H}^{\circ}\right)$. By virtue of the shape and location of $\mathcal{O}$ in the plane, the number of interior lattice points of $\mathcal{W}$ and $\mathcal{W}^{\top}$ satisfy the inequality $\operatorname{ord}\left(\mathcal{W}^{\circ}\right) \leqslant \operatorname{ord}\left(\mathcal{W}^{\top}\right)$. The Milnor-Jung formula implies $\mu_{\mathrm{alg}}(f)=2\left(\operatorname{ord}\left(\mathcal{W}^{\circ}\right)+\operatorname{ord}\left(\mathcal{H}^{\circ}\right)\right)-r(f)+1$.

In two dimensions, for any positive weights,

$$
\begin{align*}
\mu_{\mathrm{nt}}(f)= & \mu_{\mathrm{alg}}(f)+\left(\frac{1}{\omega_{1}}-1\right)\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{2}}\right)-\left\{\frac{1}{\omega_{2}}\right\}\right)+\left(\frac{1}{\omega_{2}}-1\right)\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right)-\left\{\frac{1}{\omega_{1}}\right\}\right) \\
& +\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right)-\left\{\frac{1}{\omega_{1}}\right\}\right)\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{2}}\right)-\left\{\frac{1}{\omega_{2}}\right\}\right) . \tag{5.101}
\end{align*}
$$

Thus,

$$
\begin{align*}
r(f)= & \operatorname{ord}\left(\mathcal{W}^{\circ}\right)-\operatorname{ord}\left(\mathcal{W}^{\top \circ}\right)+\operatorname{ord}\left(\mathcal{H}^{\circ}\right)+1 \\
& +\left(\frac{1}{\omega_{1}}-1\right)\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{2}}\right)-\left\{\frac{1}{\omega_{2}}\right\}\right)+\left(\frac{1}{\omega_{2}}-1\right)\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right)-\left\{\frac{1}{\omega_{1}}\right\}\right) \\
& +\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{1}}\right)-\left\{\frac{1}{\omega_{1}}\right\}\right)\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{+}\left(\frac{1}{\omega_{2}}\right)-\left\{\frac{1}{\omega_{2}}\right\}\right), \tag{5.102}
\end{align*}
$$

which is the claimed identity.
Corollary 5.31. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, quasi-BrieskornPham polynomial, then the generalized branch number satisfies

$$
\begin{equation*}
r(f)=\operatorname{ord}\left(\mathcal{H}^{\circ}\right)+1 \tag{5.103}
\end{equation*}
$$

Proof. In this case, $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{nt}}(f)$ and the orthotopal $n$-polytope $\mathcal{O}(f)$ is integral, so $\operatorname{ord}\left(\mathcal{W}^{\circ}\right)=\operatorname{ord}\left(\mathcal{W}^{\top \circ}\right)$ and $\Delta(\boldsymbol{\omega})=0$. Alternatively, by Proposition 5.20, the following difference implies the claim:

$$
\begin{align*}
0 & =\mu_{\mathrm{nt}}(f)-\mu_{\mathrm{alg}}(f)  \tag{5.104}\\
& =\left(\operatorname{ord}\left(\mathcal{W}^{\top \circ}\right)-\operatorname{ord}\left(\mathcal{W}^{\circ}\right)\right)-\operatorname{ord}\left(\mathcal{H}^{\circ}\right)+r(f)-1 \tag{5.105}
\end{align*}
$$

REMARK 5.5.8. The equality $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{nt}}(f)$ is sufficient but not necessary to ensure $r(f)=\operatorname{ord}\left(\mathcal{H}^{\circ}\right)+1$. For instance, with the weights $\left\{\frac{2}{7}, \frac{1}{15}\right\}, \mu_{\mathrm{alg}}(f)=$ 35 and $\mu_{\mathrm{nt}}(f)=42$, while $r(f)=2$ and $\operatorname{ord}\left(\mathcal{H}^{\circ}\right)=1$.

REMARK 5.5.9. Consider the $T_{6,4}$ link. Equations (5.7ob) and (5.82c) with $\{a, b\}=\{6,4\}$ yield $u\left(T_{6,4}\right)=10-\sum_{i=1}^{6}\left\{4\left(1-\frac{i}{6}\right)\right\}=8$, which coincides with the 8 positive lattice points of $\mathcal{W}(f)$, and $r(f)=\operatorname{GCD}(6,4)=2$, the number of components of $T_{6,4}$, which is one more than the number of positive lattice points on the hypotenuse $\mathcal{H}(f)$ (see Figure 5.3).


Figure 5.3. The Positive Lattice Points in an Integral Weight Polytope

Corollary 5.32. For positive integers $a$ and $b$,

$$
\begin{equation*}
\sum_{i=1}^{a-1} \chi_{\mathbb{R} / \mathbb{Z}}^{+}\left(b\left(1-\frac{i}{a}\right)\right)=a-\operatorname{GCD}(a, b) \tag{5.106}
\end{equation*}
$$

Proof. Suppose $f$ is quasi-Brieskorn-Pham with inverse exponents $a$ and b. Then $r(f)=\operatorname{ord}\left(\mathcal{H}^{\circ}\right)+1$, which implies

$$
\begin{align*}
\sum_{i=1}^{a-1} \chi_{\mathbb{R} / \mathbb{Z}}^{+}\left(b\left(1-\frac{i}{a}\right)\right) & =2 \sum_{i=1}^{a}\left\{b\left(1-\frac{i}{a}\right)\right\}  \tag{5.107}\\
& =a-\operatorname{GCD}(a, b) \tag{5.108}
\end{align*}
$$

5.5.3. Topological Determinacy, Revisited. Recall that if $f:\left(\mathbb{C}^{n}, 0\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial with an isolated critical point at the origin, then in the neighborhood of the origin there is a positive constant $\varepsilon$ and smallest, positive exponent, the Łojasiewicz exponent, $\ell=\ell_{0}(f)$, such that $|\partial f| \geqslant \varepsilon|z|^{\ell}$. In 2010, Tan, Yau and Zuo [452] prove that $\ell_{0}(f)$ depends only on the weights and is given by $\max \left\{\frac{1}{\omega_{1}}-1, \ldots, \frac{1}{\omega_{n}}-1\right\}$. Teissier proves that $\ell_{0}(f)+1$ is the maximal polar invariant of $f$ and depends only on its topological type [454]. The integer $\left\lfloor\ell_{0}(f)\right\rfloor+1$ is known as the topological determinacy order of $f$.

Proposition 5.33. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial with weights satisfying

$$
\begin{equation*}
\frac{\omega_{1}}{\omega_{2}}\left(\left\{\frac{1}{\omega_{1}}\right\}-\left\{\frac{1}{\omega_{1}}\right\}^{2}\right) \leqslant 2 \sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\} \tag{5.109}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\omega_{2}}{\omega_{1}}\left(\left\{\frac{1}{\omega_{2}}\right\}-\left\{\frac{1}{\omega_{2}}\right\}^{2}\right) \leqslant 2 \sum_{i=1}^{\left\lfloor 1 / \omega_{2}\right\rfloor}\left\{\frac{1-i \omega_{2}}{\omega_{1}}\right\} \tag{5.110}
\end{equation*}
$$

then the generalized branch number bounds the topological determinacy order,

$$
\begin{equation*}
r(f) \leqslant\left\lfloor\ell_{0}(f)\right\rfloor+1 \tag{5.111}
\end{equation*}
$$

Proof. The inequalities imply $r(f) \leqslant \min \left\{\frac{1}{\omega_{1}}, \frac{1}{\omega_{2}}\right\}$. By the Proposition 5.28,

$$
\begin{align*}
r(f)-1 & \leqslant \min \left\{\frac{1}{\omega_{1}}, \frac{1}{\omega_{2}}\right\}-1  \tag{5.112}\\
& \leqslant \max \left\{\frac{1}{\omega_{1}}, \frac{1}{\omega_{2}}\right\}-1  \tag{5.113}\\
& =\ell_{0}(f) \tag{5.114}
\end{align*}
$$

as claimed. Since $r(f)$ is an integer, the claimed inequality follows.
The aforementioned bound is sharp.

Corollary 5.34. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, quasi-BrieskornPham polynomial, then the generalized branch number bounds the topological determinacy order,

$$
\begin{equation*}
r(f) \leqslant\left\lfloor\ell_{0}(f)\right\rfloor+1 \tag{5.115}
\end{equation*}
$$

Proof. In the case that the weights of $f$ are inverse integers, then inequalities above imply non-negativity of two fractional part summations, which is
trivially true by the non-negativity of the fractional part function on the nonnegative real axis, that is, $\{x\} \geqslant 0$ for $x \geqslant 0$.
5.5.4. Milnor Conjecture. In §10 in [310], Milnor proves a relation between the dimension of the local algebra and the number of double points and branching number of a complex plane curve. Recall that a square-free polynomial is a polynomial with no repeated roots. Let $f$ be a square-free polynomial of two complex variables. Let $C=f^{-1}(0)$ denote the corresponding hypersurface with $\delta=\delta_{C}$ double points at the origin and $r=r_{C}$ number of local analytic branches passing through the origin. Then

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=2 \delta-r+1 \tag{5.116}
\end{equation*}
$$

Equation (5.116) is also known as the Milnor-Jung formula. In the same chapter, Milnor conjectures that $\delta$ is the unknotting number of the corresponding algebraic link and, therefore, also the genus of the associated Milnor fiber.

In 1992, Kronheimer and Mrowka succeeded in proving the Milnor Conjecture as a consequence of proving the genus minimizing property of complex curves in a K3 surface using Donaldson invariants [247, 248]. Subsequently, in 1994, Kronheimer and Mrowka proved the Thom Conjecture on genusminimizing complex curves in the projective plane using Seiberg-Witten theory. It follows that if $K_{f}$ is a knot, i.e., $r=1$, then

$$
\begin{equation*}
u\left(K_{f}\right)=\delta(f)=\frac{1}{2} \mu=\frac{1}{2} \operatorname{deg} \Delta_{h_{*}}(t)=g\left(K_{f}\right)=g\left(F_{f, 0}\right) \tag{5.117}
\end{equation*}
$$

If $K_{f}$ is a link, i.e., $r>1$, then

$$
\begin{align*}
& u\left(K_{f}\right)=\delta(f)=\frac{1}{2}(\mu+r-1)  \tag{5.118}\\
& g\left(K_{f}\right) \geqslant \frac{1}{2} \mu=\frac{1}{2} \operatorname{deg} \Delta_{h_{*}}(t) \tag{5.119}
\end{align*}
$$

Remark 5.5.10. With regard to torus link invariants, the branch number of the Brieskorn-Pham singularity $f=x^{p}+y^{q}$ is precisely GCD $(p, q)$, which coincides with the number of components of the corresponding torus link $T_{p, q}$. Equation (5.70b) implies

$$
\begin{equation*}
u\left(T_{p, q}\right)=\frac{1}{2}(p q-p-q+\operatorname{GCD}(p, q)) \tag{5.120}
\end{equation*}
$$

while, in particular, if $p$ and $q$ are coprime,

$$
\begin{equation*}
u\left(T_{p, q}\right)=g\left(T_{p, q}\right)=\frac{1}{2}(p-1)(q-1), \tag{5.121}
\end{equation*}
$$

which is one-half of the corresponding algebraic index or, equivalently, one-half of the degree of the corresponding Alexander polynomial $\Delta_{T_{p, q}}(t)$.

Remark 5.5.11. In Example 2 following Theorem 7.3 of [312], Milnor gives two complex analytic germs over $\mathbb{C}^{3}$ which have isotopic links (hence equal link invariants) but different algebraic indices (hence non-diffeomorphic fibers), namely, $f=x^{2}+y^{9}+z^{18}$ (which corresponds to the 2-fold cyclic branched covering over the torus link $T_{9,18}$ ) and $g=x^{3}+y^{5}+z^{15}$ (which corresponds to the 3 -fold cyclic branched covering over the torus link $T_{5,15}$ ) with $\mu(f)=$ $2^{3} \cdot 17=136$ and $\mu(g)=2^{4} \cdot 7=112$, respectively.

### 5.6. Arithmetic and Geometric Genera

5.6.1. Arithmetic Genus. Let $V \subset \mathbb{P}^{n}$ be a non-singular, irreducible, projective algebraic variety of dimension $n$ defined by an ideal $I(V)$ of polynomials which vanish on $V$. Let $P_{V}(t)$ denote the Hilbert polynomial of the coordinate ring $\mathbb{C}\left[z_{0}, \ldots, z_{n}\right] / I(V)$. Hilbert proved that if $C$ is a non-singular, complex plane curve, then $g(C)=P_{C}(0)-1$, where $P_{C}(t)$ is the Hilbert polynomial of $C$ [1], [193], [194]. The arithmetic genus $p_{a}(V)$ is the normalized constant term of $P_{V}(0)$, namely, $p_{a}(V)=(-1)^{n}\left(P_{V}(0)-1\right)$ [1], [162], [177]. The arithmetic genus does not depend on the projective embedding of $V$.

Remark 5.6.1. Given a projective hypersurface $V \subset \mathbb{P}^{n}$ of degree $d$, the Hilbert polynomial of $V$ is $P_{V}(t)=\binom{t+n}{n}-\binom{t-d+n}{n}$ [ $\mathbf{1}$ ]. Thus, the arithmetic genus $p_{a}(V)=(-1)^{n}\left(P_{V}(0)-1\right)=\binom{d-1}{n}$, which coincides with the genus $g(V)$.

Remark 5.6.2. Given a projective complete intersection $V \subset \mathbb{P}^{n}$ defined by the ideal $I(V)=\left(f_{1}, \ldots, f_{m}\right)$, where $\operatorname{deg} f_{i}=a_{i} \in \mathbb{N}$, the Hilbert polynomial of $V$ is

$$
\begin{equation*}
P_{V}(t)=\binom{t+n}{n}+\sum_{k=1}^{m}(-1)^{k} \sum_{1 \leqslant i_{1}<\cdots<i_{k} \leqslant m}\binom{t-\left(a_{i}+\cdots+a_{k}\right)+n}{n} \tag{5.122}
\end{equation*}
$$

Thus, the arithmetic genus is the alternating summation

$$
\begin{equation*}
p_{a}(V)=\sum_{k=1}^{m}(-1)^{n-k} \sum_{1 \leqslant i_{1}<\cdots<i_{k} \leqslant m}\binom{-\left(a_{i}+\cdots+a_{k}\right)+n}{n}, \tag{5.123}
\end{equation*}
$$

where $\binom{-n}{k}=(-1)^{k}\binom{n+k-1}{k}$ for $n \in \mathbb{Z}$ and $k \in \mathbb{Z}_{\geqslant 0}$. Taking $m=1$ and $\operatorname{deg} f_{1}=d$ yields the arithmetic genus computed in Remark (5.6.1).

Let $\left(X, \mathcal{O}_{X}\right)$ denote a topological space $X$ with a structure sheaf $\mathcal{O}_{X}$ of rings. Suppose $X$ be a compact complex manifold of (complex) dimension $n$, and let $\chi\left(X, \mathcal{O}_{X}\right)$ denote the Euler characteristic (in the coherent cohomology) of the structure sheaf $\mathcal{O}_{X}$, which is equal to the holomorphic Euler characteristic of the trivial line bundle of $X$. Let $h^{p, q}=\operatorname{dim}_{C} H^{p}\left(X, \mathcal{O}_{X}\right)$ denote the $(p, q)$-Hodge number of $X$. The (Severi-Kodaira-Spencer) arithmetic genus $p_{a}(X)$ is the alternating summation of hodge numbers,

$$
\begin{align*}
p_{a}(X) & =\sum_{k=1}^{n}(-1)^{n-k} h^{k, 0}  \tag{5.124}\\
& =(-1)^{n} \tilde{\chi}\left(X, \mathcal{O}_{X}\right) \tag{5.125}
\end{align*}
$$

which generalizes the arithmetic genus ${ }^{*}$ for non-singular, irreducible, projective algebraic varieties. Since $\tilde{\chi}\left(X, \mathcal{O}_{X}\right)$ is a birational invariant, so is $p_{a}(X)$. See [162] for related details.
5.6.2. Geometric Genus. Let $\Omega^{q}$ denote the sheaf of holomorphic $q$ forms on $X$. Define the geometric genus $p_{g}(X)$ as the hodge number $h^{n, 0}$, i.e., the complex dimension of the sheaf cohomology group $H^{0}\left(X, \Omega^{n}\right) \cong H^{n}(X, \Omega)$, the number of linearly independent holomorphic (top) $n$-forms on $X$.
*In his book [203], Hirzebruch defines $\chi\left(X, \mathcal{O}_{X}\right)$ as the arithmetic genus.

The geometric genus is the first of the series of plurigenera and is closely related to the arithmetic genus. However, while the arithmetic and geometric genera are both integers, only the geometric genus is a priori non-negative.

Proposition 5.35. If $X$ is a smooth, projective algebraic complex curve, then the arithmetic and geometric genera coincide,

$$
\begin{equation*}
p_{a}(X)=p_{g}(X) \tag{5.126}
\end{equation*}
$$

Remark 5.6.3. The identity of Proposition 5.35 holds even for smooth, projective algebraic curve over any algebraically closed field.

REMARK 5.6.4. For a projective plane curve $C$ with singular points $\Sigma(C)$, the aforementioned genera are related by the following formula,

$$
\begin{equation*}
p_{g}(C)=p_{a}(C)-\sum_{p \in \Sigma(C)} \delta_{p}(C) \tag{5.127}
\end{equation*}
$$

where $\delta_{p}(C)=\operatorname{dim} \tilde{R}_{p} / R_{p}$ denotes the local delta invariant of $C$ at $p$ where $R_{p}$ is the local ring of $C$ at $p$ and $\tilde{R}_{p}$ is its normalization [104].

In the context of complex analytic germs in $\mathbb{C}^{3}$ with isolated critical points at the origin and small arithmetic genera, Yau proves an invariance of the corresponding multiplicities [487].

Proposition 5.36 (Yau, [487]). If $f, g:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ are non-degenerate complex analytic germs such that $\left(\mathbb{C}^{3}, V_{f, 0}\right)$ and $\left(\mathbb{C}^{3}, V_{g, 0}\right)$ are locally homeomorphic at the origin and $p_{a}(f) \leqslant 2$, then the multiplicities of $f$ and $g$ coincide, i.e., $v(f)=v(g)$.
5.6.2.1. Genera of Quasihomogeneous Hypersurfaces. Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a non-degenerate, weighted homogeneous polynomial with weights $\left\{\omega_{i}\right\} \subset$ $\mathbf{Q} \cap(0,1)$. Suppose $n>2$ and consider the quasihomogeneous hypersurface $V_{f, 0}=f^{-1}(0)$ and an arbitrary resolution $\pi: \tilde{V}_{f, 0} \rightarrow V_{f, 0}$. The arithmetic genus $p_{a}(f)=p_{a}\left(\tilde{V}_{f, 0}\right)$ and geometric genus $p_{g}(f)=p_{g}\left(\tilde{V}_{f, 0}\right)=\operatorname{dim}_{\mathbb{C}} H^{n-1}\left(\tilde{V}_{f, 0}, \Omega\right)$ do not depend on the resolution.

Merle and Teissier [301] (and perhaps also Watanabe) gave the following formula for the geometric genus $p_{g}(f)$ of a weighted homogeneous hypersurface $V_{f, 0}$ as the size of the solution set of a Diophantine inequality determined by the integral weights $\left\{q_{i}\right\}$ and weighted degree $d$,

$$
\begin{align*}
p_{g}(f) & =\left|\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{Z}_{\geqslant 0}^{n+1} \mid \sum_{i=0}^{n} q_{i} x_{i} \leqslant d-\sum_{i=0}^{n} q_{i}\right\}\right|  \tag{5.128}\\
& =\left|\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{N}^{n+1} \mid \sum_{i=0}^{n} q_{i} x_{i} \leqslant d\right\}\right| \tag{5.129}
\end{align*}
$$

Proposition 5.37. Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a non-degenerate weighted homogeneous polynomial with integral weights $\left\{q_{0}, \ldots, q_{n}\right\}$ and weighted degree $d$. Consider the rational polytope $\mathcal{P}=\mathbf{c o n v}\left\{\mathbf{0}, a_{0} \mathbf{e}_{0}, \ldots, a_{n} \mathbf{e}_{n}\right\}$, where $a_{i}$ satisfies $q_{i} a_{i}=d-\sum_{k=0}^{n} q_{k}$. The geometric genus $p_{g}\left(f_{t}\right)$ of the $t$-dilate $f_{t}$ is equal to the Ehrhart function $\mathscr{L}_{\mathcal{P}}(t)=\left|t \mathcal{P} \cap \mathbb{Z}_{\geqslant 0}^{n+1}\right|$ of the $t$-dilate of $\mathcal{P}$.

Proof. The polytope $\mathcal{P}$ admits the equivalent definition as the locus

$$
\begin{equation*}
\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{R}_{\geqslant 0}^{n+1} \left\lvert\, \sum_{i=0}^{n} \frac{x_{i}}{a_{i}} \leqslant 1\right.\right\} . \tag{5.130}
\end{equation*}
$$

Thus, the Ehrhart function $\mathscr{L}_{\mathcal{P}}(t)$ counts the cardinality of the set $\left\{\left(x_{0}, \ldots, x_{n}\right) \in\right.$ $\left.\mathbb{Z}_{\geqslant 0}^{n+1} \left\lvert\, \sum_{i=0}^{n} \frac{x_{i}}{a_{i}} \leqslant t\right.\right\}$. Compare this to equation (5.128).
5.6.2.2. A Linear Diophantine Inequality and the Geometric Genus. A closed analytic form of the number $N_{\mathbf{a}}(R)$ of non-negative integral solutions of the Diophantine inequality $0 \leqslant a_{1} x_{1}+\cdots+a_{n} x_{n} \leqslant R$ with relatively prime, positive integral coefficients $\left\{a_{i}\right\}$ and positive bounding integer $R$ has recently been proposed by Mahmoudvand, Hassani, Farzaneh and Howell [282].

The essential step in their proof is to introduce an additional variable $x_{n+1}$, transform the inequality into the equality $a_{1} x_{1}+\cdots+a_{n} x_{n}+x_{n+1}=R$ and simply enumerate the non-negative solutions of the transformed equation inductively as a function of $n$, as the number of solutions of the equality is bijective with those of the inequality. The number of said solutions is computed using the iterated summation,

$$
\begin{equation*}
N_{\mathbf{a}}(R)=\sum_{i_{1}=0}^{\left\lfloor R / a_{1}\right\rfloor\left\lfloor\left(R-a_{1} i_{1}\right) / a_{2}\right\rfloor} \sum_{i_{2}=0}^{\left\lfloor\left(R-\sum_{k=1}^{n-1} a_{k} i_{k}\right) / a_{n}\right\rfloor} \sum_{i_{n}=0} 1, \tag{5.131}
\end{equation*}
$$

where $\lfloor\cdot\rfloor$ denotes the floor function. In Volume 2, we generalize this formula to positive real coefficients. Combining this result with that of Merle, Teissier and Watanabe [301], we determine an explicit formula for the geometric genus in terms of the weights.

Proposition 5.38. Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a non-degenerate, weighted homogeneous polynomial with integral weights $\left\{q_{0}, \ldots, q_{n}\right\}$ and weighted degree $d$.

Define the Gorenstein parameter $R=d-\sum_{i=0}^{n} q_{i}$. The geometric genus $p_{g}(f)=$ $p_{g}\left(V_{f, 0}\right)$ is given by the iterated summation

$$
\begin{equation*}
p_{g}(f)=\sum_{i_{0}=0}^{\left\lfloor R / q_{0}\right\rfloor} \cdots \sum_{i_{n}=0}^{\left\lfloor\left(R-\sum_{k=0}^{n-1} q_{k} i_{k}\right) / q_{n}\right\rfloor} 1 \tag{5.132}
\end{equation*}
$$

Proof. Apply equation (5.131) to equation (5.128).

REMARK 5.6.5. If $\mathcal{W}$ is an elementary 3-simplex, then the number of positive lattice points is $\mathscr{L}_{\mathcal{W}}\left(t ; \mathbb{N}^{3}\right)=\binom{t}{3}$. More generally, if given a homogeneous polynomial $f=\sum_{i=0}^{n} z_{i}^{d}$ of degree $d$, then the corresponding polytope is the $d$-dilate of an $(n+1)$-simplex, and the geometric genus is

$$
\begin{equation*}
p_{g}\left(f_{t}\right)=\frac{1}{(n+1)!} \sum_{k=0}^{n+1} s(n+1, k)(d t)^{k}=\binom{d t}{n+1} \tag{5.133}
\end{equation*}
$$

Remark 5.6.6. To determine the asymptotic behavior of the geometric genus of a weighted homogeneous polynomial of $n+1$ indeterminates, we employ the Euler-Maclaurin summation formula,

$$
\begin{align*}
p_{g}(f) & =\sum_{i_{0}=0}^{\left\lfloor R / q_{0}\right\rfloor} \cdots \sum_{i_{n}=0}^{\left\lfloor\left(R-\sum_{k=0}^{n-1} i_{k} / q_{k}\right) / q_{n}\right\rfloor} 1  \tag{5.134}\\
& =\frac{R^{n+1}}{(n+1)!q_{0} \cdots q_{n}}+O\left(R^{n}\right) \tag{5.135}
\end{align*}
$$

which indicates that $p_{g}(f)=O\left(d^{n+1}\right)$.

Proposition 5.39 (Steenbrink, [437]). If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial with spectrum $\operatorname{Sp}(f)$, then

$$
\begin{equation*}
p_{g}(f)=\left|\operatorname{Sp}(f) \cap Q_{\leqslant 1}\right| . \tag{5.136}
\end{equation*}
$$

Define the generalized geometric genus

$$
p_{g}(f)= \begin{cases}\operatorname{dim}_{\mathbb{C}} R^{n-1} \rho_{*} \mathcal{O}_{\tilde{V}_{f, 0}} & n \geqslant 2  \tag{5.137}\\ \operatorname{dim}_{\mathbb{C}} \rho_{*} \mathcal{O}_{\tilde{V}_{f, 0}} / \mathcal{O}_{V_{f, 0}} & n=1\end{cases}
$$

where, for $n=1, p_{g}(f)=\delta(f)$, the delta invariant [410]. The definitions of the ordinary and generalized geometric genus coincide for $n \geqslant 2$.

Proposition 5.40 (Durfee, [118]; Saito, [410]). If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous singularity, then the generalized geometric genus of $f$ satisfies

$$
2 p_{g}(f)=\left\{\begin{array}{cc}
\varsigma_{+}+\varsigma_{0} & n=2  \tag{5.138}\\
\varsigma_{+}+2 \varsigma_{0}+\varsigma_{-} & n=1
\end{array}\right.
$$

where $\varsigma_{+}, \varsigma_{0}$ and $\varsigma_{-}$are the number of positive, zero and negative eigenvalues of the corresponding intersection form $S$.

Remark 5.6.7. For $n=1$, since $p_{g}(f)=\delta(f)$ and $\mu_{\mathrm{alg}}(f)=\varsigma_{+}+\varsigma_{0}+\varsigma_{-}$, it follows by the Milnor-Jung formula that the branch number satisfies $r(f)=$ $\varsigma_{0}+1$.

The previous remark, Proposition 5.30 and Corollary 5.31 imply the following result.

Corollary 5.41. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial with reduced weights $\boldsymbol{\omega}=\left\{\omega_{1}, \omega_{2}\right\}$, then

$$
\begin{equation*}
\varsigma_{0}=\operatorname{ord}\left(\mathcal{H}^{\circ}\right)+\Delta(\boldsymbol{\omega}), \tag{5.139}
\end{equation*}
$$

where $\mathcal{H}^{\circ}$ is the interior of the hypotenuse of the weight polytope $\mathcal{W}(f)$ and $\Delta(\boldsymbol{\omega})$ is defined in Proposition 5.30. In particular, if $f$ is a quasi-Brieskorn-Pham polynomial, then

$$
\begin{equation*}
\varsigma_{0}=\operatorname{ord}\left(\mathcal{H}^{\circ}\right) \tag{5.140}
\end{equation*}
$$

Thus, for $n=1$, the eigenvalue signature $\left(\varsigma_{+}, \varsigma_{0}, \varsigma_{-}\right)$of the corresponding intersection form $S$ admits the following exact representation in terms of the weights,

$$
\begin{align*}
\varsigma_{0} & =\frac{1}{\omega_{1}}-1+\frac{\omega_{1}}{\omega_{2}}\left(\left\{\frac{1}{\omega_{1}}\right\}-\left\{\frac{1}{\omega_{1}}\right\}^{2}\right)-2 \sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}  \tag{5.141}\\
& =\frac{1}{\omega_{2}}-1+\frac{\omega_{2}}{\omega_{1}}\left(\left\{\frac{1}{\omega_{2}}\right\}-\left\{\frac{1}{\omega_{2}}\right\}^{2}\right)-2 \sum_{i=1}^{\left\lfloor 1 / \omega_{2}\right\rfloor}\left\{\frac{1-i \omega_{2}}{\omega_{1}}\right\} \tag{5.142}
\end{align*}
$$

and, since the signature is zero, $\varsigma_{+}=\varsigma_{-}$, so

$$
\begin{align*}
\varsigma_{ \pm}=1 & -\frac{1}{\omega_{1}}-\frac{1}{2 \omega_{2}}+\frac{1}{2 \omega_{1} \omega_{2}}-\frac{\omega_{1}}{2 \omega_{2}}\left(\left\{\frac{1}{\omega_{1}}\right\}+\left\{\frac{1}{\omega_{1}}\right\}^{2}\right) \\
& +\sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}  \tag{5.143}\\
=1 & -\frac{1}{\omega_{2}}-\frac{1}{2 \omega_{1}}+\frac{1}{2 \omega_{1} \omega_{2}}-\frac{\omega_{2}}{2 \omega_{1}}\left(\left\{\frac{1}{\omega_{2}}\right\}+\left\{\frac{1}{\omega_{2}}\right\}^{2}\right) \\
& +\sum_{i=1}^{\left\lfloor 1 / \omega_{2}\right\rfloor}\left\{\frac{1-i \omega_{2}}{\omega_{1}}\right\} . \tag{5.144}
\end{align*}
$$

REmARK 5.6.8. Suppose $f$ is a quasi-Brieskorn-Pham singularity with inverse weights $\{p, q\}$. Then $\varsigma_{0}=\operatorname{GCD}(p, q)-1$ and

$$
\begin{align*}
\varsigma_{ \pm} & =1-p-\frac{q}{2}+\frac{p q}{2}+\sum_{i=1}^{p}\left\{q-\frac{i q}{p}\right\}  \tag{5.145}\\
& =1-q-\frac{p}{2}+\frac{p q}{2}+\sum_{i=1}^{q}\left\{p-\frac{i p}{q}\right\}  \tag{5.146}\\
& =\frac{1}{2}(p q-p-q-\operatorname{GCD}(p, q))+1 \tag{5.147}
\end{align*}
$$

Corollary 5.42. For $p, q \in \mathbb{N}$,

$$
\begin{align*}
& \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \operatorname{sgn}^{ \pm} \sin \left(\pi\left(\frac{i}{p}+\frac{j}{q}\right)\right)=\frac{1}{2}(p q-p-q-\operatorname{GCD}(p, q))+1  \tag{5.148}\\
& \sum_{i=1}^{p-1} \sum_{j=1}^{q-1} \operatorname{sgn}^{0} \sin \left(\pi\left(\frac{i}{p}+\frac{j}{q}\right)\right)=\operatorname{GCD}(p, q)-1 \tag{5.149}
\end{align*}
$$

In particular, for $d \in \mathbb{N}$,

$$
\begin{align*}
& \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \operatorname{sgn}^{ \pm} \sin \left(\pi\left(\frac{i+j}{d}\right)\right)=\binom{d}{2}  \tag{5.150}\\
& d-1  \tag{5.151}\\
& \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} \operatorname{sgn}^{0} \sin \left(\pi\left(\frac{i+j}{d}\right)\right)=d-1
\end{align*}
$$

### 5.7. Geometric Genus of Weighted Homogeneous Surface Singularities

Although equation (5.132) gives an effective means of computing the geometric genus, it does little to illustrate the curious behavior of the geometric genus as a function of the weights. In short, while this formula is interesting, it is nevertheless useless.

An equivalent way to compute the geometric genus is to count the number of positive integer solutions $\left(x_{0}, \ldots, x_{n}\right)$ of $\sum_{i=0}^{n} \omega_{i} x_{i} \leqslant 1$ with rational weights $\left\{\omega_{i}\right\}$, that is, the Ehrhart function of the positive part of the weight polytope $\mathcal{W}(f)$. It follows that the geometric genus is a higher dimensional analogue of the delta invariant of a complex plane curve.

By simplifying the Ehrhart function counting the positive lattice points of the weight polytope, the geometric genus is expressible exactly in terms of the weights as a quasi-polynomial of degree $n+1$ in the dilation variable $t$. That is, after simplifying the iterated summation, repeatedly using the identity $\lfloor x\rfloor=$ $x-\{x\}$ for $x \in \mathbb{R}_{\geqslant 0}$ and using elementary summation identities, one arrives at an expression of the form $p_{g}\left(f_{t}\right)=\sum_{k=0}^{n+1} c_{k}(t) t^{k}$, where $c_{k}(t)$ are lattice-periodic $Q$-valued functions.

Proposition 5.43. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate, weighted homogeneous polynomial with weight polytope $\mathcal{W}(f)=\boldsymbol{\operatorname { c o n v }}\left\{\mathbf{0}, \frac{\mathbf{e}_{0}}{\omega_{0}}, \ldots, \frac{\mathbf{e}_{n}}{\omega_{n}}\right\}$, then $p_{g}\left(f_{t}\right)$ is a quasi-polynomial of degree $n+1$ in the variable $t$ and satisfies

$$
\begin{equation*}
\lim _{t \rightarrow \infty} t^{-n-1} p_{g}\left(f_{t}\right)=\frac{1}{(n+1)!} \prod_{i=0}^{n} \frac{1}{\omega_{i}} . \tag{5.152}
\end{equation*}
$$

Proof. The claim follows from the representation

$$
\begin{align*}
p_{g}\left(f_{t}\right) & =\left|t \mathcal{W}(f) \cap \mathbb{N}^{n+1}\right|  \tag{5.153}\\
& =\sum_{i_{0}=1}^{\left\lfloor t / \omega_{0}\right\rfloor} \cdots \sum_{i_{n}=1}^{\left\lfloor\left(t-\sum_{k=0}^{n-1} i_{k} \omega_{k}\right) / \omega_{n}\right\rfloor} 1, \tag{5.154}
\end{align*}
$$

which is a quasi-polynomial of degree $n+1$ in $t$ with leading coefficient equal to the $(n+1)$-content of the weight polytope, $v i z ., \operatorname{vol}_{n+1} \mathcal{W}(f)=\frac{1}{(n+1)!} \prod_{i=0}^{n} \frac{1}{\omega_{i}}$.

In Volume 2, we compute the Ehrhart function $\mathscr{L}_{\mathcal{P}}\left(t, \mathbb{N}^{3}\right)=\left|t \mathcal{P} \cap \mathbb{N}^{3}\right|$, where $\mathcal{P}=\boldsymbol{\operatorname { c o n v }}\left\{\mathbf{0}, a_{1} \mathbf{e}_{1}, a_{2} \mathbf{e}_{2}, a_{3} \mathbf{e}_{3}\right\}$, a real trirectangular tetrahedron. As a
consequence, we prove an exact expression for the geometric genus of nondegenerate, weighted homogeneous surface singularities.

Proposition 5.44. The geometric genus $p_{g}\left(f_{t}\right)$ of the (positive integral) t-dilate of a non-degenerate, weighted homogeneous surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with weights $\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}$ admits the following representation,

$$
\begin{align*}
p_{g}\left(f_{t}\right)= & \sum_{i=1}^{\left.\left\lfloor t / \omega_{1}\right\rfloor\left\lfloor\left(t-i \omega_{1}\right) / \omega_{2}\right\rfloor\left(t-i \omega_{1}-j \omega_{2}\right) / \omega_{3}\right\rfloor} \sum_{k=1} 1  \tag{5.155a}\\
= & \left(\frac{1}{6 \omega_{1} \omega_{2} \omega_{3}}\right) t^{3}-\frac{1}{4 \omega_{3}}\left(\frac{1}{\omega_{1}}+\frac{1}{\omega_{2}}\right) t^{2}+\frac{1}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{3 \omega_{2}}\right) t \\
& -\frac{\omega_{1}}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{3 \omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\}+\frac{\omega_{1}}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{\omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\}^{2}-\frac{\omega_{1}^{2}}{6 \omega_{2} \omega_{3}}\left\{\frac{t}{\omega_{1}}\right\}^{3} \\
& +\frac{\omega_{2}}{2 \omega_{3}} \sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor}\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\left(1-\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\right) \\
& -\sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor\left\lfloor\left(t-i \omega_{1}\right) / \omega_{2}\right\rfloor} \sum_{j=1}\left\{\frac{t-i \omega_{1}-j \omega_{2}}{\omega_{3}}\right\} \tag{5.155~b}
\end{align*}
$$

Remark 5.7.1. As the corresponding Ehrhart function is defined on a larger domain, equation ( 5.155 b ) holds for any permutation of the weights and for real $t$-dilates of weighted homogeneous, finite Puiseux series.

Alternatively, the geometric genus $p_{g}\left(f_{t}\right)$ may be defined by equation (5.155b) for all real $t \geqslant 0$. In fact, any singularity invariant which depends explicitly on an Ehrhart function of the weight polytope admits a similar generalization.

Proposition 5.45. A non-degenerate, weighted homogeneous polynomial over $\mathbb{C}^{n+1}$ with weights that the sum greater unity has vanishing geometric genus.

Proof. If the Gorenstein parameter $R<0$, then there are no solutions for the Diophantine inequality $\sum_{i=0}^{n} q_{i} x_{i} \leqslant R$. Since $R=d-\sum_{i=0}^{n} q_{i}$, the negativity of $R$ is equivalent to the $\sum_{i=0}^{n} q_{i}>d$, that is, $\sum_{i=0}^{n} \omega_{i}>1$. Therefore, $\mathcal{W}=$ $\operatorname{conv}\left\{0, \frac{\mathbf{e}_{0}}{\omega_{0}}, \ldots, \frac{\mathbf{e}_{n}}{\omega_{n}}\right\}$ is a rational, orthotopal simplicial polytope that does not intersect any positive lattice points.

Remark 5.7.2. The converse of Proposition 5.45 for $n=2$ was shown by Yoshinaga and Watanabe [494]. More generally, a normal two-dimensional singularity has zero geometric genus if and only if it is rational [482].

Recall that the generalized geometric genus is well-defined for weighted homogeneous, finite Puiseux series.

Proposition 5.46. Given a square-free, non-degenerate weighted homogeneous polynomial $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with weights $\left\{\omega_{1}, \omega_{2}\right\}$, let $\tilde{f}:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be any weighted homogeneous polynomial with weights $\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}$. The geometric genus of $\tilde{f}$ and the delta invariant of $f$ satisfy the identity,

$$
\begin{equation*}
p_{g}\left(\tilde{f}_{t+\omega_{3}}\right)=p_{g}\left(\tilde{f}_{t}\right)+\delta\left(f_{t}\right) \tag{5.156}
\end{equation*}
$$

In particular, if $\omega_{3}=1$, then

$$
\begin{equation*}
p_{g}\left(\tilde{f}_{t+1}\right)=\sum_{l=1}^{t} \delta\left(f_{l}\right) \tag{5.157}
\end{equation*}
$$

Proof. Write $\delta(t)=\delta\left(f_{t}\right)$ and $\tilde{p}_{g}(t)=p_{g}\left(\tilde{f}_{t}\right)$. Observe the recurrence relation

$$
\begin{equation*}
\tilde{p}_{g}(t)=\sum_{i=1}^{\left\lfloor t / \omega_{3}\right\rfloor} \delta\left(t-i \omega_{3}\right) \tag{5.158}
\end{equation*}
$$

which implies the difference and telescoping summation

$$
\begin{align*}
\tilde{p}_{g}\left(t+\omega_{3}\right)-\tilde{p}_{g}(t) & =\sum_{i=1}^{\left\lfloor t / \omega_{3}\right\rfloor+1} \delta\left(t-(i-1) \omega_{3}\right)-\sum_{i=1}^{\left\lfloor t / \omega_{3}\right\rfloor} \delta\left(t-i \omega_{3}\right)  \tag{5.159a}\\
& =\sum_{i=2}^{\left\lfloor t / \omega_{3}\right\rfloor} \delta\left(t-i \omega_{3}\right)-\sum_{i=1}^{\left\lfloor t / \omega_{3}\right\rfloor} \delta\left(t-i \omega_{3}\right)  \tag{5.159b}\\
& =\delta(t) \tag{5.159c}
\end{align*}
$$

If $\omega_{3}=1$, then $p_{g}(1)=0$, as the geometric genus of any weighted homogeneous polynomial with weights that sum greater than unity vanishes by Proposition 5.45. Moreover,

$$
\begin{align*}
\tilde{p}_{g}(t+1) & =\tilde{p}_{g}(t)+\delta(t)  \tag{5.160}\\
& =\tilde{p}_{g}(1)+\sum_{l=1}^{t} \delta(l) \tag{5.161}
\end{align*}
$$

which implies the claimed identity.
Remark 5.7.3. Consider $f=x^{3}+x y^{3}$ over $\mathbb{C}^{2}$ and $\tilde{f}=\Sigma f=x^{3}+x y^{3}+z^{2}$ over $\mathbb{C}^{3}$. The weights of $\tilde{f}$ are $\left\{\frac{1}{3}, \frac{2}{9}, \frac{1}{2}\right\}$, so $\tilde{f}_{3 / 2}=x^{9 / 2}+x^{3 / 2} y^{9 / 2}+z^{3}$. One
computes $p_{g}(\tilde{f})=0$ and $p_{g}\left(\tilde{f}_{3 / 2}\right)=4$. By equation (5.74),

$$
\begin{equation*}
\delta\left(f_{t}\right)=\frac{27}{4} t^{2}-\frac{3}{2} t-\sum_{i=1}^{3 t}\left\{\frac{9}{2}\left(t-\frac{i}{3}\right)\right\} \tag{5.162}
\end{equation*}
$$

so $\delta(f)=\frac{27}{4}-\frac{9}{4}-\frac{1}{2}=4$.

Remark 5.7.4. Consider two quasi-Brieskorn-Pham polynomials $\tilde{f}$ and $f$ with inverse weights $\{k a, k b, k\} \subset \mathbb{N}$ and $\{a, b\}$, respectively. Equations (5.7ob) and (5.157) imply

$$
\begin{align*}
p_{g}(f)= & \sum_{l=1}^{k-1} \delta\left(f_{l}\right)  \tag{5.163a}\\
= & \frac{1}{2} \sum_{l=1}^{k-1} l^{2} a b-l a-l b+l \operatorname{GCD}(a, b)  \tag{5.163b}\\
= & \frac{a b}{6} k^{3}-\frac{1}{4}(a b+a+b-\operatorname{GCD}(a, b)) k^{2} \\
& \quad+\frac{1}{4}\left(\frac{a b}{3}+a+b-\operatorname{GCD}(a, b)\right) k \tag{5.163c}
\end{align*}
$$

where the quadratic coefficient is precisely $-\frac{1}{2} \delta(f)$.

### 5.8. Durfee Conjecture

Recall that a complex analytic germ is a non-degenerate if and only if it possesses and isolated critical point at the origin. Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, $V_{f, \kappa}=f^{-1}(\kappa)$ and $\bar{F}_{f, 0} \cong{ }_{d} V_{f, \kappa} \cap \bar{B}_{\varepsilon}$ for sufficiently small $0<\varepsilon<\kappa$ denote a non-degenerate, complex analytic germ, the corresponding hypersurface and (closed) Milnor fiber, respectively. Denote by $\sigma\left(F_{f, 0}\right)$ the signature of $F_{f, 0}$ and
by $p_{g}(f)=\operatorname{dim}_{\mathbb{C}} H^{n-1}\left(\tilde{V}_{f, 0}, \mathcal{O}_{\tilde{V}_{f, 0}}\right)$ the geometric genus (or first plurigenus) of any minimal resolution $\tilde{V}_{f, 0}$. In [118], Durfee conjectured that non-degenerate, weighted homogeneous, surface singularities $(n=2)$ satisfy

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right) \leqslant 0 \quad \text { and } \quad 6 p_{g}(f) \leqslant \mu_{\mathrm{alg}}(f) \tag{5.164}
\end{equation*}
$$

with equality of the latter inequality only in the case $\mu_{\mathrm{alg}}(f)=0$. In the case that $f$ is non-degenerate, strict positivity of $\mu$ is known for $n \geqslant 0$. In 1993, Xu and Yau sharpen and prove the Durfee Conjecture for surface singularities.

Proposition $5.47(\mathrm{Xu}, \mathrm{Yau},[478])$. Let $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a nondegenerate, weighted homogeneous germ, and $V_{f, 0}=f^{-1}(0)$ its corresponding hypersurface. Let $\mu_{\mathrm{alg}}(f), \tau(f), p_{g}(f), v(f)$ denote the algebraic index and Tjurina numbers, geometric genus and multiplicity of $f$, respectively. Then

$$
\begin{equation*}
6 p_{g}(f) \leqslant \mu_{\mathrm{alg}}(f)-v(f)+1 \tag{5.165}
\end{equation*}
$$

with equality if and only if $V_{f, 0}$ is defined by a homogeneous polynomial. Moreover, if $\sigma\left(F_{f, 0}\right)$ denotes the signature of the Milnor fiber of $f$, then

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right) \leqslant-\frac{1}{3} \mu_{\mathrm{alg}}(f)-\frac{2}{3}(v(f)-1) . \tag{5.166}
\end{equation*}
$$

Furthermore, if $f$ is simply a two-dimensional surface singularity with isolated critical point at the origin, Xu and Yau (based on prior work of Saito) give a coordinate-free characterization of homogeneity. That is, if $\mu_{\mathrm{alg}}(f)=\tau(f)$
and $6 p_{g}(f)=\mu_{\text {alg }}(f)-v(f)+1$, then $f$ is a homogeneous polynomial after a biholomorphic change of variables.

In the same paper, Durfee conjectures the following generalization for nondegenerate, weighted homogeneous singularities over $\mathbb{C}^{n+1}$,

$$
\begin{equation*}
(n+1)!p_{g}(f) \leqslant \mu_{\mathrm{alg}}(f) \tag{5.167}
\end{equation*}
$$

with equality if and only if $\mu_{\mathrm{alg}}(f)=0$. Yau later sharpened the conjecture.

Conjecture 5.48 (Durfee, Yau). Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a nondegenerate, weighted homogeneous polynomial. Let $\mu(f), p_{g}(f)$ and $v(f)$ denote the algebraic index, geometric genus and multiplicity of $f$, respectively. Then

$$
\begin{equation*}
(n+1)!p_{g}(f) \leqslant \mu_{\mathrm{alg}}(f)-(v(f)-1)^{n+1}+v(f)(v(f)-1) \cdots(v(f)-n) \tag{5.168}
\end{equation*}
$$

with equality if and only if $f$ is homogeneous.

Conjecture 5.48 is true for $n=2$ [478] and $n=3$ [269]. Sękalski [423] proves that the multiplicity of a weighted homogeneous polynomial depends only on its weights, $v(f)=\min \left\{k \in \mathbb{N} \left\lvert\, k \geqslant \min \left\{\frac{1}{\omega_{i}}\right\}\right.\right\}$.

Building upon earlier work [476], Yau and Zhang [488] estimate the number of non-negative and positive lattice points in simplicial $n$-polytopes and establish equation (5.167). Conjecture 5.48 remains open for $n>3$. Based on the Durfee and Yau Conjectures, Lin and Yau (and independently by Granville) [270] conjectured rough and sharp upper bounds on the number of non-negative and positive lattice points in simplicial $(n+1)$-polytopes.

Conjecture 5.49 (Granville, Lin, Yau). Let $P_{n}$ denote the number of positive lattice points in the real tetrahedron $T_{n}=\boldsymbol{\operatorname { c o n v }}\left\{\mathbf{0}, a_{1} \mathbf{e}_{1}, \ldots, a_{n} \mathbf{e}_{n}\right\}$, and let $s(n, k)$ be the $(n, k)^{\text {th }}$-Stirling number of the first kind. Then, for $n \geqslant 3$, the following is true:

1. (Rough GLY Estimate) For $a_{1} \geqslant \cdots \geqslant a_{n} \geqslant 1$,

$$
\begin{equation*}
n!P_{n} \leqslant \prod_{i=1}^{n}\left(a_{i}-1\right) \tag{5.169}
\end{equation*}
$$

2. (Sharp GLY Estimate) For $a_{1} \geqslant \cdots \geqslant a_{n} \geqslant 1$,

$$
\begin{equation*}
n!P_{n} \leqslant A_{0}^{n}+\frac{s(n, n-1)}{n} A_{1}^{n}+\sum_{k=1}^{n-2} \frac{s(n, n-k-1)}{\binom{n-1}{k}} A_{k}^{n-1} \tag{5.170}
\end{equation*}
$$

where $A_{k}^{n}=a_{n} A_{k}^{n-1}+A_{k-1}^{n-1}$ and

$$
\begin{equation*}
A_{k}^{n}=\left(\prod_{i=1}^{n} a_{i}\right)\left(\sum_{1 \leqslant i_{1} \leqslant \cdots \leqslant i_{k} \leqslant n}\left(a_{i_{1}} \cdots a_{i_{k}}\right)^{-1}\right) \tag{5.171}
\end{equation*}
$$

The rough GLY estimate was proven by Yau and Zhang [488]. The sharp GLY estimate is true for $3 \leqslant n \leqslant 6$ [270, 271, 476, 479]. However, the GLY Conjecture is not true in general; Wang and Yau [468] exhibit a counter-example to the sharp GLY estimate for $n=7$.

Proposition 5.50. The Durfee Conjecture is true for any non-degenerate, nonhomogeneous, weighted homogeneous polynomial $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with sufficiently small weights.

Proof. The exact expression of the geometric genus in terms of the reduced weights of $f$ and the corresponding algebraic index is simply

$$
\begin{align*}
p_{g}\left(f_{t}\right)= & \frac{1}{6} \mu_{\mathrm{alg}}\left(f_{t}\right)+\frac{1}{6}\left(\frac{1}{\omega_{1} \omega_{2}}-\frac{1}{2 \omega_{2} \omega_{3}}-\frac{1}{2 \omega_{3} \omega_{1}}\right) t^{2} \\
& -\frac{1}{6}\left(\frac{1}{\omega_{1}}+\frac{1}{\omega_{2}}-\frac{1}{2 \omega_{3}}-\frac{\omega_{1}}{2 \omega_{2} \omega_{3}}\right) t \\
& +\frac{1}{6}-\frac{\omega_{1}}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{3 \omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\} \\
& +\frac{\omega_{1}}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{\omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\}^{2}-\frac{\omega_{1}^{2}}{6 \omega_{2} \omega_{3}}\left\{\frac{t}{\omega_{1}}\right\}^{3} \\
& +\frac{\omega_{2}}{2 \omega_{3}} \sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor}\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\left(1-\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\right) \\
& -\sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor\left\lfloor\left(t-i \omega_{1}\right) / \omega_{2}\right\rfloor} \sum_{j=1}\left\{\frac{t-i \omega_{1}-j \omega_{2}}{\omega_{3}}\right\} . \tag{5.172}
\end{align*}
$$

The conjecture is equivalent to the following fractional part summation inequality,

$$
\begin{align*}
\sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor} \sum_{j=1}^{\left\lfloor\left(t-i \omega_{1}\right) / \omega_{2}\right\rfloor}\left\{\frac{t-i \omega_{1}-j \omega_{2}}{\omega_{3}}\right\} & \geqslant \frac{1}{6}\left(\frac{1}{\omega_{1} \omega_{2}}-\frac{1}{2 \omega_{2} \omega_{3}}-\frac{1}{2 \omega_{3} \omega_{1}}\right) t^{2} \\
& -\frac{1}{6}\left(\frac{1}{\omega_{1}}+\frac{1}{\omega_{2}}-\frac{1}{2 \omega_{3}}-\frac{\omega_{1}}{2 \omega_{2} \omega_{3}}\right) t \\
& +\frac{1}{6}-\frac{\omega_{1}}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{3 \omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\} \\
& +\frac{\omega_{1}}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{\omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\}^{2}-\frac{\omega_{1}^{2}}{6 \omega_{2} \omega_{3}}\left\{\frac{t}{\omega_{1}}\right\}^{3} \\
& +\frac{\omega_{2}}{2 \omega_{3}} \sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor}\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\left(1-\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\right) \tag{5.173}
\end{align*}
$$

Without loss of generality, one may assume $0<\omega_{3} \leqslant \omega_{2} \leqslant \omega_{1} \leqslant \frac{1}{2}$. Assuming the ordered weights are not identical (which occurs only in the homogeneous case, c.f. Proposition 5.51 ), one has $\omega_{1}+\omega_{2}>2 \omega_{3}$ and the coefficient of the quadratic power of $t$ in the inequality (5.173) is negative definite. Proving that this term dominates the right side for $t \geqslant 1$ would imply the conjecture since the left side is non-negative for $t \geqslant 1$. It is therefore sufficient to prove the inequality

$$
\begin{align*}
\frac{\omega_{2}}{2 \omega_{3}} \sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor}\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\left(1-\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\right) \leqslant & -\frac{1}{6}\left(\frac{1}{\omega_{1} \omega_{2}}-\frac{1}{2 \omega_{2} \omega_{3}}-\frac{1}{2 \omega_{3} \omega_{1}}\right) t^{2} \\
& +\frac{1}{6}\left(\frac{1}{\omega_{1}}+\frac{1}{\omega_{2}}-\frac{1}{2 \omega_{3}}-\frac{\omega_{1}}{2 \omega_{2} \omega_{3}}\right) t \\
& -\frac{1}{6}+\frac{\omega_{1}}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{3 \omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\} \\
& -\frac{\omega_{1}}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{\omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\}^{2}+\frac{\omega_{1}^{2}}{6 \omega_{2} \omega_{3}}\left\{\frac{t}{\omega_{1}}\right\}^{3} \tag{5.174}
\end{align*}
$$

Note that $-\frac{1}{6}\left(\frac{1}{\omega_{1} \omega_{2}}-\frac{1}{2 \omega_{2} \omega_{3}}-\frac{1}{2 \omega_{3} \omega_{1}}\right)>0$. The left side is bounded from above by

$$
\begin{align*}
\frac{\omega_{2}}{2 \omega_{3}} \sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor}\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\left(1-\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\right) & \leqslant \frac{\omega_{2}}{8 \omega_{3}}\left\lfloor\frac{t}{\omega_{1}}\right\rfloor  \tag{5.175}\\
& =\frac{\omega_{2}}{8 \omega_{1} \omega_{3}} t-\frac{\omega_{2}}{8 \omega_{3}}\left\{\frac{t}{\omega_{1}}\right\} \tag{5.176}
\end{align*}
$$

which proves the conjecture for sufficiently large $t$.

REMARK 5.8.1. If $\omega_{1}=\omega_{2}=\frac{1}{2}$, then the fractional part summation vanishes at $t=1$ and the Durfee Conjecture is equivalent to

$$
\begin{align*}
0 & \leqslant-\frac{1}{6}\left(4-\frac{2}{\omega_{3}}\right)+\frac{1}{6}\left(4-\frac{1}{\omega_{3}}\right)-\frac{1}{6}  \tag{5.177}\\
& =\frac{1}{6}\left(\frac{1}{\omega_{3}}-1\right) \tag{5.178}
\end{align*}
$$

which is true for $0<\omega_{3} \leqslant 1$. More generally, if $\frac{1}{\omega_{1}}=a, \frac{1}{\omega_{2}}=b \in \mathbb{N}, a \leqslant b$ and $a$ divides $b$, then all fractional parts vanish, and we have

$$
\begin{equation*}
0 \leqslant-\frac{1}{6}\left(a b-\frac{b}{2 \omega_{3}}-\frac{a}{2 \omega_{3}}\right)+\frac{1}{6}\left(a+b-\frac{1}{2 \omega_{3}}-\frac{b}{2 a \omega_{3}}\right)-\frac{1}{6} \tag{5.179}
\end{equation*}
$$

which simplifies to the bound $\frac{a+b}{2 a(b-1)} \geqslant \omega_{3}$. However, by assumption $\omega_{3} \leqslant \omega_{2}=$ $\frac{1}{b} \leqslant \frac{a+b}{2 a(b-1)}$, since $a \leqslant b$, so the Durfee conjecture is also true for this case. $\Delta$

We treat now the homogeneous case.
Proposition 5.51. The Durfee Conjecture is true for homogeneous polynomials of degree $d \geqslant 1$ over $\mathbb{C}^{3}$.

Proof. Consider the homogeneous case $\omega_{1}=\omega_{2}=\omega_{3}=\frac{1}{d}$, where $d \geqslant 2$. Equation (5.173) reduces to

$$
\begin{gather*}
\sum_{i=1}^{\lfloor d t\rfloor} \sum_{j=1}^{\mid d t-i\rfloor}\{d t-i-j\} \geqslant-\frac{1}{6}(d t)+\frac{1}{6}-\frac{1}{3}\{d t\}+\frac{1}{2}\{d t\}^{2}-\frac{1}{6}\{d t\}^{3} \\
 \tag{5.180}\\
+\frac{1}{2} \sum_{i=1}^{\lfloor d t\rfloor}\{d t-i\}(1-\{d t-i\}),
\end{gather*}
$$

and since the fractional part summations vanish, the Durfee Conjecture is equivalent to

$$
\begin{equation*}
0 \leqslant \frac{1}{6}(d t)-\frac{1}{12}<\frac{1}{6}(d t)-\frac{1}{6}+\frac{1}{3}\{d t\}-\frac{1}{2}\{d t\}^{2}+\frac{1}{6}\{d t\}^{3}, \tag{5.181}
\end{equation*}
$$

which is (trivially) true for $t \geqslant 1$.
Precise estimates of iterated fractional-part summations would, of course, yield even sharper bounds than that which the Durfee Conjecture suggests. To apply some of the other identities that we develop in Volume 2, we give a proof of a special case.

Proposition 5.52. Let $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a non-degenerate weighted homogeneous polynomial. If $f$ has weights $\left\{\frac{1}{a}, \frac{1}{b}, \omega_{3}\right\}$ such that $a$ and $b$ are coprime integers (each greater than or equal to 2 ) and $\omega_{3} \leqslant \frac{a+b+1}{2 a b}$, then $f$ satisfies the Durfee Conjecture.

Proof. If $\omega_{1}=\frac{1}{a}$ and $\omega_{2}=\frac{1}{b}$ are inverse integers, then

$$
\begin{align*}
p_{g}\left(f_{t}\right)-\frac{1}{6} \mu_{\mathrm{alg}}\left(f_{t}\right)= & \frac{1}{6}\left(a b-\frac{b}{2 \omega_{3}}-\frac{a}{2 \omega_{3}}\right) t^{2}-\frac{1}{6}\left(a+b-\frac{1}{2 \omega_{3}}-\frac{b}{2 a \omega_{3}}\right) t \\
& +\frac{1}{6}+\frac{1}{2 b \omega_{3}} \sum_{i=1}^{a t}\left\{b\left(t-\frac{i}{a}\right)\right\}\left(1-\left\{b\left(t-\frac{i}{a}\right)\right\}\right) \\
& -\sum_{i=1}^{a t} \sum_{j=1}^{[b(t-i / a)]}\left\{\frac{t-i / a-j / b}{\omega_{3}}\right\}  \tag{5.182}\\
= & \frac{1}{6}\left(a b-\frac{b}{2 \omega_{3}}-\frac{a}{2 \omega_{3}}\right) t^{2}-\frac{1}{6}\left(a+b-\frac{1}{2 \omega_{3}}-\frac{b}{2 a \omega_{3}}\right) t+\frac{1}{6} \\
& +\frac{\operatorname{GCD}(a, b)^{2}}{2 a b \omega_{3}}\left(\left(\sum_{l=0}^{a / \operatorname{GCD}(a, b)-1} l\right)-\frac{\operatorname{GCD}(a, b)}{a}\left(\sum_{l=0}^{a / \operatorname{GCD}(a, b)-1} l^{2}\right)\right) t \\
& -\sum_{i=1}^{a t} \sum_{j=1}^{[b(t-i / a)\rfloor}\left\{\frac{t-i / a-j / b}{\omega_{3}}\right\}, \tag{5.183}
\end{align*}
$$

since

$$
\begin{align*}
\sum_{i=1}^{a t}\left\{b\left(t-\frac{i}{a}\right)\right\}^{k} & =\sum_{i=1}^{a t}\left\{\frac{b i}{a}\right\}^{k}  \tag{5.184}\\
& =\frac{\operatorname{GCD}(a, b)^{k+1}}{a^{k}}\left(\sum_{l=0}^{a / \operatorname{GCD}(a, b)-1} l^{k}\right) t \tag{5.185}
\end{align*}
$$

as shown in Volume 2. If $a$ and $b$ are coprime integers, then

$$
\begin{align*}
p_{g}\left(f_{t}\right)-\frac{1}{6} \mu_{\mathrm{alg}}\left(f_{t}\right)=\frac{1}{6} & \left(a b-\frac{b}{2 \omega_{3}}-\frac{a}{2 \omega_{3}}\right) t^{2}-\frac{1}{6}\left(a+b-\frac{1}{2 \omega_{3}}-\frac{b}{2 a \omega_{3}}\right) t+\frac{1}{6} \\
& +\frac{1}{2 a b \omega_{3}}\left(\frac{a}{2}(a-1)-\frac{1}{6}(a-1)(2 a-1)\right) t \\
& -\sum_{i=1}^{a t} \sum_{j=1}^{\lfloor b(t-i / a)\rfloor}\left\{\frac{t-i / a-j / b}{\omega_{3}}\right\} . \tag{5.186}
\end{align*}
$$

For $t=1$, the right side simplifies to

$$
\begin{align*}
& p_{g}(f)-\frac{1}{6} \mu_{\mathrm{alg}}(f)=\frac{1}{6}(a-1)(b-1)\left(1-\frac{a+b+1}{2 a b \omega_{3}}\right) \\
&-\sum_{i=1}^{a} \sum_{j=1}^{\lfloor b(1-i / a)\rfloor}\left\{\frac{1-i / a-j / b}{\omega_{3}}\right\} . \tag{5.187}
\end{align*}
$$

If $\omega_{3}=\frac{c}{d}$ is in reduced form, then the Durfee Conjecture is equivalent to the following fractional part summation inequality,

$$
\begin{equation*}
\sum_{i=1}^{a} \sum_{j=1}^{\lfloor b(1-i / a)\rfloor}\left\{\frac{d(1-i / a-j / b)}{c}\right\} \geqslant \frac{1}{6}(a-1)(b-1)\left(1-\frac{d(a+b+1)}{2 a b c}\right) \tag{5.188}
\end{equation*}
$$

By assumption $\frac{c}{d} \leqslant \frac{a+b+1}{2 a b}$, so the right side is non-positive. The fractional part summation is non-negative, zero only in the case that $a, b$ and $c$ divide $d$, so the claimed inequality is (trivially) true.
$\operatorname{RemaRK}$ 5.8.2. In particular, if $\{a, b\} \in\{\{2,2\},\{2,3\}\}$, then one requires $\omega_{3} \leqslant \frac{5}{8}$ or $\omega_{3} \leqslant \frac{1}{2}$, respectively, both of which are true by non-degeneracy.

Hence, the Durfee Conjecture is true for non-degenerate, weighted homogeneous polynomials with weights $\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{c}\right\}$ for $c \in \mathbb{N}_{>1}$ or $\left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{d}\right\}$ for $d \in \mathbb{N}_{>3}$. The former has trivial geometric genus, while the latter does not, q.v., Proposition 6.19.

Proposition 5.53. Provided that the summation of the reduced weights is less than unity, there are positive constants $A$ and $B$, depending only on $n$, such that

$$
\begin{equation*}
A p_{g} \leqslant \mu_{\mathrm{alg}}(f) \leqslant B p_{g} \tag{5.189}
\end{equation*}
$$

Proof. See Volume 3.

### 5.9. Signature of Weighted Homogeneous Surface Singularities

Proposition 5.54. Let $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a square-free, weighted homogeneous polynomial. The following identity holds,

$$
\begin{equation*}
\delta(f)=\varsigma_{+}=\zeta_{-} \quad \text { and } \quad r(f)=\varsigma_{0}+1 \tag{5.190}
\end{equation*}
$$

where $\varsigma_{+}, \varsigma_{-}$and $\varsigma_{0}$ are the number of positive, negative and zero eigenvalues of the intersection form $S$, respectively.

Proof. Since $n$ is odd, the signature of the corresponding Milnor fiber vanishes, i.e., $\sigma\left(F_{f, 0}\right)=\varsigma_{+}-\varsigma_{-}=0$, so $\varsigma_{+}=\varsigma_{-}$. Therefore, $\mu_{\mathrm{alg}}(f)=2 \varsigma_{+}+\varsigma_{0}$. The Milnor-Jung formula implies $2 \varsigma_{+}+\zeta_{0}=2 \delta(f)-r(f)+1$. By counting
lattice points,

$$
\begin{equation*}
\zeta_{+}=\sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor} \sum_{j=1}^{\left\lfloor\left(1-i \omega_{1}\right) / \omega_{2}\right\rfloor} 1=\delta(f) \tag{5.191}
\end{equation*}
$$

which completes the proof.

Proposition 5.55. Given a weighted homogeneous surface singularity
$f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with weights $\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}$, intersection form invariants $\left\{\varsigma_{+}, \zeta_{0}, \zeta_{-}\right\}$, geometric genus $p_{g}(f)$, signature $\sigma\left(F_{f, 0}\right)$ and algebraic index $\mu_{\mathrm{alg}}(f)$,

$$
\begin{equation*}
4 p_{g}(f)=\sigma\left(F_{f, 0}\right)+\mu_{\mathrm{alg}}(f)+\varsigma_{0} \tag{5.192}
\end{equation*}
$$

Proof. The claimed identity follows from the following identities:

$$
\begin{align*}
& \mu_{\mathrm{alg}}(f)=\varsigma_{+}+\varsigma_{0}+\varsigma_{-}  \tag{5.193}\\
& 2 p_{g}(f)=\varsigma_{+}+\varsigma_{0}  \tag{5.194}\\
& \sigma\left(F_{f, 0}\right)=\varsigma_{+}-\varsigma_{-} . \tag{5.195}
\end{align*}
$$

Proposition 5.56. The signature $\sigma\left(F_{f, 0}\right)$ of the Milnor fiber $F_{f, 0}$ of a weighted homogeneous surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with integral weights $\left\{q_{1}, q_{2}, q_{3}\right\}$,
weighted degree $d$ and intersection form invariants $\left\{\varsigma_{+}, \varsigma_{0}, \zeta_{-}\right\}$is given by

$$
\begin{align*}
\sigma\left(F_{f, 0}\right)= & 1-\varsigma_{0}-\frac{d^{3}}{3 q_{1} q_{2} q_{3}}+\frac{d^{2}}{q_{1} q_{2}}+\left(\frac{q_{1}}{3 q_{2} q_{3}}-\frac{1}{q_{1}}-\frac{1}{q_{2}}\right) d \\
& -\frac{q_{1}}{q_{3}}\left(1+\frac{q_{1}}{3 q_{2}}\right)\left\{\frac{d}{q_{1}}\right\} \\
& +\frac{q_{1}}{q_{3}}\left(1+\frac{q_{1}}{q_{2}}\right)\left\{\frac{d}{q_{1}}\right\}^{2}-\frac{2 q_{1}^{2}}{3 q_{2} q_{3}}\left\{\frac{d}{q_{1}}\right\}^{3} \\
& +\frac{2 q_{2}}{q_{3}} \sum_{i=1}^{\left\lfloor d / q_{1}\right\rfloor}\left\{\frac{d-i q_{1}}{q_{2}}\right\}\left(1-\left\{\frac{d-i q_{1}}{q_{2}}\right\}\right) \\
& -4 \sum_{i=1}^{\left\lfloor d / q_{1}\right\rfloor\left\lfloor\left(d-i q_{1}\right) / q_{2}\right\rfloor} \sum_{j=1}^{\left\{\frac{d-i q_{1}-j q_{2}}{q_{3}}\right\},} \tag{5.196}
\end{align*}
$$

where $\varsigma_{0}=\frac{d^{2}}{q_{1} q_{2} q_{3}}-\sum_{1 \leqslant i<j \leqslant 3} \frac{d}{\operatorname{LCM}\left(q_{i}, q_{j}\right)}+\sum_{1 \leqslant i \leqslant 3} \frac{\operatorname{GCD}\left(d, q_{i}\right)}{q_{i}}-1$.
Proof. If $f$ has reduced weights $\left\{\omega_{1}, \omega_{2}, \omega_{3}\right\}$, then

$$
\begin{align*}
p_{g}\left(f_{t}\right)= & \left(\frac{1}{6 \omega_{1} \omega_{2} \omega_{3}}\right) t^{3}-\frac{1}{4 \omega_{3}}\left(\frac{1}{\omega_{1}}+\frac{1}{\omega_{2}}\right) t^{2}+\frac{1}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{3 \omega_{2}}\right) t \\
& -\frac{\omega_{1}}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{3 \omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\}+\frac{\omega_{1}}{4 \omega_{3}}\left(1+\frac{\omega_{1}}{\omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\}^{2} \\
& -\frac{\omega_{1}^{2}}{6 \omega_{2} \omega_{3}}\left\{\frac{t}{\omega_{1}}\right\}^{3}+\frac{\omega_{2}}{2 \omega_{3}} \sum_{i=1}^{\left.t / \omega_{1}\right\rfloor}\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\left(1-\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\right) \\
& -\sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor\left\lfloor\left(t-i \omega_{1}\right) / \omega_{2}\right\rfloor} \sum_{j=1}\left\{\frac{t-i \omega_{1}-j \omega_{2}}{\omega_{3}}\right\} . \tag{5.197}
\end{align*}
$$

## By Proposition 5.55,

$$
\begin{align*}
\sigma\left(F_{f_{t}, 0}\right)= & 1-\varsigma_{0}-\frac{1}{3 \omega_{1} \omega_{2} \omega_{3}} t^{3}+\frac{1}{\omega_{1} \omega_{2}} t^{2}+\left(\frac{\omega_{1}}{3 \omega_{2} \omega_{3}}-\frac{1}{\omega_{1}}-\frac{1}{\omega_{2}}\right) t \\
& -\frac{\omega_{1}}{\omega_{3}}\left(1+\frac{\omega_{1}}{3 \omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\}+\frac{\omega_{1}}{\omega_{3}}\left(1+\frac{\omega_{1}}{\omega_{2}}\right)\left\{\frac{t}{\omega_{1}}\right\}^{2}-\frac{2 \omega_{1}^{2}}{3 \omega_{2} \omega_{3}}\left\{\frac{t}{\omega_{1}}\right\}^{3} \\
& +\frac{2 \omega_{2}}{\omega_{3}} \sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor}\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\left(1-\left\{\frac{t-i \omega_{1}}{\omega_{2}}\right\}\right) \\
& -4 \sum_{i=1}^{\left\lfloor t / \omega_{1}\right\rfloor\left\lfloor\left(t-i \omega_{1}\right) / \omega_{2}\right\rfloor} \sum_{j=1}\left\{\frac{t-i \omega_{1}-j \omega_{2}}{\omega_{3}}\right\} . \tag{5.198}
\end{align*}
$$

According to Durfee, $H_{1}(E) \cong H_{1}\left(K_{f}\right)$ and $\varsigma_{0}=\operatorname{rank} H_{1}\left(K_{f}\right)$ [118]. Orlik and Wagreich compute the (arithmetic) genus of an arbitrary weighted homogeneous surface singularity (Corollary 5.4, [363]; Proposition 3, [364]; also [362]). Given integral weights $\left\{q_{1}, q_{2}, q_{3}\right\}$ and weighted degree $d$, the genus of $V_{f, 0}^{\times} / \mathbb{C}^{\times} \simeq V_{f, 0}^{\times} / S^{1}$, equal to the base-orbifold, is given by

$$
\begin{equation*}
g=\frac{1}{2}\left(\frac{d^{2}}{q_{1} q_{2} q_{3}}-\sum_{1 \leqslant i<j \leqslant 3} \frac{d}{\operatorname{LCM}\left(q_{i}, q_{j}\right)}+\sum_{1 \leqslant i \leqslant 3} \frac{\operatorname{GCD}\left(d, q_{i}\right)}{q_{i}}-1\right) . \tag{5.199}
\end{equation*}
$$

Finally, $\varsigma_{0}=2 g$.

Remark 5.9.1. In terms of the reduced weights,

$$
\begin{align*}
\sigma\left(F_{f, 0}\right)= & 1-\zeta_{0}-\frac{1}{3 \omega_{1} \omega_{2} \omega_{3}}+\frac{1}{\omega_{1} \omega_{2}}+\left(\frac{\omega_{1}}{3 \omega_{2} \omega_{3}}-\frac{1}{\omega_{1}}-\frac{1}{\omega_{2}}\right)-\frac{\omega_{1}}{\omega_{3}}\left(1+\frac{\omega_{1}}{3 \omega_{2}}\right)\left\{\frac{1}{\omega_{1}}\right\} \\
& +\frac{\omega_{1}}{\omega_{3}}\left(1+\frac{\omega_{1}}{\omega_{2}}\right)\left\{\frac{1}{\omega_{1}}\right\}^{2}-\frac{2 \omega_{1}^{2}}{3 \omega_{2} \omega_{3}}\left\{\frac{1}{\omega_{1}}\right\}^{3}+\frac{2 \omega_{2}}{\omega_{3}} \sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor}\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}\left(1-\left\{\frac{1-i \omega_{1}}{\omega_{2}}\right\}\right) \\
& -4 \sum_{i=1}^{\left\lfloor 1 / \omega_{1}\right\rfloor\left\lfloor\left(1-i \omega_{1}\right) / \omega_{2}\right\rfloor} \sum_{j=1}\left\{\frac{1-i \omega_{1}-j \omega_{2}}{\omega_{3}}\right\} . \tag{5.200}
\end{align*}
$$

REMARK 5.9.2. Suppose $f$ is a homogeneous surface singularity with degree $d$. A straightforward computation shows that $\zeta_{0}=(d-1)(d-2)$ and

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=-\frac{1}{3}(d-1)\left(d^{2}+d-3\right) \tag{5.201}
\end{equation*}
$$

which is divisible by 8 if and only if $d \equiv 1 \bmod 8$.

This concludes our remarks on some combinatorial structures at the foundation of singularities of complex analytic germs. We proceed to a discussion of some arithmetic facets.

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## Chapter 6

# Arithmetic Structure of Isolated Singularities 

Die Mathematik ist die Königin der Wissenschaften und die Zahlentheorie ist die Königin der Mathematik.* - C. F. Gauss

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In this chapter we compute the signature, geometric genus and other related invariants of Brieskorn-Pham 3-manifolds. In particular, we compute the signature of a torus link.

[^34]
### 6.1. Signature of Torus Links

The reader is encouraged to consult $[235,335]$ or any similar reference on the fundamentals of knot and links and related numerical invariants.

Much is known of the signature of torus links. For instance, $T_{p, q}$ for $p, q>1$ is not amphichiral and therefore $\sigma\left(T_{p, q}\right)$ does not vanish (Theorem 7.4.2, [335]). In particular, if $p$ and $q$ are coprime and odd, then $\sigma\left(T_{p, q}\right) \equiv 0 \bmod 8$. More generally, for a torus knot $T_{p, q}$ if $\Delta_{T_{p, q}}(-1)=1$, then said congruence is true [335].

Proposition 6.1. The signature of the torus links satisfies the following:

1. $\sigma\left(T_{q, p}\right)=\sigma\left(T_{p, q}\right)$;
2. $\sigma\left(T_{-p, q}\right)=\sigma\left(T_{p,-q}\right)=-\sigma\left(T_{p, q}\right)$; and,
3. $\sigma\left(T_{-p,-q}\right)=\sigma\left(T_{p, q}\right)$.

Proof. For $p, q \in \mathbb{N}, T_{p, q} \simeq T_{q, p}, T_{p, q} \simeq T_{-p,-q} \simeq-T_{p, q}, T_{p,-q} \simeq T_{p, q}^{*}[74]$.
Murasugi computed the signature of various families of torus links.

Proposition 6.2 (Murasugi, [334]). For $m \in \mathbb{N}$,

1. $\sigma\left(T_{3,3 m}\right)=-4 m ;$
2. $\sigma\left(T_{3,6 m+1}\right)=-8 m$;
3. $\sigma\left(T_{3,6 m+2}\right)=-8 m-2$;
4. $\sigma\left(T_{3,6 m+4}\right)=-8 m-6$;
5. $\sigma\left(T_{3,6 m+5}\right)=-8 m-8$;
6. $\sigma\left(T_{4,4 m+1}\right)=-8 m$;
7. $\sigma\left(T_{4,4 m+3}\right)=-8 m-6$; and,
8. $\sigma\left(T_{4,2 m}\right)=-4 m+1$.

Proof. See Propositions 9.1 and 9.2 in [334].
Gordon, Litherland and Murasugi gave an algorithm to compute the signature of an arbitrary torus link.

Proposition 6.3 (Gordon, Litherland, Murasugi, [156]). Let $\sigma(p, q)=$ $-\sigma\left(T_{p, q}\right)$ for $p, q \in \mathbb{N}$. The following algorithm computes $\sigma(p, q)$ recursively:

1. If $0<2 q<p$, then
if $q>0$ is odd, then $\sigma(p, q)=\sigma(p-2 q, q)+q^{2}-1$; or,
if $q>0$ is even, then $\sigma(p, q)=\sigma(p-2 q, q)+q^{2}$;
2. $\sigma(2 q, q)=q^{2}-1$;
3. If $0<q \leqslant p<2 q$, then
if $q>0$ is odd, then $\sigma(p, q)+\sigma(2 q-p, q)=q^{2}-1$; or,
if $q>0$ is even, then $\sigma(p, q)+\sigma(2 q-p, q)=q^{2}-2$; and,
4. $\sigma(p, q)=\sigma(q, p), \sigma(p, 1)=0$ and $\sigma(p, 2)=p-1$.


Figure 6.1. The Hopf Link $\left(T_{2,2}\right)$

Remark 6.1.1. Suppose $p=q$ is even. With the stated algorithm, one computes $\sigma(p, p)+\sigma(2 p-p, p)=p^{2}-2$. That is, $-\sigma\left(T_{p, p}\right)=\sigma(p, p)=\frac{1}{2} p^{2}-$

1. Thus, for example, the Hopf link $T_{2,2}$ (Figure 6.1) has signature $\sigma\left(T_{2,2}\right)=$ -1 .

Based on work of Brieskorn, the Hirzebruch signature of the closure of the Milnor fiber $\bar{F}_{f, 0}$ of the stabilized, Brieskorn-Pham singularity $f=z_{0}^{p}+z_{1}^{q}+$ $\sum_{i=2}^{n} z_{i}^{2}$, where $p$ and $q$ odd and coprime and $n$ is even, coincides with the signature of the torus knot $T_{p, q}$ up to sign $(-1)^{n / 2+1}[201,202]$. In particular, for $p$ and $q$ odd and coprime, Hirzebruch computes

$$
\begin{equation*}
\sigma\left(T_{p, q}\right)=-\frac{1}{2}(p-1)(q-1)-2\left(N_{p, q}+N_{q, p}\right) \tag{6.1}
\end{equation*}
$$

where $N_{p, q}$ is the number of lattice points intersecting the wedge,

$$
\begin{equation*}
\left|\left\{(x, y) \in \mathbb{N} \left\lvert\, 1 \leqslant x \leqslant \frac{1}{2}(p-1)\right., 1 \leqslant y \leqslant \frac{1}{2}(q-1),-\frac{1}{2 q}<\frac{x}{p}-\frac{y}{q}<0\right\}\right| . \tag{6.2}
\end{equation*}
$$

Litherland [273] computes the signature as the difference $\sigma\left(T_{p, q}\right)=\varsigma_{+}-\varsigma_{-}$, where

$$
\begin{align*}
& \varsigma_{+}=\left|\left\{(x, y) \in \mathbb{N}^{2} \mid 0<x<p, 0<y<q, 0<\frac{x}{p}+\frac{y}{q}<\frac{1}{2}\right\}\right| \\
& \quad+\left|\left\{(x, y) \in \mathbb{N}^{2} \mid 0<x<p, 0<y<q, \frac{3}{2}<\frac{x}{p}+\frac{y}{q}<2\right\}\right|  \tag{6.3}\\
& \varsigma_{-}=\left|\left\{(x, y) \in \mathbb{N}^{2} \mid 0<x<p, 0<y<q, \frac{1}{2}<\frac{x}{p}+\frac{y}{q}<\frac{3}{2}\right\}\right| . \tag{6.4}
\end{align*}
$$

Based on these formulas, Ait Nouh and Yasuhara [8] provide the following summatory representation which they use to compute a simple upper bound on the signature.

Proposition 6.4 (Ait Nouh, Yasuhara, [8]). For positive integers $p$ and $q$ with $p<q$, the signature of the torus link $T_{p, q}$ is given by

$$
\begin{align*}
\sigma\left(T_{p, q}\right)= & \left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)(q-1) \\
& +2 \sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\lfloor\frac{(p-2 k) q}{2 p}\right\rfloor-\left\lfloor\frac{\left(3 p-2\left\lfloor\frac{p}{2}\right\rfloor-2 k\right) q}{2 p}\right\rfloor . \tag{6.5}
\end{align*}
$$

Proof. See Proposition 2.1 in [8].

Remark 6.1.2. Equation (6.5) holds also if $p \geqslant q$.

Corollary 6.5 (Ait Nouh, Yasuhara, [8]). For positive integers $p$ and $q$ with $p<q$, the signature of the torus link $T_{p, q}$ satisfies the bound,

$$
\begin{equation*}
\sigma\left(T_{p, q}\right) \leqslant-2\left\lfloor\frac{p}{2}\right\rfloor\left\lfloor\frac{q}{2}\right\rfloor . \tag{6.6}
\end{equation*}
$$

Proof. See Corollary 2.2 in [8].

Corollary 6.6 (Ait Nouh, Yasuhara, [8]). For an odd integer $p$ and even integer $r$ with $2 \leqslant r<p$, the signature of the torus knot $T_{p, p+r}$ is given by

$$
\begin{align*}
\sigma\left(T_{p, p+r}\right)=- & \frac{1}{2}(p-1)(p+r+1) \\
& +2 \sum_{k=1}^{r / 2}\left\lfloor\frac{(2 k-1) p}{2 r}\right\rfloor-\left\lfloor\frac{(2 k-1) p+r}{2 r}\right\rfloor . \tag{6.7}
\end{align*}
$$

Proof. See Proposition 2.3 in [8].

Lemma 6.7. For any positive integer $m$, the following quasi-polynomial

$$
\begin{equation*}
f(m)=-2 m(m+2)-1+(-1)^{m} \tag{6.8}
\end{equation*}
$$

is divisible by 8.

Proof. We prove the claim by induction. It is clear that the claim is true for $m=1$. Suppose it true for some for some auxiliary integer $l>1$. We show that it remains true for $l+1$,

$$
\begin{align*}
f(l+1) & =-2 l^{2}-8 l-7+(-1)^{l+1}  \tag{6.9}\\
& =f(l)-4 l-6-2(-1)^{l}  \tag{6.10}\\
& =\left\{\begin{array}{cc}
4 l+4 \bmod 8 & l \text { odd } \\
4 l \bmod 8 & l \text { even. }
\end{array}\right. \tag{6.11}
\end{align*}
$$

which implies the claim for $m \geqslant 1$.

Remark 6.1.3. Using the identity $\lfloor x\rfloor=x-\{x\}$ for $x \in \mathbb{R}_{\geqslant 0}$, equation (6.7) may be written as

$$
\begin{align*}
\sigma\left(T_{p, p+r}\right)=- & \frac{1}{2}(p-1)(p+r+1)-\frac{r}{2} \\
& +2 \sum_{k=1}^{r / 2}\left\{\frac{1}{2}-\frac{p}{2 r}+\frac{k p}{r}\right\}-\left\{-\frac{p}{2 r}+\frac{k p}{r}\right\} . \tag{6.12}
\end{align*}
$$

In particular, if $r=2$, a closed form expression is given by

$$
\begin{align*}
\sigma\left(T_{p, p+2}\right) & =-\frac{1}{2}(p-1)(p+3)-1+2\left\{\frac{1}{2}+\frac{p}{4}\right\}-2\left\{\frac{p}{4}\right\}  \tag{6.13}\\
& =-\frac{1}{2}(p-1)(p+3)-1+(-1)^{\binom{p}{2}} \tag{6.14}
\end{align*}
$$

which is a quasi-polynomial of degree 2 in $p$. For example, $\sigma\left(T_{3,5}\right)=-8$, $\sigma\left(T_{5,7}\right)=-32$ and $\sigma\left(T_{7,9}\right)=-64$. Since $p$ is assumed odd, in general, for $p=2 m+1$,

$$
\begin{equation*}
\sigma\left(T_{2 m+1,2 m+3}\right)=-2 m(m+2)-1+(-1)^{m} \equiv 0 \bmod 8 \tag{6.15}
\end{equation*}
$$

by Lemma 6.7.
6.1.1. Dedekind Sum Function. The Dedekind sum function $\mathfrak{s :} \mathbb{N}^{2} \rightarrow \mathbb{Q}$ is defined on coprime positive integers by

$$
\begin{align*}
\mathfrak{s}(p, q) & =\frac{1}{4 q} \sum_{k=1}^{q-1} \cot \left(\frac{\pi k}{q}\right) \cot \left(\frac{\pi k p}{q}\right)  \tag{6.16}\\
& =\frac{1}{4}-\frac{1}{4 q}-\frac{1}{q} \sum_{\substack{\zeta q=1 \\
\zeta \neq 1}} \frac{1}{\left(1-\zeta^{p}\right)(1-\zeta)}, \tag{6.17}
\end{align*}
$$

where the summation is performed over the non-trivial $q^{\text {th }}$-roots of unity.

Remark 6.1.4. The Dedekind Reciprocity Law [16] states

$$
\begin{equation*}
\mathfrak{s}(p, q)+\mathfrak{s}(q, p)=\frac{1}{12}\left(\frac{p}{q}+\frac{1}{p q}+\frac{q}{p}\right)-\frac{1}{4}, \tag{6.18}
\end{equation*}
$$

which implies the congruence

$$
\begin{equation*}
12 p q(\mathfrak{s}(p, q)+\mathfrak{s}(q, p)) \equiv p^{2}+q^{2}+1 \bmod p q . \tag{6.19}
\end{equation*}
$$

We refer the reader to Volume 2 for a detailed treatment of the Dedekind sum functions in the context of lattice point enumeration in trirectangular tetrahedra. For a general reference on Dedekind sum functions, see [392].

Also using lattice point enumeration methods, Borodzik and Oleszkiewic [56] compute the signature of certain torus knots in terms of Dedekind sum functions. We generalize their work to all torus links in §6.6.

Proposition 6.8 (Borodzik, Oleszkiewic, [56]). For odd and coprime integers $p, q>0$,

$$
\begin{equation*}
\sigma\left(T_{p, q}\right)=\frac{1}{6 p q}+\frac{2 p}{3 q}+\frac{2 q}{3 p}-\frac{p q}{2}-4(\mathfrak{s}(2 p, q)+\mathfrak{s}(2 q, p))-1 \tag{6.20}
\end{equation*}
$$

while for $p$ odd and $q>2$ even,

$$
\begin{equation*}
\sigma\left(T_{p, q}\right)=-\frac{p q}{2}+1+4 \mathfrak{s}(2 p, q)-8 \mathfrak{s}(p, q) . \tag{6.21}
\end{equation*}
$$

Proof. See Proposition 4.1 in [56].
Utilizing the Dedekind sum representation, we prove the following upper and lower bounds, the former of which is often superior to Corollary 6.5.

Proposition 6.9. For odd and coprime integers $p, q>0$,

$$
\begin{align*}
& \sigma\left(T_{p, q}\right) \leqslant\left\lfloor-\frac{p q}{2}+\frac{1}{6 p q}+\frac{2 p+2}{3 q}+\frac{2 q+2}{3 p}+\frac{p+q}{3}-3\right]  \tag{6.22}\\
& \sigma\left(T_{p, q}\right) \geqslant\left[-\frac{p q}{2}+\frac{1}{6 p q}+\frac{2 p-2}{3 q}+\frac{2 q-2}{3 p}-\frac{p+q}{3}+1\right] \tag{6.23}
\end{align*}
$$

while for $p$ odd and $q>2$ even,

$$
\begin{equation*}
\left\lceil-\frac{p q}{2}-q-\frac{2}{q}+4\right\rceil \leqslant \sigma\left(T_{p, q}\right) \leqslant\left\lfloor-\frac{p q}{2}+q+\frac{2}{q}-2\right\rfloor . \tag{6.24}
\end{equation*}
$$

Proof. Combine Proposition 6.8 with the elementary sharp bounds,

$$
\begin{equation*}
-\mathfrak{s}(1, q) \leqslant \mathfrak{s}(p, q) \leqslant \mathfrak{s}(1, q)=\frac{q}{12}+\frac{1}{6 q}-\frac{1}{4} \tag{6.25}
\end{equation*}
$$

REmARK 6.1.5. The bounds of equation (6.22) are sharp. Consider the torus knots $T_{3,5}$ and $T_{3,7}$. Corollary 6.5 gives $\sigma\left(T_{3,5}\right) \leqslant-4$ and $\sigma\left(T_{3,7}\right) \leqslant-6$, while Proposition 6.9, yields the lower and upper bounds $-8 \leqslant \sigma\left(T_{3,5}\right) \leqslant-6$ and $-11 \leqslant \sigma\left(T_{3,7}\right) \leqslant-8$. Since the signatures must be even, it follows that $\sigma\left(T_{3,5}\right) \in\{-8,-6\}$ and $\sigma\left(T_{3,7}\right) \in\{-10,-8\}$. Moreover, since the signatures must be divisible by 8 , it follows that $\sigma\left(T_{3,5}\right)=\sigma\left(T_{3,7}\right)=-8$. This value may be verified by Proposition 6.3.

REMARK 6.1.6. The upper bound of equation (6.24) appears to be weaker than that of Corollary 6.5. Consider the torus knot $T_{1001,1160}$. Corollary 6.5 yields the upper bound $\sigma\left(T_{1001,1160}\right) \leqslant-580000$, while Proposition 6.9 yields
the lower and upper bounds $-581496 \leqslant \sigma\left(T_{1001,1160}\right) \leqslant-579502$. Proposition 6.3 yields the value $\sigma\left(T_{1001,1160}\right)=-580496$.
6.1.2. Exact Representation of the Signature. Since the validity of these formulas extends to all pairs of positive integers $(p, q)$, the signature of infinitely many torus (multi-component) links may be computed in closed form.

First, we correct* and generalize a classical result on fractional part summations.

Proposition 6.10. For $\alpha \in \mathbb{R}, m \in \mathbb{N}$ and $n \in \mathbb{Z}$,

$$
\begin{align*}
\sum_{k=0}^{m-1}\left\{\alpha+\frac{k n}{m}\right\}= & \frac{1}{2}(m-\operatorname{GCD}(m, n))-\sum_{k=0}^{m-1} \chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha+\frac{k n}{m}\right) \\
& +\operatorname{GCD}(m, n)\left(\left\{\frac{\alpha m}{\operatorname{GCD}(m, n)}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\frac{\alpha m}{\operatorname{GCD}(m, n)}\right)\right), \tag{6.26}
\end{align*}
$$

where $\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}$denotes the characteristic function of the punctured half-line $\mathbb{R}_{\leqslant 0} \backslash \mathbb{Z}$.
Proof. We follow the proof of Theorem 1 in [140] rather closely and modify it as needed. Begin with the classical identity

$$
\begin{equation*}
\sum_{k=0}^{l-1}\left\lfloor\alpha+\frac{k}{l}\right\rfloor=\lfloor\alpha l\rfloor . \tag{6.27}
\end{equation*}
$$

*Theorem 1 in [140] claims the fractional part sum identity,

$$
\sum_{k=0}^{m-1}\left\{\alpha+\frac{k n}{m}\right\}=\frac{1}{2}(m-\operatorname{GCD}(m, n))+\operatorname{GCD}(m, n)\left\{\frac{\alpha m}{\operatorname{GCD}(m, n)}\right\},
$$

where $\alpha \in \mathbb{R}, m \in \mathbb{N}_{>1}$ and $n \in \mathbb{Z}$. Perhaps some other definition of the fractional part was used. Assuming the identity $\{-|x|\}=-\{|x|\}$ for $\mathbb{R}$, counter-examples abound show that this identity is not correct when $\alpha \in \mathbb{R}_{<0}$.
where $\alpha \in \mathbb{R}$ and $l \in \mathbb{N}$. The floor function can be written in terms of the fractional part, but care must be taken when the argument is potentially negative, viz., $\lfloor x\rfloor=x-\{x\}-\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}(x)$. One then derives the identity

$$
\begin{equation*}
\sum_{k=0}^{l-1}\left\{\alpha+\frac{k}{l}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha+\frac{k}{l}\right)=\frac{1}{2}(l-1)+\{\alpha l\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}(\alpha l) \tag{6.28}
\end{equation*}
$$

For any integer $c$ coprime to $k$, the map $k \rightarrow c k$ is a bijection on the complete residue set of non-negative integers $\{0, \ldots, k-1\}$ modulo $k$. Thus,

$$
\begin{equation*}
\sum_{k=0}^{l-1}\left\{\alpha+\frac{k c}{l}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha+\frac{k c}{l}\right)=\frac{1}{2}(l-1)+\{\alpha l\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}(\alpha l) . \tag{6.29}
\end{equation*}
$$

Taking $c=\frac{n}{\operatorname{GCD}(m, n)}$ and $l=\frac{m}{\operatorname{GCD}(m, n)}$,

$$
\begin{align*}
\sum_{k=0}^{m / \operatorname{GCD}(m, n)-1}\left\{\alpha+\frac{k n}{m}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha+\frac{k n}{m}\right)= & \frac{1}{2}\left(\frac{m}{\operatorname{GCD}(m, n)}-1\right)+\left\{\frac{\alpha m}{\operatorname{GCD}(m, n)}\right\} \\
& +\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\frac{\alpha m}{\operatorname{GCD}(m, n)}\right) \tag{6.30}
\end{align*}
$$

However, by the periodicity of the fractional part function,

$$
\begin{equation*}
\sum_{k=0}^{m-1}\left\{\alpha+\frac{k n}{m}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha+\frac{k n}{m}\right)=d \sum_{k=0}^{m / d-1}\left\{\alpha+\frac{k n}{m}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha+\frac{k n}{m}\right) \tag{6.31}
\end{equation*}
$$

where $d=\operatorname{GCD}(m, n)$. Substituting this identity into equation (6.30) completes the proof of the claim.

To better understand the scope of the ensuing discussion, we characterize the set of integers $\left\{(p, q) \in \mathbb{N}^{2} \left\lvert\, p<q \wedge \frac{q}{p}\left(\left\lfloor\frac{p-1}{2}\right\rfloor+1\right) \in \mathbb{N}\right.\right\}$, which we shall hereafter refer to as the admissible set of indices.

In Proposition 5.1 in [56], it is shown that $\sigma\left(T_{p, q}\right)$ is not a rational function of $p$ and $q$ in the case that $p$ and $q$ are odd and coprime integers. We generalize this result.

Lemma 6.11. Given positive integers $p$ and $q$ with $p \leqslant q$, the rational $\frac{q}{p}\left(\left\lfloor\frac{p-1}{2}\right\rfloor+\right.$ 1) is integral if and only if $p$ divides $q$ or both $p$ and $q$ are even.

Proof. Clearly, if $p$ divides $q$ (including the case $p=1$ ), then the integrality condition is satisfied. If $p$ is even (and does not divide $q$ ), then $\frac{q}{p}\left(\left\lfloor\frac{p-1}{2}\right\rfloor+1\right)=\frac{q}{2}$, so if $q$ is even, then the integrality condition is satisfied; otherwise, if $q$ is odd, it is violated. If $p$ is odd (and does not divide $q$ ), then $\frac{q}{p}\left(\left\lfloor\frac{p-1}{2}\right\rfloor+1\right)=\frac{(p+1) q}{2 p}$, so the integrality condition is violated regardless of the parity of $q$.

Proposition 6.12. For positive integers $p$ and $q$ such that $p \leqslant q$, if $p$ divides $q$ or if $p$ and $q$ are even, then the signature of the torus link $T_{p, q}$ is given by

$$
\begin{equation*}
\sigma\left(T_{p, q}\right)=(q-1)\left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)-\frac{2 q}{p}\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right) . \tag{6.32}
\end{equation*}
$$

Equivalently, for any admissible pair of indices,

$$
\begin{align*}
& \sum_{k=0}^{m_{p, q}-1}\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha_{p, q}+\frac{k n_{p, q}}{m_{p, q}}\right)-\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha_{p, q}^{\prime}+\frac{k n_{p, q}}{m_{p, q}}\right)\right) \\
& \quad=\left\{\alpha_{p, q}^{\prime}\right\}-\left\{\alpha_{p, q}\right\}-\gamma_{p, q}\left(\left\{\frac{\alpha_{p, q}^{\prime} m_{p, q}}{\gamma_{p, q}}\right\}-\left\{\frac{\alpha_{p, q} m_{p, q}}{\gamma_{p, q}}\right\}\right) \\
& \quad-\gamma_{p, q}\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\frac{\alpha_{p, q}^{\prime} m_{p, q}}{\gamma_{p, q}}\right)-\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\frac{\alpha_{p, q} m_{p, q}}{\gamma_{p, q}}\right)\right), \tag{6.33}
\end{align*}
$$

where $\alpha_{p, q}=\frac{q}{2}-n_{p, q}, \alpha_{p, q}^{\prime}=\frac{3 q}{2}-\frac{q}{p}\left\lfloor\frac{p}{2}\right\rfloor-n_{p, q}, n_{p, q}=\frac{q}{p} m_{p, q}, m_{p, q}=\left\lfloor\frac{p-1}{2}\right\rfloor+1$ and $\gamma_{p, q}=\operatorname{GCD}\left(m_{p, q}, n_{p, q}\right)$.

Proof. For positive integers $p$ and $q$ with $p \leqslant q$, the signature may be written in terms of fractional part summations,

$$
\begin{align*}
\sigma\left(T_{p, q}\right)= & \left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)(q-1) \\
& +2 \sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\lfloor\frac{(p-2 k) q}{2 p}\right\rfloor-\left\lfloor\frac{\left(3 p-2\left\lfloor\frac{p}{2}\right\rfloor-2 k\right) q}{2 p}\right\rfloor  \tag{6.34}\\
= & (q-1)\left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)-\frac{2 q}{p}\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right) \\
& +2 \sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{\left(3 p-2\left\lfloor\frac{p}{2}\right\rfloor-2 k\right) q}{2 p}\right\}-\left\{\frac{(p-2 k) q}{2 p}\right\} . \tag{6.35}
\end{align*}
$$

Suppose $p$ divides $q\left(\left\lfloor\frac{p-1}{2}\right\rfloor+1\right)$. By Proposition 6.10,

$$
\begin{align*}
\sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{(p-2 k) q}{2 p}\right\}= & \sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{q}{2}-\frac{q}{p}\left(\left\lfloor\frac{p-1}{2}\right\rfloor+1\right)+\frac{k q}{p}\right\}  \tag{6.36}\\
= & \sum_{k=1}^{m_{p, q}-1}\left\{\alpha_{p, q}+\frac{k n_{p, q}}{m_{p, q}}\right\}  \tag{6.37}\\
= & \frac{1}{2}\left(m_{p, q}-\gamma_{p, q}\right)-\left\{\alpha_{p, q}\right\} \\
& +\gamma_{p, q}\left(\left\{\frac{\alpha_{p, q} m_{p, q}}{\gamma_{p, q}}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\frac{\alpha_{p, q} m_{p, q}}{\gamma_{p, q}}\right)\right) \\
& -\sum_{k=0}^{m_{p, q}-1} \chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha_{p, q}+\frac{k n_{p, q}}{m_{p, q}}\right) \tag{6.38}
\end{align*}
$$

where $\alpha_{p, q}=\frac{q}{2}-\frac{q}{p}\left(\left\lfloor\frac{p-1}{2}\right\rfloor+1\right), m_{p, q}=\left\lfloor\frac{p-1}{2}\right\rfloor+1, n_{p, q}=\frac{q}{p}\left(\left\lfloor\frac{p-1}{2}\right\rfloor+1\right)$ and $\gamma_{p, q}=\operatorname{GCD}\left(m_{p, q}, n_{p, q}\right)$. Similarly,

$$
\begin{align*}
\sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{\left(3 p-2\left\lfloor\frac{p}{2}\right\rfloor-2 k\right) q}{2 p}\right\}= & \sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{3 q}{2}-\frac{q}{p}\left(\left\lfloor\frac{p-1}{2}\right\rfloor+\left\lfloor\frac{p}{2}\right\rfloor+1\right)+\frac{k q}{p}\right\} \\
= & \sum_{k=1}^{m_{p, q}-1}\left\{\alpha_{p, q}^{\prime}+\frac{k n_{p, q}}{m_{p, q}}\right\}  \tag{6.39}\\
= & \frac{1}{2}\left(m_{p, q}-\gamma_{p, q}\right)-\left\{\alpha_{p, q}^{\prime}\right\} \\
& +\gamma_{p, q}\left(\left\{\frac{\alpha_{p, q}^{\prime} m_{p, q}}{\gamma_{p, q}}\right\}+\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\frac{\alpha_{p, q}^{\prime} m_{p, q}}{\gamma_{p, q}}\right)\right) \\
& -\sum_{k=0}^{m_{p, q}-1} \chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha_{p, q}^{\prime}+\frac{k n_{p, q}}{m_{p, q}}\right) \tag{6.40}
\end{align*}
$$

where $\alpha_{p, q}^{\prime}=\frac{3 q}{2}-\frac{q}{p}\left(\left\lfloor\frac{p-1}{2}\right\rfloor+\left\lfloor\frac{p}{2}\right\rfloor+1\right)$. Note that $\alpha_{p, q}$ and $\alpha_{p, q}^{\prime}$ may assume negative values for certain indices. Combining terms, therefore,

$$
\begin{align*}
\sigma\left(T_{p, q}\right)= & (q-1)\left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)-\frac{2 q}{p}\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right)+2\left\{\alpha_{p, q}\right\}-2\left\{\alpha_{p, q}^{\prime}\right\} \\
& +2 \gamma_{p, q}\left(\left\{\frac{\alpha_{p, q}^{\prime} m_{p, q}}{\gamma_{p, q}}\right\}-\left\{\frac{\alpha_{p, q} m_{p, q}}{\gamma_{p, q}}\right\}\right) \\
& +2 \gamma_{p, q}\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\frac{\alpha_{p, q}^{\prime} m_{p, q}}{\gamma_{p, q}}\right)-\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\frac{\alpha_{p, q} m_{p, q}}{\gamma_{p, q}}\right)\right) \\
& +2 \sum_{k=0}^{m_{p, q}-1}\left(\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha_{p, q}+\frac{k n_{p, q}}{m_{p, q}}\right)-\chi_{\mathbb{R} \backslash \mathbb{Z}}^{-}\left(\alpha_{p, q}^{\prime}+\frac{k n_{p, q}}{m_{p, q}}\right)\right) \tag{6.41}
\end{align*}
$$

holds for positive integers $p$ and $q$ such that $p \leqslant q$, if $\frac{q}{p}\left(\left\lfloor\frac{p-1}{2}\right\rfloor+1\right)$ is an integer. By Lemma 6.11, these values of $p$ and $q$ are precisely the admissible set of indices.

Consider the two fractional part summations

$$
\begin{equation*}
\sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{\left(3 p-2\left\lfloor\frac{p}{2}\right\rfloor-2 k\right) q}{2 p}\right\} \text { and } \sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{(p-2 k) q}{2 p}\right\} \tag{6.42}
\end{equation*}
$$

There are two cases. In the first case, when $p$ divides $q$, the two lists of fractional parts are identical and in the same order. Given that

$$
\begin{equation*}
\frac{\left(3 p-2\left\lfloor\frac{p}{2}\right\rfloor-2 k\right) q}{2 p}=\frac{3 q}{2}-\frac{q}{p}\left\lfloor\frac{p}{2}\right\rfloor-\frac{k q}{p} \quad \text { and } \quad \frac{(p-2 k) q}{2 p}=\frac{q}{2}-\frac{k q}{p} \tag{6.43}
\end{equation*}
$$

it is clear that if $p$ divides $q$ and $q$ is even, then the arguments of the fractional parts are integers, and the summands vanish identically. If $p$ divides $q$ and $q$ is
odd, then the difference of fractional parts summands vanishes by the periodicity of the fractional part function and the identity $\left\{\frac{3 q}{2}\right\}=\left\{\frac{q}{2}\right\}$.

In the second case, when $p$ and $q$ are even, the two lists of fractional parts are identical but in reverse order. For this case, let $p=2 a$ and $q=2 b$. Then

$$
\begin{align*}
\sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{(p-2 k) q}{2 p}\right\} & =\sum_{k=1}^{a-1}\left\{b\left(1-\frac{k}{a}\right)\right\}  \tag{6.44}\\
& =\frac{1}{2}(a-\operatorname{GCD}(a, b)) \tag{6.45}
\end{align*}
$$

However,

$$
\begin{align*}
\sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{\left(3 p-2\left\lfloor\frac{p}{2}\right\rfloor-2 k\right) q}{2 p}\right\} & =\sum_{k=1}^{a-1}\left\{b\left(2+\frac{k}{a}\right)\right\}  \tag{6.46}\\
& =\sum_{k=1}^{a-1}\left\{b\left(2+\frac{(a-1+1-k)}{a}\right)\right\}  \tag{6.47}\\
& =\sum_{k=1}^{a-1}\left\{b\left(1-\frac{k}{a}\right)\right\} \tag{6.48}
\end{align*}
$$

so the difference of fractional parts cancels.

Remark 6.1.7. Although they do not satisfy the requisite divisibility, there are many sporadic pairs of indices with which Proposition 6.12 correctly computes the corresponding signature. For example, equation (6.32) computes the signature of elementary torus links, namely, $\sigma\left(T_{2, q}\right)=1-q$, for $q \geqslant 1$.

Corollary 6.13. For positive integers $1<p \leqslant q$, if $p$ and $q$ are even, then the signature of the torus link $T_{p, q}$ is odd. If $p$ is odd and divides $q$, then the signature of the torus link $T_{p, q}$ is even.

Proof. On any admissible set of indices with $p$ and $q$ even, the integer $-\frac{2 q}{p}\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right)$ is even, and the integer $(q-1)\left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)$ is odd. On any admissible set of indices with $p$ odd and dividing $q$, then both the integers $-\frac{2 q}{p}\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right)$ and $(q-1)\left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)$ are even.

Remark 6.1.8. The remaining case is $p \leqslant q$ odd and $1<\operatorname{GCD}(p, q)<p$, which requires new methods and will be handled in the sequel.

Remark 6.1.9. As in Proposition 6.2, consider the torus links $T_{3,3 m}$ and $T_{4,2 m}$, where $m \geqslant 1$. By equation (6.32),

$$
\begin{align*}
\sigma\left(T_{3,3 m}\right) & =(3 m-1)(2-2)-\frac{6 m}{3}\left\lfloor\frac{3-1}{2}\right\rfloor\left(3-\left\lfloor\frac{3}{2}\right\rfloor\right)  \tag{6.49}\\
& =-4 m \tag{6.50}
\end{align*}
$$

and

$$
\begin{align*}
\sigma\left(T_{4,2 m}\right) & =(2 m-1)\left(4-1-2\left\lfloor\frac{4}{2}\right\rfloor\right)-\frac{4 m}{4}\left\lfloor\frac{4-1}{2}\right\rfloor\left(4-\left\lfloor\frac{4}{2}\right\rfloor\right)  \tag{6.51}\\
& =1-4 m \tag{6.52}
\end{align*}
$$

Corollary 6.14. For positive integers $p$ and $r$,

$$
\begin{align*}
\sigma\left(T_{p, r p}\right) & =\left\{\begin{array}{cc}
1-\frac{r}{2} p^{2} & \text { peven } \\
-\frac{r}{2}(p-1)(p+1) & \text { podd }
\end{array}\right.  \tag{6.53}\\
& \equiv\left\{\begin{array}{ll}
0 & \text { reven, podd or rodd }, p \equiv\{1,7\} \bmod 8 \\
1 & p \equiv 0 \bmod 4 \operatorname{or} r \equiv 0 \bmod 4, p \equiv 2 \bmod 4 \\
3 & r \equiv 3 \bmod 4, p \equiv 2 \bmod 4 \\
4 & r o d d, p \equiv\{3,5\} \bmod 8 \\
5 & r \equiv p \equiv 2 \bmod 4 \\
7 & r \equiv 1 \bmod 4, p \equiv 2 \bmod 4
\end{array}\right\} \bmod 8 . \tag{6.54}
\end{align*}
$$

Proof. By Proposition 6.12,

$$
\begin{align*}
\sigma\left(T_{p, r p}\right) & =(r p-1)\left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)-2 r\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right)  \tag{6.55}\\
& =(r p-1)\left(2\left\{\frac{p}{2}\right\}-1\right)-2 r\left\lfloor\frac{p-1}{2}\right\rfloor\left(\frac{p}{2}+\left\{\frac{p}{2}\right\}\right)  \tag{6.56}\\
& =\left\{\begin{array}{cl}
1-\frac{r}{2} p^{2} & p \text { even } \\
-\frac{r}{2}(p-1)(p+1) & p \text { odd }
\end{array}\right. \tag{6.57}
\end{align*}
$$

as $\lfloor x\rfloor=x-\{x\}$ for real $x \geqslant 0$ and $\left\lfloor\frac{p-1}{2}\right\rfloor=\frac{p}{2}-1$ for any even integer $p$. By a case-by-case analysis, it is straightforward to show that $1-\frac{r}{2} p^{2}$ is not divisible by 8 for any values of $r$ or even $p$. However, $-\frac{r}{2}(p-1)(p+1)$ is divisible by 8 if $r=2 s$ and $p=2 l-1$ with $s, l \in \mathbb{N}$, as $-\frac{r}{2}(p-1)(p+1)=-4 s l(l-1)$ and
$l(l-1)$ is even. Suppose $r=2 s-1$ and $p=8 l+m$ with $s, l, m \in \mathbb{N}$. Then

$$
\begin{equation*}
-\frac{r}{2}(p-1)(p+1) \equiv\left(s-\frac{1}{2}\right)\left(1-m^{2}\right) \bmod 8 \tag{6.58}
\end{equation*}
$$

which is congruent to $0 \bmod 8$ if and only if $m \in\{1,7\}$. All other cases are handled similarly, so the details are omitted.

REmARK 6.1.10. The signature of the torus link $T_{p, r p}$ was computed in [151].

Corollary 6.15. For positive integers $p$ and $q$,

$$
\begin{equation*}
\sigma\left(T_{2 p, 2 q}\right)=1-2 p q . \tag{6.59}
\end{equation*}
$$

In particular, $\sigma\left(T_{2 p, 2 q}\right)$ is not divisible by 8 .

Proof. Observe

$$
\begin{align*}
\sigma\left(T_{2 p, 2 q}\right) & =(2 q-1)(2 p-1-2\lfloor p\rfloor)-\frac{2 q}{p}\left\lfloor\frac{2 p-1}{2}\right\rfloor(2 p-\lfloor p\rfloor)  \tag{6.60}\\
& =1-2 q-2 q\left\lfloor\frac{2 p-1}{2}\right\rfloor  \tag{6.61}\\
& =1-2 p q . \tag{6.62}
\end{align*}
$$

Remark 6.1.11. Observe $\sigma\left(T_{2,2 r}\right)=1-2 r$. Although it is generally true that $\sigma\left(T_{p, q}\right)=\sigma\left(T_{q, p}\right)$ for any positive integers $p$ and $q$, equation (6.32) is not $a$ priori symmetric in $p$ and $q$ in general. However, equation (6.32) does compute
the correct signature for a number of cases, namely,

$$
\begin{align*}
\sigma\left(T_{2 r, 2}\right) & =\left(2 r-1-2\left\lfloor\frac{2 r}{2}\right\rfloor\right)-\frac{4}{2 r}\left\lfloor\frac{2 r-1}{2}\right\rfloor\left(2 r-\left\lfloor\frac{2 r}{2}\right\rfloor\right)  \tag{6.63}\\
& =-2\left\lfloor r-\frac{1}{2}\right\rfloor-1  \tag{6.64}\\
& =1-2 r \tag{6.65}
\end{align*}
$$

both consistent with the last statement of Proposition 6.3. Similarly,

$$
\begin{align*}
\sigma\left(T_{p, 2 p}\right) & =(2 p-1)\left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)-4\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right)  \tag{6.66}\\
& =1-p^{2} \tag{6.67}
\end{align*}
$$

consistent with the second statement of Proposition 6.3. Finally, for $p$-fold Hopf links,

$$
\begin{align*}
\sigma\left(T_{p, p}\right) & =(p-1)\left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)-2\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\frac{p}{2}\right)  \tag{6.68}\\
& =-\frac{1}{2} p^{2}+\frac{3}{4}+\frac{1}{4}(-1)^{p} \tag{6.69}
\end{align*}
$$

$$
\equiv\left\{\begin{array}{rl}
0 \bmod 8 & p \equiv\{1,7\} \bmod 8  \tag{6.70}\\
1 \bmod 8 & p \equiv 0 \bmod 4 \\
4 \bmod 8 & p \equiv\{3,5\} \bmod 8 \\
7 \bmod 8 & p \equiv 2 \bmod 4
\end{array}\right.
$$

consistent with the third statement of Proposition 6.3.

In order to complete our study of the signature of torus links, we require an invariant of quasi-Brieskorn-Pham singularities, namely, the geometric genus, as introduced in §5.6.

### 6.2. Geometric Genus of Quasi-Brieskorn-Pham Singularities

Definition 6.16. Let $\left(a_{0}, \ldots, a_{n}\right)$ denote the equivalence class of quasi-Brieskorn-Pham singularities with inverse weights $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}$. Define

$$
\begin{equation*}
\mathcal{B} \mathcal{P}_{n, m}=\left\{\left(a_{0}, \ldots, a_{n}\right) \in \mathbb{N}^{n+1} \mid p_{g}\left(a_{0}, \ldots, a_{n}\right)=m\right\} \tag{6.71}
\end{equation*}
$$

as the class of quasi-Brieskorn-Pham polynomials in $n+1$ complex variables with corresponding geometric genus equal to $m$.

### 6.2.1. Zero Geometric Genus.

Proposition 6.17. The class $\mathcal{B} \mathcal{P}_{2,0}$ can be partitioned into five subclasses, two of which are infinite in size, corresponding to the five simple singularities, namely, $A_{k}, D_{k}, E_{6}, E_{7}$ and $E_{8}$.

Proof. The Diophantine inequality $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}>1$ has only five integral solutions classes (up to permutation). These are $(1, b, c)$ for $b, c \in \mathbb{N},(2,2, c)$ for $c \in \mathbb{N},(2,3,3),(2,3,4)$ and $(2,3,5)$. Without loss of generality, we may order the integers $a \leqslant b \leqslant c$. If $a=1$, then $\frac{1}{b}+\frac{1}{c}>0$, which has any positive integral pair $(b, c)$ as a solution. If $a=b=2$, then $\frac{1}{c}>0$, which has any positive integer $c$ as a solution. If $2=a<b \leqslant c$, then $\frac{1}{b}+\frac{1}{c}>\frac{1}{2}$, which has only three solutions $(3,3),(3,4)$ and $(3,5)$.

### 6.2.2. Non-zero Geometric Genus.

Proposition 6.18. The class $\mathcal{B} \mathcal{P}_{2, m}$ is finite for all $m \geqslant 1$.

Proof. For any $m \geqslant 1$, suppose on the contrary that $\mathcal{B} \mathcal{P}_{2, m}$ is countably infinite. This would then imply that there are infinitely many integral tetrahedra with a fixed number $m$ of positive lattice points, which is absurd.

Remark 6.2.1. The finiteness above is special to $\mathcal{B} \mathcal{P}_{2, m}$, as $\mathcal{B} \mathcal{P}_{n, m}$ is countably infinite for any $n>2$ and $m \geqslant 1$.

The next result proves that the set $\mathcal{B} \mathcal{P}_{2, \geqslant 1}$ of quasi-Brieskorn-Pham singularities with geometric genus at least equal to 1 has at least 3 infinite families of singularities.

Proposition 6.19. The following quasi-Brieskorn-Pham surface singularities

1. $\left(2,3,6 k_{1}+\ell_{1}\right)$ with $\ell_{1}=0, \ldots, 5$ and $k_{1} \geqslant 1$;
2. $\left(2,4,4 k_{2}+\ell_{2}\right)$ with $\ell_{2}=0, \ldots, 3$ and $k_{2} \geqslant 1$; and,
3. $\left(3,3,3 k_{3}+\ell_{3}\right)$ with $\ell_{3}=0, \ldots, 2$ and $k_{3} \geqslant 1$
have geometric genera $k_{1}, k_{2}, k_{3}$, respectively.

Proof. The first case is the most straight-forward, and the proof for it is similar to those for the last two cases. The equation for the number of nonnegative solutions of the Diophantine inequality enjoys a permutation symmetry under the coefficients, so $\left\{q_{1}, q_{2}, q_{3}\right\} \mapsto\left\{q_{\pi(1)}, q_{\pi(2)}, q_{\pi(3)}\right\}$ is a symmetry
of the closed form solution for the geometric genus of quasi-Brieskorn-Pham surface singularities $\left(a_{1}, a_{2}, a_{3}\right)$.

Define $d=a_{1} a_{2} a_{3}, q_{i}=\frac{d}{a_{i}}$ and $R=d-\sum_{i} q_{i}$. Without loss of generality, we may choose the ordering $q_{3} \leqslant q_{2} \leqslant q_{1}$. For the first statement, we can choose $d=6\left(6 k_{1}+\ell_{1}\right), q_{1}=3\left(6 k_{1}+\ell_{1}\right), q_{2}=2\left(6 k_{1}+\ell_{1}\right)$ and $q_{3}=6$. Then $R=$ $6\left(6 k_{1}+\ell_{1}\right)-5\left(6 k_{1}+\ell_{1}\right)-6=6\left(k_{1}-1\right)+\ell_{1}$. Since $\left\lfloor\frac{R}{q_{1}}\right\rfloor=\frac{6\left(k_{1}-1\right)+\ell_{1}}{3\left(6 k_{1}+\ell_{1}\right)}=0$ and $\left\lfloor\frac{\left(R-q_{1} i_{1}\right)}{q_{2}}\right\rfloor=\frac{6\left(k_{1}-1\right)+\ell_{1}}{2\left(6 k_{1}+\ell_{1}\right)}=0$ for $k_{1}, \ell_{1} \geqslant 1$, the only summation that does not collapse is the innermost one:

$$
\begin{align*}
p_{g}\left(\left\{2,3,6 k_{1}+\ell_{1}\right\}\right) & =\sum_{i_{1}=0}^{\left\lfloor R / q_{1}\right\rfloor} \sum_{i_{2}=0}^{\left\lfloor\left(R-q_{1} i_{1}\right) / q_{2}\right\rfloor} \sum_{i_{3}=0}^{\left\lfloor\left(R-q_{1} i_{1}-q_{2} i_{2}\right) / q_{3}\right\rfloor} 1  \tag{6.72a}\\
& =\sum_{i_{3}=0}^{\left\lfloor R / q_{3}\right\rfloor} 1=\left\lfloor\frac{R}{q_{3}}\right\rfloor+1=\left\lfloor\frac{6\left(k_{1}-1\right)+\ell_{1}}{6}\right\rfloor+1  \tag{6.72b}\\
& =k_{1} \tag{6.72c}
\end{align*}
$$

for $\ell_{1} \in\{0,1,2,3,4,5\}$, which implies the claim of the first statement. Similarly, for the second statement we have $d=8\left(4 k_{2}+\ell_{2}\right), q_{1}=4\left(4 k_{2}+\ell_{2}\right), q_{2}=$ $2\left(4 k_{2}+\ell_{2}\right), q_{3}=8$ and $R=8\left(k_{2}-1\right)+2 \ell_{2}$. Hence,

$$
\begin{align*}
p_{g}\left(\left\{2,4,4 k_{2}+\ell_{2}\right\}\right) & =\sum_{i_{1}=0}^{\left\lfloor R / q_{1}\right\rfloor} \sum_{i_{2}=0}^{\left\lfloor\left(R-q_{1} i_{1}\right) / q_{2}\right\rfloor\left\lfloor\left(R-q_{1} i_{1}-q_{2} i_{2}\right) / q_{3}\right\rfloor} \sum_{i_{3}=0} 1  \tag{6.73a}\\
& =\sum_{i_{3}=0}^{\left\lfloor R / q_{3}\right\rfloor} 1=\left\lfloor\frac{R}{q_{3}}\right\rfloor+1=\left\lfloor\frac{4\left(k_{2}-1\right)+\ell_{2}}{4}\right\rfloor+1  \tag{6.73b}\\
& =k_{2} \tag{6.73c}
\end{align*}
$$

for $\ell_{2} \in\{0,1,2,3\}$. Finally, for the last statement, we have $d=9\left(3 k_{3}+\ell_{3}\right)$, $q_{1}=q_{2}=3\left(3 k_{3}+\ell_{3}\right), q_{3}=9$ and $R=9\left(k_{3}-1\right)+3 \ell_{3}$. Hence,

$$
\begin{align*}
p_{g}\left(\left\{3,3,3 k_{3}+\ell_{3}\right\}\right) & =\sum_{i_{1}=0}^{\left\lfloor R / q_{1}\right\rfloor} \sum_{i_{2}=0}^{\left\lfloor\left(R-q_{1} i_{1}\right) / q_{2}\right\rfloor}\left\lfloor\sum_{i_{3}=0}^{\left\lfloor\left(R-q_{1} i_{1}-q_{2} i_{2}\right) / q_{3}\right\rfloor} 1\right.  \tag{6.74a}\\
& =\sum_{i_{3}=0}^{\left\lfloor R / q_{3}\right\rfloor} 1=\left\lfloor\frac{R}{q_{3}}\right\rfloor+1=\left\lfloor\frac{9\left(k_{3}-1\right)+\ell_{3}}{9}\right\rfloor+1  \tag{6.74b}\\
& =k_{3} \tag{6.74c}
\end{align*}
$$

for $\ell_{3} \in\{0,1,2\}$. This completes the proof.
The next result includes divisibility as a criterion.

Proposition 6.20. For a given non-degenerate, quasi-Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with integral weights $\left\{q_{1}, q_{2}, q_{3}\right\}$ and weighted degree $d$ satisfying $\operatorname{GCD}\left(q_{2}, q_{3}\right)=1, d<2 q_{1}+q_{2}+q_{3},\left\lfloor\frac{R}{q_{2}}\right\rfloor=q_{3}-1$, where $R=d-$ $\sum_{i=1}^{3} q_{i}$ and $q_{3} \mid q_{1}+q_{2}$, then

$$
\begin{equation*}
p_{g}(f)=d+1-q_{1}-\frac{1}{2}\left(q_{2}+1\right)\left(q_{3}+1\right) \tag{6.75}
\end{equation*}
$$

Proof. The second assumption implies

$$
\begin{align*}
p_{g}(f) & =\sum_{k=0}^{\left\lfloor\frac{R}{q_{2}}\right\rfloor}\left(\left\lfloor\frac{R-k q_{2}}{q_{3}}\right\rfloor+1\right)  \tag{6.76}\\
& =\left(q_{3}\left(\frac{R}{q_{3}}\right)-\sum_{k=0}^{q_{3}-1}\left\lceil\frac{k q_{2}}{q_{3}}\right\rfloor\right)+q_{3} \tag{6.77}
\end{align*}
$$

by the third and last assumptions (which implies $q_{3} \mid R$ ). Finally,

$$
\begin{align*}
p_{g}(f) & =R+q_{3}-\sum_{k=0}^{q_{3}-1}\left[\frac{k q_{2}}{q_{3}}\right\rceil  \tag{6.78a}\\
& =R+q_{3}-\sum_{k=1}^{q_{3}-1}\left(\left\lfloor\frac{k q_{2}}{q_{3}}\right\rfloor+1\right)  \tag{6.78b}\\
& =R+1-\frac{1}{2}\left(q_{2}-1\right)\left(q_{3}-1\right) \tag{6.78c}
\end{align*}
$$

by the identity $2 \sum_{k=0}^{q-1}\left\lfloor\frac{k p}{q}\right\rfloor=(p-1)(q-1)$, assuming $\operatorname{GCD}(p, q)=1$, which is ensured by the first assumption. To conclude the proof, simply substitute $R=d-\sum_{i=1}^{3} q_{i}$.

The next few results are applicable to special classes of quasi-BrieskornPham singularities satisfying certain arithmetic constraints.

Proposition 6.21. If a quasi-Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with inverse weights $\left\{a_{1}, a_{2}, a_{3}\right\}$ satisfies

$$
\begin{equation*}
1<\frac{2}{a_{1}}+\frac{1}{a_{2}}+\frac{1}{a_{3}} \quad \text { and } \quad 1<\frac{1}{a_{1}}+\frac{2}{a_{2}}+\frac{1}{a_{3}} \tag{6.79}
\end{equation*}
$$

then the geometric genus is

$$
\begin{equation*}
p_{g}(f)=\left\lfloor a_{3}\left(1-\frac{1}{a_{1}}-\frac{1}{a_{2}}\right)\right\rfloor . \tag{6.8o}
\end{equation*}
$$

In particular, $p_{g}(f)<a_{3}$.

Proof. The proof is obvious once one notices that all summations are trivial save the innermost one, since the assumptions imply $R=d-q_{1}-q_{2}-q_{3}<$ $q_{1}$ and $R<q_{2}$. Hence, $\left\lfloor\frac{R}{q_{1}}\right\rfloor=\left\lfloor\frac{R}{q_{2}}\right\rfloor=0$ and therefore

$$
\begin{align*}
p_{g}(f) & =\sum_{i_{1}=0}^{\left\lfloor R / q_{1}\right\rfloor} \sum_{i_{2}=0}^{\left\lfloor\left(R-q_{1} i_{1}\right) / q_{2}\right\rfloor} \sum_{i_{3}=0}^{\left\lfloor\left(R-q_{1} i_{1}-q_{2} i_{2}\right) / q_{3}\right\rfloor} 1  \tag{6.81}\\
& =\sum_{i_{3}=0}^{\left\lfloor R / q_{3}\right\rfloor} 1=\left\lfloor\frac{R}{q_{3}}\right\rfloor+1 . \tag{6.82}
\end{align*}
$$

Finally, substitute $R=\operatorname{LCM}\left(a_{1}, a_{2}, a_{3}\right)\left(1-\sum_{i=1}^{3} \frac{1}{a_{i}}\right)$ and $q_{3}=\frac{1}{a_{3}} \operatorname{LCM}\left(a_{1}, a_{2}, a_{3}\right)$.

Remark 6.2.2. There are infinitely many triples $\left(a_{1}, a_{2}, a_{3}\right)$ which satisfy the criteria of Proposition 6.21. For instance, the class $\left(2,2, a_{3}\right)$ has zero geometric genus, while that of the class $\left(2,3, a_{3}\right)$ is $\left\lfloor\frac{a_{3}}{6}\right\rfloor$.

Proposition 6.22. If a quasi-Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with inverse weights $\left\{a_{1}, a_{2}, a_{3}\right\}$ satisfies

$$
\begin{equation*}
1<\frac{2}{a_{1}}+\frac{1}{a_{2}}+\frac{1}{a_{3}} \quad \text { and } \quad \frac{2}{a_{2}} \leqslant 1-\frac{1}{a_{1}}-\frac{1}{a_{3}}<\frac{3}{a_{2}} \tag{6.83}
\end{equation*}
$$

then the corresponding geometric genus is

$$
\begin{equation*}
p_{g}(f)=\left\lfloor a_{3}\left(1-\frac{1}{a_{1}}-\frac{1}{a_{2}}\right)\right\rfloor+\left\lfloor a_{3}\left(1-\frac{1}{a_{1}}-\frac{2}{a_{2}}\right)\right\rfloor . \tag{6.84}
\end{equation*}
$$

In particular, $p_{g}(f)<2 a_{3}$.

Proof. The result follows from the identity $p_{g}(f)=\left\lfloor\frac{R}{q_{3}}\right\rfloor+\left\lfloor\frac{R-q_{2}}{q_{3}}\right\rfloor+2$.
Remark 6.2.3. Numerical experiments suggest the following:

1. There are no quasi-Brieskorn-Pham polynomials in $\mathbb{C}^{3}$ which satisfy the two inequalities and are of the form $2=a_{1}=a_{2} \leqslant a_{3}$;
2. There are no quasi-Brieskorn-Pham polynomials in $\mathbb{C}^{3}$ which satisfy the two inequalities and are of the form $a_{1}=2, a_{2}=3 \leqslant a_{3}$;
3. There are no quasi-Brieskorn-Pham polynomials in $\mathbb{C}^{3}$ which satisfy the two inequalities and are of the form $a_{1}=2, a_{2}=4 \leqslant a_{3}$;
4. There are infinitely many quasi-Brieskorn-Pham polynomials in $\mathbb{C}^{3}$ which satisfy the two inequalities and are of the form $a_{1}=2, a_{2}=5 \leqslant$ $a_{3}$ or $a_{1}=2, a_{2}=6 \leqslant a_{3} ;$
5. There are 7 quasi-Brieskorn-Pham polynomials in $\mathbb{C}^{3}$ which satisfy the two inequalities and are of the form $a_{1}=2, a_{2}=7 \leqslant a_{3}$. These are $(2,7, k)$ with $k \in\{7,8,9,10,11,12,13\} ;$
6. There are no quasi-Brieskorn-Pham polynomials in $\mathbb{C}^{3}$ which satisfy the two inequalities and are of the form $a_{1}=2,8 \leqslant a_{2} \leqslant a_{3}$;
7. There are 9 quasi-Brieskorn-Pham polynomials in $\mathbb{C}^{3}$ which satisfy the two inequalities and are of the form $3 \leqslant a_{1} \leqslant a_{2} \leqslant a_{3}$ are the following 9: $(3,4, l)$ with $l \in\{6,7,8,9,10,11\}$ and $(3,5, k)$ with $k \in\{5,6,7\}$;
8. There are no quasi-Brieskorn-Pham polynomials in $\mathbb{C}^{3}$ which satisfy the two inequalities and are of the form $a_{1}=3,6 \leqslant a_{2} \leqslant a_{3}$; and,
9. There are no quasi-Brieskorn-Pham polynomials in $\mathbb{C}^{3}$ which satisfy the two inequalities above and are of the form $4 \leqslant a_{1} \leqslant a_{2} \leqslant a_{3}$.

These nine cases cover all possibilities and are special cases of the following result.

Proposition 6.23. If a quasi-Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with inverse weights $\left\{a_{1}, a_{2}, a_{3}\right\}$ satisfies

$$
\begin{equation*}
1<\frac{2}{a_{1}}+\frac{1}{a_{2}}+\frac{1}{a_{3}} \tag{6.85}
\end{equation*}
$$

then the corresponding geometric genus is

$$
\begin{equation*}
p_{g}(f)=1+\left\lfloor\frac{R}{q_{2}}\right\rfloor+\sum_{k=0}^{\left\lfloor\frac{R}{q_{2}}\right\rfloor}\left\lfloor\frac{R-k q_{2}}{q_{3}}\right\rfloor \tag{6.86}
\end{equation*}
$$

Proof. The assumption implies $d<2 q_{1}+q_{2}+q_{3}$, hence $\left\lfloor\frac{R}{q_{1}}\right\rfloor=0$. The outer sum collapses and the inner sums simplify.

In Appendix D, we tabulate those quasi-Brieskorn-Pham singularities by geometric genera no greater than 25 .
6.2.3. Geometric Genus of Quasi-Brieskorn-Pham Singularities. Recall that a polynomial is Brieskorn-Pham if and only if it is of the form $f=\sum_{i=0}^{n} z_{i}^{a_{i}}$ with exponents $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}$, and quasi-Brieskorn-Pham if and only if it is a non-degenerate weighted homogeneous with an integral weight polytope.

If $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is quasi-Brieskorn-Pham, then the corresponding weight polytope $\mathcal{W}(f)$ is an integral trirectangular tetrahedron, and the geometric genus $p_{g}\left(f_{t}\right)$, counting the number of positive lattice points intersecting $\mathcal{W}(f)$, then coincides with the Ehrhart quasi-polynomial $\mathscr{L}_{\mathcal{W}}\left(t ; \mathbb{N}^{3}\right)$, namely, for $t \in \mathbb{N}$,

$$
\begin{align*}
p_{g}\left(f_{t}\right)= & \sum_{i=1}^{a t} \sum_{j=1}^{\lfloor b(t-i / a)\rfloor} \sum_{k=1}^{\lfloor c(t-i / a-j / b)\rfloor} 1  \tag{6.87a}\\
= & \frac{a b c}{6} t^{3}-\frac{(a+b) c}{4} t^{2}+\left(\frac{c}{4}+\frac{b c}{12 a}\right) t+\frac{c}{2 b} \sum_{i=1}^{a t}\left\{b\left(t-\frac{i}{a}\right)\right\} \\
& \quad-\frac{c}{2 b} \sum_{i=1}^{a t}\left\{b\left(t-\frac{i}{a}\right)\right\}^{2}-\sum_{i=1}^{a t} \sum_{j=1}^{\lfloor b(t-i / a)\rfloor}\left\{c\left(t-\frac{i}{a}-\frac{j}{b}\right)\right\} . \tag{6.87b}
\end{align*}
$$

Let $a, b$ and $c$ be positive integers with no common factor, i.e., $\operatorname{GCD}(a, b, c)=$ 1. Define $a^{\prime}=\operatorname{GCD}(b, c), b^{\prime}=\operatorname{GCD}(c, a), c^{\prime}=\operatorname{GCD}(a, b), d=a^{\prime} b^{\prime} c^{\prime}, l=a+b+c$ and $l^{\prime}=a^{\prime}+b^{\prime}+c^{\prime}$. In Volume 2 of this work, we prove that the number of positive lattice points in the (integral) $t$-dilate of the lattice tetrahedron $\mathcal{W}=$ $\operatorname{conv}\left\{\mathbf{0}, a \mathbf{e}_{1}, b \mathbf{e}_{2}, c \mathbf{e}_{3}\right\}$ simplifies

$$
\begin{gather*}
\mathscr{L}_{\mathcal{W}}\left(t ; \mathbb{N}^{3}\right)=\frac{a b c}{6} t^{3}-\frac{(a+b) c}{4} t^{2}+\left(\frac{c}{4}+\frac{b c}{12 a}+\frac{c\left(a+c^{\prime}\right)\left(a-c^{\prime}\right)}{12 a b}\right) t \\
-\sum_{i=1}^{a t} \sum_{j=1}^{[b(t-i / a)\rfloor}\left\{c\left(t-\frac{i}{a}-\frac{j}{b}\right)\right\} . \tag{6.88}
\end{gather*}
$$

Although not at all obvious, equation (6.88) is a polynomial of degree 3 .

Remark 6.2.4. A simple and sharp upper bound for the geometric genus of quasi-Brieskorn-Pham surface singularities is given by utilizing the nonnegativity of the fractional part summation, namely,

$$
\begin{equation*}
p_{g}\left(f_{t}\right) \leqslant\left\lfloor\frac{a b c}{6} t^{3}-\frac{(a+b) c}{4} t^{2}+\left(\frac{c}{4}+\frac{b c}{12 a}+\frac{c\left(a+c^{\prime}\right)\left(a-c^{\prime}\right)}{12 a b}\right) t\right\rfloor, \tag{6.89}
\end{equation*}
$$

which is an equality if $a$ and $b$ divide $c$ (and therefore stronger than the classical Durfee-Yau Inequality, $6 p_{g} \leqslant \mu_{\mathrm{alg}}$ ). The exponents $(2,3,6),(2,4,4)$ and $(3,3,3)$ are such examples. Any weighted homogeneous polynomial with weight sum equal to unity is exactly one of these three cases [361].

Define the symmetric, three-term Dedekind sum $\mathfrak{S}(a, b, c ; d)$ as

$$
\begin{align*}
\mathfrak{S}(a, b, c ; d) & =a^{\prime} \mathfrak{s}\left(\frac{b c}{d}, \frac{a a^{\prime}}{d}\right)+b^{\prime} \mathfrak{s}\left(\frac{a c}{d}, \frac{b b^{\prime}}{d}\right)+c^{\prime} \mathfrak{s}\left(\frac{a b}{d}, \frac{c c^{\prime}}{d}\right)  \tag{6.90}\\
& =a^{\prime} \mathfrak{s}\left(\frac{b c}{d}, \frac{a}{b^{\prime} c^{\prime}}\right)+b^{\prime} \mathfrak{s}\left(\frac{a c}{d}, \frac{b}{a^{\prime} c^{\prime}}\right)+c^{\prime} \mathfrak{s}\left(\frac{a b}{d}, \frac{c}{a^{\prime} b^{\prime}}\right) . \tag{6.91}
\end{align*}
$$

In Volume 2, we prove that the fractional part summation in equation (6.88) admits the following exact representation in terms of Dedekind sum functions,

$$
\begin{equation*}
\sum_{i=1}^{a t} \sum_{j=1}^{[b(t-i / a)]}\left\{c\left(t-\frac{i}{a}-\frac{j}{b}\right)\right\}=\frac{a b-d}{4} t^{2}+(\mathfrak{S}(a, b, c ; d)+\gamma) t \tag{6.92}
\end{equation*}
$$

where $\gamma=\frac{1}{4}\left(a^{\prime}+b^{\prime}+c^{\prime}-a-b\right)-\frac{a^{2} b^{2}+c^{2}\left(c^{\prime}\right)^{2}+d^{2}}{12 a b c}$.
Proposition 6.24. Given $a, b, c, t \in \mathbb{N}$ with $\operatorname{GCD}(a, b, c)=1$, the geometric genus of the (positive integral) $t$-dilate of the quasi-Brieskorn-Pham polynomial $f$ with
weights $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$ is the degree 3 polynomial

$$
\begin{align*}
p_{g}\left(f_{t}\right)=\frac{a b c}{6} & t^{3}-\frac{1}{4}(a b+b c+c a-d) t^{2} \\
& +\left(\frac{1}{4}\left(l-l^{\prime}\right)+\frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c}\right)-\mathfrak{S}(a, b, c ; d)\right) t \tag{6.93}
\end{align*}
$$

where $l=a+b+c, a^{\prime}=\operatorname{GCD}(b, c), b^{\prime}=\operatorname{GCD}(a, c), c^{\prime}=\operatorname{GCD}(a, b), l^{\prime}=a^{\prime}+b^{\prime}+c^{\prime}$ and $d=a^{\prime} b^{\prime} c^{\prime}$.

Proof. See Volume 2.

Remark 6.2.5. Suppose $f$ is a quasi-Brieskorn Pham polynomial with inverse weights $\{a, b, 2\}$, where $a$ and $b$ are odd and coprime. By Proposition 6.24,

$$
\begin{equation*}
p_{g}(f)=\frac{1}{24 a b}+\frac{1}{6}\left(\frac{a}{b}+\frac{b}{a}\right)-\frac{a+b}{4}+\frac{a b}{8}-\mathfrak{s}(2 a, b)-\mathfrak{s}(2 b, a), \tag{6.94}
\end{equation*}
$$

which yields a reciprocity law (unlike that of Dedekind),

$$
\begin{align*}
\mathfrak{s}(2 a, b)+\mathfrak{s}(2 b, a) & =-p_{g}(f)+\frac{1}{24 a b}+\frac{1}{6}\left(\frac{a}{b}+\frac{b}{a}\right)-\frac{a+b}{4}+\frac{a b}{8}  \tag{6.95}\\
& =\frac{1}{24 a b}+\frac{1}{6}\left(\frac{a}{b}+\frac{b}{a}\right)-\frac{a+b}{4}+\frac{a b}{8}-\sum_{i=1}^{\lfloor a / 2\rfloor}\left\lfloor\frac{b}{2}-\frac{b i}{a}\right\rfloor . \tag{6.96}
\end{align*}
$$

See the proof of Proposition 6.56 for a simplification of $p_{g}(f)$ into the stated summation of floor functions.

However, one need not only restrict attention to those integral triples that have no factor in common.

Proposition 6.25. Given $a, b, c, t \in \mathbb{N}$, the geometric genus of the $t$-dilate of the quasi-Brieskorn-Pham polynomial $f$ with inverse weights $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$ is the degree 3 polynomial

$$
\begin{align*}
p_{g}\left(f_{t}\right)=\frac{a b c}{6} & t^{3}-\frac{1}{4}\left(a b+b c+c a-\frac{d}{\tau}\right) t^{2} \\
& +\left(\frac{1}{4}\left(l-l^{\prime}\right)+\frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right)-\mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)\right) t \tag{6.97}
\end{align*}
$$

where $\tau=\operatorname{GCD}(a, b, c), l=a+b+c, a^{\prime}=\operatorname{GCD}(b, c), b^{\prime}=\operatorname{GCD}(a, c), c^{\prime}=$ $\operatorname{GCD}(a, b), l^{\prime}=a^{\prime}+b^{\prime}+c^{\prime}$ and $d=a^{\prime} b^{\prime} c^{\prime}$.

Proof. Define $\tilde{f}$ to be the quasi-Brieskorn-Pham polynomial with inverse weights $\{\tilde{a}, \tilde{b}, \tilde{c}\}$, where $\tilde{a}=\frac{a}{\tau}, \tilde{b}=\frac{b}{\tau}$ and $\tilde{c}=\frac{c}{\tau}$. Since $\operatorname{GCD}(\tilde{a}, \tilde{b}, \tilde{c})=1$, it follows that

$$
\begin{align*}
& p_{g}(f)=p_{g}\left(\tilde{f}_{\tau}\right)  \tag{6.98}\\
& =\frac{\tilde{a} \tilde{b} \tilde{6}}{6} \tau^{3}-\frac{1}{4}(\tilde{a} \tilde{b}+\tilde{b} \tilde{c}+\tilde{c} \tilde{a}-\tilde{d}) \tau^{2} \\
& +\left(\frac{1}{4}\left(\tilde{l}-\tilde{l}^{\prime}\right)+\frac{1}{12}\left(\frac{\tilde{b} \tilde{b}}{\tilde{c}}+\frac{\tilde{b} \tilde{c}}{\tilde{a}}+\frac{\tilde{c} \tilde{a}}{\tilde{b}}+\frac{\tilde{d}^{2}}{\tilde{a} \tilde{b} \tilde{c}}\right)-\mathfrak{S}(\tilde{a}, \tilde{b}, \tilde{c} ; \tilde{d})\right) \tau  \tag{6.99}\\
& =\frac{a b c}{6}-\frac{1}{4}\left(a b+b c+c a-\tilde{d} \tau^{2}\right) \\
& +\frac{1}{4}\left(l-\tilde{l}^{\prime} \tau\right)+\frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{\tilde{d}^{2} \tau}{\tilde{a} \tilde{b} \tilde{c}}\right)-\tau \mathfrak{S}(\tilde{a}, \tilde{b}, \tilde{c} ; \tilde{d})  \tag{6.100}\\
& =\frac{a b c}{6}-\frac{1}{4}\left(a b+b c+c a-\frac{d}{\tau}\right) \\
& +\frac{1}{4}\left(l-l^{\prime}\right)+\frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right)-\mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right), \tag{6.101}
\end{align*}
$$

by Proposition 6.24, where $\tilde{a}^{\prime}=\operatorname{GCD}(\tilde{b}, \tilde{c}), \tilde{b}^{\prime}=\operatorname{GCD}(\tilde{c}, \tilde{a}), \tilde{c}^{\prime}=\operatorname{GCD}(\tilde{a}, \tilde{b})$,

$$
\tilde{d}=\tilde{a}^{\prime} \tilde{b}^{\prime} \tilde{c}^{\prime}, \tilde{l}=\tilde{a}+\tilde{b}+\tilde{c} \text { and } \tilde{l}^{\prime}=\tilde{a}^{\prime}+\tilde{b}^{\prime}+\tilde{c}^{\prime} \text { and }
$$

$$
\begin{equation*}
\mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)=a^{\prime} \mathfrak{s}\left(\frac{b c \tau}{d}, \frac{a a^{\prime} \tau}{d}\right)+b^{\prime} \mathfrak{s}\left(\frac{a c \tau}{d}, \frac{b b^{\prime} \tau}{d}\right)+c^{\prime} \mathfrak{s}\left(\frac{a b \tau}{d}, \frac{c c^{\prime} \tau}{d}\right) \tag{6.102}
\end{equation*}
$$

The last equality follows from the identity $\operatorname{GCD}(p r, q r)=r \operatorname{GCD}(p, q)$. Finally, take $a^{*}=a t, b^{*}=b t$ and $c^{*}=c t$ and simplify.

Remark 6.2.6. Suppose $f$ is a quasi-Brieskorn Pham polynomial with inverse weights $\{a, b, 2\}$, where $a, b \in \mathbb{N}$. Then

$$
\begin{align*}
p_{g}(f)= & \frac{a b}{3}-\frac{1}{4}\left(a b+2 a+2 b-\frac{d}{\tau}\right) \\
& +\frac{1}{4}\left(a+b+2-l^{\prime}\right)+\frac{1}{12}\left(\frac{a b}{2}+\frac{2 a}{b}+\frac{2 b}{a}+\frac{d^{2}}{2 a b \tau^{2}}\right)-\mathfrak{S}\left(a, b, 2 ; \frac{d}{\tau}\right)  \tag{6.103}\\
= & \frac{d^{2}}{24 a b \tau^{2}}+\frac{1}{6}\left(\frac{a}{b}+\frac{b}{a}\right)-\frac{1}{4}\left(a+b+l^{\prime}-\frac{d}{\tau}\right)+\frac{1}{2}+\frac{a b}{8}-\mathfrak{S}\left(a, b, 2 ; \frac{d}{\tau}\right) . \tag{6.104}
\end{align*}
$$

### 6.2.4. Bounds for the Geometric Genus.

Proposition 6.26. Given $a, b, c, t \in \mathbb{N}$ with $\operatorname{GCD}(a, b, c)=1$, the geometric genus of the $t$-dilate of the quasi-Brieskorn-Pham polynomial $f$ with weights $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$ satisfies the sharp lower and upper bounds:

$$
\begin{equation*}
\left\lceil\frac{a b c}{6} t^{3}-\frac{1}{4} \kappa t^{2}+\eta_{-} t\right\rceil \leqslant p_{g}\left(f_{t}\right) \leqslant\left\lfloor\frac{a b c}{6} t^{3}-\frac{1}{4} \kappa t^{2}+\eta_{+} t\right\rfloor \tag{6.105}
\end{equation*}
$$

where $\kappa=a b+b c+c a-d$ and the linear coefficients $\eta_{+}$and $\eta_{-}$are given by

$$
\begin{align*}
\eta_{-}= & \frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{a c}{b}+\frac{d^{2}}{a b c}\right)-\frac{1}{12 d}\left(a\left(a^{\prime}\right)^{2}+b\left(b^{\prime}\right)^{2}+c\left(c^{\prime}\right)^{2}\right) \\
& \quad-\frac{d}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{1}{4} l  \tag{6.106}\\
\eta_{+}= & \frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{a c}{b}+\frac{d^{2}}{a b c}\right)+\frac{1}{12 d}\left(a\left(a^{\prime}\right)^{2}+b\left(b^{\prime}\right)^{2}+c\left(c^{\prime}\right)^{2}\right) \\
& +\frac{d}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{1}{4} l-\frac{1}{2} l^{\prime}, \tag{6.107}
\end{align*}
$$

respectively.

Proof. The Dedekind sum function admits the following sharp lower and upper bounds [390],

$$
\begin{equation*}
-\mathfrak{s}(1, q) \leqslant \mathfrak{s}(p, q) \leqslant \mathfrak{s}(1, q)=\frac{q}{12}+\frac{1}{6 q}-\frac{1}{4}, \tag{6.108}
\end{equation*}
$$

provided that $p$ and $q$ are coprime. From this one infers the bounds

$$
\begin{equation*}
-R(a, b, c ; d) \leqslant \mathfrak{S}(a, b, c ; d) \leqslant R(a, b, c ; d) \tag{6.109}
\end{equation*}
$$

where

$$
\begin{equation*}
R(a, b, c ; d)=\frac{1}{12 d}\left(a\left(a^{\prime}\right)^{2}+b\left(b^{\prime}\right)^{2}+c\left(c^{\prime}\right)^{2}\right)+\frac{d}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{1}{4} l^{\prime} \tag{6.110}
\end{equation*}
$$

where $l^{\prime}=a^{\prime}+b^{\prime}+c^{\prime}$. The lower and upper bounds now follow by substituting $R(a, b, c ; d)$ (with appropriate sign) in place of $\mathfrak{S}(a, b, c ; d)$ in equation (6.93).

For weighted homogeneous surface singularities, the Durfee-Yau Conjecture asserts $6 p_{g}(f) \leqslant \mu_{\mathrm{alg}}(f)-v(f)+1$, where $v(f)=\left\lceil\min _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}\right\}\right\rceil$.

Proposition 6.27. The Durfee-Yau Conjecture is true for quasi-BrieskornPham singularities with pairwise coprime exponents with multiplicity greater than or equal to 3. In particular, the following stronger bound is true

$$
\begin{equation*}
6 p_{g}(f) \leqslant \mu_{\mathrm{alg}}(f)-6 \tag{6.111}
\end{equation*}
$$

Proof. First, assume that $a, b, c \geqslant 2$ and $\operatorname{Gcd}(a, b, c)=1$. Recall that

$$
\begin{equation*}
\mu_{\mathrm{alg}}\left(f_{t}\right)=(a t-1)(b t-1)(c t-1) . \tag{6.112}
\end{equation*}
$$

Consider the following difference polynomial

$$
\begin{align*}
P(t)= & \frac{1}{6}(a t-1)(b t-1)(c t-1)-\lambda \\
& -\left(\frac{a b c}{6} t^{3}-\frac{1}{4}(a b+b c+c a-d) t^{2}+\eta+t\right)  \tag{6.113}\\
= & \alpha t^{2}+\beta t-\left(\lambda+\frac{1}{6}\right), \tag{6.114}
\end{align*}
$$

where $\alpha=\frac{1}{12}(a b+b c+c a-d)$ and

$$
\begin{align*}
\beta=\frac{1}{2} l^{\prime} & -\frac{1}{12}\left(l+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c}\right)-\frac{1}{12 d}\left(a\left(a^{\prime}\right)^{2}+b\left(b^{\prime}\right)^{2}+c\left(c^{\prime}\right)^{2}\right) \\
& +\frac{d}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) . \tag{6.115}
\end{align*}
$$

Proposition 6.26 implies the inequality $6\left(p_{g}(f)+\lambda\right) \leqslant \mu_{\mathrm{alg}}(f)$ if and only if $P(1)=\alpha+\beta-\lambda-\frac{1}{6} \geqslant 0$, or

$$
\begin{align*}
(a b+b c+c a)+6\left(a^{\prime}+b^{\prime}+c^{\prime}\right) \geqslant & \left(a+b+c+3 d+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c}\right) \\
& +\frac{1}{d}\left(a\left(a^{\prime}\right)^{2}+b\left(b^{\prime}\right)^{2}+c\left(c^{\prime}\right)^{2}\right) \\
& +2 d\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+12 \lambda+2 \tag{6.116}
\end{align*}
$$

Now suppose further that $a, b$ and $c$ are pairwise coprime and each not less than 3. Thus, $\mu(f) \geqslant 3$ [423]. Then $a^{\prime}=b^{\prime}=c^{\prime}=d=1$, and the inequality of equation (6.116) simplifies to

$$
\begin{align*}
(a b+b c+c a)+13 \geqslant & \left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{1}{a b c}\right) \\
& +2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+12 \lambda \tag{6.117}
\end{align*}
$$

which we now prove. Observe that

$$
\begin{equation*}
0 \leqslant\left(\frac{2}{3} b-2\right) a+\left(\frac{2}{3} c-2\right) b+\left(\frac{2}{3} a-2\right) c+\frac{296}{27}-12 \lambda . \tag{6.118}
\end{equation*}
$$

provided that $\lambda \leqslant \frac{74}{81}$. Upon separating the signed terms,

$$
\begin{align*}
& a b+b c+c a+13 \geqslant\left(\frac{a b}{3}+\frac{b c}{3}+\frac{c a}{3}+\frac{1}{27}\right)+2(a+b+c+1)+12 \lambda  \tag{6.119a}\\
& \geqslant\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{1}{a b c}\right) \\
&+2\left(a+b+c+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+12 \lambda \tag{6.119b}
\end{align*}
$$

which proves the inequality. Finally, since $p_{g}(f)$ and $\mu_{\mathrm{alg}}(f)$ are positive integers, then one has the slightly improved bound $6 p_{g}(f) \leqslant \mu_{\mathrm{alg}}(f)-6$ since $\lceil\lambda\rceil=1$.

REmARK 6.2.7. Using similar methods, one can prove the even stronger inequality

$$
\begin{equation*}
6 p_{g}(f) \leqslant \mu_{\mathrm{alg}}(f)-\left\lceil\frac{1}{2}(a+b+c)+\frac{735}{128}\right\rceil \tag{6.120}
\end{equation*}
$$

provided that $a, b$ and $c$ are pairwise coprime and each not less than 4 . For example, for weights $\left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$, the first bound gives $18 \leqslant 24$ versus $12 \leqslant 24$ for the standard conjecture. For weights $\left\{\frac{1}{4}, \frac{1}{5}, \frac{1}{7}\right\}$, the second bound gives $62 \leqslant 72$ versus $48 \leqslant 72$ for the standard conjecture.

Though not nearly as elementary as the previous upper bound, a substantially stronger upper bound is the following.

Proposition 6.28. Given $a, b, c, t \in \mathbb{N}$ with $\operatorname{Gcd}(a, b, c)=1$, the geometric genus of the $t$-dilate of the quasi-Brieskorn-Pham polynomial $f$ with weights $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$ satisfies the following identity,

$$
\begin{equation*}
6 p_{g}(f) \leqslant \mu_{\mathrm{alg}}(f)-\frac{1}{2}(a b+b c+c a)+\frac{3}{2} d+6 \eta_{+}-l+1 \tag{6.121}
\end{equation*}
$$

where $\eta_{+}$is given by equation (6.107).

Proof. This is, of course, a special case of the polynomial inequality

$$
\begin{equation*}
6 p_{g}\left(f_{t}\right) \leqslant \mu_{\mathrm{alg}}\left(f_{t}\right)-\frac{1}{2}(a b+b c+c a-3 d) t^{2}+\left(6 \eta_{+}-l\right) t+1 \quad t \in \mathbb{N} \tag{6.122}
\end{equation*}
$$

To prove the Durfee Conjecture, we need to the prove the following inequality:

$$
\begin{equation*}
(a b+b c+c a)+2 l \geqslant 3 d+12 \eta_{+}+2 \tag{6.123}
\end{equation*}
$$

where

$$
\begin{align*}
\eta_{+}= & \frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{a c}{b}+\frac{d^{2}}{a b c}\right)+\frac{1}{12 d}\left(a\left(a^{\prime}\right)^{2}+b\left(b^{\prime}\right)^{2}+c\left(c^{\prime}\right)^{2}\right) \\
& +\frac{d}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{1}{4} l-\frac{1}{2} l^{\prime}  \tag{6.124a}\\
\leqslant & \frac{1}{24}\left(a b+b c+a c+\frac{d^{2}}{4}\right)+\frac{1}{12}\left(a a^{\prime}+b b^{\prime}+c c^{\prime}\right) \\
& +\frac{1}{4}(a+b+c+d)-\frac{1}{2}\left(a^{\prime}+b^{\prime}+c^{\prime}\right) \tag{6.124b}
\end{align*}
$$

It is now a trivial matter to prove the Durfee Conjecture for pairwise coprime integers $a, b, c \geqslant 2$ such that $\operatorname{GCD}(a, b, c)=1$. We need only to the prove the following inequality

$$
\begin{equation*}
-\frac{1}{2}(a b+b c+c a)+6 \eta_{+}-l+\frac{5}{2} \leqslant 0 \tag{6.125}
\end{equation*}
$$

Suppose $a, b, c$ are pairwise coprime and recall that $l=a+b+c$. Then $a^{\prime}=b^{\prime}=$ $c^{\prime}=d=1$ and

$$
\begin{align*}
12 \eta_{+}+5 & =\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{1}{a b c}\right)+4 l+2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-13  \tag{6.126a}\\
& \leqslant \frac{1}{2}(a b+b c+c a)+4 l-\frac{79}{8} . \tag{6.126b}
\end{align*}
$$

Finally, the desired inequality

$$
\begin{equation*}
\frac{1}{2}(a b+b c+c a)+4 l-\frac{79}{8}<(a b+b c+c a)+2 l \tag{6.127}
\end{equation*}
$$

or, equivalently, $(a b+b c+c a)-4(a+b+c)>-\frac{79}{4}=-19 \frac{3}{4}$ which is true and follows from the following fact: The inhomogeneous quadratic form

$$
\begin{equation*}
Q(a, b, c)=(a b+b c+c a)-4(a+b+c) \tag{6.128}
\end{equation*}
$$

achieves a minimum value of $Q(2,2,2)=-12$ on $\mathbb{N}_{>1}^{3}$.
6.2.4.1. Geometric Genus of Quasi-Brieskorn-Pham Singularities. In this section, we prove an identity relating the algebraic index and geometric genus of a quasi-Brieskorn-Pham singularity with pairwise coprime inverse weights.

Proposition 6.29. Given $a, b, c \in \mathbb{N}$ with $\operatorname{GCD}(a, b, c)=1$, the geometric genus $p_{g}(f)$ and algebraic index $\mu_{\mathrm{ag}}(f)$ of the quasi-Brieskorn-Pham polynomial $f$
with weights $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$ satisfy the following identity

$$
\begin{align*}
6 p_{g}(f)= & \mu_{\mathrm{a} \lg }(f)-\frac{3}{2}\left(l^{\prime}-d\right)+\frac{1}{2}\left(l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c}\right) \\
& -6 \mathfrak{S}(a, b, c ; d)+1 \tag{6.129}
\end{align*}
$$

where $l=a+b+c, l^{\prime}=a^{\prime}+b^{\prime}+c^{\prime}, d=a^{\prime} b^{\prime} c^{\prime}$ and

$$
\begin{equation*}
\mathfrak{S}(a, b, c ; d)=a^{\prime} \mathfrak{s}\left(\frac{b c}{d}, \frac{a a^{\prime}}{d}\right)+b^{\prime} \mathfrak{s}\left(\frac{a c}{d}, \frac{b b^{\prime}}{d}\right)+c^{\prime} \mathfrak{s}\left(\frac{a b}{d}, \frac{c c^{\prime}}{d}\right) . \tag{6.130}
\end{equation*}
$$

Proof. The claim follows from Proposition 6.24 and the identity

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=(a-1)(b-1)(c-1) . \tag{6.131}
\end{equation*}
$$

### 6.3. Delta Invariant and Geometric Genus

Corollary 6.30. Given a non-degenerate, square-free quasi-Brieskorn-Pham polynomial $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with inverse weights $\{a, b\}$, then

$$
\begin{equation*}
\delta(f)=p_{g}\left(\Sigma f_{2}\right) \tag{6.132}
\end{equation*}
$$

Proof. Let $\tilde{f}:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be any quasi-Brieskorn-Pham surface singularity with inverse weights $\{a, b, 1\}$. By Proposition $5 \cdot 46$, one has $\delta\left(f_{t}\right)=$ $p_{g}\left(\tilde{f}_{t+1}\right)-p_{g}\left(\tilde{f}_{t}\right)$. For $t=1, p_{g}(\tilde{f})=0$ and $\tilde{f}_{2} \simeq \Sigma f_{2}$. The claim now follows.

Remark 6.3.1. Let $f$ be a quasi-Brieskorn-Pham singularity with inverse weights $\{a, b\}$. Then $\Sigma f_{2}$ is a quasi-Brieskorn-Pham singularity with inverse
weights $\{2 a, 2 b, 2\}$. Since the geometric genus of the latter singularity coincides with the delta invariant of the former singularity,

$$
\begin{align*}
\delta(f)= & \frac{(\operatorname{GCD}(a, b))^{2}}{6 a b}+\frac{1}{6}\left(\frac{a}{b}+\frac{b}{a}\right)+\frac{1}{2}(a b-a-b+\operatorname{GCD}(a, b)-1) \\
& -2\left(\mathfrak{s}\left(\frac{b}{\operatorname{GCD}(a, b)}, \frac{a}{\operatorname{GCD}(a, b)}\right)+\mathfrak{s}\left(\frac{a}{\operatorname{GCD}(a, b)}, \frac{b}{\operatorname{GCD}(a, b)}\right)\right), \tag{6.133}
\end{align*}
$$

and, together with the identity $\delta(f)=\frac{1}{2}(a b-a-b+\operatorname{GCD}(a, b))$, implies the Dedekind Reciprocity Law, q.v. Remark 6.1.4,

$$
\begin{equation*}
\mathfrak{s}\left(\frac{b}{\operatorname{GCD}(a, b)}, \frac{a}{\operatorname{GCD}(a, b)}\right)+\mathfrak{s}\left(\frac{a}{\operatorname{GCD}(a, b)}, \frac{b}{\operatorname{GCD}(a, b)}\right)=\frac{1}{12}\left(\frac{a}{b}+\frac{(\operatorname{GCD}(a, b))^{2}}{a b}+\frac{b}{a}\right)-\frac{1}{4} . \tag{6.134}
\end{equation*}
$$

Corollary 6.31. Let $f:\left(\mathbb{C}^{4}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a quasi-Brieskorn-Pham singularity with inverse weights $\{k a, k b, k c, k\} \subset \mathbb{N}^{4}$. Then the geometric genus $p_{g}(f)$ is given by $p_{g}(f)=A k^{4}+B k^{3}+C k^{2}+D k$, where $A=\frac{a b c}{24}$,

$$
\begin{align*}
& B=-\frac{1}{12}\left(a b+b c+c a-\frac{d}{\tau}\right)-\frac{a b c}{12}  \tag{6.135}\\
& C=\frac{a b c}{24}+\frac{1}{8}\left(a b+b c+c a-\frac{d}{\tau}\right)+E  \tag{6.136}\\
& D=-\frac{1}{24}\left(a b+b c+c a-\frac{d}{\tau}\right)-E  \tag{6.137}\\
& E=\frac{1}{2}\left(\frac{1}{4}\left(l-l^{\prime}\right)+\frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right)-\mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)\right) . \tag{6.138}
\end{align*}
$$

Proof. Let $h:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ denote the quasi-Brieskorn-Pham surface singularity with inverse exponents $\{a, b, c\}$ and geometric genus $p_{g}(t)=p_{g}\left(h_{t}\right)$. The claim follows from the relation $p_{g}(f)=\sum_{i=1}^{k} p_{g}(k-i)$.

### 6.4. Three-Term Symmetric Dedekind Sum Function

With the computation of the geometric genus of a quasi-Brieskorn-Pham polynomial, the Durfee conjecture implies and is implied by the inequality

$$
\begin{equation*}
\mathfrak{S}(a, b, c ; d) \geqslant-\frac{1}{4}\left(l^{\prime}-d\right)+\frac{1}{12}\left(l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c}\right)+\frac{1}{6} . \tag{6.139}
\end{equation*}
$$

Therefore, any improvement upon the original Durfee conjecture would yield a sharper lower bound for the Dedekind sum $\mathfrak{S}(a, b, c ; d)$ and vice versa.

Corollary 6.32. If $a, b$ and $c$ are pairwise coprime integers with no factor in common, then the Dedekind sum $\mathfrak{S}(a, b, c ; d)$ is bounded from below,

$$
\begin{equation*}
\mathfrak{S}(a, b, c ; d) \geqslant \frac{1}{12}\left(a+b+c-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{1}{a b c}\right)-\frac{1}{3} . \tag{6.140}
\end{equation*}
$$

Remark 6.4.1. The bound of Corollary 6.32 is sharp. If any of the integers $a, b$ or $c$ is equal to 1 , then

$$
\begin{align*}
\mathfrak{S}(a, b, 1 ; 1) & \geqslant \frac{1}{12}\left(1+\frac{b}{a}+\frac{a}{b}+\frac{1}{a b}\right)-\frac{1}{3}  \tag{6.141}\\
& =\frac{1}{12}\left(\frac{b}{a}+\frac{a}{b}+\frac{1}{a b}\right)-\frac{1}{4}, \tag{6.142}
\end{align*}
$$

which is an equality by the classical Dedekind Reciprocity Law.

Remark 6.4.2. Corollary 6.32 is often weaker than the bound

$$
\begin{align*}
\mathfrak{S}(a, b, c ; d) & \geqslant-\mathfrak{s}(1, a)-\mathfrak{s}(1, b)-\mathfrak{s}(1, c)  \tag{6.143}\\
& =-\frac{1}{12}(a+b+c)-\frac{1}{6}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{3}{4} \tag{6.144}
\end{align*}
$$

6.4.1. Three-Term Dedekind Reciprocity Law. Let $N_{\mathcal{T}}$ denote the number of positive lattice points in the tetrahedron $\mathcal{T}=\boldsymbol{\operatorname { c o n v }}\left\{\mathbf{0}, a \mathbf{e}_{1}, b \mathbf{e}_{2}, c \mathbf{e}_{3}\right\}$.

Corollary 6.33. For pairwise coprime positive integers $a, b$ and $c$ with no factor in common,

$$
\begin{align*}
& \mathfrak{S}(a, b, c ; d)=\frac{1}{6} a b c-\frac{1}{4}(a b+b c+c a)+\frac{1}{4}(a+b+c) \\
&+\frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{a c}{b}+\frac{1}{a b c}\right)-N_{\mathcal{T}}-\frac{1}{2} \tag{6.145}
\end{align*}
$$

Proof. Clearly, $p_{g}(f)=N_{\mathcal{T}}$. Observe

$$
\begin{align*}
6 p_{g}(f)= & \mu_{\mathrm{alg}}(f)+\frac{1}{2}\left(l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{1}{a b c}\right) \\
& -6 \mathfrak{S}(a, b, c ; d)-2 \tag{6.146}
\end{align*}
$$

which can be rearranged to yield the claimed identity.
6.4.2. New Integrality of the Dedekind Sum. We can now state a few results which are of interest in Number Theory. Given the integrality of $\mu$ and $p_{g}$, we have the following Dedekind sum integrality.

Proposition 6.34. For $a, b, c \in \mathbb{N}$ with $\operatorname{Gcd}(a, b, c)=1$,

$$
\begin{equation*}
6 \mathfrak{S}(a, b, c ; d)-\frac{1}{2} L+\frac{3}{2}\left(l^{\prime}-d\right) \in \mathbb{Z} \tag{6.147}
\end{equation*}
$$

where $L=l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c}$.
6.4.3. New Congruences for the Dedekind Sum. Recall that for coprime positive integer $b$ and $c$, the Dedekind Reciprocity Law states

$$
\begin{equation*}
\mathfrak{s}(b, c)+\mathfrak{s}(c, b)=\frac{1}{12}\left(\frac{b}{c}+\frac{1}{b c}+\frac{c}{b}\right)-\frac{1}{4} \tag{6.148}
\end{equation*}
$$

which implies the congruence

$$
\begin{equation*}
12 b c(\mathfrak{s}(b, c)+\mathfrak{s}(c, b)) \equiv 1+b^{2}+c^{2}+1 \bmod b c \tag{6.149}
\end{equation*}
$$

As one might have already anticipated, a generalized three-term reciprocity law satisfied by the Dedekind sum is precisely the identity proved in Proposition 6.29,

$$
\begin{align*}
\mathfrak{S}(a, b, c ; d)= & \frac{1}{12}\left(l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c}\right)-\frac{1}{4}\left(l^{\prime}-d\right) \\
& +\frac{1}{6}\left(\mu_{\mathrm{alg}}(f)-p_{g}(f)+1\right)  \tag{6.150a}\\
= & \frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c}\right)+\frac{1}{4}\left(l-l^{\prime}-a b-b c-c a\right) \\
& +\frac{1}{6}\left(a b c-p_{g}(f)-1\right) \tag{6.15ob}
\end{align*}
$$

Remark 6.4.3. Suppose $a=1$. Then $\mu_{\mathrm{alg}}(f)=p_{g}(f)=0, d=\operatorname{GCD}(b, c)$, $l^{\prime}=\operatorname{GCD}(b, c)+2$ and equation (6.150a) simplifies

$$
\begin{equation*}
\mathfrak{s}\left(\frac{b}{d}, \frac{c}{d}\right)+\mathfrak{s}\left(\frac{c}{d}, \frac{b}{d}\right)=\mathfrak{S}(1, b, c ; d)=\frac{1}{12}\left(\frac{b}{c}+\frac{d^{2}}{b c}+\frac{c}{b}\right)-\frac{1}{4}, \tag{6.151}
\end{equation*}
$$

which is equivalent to the classical reciprocity law as applied to coprime integers $\frac{b}{d}$ and $\frac{c}{d}$. In $\S 6.5$, we prove another reciprocity law.

Proposition 6.35. For $a, b, c \in \mathbb{N}$ with no factor in common,

$$
\begin{equation*}
12 a b c \mathfrak{S}(a, b, c ; d) \equiv(a b)^{2}+(b c)^{2}+(c a)^{2}+d^{2} \bmod a b c . \tag{6.152}
\end{equation*}
$$

Proof. Multiply equation (6.129) by $2 a b c$ and the residue modulo $a b c$.
Remark 6.4.4. For $b, c \in \mathbb{N}$ with $b$ and $c$ odd and coprime,

$$
\begin{equation*}
24 b c \mathfrak{S}(2, b, c) \equiv(b-1)(b-5)+(c-1)(c-5) \bmod 24 \tag{6.153}
\end{equation*}
$$

For example ${ }^{*}$, taking $c=3$, it follows that

$$
\begin{equation*}
72 c \mathfrak{S}(2,3, c)+4 \equiv(c-1)(c-5) \bmod 24 \tag{6.154}
\end{equation*}
$$

and, therefore,

$$
72 c \mathfrak{S}(2,3, c) \bmod 24 \equiv\left\{\begin{array}{cl}
20 & c \equiv\{1,5\} \bmod 12  \tag{6.155}\\
8 & c \equiv\{7,11\} \bmod 12
\end{array}\right.
$$

*Explicitly in terms of a sum of cotangents,

$$
2 \sqrt{3} c \csc \left(\frac{4 \pi c}{3}\right)+18 \sum_{k=1}^{c-1} \cot \left(\frac{\pi k}{c}\right) \cot \left(\frac{6 \pi k}{c}\right)+4 \equiv(c-1)(c-5) \bmod 24
$$

Corollary 6.36. For $a, b, c \in \mathbb{N}$ with no factor in common,

$$
\begin{align*}
12 a b c \mathfrak{S}(a, b, c ; d) & \equiv(b c)^{2}+(c a)^{2}+d^{2} \bmod a b  \tag{6.156a}\\
& \equiv(a b)^{2}+(c a)^{2}+d^{2} \bmod b c  \tag{6.156b}\\
& \equiv(a b)^{2}+(b c)^{2}+d^{2} \bmod c a \tag{6.156c}
\end{align*}
$$

Corollary 6.37. For $a, b, c \in \mathbb{N}$ with no factor in common,

$$
\begin{align*}
12 a b c \mathfrak{S}(a, b, c ; d) & \equiv(b c)^{2}+d^{2} \bmod a  \tag{6.157a}\\
& \equiv(c a)^{2}+d^{2} \bmod b  \tag{6.157b}\\
& \equiv(a b)^{2}+d^{2} \bmod c \tag{6.157c}
\end{align*}
$$

In fact, infinitely many higher congruences are known.*
6.4.3.1. Higher Congruences. For $a_{1}, \ldots, a_{n} \in \mathbb{N}$ such that $\operatorname{GCD}\left(a_{1}, \ldots, a_{n}\right)=1$, let $a_{i}^{\prime}=\operatorname{GCD}\left(a_{1}, \ldots, \hat{a}_{i}, \ldots, a_{n}\right)$ and $a_{n+1}=a_{1}^{\prime} \cdots a_{n}^{\prime}$. Define the symmetric summation

$$
\begin{equation*}
\mathfrak{S}\left(a_{1}, \ldots, a_{n} ; a_{n+1}\right)=\sum_{i=1}^{n} a_{i}^{\prime} \mathfrak{s}\left(\frac{a_{1} \cdots \hat{a}_{i} \cdots a_{n}}{a_{n+1}}, \frac{a_{i}}{a_{1}^{\prime} \cdots \hat{a}_{i}^{\prime} \cdots a_{n}^{\prime}}\right) . \tag{6.158}
\end{equation*}
$$

${ }^{*}$ Define $a^{\prime}=\operatorname{GCD}(b, c, d), b^{\prime}=\operatorname{GCD}(a, c, d), c^{\prime}=\operatorname{GCD}(a, b, d), d^{\prime}=\operatorname{GCD}(a, b, c)$ and $e=a^{\prime} b^{\prime} c^{\prime} d^{\prime}$. Similarly, define the 4 -term summation

$$
\mathfrak{S}(a, b, c, d ; e)=a^{\prime} \mathfrak{s}\left(\frac{b c d}{e}, \frac{a}{b^{\prime} c^{\prime} d^{\prime}}\right)+b^{\prime} \mathfrak{s}\left(\frac{a c d}{e}, \frac{b}{a^{\prime} c^{\prime} d^{\prime}}\right)+c^{\prime} \mathfrak{s}\left(\frac{a b d}{e}, \frac{c}{a^{\prime} b^{\prime} d^{\prime}}\right)+d^{\prime} \mathfrak{s}\left(\frac{a b c}{e}, \frac{d}{a^{\prime} b^{\prime} c^{\prime}}\right) .
$$

Given the unique form of the 3 -term congruence, it is reasonable to conjecture the following 4 -term congruence:

$$
12 a b c d \mathfrak{S}(a, b, c, d) \equiv(a b c)^{2}+(a b d)^{2}+(a c d)^{2}+(b c d)^{2}+e^{2} \bmod a b c d .
$$

However, as stated, this congruence does not hold. It is true for pairwise coprime integers $a, b, c, d$ such that $a^{\prime}=b^{\prime}=c^{\prime}=d^{\prime}=e=1$.

Proposition 6.38. For pairwise coprime $\left\{a_{1}, \ldots, a_{n}\right\} \subset \mathbb{N}$,

$$
\begin{equation*}
12 a_{1} \cdots a_{n} \mathfrak{S}\left(a_{1}, \ldots, a_{n} ; a_{n+1}\right) \equiv 1+\sum_{i=1}^{n}\left(a_{1} \cdots \hat{a}_{i} \cdots a_{n}\right)^{2} \bmod a_{1} \cdots a_{n} \tag{6.159}
\end{equation*}
$$

We defer the proof of Proposition 6.38.

### 6.5. Signature of Brieskorn-Pham Manifolds

Consider a Brieskorn-Pham fiber $F_{f, 0} \cong_{d}\left\{\left(z_{0}, \ldots, z_{n}\right) \in \mathbb{C}^{n+1} \mid \sum_{i=0}^{n} z_{i}^{a_{i}}=1\right\}$. The signature $\sigma\left(F_{f, 0}\right)$ can be computed in a number of equivalent ways. Firstly, Brieskorn [63] proves $\sigma\left(F_{f, 0}\right)=\varsigma_{+}-\varsigma_{-}$, where

$$
\begin{aligned}
& \varsigma_{+}=\left|\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{N}^{n+1} \left\lvert\, 0<x_{i}<a_{i} \wedge 0<\sum_{i=0}^{n} \frac{x_{i}}{a_{i}}<1 \bmod 2\right.\right\}\right| \\
& \varsigma_{-}=\left|\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{N}^{n+1} \left\lvert\, 0<x_{i}<a_{i} \wedge 1<\sum_{i=0}^{n} \frac{x_{i}}{a_{i}}<2 \bmod 2\right.\right\}\right|,
\end{aligned}
$$

which generalizes the signature of the torus link $T_{p, q}$. Secondly, Hirzebruch [201] gives the identity

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=2 \sum_{k_{0}=1}^{a_{0}-1} \cdots \sum_{k_{n}=1}^{a_{n}-1}\left(\left(\frac{1}{2}+\sum_{i=0}^{n} \frac{k_{i}}{2 a_{i}}\right)\right)-\left(\left(\sum_{i=0}^{n} \frac{k_{i}}{2 a_{i}}\right)\right), \tag{6.16o}
\end{equation*}
$$

where $((x))=x-\lfloor x\rfloor-\frac{1}{2}$ if $x \notin \mathbb{Z}$ and 0 otherwise. Thirdly, Zagier gives the identity

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=\frac{(-1)^{n / 2}}{N} \sum_{j=0}^{N-1} \cot \left(\frac{\pi(2 j+1)}{2 N}\right) \prod_{i=0}^{n+1} \cot \left(\frac{\pi(2 j+1)}{2 a_{i}}\right) \tag{6.161}
\end{equation*}
$$

where $N=\operatorname{LCm}\left(a_{0}, \ldots, a_{n}\right)$. As Hirzebruch mentions, an identity attributed to Eisenstein (p. 276, [391]) can be used to prove the equivalence of the last two forms,

$$
\begin{equation*}
\left(\left(\frac{p}{q}\right)\right)=\frac{i}{2 q} \sum_{k=1}^{q-1} \cot \left(\frac{\pi k}{q}\right) e^{2 \pi i k p / q} . \tag{6.162}
\end{equation*}
$$

Lastly, Hirzebruch and Zagier [204] prove the identity

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=\sum_{k \geqslant 0}(-1)^{k} \sum_{\substack{k_{0}=1 \\ k<\sum_{i=0}^{n}}}^{a_{0}-1} \cdots \sum_{k_{n}=1}^{a_{i}} 1 . \tag{6.163}
\end{equation*}
$$

Proposition 6.39. Let $F_{f, 0} \cong{ }_{d} f^{-1}(1)$ denote the fiber of the Brieskorn-Pham singularity $f$ with exponents $\left\{a_{0}, \ldots, a_{n}\right\}$. The signature of the manifold $F_{f, 0}$ is identically zero if $n$ is odd or if $a_{i}=1$ for some $0 \leqslant i \leqslant n$ and otherwise bounded by

$$
\begin{equation*}
\left|\sigma\left(F_{f, 0}\right)\right| \leqslant \mu_{\mathrm{alg}}(f) \tag{6.164}
\end{equation*}
$$

Proof. The signature is zero for smooth manifolds whose dimension is not a multiple of 4 . The Brieskorn-Pham fiber $F_{f, 0}$ is a smooth $2 n$-dimensional manifold. Thus, the signature $\sigma\left(F_{f, 0}\right)$ is possibly non-zero if and only if $n$ is even.

For $x \in \mathbb{R}$, one has the fractional part identity

$$
\begin{equation*}
\left(\left(x+\frac{1}{2}\right)\right)=((x))+\frac{1}{2} \operatorname{sign} \sin (2 \pi x), \tag{6.165}
\end{equation*}
$$

which implies the identity

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=\sum_{k_{0}=1}^{a_{0}-1} \cdots \sum_{k_{n}=1}^{a_{n}-1} \operatorname{sign} \sin \left(\pi \sum_{i=0}^{n} \frac{k_{i}}{a_{i}}\right), \tag{6.166}
\end{equation*}
$$

by equation (6.16o) and the bound

$$
\begin{align*}
\left|\sigma\left(F_{f, 0}\right)\right| & \leqslant \sum_{k_{0}=1}^{a_{0}-1} \cdots \sum_{k_{n}=1}^{a_{n}-1} 1  \tag{6.167}\\
& =\prod_{i=0}^{n}\left(a_{i}-1\right) . \tag{6.168}
\end{align*}
$$

If $a_{i}=1$ for any $0 \leqslant i \leqslant n$, then it follows that $\sigma\left(F_{f, 0}\right)=0$.
6.5.o.2. Signature and Dedekind Sums. Recall the Dedekind sum

$$
\begin{equation*}
\mathfrak{s}(a, b)=\frac{1}{4 b} \sum_{k=1}^{b-1} \cot \left(\frac{\pi k}{b}\right) \cot \left(\frac{\pi a k}{b}\right) \tag{6.169}
\end{equation*}
$$

for coprime $a, b \in \mathbb{N}$. For pairwise coprime $\left\{a_{0}, \ldots, a_{n}\right\}$, then the signature admits the following representation [343],

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=-1+\frac{1}{3 N}\left(1-(n-1) N^{2}+\sum_{k=0}^{n} b_{k}^{2}\right)-4 \sum_{k=0}^{n} \mathfrak{s}\left(b_{k}, a_{k}\right) \tag{6.170}
\end{equation*}
$$

where $N=a_{0} \cdots a_{n}$ and $b_{k}=\frac{N}{a_{k}}$.
Proposition 6.40. Given a Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with pairwise coprime exponents $\{a, b, c\} \subset \mathbb{N}$ and fiber $F_{f, 0}$, the signature of
$F_{f, 0}$ and the geometric genus and algebraic index of $f$ satisfy

$$
\begin{equation*}
4 p_{g}(f)=\sigma\left(F_{f, 0}\right)+\mu_{\mathrm{alg}}(f) \tag{6.171}
\end{equation*}
$$

Proof. By Proposition 6.29,

$$
\begin{align*}
6 p_{g}(f)= & \mu_{\mathrm{alg}}(f)+\frac{1}{2}\left(l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{1}{a b c}\right) \\
& -6 \mathfrak{S}(a, b, c ; d)-2 \tag{6.172}
\end{align*}
$$

where $l=a+b+c$ and $\mathfrak{S}(a, b, c ; 1)=\mathfrak{s}(b c, a)+\mathfrak{s}(a c, b)+\mathfrak{s}(a b, c)$. By equation (6.170),

$$
\begin{align*}
\sigma\left(F_{f, 0}\right)=- & 1+\frac{1}{3 a b c}\left(1-(a b c)^{2}+(a b)^{2}+(b c)^{2}+(c a)^{2}\right) \\
& -4 \mathfrak{S}(a, b, c ; d) . \tag{6.173}
\end{align*}
$$

Combining the two expressions yields the claimed identity.
REMARK 6.5.1. Equation (6.171) holds if $K_{f}$ is a rational homology sphere [343]. and, more generally, for any weighted homogeneous polynomial $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with intersection form invariants $\left\{\varsigma_{+}, \varsigma_{0}, \varsigma_{-}\right\}$, where $\varsigma_{0}=0$.

Corollary 6.41. If a Brieskorn-Pham singularity $f$ has pairwise coprime exponents, then the signature of its fiber is divisible by 4.

Proof. At most one exponent is even, so $\mu_{\mathrm{alg}}(f)$ is divisible by 4 . Equation (6.171) implies the claimed divisibility.

Corollary 6.42. If a Brieskorn-Pham singularity $f$ has pairwise coprime exponents $\{a, b, c\} \subset \mathbb{N}$, then the corresponding geometric genus and algebraic index satisfy

$$
\begin{equation*}
p_{g}(f) \equiv \frac{1}{4} \mu_{\mathrm{alg}}(f) \bmod 2 \tag{6.174}
\end{equation*}
$$

Proof. Since $\sigma\left(F_{f, 0}\right)=8 \lambda(\Sigma(a, b, c))$, then $p_{g}(f)=2 \lambda(\Sigma(a, b, c))+\frac{1}{4} \mu_{\mathrm{alg}}(f)$ by equation (6.171). At most one exponent is even, so $\mu_{\mathrm{alg}}(f)$ is divisible by 4 .
6.5.0.3. Quadratic Reciprocity Law. The Law of Quadratic Reciprocity states that given two odd, distinct primes, $p$ and $q$, the congruence quadratic equation $x^{2} \equiv p \bmod q$ is solvable in $\mathbb{Z}$ if and only if $x^{2} \equiv \varepsilon q \bmod p$ is solvable in $\mathbb{Z}$, where $\varepsilon$ is the sign of the congruence class of $q$ modulo 4 . Whether said congruence quadratic equations have solutions or not is encoded within the Legendre symbols $\left(\frac{p}{q}\right)$ and $\left(\frac{q}{p}\right)$. The Law of Quadratic Reciprocity is succinctly written

$$
\begin{equation*}
\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{(p-1)(q-1) / 4} . \tag{6.175}
\end{equation*}
$$

Jacobi generalized the Law of Quadratic Reciprocity to odd, coprime integers.
In §6.2.3, the geometric genus of quasi-Brieskorn-Pham singularities is computed in terms of Dedekind sums by counting positive lattice points intersecting certain integral trirectangular tetrahedra, the corresponding weight polytopes. The next result relates these two objects.

Corollary 6.43. If $p$ and $q$ are odd and coprime positive integers, then the Law of Quadratic Reciprocity (of Jacobi symbols) is equivalent to the following identity,

$$
\begin{equation*}
\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{p_{g}(f)} \tag{6.176}
\end{equation*}
$$

where $p_{g}(f)$ is the geometric genus of the Brieskorn-Pham surface singularity

$$
\begin{equation*}
f=x^{p}+y^{q}+z^{2}, \tag{6.177}
\end{equation*}
$$

where

$$
\begin{equation*}
p_{g}(f)=\frac{1}{24 p q}+\frac{1}{6}\left(\frac{p}{q}+\frac{q}{p}\right)-\frac{p+q}{4}+\frac{p q}{8}-\mathfrak{s}(2 p, q)-\mathfrak{s}(2 q, p) . \tag{6.178}
\end{equation*}
$$

Proof. Corollary 6.42, the identity $\mu_{\mathrm{alg}}(f)=(p-1)(q-1)$ and the quadratic reciprocity law $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{(p-1)(q-1) / 4}$ imply the claim.

Remark 6.5.2. As shown in [324], if $p$ and $q$ are distinct odd prime congruent to $1 \bmod 4$, then one facet of the prime-knot analogy is the fascinating identity

$$
\begin{equation*}
\left(\frac{p^{*}}{q}\right)=(-1)^{1 \mathrm{k}(p, q)} \quad p^{*}=(-1)^{(p-1) / 2} p \tag{6.179}
\end{equation*}
$$

where $\operatorname{lk}(p, q)$ is to be regarded as the linking number of the knots arising from the primes $p$ and $q$. One recognizes that the primes are, in fact, nothing other than the exponents of a Brieskorn-Pham polynomial, $f=x^{p}+y^{q}+z^{2}$. It is plausible that the prime-knot analogy gives rise to an integer-link generalization by this observation.
6.5.1. Casson Invariant for Brieskorn-Pham 3-Manifolds. The signature of certain Brieskorn-Pham manifolds is divisible by 8 and depends only on a related invariant, the Casson Invariant. For relevant definitions, see Chapter 1 in [417].

For pairwise coprime $\left\{a_{0}, \ldots, a_{n}\right\}$, the Casson invariant of the 3-manifold $\Sigma\left(a_{0}, \ldots, a_{n}\right)$ is computed in terms of Dedekind sums,

$$
\begin{align*}
\lambda\left(\Sigma\left(a_{0}, \ldots, a_{n}\right)\right)=- & \frac{1}{8}+\frac{1}{24 N}\left(1-(n-1) N^{2}+\sum_{k=0}^{n} b_{k}^{2}\right) \\
& -\frac{1}{2} \sum_{k=0}^{n} \mathfrak{s}\left(b_{k}, a_{k}\right) \tag{6.180}
\end{align*}
$$

where $N=a_{0} \cdots a_{n}$ and $b_{k}=\frac{N}{a_{k}}$. Specializing to integral homology 3-spheres, we have the following characterization of the Casson invariant [417]:

1. $\lambda\left(S^{3}\right)=0$;
2. $\lambda(\Sigma)=-\lambda(\Sigma)$, where $-\Sigma$ denotes $\Sigma$ with the opposite orientation;
3. $\lambda\left(\Sigma \# \Sigma^{\prime}\right)=\lambda(\Sigma)+\lambda\left(\Sigma^{\prime}\right)$;
4. Let $K$ be a knot in $\Sigma$. Let $\Sigma+\frac{1}{n} K$ be the $\frac{1}{n}$-surgery on $K$. The difference $\lambda\left(\Sigma+\frac{1}{n} K\right)-\lambda\left(\Sigma+\frac{1}{n+1} K\right)$ does not depend on $n$; and,
5. If $\Sigma$ is the boundary of a spin 4-manifold $F$ with signature $\sigma(F)$, then

$$
\begin{equation*}
4 \lambda(\Sigma) \equiv \sigma(F) \bmod 16 \tag{6.181}
\end{equation*}
$$

REmark 6.5.3. If $\lambda(\Sigma) \neq 0$, then $\Sigma \# \Sigma$ and $\Sigma \#(-\Sigma)$ are not homeomorphic.

Proposition 6.44 (Neumann, Wall, [343]). Let $\Sigma\left(a_{0}, \ldots, a_{n}\right)$ denote a Brieskorn-Pham 3-manifold, the link of the Brieskorn-Pham singularity $f:\left(C^{3}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with Milnor fiber $F_{f, 0}$ and pairwise coprime exponents $\left\{a_{0}, \ldots, a_{n}\right\}$. The Casson invariant of $\Sigma\left(a_{0}, \ldots, a_{n}\right)$ is determined by the signature of $F_{f, 0}$, namely,

$$
\begin{equation*}
\lambda\left(\Sigma\left(a_{0}, \ldots, a_{n}\right)\right)=\frac{1}{8} \sigma\left(F_{f, 0}\right) . \tag{6.182}
\end{equation*}
$$

In particular, $\sigma\left(F_{f, 0}\right)$ is divisible by 8.
6.5.1.1. Laufer's Formula. In [256], Laufer relates the algebraic index and geometric genus to certain invariants of the minimal resolution of a complex hypersurface.

Proposition 6.45 (Laufer,[256]). Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbf{C}, 0)$ with hypersurface $V_{f, 0}=f^{-1}(0)$ and any minimal resolution given by a proper, analytic, surjective map $\pi:\left(\tilde{V}_{f, 0}, E\right) \rightarrow\left(V_{f, 0}, 0\right)$ with exceptional locus $E=\pi^{-1}(\mathbf{0})$ such that $\tilde{V}_{f, 0} \backslash E \rightarrow V_{f, 0}^{\times}$is an analytic isomorphism and $\pi^{-1}\left(V_{f, 0}^{\times}\right)$ is dense in $\tilde{V}_{f, 0}$. Then

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=12 p_{g}(f)+\tilde{\chi}(E)+K^{2}, \tag{6.183}
\end{equation*}
$$

where $\tilde{\chi}(E)=\chi(E)-1$ is the reduced (topological) Euler characteristic of $E$ and $K^{2}$ is the self-intersection number of the canonical divisor on $\tilde{V}_{f, 0}$.

Proof. See Chapter IV in [420].

Remark 6.5.4. As a consequence of Proposition 6.45,

$$
\begin{equation*}
\tilde{\chi}(E)+K^{2} \equiv \mu_{\mathrm{alg}}(f) \bmod 12 \tag{6.184}
\end{equation*}
$$

Remark 6.5.5. As a consequence of Propositions 6.40 and 6.45 , the signature of the fiber $F_{f, 0}$ of a Brieskorn-Pham surface singularity satisfies the following identity

$$
\begin{equation*}
3 \sigma\left(F_{f, 0}\right)+\tilde{\chi}(E)+K^{2}+2 \mu_{\mathrm{alg}}(f)=0 \tag{6.185}
\end{equation*}
$$

By Proposition 6.44, for pairwise coprime $a, b, c \in \mathbb{N}$, the Casson invariant $\lambda(\Sigma(a, b, c))$ satisfies the identity

$$
\begin{equation*}
24 \lambda(\Sigma(a, b, c))+\tilde{\chi}(E)+K^{2}+2 \mu_{\mathrm{alg}}(f)=0 \tag{6.186}
\end{equation*}
$$

which implies the stronger congruence

$$
\begin{equation*}
\tilde{\chi}(E)+K^{2} \equiv 22 \mu_{\mathrm{alg}}(f) \bmod 24 \tag{6.187}
\end{equation*}
$$

$\triangle$

Remark 6.5.6. Consider the family of surface singularities specified by $f_{l}=$ $x^{6 l+5}+y^{3}+z^{2}$ for $l \geqslant 0$. Observe $\mu_{\mathrm{alg}}\left(f_{l}\right)=12 l+8$ and $g(\Sigma(6 l+5,3,2))=0$ for $l \geqslant 0$, since the exponents are pairwise coprime. Yau computes $K^{2}=-l$ for
$l \geqslant 0$ [483], [484]. By Proposition 6.19, $p_{g}\left(f_{l}\right)=l$ for $l \geqslant 0$. By Proposition 6.46,

$$
\begin{align*}
\sigma\left(F_{f_{l}, 0}\right) & =4 p_{g}\left(f_{l}\right)-\mu_{\mathrm{alg}}\left(f_{l}\right)  \tag{6.188}\\
& =4 l-(12 l+8)  \tag{6.189}\\
& =-8 l-8 \tag{6.190}
\end{align*}
$$

which is divisible by 8 for $l \geqslant 0$. Hence, $\lambda(\Sigma(6 l+11,3,2))=-l-1$. Moreover,

$$
\begin{align*}
\tilde{\chi}(E) & =\mu_{\mathrm{alg}}\left(f_{l}\right)-12 p_{g}\left(f_{l}\right)-K^{2}  \tag{6.191}\\
& =12 l+8-12 l+l  \tag{6.192}\\
& =l+8 . \tag{6.193}
\end{align*}
$$

Thus, the factor $\tilde{\chi}(E)+K^{2}=8$, which is independent of $l$, satisfies the congruence in equation (6.187), as $22(12 l+8) \equiv 8 \bmod 24$ for $l \geqslant 0$.

Proposition 6.46. Given a Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with exponents $\{a, b, c\} \subset \mathbb{N}$ and fiber $F_{f, 0}$, the signature of $F_{f, 0}$ and the geometric genus and algebraic index of $f$ satisfy

$$
\begin{equation*}
4 p_{g}(f)=\sigma\left(F_{f, 0}\right)+\mu_{\mathrm{alg}}(f)+2 g\left(\Sigma(a, b, c) / S^{1}\right) \tag{6.194}
\end{equation*}
$$

where $g\left(\Sigma(a, b, c) / S^{1}\right)$ is the (base-orbifold) genus of the Seifert fibration on $\Sigma(a, b, c)$.

Proof. As above, let $N=\operatorname{Lcm}(a, b, c)$,

$$
\begin{array}{lll}
b_{0}=\frac{\operatorname{LCM}(a, b, c)}{a} & c_{0}=\frac{\operatorname{LCM}(a, b, c)}{\operatorname{LCM}(b, c)} & d_{0}=\frac{b c}{\operatorname{LCM}(b, c)}=\operatorname{GCD}(b, c) \\
b_{1}=\frac{\operatorname{LCM}(a, b, c)}{b} & c_{1}=\frac{\operatorname{LCM}(a, b, c)}{\operatorname{LCM}(a, c)} \quad d_{1}=\frac{a c}{\operatorname{LCM}(a, c)}=\operatorname{GCD}(a, c) \\
b_{2}=\frac{\operatorname{LCM}(a, b, c)}{c} & c_{2}=\frac{\operatorname{LCM}(a, b, c)}{\operatorname{LCM}(a, b)} \quad & d_{2}=\frac{a b}{\operatorname{LCM}(a, b)}=\operatorname{GCD}(a, b) \tag{6.197}
\end{array}
$$

Thus, the corresponding signature is simply

$$
\begin{align*}
\sigma\left(F_{f, 0}\right)=- & +\frac{a b c}{3 N^{2}}\left(1-N^{2}+b_{0}^{2}+b_{1}^{2}+b_{2}^{2}\right) \\
& -4\left(d_{0} \mathfrak{s}\left(b_{0}, c_{0}\right)+d_{1} \mathfrak{s}\left(b_{1}, c_{1}\right)+d_{2} \mathfrak{s}\left(b_{2}, c_{2}\right)\right) \tag{6.198}
\end{align*}
$$

Let $l=a+b+c, l^{\prime}=\operatorname{GCD}(a, b)+\operatorname{GCD}(b, c)+\operatorname{GCD}(a, c)$ and

$$
\begin{equation*}
\frac{d}{\tau}=\frac{\operatorname{GCD}(a, b) \operatorname{GCD}(b, c) \operatorname{GCD}(a, c)}{\operatorname{GCD}(a, b, c)} \tag{6.199}
\end{equation*}
$$

For $a, b, c \in \mathbb{N}$ with $\tau=\operatorname{GcD}(a, b, c)$,

$$
\begin{align*}
6 p_{g}(f)= & \mu_{\mathrm{alg}}(f)-\frac{3}{2}\left(l^{\prime}-\frac{d}{\tau}\right)+\frac{1}{2}\left(l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right) \\
& -6 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)+1 \tag{6.200}
\end{align*}
$$

Since $\mathfrak{S}\left(a, b, c, \frac{d}{\tau}\right)=d_{0} \mathfrak{s}\left(b_{0}, c_{0}\right)+d_{1} \mathfrak{s}\left(b_{1}, c_{1}\right)+d_{2} \mathfrak{s}\left(b_{2}, c_{2}\right)$,

$$
\begin{align*}
& 3 \sigma\left(F_{f, 0}\right)= 12 p_{g}(f)-2 \mu_{\mathrm{alg}}(f)-5+\frac{a b c}{N^{2}}\left(1-N^{2}+b_{0}^{2}+b_{1}^{2}+b_{2}^{2}\right) \\
&+3\left(l^{\prime}-\frac{d}{\tau}\right)-\left(l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right)  \tag{6.201}\\
&=12 p_{g}(f)-2 \mu_{\mathrm{alg}}(f)-5+\frac{a b c}{N^{2}}+\left(\frac{b c}{a}+\frac{a c}{b}+\frac{a b}{c}-a b c\right) \\
&+3\left(l^{\prime}-\frac{d}{\tau}\right)-\left(l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right)  \tag{6.202}\\
&=12 p_{g}(f)-2 \mu_{\mathrm{alg}}(f)-5+\frac{a b c}{N^{2}}-\frac{d^{2}}{a b c \tau^{2}}+3\left(l^{\prime}-\frac{d}{\tau}\right) \\
& \quad a-b-c+a b+b c+c a-a b c . \tag{6.203}
\end{align*}
$$

$$
\begin{equation*}
=12 p_{g}(f)-3 \mu_{\mathrm{alg}}(f)-6+\frac{a b c}{N^{2}}-\frac{d^{2}}{a b c \tau^{2}}+3\left(l^{\prime}-\frac{d}{\tau}\right) . \tag{6.204}
\end{equation*}
$$

Collecting the various terms and simplifying yields

$$
\begin{align*}
\sigma\left(F_{f, 0}\right)+\mu_{\mathrm{alg}}(f) & =4 p_{g}(f)+\frac{1}{3}\left(\frac{a b c}{N^{2}}-\frac{d^{2}}{a b c \tau^{2}}\right)+l^{\prime}-\frac{d}{\tau}-2  \tag{6.205}\\
& =4 p_{g}(f)+l^{\prime}-\frac{d}{\tau}-2 \tag{6.206}
\end{align*}
$$

since

$$
\begin{equation*}
\frac{a b c}{\operatorname{LCM}(a, b, c)}=\frac{\operatorname{GCD}(a, b) \operatorname{GCD}(a, c) \operatorname{GCD}(b, c)}{\operatorname{GCD}(a, b, c)} . \tag{6.207}
\end{equation*}
$$

Finally, for the Brieskorn-Pham manifold $\Sigma(a, b, c)$, one has

$$
\begin{equation*}
g\left(\Sigma(a, b, c) / S^{1}\right)=\frac{1}{2}\left(\frac{d}{\tau}-l^{\prime}\right)+1 . \tag{6.208}
\end{equation*}
$$

Hence, $-2 g\left(\Sigma(a, b, c) / S^{1}\right)=l^{\prime}-\frac{d}{\tau}-2$.

REmARK 6.5.7. Consider the surface singularity $f_{l}=x^{6 l+3}+y^{3}+z^{2}$ for $l \geqslant$ 0 . Observe $\mu_{\mathrm{alg}}\left(f_{l}\right)=12 l+4$ and $g(\Sigma(6 l+3,3,2))=\frac{1}{2}(3-3-1-1)+1=0$ for $l \geqslant 0$, while the exponents are not pairwise coprime. Yau computes $K^{2}=-l$ and $\tilde{\chi}(E)=l+4$ for $l \geqslant 0$ [483], [484]. By Proposition 6.19, $p_{g}\left(f_{l}\right)=l$ for $l \geqslant 0$. By Proposition 6.46,

$$
\begin{align*}
\sigma\left(F_{f_{l}, 0}\right) & =4 p_{g}\left(f_{l}\right)-\mu_{\mathrm{alg}}\left(f_{l}\right)-2 g(\Sigma(6 l+3,3,2))  \tag{6.209}\\
& =4 l-(12 l+4)  \tag{6.210}\\
& =-8 l-4 \tag{6.211}
\end{align*}
$$

which is divisible by 4 (but not 8 ) for $l \geqslant 0$.

REmaRK 6.5.8. Consider the surface singularity $f_{l, k}=x^{3 l+k}+y^{3}+z^{3}$ for $l \geqslant$ 1 and $k \in\{0,1,2\}$. Observe $\mu_{\mathrm{alg}}\left(f_{l, k}\right)=12 l+4 k-4$ and $g\left(\Sigma(3 l+k, 3,3) / S^{1}\right)=$ $\delta_{k, 0}$. By Proposition 6.19, $p_{g}\left(f_{l, k}\right)=l$. By Proposition 6.46,

$$
\begin{align*}
\sigma\left(F_{f_{l, k}, 0}\right) & =4 p_{g}\left(f_{l, k}\right)-\mu_{\mathrm{alg}}\left(f_{l, k}\right)-2 g(\Sigma(3 l+k, 3,3))  \tag{6.212}\\
& =4 l-(12 l+4 k-4)-2 \delta_{k, 0}  \tag{6.213}\\
& =-8 l-4 k-2 \delta_{k, 0}+4 \tag{6.214}
\end{align*}
$$

which is divisible by 2,8 and 4 for $k=0,1$ and 2 , respectively. Moreover,

$$
\begin{align*}
\tilde{\chi}\left(E_{l, k}\right)+K_{l, k}^{2} & =\mu_{\mathrm{alg}}\left(f_{l, k}\right)-12 p_{g}\left(f_{l, k}\right)  \tag{6.215}\\
& =(12 l+4 k-4)-12 l  \tag{6.216}\\
& =4 k-4 \tag{6.217}
\end{align*}
$$

which is independent of $l$. Yau computes $K_{l, 1}^{2}=-3 l-1$ and $K_{l, 2}^{2}=-3 l$ [483], [484]. Thus, $\tilde{\chi}\left(E_{l, 1}\right)=3 l+1$ and $\tilde{\chi}\left(E_{l, 2}\right)=3 l+4$.

Problem 6.5.1. Compute $\tilde{\chi}\left(E_{1,0}\right)$ and $K_{l, 0}^{2}$ as functions of $l$.

Remark 6.5.9. For a homogeneous, Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with exponents $\{d, d, d\}$, one has $p_{g}(f)=\binom{d}{3}, \mu_{\mathrm{alg}}(f)=$ $(d-1)^{3}$ and $g\left(\Sigma(d, d, d) / S^{1}\right)=\binom{d-1}{2}$. By Proposition 6.46,

$$
\begin{align*}
\sigma\left(F_{f, 0}\right) & =4\binom{d}{3}-(d-1)^{3}-2\binom{d-1}{2}  \tag{6.218}\\
& =-\frac{1}{3}(d-1)\left(d^{2}+d-3\right) \tag{6.219}
\end{align*}
$$

which establishes the existence of a Brieskorn-Pham manifold with a signature that is divisible by any positive integer. In particular, if $d \geqslant 3$ is odd, then $2^{\lambda(d-2)}$ divides $\sigma\left(F_{f, 0}\right)$, where $\lambda(n)=\sum_{\text {odd } d \mid n} \mu(d) \tau\left(\frac{n}{d}\right)$ is the 2 -adic valuation of $2 n$ (A001511).

In fact, for certain values of $d$, the signature is divisible by any power of 2 . That is, if $d=2^{m}+1 \geqslant 3$ for $m \in \mathbb{N}$,

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=-\frac{1}{3} 2^{m}\left(4^{m}+3 \cdot 2^{m}-1\right) \tag{6.220}
\end{equation*}
$$

In contrast to Example 2.6 in [118], this calculation does not require a minimal resolution of the singularity.

Proposition 6.47. For Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with exponents $\{a, b, c\} \subset \mathbb{N}$, the following signature inequality holds,

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right) \leqslant-\frac{1}{3} \mu_{\mathrm{alg}}(f)-\frac{2}{3} \min \{a-1, b-1, c-1\}-2 g\left(\Sigma(a, b, c) / S^{1}\right) \tag{6.221}
\end{equation*}
$$

Proof. The Durfee-Yau Theorem (Proposition 5.48) is true for BrieskornPham surface singularities. When combined with Proposition 6.46, the following signature inequality holds,

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right) \leqslant-\frac{1}{3} \mu_{\mathrm{alg}}(f)-\frac{2}{3}(v(f)-1)-2 g\left(\Sigma(a, b, c) / S^{1}\right) \tag{6.222}
\end{equation*}
$$

where $v(f)=\min \{a, b, c\}$, which improves the inequality of equation (5.166) by a strictly non-positive factor $-2 g\left(\Sigma(a, b, c) / S^{1}\right)$.

Remark 6.5.10. Proposition 6.47 is strict if and only if $f$ is not homogeneous.

Proposition 6.48. If the signature of the Milnor fiber of a Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with exponents $\{a, b, c\} \subset \mathbb{N}$ satisfies

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=-\frac{1}{3} \mu_{\mathrm{alg}}(f)-\frac{2}{3} \min \{a-1, b-1, c-1\}-2 g\left(\Sigma(a, b, c) / S^{1}\right) \tag{6.223}
\end{equation*}
$$

then $f$ is a homogeneous polynomial of degree $a=b=c=d$ and

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=-\frac{1}{3}(d-1)\left(d^{2}+d-3\right) \tag{6.224}
\end{equation*}
$$

Proof. Recall that if $\mu_{\mathrm{alg}}(f)=\tau(f)$ and $6 p_{g}(f)=\mu_{\mathrm{alg}}(f)-v(f)+1$, then $f$ is a homogeneous polynomial after a biholomorphic change of variables by Proposition 5.47. Since $f$ is Brieskorn-Pham, it is also weighted homogeneous, so the first identity is satisfied. Combining the latter equation with that of Proposition 6.46 yields

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right)=-\frac{1}{3} \mu_{\mathrm{alg}}(f)-\frac{2}{3}(v(f)-1)-2 g\left(\Sigma(a, b, c) / S^{1}\right) \tag{6.225}
\end{equation*}
$$

where $v(f)=\min \{a, b, c\}$. Finally, the signature of a homogeneous surface singularity of degree $d$ is computed in Remark 6.5.9.

Remark 6.5.11. Proposition 6.48 proves that equation (6.221) is sharp. $\triangle$
Corollary 6.49. For any Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with exponents $\{a, b, c\} \subset \mathbb{N}$, the signature of the Milnor fiber $F_{f, 0}$ and algebraic index of $f$ satisfy the congruence

$$
\begin{equation*}
\sigma\left(F_{f, 0}\right) \equiv \mu_{\mathrm{alg}}(f) \bmod 2 \tag{6.226}
\end{equation*}
$$

In particular, $\sigma\left(F_{f, 0}\right)$ is odd if and only if $a, b$ and $c$ are even.
Corollary 6.50. If $a, b$ and $c$ are even, then $\lambda(\Sigma(a, b, c)) \neq \frac{1}{8} \sigma\left(F_{f, 0}\right)$. In particular, the Milnor fiber $F_{f, 0}$ is not a spin 4-manifold.

Combining Laufer's formula with Proposition 6.46 yields the following identity.

Corollary 6.51. For any Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with exponents $\{a, b, c\} \subset \mathbb{N}$, the following identity holds:

$$
\begin{equation*}
3 \sigma\left(F_{f, 0}\right)+\tilde{\chi}(E)+K^{2}+2 \mu_{\mathrm{alg}}(f)+6 g\left(\Sigma(a, b, c) / S^{1}\right)=0 \tag{6.227}
\end{equation*}
$$

Proposition 6.52 (Yau, [481]). Let $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a complex analytic function with an isolated singularity at the origin. Let $\pi: \tilde{V} \rightarrow V$ be a resolution of $V$ and $E=\pi^{-1}(\mathbf{0})$ have s components and topological Euler characteristic $\chi(E)$. Let $K^{2}$ be the canonical divisor on $\tilde{V}$, and let $\Omega^{1}$ be the sheaf of germs of holomorphic 1-forms on $\tilde{V}$. Then the following bounds hold:

$$
\begin{align*}
& \sigma\left(F_{f, 0}\right) \leqslant-s+2 p_{g}(f)+\operatorname{dim} H^{1}\left(\tilde{V}, \Omega^{1}\right)  \tag{6.228}\\
& \mu_{\mathrm{alg}}(f) \geqslant \tilde{\chi}(E)+2 p_{g}(f)-\operatorname{dim} H^{1}\left(\tilde{V}, \Omega^{1}\right) \tag{6.229}
\end{align*}
$$

and the following identities hold:

$$
\begin{align*}
& \mu_{\mathrm{alg}}(f)=\tilde{\chi}(E)-\sigma\left(F_{f, 0}\right)-s+4 p_{g}(f)  \tag{6.230}\\
& \sigma\left(F_{f, 0}\right)=-K^{2}-s-8 p_{g}(f) \tag{6.231}
\end{align*}
$$

Proof. See Theorems 3.1 and 3.2 in [481].

Corollary 6.53. Let $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a Brieskorn-Pham surface singularity. Let $\pi: \tilde{V} \rightarrow V$ be a resolution of $V$ and $E=\pi^{-1}(\mathbf{0})$ have s components and topological Euler characteristic $\chi(E)$. Let $K^{2}$ be the canonical divisor on $\tilde{V}$, and let $\Omega^{1}$ be the sheaf of germs of holomorphic 1-forms on $\tilde{V}$. Then the following identity holds:

$$
\begin{align*}
\tilde{\chi}(E) & =s-2 g\left(\Sigma(a, b, c) / S^{1}\right)  \tag{6.232}\\
-K^{2} & =s+12 p_{g}(f)-\mu_{\mathrm{alg}}(f)-2 g\left(\Sigma(a, b, c) / S^{1}\right) \tag{6.233}
\end{align*}
$$

and the following bound holds:

$$
\begin{equation*}
\operatorname{dim} H^{1}\left(\tilde{V}, \Omega^{1}\right) \geqslant-K^{2}-10 p_{g}(f) \tag{6.234}
\end{equation*}
$$

Remark 6.5.12. Compare equation (6.232) to the identity $\chi\left(F_{f, 0}\right)=r-2 \delta$, where $f$ is a squarefree, non-degenerate weighted homogeneous singularity with $r$ branches and $\delta$ double points.

Remark 6.5.13. The integer $s$ is invariant under homeomorphisms. It is also known that $\sigma(\tilde{V})=-s[118]$. Thus, $\sigma(\tilde{V})=2 g\left(\Sigma(a, b, c) / S^{1}\right)-\tilde{\chi}(E)$, which implies the congruence

$$
\begin{equation*}
\sigma(\tilde{V}) \equiv \tilde{\chi}(E) \bmod 2 \tag{6.235}
\end{equation*}
$$

REmark 6.5.14. Explicitly, one has

$$
\begin{aligned}
\tilde{\chi}(E)=s & +l^{\prime}-\frac{d}{\tau}-2 \\
-K^{2}=s & +2(a-1)(b-1)(c-1)-a b c+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b} \\
& +\frac{2 d}{\tau}-2 l^{\prime}+\frac{d^{2}}{a b c \tau^{2}}-12 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)+1,
\end{aligned}
$$

where $l^{\prime}=\operatorname{GCD}(a, b)+\operatorname{GCD}(b, c)+\operatorname{GCD}(a, c)$ and $\frac{d}{\tau}=\frac{\operatorname{GCD}(a, b) \operatorname{GCD}(b, c) \operatorname{GCD}(a, c)}{\operatorname{GCD}(a, b, c)}$. For odd, coprime $p$ and $q$,

$$
\begin{aligned}
-K^{2}=s & +2(p-1)(q-1)-2 p q+\frac{2 p}{q}+\frac{p q}{2}+\frac{2 q}{p} \\
& +\frac{2 d}{\tau}-2 l^{\prime}+\frac{d^{2}}{2 p q \tau^{2}}-12 \mathfrak{S}\left(p, q, 2 ; \frac{d}{\tau}\right)+1
\end{aligned}
$$

where $l^{\prime}=\operatorname{GCD}(p, 2)+\operatorname{GCD}(p, q)+\operatorname{GCD}(2, q)$ and $\frac{d}{\tau}=\frac{\operatorname{GCD}(p, 2) \operatorname{GCD}(p, q) \operatorname{GCD}(2, q)}{\operatorname{GCD}(p, q, 2)}$.

Remark 6.5.15. As in Remark 6.5.7, if $f=x^{6 l+3}+y^{3}+z^{2}$ over $\mathbb{C}^{3}$ for $l \geqslant 0$, then $K^{2}=-l$. By equation (6.233), $K^{2}=4-s$, so $s=l+4$.

Remark 6.5.16. As in Remark 6.5.8, if $f=x^{3 l+k}+y^{3}+z^{3}$ over $\mathbb{C}^{3}$ for $l \geqslant 1$, then $K_{l, 1}^{2}=-3 l-1$ and $K_{l, 2}^{2}=-3 l$. By equation (6.233), $K_{l, 1}^{2}=-s_{l, 1}$, so $s_{l, 1}=3 l+1$. Similarly, $K_{l, 2}^{2}=4-s_{l, 2}$, so $s_{l, 2}=3 l+4$. We compute also $K_{l, 0}^{2}=-2-s_{l, 0}$ and $\chi\left(\tilde{E}_{l, 0}\right)=s_{l, 0}-2$.

Problem 6.5.2. Compute $s_{l, 0}$ as a function of $l$.

Proposition 6.54 (Yau, [487]). Let $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a non-degenerate surface singularity with hypersurface $V_{f}$ and with multiplicity $v(f)$. Let $K$ be the canonical divisor on a minimal resolution of $V_{f}$ at the origin. Then

$$
\begin{equation*}
-K^{2} \geqslant v(f)(v(f)-1)(v(f)-3)+2 \tag{6.236}
\end{equation*}
$$

REMARK 6.5.17. For a homogeneous polynomial $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with exponents $\{d, d, d\}$, one has

$$
\begin{equation*}
\tilde{\chi}(E)+K^{2}=-d^{3}+3 d^{2}-d-1 \tag{6.237}
\end{equation*}
$$

As $g\left(\Sigma(d, d, d) / S^{1}\right)=\binom{d-1}{2}$, so $\tilde{\chi}(E)=s-(d-1)(d-2)$ and $K^{2}=d(d-2)^{2}+$ $1-s$. By Proposition 6.54,

$$
\begin{equation*}
s-d(d-2)^{2}-1=-K^{2} \geqslant d(d-1)(d-3)+2 \tag{6.238}
\end{equation*}
$$

and, therefore, $s \geqslant 2 d^{3}-8 d^{2}+7 d+3$.

Proposition 6.55. Let $f$ and $g$ be Brieskorn-Pham surface singularities. If the corresponding Brieskorn-Pham manifolds $\Sigma_{f}$ and $\Sigma_{g}$ are homeomorphic, then the
following equalities hold:

$$
\begin{align*}
\mu_{\mathrm{alg}}(f)-12 p_{g}(f) & =\mu_{\mathrm{alg}}(g)-12 p_{g}(g)  \tag{6.239}\\
4 p_{g}(f)-\sigma\left(F_{f, 0}\right)-\mu_{\mathrm{alg}}(f) & =4 p_{g}(g)-\sigma\left(F_{g, 0}\right)-\mu_{\mathrm{alg}}(g)  \tag{6.240}\\
3 \sigma\left(F_{f, 0}\right)+2 \mu_{\mathrm{alg}}(f) & =3 \sigma\left(F_{g, 0}\right)+2 \mu_{\mathrm{alg}}(g)  \tag{6.241}\\
\sigma\left(F_{f, 0}\right)+8 p_{g}(f) & =\sigma\left(F_{g, 0}\right)+8 p_{g}(g) . \tag{6.242}
\end{align*}
$$

from which one infers $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{alg}}(g)$ if and only if $p_{g}(f)=p_{g}(g)$ if and only if $\sigma\left(F_{f, 0}\right)=\sigma\left(F_{g, 0}\right)$. Otherwise, the following congruences hold:

$$
\begin{align*}
\mu_{\mathrm{alg}}(f) & \equiv \mu_{\mathrm{alg}}(g) \bmod 12  \tag{6.243}\\
4 p_{g}(f)-\sigma\left(F_{f, 0}\right) & \equiv 4 p_{g}(g)-\sigma\left(F_{g, 0}\right) \bmod 12  \tag{6.244}\\
\sigma\left(F_{f, 0}\right) & \equiv \sigma\left(F_{g, 0}\right) \bmod 8 \tag{6.245}
\end{align*}
$$

Proof. Proposition 6.45 implies the first identity. The second and third identities follow from Proposition 6.46 and Corollary 6.51, respectively. Combining the first and second or third identities yields the last identity. The other congruences follow similarly.

Remark 6.5.18. The topology of $V_{f, 0}$ does not determine the geometric genus, the algebraic index or the signature. For example, the hypersurfaces defined by $f=x^{3}+y^{5}+z^{15}$ and $g=x^{2}+y^{9}+z^{18}$ are homeomorphic and have different algebraic indices, namely, $\mu_{\mathrm{alg}}(f)=112$ and $\mu_{\mathrm{alg}}(g)=136$, both even. It follows from equation (6.183) that the corresponding geometric
genera must also differ, as the combination $\tilde{\chi}(E)+K^{2}$ is invariant under homeomorphisms, in this case, equal to -56 . Observe that $-56 \equiv 112 \equiv 136 \equiv 4$ $\bmod$ 12. By Proposition 6.97, $p_{g}(f)=14$ and $p_{g}(g)=16$. By equation (6.208), $g\left(\Sigma(3,5,15) / S^{1}\right)=g\left(\Sigma(2,9,18) / S^{1}\right)=4$. By Proposition 6.46, $\sigma\left(F_{f, 0}\right)=-64$ and $\sigma\left(F_{g, 0}\right)=-80$, both even.

Consider $h=x^{2}+y^{10}+z^{10}$ and $r=x^{2}+y^{10}+z^{20}$. Although the corresponding base-orbifold genera are equal to $4, V_{h, 0}$ is not homeomorphic to $V_{f, 0} \cong V_{g, 0}$ or $V_{r, 0}$ as $\mu_{\mathrm{alg}}(h)=81, \mu_{\mathrm{alg}}(r)=171, p_{g}(h)=10, p_{g}(r)=20, \sigma\left(F_{h, 0}\right)=-49$ and $\sigma\left(F_{r, 0}\right)=-99$ and the required congruences do not hold.

### 6.6. Signature of Torus Links, Revisited

Proposition 6.56. For $p, q \in \mathbb{N}$, the signature of a torus link $T_{p, q}$ is given by

$$
\begin{align*}
\sigma\left(T_{p, q}\right)=- & (p-1)(q-1)+\frac{2 q}{p}\left(p-1-\left\lfloor\frac{p}{2}\right\rfloor\right)\left\lfloor\frac{p}{2}\right\rfloor+l^{\prime}-\frac{d}{\tau}-2 \\
& -4 \sum_{i=1}^{\lfloor p / 2\rfloor}\left\{\frac{q}{2}-\frac{q i}{p}\right\} \tag{6.246}
\end{align*}
$$

where $l^{\prime}=\operatorname{GCD}(p, 2)+\operatorname{GCD}(p, q)+\operatorname{GCD}(2, q)$ and $\frac{d}{\tau}=\frac{\operatorname{GCD}(p, 2) \operatorname{GCD}(p, q) \operatorname{GCD}(2, q)}{\operatorname{GCD}(p, q, 2)}$.

Proof. If $f$ has exponents $\{2, p, q\}$, then $\Sigma(p, q, 2)$ is the 2 -fold branched cyclic covering over the torus link $T_{p, q}$. In particular, $\sigma\left(F_{f, 0}\right)=\sigma\left(T_{p, q}\right)$. Moreover, the geometric genus $p_{g}(f)$ can be computed by the iterated summation

$$
\begin{align*}
p_{g}(f) & =\sum_{i=1}^{2} \sum_{j=1}^{\lfloor p(1-i / 2)\rfloor} \sum_{k=1}^{\lfloor q(1-i / 2-j / p)\rfloor} 1  \tag{6.247}\\
& =\sum_{i=1}^{\lfloor p / 2\rfloor} \sum_{j=1}^{\lfloor q(1 / 2-i / p)\rfloor} 1  \tag{6.248}\\
& =\sum_{i=1}^{\lfloor p / 2\rfloor}\left\lfloor\frac{q}{2}-\frac{q i}{p}\right\rfloor  \tag{6.249}\\
& =\frac{q}{2 p}\left(p-1-\left\lfloor\frac{p}{2}\right\rfloor\right)\left\lfloor\frac{p}{2}\right\rfloor-\sum_{i=1}^{\lfloor p / 2\rfloor}\left\{\frac{q}{2}-\frac{q i}{p}\right\} . \tag{6.250}
\end{align*}
$$

Remark 6.6.1. Let $f=x^{p}+y^{q}$ and $\tilde{f}=x^{2 p}+y^{2 q}+z^{2}$. By equations (5.70b),

$$
\begin{align*}
\sigma\left(T_{2 p, 2 q}\right)= & 4 \delta(f)-\mu_{\mathrm{alg}}(f)-2 g\left(\Sigma(2 p, 2 q, 2) / S^{1}\right)  \tag{6.251}\\
= & 2(p q-p-q+\operatorname{GCD}(p, q)) \\
& -(2 p-1)(2 q-1)-2 \mathrm{GCD}(p, q)+2  \tag{6.252}\\
= & 1-2 p q . \tag{6.253}
\end{align*}
$$

Thus,

$$
\begin{align*}
\operatorname{rank} H_{1}\left(T_{2 p, 2 q}\right)= & 4 \delta(f)-\sigma\left(T_{2 p, 2 q}\right)-\mu_{\mathrm{ag}}(f)  \tag{6.254}\\
= & 2(p q-p-q+\operatorname{GCD}(p, q)) \\
& -(1-2 p q)-(p-1)(q-1)  \tag{6.255}\\
= & 3 p q-p-q+2 \operatorname{GCD}(p, q)-2 \tag{6.256}
\end{align*}
$$

In particular, $\operatorname{rank} H_{1}\left(T_{2 p, 2 p}\right)=3 p^{2}-2$.

Corollary 6.57. The parity of the signature of a torus link is opposite that of the number of its components. In particular, the signature of a torus knot is even.

Proof. Let $f=x^{p}+y^{q}$ with $p, q \in \mathbb{N}$. The Milnor-Jung formula implies

$$
\begin{align*}
4 p_{g}(\Sigma f) & =\sigma\left(F_{\Sigma f, 0}\right)+\mu_{\mathrm{alg}}(\Sigma f)+2 g\left(\Sigma(p, q, 2) / S^{1}\right)  \tag{6.257}\\
& =\sigma\left(T_{p, q}\right)+2 \delta(f)-\operatorname{GCD}(p, q)+1+2 g\left(\Sigma(p, q, 2) / S^{1}\right) \tag{6.258}
\end{align*}
$$

where $\mu_{\mathrm{alg}}(\Sigma f)=\mu_{\mathrm{alg}}(f)=(p-1)(q-1)$ and $\sigma\left(F_{\Sigma f, 0}\right)=\sigma\left(T_{p, q}\right)$. Thus, for $p, q \in \mathbb{N}$, the signature of a torus link $T_{p, q}$ satisfies

$$
\begin{equation*}
\sigma\left(T_{p, q}\right) \equiv(p-1)(q-1) \equiv \operatorname{GCD}(p, q)-1 \bmod 2 \tag{6.259}
\end{equation*}
$$

Corollary 6.58. If $p$ and $q$ are coprime positive integers, then

$$
\begin{equation*}
\sigma\left(T_{p, q}\right)=-(p-1)(q-1)+\frac{2 q}{p}\left(p-1-\left\lfloor\frac{p}{2}\right\rfloor\right)\left\lfloor\frac{p}{2}\right\rfloor-4 \sum_{i=1}^{\lfloor p / 2\rfloor}\left\{\frac{q}{2}-\frac{q i}{p}\right\} \tag{6.260}
\end{equation*}
$$

Moreover, $\sigma\left(T_{p, 1}\right)=\sigma\left(T_{1, q}\right)=0, \sigma\left(T_{p, 2}\right)=1-p$ and $\sigma\left(T_{2, q}\right)=1-q$ for $p, q \in$ $\mathbb{N}_{>1}$.

Proof. Equation (6.171) implies

$$
\begin{align*}
\sigma\left(T_{p, q}\right) & =-(p-1)(q-1)+4 \sum_{i=1}^{\lfloor p / 2\rfloor} \sum_{j=1}^{\lfloor q(1 / 2-i / p)\rfloor} 1  \tag{6.261}\\
& =-(p-1)(q-1)+4 \sum_{i=1}^{\lfloor p / 2\rfloor}\left\lfloor\frac{q}{2}-\frac{q i}{p}\right\rfloor \tag{6.262}
\end{align*}
$$

which simplifies to the the claimed identities upon use of the identity $\lfloor x\rfloor=$ $x-\{x\}$.

Corollary 6.59. If $p$ and $q$ are odd and coprime positive integers, then the signature $\sigma\left(T_{p, q}\right)$ is divisible by 4.

Proof. Since $p$ and $q$ are odd, then $(p-1)(q-1)$ is divisible by 4 . Equation (6.261) then implies the claim.

Remark 6.6.2. For coprime positive integers $p$ and $q$, the identity

$$
\begin{align*}
\sum_{i=1}^{\lfloor p / 2\rfloor} \sum_{j=1}^{\lfloor q(1 / 2-i / p)\rfloor} 1 & +\sum_{i=1}^{\lfloor p / 2\rfloor} \sum_{j=1}^{\lfloor q(1 / 2+i / p)\rfloor} 1  \tag{6.263}\\
& =\left\{\begin{array}{cc}
\frac{1}{2}(p-1)(q-1) & p, q \text { odd or } p \text { odd, } q \text { even } \\
\frac{1}{2} p(q-1)+1 & p \text { even, } q \text { odd }
\end{array}\right. \tag{6.264}
\end{align*}
$$

and a symmetry argument imply the identity

$$
\begin{equation*}
\sigma\left(T_{p, q}\right)=-4 \sum_{i=1}^{\lfloor p / 2\rfloor} \sum_{j=1}^{\lfloor q i / p+1 / 2\rfloor} 1 \tag{6.265}
\end{equation*}
$$

which implies that $\sigma\left(T_{p, q}\right)$ is non-positive and divisible by 4 .

Corollary 6.60. For coprime positive integers $p$ and $q$,

$$
\begin{align*}
& \sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{\left(3 p-2\left\lfloor\frac{p}{2}\right\rfloor-2 k\right) q}{2 p}\right\}+\sum_{k=1}^{\lfloor p / 2\rfloor}\left\{\frac{(p-2 k) q}{2 p}\right\}=  \tag{6.266}\\
& \quad \frac{1}{p}\left(q\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right)-\left(p-1-\left\lfloor\frac{p}{2}\right\rfloor\right)\left(p(q-1)-q\left\lfloor\frac{p}{2}\right\rfloor\right)\right) . \tag{6.267}
\end{align*}
$$

Proof. The claimed identity follows by equating two equivalent expressions for the signature of the torus knot $T_{p, q}$, namely,

$$
\begin{align*}
\sigma\left(T_{p, q}\right)= & (q-1)\left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)-\frac{2 q}{p}\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right) \\
& +2 \sum_{k=1}^{\lfloor(p-1) / 2\rfloor}\left\{\frac{\left(3 p-2\left\lfloor\frac{p}{2}\right\rfloor-2 k\right) q}{2 p}\right\}-\left\{\frac{(p-2 k) q}{2 p}\right\} \tag{6.268}
\end{align*}
$$

and

$$
\begin{equation*}
\sigma\left(T_{p, q}\right)=-(p-1)(q-1)+\frac{2 q}{p}\left(p-1-\left\lfloor\frac{p}{2}\right\rfloor\right)\left\lfloor\frac{p}{2}\right\rfloor-4 \sum_{i=1}^{\lfloor p / 2\rfloor}\left\{\frac{q}{2}-\frac{q i}{p}\right\} \tag{6.269}
\end{equation*}
$$

and using the fact that $\left\lfloor\frac{p-1}{2}\right\rfloor+1 \geqslant\left\lfloor\frac{p}{2}\right\rfloor$ with equality when $p$ is even.
Since the geometric genus of the Brieskorn-Pham singularity $f=x^{p}+y^{q}+$ $z^{2}$ has a representation in terms of Dedekind sum functions, a closed form expression of the signature of torus links may be given. The following results generalize the related results in [341] and [56].

Proposition 6.61. For $p, q \in \mathbb{N}$, the signature of a torus link $T_{p, q}$ is given by

$$
\begin{align*}
\sigma\left(T_{p, q}\right)= & \frac{2 p q}{3 \mathrm{LCM}(p, q, 2)^{2}}-\frac{p q}{2}+\frac{2 q}{3 p}+\frac{2 p}{3 q}-1 \\
& \quad-4\left(p^{\prime} \mathfrak{s}\left(\frac{2 q \tau}{d}, \frac{p p^{\prime} \tau}{d}\right)+q^{\prime} \mathfrak{s}\left(\frac{2 p \tau}{d}, \frac{q q^{\prime} \tau}{d}\right)+r \mathfrak{s}\left(\frac{p q \tau}{d}, \frac{2 r \tau}{d}\right)\right), \tag{6.270}
\end{align*}
$$

where $p^{\prime}=\operatorname{GCD}(2, q), q^{\prime}=\operatorname{GCD}(p, 2), r=\operatorname{GCD}(p, q)$ and

$$
\begin{equation*}
\frac{d}{\tau}=\frac{\operatorname{GCD}(p, 2) \operatorname{GCD}(p, q) \operatorname{GCD}(2, q)}{\operatorname{GCD}(p, q, 2)} \tag{6.271}
\end{equation*}
$$

Proof. The representation of $\sigma\left(T_{p, q}\right)$ in terms of Dedekind sum functions follows from Proposition 6.46.

Corollary 6.62. The signature $\sigma\left(T_{p, q}\right)$ is not a rational function of $p$ and $q$.

Proof. Proposition 5.1 in [56] is the claim for $p$ and $q$ odd and coprime. Proposition (6.12) implies the claim when $p$ and $q$ are even or $p$ divides $q$. Proposition 6.61 implies the claim for all other cases.

Corollary 6.63. For odd coprime $p$ and $q$,

$$
\begin{equation*}
\sigma\left(T_{p, q}\right)=\frac{1}{6 p q}+\frac{2 q}{3 p}+\frac{2 p}{3 q}-\frac{p q}{2}-4(\mathfrak{s}(2 q, p)+\mathfrak{s}(2 p, q))-1 . \tag{6.272}
\end{equation*}
$$

Remark 6.6.3. Hirzebruch and Zagier prove equations (6.272) and (6.261) using different methods (Chapter II, §5, Theorem 2 and Theorem 3, [204]).

## 6.6.o.2. Bounds on the Signature of Torus Links.

Proposition 6.64. For $p, q \in \mathbb{N}_{>1}$, the signature of the torus link $T_{p, q}$ satisfies the sharp bounds

$$
\begin{align*}
\sigma\left(T_{p, q}\right) \leqslant & \left\lfloor\frac{2 p q}{3 \mathrm{LCM}(p, q, 2)^{2}}-\frac{p q}{2}+\frac{2 q}{3 p}+\frac{2 p}{3 q}-1\right. \\
& \left.+\frac{\tau}{3 d}\left(p\left(p^{\prime}\right)^{2}+q\left(q^{\prime}\right)^{2}+2 r^{2}\right)+\frac{2 d}{3 \tau}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{2}\right)-l^{\prime}\right\rfloor . \tag{6.273}
\end{align*}
$$

and

$$
\begin{align*}
\sigma\left(T_{p, q}\right) \geqslant & {\left[\frac{2 p q}{3 \mathrm{LCM}(p, q, 2)^{2}}-\frac{p q}{2}+\frac{2 q}{3 p}+\frac{2 p}{3 q}-1\right.} \\
& \left.-\frac{\tau}{3 d}\left(p\left(p^{\prime}\right)^{2}+q\left(q^{\prime}\right)^{2}+2 r^{2}\right)-\frac{2 d}{3 \tau}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{2}\right)+l^{\prime}\right], \tag{6.274}
\end{align*}
$$

where $p^{\prime}=\operatorname{GCD}(2, q), q^{\prime}=\operatorname{GCD}(p, 2), r=\operatorname{GCD}(p, q), l^{\prime}=p^{\prime}+q^{\prime}+r$ and

$$
\begin{equation*}
\frac{d}{\tau}=\frac{\operatorname{GCD}(p, 2) \operatorname{GCD}(p, q) \operatorname{GCD}(2, q)}{\operatorname{GCD}(p, q, 2)} \tag{6.275}
\end{equation*}
$$

Proof. For coprime integers $b$ and $c$, the following bounds hold:

$$
\begin{equation*}
-\mathfrak{s}(1, c) \leqslant \mathfrak{s}(b, c) \leqslant \mathfrak{s}(1, c)=\frac{c}{12}+\frac{1}{6 c}-\frac{1}{4} . \tag{6.276}
\end{equation*}
$$

Since $\operatorname{GCD}\left(\frac{2 q \tau}{d}, \frac{p p^{\prime} \tau}{d}\right)=\operatorname{GCD}\left(\frac{2 p \tau}{d}, \frac{q q^{\prime} \tau}{d}\right)=\operatorname{GCD}\left(\frac{p q \tau}{d}, \frac{2 r \tau}{d}\right)=1$, then for $p, q \in \mathbb{N}$,

$$
\begin{align*}
\mathfrak{S}\left(p, q, 2 ; \frac{d}{\tau}\right) & \leqslant p^{\prime} \mathfrak{s}\left(1, \frac{p p^{\prime} \tau}{d}\right)+q^{\prime} \mathfrak{s}\left(1, \frac{q q^{\prime} \tau}{d}\right)+r \mathfrak{s}\left(1, \frac{2 r \tau}{d}\right)  \tag{6.277}\\
& =p^{\prime}\left(\frac{p p^{\prime} \tau}{12 d}+\frac{d}{6 p p^{\prime} \tau}-\frac{1}{4}\right)+q^{\prime}\left(\frac{q q^{\prime} \tau}{12 d}+\frac{d}{6 q q^{\prime} \tau}-\frac{1}{4}\right)+r\left(\frac{r \tau}{6 d}+\frac{d}{12 r \tau}-\frac{1}{4}\right)
\end{align*}
$$

and

$$
\begin{aligned}
\mathfrak{S}\left(p, q, 2 ; \frac{d}{\tau}\right) & \geqslant-p^{\prime} \mathfrak{s}\left(1, \frac{p p^{\prime} \tau}{d}\right)-q^{\prime} \mathfrak{s}\left(1, \frac{q q^{\prime} \tau}{d}\right)-r \mathfrak{s}\left(1, \frac{2 r \tau}{d}\right) \\
& =-p^{\prime}\left(\frac{p p^{\prime} \tau}{12 d}+\frac{d}{6 p p^{\prime} \tau}-\frac{1}{4}\right)-q^{\prime}\left(\frac{q q^{\prime} \tau}{12 d}+\frac{d}{6 q q^{\prime} \tau}-\frac{1}{4}\right)-r\left(\frac{r \tau}{6 d}+\frac{d}{12 r \tau}-\frac{1}{4}\right)
\end{aligned}
$$

Remark 6.6.4. The bounds are sharp. For odd $p, \sigma\left(T_{p, p}\right)=\frac{1}{2}\left(1-p^{2}\right)$. However,

$$
\begin{align*}
\sigma\left(T_{p, p}\right) & \leqslant\left\lfloor\frac{1}{2}-\frac{p^{2}}{2}+\frac{1}{3 p}\left(2 p+2 p^{2}\right)+\frac{2 p}{3}\left(\frac{2}{p}+\frac{1}{2}\right)-p-2\right\rfloor  \tag{6.279}\\
& =\frac{1}{2}\left(1-p^{2}\right) . \tag{6.28o}
\end{align*}
$$

Similarly,

$$
\begin{align*}
\sigma\left(T_{p, p}\right) & \geqslant\left\lceil\frac{1}{2}-\frac{p^{2}}{2}-\frac{1}{3 p}\left(2 p+2 p^{2}\right)-\frac{2 p}{3}\left(\frac{2}{p}+\frac{1}{2}\right)+p+2\right]  \tag{6.281}\\
& =\frac{1}{2}\left(1-p^{2}\right) . \tag{6.282}
\end{align*}
$$

A similar phenomenon occurs when $p$ is even, but it is not limited to this particular family of torus links. For example, the signatures $\sigma\left(T_{69,92}\right)=-3174$, $\sigma\left(T_{35,7}\right)=-120, \sigma\left(T_{47,47}\right)=-1104$ achieve the corresponding lower and upper bounds.

A simple, but weaker, inequality follows from Proposition 6.47.
Corollary 6.65. For $p, q \in \mathbb{N}$, the signature of the torus link $T_{p, q}$ satisfies the upper bound

$$
\begin{equation*}
\sigma\left(T_{p, q}\right) \leqslant\left\lfloor-\frac{1}{3}(p-1)(q-1)-\frac{8}{3}-\frac{d}{\tau}+l^{\prime}\right\rfloor, \tag{6.283}
\end{equation*}
$$

where $l^{\prime}=\operatorname{GCD}(p, 2)+\operatorname{GCD}(p, q)+\operatorname{GCD}(2, q)$ and $\frac{d}{\tau}=\frac{\operatorname{GCD}(p, 2) \operatorname{GCD}(p, q) \operatorname{GCD}(2, q)}{\operatorname{GCD}(p, q, 2)}$.
Proof. The inequality of equation (6.221) implies the bound.
Remark 6.6.5. In particular, the inequality is an equality only for $T_{2,2}$; otherwise, it is a strict inequality.

Remark 6.6.6. Since $3 \leqslant l^{\prime} \leqslant 4+\operatorname{GCD}(p, q)$ and $\frac{\operatorname{GCD}(p, q)}{2} \leqslant \frac{d}{\tau} \leqslant 4 \operatorname{GCD}(p, q)$,

$$
\begin{equation*}
\sigma\left(T_{p, q}\right) \leqslant\left\lfloor-\frac{2}{3}\left(u\left(T_{p, q}\right)-2\right)+\frac{1}{2} \operatorname{gcd}(p, q)\right\rfloor . \tag{6.284}
\end{equation*}
$$

Corollary 6.66. If $p$ and $q$ are coprime and of different parity, then

$$
\sigma\left(T_{p, q}\right)=\frac{2}{3 p q}-\frac{p q}{2}+\frac{2 q}{3 p}+\frac{2 p}{3 q}-1-4 \cdot \begin{cases}\mathfrak{s}\left(q, \frac{p}{2}\right)+2 \mathfrak{s}(p, q) & p \text { even }  \tag{6.285}\\ \mathfrak{s}\left(p, \frac{q}{2}\right)+2 \mathfrak{s}(q, p) & q \text { even. }\end{cases}
$$

Proposition 6.67. For a weighted homogeneous surface singularity
$f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with Milnor fiber $F_{f, 0}$ and (boundary) algebraic link $K_{f}=\partial F_{f, 0}$,

$$
\begin{equation*}
\operatorname{rank} H_{1}\left(K_{f}\right)=4 p_{g}(f)-\sigma\left(F_{f, 0}\right)-\mu_{\mathrm{alg}}(f) \tag{6.286}
\end{equation*}
$$

Proof. By Propositions 3.21 and 5.40 ,

$$
\begin{align*}
& \sigma\left(F_{f, 0}\right)=\varsigma_{+}-\varsigma_{-}  \tag{6.287}\\
& \mu_{\mathrm{alg}}(f)=\varsigma_{+}+\varsigma_{0}+\varsigma_{-}  \tag{6.288}\\
& 2 p_{g}(f)=\varsigma_{+}+\varsigma_{0} \tag{6.289}
\end{align*}
$$

and one computes $4 p_{g}(f)-\sigma\left(F_{f, 0}\right)-\mu_{\mathrm{alg}}(f)=\varsigma_{0}$, which is the rank of the homology group $H_{1}(E) \cong H_{1}\left(K_{f}\right)$ [118].

Corollary 6.68. The following identities hold:

$$
\begin{align*}
2 g\left(\Sigma(a, b, c) / S^{1}\right) & =\varsigma_{0}  \tag{6.290}\\
\tilde{\chi}(E) & =s-\varsigma_{0}  \tag{6.291}\\
-K^{2} & =s+5 \zeta_{+}+4 \varsigma_{0}-\varsigma_{-} \tag{6.292}
\end{align*}
$$

which imply the congruences

$$
\begin{align*}
\tilde{\chi}(E) & \equiv s \bmod 2  \tag{6.293}\\
-K^{2} & \equiv s+\sigma\left(F_{f, 0}\right) \bmod 4 \tag{6.294}
\end{align*}
$$

Proof. Applying Proposition 6.46 yields the first two identities. Corollary 6.53 yields the last identity. The congruences follow immediately, and the details are omitted.

Corollary 6.69. For $a, b, c \in \mathbb{N}$, the number of zero and positive eigenvalues of the intersection form $S$ corresponding to a Brieskorn-Pham 3-manifold $\Sigma(a, b, c)$ are both even. Moreover,

$$
\begin{equation*}
\operatorname{rank} H_{1}(E)=\operatorname{rank} H_{1}(\Sigma(a, b, c))=2 g\left(\Sigma(a, b, c) / S^{1}\right) \tag{6.295}
\end{equation*}
$$

In particular, the fundamental group $\pi_{1}(\Sigma(a, b, c))$ is not infinite solvable and may be finite or nilpotent.

Proof. Let $M$ be a Seifert manifold with invariants $\left\{g ;\left(\alpha_{i}, \beta_{i}\right)\right\}$. According to Neumann and Raymond, Seifert proved $H_{1}(M) \cong \mathbb{Z}^{2 g} \oplus$ coker $S$ (Theorem 4.1, [342]), where $S$ is the matrix

$$
S=\left(\begin{array}{cccccc}
1 & 1 & 1 & \cdots & 1 & 0  \tag{6.296}\\
\alpha_{1} & 0 & 0 & \cdots & 0 & \beta_{1} \\
0 & \alpha_{2} & 0 & \cdots & 0 & \beta_{2} \\
0 & 0 & \ddots & \cdots & 0 & \vdots \\
0 & 0 & 0 & \cdots & \alpha_{m} & \beta_{m}
\end{array}\right)
$$

If $M=\Sigma(a, b, c)$, then coker $S$ is trivial. Thus, rank $H_{1}(\Sigma(a, b, c))=2 g$.
The evenness of $\varsigma_{+}+\varsigma_{0}$ follows from Proposition 4.14 in [437] (proving a conjecture of Arnol'd). The evenness of $s_{0}$ follows from Corollary 6.68, which
implies the evenness of $\varsigma_{+}$and the claimed identity. In particular, rank $H_{1}(E)=$ rank $H_{1}(\Sigma(a, b, c))$ is even. Finally, the equivalence $\varsigma_{0}=\varsigma_{+}=1$ if and only if $\pi_{1}\left(V_{f, 0}^{\times}\right) \cong \pi_{1}(\Sigma(a, b, c))$ is infinite solvable and not nilpotent (Proposition 3.3, [118]). As both $\varsigma_{+}$and $\varsigma_{0}$ must be even, this fundamental group type does not occur for any Brieskorn-Pham 3-manifold.

Remark 6.6.7. Milnor classified the fundamental group $\pi_{1}(\Sigma(a, b, c))$ according to the sign of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1$ [312], q.v., Proposition 4.55.

Proposition 6.70. Given a quasi-Brieskorn-Pham surface singularity $f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with inverse exponents $\{a, b, c\} \subset \mathbb{N}$, then the eigenvalue signature $\left(\varsigma_{+}, \varsigma_{0}, \zeta_{-}\right)$of the corresponding intersection form $S$ satisfies

$$
\begin{align*}
& \varsigma_{+}=\frac{a b c}{3}+\frac{1}{2}\left(l+l^{\prime}-a b-b c-c a-\frac{d}{\tau}\right) \\
&+\frac{1}{6}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right)-2 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)-2  \tag{6.297}\\
& \varsigma_{0}= \frac{d}{\tau}-l^{\prime}+2  \tag{6.298}\\
& \zeta_{-}= \frac{2 a b c}{3}+\frac{1}{2}\left(l+l^{\prime}-a b-b c-c a-\frac{d}{\tau}\right) \\
& \quad-\frac{1}{6}\left(\frac{a b}{c}+\frac{a c}{b}+\frac{b c}{a}+\frac{d^{2}}{a b c \tau^{2}}\right)+2 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)-1 . \tag{6.299}
\end{align*}
$$

Proof. The general formula for the geometric genus corresponding to a quasi-Brieskorn-Pham polynomial with inverse weights $\{a, b, c\}$ implies the
identity

$$
\begin{align*}
\varsigma_{+}= & 2 p_{g}(f)-\varsigma_{0}  \tag{6.300}\\
=\frac{a b c}{3} & +\frac{1}{2}\left(l+l^{\prime}-a b-b c-c a-\frac{d}{\tau}\right) \\
& +\frac{1}{6}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right)-2 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)-2 . \tag{6.301}
\end{align*}
$$

Since $\mu_{\mathrm{alg}}(f)=\varsigma_{+}+\varsigma_{0}+\varsigma_{-}$, by equation (6.200),

$$
\begin{align*}
& \mathcal{S}_{-}= \mu_{\mathrm{alg}}(f)-2 p_{g}(f)  \tag{6.302}\\
&=\frac{2 a b c}{3}+\frac{1}{2}\left(l+l^{\prime}-a b-b c-c a-\frac{d}{\tau}\right) \\
&-\frac{1}{6}\left(\frac{a b}{c}+\frac{a c}{b}+\frac{b c}{a}+\frac{d^{2}}{a b c \tau^{2}}\right)+2 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)-1 . \tag{6.303}
\end{align*}
$$

Remark 6.6.8. Thus, for example,

$$
\begin{align*}
& \varsigma_{+}+\varsigma_{-}=a b c+\left(l+l^{\prime}-a b-b c-c a-\frac{d}{\tau}\right)-3  \tag{6.304}\\
& \varsigma_{-}+\varsigma_{0}=\frac{2 a b c}{3}+\frac{1}{2}\left(l-l^{\prime}-a b-b c-c a+\frac{d}{\tau}\right) \\
& \quad-\frac{1}{6}\left(\frac{a b}{c}+\frac{a c}{b}+\frac{b c}{a}+\frac{d^{2}}{a b c \tau^{2}}\right)+2 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)+1 . \tag{6.305}
\end{align*}
$$

Corollary 6.71. Given a quasi-Brieskorn-Pham surface singularity
$f:\left(\mathbb{C}^{3}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with inverse exponents $\{a, b, c\} \subset \mathbb{N}$, then

$$
\begin{equation*}
\zeta_{0}=2 \operatorname{ord} \triangle^{\circ}(\mathcal{W}), \tag{6.306}
\end{equation*}
$$

where $\triangle^{\circ}(\mathcal{W})$ is the interior of the 2 -simplex opposite the origin of the weight polytope $\mathcal{W}(f)$. In particular,

$$
\begin{equation*}
g\left(\Sigma(a, b, c) / S^{1}\right)=\operatorname{ord} \triangle^{\circ}(\mathcal{W}) . \tag{6.307}
\end{equation*}
$$

Proof. The number of lattice points intersecting $\triangle^{\circ}(t \mathcal{W})$ is equal to the number of positive integral solutions of the Diophantine equation $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=$ $t$, which is $\frac{1}{2}\left(\frac{d}{\tau} t^{2}-l^{\prime} t\right)+1$. This integer also coincides with the genus of the corresponding base-orbifold, $g\left(\Sigma(a t, b t, c t) / S^{1}\right)$. Taking $t=1$ yields the claim.

Conjecture 6.72. Given a quasi-Brieskorn-Pham singularity $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ (C,0), for $n>2$, there is an rational $N_{n}$ depending only on $n$, such that

$$
\begin{equation*}
\varsigma_{0}=N_{n} \operatorname{ord} \triangle^{\circ}(\mathcal{W}), \tag{6.308}
\end{equation*}
$$

where $\triangle^{\circ}(\mathcal{W})$ is the interior of the $n$-simplex opposite the origin of the weight polytope $\mathcal{W}(f)$.

Problem 6.6.1. Determine $N_{n}$ as a function of $n$.

### 6.6.1. Dedekind Sum Identities.

Proposition 6.73. For $p, q \in \mathbb{N}$,

$$
\begin{align*}
\mathfrak{S}\left(p, q, 2 ; \frac{d}{\tau}\right)= & \frac{p q}{6 \mathrm{LCM}(p, q, 2)^{2}}-\frac{p+q}{4}+\frac{p q}{8}+\frac{1}{6}\left(\frac{q}{p}+\frac{p}{q}\right) \\
& -\frac{q}{2 p}\left(p-1-\left\lfloor\frac{p}{2}\right\rfloor\right)\left\lfloor\frac{p}{2}\right\rfloor-\frac{1}{4}\left(l^{\prime}-\frac{d}{\tau}\right)+\frac{1}{2} \\
& +\sum_{i=1}^{\lfloor p / 2\rfloor}\left\{\frac{q}{2}-\frac{q i}{p}\right\} \tag{6.309}
\end{align*}
$$

where $l^{\prime}=\operatorname{GCD}(p, 2)+\operatorname{GCD}(p, q)+\operatorname{GCD}(2, q)$ and $\frac{d}{\tau}=\frac{\operatorname{GCD}(p, 2) \operatorname{GCD}(p, q) \operatorname{GCD}(2, q)}{\operatorname{GCD}(p, q, 2)}$.
Proof. Combine the two equivalent representations of the signature of the torus link $T_{p, q}$, namely,

$$
\begin{align*}
\sigma\left(T_{p, q}\right)= & -(p-1)(q-1)+\frac{2 q}{p}\left(p-1-\left\lfloor\frac{p}{2}\right\rfloor\right)\left\lfloor\frac{p}{2}\right\rfloor+l^{\prime}-\frac{d}{\tau}-2 \\
& -4 \sum_{i=1}^{\lfloor p / 2\rfloor}\left\{\frac{q}{2}-\frac{q i}{p}\right\}  \tag{6.310}\\
= & \frac{2 p q}{3 \operatorname{LCM}(p, q, 2)^{2}}-\frac{p q}{2}+\frac{2 q}{3 p}+\frac{2 p}{3 q}-1-4 \mathfrak{S}\left(p, q, 2 ; \frac{d}{\tau}\right) . \tag{6.311}
\end{align*}
$$

Corollary 6.74. If $p \leqslant q$ and $p$ divides $q$ or if $p$ and $q$ are even, then

$$
\begin{align*}
\mathfrak{S}\left(p, q, 2 ; \frac{d}{\tau}\right)=- & \frac{1}{2}+\frac{p+q}{4}+\frac{1}{6}\left(\frac{p}{q}+\frac{q}{p}\right)-\frac{3 p q}{8}+\frac{p q}{6 \mathrm{LCM}(p, q, 2)^{2}} \\
& +\frac{q}{2 p}\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right)+\frac{1}{2}(q-1)\left\lfloor\frac{p}{2}\right\rfloor . \tag{6.312}
\end{align*}
$$

Proof. Combining Corollary 6.12 and Proposition 6.56 yields

$$
\begin{align*}
\mathfrak{S}\left(p, q, 2 ; \frac{d}{\tau}\right)=- & \frac{1}{4}(q-1)\left(p-1-2\left\lfloor\frac{p}{2}\right\rfloor\right)+\frac{q}{2 p}\left\lfloor\frac{p-1}{2}\right\rfloor\left(p-\left\lfloor\frac{p}{2}\right\rfloor\right)+\frac{p q}{6 \mathrm{LCM}(p, q, 2)^{2}} \\
& -\frac{p q}{8}+\frac{q}{6 p}+\frac{p}{6 q}-\frac{1}{4} \tag{6.313}
\end{align*}
$$

which simplifies to the claimed identity.
6.6.2. Generalized Dedekind Reciprocity Law, Revisited. We now make a few remarks concerning the Dedekind sum function. In general, for $a, b, c \in \mathbb{N}$, we have shown

$$
\begin{align*}
\mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)=\frac{a b c}{6} & -\frac{1}{4}\left(a b+b c+c a-\frac{d}{\tau}\right) \\
& +\frac{1}{4}\left(l-l^{\prime}\right)+\frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right)  \tag{6.314}\\
& -\sum_{i=1}^{a} \sum_{j=1}^{\lfloor b(1-i / a)\rfloor} \sum_{k=1}^{\lfloor c(1-i / a-j / b)\rfloor} 1 . \tag{6.315}
\end{align*}
$$

Corollary 6.75. For $a, b, c \in \mathbb{N}$,

$$
\begin{equation*}
12 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)-\frac{a b}{c}-\frac{a c}{b}-\frac{b c}{a}-\frac{d^{2}}{a b c \tau^{2}} \in \mathbb{Z} \tag{6.316}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
12 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)-\frac{a b}{c}-\frac{a c}{b}-\frac{b c}{a}+a b c-\frac{d^{2}}{a b c \tau^{2}} \in 3 \mathbb{Z} \tag{6.317}
\end{equation*}
$$

Proof. Proposition 6.46 and Laufer's formula imply

$$
\begin{align*}
\tilde{\chi}(E)+K^{2} & =-\mu_{\mathrm{alg}}(f)+6(1-g)-\left(l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right) \\
& +12 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)-2 \tag{6.318}
\end{align*}
$$

where $g=g\left(\Sigma(a, b, c) / S^{1}\right)$. Hence, the factor

$$
\begin{align*}
L(a, b, c)= & \tilde{\chi}(E)+K^{2}+2 \mu_{\mathrm{alg}}(f)+6 g-6  \tag{6.319}\\
= & \mu_{\mathrm{alg}}(f)-\left(l-a b-b c-c a+\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right) \\
& \quad+12 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)-2 \\
= & 12 \mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)-\frac{a b}{c}-\frac{a c}{b}-\frac{b c}{a}+a b c-\frac{d^{2}}{a b c \tau^{2}}-3 \tag{6.320}
\end{align*}
$$

is an integer divisible by 3 by Corollary 6.51 .

Remark 6.6.9. Taking $c=1$ yields

$$
\begin{equation*}
12 \mathfrak{S}(a, b, 1 ; \operatorname{GCD}(a, b))-\frac{a}{b}-\frac{b}{a}-\frac{\operatorname{GCD}(a, b)}{\operatorname{LCM}(a, b)} \in 3 \mathbb{Z} \tag{6.321}
\end{equation*}
$$

which is very nearly the Dedekind Reciprocity Law. Equivalently, for coprime, positive integers $a$ and $b$, there is an integer $3 k$ such that

$$
\begin{equation*}
\mathfrak{s}(a, b)+\mathfrak{s}(b, a)=\frac{1}{12}\left(\frac{a}{b}+\frac{1}{a b}+\frac{b}{a}\right)+\frac{k}{4} . \tag{6.322}
\end{equation*}
$$

The Dedekind Reciprocity Law is equivalent is recovered by taking $k=1$.
We generalize.

Proposition 6.76. The Dedekind sum function satisfies

$$
\begin{align*}
\mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)= & \frac{\operatorname{area}\left(\triangle_{a, b, c}\right)^{2}}{18 \operatorname{vol}\left(\mathcal{T}_{a, b, c}\right)}-\frac{\chi(\Sigma(a, b, c))}{12}-\frac{1}{4} \\
& \quad-p_{g}(f)+\frac{1}{4} \mu_{\mathrm{alg}}(f)+\frac{1}{2} g\left(\Sigma(a, b, c) / S^{1}\right)-\frac{1}{2} \operatorname{vol}\left(\mathcal{T}_{a, b, c}\right), \tag{6.323}
\end{align*}
$$

where $\mathcal{T}_{a, b, c}=\mathbf{\operatorname { c o n v }}\left\{\mathbf{0}, a \mathbf{e}_{1}, b \mathbf{e}_{2}, c \mathbf{e}_{3}\right\}$ and $\triangle_{a, b, c}$ is the face opposite the origin.

Proof. Let $f$ be a quasi-Brieskorn-Pham surface singularity with inverse weights $\{a, b, c\} \subset \mathbb{N}$. Recall $\chi(\Sigma(a, b, c))=-\frac{d^{2}}{a b c \tau^{2}}$. By Proposition 6.46,

$$
\begin{align*}
\mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)= & \frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}-\chi(\Sigma(a, b, c))\right)-\frac{1}{4} \\
& \quad-p_{g}(f)+\frac{1}{4} \mu_{\mathrm{alg}}(f)+\frac{1}{2} g\left(\Sigma(a, b, c) / S^{1}\right)-\frac{1}{2} \operatorname{vol}\left(\mathcal{T}_{a, b, c}\right)  \tag{6.324}\\
= & \frac{\operatorname{area}\left(\triangle_{a, b, c}\right)^{2}}{18 \operatorname{vol}\left(\mathcal{T}_{a, b, c}\right)}-\frac{\chi(\Sigma(a, b, c))}{12}-\frac{1}{4} \\
& \frac{1}{4} \mu_{\mathrm{alg}}(f)+\frac{1}{2} g\left(\Sigma(a, b, c) / S^{1}\right)-\frac{1}{2} \operatorname{vol}\left(\mathcal{T}_{a, b, c}\right)-p_{g}(f), \tag{6.325}
\end{align*}
$$

where $\mathcal{T}_{a, b, c}$ is the weight polytope of $f$ with volume $\operatorname{vol}\left(\mathcal{T}_{a, b, c}\right)=\frac{a b c}{6}$ and $\triangle_{a, b, c}$ is the face opposite the origin with area satisfying de Gua's Theorem,

$$
\begin{equation*}
\operatorname{area}\left(\triangle_{a, b, c}\right)^{2}=\frac{1}{4}\left(a^{2} b^{2}+b^{2} c^{2}+c^{2} a^{2}\right) \tag{6.326}
\end{equation*}
$$

Remark 6.6.10. Equation (6.323) involves ingredients from geometry (area and volume), algebra (dimension of the local algebra), combinatorics (geometric genus) and topology (genus and Euler characteristic).
6.6.3. Generalized Dedekind Sum Congruences. In this section, we generalize classical congruences satisfied by the Dedekind sum function.

Proposition 6.77. For $a_{0}, \ldots, a_{n} \in \mathbb{N}$, let $N^{\prime}=a_{0} \cdots a_{n}$,

$$
\begin{align*}
N & =\operatorname{LCM}\left(a_{0}, \ldots, a_{n}\right)  \tag{6.327}\\
b_{k} & =\frac{\operatorname{LCM}\left(a_{0}, \ldots, a_{n}\right)}{a_{k}}  \tag{6.328}\\
c_{k} & =\frac{\operatorname{LCM}\left(a_{0}, \ldots, a_{n}\right)}{\operatorname{LCM}\left(a_{0}, \ldots, \hat{a}_{k}, \ldots, a_{n}\right)}  \tag{6.329}\\
d_{k} & =\frac{a_{0} \ldots \hat{a}_{k} \cdots a_{n}}{\operatorname{LCM}\left(a_{0}, \ldots, \hat{a}_{k}, \ldots, a_{n}\right)} . \tag{6.330}
\end{align*}
$$

Then

$$
\begin{equation*}
12 N \sum_{k=0}^{n} d_{k} \mathfrak{s}\left(b_{k}, c_{k}\right) \equiv \frac{N^{\prime}}{N}\left(1+\sum_{k=0}^{n} b_{k}^{2}\right) \bmod N \tag{6.331}
\end{equation*}
$$

Proof. According to [343], the signature of the Milnor fiber of a BrieskornPham 3-manifold $\Sigma\left(a_{0}, \ldots, a_{n}\right)$ is computed as

$$
\begin{align*}
\sigma\left(F_{f, 0}\right)=-1 & +\frac{N^{\prime}}{3 N^{2}}\left(1-(n-1) N^{2}+\sum_{k=0}^{n} b_{k}^{2}\right) \\
& -4 \sum_{k=0}^{n} d_{k} \mathfrak{s}\left(b_{k}, c_{k}\right) \tag{6.332}
\end{align*}
$$

Therefore,

$$
\begin{align*}
3 N \sigma\left(F_{f, 0}\right)=- & 3 N+\frac{N^{\prime}}{N}\left(1-(n-1) N^{2}+\sum_{k=0}^{n} b_{k}^{2}\right) \varsigma_{0} \\
& -12 N \sum_{k=0}^{n} d_{k} \mathfrak{s}\left(b_{k}, c_{k}\right) . \tag{6.333}
\end{align*}
$$

Since the signature is an integer, the following congruence follows,

$$
\begin{equation*}
12 N \sum_{k=0}^{n} d_{k} \mathfrak{s}\left(b_{k}, c_{k}\right) \equiv \frac{N^{\prime}}{N}\left(1+\sum_{k=0}^{n} b_{k}^{2}\right) \bmod N \tag{6.334}
\end{equation*}
$$

Proof of Proposition 6.38. Observe

$$
\begin{equation*}
3 N \sigma\left(F_{f, 0}\right)=-3 N+\left(1-(n-1) N^{2}+\sum_{k=0}^{n} b_{k}^{2}\right)-12 N \sum_{k=0}^{n} \mathfrak{s}\left(b_{k}, a_{k}\right) \tag{6.335}
\end{equation*}
$$

Hence, since the signature is an integer,

$$
\begin{equation*}
12 N \sum_{k=0}^{n} \mathfrak{s}\left(b_{k}, a_{k}\right) \equiv 1+\sum_{k=0}^{n} b_{k}^{2} \bmod N \tag{6.336}
\end{equation*}
$$

which is the claimed congruence.

### 6.7. Characteristic and Cyclotomic Polynomials

In $\S 4$ of [143], Glasby studied cyclotomic field extensions of $Q$ and (polynomial) tensor products of cyclotomic polynomials. We use this analysis to
compute the characteristic polynomial of the quasi-Brieskorn-Pham singularity as a product of cyclotomic polynomials.

Proposition 6.78. The characteristic polynomial of a non-degenerate, quasi-Brieskorn-Pham singularity $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with inverse weights $\left\{a_{0}, \ldots, a_{n}\right\}$ is the following product of cyclotomic polynomials,

$$
\begin{align*}
\Delta_{f}(t)= & \Phi_{1}(t)^{(-1)^{n+1}} \\
& \cdot \prod_{k=1}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \prod_{d \mid \operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)} \Phi_{d}(t)^{(-1)^{n-k+1} N\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)} \tag{6.337}
\end{align*}
$$

where $N\left(a_{0}, \ldots, a_{n}\right)$ is defined in Proposition 4.71. In particular, if $f$ is BrieskornPham with exponents $\left\{a_{0}, \ldots, a_{n}\right\}$, then

$$
\begin{align*}
\Delta_{f}(t) & =\bigotimes_{i=0}^{n} \prod_{1<d \mid a_{i}} \Phi_{d}(t)  \tag{6.338}\\
& =\prod_{1<d_{0} \mid a_{0}} \cdots \prod_{1<d_{n} \mid a_{n}} \Phi_{d_{0}}(t) \cdots \Phi_{d_{n}}(t) \tag{6.339}
\end{align*}
$$

Proof. Recall the identities $\left(t^{m}-1\right) \otimes\left(t^{n}-1\right)=\left(t^{\operatorname{LCM}(m, n)}-1\right)^{\operatorname{GCD}(m, n)}$ and $t^{n}-1=\prod_{d \mid n} \Phi_{d}(t)$. Combining these, we find

$$
\begin{align*}
\prod_{d \mid \mathrm{LCM}(m, n)} \Phi_{d}(t)^{\mathrm{GCD}(m, n)} & =\left(t^{\mathrm{LCM}(m, n)}-1\right)^{\mathrm{GCD}(m, n)} \\
& =\prod_{d \mid m} \Phi_{d}(t) \otimes \prod_{d^{\prime} \mid n} \Phi_{d^{\prime}}(t)  \tag{6.341}\\
& =\prod_{d \mid m} \prod_{d^{\prime} \mid n} \Phi_{d}(t) \otimes \Phi_{d^{\prime}}(t) \tag{6.342}
\end{align*}
$$

Thus, by Proposition 4.72,

$$
\begin{align*}
\Delta_{f}(t) & =(t-1)^{(-1)^{n+1}} \prod_{k=1}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n}\left(t^{\mathrm{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)}-1\right)^{(-1)^{n-k+1} N\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)} \\
& =\Phi_{1}(t)^{(-1)^{n+1}} \prod_{k=1}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n d \mid \operatorname{LCM}\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)} \Phi_{d}(t)^{(-1)^{n-k+1} N\left(a_{i_{1}}, \ldots, a_{i_{k}}\right)}, \tag{6.343}
\end{align*}
$$

which implies the claim. With $f=z^{a}$ for some $a \in \mathbb{N}$,

$$
\begin{equation*}
\Delta_{(a)}(t)=\prod_{1<d \mid a} \Phi_{d}(t) \tag{6.345}
\end{equation*}
$$

Thus, if $f=\square_{i=0}^{n} f_{i}$, where $f_{i}=z^{a_{i}}$, then

$$
\begin{align*}
\Delta_{f}(t) & =\bigotimes_{i=0}^{n} \Delta_{f_{i}}(t)  \tag{6.346}\\
& =\bigotimes_{i=0}^{n} \prod_{1<d \mid a_{i}} \Phi_{d}(t) \tag{6.347}
\end{align*}
$$

by the Sebastiani-Thom equivalence.
The following result is classical; our proof is new.

Corollary 6.79. The Euler totient function satisfies the identity

$$
\begin{equation*}
k=\sum_{d \mid k} \varphi(k) \tag{6.348}
\end{equation*}
$$

In particular, $\varphi(k) \leqslant k-1$ with equality if and only if $k$ is prime.
Proof. Consider the singularity $f=z^{k}$ with $k \in \mathbb{N}$. The characteristic polynomial is the product $\Delta_{f}(t)=\prod_{1<d \mid k} \Phi_{d}(t)$, and the algebraic index is $k-1$. Thus, since $\mu_{\mathrm{alg}}(f)=\operatorname{deg} \Delta_{f}(t)$ and $\varphi(d)=\operatorname{deg} \Phi_{d}(t)$, one has $k-1=$ $\sum_{1<d \mid k} \varphi(d)$, which implies the claim.

Define the $q$-integer $[n]_{q}=\frac{q^{n}-1}{q-1}$.
Corollary 6.80. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a homogeneous polynomial of degree $D$, then the characteristic polynomial is the following product of cyclotomic
polynomials

$$
\begin{equation*}
\Delta_{f}(t)=\prod_{d \mid D} \Phi_{d}(t)^{(-1)^{n}\left([n+1]_{1-D}-\delta_{d, 1}\right)} \tag{6.349}
\end{equation*}
$$

with degree $\mu_{\mathrm{alg}}(f)=(D-1)^{n+1}$.
Proof. By Proposition 6.78,

$$
\begin{align*}
\Delta_{f}(t) & =\Phi_{1}(t)^{(-1)^{n+1}} \cdot \prod_{k=1}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \prod_{d \mid D} \Phi_{d}(t)^{(-1)^{n-k+1} D^{k-1}}  \tag{6.350}\\
& =\Phi_{1}(t)^{(-1)^{n+1}} \cdot \prod_{d \mid D} \Phi_{d}(t)^{\sum_{k=1}^{n+1}(-1)^{n-k+1}\binom{n+1}{k} D^{k-1}}  \tag{6.351}\\
& =\prod_{d \mid D} \Phi_{d}(t)^{(-1)^{n}\left(1-(1-D)^{n+1}\right) / D+(-1)^{n+1} \delta_{d, 1}}  \tag{6.352}\\
& =\prod_{d \mid D} \Phi_{d}(t)^{(-1)^{n}\left([n+1]_{1-D}-\delta_{d, 1}\right)}, \tag{6.353}
\end{align*}
$$

where the exponent $a_{n+1, D}=\frac{(-1)^{n}}{D}\left(1-(1-D)^{n+1}\right)$, and, for $n \geqslant 0$, may be written as $a_{n+1, D}=(-1)^{n}[n+1]_{1-D}$. Moreover, since $\varphi(n)=\operatorname{deg} \Phi_{n}(t)$, one has

$$
\begin{align*}
\mu_{\mathrm{alg}}(f) & =\operatorname{deg} \Delta_{f}(t)  \tag{6.354}\\
& =(-1)^{n} \sum_{d \mid D}\left([n+1]_{1-D}-\delta_{d, 1}\right) \varphi(d)  \tag{6.355}\\
& =(-1)^{n}\left([n+1]_{1-D} D-1\right)  \tag{6.356}\\
& =(D-1)^{n+1}, \tag{6.357}
\end{align*}
$$

as claimed.

REmARK 6.7.1. As a function of $D \geqslant 2, a_{n+1, D}=\sum_{k=0}^{n}(-1)^{k+n}(D-1)^{k}$, and $\left\{a_{n+1, D}\right\}_{D \geqslant 2}$ counts the number of walks of length $n+1$ between two distinct vertices on the complete graph $K_{D}$ (e.g., A001477, A002061, A062158 and A060884). Figure 6.2 illustrates seven walks of length three $(n=2)$ on the complete graph $K_{4}(D=4)$.


Figure 6.2. Seven Walks of Length Three on the Complete Graph $K_{4}$

Remark 6.7.2. As a function of $n$, the exponent $a_{n+1, D}$ satisfies the secondorder, linear recurrence relation $a_{n, D}=(D-2) a_{n-1, D}+(D-1) a_{n-2, D}$ with $a_{0, D}=0$ and $a_{1, D}=1$. For example, $\left\{a_{n, 3}\right\}_{n \geqslant 0}$ enumerates the Jacobsthal sequence, which counts the number of perfect matchings of a $2 \times n$ modified grid graph with squares, i.e., $C_{4}$ graphs, replaced by tetrahedra, i.e., $K_{4}$ graphs (A001045).

Corollary 6.81. For a non-degenerate, quasi-Brieskorn-Pham singularity $f$ with distinct prime inverse weights $\left\{p_{0}, \ldots, p_{n}\right\}$, the corresponding characteristic polynomial $\Delta_{f}(t)$ is the cyclotomic polynomial $\Phi_{N}(t)$, where $N=p_{0} \cdots p_{n}$. In particular, $\mu_{\mathrm{alg}}(f)=\varphi(N)$, the Euler totient function. If $f$ is Brieskorn-Pham, then $K_{f} \simeq \Sigma\left(p_{0}, \ldots, p_{n}\right)$ is a topological sphere if and only if $N$ is not prime.

Proof. Let $\mu$ and $\varphi=\mathrm{id} * \mu$ denote the standard Möbius function and Euler totient function, respectively. If $\mathbb{B}$ is a finite basis of distinct primes $P$ with conductor $N=\prod_{p \in P} p$, then $\mu_{\mathbb{B}}=\mu$ on the set of divisors of the squarefree integer $N$. Since the $P$-divisors of $N$ coincide with ordinary (unitary) divisors,

$$
\begin{equation*}
\Delta_{f}(t)=\prod_{d \mid N}\left(t^{d}-1\right)^{\mu(N / d)}=\prod_{d \mid N}\left(t^{N / d}-1\right)^{\mu(d)}=\Phi_{N}(t), \tag{6.358}
\end{equation*}
$$

which is the $N^{\text {th }}$-cyclotomic polynomial. The degree of $\Delta_{f}(t)$ follows from convolution identity

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\operatorname{deg} \Delta_{f}(t)=\sum_{d \mid N} d \mu\left(\frac{N}{d}\right)=\sum_{d \mid N} \frac{N}{d} \mu(d)=\varphi(N) . \tag{6.359}
\end{equation*}
$$

By Proposition ??, if $f$ is Brieskorn-Pham, then $K_{f}$ is the Brieskorn-Pham manifold $\Sigma\left(p_{0}, \ldots, p_{n}\right)$, which is a topological sphere if and only if $\Delta_{f}(1)= \pm 1$. The classical identity $n=\prod_{1<d \mid n} \Phi_{n}(1)$ for $n \in \mathbb{N}$ implies $\Phi_{n}(1)=p$ if and only if $n$ is a power of a prime $p$, and $\Phi_{n}(1)=1$ if and only if $\omega(n)>1$, where $\omega(n)$ denotes the number of prime factors of $n$. That is, $\Phi_{n}(1)=\operatorname{GCD}\{p\}_{p \mid n}$.

One infers that the algebraic index of a quasi-Brieskorn-Pham polynomial generalizes the Euler totient function, and is not, in the conventional sense, a multiplicative arithmetic function. However, by the Sebastiani-Thom equivalence, the algebraic index does satisfy a multiplicative identity.

### 6.8. Abstract Arithmetic

Let $D(k)$ denote the set of divisors of $k \in \mathbb{N}$. Denote by $M D\left(k_{1}, \ldots, k_{m}\right)$ the set of mixed divisors of the integer $k_{1} \cdots k_{m}$ such that each divisors has at least one prime divisor from each of the integers $\left\{k_{1}, \ldots, k_{m}\right\}$. In particular, if $k=p_{1}^{r_{1}} \cdots p_{n}^{r_{m}}$, then $M D\left(p_{1}^{r_{1}}, \ldots, p_{n}^{r_{m}}\right)=D(k)$.

Remark 6.8.1. Consider three coprime integers, say, $20=2^{2} \cdot 5,3$ and 7 . Since $\tau(20)=6$ and $\tau(3)=\tau(7)=2$, there are five mixed divisors of the product 420, namely, $42=2 \cdot 3 \cdot 7,84=2^{2} \cdot 3 \cdot 7,105=5 \cdot 3 \cdot 7,210=(2 \cdot 5) \cdot 3 \cdot 7$ and $420=\left(2^{2} \cdot 5\right) \cdot 3 \cdot 7$. Thus, $M D(20,3,7)=\{42,84,105,210,420\}$.

Recall the Dirichlet convolution of the arithmetic functions $f, g: \mathbb{N} \rightarrow \mathbb{C}$ is the $\operatorname{sum}(f * g)(k)=\sum_{d \mid k} f(d) g\left(\frac{k}{d}\right)$, which is associative and commutative. For example, $\mathrm{id} * \mu=\varphi$ and $\varphi * 1=\mathrm{id}$. The set of arithmetic functions $\mathcal{D}$ equipped with pointwise addition and Dirichlet convolution is a commutative ring.

Define the arithmetic function $\mu_{\mathbb{B}}(k)$ to be $(-1)^{l}$ if $k$ can be factored into a $\mathbb{B}$-squarefree product of $l$ elements of an integral basis $\mathbb{B} \subset \mathbb{N}_{>1}$ and 0 otherwise. Define the generalized Euler totient function $\varphi_{\mathbb{B}}$ with respect to the basis $\mathbb{B}$ as $\varphi(k)=\sum_{\left.d\right|_{\mathbb{B}} k} d \mu_{\mathbb{B}}\left(\frac{k}{d}\right)$, where $\left.d\right|_{\mathbb{B}} k$ denotes a restricted divisor of $k$ with
respect to the basis $\mathbb{B}$. The reader is referred to Volume 3 for a discussion of arithmetic and analytic number theory with finite integral bases.

Remark 6.8.2. Consider the example given in Remark 6.8.7. An elementary computation yields $M D(3,4,5)=\{30,60\}$, so $\Delta_{(3,4,5)}(t)=\Phi_{30}(t) \Phi_{60}(t)$. The $\{3,4,5\}$-divisors of 60 are $\{1,3,4,5,12,15,20,60\}$ and, therefore,

$$
\begin{equation*}
\varphi_{\{3,4,5\}}(60)=-1+3+4+5-12-15-20+60=24 \tag{6.360}
\end{equation*}
$$

which coincides with $\operatorname{deg} \Delta_{(3,4,5)}(t)$.
The previous remark suggests some general structure.
Proposition 6.82. For a quasi-Brieskorn-Pham singularity $f$ with pairwise coprime inverse weights $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}$, the corresponding characteristic polynomial is the product

$$
\begin{align*}
\Delta_{f}(t) & =\Phi_{1}(t)^{(-1)^{n+1}} \prod_{k=1}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \prod_{d \mid a_{i_{1}} \cdots a_{i_{k}}} \Phi_{d}(t)^{(-1)^{n-k+1}}  \tag{6.361}\\
& =\prod_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} \Phi_{d}(t) \tag{6.362}
\end{align*}
$$

with degree

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} \varphi(d)=\varphi_{\mathbb{B}}(N) \tag{6.363}
\end{equation*}
$$

where $\varphi$ is the Euler totient function, $\varphi_{\mathbb{B}}$ is the generalized Euler totient function with respect to the basis $\mathbb{B}=\left\{a_{0}, \ldots, a_{n}\right\}$ and $N=a_{0} \cdots a_{n}$.

Proof. For a basis of pairwise coprime exponents $\mathbb{B}=\left\{a_{0}, \ldots, a_{n}\right\}$, the characteristic polynomial is especially simple to compute since the least common multiple is the standard product and all greatest common divisors are equal to 1 . By Corollary $4 \cdot 74$,

$$
\begin{equation*}
\Delta_{f}(t)=\prod_{\left.d\right|_{\mathbb{B}} N}\left(t^{d}-1\right)^{\mu_{\mathbb{B}}(N / d)} \tag{6.364}
\end{equation*}
$$

where the conductor of $\mathbb{B}$ is $N=a_{0} \cdots a_{n}$. Observe the identity

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\operatorname{deg} \Delta_{f}(t)=\sum_{\left.d\right|_{\mathbb{B}} N} d \mu_{\mathbb{B}}\left(\frac{N}{d}\right)=\varphi_{\mathbb{B}}(N) \tag{6.365}
\end{equation*}
$$

REMARK 6.8.3. If any of the integers $a_{0}, \ldots, a_{n}$ is equal to 1 , then the numerator and denominator of equation(6.361) cancel, thereby yielding $\Delta_{\left(a_{0}, \ldots, a_{n}\right)}(t)=$ 1. Equivalently, $M D\left(a_{0}, \ldots, a_{n}\right)=\varnothing$ if and only if $a_{i}=1$ for some $0 \leqslant i \leqslant n . \quad \Delta$

REMARK 6.8.4. For coprime integers $p, q \in \mathbb{N}$,

$$
\begin{align*}
\Delta_{T_{p, q}}(t) & =\frac{\prod_{1<d \mid p q} \Phi_{d}(t)}{\prod_{1<d \mid p} \Phi_{d}(t) \prod_{1<d \mid q} \Phi_{d}(t)}  \tag{6.366}\\
& =\prod_{d \in M D(p, q)} \Phi_{d}(t) . \tag{6.367}
\end{align*}
$$

Since each cyclotomic polynomial is irreducible, this product is a complete factorization of the reduced Alexander polynomial of the torus link.

Remark 6.8.5. Consider a Brieskorn-Pham polynomial $f$ with inverse weights $a=7$ and $b=247=13 \cdot 19$. By Proposition 6.82,

$$
\begin{equation*}
\Delta_{T_{7,247}}(t)=\Phi_{91}(t) \Phi_{133}(t) \Phi_{1729}(t) \tag{6.368}
\end{equation*}
$$

as $M D(7,247)=\{91,133,1729\}$, and

$$
\begin{align*}
\mu_{\mathrm{alg}}(f) & =\varphi(91)+\varphi(133)+\varphi(1729)  \tag{6.369}\\
& =72+108+1296  \tag{6.370}\\
& =1476  \tag{6.371}\\
& =(7-1)(13 \cdot 19-1) \tag{6.372}
\end{align*}
$$

The reduced Alexander polynomials of the torus links $T_{p, q}$ for $2 \leqslant p \leqslant q \leqslant 10$ is tabulated in Appendix A (Table A.13).

Remark 6.8.6. For coprime integers $\{a, b, c\} \in \mathbb{N}$,

$$
\begin{align*}
\Delta_{(a, b, c)}(t) & =\frac{\prod_{1<d \mid a} \Phi_{d}(t) \prod_{1<d \mid b} \Phi_{d}(t) \prod_{1<d \mid c} \Phi_{d}(t) \prod_{1<d \mid a b c} \Phi_{d}(t)}{\prod_{1<d \mid a b} \Phi_{d}(t) \prod_{1<d \mid b c} \Phi_{d}(t) \prod_{1<d \mid c a} \Phi_{d}(t)}  \tag{6.373}\\
& =\prod_{d \in M D(a, b, c)} \Phi_{d}(t) \tag{6.374}
\end{align*}
$$

Corollary 6.83. For a non-degenerate, quasi-Brieskorn-Pham singularity $f$ with distinct prime-power, inverse weights $\left\{p_{0}^{r_{0}}, \ldots, p_{n}^{r_{n}}\right\}$, the corresponding characteristic polynomial is the product of cyclotomic polynomials,

$$
\begin{align*}
\Delta_{f}(t) & =\prod_{1 \leqslant i_{0} \leqslant r_{0}} \cdots \prod_{1 \leqslant i_{n} \leqslant r_{n}} \Phi_{p_{0}^{i_{0} \ldots p_{n}^{i_{n}}}}(t)  \tag{6.375}\\
& = \begin{cases}\Phi_{1}(t) & N=1 \\
\prod_{d \in \psi^{\circ}\left(N ;\left\{p_{0}, \ldots, p_{n}\right\}\right)} \Phi_{d}(t) & N=p_{0}^{r_{0}} \cdots p_{n}^{r_{n}}>1\end{cases} \tag{6.376}
\end{align*}
$$

with $\Omega_{\omega(N)}(N)=\prod_{i=0}^{n} r_{i}$ terms in the product and degree $\mu_{\mathrm{alg}}(f)=\varphi^{*}(N)$, where $\varphi^{*}$ is the unitary Euler totient function. If $f$ is Brieskorn-Pham, then $K_{f} \simeq$ $\Sigma\left(p_{0}^{r_{0}}, \ldots, p_{n}^{r_{n}}\right)$ is a topological $(2 n-1)$-sphere if $n>2$.

Proof. We merely mention that $\Phi_{k}(1)$ equals 1 unless the $k$ is a prime power.

REmark 6.8.7. Consider a non-degenerate, quasi-Brieskorn-Pham polynomial with inverse weights $\{3,4,5\}$. By equation (4.182),

$$
\begin{align*}
\Delta_{(3,4,5)}(t)= & \frac{\left(t^{60}-1\right)\left(t^{3}-1\right)\left(t^{4}-1\right)\left(t^{5}-1\right)}{\left(t^{12}-1\right)\left(t^{20}-1\right)\left(t^{15}-1\right)(t-1)}  \tag{6.377}\\
= & 1+t+t^{2}-t^{4}-2 t^{5}-2 t^{6}-t^{7}+t^{9}+t^{10}+t^{11}+t^{12} \\
& +t^{13}+t^{14}+t^{15}-t^{17}-2 t^{18}-2 t^{19}-t^{20}+t^{22}+t^{23}+t^{24} \tag{6.378}
\end{align*}
$$

By Corollary $6.83, N=60$, and $\psi^{\circ}(60 ;\{2,3,5\})=\{1,30,60\}$. Hence, the product of cyclotomic polynomials consists of $\Omega_{3}(60)=2$ terms, namely,
$\Delta_{(3,4,5)}(t)=\Phi_{30}(t) \Phi_{60}(t)$, where $\Phi_{30}(t)=1+t-t^{3}-t^{4}-t^{5}+t^{7}+t^{8}$ and $\Phi_{60}(t)=\Phi_{30}\left(t^{2}\right)$. Moreover, $\varphi^{*}(60)=(4-1)(3-1)(5-1)=24$, which coincides with $\operatorname{deg} \Delta_{(3,4,5)}(t)$. Thus, $\Delta_{(3,4,5)}(1)=\Phi_{30}(1)^{2}=1$, so $\Sigma(3,4,5)$ is a integral homology 3-sphere. Furthermore, $\Delta_{(3,4,5)}(-1)=\Phi_{30}(-1) \Phi_{60}(1)=1$, so the Arf-Kervaire invariant of $\Sigma(3,4,5)$ is 0 and, therefore, $\Sigma(3,4,5)$ is diffeomorphic to $S^{3}$.

REmARK 6.8.8. Suppose $k=2^{r} p_{k_{1}} \cdots p_{k_{n}}$, where $r \in \mathbb{N}$ and $p_{k_{i}}=2^{2^{k_{i}}}+1$ is a Fermat prime. Corollary 6.83 implies

$$
\begin{equation*}
\Delta_{f}(t)=\prod_{1 \leqslant i \leqslant r} \Phi_{2^{i} p_{k_{1}} \cdots p_{k_{n}}}(t) \tag{6.379}
\end{equation*}
$$

and

$$
\begin{align*}
\mu_{\mathrm{alg}}(f) & =\sum_{i=1}^{r} \varphi\left(2^{i} p_{k_{1}} \cdots p_{k_{n}}\right)  \tag{6.380}\\
& =\left(\sum_{i=1}^{r} 2^{i-1}\right) 2^{\sum_{i=1}^{n} 2^{k_{i}}}  \tag{6.381}\\
& =2^{\sum_{i=1}^{n} 2^{k_{i}}}\left(2^{r}-1\right) . \tag{6.382}
\end{align*}
$$

At present, five Fermat primes are known, namely, 3, 5, 17, 257 and 65537.

Corollary 6.84. For coprime integers $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}$,

$$
\begin{equation*}
\bigotimes_{i=0}^{n} \sum_{1<d \mid a_{i}} \Phi_{d}(t)=\prod_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} \Phi_{d}(t) . \tag{6.383}
\end{equation*}
$$

In particular, for distinct primes $\left\{p_{0}, \ldots, p_{n}\right\} \subset \mathbb{P}$,

$$
\begin{equation*}
\bigotimes_{i=0}^{n} \Phi_{p_{i}}(t)=\Phi_{p_{0} \cdots p_{n}}(t) . \tag{6.384}
\end{equation*}
$$

Proposition 6.85. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a multiplicative arithmetic function. For any set of pairwise coprime integers $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}$, the following identity holds,

$$
\begin{equation*}
\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} f(d)=\prod_{i=0}^{n}\left((f * 1)\left(a_{i}\right)-1\right) . \tag{6.385}
\end{equation*}
$$

or, equivalently,

$$
\begin{equation*}
\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)}(f * \mu)(d)=\prod_{i=0}^{n}\left(f\left(a_{i}\right)-1\right) \tag{6.386}
\end{equation*}
$$

Proof. Expand the right side, use the multiplicativity of $f * 1$ and the Principle of Inclusion-Exclusion. To prove the second identity from the first, observe that $f * \mu$ is a multiplicative arithmetic function and apply the identity

$$
\begin{equation*}
(f * \mu) * 1=f *(\mu * 1)=f * \varepsilon=f . \tag{6.387}
\end{equation*}
$$

Corollary 6.86. For coprime integers $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}$,

$$
\begin{align*}
\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} \varphi(d) & =\prod_{i=0}^{n}\left(a_{i}-1\right)  \tag{6.388}\\
\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} 1 & =\prod_{i=0}^{n}\left(\tau\left(a_{i}\right)-1\right) . \tag{6.389}
\end{align*}
$$

Proof 1. By Proposition 6.82,

$$
\begin{equation*}
\mu_{\mathrm{alg}}(f)=\operatorname{deg} \Delta_{f}(t)=\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} \varphi(d) \tag{6.390}
\end{equation*}
$$

as $\operatorname{deg} \Phi_{n}(t)=\varphi(n)$ for $n \in \mathbb{N}$.
To compute the cardinality of $M D\left(a_{0}, \ldots, a_{n}\right)$, observe the recurrence relation $M D\left(a_{0}, \ldots, a_{n}\right)=M D\left(a_{0}, \ldots, a_{n-1}\right) \cdot\left(D\left(a_{n}\right) \backslash\{1\}\right)$, where the product of sets is the set of the implied product of their corresponding elements. Since there $\tau(a)-1$ elements in $D(a) \backslash\{1\}$, where $\tau(a)=\prod_{p \mid a} \operatorname{ord}_{a}(p)$ is the number of divisors function, it follows that

$$
\begin{equation*}
\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} 1=\left|M D\left(a_{0}, \ldots, a_{n}\right)\right|=\prod_{i=0}^{n}\left(\tau\left(a_{i}\right)-1\right) \tag{6.391}
\end{equation*}
$$

Since the weights of $f$ are $\left\{\frac{1}{a_{0}}, \ldots, \frac{1}{a_{n}}\right\}$, one has $\mu_{\mathrm{alg}}(f)=\prod_{i=0}^{n}\left(a_{i}-1\right)$.

Proof 2. The proof of the second identity is identical to that given in the first proof. As for the first identity, one has

$$
\begin{align*}
\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} \varphi(d) & =\sum_{k=0}^{n}(-1)^{n+1-k} \sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} \sum_{d \mid D\left(a_{i_{1}} \cdots a_{i_{k}}\right)} \varphi(d)  \tag{6.392}\\
& =\sum_{k=0}^{n}(-1)^{n+1-k} \sum_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n} a_{i_{1}} \cdots a_{i_{k}}  \tag{6.393}\\
& =\sum_{k=0}^{n}(-1)^{n+1-k} e_{k}\left(a_{0}, \ldots, a_{n}\right)  \tag{6.394}\\
& =\prod_{k=0}^{n}\left(a_{k}-1\right) \tag{6.395}
\end{align*}
$$

since $\varphi * 1=\mathrm{id}$.

Remark 6.8.9. Taking $f=1$ and $f=\mathrm{id} * \mu=\varphi$ in Proposition 6.85 yields the identities of Corollary 6.86, since $\tau=1 * 1$ and $\varepsilon=\mu * 1$, respectively. Similarly, consider $f=\mu$, the Möbius function. Since $\varepsilon=\mu * 1$, then

$$
\begin{align*}
\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} \mu(d) & =\prod_{i=0}^{n}\left(\varepsilon\left(a_{i}\right)-1\right)  \tag{6.396}\\
& =\left\{\begin{array}{cl}
(-1)^{n+1} & \left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}_{>1} \\
0 & a_{i}=1 \text { for some } 0 \leqslant i \leqslant n
\end{array}\right. \tag{6.397}
\end{align*}
$$

Finally, consider $f=\mathrm{id}_{\ell}$, the identity power function. Since $\sigma_{\ell}=\mathrm{id}_{\ell} * 1$, then

$$
\begin{equation*}
\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} d^{\ell}=\prod_{i=0}^{n}\left(\sigma_{\ell}\left(a_{i}\right)-1\right) \quad \ell \in \mathbb{N} \tag{6.398}
\end{equation*}
$$

Corollary 6.87. Let $f$ be a quasi-Brieskorn-Pham singularity with pairwise coprime inverse weights $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}$. The number of irreducible polynomials in the factorization of $\Delta_{f}(t)$ depends only on the signature of the prime factorization of the conductor $N=a_{0} \cdots a_{n}$.

Proof. The characteristic polynomial $\Delta_{f}(t)$ is a product of cyclotomic polynomials, which are irreducible. By Corollary 6.86, the number of said irreducible polynomials is the product $\prod_{i=0}^{n}\left(\tau\left(a_{i}\right)-1\right)$, which depends only on the signature of $N$.

Corollary 6.88. For any set of positive integers $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}_{>1}$, there is a set of non-negative integers $\left\{c_{d}\right\}$, not all zero, such that

$$
\begin{equation*}
\sum_{d \mid \mathrm{LCM}\left(a_{0}, \cdots, a_{n}\right)} c_{d} \varphi(d)=\prod_{i=0}^{n}\left(a_{i}-1\right) \tag{6.399}
\end{equation*}
$$

In particular, if $\left\{a_{0}, \ldots, a_{n}\right\}$ is a set of pairwise coprime integers, then $c_{d}=1$ if and only if $d \in M D\left(a_{0}, \ldots, a_{n}\right)$ and zero otherwise.

Remark 6.8.10. Corollary 6.88 generalizes Corollary 6.79.

The question then arises whether or not other arithmetic functions can be represented in a similar manner, that is, as the algebraic index of a singularity.

Corollary 6.89. Given a multiplicative arithmetic function $f: \mathbb{N} \rightarrow \mathbb{N}$ and a positive integer $k=p_{0}^{r_{0}} \ldots p_{n}^{r_{n}} \in \mathbb{N}$, where $p_{i}$ is the $i^{\text {th }}$-prime factor of $k$, let
$f_{k}:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a non-degenerate, quasi-Brieskorn-Pham singularity with inverse weights $\left\{f\left(p_{i}^{r_{i}}\right)+1\right\}_{i=0}^{n}$. Then the algebraic index of $f_{k}$ is precisely the value $f(k)$.

Proof. The formula for the algebraic index of a quasi-Brieskorn-Pham singularity in terms of the weights implies

$$
\begin{equation*}
\mu_{\mathrm{alg}}\left(f_{k}\right)=\prod_{i=0}^{n}\left(f\left(p_{i}^{r_{i}}\right)+1-1\right)=f(k) \tag{6.400}
\end{equation*}
$$

by the multiplicativity of $f$.

Remark 6.8.11. More generally, write $k=d_{0} \cdots d_{m}$, where the divisors $d_{0}, \ldots, d_{m}$ are pairwise coprime, and let $f_{k}:\left(\mathbb{C}^{m+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a quasi-Brieskorn-Pham singularity with inverse weights $\left\{f\left(d_{i}\right)+1\right\}_{i=0}^{m}$. Then $\mu_{\mathrm{alg}}\left(f_{k}\right)=f(k)$.

Remark 6.8.12. Let $f=\varphi * \mathrm{id}=(\mathrm{id} * \mathrm{id}) * \mu=(\mathrm{id} \cdot \tau) * \mu$. Then

$$
\begin{equation*}
\mu_{\mathrm{alg}}\left(f_{k}\right)=(\varphi * \mathrm{id})(k)=\sum_{l=1}^{k} \operatorname{GCD}(l, k) . \tag{6.401}
\end{equation*}
$$

Associate to each multiplicative arithmetic function $f: \mathbb{N} \rightarrow \mathbb{N}$ a family of rational functions $\left\{\Phi_{n}^{(f)}(t)\right\}_{n \in \mathbb{N}}$ such that

$$
\begin{equation*}
t^{f(n)}-1=\prod_{d \mid n} \Phi_{d}^{(f)}(t) \tag{6.402}
\end{equation*}
$$

By Möbius inversion,

$$
\begin{equation*}
\Phi_{n}^{(f)}(t)=\prod_{d \mid n}\left(t^{f(d)}-1\right)^{\mu(n / d)} \in \mathbb{Z}(t) \tag{6.403}
\end{equation*}
$$

Recall that the degree of a rational function as the difference of the degrees of the numerator and denominator polynomials. Thus,

$$
\begin{equation*}
\operatorname{deg} \Phi_{n}^{(f)}(t)=\sum_{d \mid n} f(d) \mu\left(\frac{n}{d}\right)=(f * \mu)(n) \tag{6.404}
\end{equation*}
$$

Taking $f=\mathrm{id}$, the identity function, yields $\Phi_{n}^{\mathrm{id}}(t)=\Phi_{n}(t)$, the $n^{\text {th }}$-cyclotomic polynomial. More generally, since

$$
\begin{align*}
\Phi_{n}^{(f)}(t) & \prod_{d \mid n} \prod_{k=1}^{f(d)}\left(t-\zeta_{f(d)}^{k}\right)^{\mu(n / d)}  \tag{6.405}\\
& =\prod_{k=1}^{f(d)} \prod_{d \mid n}\left(t-\zeta_{f(d)}^{k}\right)^{\mu(n / d)} \tag{6.406}
\end{align*}
$$

if there is an $n \in \mathbb{N}$ such that $f(d)=d$ for each divisor of $n$, then $\Phi_{n}^{(f)}(t)=$ $\Phi_{n}(t)$.

Proposition 6.90. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a multiplicative arithmetic function. Given pairwise coprime integers $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}$, the rational function

$$
\begin{align*}
\Delta^{(f)}(t) & =\prod_{k_{0}=1}^{f\left(a_{0}\right)-1} \cdots \prod_{k_{n}=1}^{f\left(a_{n}\right)-1}\left(t-\zeta_{f\left(a_{0}\right)}^{k_{0}} \cdots \zeta_{f\left(a_{n}\right)}^{k_{n}}\right)  \tag{6.407}\\
& =\Phi_{1}^{(f)}(t)^{(-1)^{n+1}} \prod_{k=1}^{n+1} \prod_{0 \leqslant i_{1}<\cdots<i_{k} \leqslant n| | a_{i_{1}} \cdots a_{i_{k}}} \Phi_{d}^{(f)}(t)^{(-1)^{n-k+1}}  \tag{6.408}\\
& =\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)} \Phi_{d}^{(f)}(t) \tag{6.409}
\end{align*}
$$

with degree

$$
\begin{equation*}
\operatorname{deg} \Delta^{(f)}(t)=\prod_{i=0}^{n}\left(f\left(a_{i}\right)-1\right)=\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)}(f * \mu)(d) \tag{6.410}
\end{equation*}
$$

Corollary 6.91. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a multiplicative arithmetic function. If $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}$ and $\left\{f\left(a_{0}\right), \ldots, f\left(a_{n}\right)\right\}$ are each sets of pairwise coprime integers, then

$$
\begin{equation*}
\sum_{d \in M D\left(a_{0}, \ldots, a_{n}\right)}(f * \mu)(d)=\sum_{d \in M D\left(f\left(a_{0}\right), \ldots, f\left(a_{n}\right)\right)} \varphi(d) . \tag{6.411}
\end{equation*}
$$

Proof. For a quasi-Brieskorn-Pham singularity $g$ with pairwise coprime inverse weights $\left\{f\left(a_{0}\right), \ldots, f\left(a_{n}\right)\right\} \subset \mathbb{N}$, the corresponding characteristic polynomial is the product

$$
\begin{equation*}
\Delta_{g}(t)=\prod_{d \in M D\left(f\left(a_{0}\right), \ldots, f\left(a_{n}\right)\right)} \Phi_{d}(t) \tag{6.412}
\end{equation*}
$$

with degree

$$
\begin{equation*}
\mu_{\mathrm{alg}}(g)=\sum_{d \in M D\left(f\left(a_{0}\right), \ldots, f\left(a_{n}\right)\right)} \varphi(d) \tag{6.413}
\end{equation*}
$$

Proposition 6.90 implies the claim.

### 6.9. Zeta Function of an Algebraic Link

Corollary 6.92. Given odd, coprime, positive integers $p$ and $q$, the Lefschetz zeta function $\zeta_{T_{p, q}}$ of the torus knot $T_{p, q}$ is a ratio of cyclotomic polynomials,

$$
\begin{align*}
\zeta_{T_{p, q}}(t) & =-\frac{\prod_{d \mid p q} \Phi_{d}(t)}{\prod_{d \mid p} \Phi_{d}(t) \prod_{d \mid q} \Phi_{d}(t)}  \tag{6.414}\\
& =-\Phi_{1}(t)^{-1} \prod_{d \in M(p, q)} \Phi_{d}(t) \tag{6.415}
\end{align*}
$$

If Proposition 4.24 extends to torus links without modification, then the following conjecture is true.

Proposition 6.93. Given positive integers $p$ and $q$, the Lefschetz zeta function $\zeta_{T_{p, q}}$ of the torus link $T_{p, q}$ is a ratio of cyclotomic polynomials,

$$
\begin{align*}
\zeta_{T_{p, q}}(t) & =(-1)^{\mu+1} \frac{\prod_{d \mid p q} \Phi_{d}(t)}{\prod_{d \mid p} \Phi_{d}(t) \prod_{d \mid q} \Phi_{d}(t)}  \tag{6.416}\\
& =(-1)^{p q-p-q} \Phi_{1}(t)^{-1} \prod_{d \in M(p, q)} \Phi_{d}(t) \tag{6.417}
\end{align*}
$$

where $\mu=(p-1)(q-1)$.
One wonders if a similar result holds for any link in $\mathbb{S}^{3}$.

Remark 6.9.1. The zeta function of the $r$-Hopf link is straightforward to compute,

$$
\begin{align*}
\zeta_{T_{r, r}}(t) & =(-1)^{(r-1)^{2}+1} \prod_{d \mid r} \Phi_{d}(t)^{-[2]_{1-r}}  \tag{6.418}\\
& =\left(1-t^{r}\right)^{r-2} \tag{6.419}
\end{align*}
$$

### 6.10. Primes and Knots

Remark 6.10.1. We mention only briefly a rather deep and beautiful connection between knots and primes [298, 324]. The fundamental observation is that the analogue of the embedding $K: S^{1} \hookrightarrow \mathbb{R}^{3}$, which represents a knot in space, in an arithmetic setting is $P: \operatorname{Spec} \mathbb{F}_{p} \hookrightarrow \operatorname{Spec} \mathbb{Z}$, where $p$ is a prime and $\mathbb{F}_{p}$ is a Galois field. The homotopy groups of $S^{1}$, namely,

$$
\pi_{k}\left(S^{1}\right) \cong\left\{\begin{array}{cc}
\{0\} & k \geqslant 2  \tag{6.420}\\
\mathbb{Z} & k=1
\end{array}\right.
$$

are dual to the étale homotopy groups

$$
\pi_{k}^{\mathrm{et}}\left(\operatorname{Spec} \mathbb{F}_{p}\right)\left\{\begin{array}{cc}
\{0\} & k \geqslant 2  \tag{6.421}\\
\hat{\mathbb{Z}} & k=1
\end{array}\right.
$$

where $\hat{Z}=\lim \mathbb{Z}_{n}$ is the profinite completion of $\mathbb{Z}$. The knot group $\pi(K)=$ $\pi_{1}\left(S^{3} \backslash K\right)$ is dual to the etale (fundamental) group $\pi_{1}^{\mathrm{et}}(\mathbb{Z} \backslash\{p\})$. Similarly, if
$N_{K, r}$ denotes the Fox completion of $M_{K, r}$, the first homology group $H_{1}\left(N_{K, r}\right) \cong$ $H_{1}\left(M_{K, \infty}\right) /\left(t^{r}-1\right) H_{1}\left(M_{K, \infty}\right)$ is analogous to a class group, and the Alexander polynomial $\Delta_{K}$ is analogous to an Iwasawa polynomial, which can used to compute the order of said homology group, q.v., Proposition 4.21.

### 6.11. Algebraic Roots

Let $\mathbb{Q}$ and $\overline{\mathbb{Q}}$ denote the field of rationals and complex algebraics, that is, the set of roots of all $\mathbb{Z}$-polynomials with finite degree, respectively.

The minimal polynomial $P(\vartheta ; x)$ of an algebraic $\vartheta$ is the unique, irreducible Qpolynomial of minimal-degree with said algebraic as a root. An algebraic integer is an algebraic with a monic minimal polynomial. An algebraic unit is an algebraic integer whose inverse is also an algebraic integer. The degree deg $\vartheta$, absolute norm $N(\vartheta)$ and absolute trace $\operatorname{Tr}(\vartheta)$ is the degree of the minimal polynomial of $\vartheta$, and the product and sum of the algebraic conjugates of $\vartheta$, respectively.

Let $K=\mathbb{Q}(\vartheta)$ be the field extension of $\mathbb{Q}$ obtained by adjoining an algebraic $\vartheta$. Denote the ring of (algebraic) integers by $\mathcal{O}_{K}$, which is a free $\mathbb{Z}$-module with an integral basis $B_{K}=\left\{b_{1}, \ldots, b_{d}\right\}$, where $d=\operatorname{deg} \vartheta$. That is, any $\alpha \in \mathcal{O}_{K}$ can be written as a $\mathbb{Z}$-linear combination of basis elements.

Definition 6.94. The algebraic root $\vartheta(f)$ of a weighted homogeneous singularity $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C})$ with non-zero weights $\left\{\omega_{i}\right\}$ and spectrum $\operatorname{Sp}(f)=\left\{\gamma_{j}\right\}$ is the algebraic number $\operatorname{Sp}\left(f ; e^{\pi i}\right)=e^{\pi i(n+1) / 2} \prod_{i=0}^{n} \cot \left(\frac{\pi \omega_{i}}{2}\right)$.

When the context is clear, we write $\vartheta$ instead of $\vartheta(f)$.

REMARK 6.11.1. If $\omega_{0}, \ldots, \omega_{n} \in \mathbb{Q} \cap(0,1)$, then $e^{-\pi i(n+1) / 2} \vartheta(f)$ is a positive, real algebraic. For example, $\vartheta(z)=0, \vartheta\left(z^{4}\right)=i(1+\sqrt{2})$ and $\vartheta\left(z^{6}\right)=i(2+$ $\sqrt{3})$.

Proposition 6.95. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a neighborhood of the origin. Assume that the complex analytic map $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is non-degenerate. Then the algebraic root of the Sebastiani-Thom sum $f_{1} \boxplus \cdots \boxplus f_{s}$ is the product of the algebraic roots of each summand,

$$
\begin{equation*}
\vartheta\left(f_{1} \boxplus \cdots \boxplus f_{s}\right)=\prod_{i=1}^{s} \vartheta\left(f_{i}\right) . \tag{6.422}
\end{equation*}
$$

Corollary 6.96. Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a weighted homogeneous polynomial with algebraic root $\vartheta(f)$. Then, for $N \in \mathbb{N}$,

$$
\begin{equation*}
\vartheta\left(\Sigma^{N} f\right)=i^{N} \vartheta(f) \tag{6.423}
\end{equation*}
$$

In particular, $\vartheta\left(\Sigma^{N} f\right)=\vartheta(f)$ if and only if $4 \mid N$.
Proof. Observe the identities $\cot \left(\frac{\pi}{4}\right)=1$ and $i^{n+4}=i^{n}$ for $n \in \mathbb{Z}$.

Definition 6.97. The minimal polynomial, absolute norm and absolute trace of a weighted homogeneous singularity is the minimal polynomial, absolute norm and absolute trace of its algebraic root.

The signature of $\mathbb{Q}(\vartheta)$ is the pair $(p, q)$, where $p$ is the number of real roots and $q$ is the number of pairs of conjugate roots of the minimal polynomial of $\vartheta$.

The discriminant $D(\vartheta)$ of a number field $\mathbb{Q}(\vartheta)$ is the determinant of the matrix $\left(\operatorname{Tr}_{\mathbb{Q}(\vartheta)} b_{i} b_{j}\right)$, where $\left\{b_{1}, \ldots, b_{d}\right\}$ is an integral basis of $\mathbb{Q}(\vartheta)$. The discriminant does not depend on the integral basis.

Definition 6.98. The signature, discriminant and class number of a weighted homogeneous singularity is the signature, discriminant and class number of the number field of its algebraic root.

Proposition 6.99. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a neighborhood of the origin. Assume that the complex analytic map $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is non-degenerate. Then the minimal polynomial of the Sebastiani-Thom sum $f_{1} \boxplus \cdots \boxplus f_{s}$ divides the tensor product of the minimal polynomials of the summands. That is, there is an integer $m \in \mathbb{N}$ and a (possibly trivial) polynomial $F(x) \in \mathbb{Z}[x]$ such that

$$
\begin{equation*}
\bigotimes_{i=1}^{s} P\left(\vartheta\left(f_{i}\right) ; x\right)=F(x) P\left(\vartheta\left(f_{1} \boxplus \cdots \boxplus f_{s}\right) ; x\right)^{m} \tag{6.424}
\end{equation*}
$$

In particular, if $4 \mid N$, then $P\left(\vartheta\left(\Sigma^{N} f\right) ; x\right)=P(\vartheta(f) ; x)$.

Proof. The claim follows immediately from Proposition 6.95, the fact that the product of algebraic roots is a root of both the minimal polynomial of the Sebastiani-Thom summation and of the tensor product of the minimal polynomials of the summands, and the fact that minimal polynomials are irreducible.

Remark 6.11.2. For example, $\vartheta\left(z^{2}\right)=i, \vartheta\left(z^{3}\right)=i \sqrt{3}$ and $\vartheta\left(z^{5}\right)=$ $i \sqrt{5+2 \sqrt{5}}$ with minimal polynomials $P\left(\vartheta\left(z^{2}\right) ; x\right)=x^{2}+1, P\left(\vartheta\left(z^{3}\right) ; x\right)=x^{2}+3$
and $P\left(\vartheta\left(z^{5}\right) ; x\right)=x^{4}+10 x^{2}+5$, respectively. Moreover,

$$
\begin{align*}
\vartheta\left(x^{2}+y^{3}+z^{5}\right) & =\vartheta\left(z^{2}\right) \vartheta\left(z^{3}\right) \vartheta\left(z^{5}\right)  \tag{6.425}\\
& =-i \sqrt{3} \sqrt{5+2 \sqrt{5}} \tag{6.426}
\end{align*}
$$

However, $P\left(\vartheta\left(z^{2}\right) \vartheta\left(z^{3}\right) \vartheta\left(z^{5}\right) ; x\right)=x^{4}+30 x^{2}+45$, so the minimal polynomial is not multiplicative over Sebastiani-Thom summations. The polynomial tensor product of the minimal polynomial of the summands is the degree-16 polynomial

$$
\begin{align*}
P\left(\vartheta\left(z^{2}\right) ; x\right) \otimes P\left(\vartheta\left(z^{3}\right) ; x\right) \otimes P\left(\vartheta\left(z^{5}\right) ; x\right) & =\prod_{i, j, k}\left(x-\alpha_{i} \beta_{j} \gamma_{k}\right)  \tag{6.427}\\
& =\left(x^{4}+30 x^{2}+45\right)^{4} \tag{6.428}
\end{align*}
$$

where $\left\{\alpha_{i}\right\},\left\{\beta_{j}\right\}$ and $\left\{\gamma_{k}\right\}$ are the roots of $P\left(\vartheta\left(z^{2}\right) ; x\right), P\left(\vartheta\left(z^{3}\right) ; x\right)$ and $P\left(\vartheta\left(z^{5}\right) ; x\right)$, respectively. Similarly, $P\left(\vartheta\left(z^{4}\right) ; x\right)=1+6 x^{2}+x^{4}$ and $P\left(\vartheta\left(x^{2}+\right.\right.$ $\left.\left.y^{3}+z^{4}\right) ; x\right)=x^{4}+18 x^{2}+9$, while $P\left(\vartheta\left(z^{2}\right) ; x\right) \otimes P\left(\vartheta\left(z^{3}\right) ; x\right) \otimes P\left(\vartheta\left(z^{4}\right) ; x\right)=$ $\left(x^{4}+18 x^{2}+9\right)^{4}$.

Define the odd von Mangoldt function,

$$
\Lambda_{\text {odd }}(n)=\left\{\begin{array}{cc}
\log p & \text { if } n \text { is an odd prime power }  \tag{6.429}\\
0 & \text { otherwise }
\end{array}\right.
$$

Recall the Euler totient function, $\varphi(n)=\prod_{p \mid n}\left(1-\frac{1}{p}\right)$, which satisfies the identities

$$
\begin{equation*}
\varphi(n m)=\frac{\varphi(n) \varphi(m) \operatorname{GCD}(n, m)}{\varphi(\operatorname{GCD}(n, m))} \quad n, m \in \mathbb{N} . \tag{6.430}
\end{equation*}
$$

Conjecture 6.100. For $n \in \mathbb{N}$,

$$
\begin{align*}
e^{\Lambda_{\mathrm{odd}}(n)} & =\delta_{n, 1}+\prod_{\substack{k=1 \\
\operatorname{GCD}(k, 2 n)=1}}^{2 n}\left(i \cot \left(\frac{\pi k}{2 n}\right)\right)  \tag{6.431}\\
& =\delta_{n, 1}+i^{\varphi(2 n)} \prod_{\substack{k=1 \\
\operatorname{GCD}(k, 2 n)=1}}^{2 n} \cot \left(\frac{\pi k}{2 n}\right)  \tag{6.432}\\
& =\frac{\operatorname{LCM}(1,3,5, \ldots, n)}{\operatorname{LCM}(1,3,5, \ldots, n-1)} \tag{6.433}
\end{align*}
$$

Proof. The first identity is left as an exercise for the reader. The following identity is straightforward to prove:

$$
\begin{align*}
e^{\Lambda_{\mathrm{odd}}(n)} & =\left\{\begin{array}{cl}
\frac{\operatorname{LCM}(1,2,3, \ldots, n)}{\operatorname{LCM}(1,2,3, \ldots, n-1)} & n \text { odd } \\
1 & \text { otherwise }
\end{array}\right.  \tag{6.434}\\
& =\frac{\operatorname{LCM}(1,3,5, \ldots, n)}{\operatorname{LCM}(1,3,5, \ldots, n-1)} \tag{6.435}
\end{align*}
$$

Proposition 6.101. Let $\vartheta=i \cot \left(\frac{\pi}{2 n}\right)$, where $n \in \mathbb{N}$. The following statements are true:

1. The real $\vartheta$ is an algebraic integer;
2. The degree of $\vartheta$ is $\varphi(2 n)$;
3. The absolute trace of $\vartheta$ is zero;
4. The absolute norm of $\vartheta$ is equal to $p$ if and only if $n$ is a power of an odd prime $p$ and 1 otherwise, that is, $N(\vartheta)=e^{\Lambda_{\text {odd }}(n)}$;
5. The algebraic $\vartheta$ is an algebraic unit if and only if $n$ is not an odd prime power;
6. The signature of $\mathbb{Q}(\vartheta)$ is trivial or purely complex, namely, $\left\{0, \frac{1}{2}(1-\right.$ $\left.\left.\delta_{n, 1}\right) \varphi(2 n)\right\} ;$ and,
7. The discriminant of $\mathbf{Q}(\vartheta)$ is the product

$$
\begin{align*}
D(\vartheta) & =\frac{(-1)^{\left(1-\delta_{n, 1}\right) \varphi(2 n) / 2} n^{\varphi(2 n)}}{\prod_{p \mid n} p^{\varphi(2 n) /(p-1)}} \cdot \begin{cases}1 & n \text { is odd } \\
2^{\varphi(2 n)} & n \text { is even }\end{cases}  \tag{6.436}\\
& =i^{\varphi(2 n)-\delta_{n, 1}}\left(\frac{n}{\prod_{2<p \mid n} \sqrt[p-1]{p}}\right)^{\varphi(2 n)} . \tag{6.437}
\end{align*}
$$

Proof. For $r \in \mathbb{Q}$, observe that $2 \cos (\pi r)=e^{\pi i r}+e^{-\pi i r}$ and $2 i \sin (\pi r)=$ $e^{\pi i r}-e^{-\pi i r}$ are algebraic integers. Since the algebraics are closed under multiplication, it follows that for $r \in \mathbb{Q} \backslash \mathbb{Z}$, the real $i \cot (\pi r)$ is algebraic. Statement 6. implies statement 3. That is, since $\vartheta$ is purely imaginary, except for $n=1$ which corresponds to the minimal polynomial $x$, the conjugates of $\vartheta$ come in complex conjugate pairs. Thus, its absolute trace, the sum of roots of the minimal polynomial $P(\vartheta ; z)$ is zero. Statement 4 . implies statement 5 ., since units are characterized by their absolute norms, namely, $N(\vartheta)= \pm 1$.

To prove statement 2., write $r=\frac{2 k}{n}$ for some $k, n \in \mathbb{N}$ such that $\operatorname{GCD}(k, n)=$ 1. Let $\zeta_{n}=e^{2 \pi i / n}$ and $\eta_{n, k}=\left(\zeta_{n}^{k}+\zeta_{n}^{-k}\right)\left(\zeta_{n}^{k}-\zeta_{n}^{k}\right)^{-1}$. Since $\cot \left(\frac{2 \pi k}{n}\right)=i \eta_{n, k}$, the
corresponding tower of field extensions from $\mathbb{Q}$ is as follows:


Since $\left[\mathbb{Q}\left(\zeta_{n}\right): \mathbb{Q}\right]=\varphi(n)$ and $\left[\mathbb{Q}\left(\zeta_{n}\right): \mathbb{Q}\left(\eta_{n, k}\right)\right]=2^{\delta_{4 \mid n}}[76]$, then it follows that

$$
\left[\mathbb{Q}\left(\cot \left(\frac{2 \pi k}{n}\right)\right): \mathbb{Q}\right]=\left\{\begin{array}{cc}
\varphi(n) & 4 \nmid n  \tag{6.439}\\
\frac{1}{4} \varphi(n) & 8 \mid n \\
\frac{1}{2} \varphi(n) & n \equiv 4 \bmod 8
\end{array}\right.
$$

Thus, $\operatorname{deg} \vartheta=2\left[\mathbb{Q}\left(\cot \left(\frac{\pi}{2 n}\right)\right): \mathbb{Q}\right]=\varphi(2 n)$. The other statements are left as exercises for the reader.

Corollary 6.102. The discriminant $D(\vartheta)$ is divisible by $2^{\varphi(2 n) \operatorname{ord}_{2}(n)}$ for even n. In particular, the discriminant $D(\vartheta)$ is divisible by 16 if $n$ is even and not equal to 2 and by 256 if a is even and not equal to 2 or 6 .

Conjecture 6.103. The minimal polynomial of the algebraic $\vartheta=i \cot \left(\frac{\pi}{2 n}\right)$, where $n \in \mathbb{N}_{>1}$, is the product

$$
\begin{equation*}
P(\vartheta ; x)=\prod_{\substack{k=1 \\ \operatorname{GCD}(k, 2 n)=1}}^{2 n}\left(x-i \cot \left(\frac{\pi k}{2 n}\right)\right) \tag{6.440}
\end{equation*}
$$

Hence, the degree of $\vartheta$ is the summation,

$$
\begin{equation*}
\sum_{\substack{k=1 \\ \operatorname{GCD}(k, 2 n)=1}}^{2 n} 1=\varphi(2 n) \tag{6.441}
\end{equation*}
$$

If $n=1$, then $P(\vartheta ; x)=x$.

Remark 6.11.3. It is most likely true that for odd coprime $n, m \in \mathbb{N}$, the minimal polynomial of the algebraic $\vartheta=i \cot \left(\frac{\pi m}{2 n}\right)$ is the product

$$
\begin{equation*}
P(\vartheta ; x)=\prod_{\substack{k=1 \\ \operatorname{GCD}(k, 2 n)=1}}^{2 n}\left(x-i \cot \left(\frac{\pi k m}{2 n}\right)\right) \tag{6.442}
\end{equation*}
$$

Based on computational evidence, we conjecture the following explicit forms of the aforementioned minimal polynomials.

Conjecture 6.104. The minimal polynomial of the algebraic $\vartheta=i \cot \left(\frac{\pi}{2 n}\right)$, where $n$ is an odd prime or a power of two, is

$$
\begin{equation*}
P(\vartheta ; x)=\sum_{k=0}^{\varphi(2 n)}\binom{n}{2 k} x^{\varphi(2 n)-2 k} \tag{6.443}
\end{equation*}
$$

If $n>1$ is not an odd prime power, then $P(\vartheta)$ is a reflexive monic polynomial of degree $\varphi(2 n)$ whose value at $\pm 1$ is a power of 2 .

Conjecture 6.105. The minimal polynomial of the algebraic $\vartheta=$ $-\cot \left(\frac{\pi}{2 n}\right) \cot \left(\frac{\pi}{2 m}\right)$, where $n, m \in \mathbb{N}_{>1}$, is the product

$$
\begin{equation*}
P(\vartheta ; x)=\prod_{\substack{k=1 \\ \operatorname{GCD}(k, 2 n m)=1}}^{\operatorname{LCm}(n, m)}\left(x+\cot \left(\frac{\pi k}{2 n}\right) \cot \left(\frac{\pi k}{2 m}\right)\right) \tag{6.444}
\end{equation*}
$$

Hence, the degree of $\vartheta$ is the summation,

$$
\begin{equation*}
\sum_{\substack{k=1 \\ \operatorname{GCD}(k, 2 n m)=1}}^{\operatorname{LCm}(n, m)} 1=\frac{\varphi(2 n m)}{2 \operatorname{GCD}(n, m)} . \tag{6.445}
\end{equation*}
$$

In particular, if $m=2$, then $\operatorname{deg} \vartheta=\varphi(n)$. If either $n$ or $m$ equals 1 , then $P(\vartheta ; x)=$ $x$.

Remark 6.11.4. The final claim of Conjecture 6.105, in case $m=2$, follows from the identity $\varphi(\operatorname{GCD}(n, 4))=\frac{\operatorname{GCD}(n, 4)}{\operatorname{GCD}(n, 2)}$, which can be proved as follows. For
$n, k \in \mathbb{N}$,

$$
\begin{equation*}
\frac{1}{\operatorname{GCD}(n, 2)}=\prod_{p \mid \operatorname{GCD}\left(n, 2^{k}\right)}\left(1-\frac{1}{p}\right) \tag{6.446}
\end{equation*}
$$

since the product is either equal to 1 or $\frac{1}{2}$ depending on whether $n$ is odd or even, respectively. Thus,

$$
\begin{align*}
\varphi\left(\operatorname{GCD}\left(n, 2^{k}\right)\right) & =\operatorname{GCD}\left(n, 2^{k}\right) \prod_{p \mid \operatorname{GCD}\left(n, 2^{k}\right)}\left(1-\frac{1}{p}\right)  \tag{6.447}\\
& =\frac{\operatorname{GCD}\left(n, 2^{k}\right)}{\operatorname{GCD}(n, 2)} \tag{6.448}
\end{align*}
$$

and, if $m=2^{k-1}$,

$$
\begin{align*}
\frac{\varphi(2 n m)}{2 \operatorname{GCD}(n, m)} & =\frac{\varphi(n) \varphi\left(2^{k-1}\right) \operatorname{GCD}(n, 2)}{2 \operatorname{GCD}\left(n, 2^{k-1}\right)}  \tag{6.449}\\
& =\varphi(n) 2^{k-2} \frac{\operatorname{GCD}(n, 2)}{\operatorname{GCD}\left(n, 2^{k-1}\right)} \tag{6.450}
\end{align*}
$$

Problem 6.11.1. Compute the minimal polynomial and degree of the algebraic $e^{\pi i(n+1) / 2} \prod_{i=0}^{n} \cot \left(\frac{\pi}{2 a_{i}}\right)$, where $\left\{a_{0}, \ldots, a_{n}\right\} \subset \mathbb{N}_{>1}$.

Weighted homogeneous singularities may be partitioned into equivalence classes by the number of complex variables and the degree of the corresponding algebraic root. For example, consider $f=z^{a}$ over $\mathbb{C}$ with $a \in \mathbb{N}$. Since weight of $f$ is $\frac{1}{a}$, then $\vartheta\left(z^{a}\right)=i \cot \left(\frac{\pi}{2 a}\right)$. Thus, by Proposition 6.101, one may partition such singularities into equivalence classes based on the value $\varphi(2 a)$,
namely,

$$
\begin{array}{r}
\{1\},\{2,3\},\{4,5,6\},\{7,9\},\{8,10,12,15\},\{11\},\{13,14,18,21\},\{16,17,20,24,30\}, \\
\{19,27\},\{22,25,33\},\{23\},\{26,28,35,36,39,42,45\},\{29\},\{31\},\{32,34,40,48,51,60\}
\end{array}
$$

and so on, where the red text denotes those exponents corresponding to an algebraic unit. Note that there are equivalence classes consisting of either only algebraic units or algebraic non-units or a mixture of the two. The following integers represent singleton equivalence classes for $1 \leqslant a \leqslant 1000$ :

$$
\begin{aligned}
& 1,11,23,29,31,47,53,59,67,71,79,81,83,103,107,121,127, \\
& 131,137,139,149,151,167,173,179,191,197,199,211,223,227, \\
& 229,239,251,263,269,271,283,293,307,311,317,331,343,347, \\
& 359,361,367,373,379,383,389,419,431,439,443,463,467,479, \\
& 491,499,503,509,523,529,547,557,563,569,571,587,599,607, \\
& 619,631,643,647,649,653,659,677,683,691,709,719,727,739 \\
& 743,751,773,787,797,809,811,823,827,839,841,853,857,859 \\
& 863,883,887,907,911,919,941,947,961,967,971,977,983,991
\end{aligned}
$$

All of the integers above are odd. Most are odd primes or proper powers of odd primes (in green text). The smallest non-prime power is $649=11 \cdot 59$.

Conjecture 6.106. The following statements are true:

1. There are no even exponents representing singleton equivalence classes; that is, each equivalence class $\left\{\left[\vartheta\left(z^{2 a}\right)\right]\right\}_{a \in \mathbb{N}}$ has at least two members;
2. There are infinitely many odd prime exponents representing singleton equivalence classes;
3. There are infinitely many proper odd-prime power exponents representing singleton equivalence classes.

A related conjecture is the following.

Conjecture 6.107 (Carmichael, [78], [79]). For each integer $n$, there is a distinct integer $m$ such that $\varphi(n)=\varphi(m)$.

REMARK 6.11.5. As the Euler totient function satisfies $\varphi(2 n)=\varphi(n)$ for $n$ odd and similar relations, it suffices to consider those integers $n \equiv 4 \bmod 8 . \quad \Delta$

REMARK 6.11.6. Ford gives a lower bound of $n \geqslant 10^{10^{10}}$ for any violation of the Carmichael Conjecture [130].

This concludes our discussion of some arithmetic aspects of complex analytic singularities. We proceed to some foundational structures.

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## Chapter 7

# Categorical Structure of Isolated Singularities 


#### Abstract

Nothing is more fruitful than these obscure analogies, these indistinct reflections of one theory into another, these furtive caresses, these inexplicable disagreements; also nothing gives the researcher greater pleasure.... The day dawns when the illusion vanishes; intuition turns to certitude; the twin theories reveal their common source before disappearing; as the Gita teaches us, knowledge and indifference are attained at the same moment. Metaphysics has become mathematics, ready to form the material for a treatise whose icy beauty no longer has the power to move us. - Andre Weil


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7.3. Monoidal Structure of the Homotopy Class of Fibers ..... 529
In previous chapters we have defined an array of numerical invariants of non-degenerate, complex analytic germs, namely, various indices involving a differential, geometric, topological, algebraic, analytic, combinatorial and arithmetic character (Table 1.1). In this chapter, we unify these invariants into a universal quantity, the Milnor number, and discuss the categorical structure of the homotopy space of singularities, Milnor fibers and their algebraic links. In particular, we prove a natural monoidal categorical structure of the homotopy class of said fibers.

### 7.1. Indices of an Isolated Singularity

Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with isolated critical point at the origin, we have defined the following numerical invariants:

1. The geometric index is the local geometric multiplicity of $f$ in a neighborhood of the origin, namely, $\mu_{\text {geom }}(f)=\left|V_{f, \kappa} \cap B_{\varepsilon}^{2 n+2}\right|$, where $\kappa \in \mathbb{C}^{\times}$is a regular value of $f$ sufficiently close to the origin and $\varepsilon>0$ is sufficiently small;
2. The differential index is the Poincare-Hopf index of the vector field $\partial f$, namely, $\mu_{\text {diff }}(f)=\operatorname{deg}_{B} \phi_{\partial f} ;$
3. The topological index is number of spheres in the homotopy type of the corresponding Milnor fiber, namely, $\mu_{\text {top }}(f)=\operatorname{rank} \tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right)$;
4. The K-theoretic index is the rank of the $n^{\text {th }}$-Grothendieck group of the corresponding Milnor fiber, namely, $\mu_{K}(f)=\operatorname{rank} \tilde{K}^{n}\left(F_{f, 0}\right)$;
5. The algebraic index is the (complex) dimension of the corresponding local algebra, namely, $\mu_{\mathrm{alg}}(f)=\operatorname{dim}_{\mathrm{C}} \mathcal{A}_{f} ;$
6. The analytic index is the residue of the logarithmic meromorphic form $\omega\left(\left.\partial f\right|_{\Omega}\right)$ of the complex analytic germ $f$ at the origin, namely, $\mu_{\text {anal }}(f)=\operatorname{Res} \omega\left(\left.\partial f\right|_{\Omega}\right) ;$
7. The combinatorial index is the mixed volume of the Kushnirenko polytope $\mathcal{K}(f)$, namely, $\mu_{\text {comb }}(f)=\operatorname{MV~} \mathcal{K}(f) ;$
8. The cohomological index is the first betti number of the infinite cyclic covering of the algebraic link $K_{f}$, namely, $\mu_{\mathrm{co}}(f)=b_{1}\left(M_{K_{f}, \infty}\right)$;
9. The arithmetic index is the number of positive integer solutions of a system of Diophantine inequalities, namely,

$$
\begin{equation*}
\mu_{\mathrm{nt}}(f)=\left|\left\{\left(x_{0}, \ldots, x_{n}\right) \in \mathbb{N}^{n+1} \mid 0<\omega_{i} x_{i}<1\right\}\right| ; \tag{7.1}
\end{equation*}
$$

10. The lattice index is the weighted lattice point summation, namely,

$$
\begin{equation*}
\mu_{\text {lat }}(f)=\sum_{p \in \mathcal{W}(f)^{\circ}} 2+\sum_{p \in \partial \mathcal{W}(f) \cap \mathbb{N}^{n+1}} 1 . \tag{7.2}
\end{equation*}
$$

### 7.2. The Milnor Number

Before we proceed to the main proposition of this section, we quote the following theorem without proof. First, define $\omega(f, g)=g \omega(f)$ where $g$ is a holomorphic function on some domain $U \subset \mathbb{C}^{n}$.

Proposition 7.1 (Griffiths, Harris, [162]). Let $U_{x}$ be a neighborhood of a point $x \in \mathbb{C}^{n}$. Given a complex analytic map $f: U_{x} \rightarrow \mathbb{C}^{n}$, where $x$ is an isolated root of the system $\left.\partial f\right|_{\Omega_{x}}=\mathbf{0}$, and any holomorphic function $g$ with domain $U \subset \mathbb{C}^{n}$, then one has the identity

$$
\begin{align*}
\operatorname{Res}_{x} \omega(f, g) & =\frac{1}{(2 \pi i)^{n}} \int_{\gamma_{f, x}} \omega(f, g)  \tag{7.3a}\\
& =\sum_{x \in f^{-1}(0) \cap U} \operatorname{mult}_{x}(f) g(x) . \tag{7.3b}
\end{align*}
$$

This implies, in particular, that the Grothendieck residue of $\omega(f)$ and geometric multiplicity mult $(f)$ coincide. See [459] or the papers [185, 186, 187] for a nice discussion of this and related results.

By the combined effort of a number of works, we have the following fact (a partial proof of Lemma 1.17).

Proposition 7.2. Given a non-degenerate, quasi-Brieskorn-Pham singularity $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, the various indices of $f$, namely,

1. (Geometric) $\mu_{\text {geom }}(f)$;
2. (Differential) $\mu_{\text {diff }}(f)$;
3. (Topological) $\mu_{\text {top }}(f)$;
4. (K-Theoretic) $\mu_{\mathrm{K}}(f)$;
5. (Algebraic) $\mu_{\mathrm{alg}}(f)$;
6. (Analytic) $\mu_{\text {anal }}(f)$;
7. (Combinatorial) $\mu_{\text {comb }}(f)$;
8. (Cohomological) $\mu_{\mathrm{co}}(f)$;
9. (Arithmetic) $\mu_{\text {arith }}(f)$; and,
10. (Lattice) $\mu_{\text {latt }}(f)$
coincide.

Proof. In [310], if given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ with isolated critical point at the origin, then $\mu_{\text {top }}(f)=\mu_{\text {geom }}(f)=\mu_{\mathrm{alg}}(f)$ is a positive integer. By Theorem 7.1, the local geometric multiplicity and the

Grothendieck residue of $\left.\partial f\right|_{\Omega}$ coincide, so $\mu_{\text {anal }}(f)=\mu_{\text {geom }}(f)$. By Proposition 1.6 in [259], $\mu_{\text {diff }}(f)=\mu_{\text {alg }}(f)$. By a theorem of Grothendieck, we have $\mu_{\mathrm{anal}}(f)=\mu_{\mathrm{alg}}(f)$ (See [428] or Chapter 5 of [21]). By Corollary 1.28, we have the isomorphism $\tilde{H}_{n}\left(F_{f, 0} ; \mathbb{Z}\right) \cong \tilde{\mathrm{K}}_{\text {top }}^{n}\left(F_{f, 0}\right)$ for any $n \geqslant 0$, so $\mu_{\text {top }}(f)=\mu_{\mathrm{K}}(f)$. A theorem of Klimek and Lesniewski [239] on the Koszul cohomology of $\mathcal{A}_{f}$ implies $\mu_{\mathrm{alg}}(f)=\mu_{\text {top }}(f)$. The fact that the local geometric multiplicity is a positive integer implies that the other Milnor numbers are also. To prove $\mu_{\mathrm{alg}}(f)=\mu_{\mathrm{co}}(f)$ simply appeal to Proposition 4.24. The remaining equivalences follow from work in previous chapters, so we omit the details.

Remark 7.2.1. Most of the previous result easily generalizes to arbitrary non-degenerate, complex analytic germs.

Define the Milnor number $\mu=\mu(f)$ of $f$ as any one of the indices discussed, or the most convenient for the purpose at hand.

### 7.3. Monoidal Structure of the Homotopy Class of Fibers

Now that we have shown that the various indices coincide, we now discuss a natural monoidal structure of the space of fibers.

It is useful to define the exponent matrix of the only constant weighted homogeneous function, 0 . Here, $\mathrm{A}_{0}=(0), f \boxtimes 0=0 \boxtimes f=0$ and $f \boxtimes 0=0 \boxtimes f=f$, which illustrates the role of 0 as an annihilator in a putative monoid of weighted
homogeneous polynomials under the Kronecker product and as an identity element in a putative monoid of weighted homogeneous polynomials under the Kronecker sum. Note, also, that $f \boxtimes z=z \boxtimes f=f$.

Since the set of diagonal matrices is closed under the operation of Kronecker sum, it follows that the set of Brieskorn-Pham polynomials equipped with $\square$ is a monoid (with a two-sided identity element 0 ).
7.3.1. The Milnor Monoid. By the preceding discussion and the fact that the wedge sum is associative over pointed spaces, we conclude that the space of the homotopy classes of Milnor fibers forms a countably-infinite, abelian, additive monoid under wedge sums, with the class of trivial fibers - those homotopy equivalent to a point - forming the identity.

Recall that a semigroup is a closed set under an associative binary operation.
Proposition 7.3. The class of complex analytic function germs about the origin forms a semigroup under the Sebastiani-Thom sum $\boxplus$.

Proof. Omitted.

Proposition 7.4. The morphism $\mu_{\mathrm{top}}:(\mathbf{M i l}, \boxplus) \rightarrow\left(\mathbb{Z}_{\geqslant 0}, \times\right)$ is a semigroup homomorphism.

Proof. The set of Milnor fibers Mil forms an ordered, abelian semigroup (or associative magma*) under the Sebastiani-Thom sum $\boxplus$. The non-negative

[^35]integers $\mathbb{Z}_{\geqslant 0}$ forms a monoid or semigroup with identity under multiplication with identity 1.

Let $\mathfrak{m}_{n}^{\text {iso }} \subset \mathfrak{m}_{n} \subset \mathcal{O}_{0, n}$ denote the ideal of holomorphic functions (of $n+1$ variables) which vanish at the origin and possess an isolated critical point there as well. Define the direct limits $\mathcal{O}_{0}=\underset{\longrightarrow}{\lim } \mathcal{O}_{0, n}$ and correspondingly $\mathfrak{m}^{\text {iso }}=$ $\xrightarrow{\lim } \mathfrak{m}_{n}^{\text {iso }}$ under the standard ordering of $\mathbb{N}$. Let $[f]$ denote the equivalence class of functions in $\mathfrak{m}^{\text {iso }}$ whose fibers have isomorphic middle reduced-homology group of finite rank, or equivalently have equal (and finite) Milnor number. This choice of equivalence classes provides a partition on the space of fibers. Denote this space of equivalence classes by $\mathfrak{F}=\left\{[f] \mid f \in \mathfrak{m}^{\text {iso }}\right\}$.

Proposition 7.5. The triple $\mathfrak{F}_{\boxplus}=\left(\mathfrak{F}, \boxplus,\left[z^{2}\right]\right)$ is an abelian homotopy monoid.
Proof. By the Sebastiani-Thom equivalence $[f]=\left[f \boxplus z^{2}\right]=[f] \boxplus\left[z^{2}\right]=$ $\left[z^{2}\right] \boxplus[f]$ for all $f \in \mathfrak{F}$. It now follows that the Sebastiani-Thom operation $\boxplus$ induces an abelian monoidal structure on the space $\mathfrak{F}$ with identity element $\left[z^{2}\right]$. Categorify.

The set of Milnor fibers Mil forms a countable, ordered, abelian semigroup under the Sebastiani-Thom sum $\boxplus$. However, the subset of weighted homogeneous Milnor fibers $\mathbf{M i l}_{w} \subset \mathbf{M i l}$ forms an ordered, abelian monoid under the Kronecker product $\boxtimes$ with $F_{z, 0} \simeq\{\bullet\}$ acting as the identity. Since $\mu_{\mathrm{alg}}\left(F_{z, 0}\right)=0$, the morphism $\mu_{\mathrm{alg}}:\left(\mathbf{M i l}_{w}, \boxtimes\right) \rightarrow\left(\mathbb{Z}_{\geqslant 0,+}\right)$ is a monoid homomorphism. It is clear
that the space of weighted homogeneous polynomials is closed under the Kronecker product with the identity $f \boxtimes z=z \boxtimes f=f$.

Proposition 7.6. The subset of weighted homogeneous Milnor fibers Mil ${ }_{w}$ forms a monoid under $\boxtimes$ with $F_{z, 0} \simeq\{\bullet\}$ as the identity.

Proof. The claim follows from closure of weighted homogeneous polynomials under $\boxtimes$, the identity $f \boxtimes z=z \boxtimes f=f$ and associativity $(f \boxtimes g) \boxtimes h=$ $f \boxtimes(g \boxtimes h)$ for all weighted homogeneous polynomials $f, g$ and $h$.

With these structures in hand, we need only categorify. We leave the details to the reader.

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## Chapter 8

# Real Structure of Isolated Singularities 

## Everything we call real is made of things that cannot be regarded as real. - Niels Bohr

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In this chapter we study isolated singularities arising from certain real analytic maps.

### 8.1. Real Isolated Singularities

8.1.1. Twisted Brieskorn-Pham Singularities. We close this chapter with some new and exciting results generalizing Milnor's work to real analytic maps.

Proposition 8.1 (Milnor, [310]). If a real analytic germ $f:\left(\mathbb{R}^{n+k}, \mathbf{0}\right) \rightarrow$ $\left(\mathbb{R}^{k}, \mathbf{0}\right)$ has an isolated critical point at the origin, then there is an $\varepsilon_{0}>0$ such that for any $0<\varepsilon<\varepsilon_{0}$, the complement of the intersection $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{n+k-1}$ fibers over $S^{1}$.

Seade [420] considers the real analytic map $f=z_{1}^{p} \bar{z}_{2}+z_{2}^{q} \bar{z}_{1}$ over $\left(\mathbb{R}^{4}, \mathbf{0}\right) \simeq$ $\left(\mathbb{C}^{2}, \mathbf{0}\right)$ and proves that the $\operatorname{map} \phi_{f}=\frac{f}{\|f\|}$ defines a fibration. Ruas, Seade and Verjovsky [429] prove that this singularity is topologically equivalent to that of
the Brieskorn-Pham polynomial $f=z_{1}^{p-1}+z_{2}^{q-1}$ and therefore has algebraic link isotopic to the torus link $T_{p-1, q-1}$. Such maps are twisted Brieskorn-Pham. The following generalization of the Join of Pham holds.

Proposition 8.2 (Ruas, Seade and Verjovsky,[429]). Suppose $f=$ $\sum_{i=0}^{n} z_{i}^{a_{i} \bar{z}_{i}^{b_{i}}}$ with integer exponents $a_{i}>b_{i} \geqslant 1$. The fiber $F_{f, 0} \simeq V_{f, 1}$ has a deformation retract homeomorphic to the join $\mathrm{C}_{a_{0}-b_{0}} \star \cdots \star \mathrm{C}_{a_{n}-b_{n}}$ and the homotopy-type of a wedge sum of $\prod_{i=0}^{n}\left(a_{i}-b_{i}-1\right) n$-spheres.

For related discussion of singularities arising from real polynomial maps, consult §11 in [310], Chapter VIII in [420], and [378].

REMARK 8.1.1. In these examples, one may write $z_{j}=r_{j} e^{i \theta_{j}}$, where $r_{0}, \ldots, r_{n} \in \mathbb{R}_{\geqslant 0}$ and $\theta_{0}, \ldots, \theta_{n} \in \mathbb{R}$, so

$$
\begin{equation*}
f=\sum_{j=0}^{n} z_{j}^{a_{j}} \bar{z}_{j}^{b_{j}}=\sum_{j=0}^{n} r_{j}^{2} e^{i\left(a_{j}-b_{j}\right) \theta_{j}} \cong \sum_{j=0}^{n} x_{j}^{a_{j}-b_{j}} \tag{8.1}
\end{equation*}
$$

where $x_{j}=e^{i \theta_{j}}$. Extending the values of each of the new coordinates from $S^{1}$ to $\mathbb{C}$, the function $\tilde{f}=f\left(x_{0}, \ldots, x_{n}\right)$ is a Laurent polynomial in the variables $\left\{x_{0}, \ldots, x_{n}\right\}$, complex analytic if and only if $a_{j}-b_{j} \geqslant 0$ and weighted homogeneous if and only if $a_{j}-b_{j} \geqslant 1$.

### 8.2. A Conjecture on Ehrhart Reciprocity

A rather curious phenomenon takes place under complex conjugation of the singularity. In this case, the multiplicity and Łojasiewicz exponent satisfy
a reciprocity relation. Let $\bar{f}$ be the conjugate singularity of a weighted homogeneous singularity $f$ with negative corresponding weights. Given the identity $\min \left\{-x_{1}, \ldots,-x_{n}\right\}=-\max \left\{x_{1}, \ldots, x_{n}\right\}$ for $\left\{x_{1}, \ldots, x_{n}\right\} \subset \mathbb{R}_{\geqslant 0}$, one has

$$
\begin{align*}
-v(\bar{f}) & =-\left[\min _{0 \leqslant i \leqslant n}\left\{-\frac{1}{\omega_{i}}\right\}\right\rceil  \tag{8.2}\\
& =-\left[-\max _{0 \leqslant i \leqslant n}\left\{\frac{1}{\omega_{i}}-1\right\}-1\right\rceil  \tag{8.3}\\
& =\left\lfloor\ell_{0}(f)\right\rfloor+1, \tag{8.4}
\end{align*}
$$

which is the degree of topological determinacy of $f, q . v$. , Definition 2.5.
Conjecture 8.3. The complex conjugation map of weighted homogeneous singularities is in some sense equivalent to the Ehrhart map $t \mathcal{W} \rightarrow(-t) \mathcal{W}$ yielding Ehrhart Reciprocity $\mathscr{L}_{\mathcal{W}}(t)=(-1)^{\operatorname{dim} \mathcal{W}} \mathscr{L}_{\mathcal{W}^{\circ}}(-t)$, where $\mathscr{L}_{\mathcal{W}}(t)=\left|t \mathcal{W} \cap \mathbb{Z}^{n+1}\right|$.

See Volumes 2.

### 8.3. Polar Weighted Homogeneity

In general, if $f=\sum_{i=1}^{m} z_{i 1}^{a_{i 1}} \bar{z}_{i 1}^{b_{i 1}} \cdots z_{i n}^{a_{i n}} \bar{z}_{i n}^{b_{i n}}$, then the associated Laurent polynomial is $\tilde{f}=\sum_{i=1}^{m} x_{i 1}^{a_{i 1}-b_{i 1}} \ldots x_{i n}^{a_{i n}-b_{i n}}$. If $\tilde{f}$ is weighted homogeneous on $\mathbb{C}^{n+1}$, then $f$ is said to be twisted weighted homogeneous on $\mathbb{C}^{n+1}$. A related generalization which subsumes twisted weighted homogeneity is polar weighted homogeneity [358].

Proposition 8.4 (Oka, [358]). Given a twisted, weighted homogeneous polynomial $f:\left(\mathbb{R}^{2 n+2}, \mathbf{0}\right) \rightarrow\left(\mathbb{R}^{2}, 0\right)$ such that $\tilde{f}:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a complex analytic,
weighted homogeneous polynomial with an isolated critical point at the origin, then there are diffeomorphisms $F_{f, 0} \cong_{d} F_{\tilde{f}, 0}$ and $K_{f} \cong_{d} K_{\tilde{f}}$.

Proof. See Theorem 10 in [358].
In closing, we remark on some recent work on mixed singularities. Based on earlier work of Rudolph [405], Pichon proves the following general result on a family of real polynomial maps.

Proposition 8.5 (Pichon,[377]). If two complex analytic maps $f, g:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ have no branch in common, then the real analytic map $f \bar{g}:\left(\mathbb{R}^{4}, \mathbf{0}\right) \rightarrow\left(\mathbb{R}^{2}, 0\right)$ has an isolated critical point at the origin if and only if $K_{f \bar{g}}=K_{f} \cup-K_{g}$ is fibered.

Proof. See Theorem 5.1 in [377].

Remark 8.3.1. Recall that $-K$ denotes $K$ with opposite orientation.

Proposition 8.6 (Gusein-Zade, et al., [171]; Pichon, Seade, [378]). Given two complex analytic germs $f, g:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ and a punctured neighborhood of the origin $U=\Omega \backslash g^{-1}(0)$, where the meromorphic function $\frac{f}{g}$ is regular on $U$, the $\operatorname{map} \phi_{f / g}=\frac{f / g}{\|f / g\|}: S_{\varepsilon}^{2 n+1} \backslash L_{f g} \rightarrow S^{1}$, where $L_{f g}=(f \cdot g)^{-1}(0) \cap S_{\varepsilon}^{2 n+1}$, is the projection of a smooth, locally trivial fiber bundle. A generic fiber $F_{f / g, \theta}=\phi^{-1}\left(e^{i \theta}\right)$ is diffeomorphic to the complex manifold $\left(\frac{f}{g}\right)^{-1}(\kappa) \cap\left(B_{\varepsilon}^{2 n+2}\right)^{\circ}$, where $\kappa \in \mathbb{C}$ is a regular value of $\frac{f}{g}$. In particular, each fiber is a parallelizable manifold with the homotopy type of a CW-complex of dimension $n$.

## Proof. See Theorem 1 in [378].

REMARK 8.3.2. Since $\frac{f \bar{g}}{\|f \overline{\mathcal{g}}\|}=\frac{f / g}{\|f / g\|}, L_{f \bar{g}}=L_{f g}$ as links without orientation.

This concludes our brief discussion on certain generalizations of complex analytic singularities to the real case. We consider now some general topological aspects of certain non-isolated singularities.

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## Chapter 9

## Topological Structure of Non-Isolated Singularities

> No man is an island, entire of itself; every man is a piece of the continent, a part of the main. If a clod be washed away by the sea, Europe is the less, as well as if a promontory were, as well as if a manor of thy friend's or of thine own were: any man's death diminishes me, because I am involved in mankind, and therefore never send to know for whom the bells tolls; it tolls for thee. - John Donne, Meditation XVII

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In this chapter we study non-isolated singularities of certain complex hypersurfaces and generalize the corresponding topological, algebraic and K-theoretic indices. We briefly review the Kato-Matsumoto Theorem and Massey's generalization of the Sebastiani-Thom equivalence to the derived category.

### 9.1. Non-Isolated Singularities

In this section we discuss briefly a few generalizations of the classical theory of complex singularities to those with higher-dimensional critical loci.
9.1.1. Higher-Dimensional Critical Loci. Recall that $\Sigma\left(V_{f, \kappa}\right)$ denotes the singular locus of the hypersurface $V_{f, \kappa}=f^{-1}(\kappa)$. Kato and Matsumoto prove
the following generalization of Milnor's construction for non-isolated singularities.

Proposition 9.1 (Kato, Matsumoto, [233]). Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, \mathbf{0})$ be a complex analytic germ with hypersurface $V_{f, 0}=f^{-1}(0)$ and Milnor fiber $F_{f, 0}=$ $\varphi_{f}^{-1}(1)$. If $\operatorname{dim} \Sigma\left(V_{f, 0}\right)=k$, then the Milnor fiber $F_{f, 0}$ is $(n-k-1)$-connected, that is, $\pi_{i}\left(F_{f, 0}\right)$ and $H_{i}\left(F_{f, 0} ; \mathbb{Z}\right)$ are trivial for $1 \leqslant i \leqslant n-k-1$.

Remark 9.1.1. If $\Sigma\left(V_{f, 0}\right)$ is a discrete, then one recovers Mi'.

Remark 9.1.2. The Kato-Matsumoto Theorem is sharp. The singularity $f=z_{0} \cdots z_{k+1}+z_{k+2}^{2}+\cdots+z_{n}^{2}$ has $\operatorname{dim} \Sigma\left(V_{f, 0}\right)=k$ and the fiber homotopy type,

$$
\begin{equation*}
\left.F_{f, 0} \simeq S^{n-k-1} \mathbb{T}^{k+1} \simeq \bigvee_{l=1}^{k+1} \bigvee^{(k+1}{ }_{l}^{k}\right) S^{l+n-k-1} \quad n \geqslant k+1 \tag{9.1}
\end{equation*}
$$

by equation (1.32). There are $k+1$ distinct wedge sums of spheres of the same dimension, whose lowest dimension is $n-k$, totaling $\sum_{l=1}^{k+1}\binom{k+1}{l}=2^{k+1}-1$ spheres,

$$
\begin{align*}
\operatorname{rank} H_{i}\left(F_{f, 0}\right) & = \begin{cases}\sum_{l=1}^{k+1}\binom{k+1}{l} \delta_{i, l+n-k-1} & 1 \leqslant i \leqslant n \\
1 & i=0\end{cases}  \tag{9.2}\\
& =\left\{\begin{array}{cl}
\binom{k+1}{i} & n-k \leqslant i \leqslant n \\
0 & 1 \leqslant i \leqslant n-k-1 \\
1 & i=0 .
\end{array}\right. \tag{9.3}
\end{align*}
$$

Although there has been substantial progress in generalizing Milnor's construction, few results exist concerning the case of complex hypersurfaces with higher-dimensional singular sets. A few notable examples include the work of Nemethi, Dimca, Lê and Massey.
9.1.2. Sebastiani-Thom Equivalence in the Derived Category. The Sebastiani-Thom isomorphism is a specialization of a rather general natural transformation (of vanishing cycles) in the derived category of bounded, constructible complexes of sheaves of product domains of complex analytic germs (Proposition 1.4, [289]). As a consequence, the join factorization holds for Milnor fibers of the Sebastiani-Thom summation $f \boxplus g$ of degenerate singularities with arbitrary singular sets $\Sigma\left(V_{f \boxplus g, 0}\right)$ about the origin [411]. This work allows for a simple generalization of Pham's formula for the Milnor number of isolated singularities of Brieskorn-Pham type, which we now describe.

According to Massey, consider the Brieskorn-Pham singularities $f=\sum_{i=1}^{n} f_{i}$, $f_{i}=z_{i}^{a_{i}}$ with a collection of local systems $\left\{L_{i}\right\}$ on $\mathbb{C}^{\times}$each of rank $\left\{d_{i}\right\}$ with monodromy isomorphisms $\left\{h_{*, i}\right\}$. The vanishing cycles $\Phi_{f_{i}}\left(\mathrm{IC}_{\mathbb{C}}^{*}\left(L_{i}\right)\right)$, functors between the derived categories of the total space and singular fiber, have nontrivial degree only in degree 0 and dimension equal to that of the nearby cycles with a correction factor accounting for the dimension of the stalk at the origin
[289], namely,

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}} \Phi_{f_{i}}\left(\mathrm{IC}_{\mathbb{C}}^{*}\left(L_{i}\right)\right)=a_{i} d_{i}-\operatorname{dim}_{\mathbb{C}} \operatorname{ker}\left(I-h_{*, i}\right) \tag{9.4}
\end{equation*}
$$

By the aforementioned natural transformation and the fact that intersection cohomology complexes are closed under external products, there is a natural transformation $\Phi_{f}\left(\mathrm{IC}_{\mathbb{C}^{n}}^{*}\left(\boxplus_{i=1}^{n} L_{i}\right)\right) \rightarrow \bigotimes_{i=1}^{n} \Phi_{f_{i}}\left(\mathrm{IC}_{\mathbb{C}}^{*}\left(L_{i}\right)\right)$ and, therefore,

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}} \Phi_{f}\left(\mathrm{IC}_{\mathbb{C}^{n}}^{*}\left(\boxplus_{i=1}^{n} L_{i}\right)\right)=\prod_{i=1}^{n}\left(a_{i} d_{i}-\operatorname{dim}_{\mathbb{C}} \operatorname{ker}\left(\mathrm{I}-h_{*, i}\right)\right) \tag{9.5}
\end{equation*}
$$

where, $1 \leqslant \operatorname{dim}_{\mathbb{C}} \operatorname{ker}\left(\mathrm{I}-h_{*, i}\right) \leqslant d_{i}$. For $d_{i}>1$ and $h_{*, i}=\mathrm{I}$ for $1 \leqslant i \leqslant n$, Massey computes* the identity

$$
\begin{equation*}
\operatorname{dim}_{\mathbb{C}} \Phi_{f}\left(\mathrm{IC}_{\mathbb{C}^{n}}^{*}\left(\boxplus_{i=1}^{n} L_{i}\right)\right)=\prod_{i=1}^{n} d_{i}\left(a_{i}-1\right) \tag{9.6}
\end{equation*}
$$

which generalizes Pham's formula.

### 9.2. Exponent Matrices, Revisited

9.2.1. Moore-Penrose Pseudo-Inverse. Let $\mathbb{F}$ be a field of characteristic 0 , $e . g ., \mathbb{R}, \mathbb{C}$, etc. In case that a matrix $A \in \mathbb{F}^{m, n}$ is not square or does not have full rank, then one typically introduces a pseudo-inverse in attempt to solve the matrix equation $A \mathbf{x}=\mathbf{b}$ [322], [370]. For this section, we refer the reader to [154] and [48].

[^36]Remark 9.2.1. A necessary and sufficient condition for the matrix equation $A \mathbf{x}=\mathbf{b}$ to be soluble is that the rank of the augmented matrix $(\mathrm{A} \mid \mathbf{b})$ be equal to that of $A$.

Let $\bar{A}$ and $A^{*}=(\overline{\mathrm{A}})^{\top}$ denote the (entry-wise) conjugate and Hermitian conjugate of $A$ in $\mathbb{F}^{m, n}$ and $\mathbb{F}^{n, m}$, respectively.

Definition 9.2. Given $A \in \mathbb{F}^{m, n}$, a Moore-Penrose pseudo-inverse $A^{+} \in \mathbb{F}^{n, m}$ satisfies the following:

1. $A A^{+} A=A$;
2. $A^{+}{A A^{+}}^{+}=A^{+}$;
3. $\left(\mathrm{AA}^{+}\right)^{*}=\mathrm{AA}^{+}$; and,
4. $\left(A^{+} A\right)^{*}=A^{+} A$.

Proposition 9.3. Given any matrix $\mathrm{A} \in \mathbb{F}^{m, n}$, then a Moore-Penrose pseudoinverse $\mathrm{A}^{+}$exists and is unique.

Proof. See [154].

REMARK 9.2.2. Consider the matrix $\mathrm{A}=\left(a_{1} \ldots a_{n}\right) \in \mathbb{C}^{1, n}$. If A is a zero vector, then $A^{+}=A^{\top}$, otherwise

$$
\mathrm{A}^{+}=\frac{1}{\sum_{i=1}^{n}\left|a_{i}\right|^{2}}\left(\begin{array}{c}
\overline{a_{1}}  \tag{9.7}\\
\vdots \\
\overline{a_{n}}
\end{array}\right)=\mathrm{A}^{*}\left(\mathrm{AA}^{*}\right)^{-1}
$$

Definition 9.4. A matrix A has full row (resp., column) rank if $m<n$ (resp., $m>n)$ and $\operatorname{rank} \mathrm{A}=m($ resp., $\operatorname{rank} \mathrm{A}=n)$. If $\operatorname{rank} \mathrm{A}=m=n$, then A has full rank*.

Proposition 9.5. If $\mathrm{A} \in \mathbb{F}^{m, n}$, then the following statements are true:

1. $\left(Z_{m, n}\right)^{+}=Z_{n, m}$;
2. $\left(A^{+}\right)^{+}=A$;
3. $\left(\mathrm{A}^{*}\right)^{+}=\left(\mathrm{A}^{+}\right)^{*}$;
4. $\operatorname{ker} A^{+} \cong \operatorname{ker} A^{*}$ and $\operatorname{im} A^{+} \cong \operatorname{im} A^{*}$;
5. $A^{+}=\lim _{\lambda \rightarrow 0^{+}}\left(\lambda I+A^{*} A\right)^{-1} A^{*}=\lim _{\lambda \rightarrow 0^{+}} A^{*}\left(\lambda I+A A^{*}\right)^{-1}$;

6a. If A has full row rank, then $\mathrm{AA}^{+}$is non-singular and $\mathrm{A}^{+}=\mathrm{A}^{*}\left(\mathrm{AA}^{*}\right)^{-1}$;
6b. If $A$ has full column rank, then $A^{+} A$ is non-singular and $A^{+}=\left(A^{*} A\right)^{-1} A^{*}$;
6c. If A has full column rank, then $\mathrm{A}^{+} \mathrm{A}=\mathrm{I}$; and,
7. If A has full rank, then $\mathrm{A}^{+}=\mathrm{A}^{-1}$.

## Proof. See [48].

A computationally fast means of computing the Moore-Penrose pseudoinverse involves singular value decompositions.

Proposition 9.6. Given any matrix $\mathrm{A} \in \mathbb{F}^{m, n}$ with singular value decomposition $\mathrm{A}=\mathrm{U} \Sigma \mathrm{V}^{*}$, where $\mathrm{U} \in \mathbb{F}^{m, m}$ is real or unitary, $\Sigma \in \mathbb{F}^{m, n}$ is real and rectangular diagonal, and $\mathrm{V} \in \mathbb{F}^{n, n}$ is real or unitary, then $\mathrm{A}^{+}=\mathrm{U} \Sigma^{+} \mathrm{V}^{*}$, where the corresponding

[^37]non-zero diagonal entries of $\Sigma$ and $\Sigma^{+}$are inverses and the corresponding zero diagonal entries are identical.

Proof. See [154] and [48].

Proposition 9.7. All solutions of the (possibly under-constrained) matrix equation $\mathbf{A x}=\mathbf{b}$ are given by the Moore-Penrose solutions, namely, $\mathbf{x}=\mathrm{A}^{+} \mathbf{b}+(\mathrm{I}-$ $\left.A^{+} A\right) \mathbf{v}$, where $\mathbf{v}$ is an arbitrary vector in $\mathbb{R}^{m}$. Moreover, if $\mathbf{x}$ is a solution of said matrix equation, then $\left\|\mathrm{A}^{+} \mathbf{b}\right\|_{2} \leqslant\|\mathbf{x}\|_{2}$, and $\mathbf{x}$ is unique if and only if $m \leqslant n$ and $\operatorname{rank} \mathrm{A}=m$, i.e., A has full column rank or full rank, in which case $\mathbf{x}=\mathrm{A}^{+} \mathbf{b}$.

Definition 9.8. If $\mathbf{x}$ solves the matrix equation $\mathrm{A} \mathbf{x}=\mathbf{b}$, then the vectors $A^{+} \mathbf{b}$ and $\left(I-A^{+} A\right) \mathbf{v}$ are the minimal part and free part of $\mathbf{x}$, respectively.

The column rank of the exponent matrix determines the number of weights.

Proposition 9.9. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial with an exponent matrix $\mathrm{A}_{f}$, then the corresponding weights of $f$ are the components of the vector $\mathrm{A}_{f}^{+} \mathbf{1}_{m}+\left(\mathrm{I}-\mathrm{A}_{f}^{+} \mathrm{A}_{f}\right) \mathbf{v}$, where $\mathbf{v}$ is an arbitrary vector in $\mathbb{R}^{n+1}$. If $\mathrm{A}_{f}$ is column rank deficient, then $f$ has a continuum of real weights whose minimal part is rational and equal to the row sums of the matrix $\mathrm{A}_{f}^{+}$. If $\mathrm{A}_{f}$ has full column rank or full rank, then the weights are unique, rational and equal to the aforementioned minimal part.

Proof. By considering the constituent monomials $z^{\mathbf{a}_{i}}=z_{0}^{a_{i 0}} \cdots z_{n}^{a_{i n}}$ comprising a weighted homogeneous polynomial, equation (3.1) implies that
$\boldsymbol{\omega}=\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ forms a solution of a system of linear equations $\left\{\mathbf{a}_{i} \cdot \boldsymbol{\omega}=1\right\}$ or equivalent matrix equation $\mathrm{A}_{f} \boldsymbol{\omega}=\mathbf{1}_{m}$, where $\mathrm{A}_{f}=\left(\mathbf{a}_{i}\right) \in \mathbb{Z}_{\geqslant 0}^{m, n}$ is the matrix of exponents of $f$. By Proposition 9.7, then $\boldsymbol{\omega}=\mathrm{A}_{f}^{+} \mathbf{1}_{m}+\left(\mathrm{I}-\mathrm{A}_{f}^{+} \mathrm{A}_{f}\right) \mathbf{v}$, where $\mathbf{v}$ is an arbitrary vector in $\mathbb{R}^{n+1}$, the minimal part given by $\mathrm{A}_{f}^{+} \mathbf{1}_{m}$. If, however, $\mathrm{A}_{f}$ has full column rank or full rank, then by Proposition 9.5, $\mathrm{A}_{f}^{+} \mathrm{A}_{f}=\mathrm{I}$, so $\boldsymbol{\omega}=\mathrm{A}_{f}^{+} \mathbf{1}_{m}$, which implies the rationality of $\omega$.

REmARK 9.2.3. Consider $f=x^{4}+x^{3} y^{2}+x^{2} y^{4}+x y^{6}+y^{8}$ over $\mathbb{C}^{2}$. Observe that

$$
\lambda^{\omega} \cdot f=\lambda^{4 \omega_{1}} x^{4}+\lambda^{3 \omega_{1}+2 \omega_{2}} x^{3} y^{2}+\lambda^{2 \omega_{1}+4 \omega_{2}} x^{2} y^{4}+\lambda^{\omega_{1}+6 \omega_{2}} x y^{6}+\lambda^{8 \omega_{2}} y^{8}
$$

equals $\lambda f(x, y)$ if and only if the following (over-determined) matrix equation

$$
\left(\begin{array}{ll}
4 & 0  \tag{9.8}\\
3 & 2 \\
2 & 4 \\
1 & 6 \\
0 & 8
\end{array}\right)\binom{\omega_{0}}{\omega_{1}}=\left(\begin{array}{l}
1 \\
1 \\
1 \\
1 \\
1
\end{array}\right)
$$

is soluble in $\mathbb{R}^{2}$. Indeed, there is a unique solution, as $A_{f}$ has full column rank, so $\mathrm{A}_{f}^{+}=\left(\mathrm{A}_{f}^{\top} \mathrm{A}_{f}\right)^{-1} \mathrm{~A}_{f}^{\top}, \mathrm{A}_{f}^{+} \mathrm{A}_{f}=\mathrm{I}$, and the Moore-Penrose solution is given by

$$
\omega=\left(\begin{array}{ccccc}
\frac{3}{20} & \frac{1}{10} & \frac{1}{20} & 0 & -\frac{1}{20}  \tag{9.9}\\
-\frac{1}{40} & 0 & \frac{1}{40} & \frac{1}{20} & \frac{3}{40}
\end{array}\right)\left(\begin{array}{l}
1 \\
1 \\
1 \\
1 \\
1
\end{array}\right)=\binom{\frac{1}{4}}{\frac{1}{8}}
$$

Thus, $f$ is weighted homogeneous with reduced weights $\omega_{0}=\frac{1}{4}$ and $\omega_{2}=\frac{1}{8}$. In fact, any $\mathbb{C}^{\times}$-linear combination of monomials of $f$ is also weighted homogeneous with the same weight system.

Remark 9.2.4. Consider $f=x^{4}+x^{2} y z^{2}$ over $\mathbb{C}^{3}$. Observe that

$$
\begin{equation*}
\lambda^{\omega} \cdot f=\lambda^{4 \omega_{1}} x^{4}+\lambda^{2 \omega_{1}+\omega_{2}+2 \omega_{2}} x^{2} y z^{2} \tag{9.10}
\end{equation*}
$$

equals $\lambda f(x, y)$ if and only if the following (under-determined) matrix equation

$$
\left(\begin{array}{lll}
4 & 0 & 0  \tag{9.11}\\
2 & 1 & 2
\end{array}\right)\left(\begin{array}{l}
\omega_{1} \\
\omega_{2} \\
\omega_{3}
\end{array}\right)=\binom{1}{1}
$$

is soluble in $\mathbb{R}^{3}$. Alas, there is no unique solution, as $\mathrm{A}_{f}$ is column rank deficient. The Moore-Penrose pseudo-inverse of $\mathrm{A}_{f}$ is

$$
\mathrm{A}_{f}^{+}=\left(\begin{array}{cc}
\frac{1}{4} & 0  \tag{9.12}\\
-\frac{1}{10} & \frac{1}{5} \\
-\frac{1}{5} & \frac{2}{5}
\end{array}\right)
$$

so the Moore-Penrose solution is given by

$$
\boldsymbol{\omega}=\left(\begin{array}{c}
\frac{1}{4}  \tag{9.13}\\
\frac{1}{10} \\
\frac{1}{5}
\end{array}\right)+\left(\begin{array}{c}
0 \\
\frac{2}{5}\left(2 v_{2}-v_{3}\right) \\
-\frac{1}{5}\left(2 v_{2}-v_{3}\right)
\end{array}\right)
$$

where $v_{2}, v_{3} \in \mathbb{R}$. The minimal part of $\omega$ is $\left\{\frac{1}{4}, \frac{1}{10}, \frac{1}{5}\right\}$, while the free part is $\left\{0, \frac{2}{5}\left(2 v_{2}-v_{3}\right),-\frac{1}{5}\left(2 v_{2}-v_{3}\right)\right\}$. Equivalently, the weights may be written $\left\{\frac{1}{4}, \frac{1}{2}-2 v, v\right\}$, where $v=\frac{1}{5}\left(1-2 v_{2}+v_{3}\right) \in \mathbb{R}$.

Proposition 9.10. A weighted homogeneous polynomial $f$ has a zero weight in some direction if and only if the corresponding row sum of $\mathrm{A}_{f}^{+}$is zero and the corresponding row of $\mathrm{I}-\mathrm{A}_{f}^{+} \mathrm{A}_{f}$ is the zero vector.

Proof. The $i^{\text {th }}$ weight is fixed and given by the $i^{\text {th }}$-entry of the minimal part $\mathrm{A}_{f}^{+} \mathbf{1}_{m}$ if and only if the $i^{\text {th }}$ entry of the free part vanishes. Since $\mathbf{v}$ is arbitrary, said vanishing occurs if and only if the $i^{\text {th }}$ row of $\mathrm{I}-\mathrm{A}_{f}^{+} \mathrm{A}_{f}$ is the zero vector $\mathbf{0}_{n}$.

REmARK 9.2.5. The polynomial $f=y+x z+z=y+z(1+x)$ over $\mathbb{C}^{3}$ has no critical points, but it is weighted homogeneous with weights $\{0,1,1\}$. The corresponding exponent matrix has full rank.

REMARK 9.2.6. The polynomial $f=x^{2} y^{3}+x^{2} y^{2}=(x y)^{2}(1+y)$ over $\mathbb{C}^{2}$ has two continua of critical points along both axes. It is weighted homogeneous with weights $\left\{\frac{1}{2}, 0\right\}$. The corresponding exponent matrix has full rank.

Unlike Brieskorn-Pham singularities, quasi-Brieskorn-Pham singularities may be degenerate. However, a degenerate weighted homogeneous polynomial does not necessarily have at least one zero weight.

Remark 9.2.7. The polynomial $f=x^{2} y^{3}+x^{3} y^{2}=(x y)^{2}(x+y)$ over $\mathbb{C}^{2}$ has two continua of critical points along both axes. It is quasi-Brieskorn-Pham with weights $\left\{\frac{1}{5}, \frac{1}{5}\right\}$. The corresponding exponent matrix has full rank.

Corollary 9.11. If a weighted homogeneous polynomial has more than one weight, then it has a continuum of weights all of which have the same minimal part.

Proof. Suppose that $\omega$ and $\omega^{\prime}$ are distinct weights of a weighted homogeneous polynomial $f$ with exponent matrix $\mathrm{A}_{f}$. Form the convex linear combination $\boldsymbol{v}=\lambda \boldsymbol{\omega}+(1-\lambda) \boldsymbol{\omega}^{\prime}$, where $\lambda \in[0,1]$. Observe

$$
\begin{align*}
\mathrm{A}_{f} \boldsymbol{v} & =\lambda \mathrm{A}_{f} \boldsymbol{\omega}+(1-\lambda) \mathrm{A}_{f} \boldsymbol{\omega}^{\prime}  \tag{9.14}\\
& =\lambda \mathbf{1}_{m}+(1-\lambda) \mathbf{1}_{m}  \tag{9.15}\\
& =\mathbf{1}_{m} \tag{9.16}
\end{align*}
$$

Moreover, the convex linear combination affects only the free part,

$$
\begin{align*}
\boldsymbol{v} & =\lambda\left(\mathrm{A}_{f}^{+} \mathbf{1}_{m}+\left(\mathrm{I}-\mathrm{A}_{f}^{+} \mathrm{A}_{f}\right) \mathbf{v}\right)+(1-\lambda)\left(\mathrm{A}_{f}^{+} \mathbf{1}_{m}+\left(\mathrm{I}-\mathrm{A}_{f}^{+} \mathrm{A}_{f}\right) \mathbf{v}^{\prime}\right)  \tag{9.17}\\
& =\mathrm{A}_{f}^{+} \mathbf{1}_{m}+\left(\mathrm{I}-\mathrm{A}_{f}^{+} \mathrm{A}_{f}\right)\left(\lambda \mathbf{v}+(1-\lambda) \mathbf{v}^{\prime}\right) \tag{9.18}
\end{align*}
$$

since $\mathrm{A}_{f}^{+}$is unique. This completes the proof.
Corollary 9.11 implies that a minimal weight is well-defined and unique for a weighted homogeneous polynomial.

Conjecture 9.12. The minimal weight is a topological invariant for weighted homogeneous polynomials.
9.2.2. Topological, K-Theoretic and Algebraic Indices, Revisited. The following remarks concern weighted homogeneous singularities with no assumed density or dimension of their putative critical points in any neighborhood of the origin. For non-isolated singularities, the fiber $F_{f, 0}$ is not necessarily a wedge sum of spheres of the same dimension, the Grothendieck groups may not coincide with the corresponding homology groups, or the local algebra may be infinite dimensional. With no special regard for the middle homology group of the corresponding fiber or the dimension of the local algebra, we define the generalized topological index of a weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ by the reduced Euler characteristic of the fiber (up to a sign depending only on
the dimension),

$$
\begin{equation*}
\bar{\mu}_{\text {top }}(f)=(-1)^{n} \tilde{\chi}\left(F_{f, 0}\right), \tag{9.19}
\end{equation*}
$$

the generalized K-theoretic index by the difference of ranks of the two fundamental Grothendieck groups,

$$
\begin{equation*}
\bar{\mu}_{K}(f)=(-1)^{n}\left(\operatorname{rank} \tilde{K}^{0}\left(F_{f, 0}\right)-\operatorname{rank} \tilde{K}^{-1}\left(F_{f, 0}\right)\right), \tag{9.20}
\end{equation*}
$$

and the generalized algebraic index by the product of the (possibly non-unique) weights,

$$
\begin{equation*}
\bar{\mu}_{\mathrm{alg}}(f)=\prod_{i=0}^{n}\left(\frac{1}{\omega_{i}}-1\right) \tag{9.21}
\end{equation*}
$$

When $f$ is non-degenerate, we have shown that these definitions coincide. Otherwise, these invariants generalize those previously defined.

Let $\omega^{*}$ denote fixed weights or fixed part of the weight $\omega$, that is, those weights which do not depend on any free parameters. Although $\omega^{*} \subset \mathrm{~A}_{f}^{+} \mathbf{1}_{m}$, it does not necessarily coincide ${ }^{*}$ with the minimal part of $f$. Define $\bar{\mu}_{\mathrm{alg}}^{*}(f)$ to be the constant term in the expansion

$$
\begin{equation*}
\bar{\mu}_{\mathrm{alg}}(f)=\bar{\mu}_{\mathrm{alg}}^{*}(f)+g(\mathbf{v}) \tag{9.22}
\end{equation*}
$$

where $g$ is a non-constant function in the field of rational fractions $\mathbb{Z}\left(v_{0}, \ldots, v_{n}\right)$.
*The polynomial of Remark 9.2.4 has minimal part $\left\{\frac{1}{4}, \frac{1}{10}, \frac{1}{5}\right\}$ and fixed part $\left\{\frac{1}{4}\right\}$.

Proposition 9.13. Given a weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$, the constant $\bar{\mu}_{\mathrm{alg}}^{*}(f)$ satisfies

$$
\bar{\mu}_{\mathrm{alg}}^{*}(f)=\left\{\begin{array}{cc}
(-1)^{n-\left|\omega^{*}\right|+1} \prod_{\omega \in \omega^{*}}\left(\frac{1}{\omega}-1\right) & \omega^{*} \text { is not empty }  \tag{9.23}\\
(-1)^{n+1} & \text { otherwise }
\end{array}\right.
$$

In particular, if $\mathrm{A}_{f}$ has full column rank or full rank, then $\bar{\mu}_{\mathrm{alg}}^{*}(f)=\bar{\mu}_{\mathrm{alg}}(f)$.

Proof. The non-fixed weights are each of the form $\frac{r_{i}}{s_{i}} g_{i}(\mathbf{v})$, where $\frac{r_{i}}{s_{i}} \in \mathbf{Q}^{\times}$ and $g_{i} \in \mathbb{Z}\left[v_{0}, \ldots, v_{n}\right]$, so

$$
\begin{align*}
\bar{\mu}_{\mathrm{alg}}(f) & =\prod_{\omega \in \omega^{*}}\left(\frac{1}{\omega}-1\right) \prod_{\omega \notin \omega^{*}}\left(\frac{1}{\omega}-1\right)  \tag{9.24}\\
& =\prod_{\omega \in \omega^{*}}\left(\frac{1}{\omega}-1\right)\left(\sum_{k=0}^{n-\left|\omega^{*}\right|-1}(-1)^{k+n-\left|\omega^{*}\right|-1} e_{k}\left(\frac{1}{\omega_{1}}, \ldots, \frac{1}{\omega_{n-\left|\omega^{*}\right|-1}}\right)\right)  \tag{9.25}\\
& =(-1)^{n-\left|\omega^{*}\right|-1} \prod_{\omega \in \omega^{*}}\left(\frac{1}{\omega}-1\right)+g(\mathbf{v}) \tag{9.26}
\end{align*}
$$

where $e_{k}$ is the $k^{\text {th }}$-elementary symmetric polynomial, which vanishes at the origin for $k>0$, and $g \in \mathbb{Z}\left(v_{0}, \ldots, v_{n}\right)$ is not constant. Finally, if $\mathrm{A}_{f}$ has full column rank or full rank, then $\mathrm{A}_{f}^{+} \mathrm{A}_{f}=\mathrm{I}$, and the free part cancels, i.e., $g(\mathbf{v})=$ 0.

Proposition 9.14. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a set of neighborhoods of the origin. Given (possibly degenerate) complex analytic maps $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ such that $f=$
$f_{1} \boxplus \cdots \boxplus f_{s}$, then

$$
\begin{equation*}
\bar{\mu}_{\mathrm{alg}}^{*}(f)=\prod_{i=1}^{s} \bar{\mu}_{\mathrm{alg}}^{*}\left(f_{i}\right) . \tag{9.27}
\end{equation*}
$$

Proof. The identity follows from the identity $\bar{\mu}_{\mathrm{alg}}(f)=\prod_{i=1}^{s} \bar{\mu}_{\mathrm{alg}}(f)$ and the definition of $\bar{\mu}_{\text {alg }}^{*}(f)$.

Conjecture 9.15. Given two weighted homogeneous polynomials $f$ and $g$ with homotopy equivalent Milnor fibers, then $\bar{\mu}_{\mathrm{alg}}^{*}(f)=\bar{\mu}_{\mathrm{alg}}^{*}(g)$. In particular, $\bar{\mu}_{\mathrm{alg}}^{*}$ is a topological invariant.

Remark 9.2.8. Consider the trivial weighted homogeneous polynomial, the constant function $f=0$, over $\mathbb{C}^{n+1}$. The fiber $F_{f, 0}$ is empty, so $\chi\left(F_{f, 0}\right)=0$ for $n \geqslant 0$. Thus, $\bar{\mu}_{\text {top }}(f)=(-1)^{n+1}(1-0)=(-1)^{n+1}$. The exponent matrix $\mathrm{A}_{f}$ is the zero vector $\mathbf{0}_{n+1}$, so it is column rank deficient. The weights of $f$ are the elements of the free vector $\mathbf{v}=\left(v_{0}, \ldots, v_{n}\right) \in \mathbb{R}^{n+1}$. Thus, the algebraic index is the product

$$
\begin{equation*}
\bar{\mu}_{\mathrm{alg}}(f)=\prod_{i=0}^{n}\left(\frac{1}{v_{i}}-1\right)=(-1)^{n+1}+g\left(v_{0}, \ldots, v_{n}\right) \tag{9.28}
\end{equation*}
$$

where $g \in \mathbb{Z}\left(v_{0}, \ldots, v_{n}\right)$ has no constant term, and $\bar{\mu}_{\mathrm{alg}}^{*}(f)=(-1)^{n+1}$.
Remark 9.2.9. Consider $f=x$ over $\mathbb{C}^{n+1}$. The fiber $F_{f, 0}$ is a point, so $\chi\left(F_{f, 0}\right)=1$ for $n \geqslant 0$. Thus, $\bar{\mu}_{\text {top }}(f)=(-1)^{n+1}(1-1)=0$. The exponent matrix $\mathrm{A}_{f}$ is the vector $\left(1 \mathbf{0}_{n}\right)$, so it is column rank deficient for $n>0$. The weights of $f$
are $\left\{1, v_{1}, \ldots, v_{n}\right\}$, where $v_{1}, \ldots, v_{n} \in \mathbb{R}$. Thus, the algebraic index vanishes, and so does $\bar{\mu}_{\text {alg }}^{*}(f)$.

REMARK 9.2.10. Consider $f=z_{0} \cdots z_{n}$. There are $\binom{n+1}{2}$ continua of critical points along orthogonal hyperplanes defined by the vanishing of any two variables. The fiber $F_{f, 0}$ is diffeomorphic to the locus

$$
\begin{equation*}
\left\{\left.\left(z_{0}, \ldots, z_{n-1}, \frac{1}{z_{0} \cdots z_{n-1}}\right) \in \mathbb{C}^{n+1} \right\rvert\, z_{0} \cdots z_{n-1} \neq 0\right\} \tag{9.29}
\end{equation*}
$$

that is, $F_{f, 0} \cong\left(\mathbb{C}^{\times}\right)^{n}$ and has the homotopy-type of an $n$-torus $\mathbb{T}^{n}=S^{1} \times \cdots \times$ $S^{1}$, which has non-trivial homology in all dimensions up to and including $n$. In fact, the $k^{\text {th }}$-homology group of $\mathbb{T}^{n}$ is free-abelian of $\operatorname{rank}\binom{n}{k}$, i.e., $H_{k}\left(\mathbb{T}^{n} ; \mathbb{Z}\right) \cong$ $\mathbb{Z}^{\binom{n}{k}}$ for $0 \leqslant k \leqslant n$, and the Euler characteristic is simply

$$
\begin{equation*}
\chi\left(F_{f, 0}\right)=\chi\left(\mathbb{T}^{n}\right)=\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}=\delta_{n, 0} \tag{9.30}
\end{equation*}
$$

which implies a generalized topological index, $\bar{\mu}_{\text {top }}(f)=(-1)^{n+1}\left(1-\delta_{n, 0}\right)$. Recall the Grothendieck groups of the $n$-torus,

$$
\tilde{K}^{p}\left(\mathbb{T}^{n}\right) \cong\left\{\begin{array}{cc}
\mathbb{Z}^{2^{n-1}-1} & p \text { even, } n \geqslant 1  \tag{9.31}\\
\mathbb{Z}^{2^{n-1}} & p \text { odd }, n \geqslant 1 \\
\{0\} & n=0
\end{array}\right.
$$

If $F_{f, 0} \simeq \mathbb{T}^{n}$, then

$$
\begin{align*}
\bar{\mu}_{K}(f) & =(-1)^{n}\left(\operatorname{rank} \tilde{K}^{0}\left(F_{f, 0}\right)-\operatorname{rank} \tilde{K}^{-1}\left(F_{f, 0}\right)\right)  \tag{9.32}\\
& =(-1)^{n+1}\left(1-\delta_{n, 0}\right) \tag{9.33}
\end{align*}
$$

The weights are not unique, however, instead given by

$$
\begin{equation*}
\left\{\frac{1}{n+1}\left(1+n v_{0}-v_{2}-\cdots-v_{n}\right), \ldots, \frac{1}{n+1}\left(1-v_{0}-\cdots-v_{n-1}+n v_{n}\right)\right\} \tag{9.34}
\end{equation*}
$$

where $v_{0}, \ldots, v_{n} \in \mathbb{R}$. Thus, $\bar{\mu}_{\text {alg }}^{*}(f)=(-1)^{n+1}\left(1-\delta_{n, 0}\right)$.
Remark 9.2.11. Consider $f=w^{2} x+y^{3}+z^{2}$ over $\mathbb{C}^{4}$. By the SebastianiThom equivalence, the fiber $F_{f, 0}$ is homotopy equivalent to the iterated join space $F_{w^{2} x, 0} \star F_{y^{3}, 0} \star F_{z^{2}, 0}$, where $F_{w^{2} x, 0} \simeq S^{2}, F_{y^{3}, 0} \simeq S^{0} \vee S^{0}$ and $F_{z^{2}, 0} \simeq S^{0}$, which has the homotopy-type (of the suspension of a join) of a wedge sum of spheres,

$$
\begin{equation*}
S\left(S^{2} \star\left(S^{0} \vee S^{0}\right)\right) \simeq\left(S^{2} \star S^{0}\right) \vee\left(S^{2} \star S^{0}\right) \simeq S^{3} \vee S^{3}, \tag{9.35}
\end{equation*}
$$

so $\chi\left(S^{3} \vee S^{3}\right)=2 \chi\left(S^{3}\right)-1=-1$ and $\bar{\mu}_{\text {top }}(f)=(-1)^{3+1}(1-(-1))=2$. Then $\partial f=\left(w^{2}, 3 y^{2}, 2 z, 2 w x\right)$, so $f$ has a continuum of critical points $\{(x, 0,0,0) \in$ $\left.\mathbb{C}^{4} \mid x \in \mathbb{C}\right\}$ and therefore is degenerate. Moreover, the rank of the exponent matrix $\mathrm{A}_{f}$ is 3 , so $\mathrm{A}_{f}$ is column rank deficient. The weights are not unique, however, instead given by

$$
\begin{equation*}
\left\{\frac{1}{5}\left(1+4 v_{0}-2 v_{3}\right), \frac{1}{3}, \frac{1}{2}, \frac{1}{5}\left(2-2 v_{0}+v_{3}\right)\right\} \quad v_{0}, v_{3} \in \mathbb{R} \tag{9.36}
\end{equation*}
$$

and the corresponding generalized algebraic index is simply

$$
\begin{equation*}
\bar{\mu}_{\mathrm{alg}}(f)=2+\frac{10}{1+4 v_{0}-2 v_{3}} \quad v_{0}, v_{3} \in \mathbb{R} \tag{9.37}
\end{equation*}
$$

and $\bar{\mu}_{\mathrm{alg}}^{*}(f)=2$.
Remark 9.2.12. Consider $f=x y z+w^{2}$ over $\mathbb{C}^{4}$. Then, by the SebastianiThom equivalence, the fiber $F_{f, 0}$ is homotopy equivalent to the suspension $S\left(\mathbb{T}^{2}\right) \cong S\left(S^{1} \times S^{1}\right)$. By equation (1.29), for two pointed spaces $X$ and $Y$, $S(X \times Y) \simeq(S X \wedge Y) \vee S X \vee S Y$, so the fiber $F_{f, 0}$ is homotopy equivalent to the suspension of a 2-torus,

$$
\begin{equation*}
S\left(S^{1} \times S^{1}\right) \simeq\left(S^{2} \wedge S^{1}\right) \vee S^{2} \vee S^{2} \cong S^{3} \vee S^{2} \vee S^{2} \tag{9.38}
\end{equation*}
$$

since $S^{n} \wedge S^{m} \cong S^{n+m}$ for $n, m \geqslant 0$. Moreover, $\chi\left(F_{f, 0}\right)=2 \chi\left(S^{2}\right)+\chi\left(S^{3}\right)-2=2$, so $\bar{\mu}_{\text {top }}(f)=(-1)^{3+1}(1-2)=-1$. The weights of $f$ are not unique, however, instead given by

$$
\begin{equation*}
\left\{\frac{1}{3}\left(1+2 v_{0}-v_{1}-v_{2}\right), \frac{1}{3}\left(1-v_{0}+2 v_{1}-v_{2}\right), \frac{1}{3}\left(1-v_{0}-v_{1}+2 v_{2}\right), \frac{1}{2}\right\} \tag{9.39}
\end{equation*}
$$

where $v_{0}, v_{1}, v_{2} \in \mathbb{R}$. Thus, $\bar{\mu}_{\mathrm{alg}}^{*}(f)=-1$.
REMARK 9.2.13. The examples above are special cases of the following. Consider $f=z_{0}^{a_{0}} \cdots z_{n}^{a_{n}}$ over $\mathbb{C}^{n+1}$, where $a_{0}, \ldots, a_{n} \in \mathbb{Z}_{\geqslant 0}$. Define $m=$ $\operatorname{GCD}\left(a_{0}, \ldots, a_{n}\right)$ and $\tilde{f}=z_{0}^{a_{0} / m} \cdots z_{0}^{a_{n} / m}$. As above, the fiber $F_{\tilde{f}, 0} \cong_{d} V_{\tilde{f}, 0}(1) \cong_{d}$ $\left(\mathbb{C}^{\times}\right)^{n}$ which has the homotopy type of $\mathbb{T}^{n}$. Thus, since there are $m m^{\text {th }}$-roots
of unity, one has $F_{f, 0} \cong_{d} \bigsqcup^{m}\left(\mathbb{C}^{\times}\right)^{n} \simeq \bigsqcup^{m} \mathbb{T}^{n}$. Now consider $\Sigma_{l} f=f \boxplus z^{l}$, where $l \in \mathbb{N}$. The fiber $F_{f \boxplus z^{l}, 0}$ is the $l$-fold cyclic suspension over $F_{f, 0}$, namely, $F_{f \boxplus z^{l}, 0} \simeq \bigvee^{l-1} S\left(F_{f, 0}\right)$, where

$$
\begin{align*}
S\left(F_{f, 0}\right) & =S\left(\bigsqcup^{m} \mathbb{T}^{n}\right)  \tag{9.40}\\
& \simeq \bigvee^{m-1} S^{1} \vee \bigvee^{m} S\left(\mathbb{T}^{n}\right)  \tag{9.41}\\
& \left.\simeq \bigvee^{m-1} S^{1} \vee \bigvee^{m}\left(\bigvee_{k=1}^{n} \bigvee^{(n}{ }_{k}^{n}\right) S^{k+1}\right) . \tag{9.42}
\end{align*}
$$

For $m \geqslant 1$,

$$
\begin{align*}
& F_{f \boxplus z^{l}, 0} \simeq \bigvee^{l-1}\left(\bigvee^{m-1} S^{1} \vee \bigvee^{m}\left(\bigvee_{k=1}^{n} \bigvee^{\binom{n}{k}} S^{k+1}\right)\right)  \tag{9.43}\\
& \simeq \bigvee^{(l-1)(m-1)} S^{1} \vee \bigvee^{(l-1) m}\left(\bigvee_{k=1}^{n} \bigvee^{(n}{ }_{k}^{n}\right)  \tag{9.44}\\
&\left.S^{k+1}\right)
\end{align*}
$$

For the special case $l=2$, then $\Sigma_{l} f=\Sigma f$, so

$$
\begin{equation*}
F_{\Sigma f, 0} \simeq \bigvee^{m-1} S^{1} \vee \bigvee^{m}\left(\bigvee_{k=1}^{n} \bigvee^{\binom{n}{k}} S^{k+1}\right) \tag{9.45}
\end{equation*}
$$

and, therefore,

$$
\begin{align*}
\chi\left(F_{\Sigma f, 0}\right) & =\chi\left(\bigvee^{m-1} S^{1}\right)+\chi\left(\bigvee^{m}\left(\bigvee_{k=1}^{n} \bigvee^{\binom{n}{k}} S^{k+1}\right)\right)-1  \tag{9.46}\\
& =(m-1) \chi\left(S^{1}\right)-(m-2)+m \chi\left(\bigvee_{k=1}^{n} \bigvee^{\binom{n}{k}} S^{k+1}\right)-(m-1)-1  \tag{9.47}\\
& =-(m-2)+m \sum_{k=1}^{n} \chi\left(\bigvee^{\binom{n}{k}} S^{k+1}\right)-m(n-1)-(m-1)-1  \tag{9.48}\\
& =m \sum_{k=1}^{n}\binom{n}{k} \chi\left(S^{k+1}\right)-2^{n} m+2  \tag{9.49}\\
& =2 m \sum_{k=1}^{n}\binom{n}{k}-2^{n} m+2  \tag{9.50}\\
& =2-m \delta_{n, 0} \tag{9.51}
\end{align*}
$$

Equivalently, the identities $\chi\left(\bigsqcup^{m} \mathbb{T}^{n}\right)=m \chi\left(\mathbb{T}^{n}\right)=m \delta_{n, 0}$ and $\chi\left(S \bigsqcup^{m} \mathbb{T}^{n}\right)=$ $2-\chi\left(\bigsqcup^{m} \mathbb{T}^{n}\right)$ imply $\chi\left(F_{\Sigma f, 0}\right)=2-m \delta_{n, 0}$. Hence, $\bar{\mu}_{\text {top }}(f)=(-1)^{n}\left(m \delta_{n, 0}-1\right)$, where $\Sigma f$ has $n+2$ variables.

For $n>0$, only one weight will be fixed, namely, $\omega_{n+2}=\frac{1}{2}$, so $\bar{\mu}_{\mathrm{alg}}^{*}(\Sigma f)=$ $(-1)^{n+1}$. For $n=0$, we recover Pham's result, $\bar{\mu}_{\text {alg }}^{*}(\Sigma f)=m-1$. Thus, in general, $\bar{\mu}_{\mathrm{alg}}^{*}(\Sigma f)=(-1)^{n}\left(m \delta_{n, 0}-1\right)$.

The invariant $\bar{\mu}_{\text {alg }}^{*}$ can assume all values in $\mathbb{Z}$.

Remark 9.2.14. Consider $f=z^{d}-x y^{d-1}$ over $\mathbb{C}^{3}$ with $d \in \mathbb{N}$. If $d=1$, then $f$ has no critical point at the origin, and the weights are $\left\{1, v_{1}, 1\right\}$, where $v_{1} \in \mathbb{R}$.

If $d>1$, then the weights are given by

$$
\begin{equation*}
\left\{\frac{1}{(d-1)^{2}+1}\left(1+(d-1)^{2} v_{0}-(d-1) v_{1}\right), \frac{1}{(d-1)^{2}+1}\left((d-1)-(d-1) v_{0}+v_{1}\right), \frac{1}{d}\right\} \tag{9.52}
\end{equation*}
$$

and $f$ is non-degenerate only for $d=2$. In all cases $\bar{\mu}_{\mathrm{alg}}^{*}(f)=(-1)^{2-1-1}(d-$ $1)=d-1$. Stabilizing $f$ with $m$ new variables adds the fixed weights $\left\{\frac{1}{2}, \ldots, \frac{1}{2}\right\}$, which yields $\bar{\mu}_{\text {alg }}^{*}\left(\Sigma^{m} f\right)=(-1)^{m}(d-1)$.

Degeneracy does not necessarily imply a non-trivial free part.

REMARK 9.2.15. Consider $f=x^{d}+y^{d}+x y z^{d-2}$ over $\mathbb{C}^{3}$ with $d \in \mathbb{N}_{>1}$. If $d=2$, then $f$ is degenerate, the weights are $\left\{\frac{1}{2}, \frac{1}{2}, v_{2}\right\}$, where $v_{2} \in \mathbb{R}$, and $\bar{\mu}_{\mathrm{alg}}^{*}(f)=-1$. If $d>2$, then $f$ is degenerate and the weights are $\left\{\frac{1}{d}, \frac{1}{d}, \frac{1}{d}\right\}$ and $\bar{\mu}_{\mathrm{alg}}(f)=\bar{\mu}_{\mathrm{alg}}^{*}(f)=(d-1)^{3}$.

The identity $\bar{\mu}_{\text {alg }}=\bar{\mu}_{\text {alg }}^{*}$ does not imply that the weights are fixed.
REMARK 9.2.16. Consider $f=x y+z^{2}$ over $\mathbb{C}^{3}$, which has an isolated critical point at the origin. The fiber $F_{f, 0} \simeq S\left(S^{1}\right) \simeq S^{2}$. Thus, $\bar{\mu}_{\text {top }}(f)=(-1)^{3}(1-2)=$ 1. The rank of the exponent matrix $A_{f}$ is 2 , so it is column rank deficient. The weights of $f$ are

$$
\begin{equation*}
\left\{\frac{1}{2}\left(1+v_{0}-v_{1}\right), \frac{1}{2}\left(1-v_{0}+v_{1}\right), \frac{1}{2}\right\} \quad v_{0}, v_{1} \in \mathbb{R} . \tag{9.53}
\end{equation*}
$$

However, the generalized algebraic index is constant

$$
\begin{equation*}
\bar{\mu}_{\mathrm{alg}}(f)=\left(\frac{2}{1+v_{0}-v_{1}}-1\right)\left(\frac{2}{1-v_{0}+v_{1}}-1\right)=1, \tag{9.54}
\end{equation*}
$$

so $\bar{\mu}_{\mathrm{alg}}^{*}(f)=1$.
Definition 9.16. A strongly degenerate, weighted homogeneous polynomial is a weighted homogeneous polynomial with at least one zero weight.

Conjecture 9.17. Given a weighted homogeneous polynomial $f$ that is not strongly degenerate, then the generalized topological, K-theoretic and fixed algebraic indices coincide, i.e.,

$$
\begin{equation*}
\bar{\mu}_{\mathrm{top}}(f)=\bar{\mu}_{K}(f)=\bar{\mu}_{\mathrm{alg}}^{*}(f) \tag{9.55}
\end{equation*}
$$

Remark 9.2.17. Conjecture 9.17 makes no claim about the dimension of the singular set of $f$ at the origin. In general, it is not true that $\bar{\mu}_{\text {top }}(f)$ is the rank of the middle homology group of $F_{f, 0}$ (which may not be homotopy equivalent to a wedge sum of spheres of the same dimension) or the rank of the middle Grothendieck group.

Corollary 9.18. If Conjecture 9.17 is true, then Conjecture 9.15 is true.

Proposition 9.19. Let $U_{\alpha} \subseteq \mathbb{C}^{n_{\alpha}}$ be a set of neighborhoods of the origin. If given weighted homogeneous polynomials $f_{\alpha}:\left(U_{\alpha}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ none strongly degenerate, then $f=f_{1} \boxplus \cdots \boxplus f_{m}$ satisfies

$$
\begin{equation*}
\bar{\mu}_{\text {top }}^{*}(f)=\prod_{i=1}^{m} \bar{\mu}_{\text {top }}\left(f_{i}\right) . \tag{9.56}
\end{equation*}
$$

Proof. The identity follows from Massey's generalization of the SebastianiThom equivalence to non-isolated singularities of arbitrary dimension, q.v., §9.1, and the multiplicative identity $\tilde{\chi}(X \star Y)=\tilde{\chi}(X) \tilde{\chi}(Y)$ for any two pointed CW complexes $X$ and $Y$.

Corollary 9.20. Given a weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ that is not strongly degenerate, then

$$
\begin{equation*}
\bar{\mu}_{\text {top }}(\Sigma f)=\bar{\mu}_{\text {top }}(f) \tag{9.57}
\end{equation*}
$$

Proposition 9.21. Given a (possibly degenerate) weighted homogeneous polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ that is not strongly degenerate such that the corresponding fiber $F_{f, 0}$ is homotopy equivalent to a wedge sum of $M$ spheres of arbitrary odd dimension and $N$ spheres of arbitrary even dimension, then

$$
\begin{equation*}
\bar{\mu}_{\mathrm{top}}(f)=(-1)^{n}(N-M) \tag{9.58}
\end{equation*}
$$

Proof. The Euler characteristic of $S^{n}$ is 2 if $n$ is even and 0 otherwise. Since the reduced Euler characteristic is additive over wedge sums, one has $\chi\left(\bigvee_{i=1}^{m} S^{n_{i}}\right)=2 N-(m-1)=N-M+1$, where $m=N+M$, and $N($ resp., $M)$
counts the even (resp., odd) spheres in the wedge sum. Thus,

$$
\begin{equation*}
\bar{\mu}_{\mathrm{top}}(f)=(-1)^{n+1}\left(1-\chi\left(F_{f, 0}\right)\right)=(-1)^{n+1}(1-(N-M+1)), \tag{9.59}
\end{equation*}
$$

as claimed.

Remark 9.2.18. In the isolated singularity case, the fiber $F_{f, 0}$ is a wedge sum of spheres whose common dimension (and therefore parity) is determined by $n$. That is, if $n$ is even, then $F_{f 0}$ is homotopy equivalent to a wedge sum of only even spheres, while if $n$ is odd, then $F_{f 0}$ is homotopy equivalent to a wedge sum of only odd spheres. In the former case, one has $\bar{\mu}_{\mathrm{alg}}(f)=(-1)^{n} N$, while in the latter, $\bar{\mu}_{\mathrm{ag}}(f)=(-1)^{n+1} M$ by Corollary 1.28. It follows that $\bar{\mu}_{\mathrm{alg}}^{*}$ generalizes $\bar{\mu}_{\mathrm{alg}}$, which supports the claim that $\bar{\mu}_{\mathrm{alg}}^{*}$ is a topological invariant.

Proposition 9.22. If $f_{i}=z_{0}^{a_{i 0}} \cdots z_{n_{i}}^{a_{i n_{i}}}$, then the Milnor fiber $F_{f, 0}$ of $f=f_{1} \boxplus$ $\cdots \boxplus f_{s}$ has the homotopy type of the iterated join space $F_{f_{1}, 0} \star \cdots \star F_{f_{r}, 0}$, where $F_{f_{i}, 0} \simeq$ $\bigsqcup^{m_{i}} \mathbb{T}^{n_{i}}, m_{i}=\operatorname{GCD}\left(a_{i 0}, \ldots, a_{i n}\right)$.

### 9.3. Non-Weighted Homogeneous Polynomials

It is a rather curious, but nonetheless potentially useful, fact that one may assign weights to any complex analytic polynomial, whether weighted homogeneous or not, that vanishes at the origin.

Definition 9.23. Given a complex analytic polynomial $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow$ $(\mathbb{C}, 0)$ with exponent matrix $\mathrm{A}_{f}$, the generalized weights $\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ of $f$ are
the elements of the vector $\mathrm{A}_{f}^{+} \mathbf{1}_{m}+\left(\mathrm{I}-\mathrm{A}_{f}^{+} \mathrm{A}_{f}\right) \mathbf{v}$, where $\mathbf{v}$ is an arbitrary vector in $\mathbb{R}^{n+1}$.

REMARK 9.3.1. The polynomial $f=x^{5}+y^{5}+x^{3} y^{3}$ over $\mathbb{C}^{2}$ is almost quasihomogeneous and not quasihomogeneous and, therefore, not weighted homogeneous, q.v., Remark 2.4.11. As $\left(I-\mathrm{A}_{f}^{+} \mathrm{A}_{f}\right)=\mathrm{Z}_{2}$, the $2 \times 2$ zero matrix, it follows that

$$
\begin{align*}
\omega & =\mathrm{A}_{f}^{+} \mathbf{1}_{3}  \tag{9.60}\\
& =\left(\begin{array}{ccc}
\frac{34}{215} & -\frac{9}{215} & \frac{3}{43} \\
-\frac{9}{215} & \frac{34}{215} & \frac{3}{43}
\end{array}\right)\left(\begin{array}{l}
1 \\
1 \\
1
\end{array}\right)  \tag{9.61}\\
& =\binom{\frac{8}{43}}{\frac{8}{43}} . \tag{9.62}
\end{align*}
$$

Therefore, the generalized weights of $f$ are $\left\{\frac{8}{43}, \frac{8}{43}\right\}$. Similarly, $g=\left(x^{4}+\right.$ y) $\left(x^{9}+y^{2}\right)$ over $\mathbb{C}^{2}$ is neither almost quasihomogeneous nor quasihomogeneous and, therefore, not weighted homogeneous. The generalized weights of $g$ are $\left\{\frac{262}{3435}, \frac{1154}{3435}\right\}$. Note that neither $f$ and $g$ satisfy a weighted homogeneous transformation law or a weighted Euler equation with these or any weights. $\Delta$

Problem 9.3.1. Determine whether or not the generalized weights of an arbitrary complex analytic polynomial which vanishes at the origin have any analytic, algebraic, geometric, combinatorial or topological significance.

This concludes our review and analysis of complex analytic singularities and certain generalizations thereof. We proceed now to surprisingly similar structures in supersymmetric quantum field theory.

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## Part 2

## Supersymmetry and Quantum Field Theory

$$
\begin{aligned}
& \text { Signed d'indication. } \\
& -\theta \rightarrow-\theta) \rightarrow+\theta+\theta+\theta)
\end{aligned}
$$

## Chapter 10

## Supersymmetry

Tiger! Tiger! burning bright
In the forest of the night,
What immortal hand or eye
Could frame thy fearful symmetry?

- William Blake


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### 10.1. The Standard Model

According to the Standard Model $(\mathrm{SM})$ with gauge group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times$ $\mathrm{U}(1)$, eight massless, spin-0 bosons (the gluons, $\{g\}$ ), three massive, spin- 1 vector bosons (the two charged and neutral weak bosons, $W^{+}, W^{-}$and $Z$ ) and a massless, spin-1 scalar boson (the photon, $\gamma$ ) mediate three fundamental interactions in the observable universe*: the strong force, the electroweak force and the electromagnetic force, respectively. Predicted in 1968 by the Glashow-WeinbergSalam Model (GWSM) with gauge group $\operatorname{SU}(2) \times \mathrm{U}(1)$ and observed indirectly

[^38]in 1973 (Gargamelle Bubble Chamber) and directly in 1983 (UAi/2) at CERN [207], the mediators of the short-range weak nuclear force, the intermediate vector bosons, $W^{ \pm}\left(m_{W}=80.385 \mathrm{GeV}\right)$ and $Z\left(m_{Z}=91.1876 \mathrm{GeV}\right)$, are responsible for the stability of all interacting matter via nuclear transmutation by beta decay and electron capture. While the gluons and photon are massless by local gauge invariance, an unbroken $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge symmetry requires massless vector bosons. The GWSM* solves this mass discrepancy and preserves renormalizability by invoking the Higgs mechanism [126, 190, 191], which purports the existence of a precursor quantum field, the elusive Higgs field, that spontaneously breaks $S U(2) \times U(1)$ and manifests as a massless scalar, the photon, three asymmetrically massive vector bosons, $W^{ \pm}$and $Z$, and a massive scalar, the Higgs boson. The resulting bosons are then available to couple with elementary fermionic particles or leptons, such as the electron $e^{-}$, muon $\mu^{-}$, tauon $\tau^{-}$ and their corresponding neutrinos $v_{e^{-}}, v_{\mu^{-}}$and $v_{\tau^{-}}$.

The GWSM alone is insufficient to predict a light Higgs boson mass, in contrast to that of the photon and the ratio of those of the weak bosons. Instead the GWSM implies that it be directly proportional to an unconstrained variable, the Higgs boson self-coupling parameter, $\lambda$, by the relation $m_{h}=\sqrt{2 \lambda} v_{h}$, where $v_{h}$
${ }^{*}$ More precisely, the GWSM postulates an $\mathrm{SU}(2)_{L} \times \mathrm{U}(1)_{Y}$ invariant Lagrangian containing four massless (precursor) scalar fields $A_{1}, A_{2}, A_{3}$ and $B$ and a single complex (Higgs) doublet $\Phi$. The Higgs field is a left-handed doublet with weak isospin $+\frac{1}{2}$ and hypercharge +1 that preserves $U(1)_{E M}$ but spontaneous breaks $S U(2)_{L} \times U(1)_{Y}$, resulting in a non-zero vacuum expectation value of the Higgs field $v_{h}$, two charged, massive vector bosons, $W^{+}$and $W^{-}$(from linear combinations of $A_{1}$ and $A_{2}$ ) and a neutral, massive vector boson, $Z$ (from linear combinations of $A_{3}$ and $B$ ), a massless photon, $\gamma\left(\right.$ from linear combinations of $A_{3}$ and $B$ ), and a massive scalar $h$, the Higgs boson.
is the vacuum expectation value of the Higgs boson. Precise muon lifetime experiments incorporating two-loop, Quantum Electrodynamic (QED) corrections yield a Fermi coupling $G_{F} \approx 1.166364(5) \times 10^{-5} \mathrm{GeV}^{-2}$ (CODATA 2010), from which one infers the value $v_{h}=\frac{1}{\sqrt[4]{2} \sqrt{G_{F}}} \approx 246.221 \mathrm{GeV}$. By imposing (renormalization group-improved) unitarity bounds on the corresponding elastic scattering amplitudes, one derives the upper bound $m_{h} \leqslant 2 \sqrt[4]{2} \sqrt{\frac{\pi}{3 G_{F}}} \approx 712.664 \mathrm{GeV}$ [260, 283]. Enhancing further the GWSM with a Yang Mills SU(3)-gauge theory, Quantum Chromodynamics (QCD), yields the SM with an additional six massive, color-charged spin- $\frac{1}{2}$ fermions or quarks (up $u$, down $d$, strange $s$, charm $c$, top $t$ and bottom $b$ ) and, with their antiparticles, conspire in pairs to form the meson families (e.g., $\pi, \eta, K, D$ and $B$ ) and in triplets* to form the baryon families (e.g., nucleons, $\Lambda, \Delta, \Sigma, \Xi$, and $\Omega$ ) through the strong interaction. However, isolated quarks or anti-quarks are believed to be essentially unobservable due to their low-energy confinement $[164,384]$ and high-energy asymptotic freedom [163] which allows only a rather weak coupling with gluons. In total, there are eighteen parameters ${ }^{\dagger}$ which determine the SM: three gauge coupling parameters, three charged lepton masses, six quark masses, three flavor mixing angles, one charge-parity (CP)-violating phase, the Higgs boson mass and vacuum

[^39]expectation value (determined by the masses of the $W^{ \pm}$and $Z$ vector bosons) [396].

While the literature is rich with theoretical proposals that engage electroweak symmetry-breaking and the Higgs mechanism* in more appealing ways, the SM is most likely the simplest and definitely the most wellunderstood. According to the SM , the three neutrinos $\left(v_{e}, v_{\mu}\right.$ and $v_{\tau}$ ) and their antiparticles are massless spin- $\frac{1}{2}$ fermions. However, experimental evidence suggest neutrino oscillations between flavor types, which a priori require massive neutrinos [102]. Coupling parameter unification (e.g., grand unification), baryon asymmetry, hierarchy problem, dark matter, naturalness, etc., are additional issues which are not addressed by SM, per se. Therefore, if one is to properly model the universe (sans gravity), the SM must be modified, extended and/or subsumed accordingly.

### 10.2. Supersymmetry

10.2.1. Coleman-Mandula Theorem. In 1967, Coleman and Mandula proved a remarkable theorem which greatly restricts the local symmetries possessed by a quantum field theory with a mass gap.

Theorem 10.1 (Coleman, Mandula, [88]). Let $G$ be an arcwise-connected symmetry group of the S-matrix (in the weak operator topology), where $S=1-$ $i(2 \pi)^{4} \delta\left(P_{\mu}-P_{\mu}^{\prime}\right) T$ such that the following conditions hold:

[^40]1. (Lorentz Invariance) G contains a subgroup locally isomorphic to the Poincaré group;
2. (Particle Finiteness) All particle types correspond to positive-energy representations of the Poincaré group. For any positive real $M$, there are finitely many particle types of mass less than M;
3. (Weak Elastic Analyticity) Elastic-scattering amplitudes are analytic functions of the center-of-mass energy s and invariant momentum transfer $t$ in some neighborhood of the physical region, except at normal thresholds;
4. (Occurrence of Scattering) Let $|p\rangle$ and $\left|p^{\prime}\right\rangle$ be any two one-particle momentum eigenstates, and let $\left|p, p^{\prime}\right\rangle$ be the two-particle state created from these. Then $T\left|p, p^{\prime}\right\rangle$ does not vanish except perhaps for certain isolated values of $s$; and
5. (An Ugly Technical Assumption) The generators of $G$, written as integral operators in momentum space, have distributions for their kernels.

Then, $G$ is necessarily locally isomorphic to the direct product of an internal symmetry group and the Poincaré group.
10.2.2. Supersymmetry. Supersymmetry (SUSY) is a conjectured symmetry of nature between integer-spin particles, the mediators of the fundamental forces, and half-integer-spin particles, the constituents of matter. In dimensions three and greater, a given Lagrangian represents a supersymmetric quantum theory if and only if there exists an infinitesimal field transformation interchanging
the integer and half-integer spin fields and admitting an equivalent representation as a graded Lie algebra of field operators.*

Given a boson $b \in \mathfrak{B}$ and fermion $f \in \mathfrak{F}$, where $\mathfrak{B}$ and $\mathfrak{F}$ are suitable Fock spaces, the images $\hat{b}=Q b$ and $\hat{f}=Q f$ of a supersymmetric charge $Q$ are the corresponding super-partners-the former a super-fermion, the latter a superboson. In theories with unbroken supersymmetry, the mass of super-partners is identical to their partners, while in those with broken supersymmetry, the mass of super-partners is comparatively larger, and may explain why no superpartners have yet been observed.

Although SUSY may have first been anticipated in the mathematical work of Frölicher and Nijenhuis [137, 138] and perhaps rediscovered by Miyazawa [318, 319], it is generally believed to have been introduced independently in the physics literature by Golfand and Likhtman [152], Volkov and Akulov [464], and Wess and Zumino [470]. In particular, Wess and Zumino introduced a renormalizable four-dimensional supersymmetric quantum field theory with cubic interaction.

Haag, Lopuszanski and Sohnius [175] generalized the Coleman-Mandula Theorem to formally include SUSY as a space-time symmetry by considering Lie super-algebras containing both commuting (even degree) and anticommuting (odd degree) generators. As a direct consequence of the addition of odd generators, certain quantum field theories circumvent the restriction of

[^41]the Coleman-Mandula Theorem and may therefore exhibit surprisingly larger spacetime symmetry groups. In this way, it was shown that supersymmetry is the most general (local) symmetry allowed in four-dimensional Minkowski space-time.
10.2.3. Supersymmetry and the Standard Model. In a supersymmetric extension of the SM, namely, the Minimally Supersymmetric Standard Model (MSSM), a type III THDM proposed by Dimopoulos and Georgi [112], the squared-mass of the light, CP even, scalar component of the Higgs field, the Higgs boson, is independently quadratically and logarithmically divergent in a sharp momentum cut-off [113]. However, certain quark-squark ${ }^{*}$ interactions provide perturbative counter-terms that dramatically suppress such divergences, which is one of the many appealing features of SUSY. In particular, the MSSM with soft SUSY-breaking (near the electroweak scale) postulates two Higgs doublets leading to five potentially observable Higgs particles: two vector bosons, $H^{+}$and $H^{-}$, two CP even scalars, $h$ and $H$, and a CP odd scalar, $A$, satisfying the following mass inequalities at tree level ${ }^{\dagger}$ : $m_{W^{ \pm}} \leqslant m_{H^{ \pm}} \leqslant m_{H}, m_{h} \leqslant m_{Z} \leqslant m_{H}$ and $m_{h} \leqslant m_{A} \leqslant m_{H^{ \pm}}$, respectively [170]. At one-loop level, the MSSM predicts an explicit upper bound on the light Higgs boson mass $m_{h}$ within the decoupling
*In the MSSM, superpartners also share gauge numbers (viz., color charge, weak isospin charge, hypercharge).
${ }^{\dagger}$ This is the lowest order in perturbation theory and considers only interactions with loopless Feynman diagrams, hence the name.
limit ${ }^{*}$ through the quartic coupling contributions from the aforementioned vector bosons and (broken supersymmetric) radiative corrections from the top-stop quark sector with mixing parameter $\alpha$,
$$
m_{h}^{2} \leqslant m_{Z}^{2}+\frac{3 G_{F}}{\sqrt{2} \pi^{2}}\left(m_{t, 1}^{4} \log \frac{m_{\tilde{t}}}{m_{t, 1}}+m_{t, 2}^{4} \alpha^{2}\left(6-3 \alpha^{2}\right)\right)
$$
where (in natural units) the pole top quark mass $m_{t}=172.9 \mathrm{GeV}$ and is given at two different energy scales, $m_{t, 1}=157 \mathrm{GeV}$ and $m_{t, 2}=150$. GeV [115]. At maximal mixing ( $\alpha=1$ ) and conjecturing $m_{\tilde{t}} \approx 1 \mathrm{TeV}$, one computes $m_{h} \leqslant 132$ GeV [172]. However, neglecting stop mixing, one computes the upper bound $m_{h} \leqslant 110 \mathrm{GeV}$ [115], which violates the LEP exclusion $m_{h}>114.4 \mathrm{GeV}$ [265].
10.2.4. Recent Discovery of a New Boson. By early 2010, groups at the Tevatron at Fermilab and the Large Hadron Collider (LHC) working independently observed curious activity in $p p$-collisions in the range $115-130 \mathrm{GeV}$. As of 2011, the CMS and ATLAS experiments at CERN improved known bounds for a light Higgs boson by exclusion to the interval $114 \mathrm{GeV} \leqslant m_{h} \leqslant 157 \mathrm{GeV}$ (at $90-95 \%$ confidence), consistent with a TeV -scale stop mass, maximal mixing in the decoupling limit and the MSSM upper bound. By mid 2012, CERN announced the observation of a new boson with a mass of approximately $125.3^{\dagger}$ GeV and decay channels consistent with those of a light Higgs boson predicted

[^42]by the SM [24]. While many anticipate a full resolution of the experimental search for a Higgs boson in the very near future, a complete physical model predicting precisely its mass remains hitherto undiscovered.

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## Chapter 11

# Twist-Regularized Wess-Zumino Model 

Arguments are to be avoided; they are always vulgar and often convincing.<br>- Oscar Wilde

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In this chapter we consider operator-valued distributions on loop space $S^{1}$ with twisted boundary conditions. The twist acts as an infrared-regulator and allows for a careful study of the massless sector of a generalized Wess-Zumino model, the $\mathrm{WZ}_{\theta, \phi}$ model.

### 11.1. Supersymmetry, Revisited

Let $\mathfrak{B}$ and $\mathfrak{F}$ denote Fock spaces of all interacting bosonic and fermionic elementary particles comprising the universe, respectively. The bosons include the photon, gluon, W, Z and Higgs bosons, their anti-particles and super-fermions. The fermions include the up, down, charm, strange, top and bottom quarks and electron, muon and tau including their corresponding neutrinos, their antiparticles and super-bosons. From a mathematical perspective, supersymmetry
is a conjectured symmetry that purports the existence of a fundamental map between particles of integral (bosons) and half-integral (fermions) spin.

Conjecture 11.1 (Supersymmetry). There is an involution $\sigma: \mathfrak{B} \rightarrow \mathfrak{F}$.
11.1.1. Graded Fock Space. Let $\mathscr{F}=\mathscr{F}^{b} \otimes \mathscr{F}^{f}$ denote the bosonicfermionic, product Fock-Hilbert space, which is the tensor product Hilbert space of the symmetric tensor $\mathbb{C}$-algebra over the even component, the bosonic Hilbert space,

$$
\begin{align*}
\mathscr{H}^{b} & \cong \bigoplus_{j=1}^{n} L^{2}(\mathbb{T}, d x) \oplus L^{2}(\mathbb{T}, d x)  \tag{11.1}\\
& \cong \bigoplus_{j=1}^{n} \ell^{2}(\hat{\mathbb{T}}, d x) \oplus \ell^{2}(\hat{\mathbb{T}}, d x) \tag{11.2}
\end{align*}
$$

and the exterior tensor $\mathbb{C}$-algebra over the odd component, the fermionic Hilbert space,

$$
\begin{align*}
\mathscr{H}^{f} & \cong \bigoplus_{j=1}^{n} L^{2}(\mathbb{T}, d x) \oplus L^{2}(\mathbb{T}, d x)  \tag{11.3}\\
& \cong \bigoplus_{j=1}^{n} \ell^{2}(\hat{\mathbb{T}}, d x) \oplus \ell^{2}(\hat{\mathbb{T}}, d x) \tag{11.4}
\end{align*}
$$

In particular, we write*

$$
\begin{align*}
& \mathscr{F}^{b}:=\operatorname{Sym}\left(\mathscr{H}^{b}\right)=\bigoplus_{k \geqslant 0}\left(\mathscr{H}^{b}\right)^{\otimes_{s} k}  \tag{11.5}\\
& \mathscr{F}^{f}:=\bigwedge\left(\mathscr{H}^{f}\right)=\bigoplus_{k \geqslant 0}\left(\mathscr{H}^{f}\right)^{\wedge k}, \tag{11.6}
\end{align*}
$$

where $\left(\mathscr{H}^{b}\right)^{\otimes_{s} k}:=\operatorname{Sym}_{j=1}^{k} \mathscr{H}^{b}$ and $\left(\mathscr{H}^{f}\right)^{\wedge k}:=\bigwedge_{j=1}^{k} \mathscr{H}^{f}$, respectively. Here, we define the zero-particle Hilbert spaces $\left(\mathscr{H}^{b}\right)^{\otimes_{s} 0}:=\mathbb{C}$ and $\left(\mathscr{H}^{f}\right)^{\wedge 0}:=\mathbb{C}$, respectively.

Define the space of even-degree elements $\mathscr{F}^{0}$ and odd-degree elements $\mathscr{F}^{1}$. In contrast to the tensor product space $\mathscr{F}$, the direct sum decomposition $\mathscr{F}=$ $\mathscr{F}^{0 \mid 1}:=\mathscr{F}^{0} \oplus \mathscr{F}^{1}$ is a $\mathbb{Z}_{2}$-graded Fock-Hilbert space. In particular, $\mathscr{H}^{0 \mid 1}:=$ $\mathscr{H}^{0} \oplus \mathscr{H}^{1}$ is a $\mathbb{Z}_{2}$-graded one-particle Hilbert space, a Hilbert super-space. Such spaces are natural ambient spaces in which to study supersymmetry.

REMARK 11.1.1. Although $\mathscr{H}^{0} \cong \mathscr{H}^{b}$ and $\mathscr{H}^{1} \cong \mathscr{H}^{f}$, in general, $\mathscr{F}^{0} \nsupseteq \mathscr{F}^{b}$ and $\mathscr{F}^{1} \not \not \equiv \mathscr{F}^{f}$.
11.1.2. Superalgebra. Let $k$ be a field and $G$ be a monoid. A $k$-algebra $\mathcal{A}$ is G-graded if it has the direct sum decomposition $\mathcal{A}=\oplus_{d \in G} \mathcal{A}_{d}$ such that for each $f \in \mathcal{A}_{d_{1}}$ and $g \in \mathcal{A}_{d_{2}}$, then $f g \in \mathcal{A}_{d_{1}+d_{2}}$. A $\mathbb{Z}_{2}$-graded algebra $\mathfrak{g}=\mathfrak{g}_{0} \oplus \mathfrak{g}_{1}$ with an involution $\Gamma$ is a (Lie) superalgebra if there is a bilinear bracket $[,, \cdot]$ : $\mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ which respects the grading and satisfies a $\mathbb{Z}_{2}$-graded Jacobi identity.
${ }^{*}$ We do not write $\mathscr{F}^{b}=\exp _{\otimes_{s}} \mathscr{H}^{b}$ and $\mathscr{F}^{b}=\exp \wedge_{\wedge} \mathscr{H}^{f}$ because the coefficients in the formal series expansion of the two tensor product exponentials suggest a normalization of the inner product in each of the tensor product Hilbert spaces that is inconsistent with our definitions.

Consider $f \in \mathfrak{g}$ and define the conjugation $f^{\Gamma}=\Gamma^{-1} f \Gamma$. If $f^{\Gamma}=f$, then $f$ is said to be even. If $f^{\Gamma}=-f$, then $f$ is said to be odd. These equalities can be restated in terms of commutator and anticommutator relations, $[f, \Gamma]=0$ and $\{f, \Gamma\}=0$, respectively.
11.1.3. $\mathbb{Z}_{2}$-Graded Lie Algebras. Define the $\mathbb{Z}_{2}$-graded commutator $[\cdot, \cdot]_{\Gamma}$ as the $\mathbb{Z}_{2}$-graded skew-symmetric Lie bracket,

$$
\begin{align*}
{[f, g]_{\Gamma} } & =f g-(-1)^{(\operatorname{deg} f)(\operatorname{deg} g)} g f  \tag{11.7}\\
& =\left\{\begin{aligned}
{[f, g]=f g-g f } & \text { if }(\operatorname{deg} f)(\operatorname{deg} g)=0 \\
\{f, g\}=f g+g f & \text { if }(\operatorname{deg} f)(\operatorname{deg} g)=1
\end{aligned}\right. \tag{11.8}
\end{align*}
$$

satisfying $[f, g]_{\Gamma}=-(-1)^{(\operatorname{deg} f)(\operatorname{deg} g)}[g, f]_{\Gamma}$. We will call the odd Lie bracket $[f, g]$ the commutator and the even Lie bracket $\{f, g\}$ the anti-commutator of $f$ and $g$, respectively.

Definition 11.2. A $\mathbb{Z}_{2}$-graded Lie algebra $\mathfrak{g}_{\Gamma}$ is a finitely-generated $\mathbb{Z}_{2^{-}}$ graded algebra over a (unital) commutative ring $R$ equipped with a $\mathbb{Z}_{2}$-graded skew symmetric endomorphism $[\because, \cdot]_{\Gamma}: \mathfrak{g}_{\Gamma} \times \mathfrak{g}_{\Gamma} \rightarrow \mathfrak{g}_{\Gamma}$ satisfying the $\mathbb{Z}_{2}$-graded Jacobi identity,

$$
\begin{aligned}
0=( & -1)^{(\operatorname{deg} h)(\operatorname{deg} f)}[f,[g, h]]+(-1)^{(\operatorname{deg} f)(\operatorname{deg} g)}[g,[h, f]] \\
& +(-1)^{(\operatorname{deg} g)(\operatorname{deg} h)}[h,[f, g]],
\end{aligned}
$$

for all $f, g, h \in \mathfrak{g}_{\Gamma}$. In particular, $[f, f]_{\Gamma}=0$ for all even elements $f$ (i.e., degree 0 ) and $\left[[g, g]_{\Gamma}, g\right]_{\Gamma}=0$ for all odd $g$ elements (i.e., degree 1) in $\mathfrak{g}_{\Gamma}$.

We now explicitly construct a set of operators which generate a (1, 1)spacetime supersymmetry algebra.
11.1.4. (1,1)-Spacetime Supersymmetry. Supersymmetry in ( 1,1 )-spacetime consists of the following data: Five degree-zero operators $H, P, K, Z$ and $I$ and three degree-one operators $Q_{+}, Q_{-}$and $\Gamma$, each acting on $\mathscr{F}$ and satisfying the following relations: $\left[Q_{+}, Q_{-}\right]_{\Gamma}=2 Z,[\Gamma, \Gamma]_{\Gamma}=2 I$,

$$
\begin{align*}
& {\left[Q_{+}, Q_{+}\right]_{\Gamma}=2[K, H+P]_{\Gamma}=4\left[K, Q_{+}\right]_{\Gamma}=2(H+P)}  \tag{11.9}\\
& {\left[Q_{-}, Q_{-}\right]_{\Gamma}=2[H-P, K]_{\Gamma}=4\left[Q_{-}, K\right]_{\Gamma}=2(H-P) .} \tag{11.10}
\end{align*}
$$

Specifically, there are two supersymmetric charges, or supercharges, $Q_{+}$and $Q_{-}$such that $Q_{+}^{2}=H+P$ and $Q_{-}^{2}=H-P$, the anti-linear involution or odd-parity operator $\Gamma$ (i.e., $\mathbb{Z}_{2}$-grading), the generator of temporal translations $H$ (i.e., Hamiltonian operator), the generator of spatial translations $P$ (i.e., momentum operator), the generator of boosts $K$, a central (charge) operator $Z$, and even-parity operator $I$ (i.e., the identity operator).

### 11.2. Supersymmetric Quantum Mechanics

In [219], Jaffe, Lesniewski and Lewenstein construct a Wess-Zumino (Holomorphic, Supersymmetric) Quantum Mechanics with an explicit supersymmetry algebra, which we now describe. Consider the following four generators of
a Clifford algebra,

$$
\gamma_{0}=\left(\begin{array}{ll}
0 & I  \tag{11.11}\\
I & 0
\end{array}\right) \quad \text { and } \quad \gamma_{j}=\left(\begin{array}{cc}
0 & i \sigma_{j} \\
-i \sigma_{j} & 0
\end{array}\right) \quad 1 \leqslant j \leqslant 3
$$

where $\sigma_{j}$ is a Pauli matrix. Define the $\mathbb{Z}_{2^{2}}$-grading operator

$$
\Gamma=\gamma_{0} \gamma_{1} \gamma_{2} \gamma_{3}=\left(\begin{array}{cc}
I & 0  \tag{11.12}\\
0 & -I
\end{array}\right) .
$$

Note that $\Gamma$ has eigenvalues $\pm 1$ (each with multiplicity 2) and that $\Gamma^{2}$ is the $4 \times 4$ identity matrix. Define two time-zero fermions: $\psi_{1}=\frac{1}{2}\left(\gamma_{0}-i \gamma_{3}\right)$ and $\psi_{2}=$ $\frac{1}{2}\left(\gamma_{1}-i \gamma_{2}\right)$ with adjoints $\bar{\psi}_{1}=\frac{1}{2}\left(\gamma_{1}+i \gamma_{2}\right)=\psi_{2}^{*}$ and $\bar{\psi}_{2}=\frac{1}{2}\left(\gamma_{0}+i \gamma_{3}\right)=\psi_{1}^{*}$, which satisfy the following canonical anticommutation relations, $\left\{\bar{\psi}_{1}, \psi_{2}\right\}=$ $\left\{\bar{\psi}_{2}, \psi_{1}\right\}=1$ and $\left\{\psi_{i}, \psi_{j}\right\}=\left\{\bar{\psi}_{i}, \bar{\psi}_{j}\right\}=0$. Consider one complex boson $\varphi$ (which may be viewed as a simple complex variable $z$ ), and define a bosonic potential $V(\varphi)=\varphi^{n}$ with $n \in \mathbb{N}_{>1}$. The Wess-Zumino Lagrangian is

$$
\begin{equation*}
\mathscr{L}=|\dot{\varphi}|^{2}+i\left(\psi_{1}^{*} \dot{\psi}_{1}+\psi_{2}^{*} \dot{\psi}_{2}\right)+\psi_{2}^{*} \psi_{1} \partial^{2} V+\psi_{1}^{*} \psi_{2}\left(\partial^{2} V\right)^{*}-|\partial V|^{2} \tag{11.13}
\end{equation*}
$$

The action is invariant under the following infinitesimal SUSY transformation,

$$
\begin{align*}
\delta \varphi & =\psi_{2}^{*} \theta  \tag{11.14}\\
\delta \bar{\varphi} & =\bar{\theta} \psi_{2}  \tag{11.15}\\
\delta \psi_{1} & =-(\partial V)^{*} \theta  \tag{11.16}\\
\delta \psi_{1}^{*} & =(\partial V) \theta  \tag{11.17}\\
\delta \psi_{2} & =i \dot{\varphi} \theta  \tag{11.18}\\
\delta \psi_{2}^{*} & =i \dot{\bar{\varphi}} \bar{\theta} \tag{11.19}
\end{align*}
$$

where $\theta$ is an arbitrary Grassman number, with corresponding conserved charges,

$$
\begin{align*}
& Q_{1}=i \psi_{2}^{*} \partial-i \psi_{1}^{*}(\partial V)^{*}  \tag{11.20}\\
& Q_{2}=i \psi_{2} \bar{\partial}+i \psi_{1}(\partial V) \tag{11.21}
\end{align*}
$$

Consider the sum $Q=Q_{1}+Q_{2}$. Then, upon squaring,

$$
\begin{align*}
Q^{2} & =-\bar{\partial} \partial-\psi_{2}^{*} \psi_{1} \partial^{2} V-\psi_{1}^{*} \psi_{2}\left(\partial^{2} V\right)^{*}+\left|\partial^{2} V\right|^{2}  \tag{11.22}\\
& =H . \tag{11.23}
\end{align*}
$$

To yield another representation, define the projection operator,

$$
\begin{align*}
Q_{-} & =i\left(\sigma_{+} \partial+\sigma_{-} \bar{\partial}\right)+\frac{1}{2}\left[\left(1+\sigma_{3}\right)(\partial V)-\left(1-\sigma_{3}\right)(\partial V)^{*}\right]  \tag{11.24}\\
& =\left(\begin{array}{cc}
\partial V & i \partial \\
i \bar{\partial} & -(\partial V)^{*}
\end{array}\right) \tag{11.25}
\end{align*}
$$

where $\sigma_{ \pm}=\frac{1}{2}\left(\sigma_{1} \pm i \sigma_{2}\right)$ and similarly

$$
Q_{+}=\left(\begin{array}{cc}
(\partial V)^{*} & i \partial  \tag{11.26}\\
i \bar{\partial} & -\partial V
\end{array}\right)=Q_{-}^{*}
$$

Note that since $\{Q, \Gamma\}=0$, we have the decomposition

$$
Q=\left(\begin{array}{cc}
0 & Q_{-}  \tag{11.27}\\
Q_{+} & 0
\end{array}\right) .
$$

Define $h_{+}=Q_{-} Q_{+}$and $h_{-}=Q_{+} Q_{-}$. Since $H=Q^{2}$, then we have the decomposition

$$
\begin{align*}
H & =\left(\begin{array}{cc}
h_{+} & 0 \\
0 & h_{-}
\end{array}\right)  \tag{11.28}\\
& =\left(\begin{array}{cc}
Q_{+}^{*} Q_{+} & 0 \\
0 & Q_{-}^{*} Q_{-}
\end{array}\right) \tag{11.29}
\end{align*}
$$

Thus, $[H, \Gamma]=0$ and we have a superalgebra.

Denote $E=\inf$ spec $H$, where $H=Q^{2}$ is a Hamiltonian in an existing superalgebra with a self-adjoint supercharge, $Q$. A supersymmetry is unbroken if $E=0$ and broken if $E>0$. If a supersymmetry is broken, then the corresponding ground state is degenerate. However, the converse is not true; the ground state may be degenerate and the corresponding supersymmetry may be unbroken. To measure supersymmetry breaking, Witten computes the Atiyah-Singer index of the supercharge $Q_{+}$[474],

$$
\begin{equation*}
\operatorname{ind}\left(Q_{+}\right)=\operatorname{dim} \operatorname{ker} Q_{+}-\operatorname{dim} \operatorname{ker} Q_{+}^{*}=\operatorname{Tr}_{\mathscr{H}} \Gamma e^{-\beta H} \tag{11.30}
\end{equation*}
$$

which is the difference of the number of bosonic and fermionic ground states.

### 11.3. The $\mathbf{W Z}_{\theta, \phi}$ Model

In the same paper [219], Jaffe, Lesniewski and Lewenstein studied the vacuum structure of said supersymmetric model of holomorphic quantum mechanics with a polynomial superpotential $V$ and calculated the Fredholm (or Witten) index* of the supercharge $Q_{+}$(satisfying $\left.Q_{+}^{2}=H+P\right)$,

$$
\begin{align*}
\operatorname{ind}\left(Q_{+}\right) & =\lim _{\beta \rightarrow \infty} \operatorname{Tr}_{\mathscr{H} b} \Gamma e^{-\beta H}  \tag{11.31a}\\
& =n_{+}-n_{-}  \tag{11.31b}\\
& =\operatorname{deg} \partial V \tag{11.31c}
\end{align*}
$$

[^43]where $n_{+}=\operatorname{ker} Q_{+}$is the number of bosonic ground states and $n_{-}=\operatorname{ker} Q_{+}^{*}$ is the number of fermionic ground states. In particular, they proved a vanishing theorem of no fermionic ground states (that is, $n_{-}=0$ ), i.e., $Q_{+}^{*}$ is injective, thus proving the non-negativity of the Witten index for this model.

Their reasoning can be summarized as follows. Consider a fermionic ground state $\Omega_{-} \in \mathscr{H}^{f}$ which satisfies $Q_{-} \Omega_{-}=h_{-} \Omega_{-}=0$. Since $h_{-}=\left(-\partial \bar{\partial}+|\partial V|^{2}\right) I$, then $(\partial V) \Omega_{-}=\bar{\partial} \Omega_{-}=0$. Hence, $\Omega_{-}=0$ and there are no fermionic ground states, i.e., $n_{-}=\operatorname{dim}$ ker $Q_{-}=0$. By proving that there are exactly $n-1$ linearly independent solutions of the equation $Q_{+} \Omega_{+}=0$, it follows that dim $\operatorname{ker} Q_{+}=$ $n-1$. Thus, $\operatorname{ind}\left(Q_{+}\right)=n-1=\operatorname{deg} \partial V$ [219].

Remark 11.3.1. Note that $\operatorname{ind}\left(Q_{+}\right) \neq 0$ is a sufficient condition for unbroken supersymmetry. Moreover, if $n_{ \pm}=0$, then $\operatorname{ind}\left(Q_{+}\right) \neq 0$ is both a necessary and sufficient condition for unbroken supersymmetry.
11.3.1. The $\mathbf{W Z}_{\theta, \phi}$ Model. In 1999, Jaffe [222] studied a twist-regularized bosonic field theory with twisted bosonic partition function

$$
\begin{equation*}
\mathfrak{Z}_{g}^{b}(\beta)=\operatorname{Tr}_{\mathscr{H}_{b}} U\left(g^{-1}\right) e^{-\beta H} \tag{11.32}
\end{equation*}
$$

where $U(g)$ is a unitary representation of a group $G$ and $H$ is a $U(g)$-invariant, self-adjoint Hamiltonian $H$ (with interaction) on a bosonic Hilbert space $\mathscr{H}^{b}$.

Here, he proved twist positivity,

$$
\begin{align*}
\mathfrak{Z}_{\gamma}^{b}(\theta, \sigma, \beta) & =\operatorname{Tr}_{\mathscr{H} b} e^{-\beta H-i \sigma P-i \theta J}  \tag{11.33a}\\
& =\prod_{i=1}^{n} \prod_{k \in \hat{\mathbb{T}}^{d}}\left|1-\gamma_{i}(k)\right|^{-2}>0, \tag{11.33b}
\end{align*}
$$

which holds for fixed $\theta, \sigma, \beta>0$ and any $g \in G$, thus implying the existence of a twisted Feynman-Kac representation of the interacting Hamiltonian, $H=H_{0}+V$.

In 2000, Jaffe [223] studied a particular version of the aforementioned bosonic field theory, as a twist-regularized, supersymmetric, generalized WessZumino model $\left(W Z_{\theta, \phi}\right)$ on a $(1,1)$-space-time torus $\mathbb{T}=S^{1} \times S^{1}$ of circumference $\ell$. Within constructive quantum field theory, the $\mathrm{WZ}_{\theta, \phi}$ model remains to date the only interacting two-dimensional supersymmetric quantum field theory that satisfies a weaker, finite-volume version of the Osterwalder-Schrader Axioms and wherein the ground-state structure is computable.

Given a weighted homogeneous potential $V$ of the scalar fields with weights $\omega_{1}, \ldots, \omega_{n} \in\left(0, \frac{1}{2}\right]$, which satisfies the condition of ellipticity,Jaffe computes the twist, boson-fermion elliptic genus* (or $\mathbb{Z}_{2}$-graded partition function) $\mathfrak{Z}^{V}: \mathbb{C} \times \mathfrak{H} \rightarrow \mathbb{C}$ of complex twist $z=\frac{1}{2 \pi}(\theta-\phi \tau)$ and space-time $\tau=\frac{1}{\ell}(\sigma+i \beta)$

[^44]parameters,
\[

$$
\begin{align*}
\mathfrak{Z}^{V}(z, \tau) & =\operatorname{Tr}_{\mathscr{F}} \Gamma e^{-\beta H-i \sigma P-i \theta J}  \tag{11.34a}\\
& =e^{i \theta \hat{\varepsilon} / 2} \prod_{i=1}^{n} \prod_{k \geqslant 0} \frac{\left(1-y^{-\left(1-\omega_{i}\right)} q^{k}\right)\left(1-y^{\left(1-\omega_{j}\right)} q^{k+1}\right)}{\left(1-y^{-\omega_{i}} q^{k}\right)\left(1-y^{\omega_{i}} q^{k+1}\right)}  \tag{11.34b}\\
& =y^{-\hat{c} / 2} \prod_{i=1}^{n} \frac{\vartheta_{1}\left(\left(1-\omega_{i}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{i} z, \tau\right)} \tag{11.34c}
\end{align*}
$$
\]

where $\hat{c}=n-2 \sum_{i=1}^{n} \omega_{i}, y=e^{2 \pi i z}$ and $q=e^{2 \pi i \tau}$. This is possible since the elliptic genus $\mathfrak{Z}^{\lambda V}$ is constant in $\lambda \in[0,1]$, and $\mathfrak{Z}^{\lambda V}$ is evaluated in the limit $\lambda \rightarrow 0$. As a result of the representation as a ratio Jacobi theta functions, the elliptic genus $\mathfrak{Z}^{V}$ is a weak Jacobi form and satisfies the following $\mathbb{Z}^{2} \ltimes \mathrm{SL}_{2}(\mathbb{Z})$ symmetry: For $\gamma_{\delta}=\left((m, n),\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)\right) \in \mathbb{Z}^{2} \ltimes \mathrm{SL}_{2}(\mathbb{Z})$ and $(z, \tau) \in \mathbb{C} \times \mathfrak{H}$, one has the transformation law

$$
\begin{equation*}
\left.\mathfrak{Z}^{V}\right|_{\gamma_{\delta}}(z, \tau)=y^{\hat{c} / 2} \mathbf{e}_{c, d}^{\hat{c}\left[c z^{2}-(2 m+1) z-a^{\prime} \tau-b^{\prime}\right]} \mathfrak{Z}^{V}(z, \tau), \tag{11.35}
\end{equation*}
$$

where $a^{\prime}=m a+n c$ and $b^{\prime}=m b+n d$ and $\mathbf{e}_{c, d}^{z}=e^{\pi i z /(c \tau+d)}$.

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## Chapter 12

## Harmonic Analysis on the Torus

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A fundamental problem with quantum field theory on a torus is the massless or zero momentum divergences.

### 12.1. Infrared Problem on $\mathbb{T}$

Consider a circle of length $\ell$, that is, the quotient space $\mathbb{T}=\mathbb{R} / \ell \mathbb{Z}$. Define $\omega(k)=\left(k^{2}+m^{2}\right)^{1 / 2}$ and set $K_{0}=\frac{2 \pi}{\ell} \mathbb{Z}$. Consider the free bosonic Hamiltonian

$$
\begin{align*}
H_{0} & =\sum_{k \in K_{0}} \omega(k) a(k)^{*} a(k)  \tag{12.1}\\
& =\sum_{k \in K_{0}} \omega(k) N(k) . \tag{12.2}
\end{align*}
$$

The trace of the heat kernel is given by

$$
\begin{align*}
\operatorname{Tr}_{\mathscr{H}} e^{-\beta H_{0}} & =\sum_{n \geqslant 0} e^{-\sum_{k \in K_{0}} n \omega(k)}  \tag{12.3}\\
& =\prod_{k \in K_{0}}\left(1-e^{-\omega(k)}\right)^{-1} \tag{12.4}
\end{align*}
$$

which is ill-defined for $m=k=0$. Therefore, one must introduce a regularization in order to study the massless or zero momentum sector of a quantum field theory on $\mathbb{T}$.

### 12.2. Harmonic Analysis on $\mathbb{T}$

Consider a 1-torus $\mathbb{T}=\mathbb{R} / \ell \mathbb{Z}$ with fixed circumference $\ell \in \mathbb{R}_{>0}$ and its (symmetric) reciprocal lattice $\hat{\mathbb{T}}=\frac{2 \pi}{\ell} \mathbb{Z}$. As an element of the Schwartz space $\mathcal{S}^{\prime}(\mathbb{T})$, recall that the Dirac measure $\delta$ on $\mathbb{T}$ has the following Fourier representation,

$$
\begin{equation*}
\delta(x)=\frac{1}{\ell} \sum_{k \in \hat{\mathbb{T}}} e^{-i k x}=\delta(-x) \tag{12.5}
\end{equation*}
$$

where the sums converge in the sense of tempered distributions.

### 12.3. Sharp Regularization

For $N \in \mathbb{N}$, let $\kappa=\frac{2 \pi}{\ell} N$. Define the finite reciprocal lattice $\hat{\mathbb{T}}_{\kappa}=\{k \in$ $\hat{\mathbb{T}}||k|<\kappa\}$ and note that $\hat{\mathbb{T}}=\lim \sup _{\mathcal{K} \rightarrow \infty} \hat{\mathbb{T}}_{\mathcal{K}}$. In a similar fashion, define a corresponding finite lattice $\mathbb{Z}_{N}=\{n \in \mathbb{Z}| | n \mid<N\}$. Clearly, $\left|\hat{\mathbb{T}}_{\mathcal{K}}\right|=\left|\mathbb{Z}_{N}\right|=$ $2 N-1$. Denote $\tilde{D}_{\kappa}(k)=\theta(\kappa-|k|)=\chi_{\hat{\mathbb{T}}_{\kappa}}(k)$, the characteristic function of
$\hat{\mathbb{T}}_{\mathcal{K}}$, and observe that $\lim _{\mathcal{\kappa} \rightarrow \infty} \tilde{D}_{\kappa}=1$. Define the Dirichlet kernel on $\mathbb{T}$ as the corresponding tempered distribution

$$
\begin{align*}
D_{\kappa}(x) & =\frac{1}{\ell} \sum_{k \in \hat{\mathbb{T}}} \tilde{D}_{\kappa}(k) e^{-i k x}  \tag{12.6}\\
& =\frac{1}{\ell} \sum_{k \in \hat{\mathbb{T}}_{\kappa}} e^{i k x}  \tag{12.7}\\
& =\frac{\sin \left((2 N-1) \frac{\pi x}{\ell}\right)}{\ell \sin \left(\frac{\pi x}{\ell}\right)}  \tag{12.8}\\
& =D_{\kappa}(-x) . \tag{12.9}
\end{align*}
$$

That $\delta=\mathrm{w}-\lim _{\kappa \rightarrow \infty} D_{\kappa}$ is self-evident. Moreover, given a test function $f$ in $\mathcal{S}(\mathbb{T})$ or tempered distribution in $\mathcal{S}^{\prime}(\mathbb{T})$ with Fourier representation

$$
\begin{equation*}
f(x)=\frac{1}{\sqrt{\ell}} \sum_{k \in \hat{\mathbb{T}}} \tilde{f}(k) e^{-i k x} \tag{12.10}
\end{equation*}
$$

we have the convolution

$$
\begin{equation*}
\left(D_{\kappa} * f\right)(x)=\left(f * D_{\kappa}\right)(x)=\int_{0}^{\ell} f(x) D_{\kappa}(x-y) d x=f_{\kappa}(x) \tag{12.11}
\end{equation*}
$$

where $f_{\kappa}$ is the sharply-regularized test function or tempered distribution corresponding to $f$ in the limit $\kappa \rightarrow \infty$ with Fourier representation

$$
\begin{align*}
f_{\kappa}(x) & =\frac{1}{\sqrt{\ell}} \sum_{k \in \hat{\mathbb{T}}} \tilde{D}_{\kappa}(k) \tilde{f}(k) e^{-i k x}  \tag{12.12}\\
& =\frac{1}{\sqrt{\ell}} \sum_{k \in \hat{\mathbb{T}}_{\kappa}} \tilde{f}(k) e^{-i k x} \tag{12.13}
\end{align*}
$$

### 12.4. Smooth Regularization

For $\Lambda \in \mathbb{R}_{>0}$, define $\tilde{\mu}_{\Lambda}(k)=\left(1+\frac{k^{2}}{\Lambda^{2}}\right)^{-1 / 2}$ and observe that $\lim _{\Lambda \rightarrow \infty} \tilde{\mu}_{\Lambda}=1$. Fix $\varepsilon>0$. If $\kappa>\Lambda$, then $\tilde{\mu}_{\Lambda}(\kappa)^{-\varepsilon} \leqslant \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} \leqslant \tilde{\mu}_{\kappa}(k)^{-\varepsilon} \leqslant 1$ for $k<|\kappa|$. In the limit $\kappa \rightarrow \infty$, we have $\tilde{\mu}_{\Lambda}(\kappa)=O\left(\kappa^{-\varepsilon}\right)$. Define the corresponding mollifier, or slow-decrease at infinity (tempered) distribution,

$$
\begin{equation*}
\mu_{\varepsilon, \Lambda}(x)=\frac{1}{\ell} \sum_{k \in \hat{\mathbb{T}}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{-i k x} \tag{12.14}
\end{equation*}
$$

also known as the JLO (Jaffe-Lesniewinski-Osterwalder) kernel. Observe that $\mu_{\varepsilon, \Lambda}$ is an approximate identity in two ways as $\delta=\lim _{\varepsilon \rightarrow 0^{+}} \mu_{\varepsilon, \Lambda}$ and

$$
\begin{align*}
\lim _{\Lambda \rightarrow \infty} \mu_{\varepsilon, \Lambda}(x) & =\lim _{\Lambda \rightarrow \infty} \frac{1}{\ell} \sum_{k \in \hat{\mathbb{T}}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{-i k x}  \tag{12.15}\\
& =\lim _{\Lambda \rightarrow \infty} \lim _{\kappa \rightarrow \infty} \frac{1}{\ell} \sum_{k \in \hat{\mathbb{T}}_{\kappa}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{-i k x}  \tag{12.16}\\
& =\lim _{\kappa \rightarrow \infty} \lim _{\Lambda \rightarrow \infty} \frac{1}{\ell} \sum_{k \in \hat{\mathbb{T}}_{\kappa}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{-i k x}  \tag{12.17}\\
& =\lim _{\kappa \rightarrow \infty} \frac{1}{\ell} \sum_{k \in \hat{\mathbb{T}}_{\kappa}} e^{-i k x}  \tag{12.18}\\
& =\delta(x) \tag{12.19}
\end{align*}
$$

Since $\lim _{\varepsilon \rightarrow \infty} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon}=\delta_{k, 0}$, it follows that $\lim _{\varepsilon \rightarrow \infty} \mu_{\varepsilon, \Lambda}=\ell^{-1}$. Also, note that $\mu_{\varepsilon, \Lambda}$ is even, real and $\ell$-periodic

$$
\begin{align*}
\mu_{\varepsilon, \Lambda}(x) & =\frac{1}{\ell} \sum_{k \in-\hat{\mathbb{T}}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{-i k x}=\frac{1}{\ell} \sum_{k \in \hat{\mathbb{T}}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{i k x}=\mu_{\varepsilon, \Lambda}(-x)  \tag{12.20}\\
& =\frac{1}{\ell} \sum_{k \in \hat{\mathbb{T}}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{-i k(-x)}=\mu_{\varepsilon, \Lambda}(x)^{*}  \tag{12.21}\\
& =\frac{1}{\ell} \sum_{n \in \mathbb{Z}} \tilde{\mu}_{\ell \Lambda / 2 \pi}(n)^{-\varepsilon} e^{-2 \pi i n x / \ell} e^{-2 \pi i n}  \tag{12.22}\\
& =\mu_{\varepsilon, \Lambda}(x+\ell) . \tag{12.23}
\end{align*}
$$

Define a second mollifier as the convolution

$$
\begin{equation*}
v_{\varepsilon, \Lambda}(x)=\left(\mu_{\varepsilon, \Lambda} * \mu_{\varepsilon, \Lambda}\right)(x)=\mu_{2 \varepsilon, \Lambda}(x) . \tag{12.24}
\end{equation*}
$$

12.4.1. $L^{p}$-norms on $\mathbb{T}$. Denote a circle $S^{1}$ of circumference $\ell$, or the 1 -torus $\mathbb{T}$, by the quotient space $\mathbb{T}=\mathbb{R} / \ell \mathbb{Z}$. For each positive integer $p$, define the $p$-norm of a function $f: \mathbb{T} \rightarrow \mathbb{C}$ by

$$
\begin{equation*}
\|f\|_{p}=\|f(x)\|_{L^{p}(\mathbb{T})}=\left(\frac{1}{\ell} \int_{0}^{\ell}|f(x)|^{p} d x\right)^{1 / p} \tag{12.25}
\end{equation*}
$$

Also, define $\|f\|_{\infty}=\sup _{x \in \mathbb{T}}|f(x)|$. Denote the space of complex-valued, measurable functions with finite $p$-norm by $L^{p}(\mathbb{T})$.

### 12.5. Dirichlet Kernels and Characteristic Functions

Define the Dirichlet kernel $D_{N}$ on $\mathbb{T}$

$$
\begin{equation*}
D_{N}(x)=\frac{1}{\ell} \sum_{n=-N+1}^{N-1} e^{2 \pi i n x / \ell}=\frac{\sin (2 N-1) \frac{\pi x}{\ell}}{\sin \frac{\pi x}{\ell}} \tag{12.26}
\end{equation*}
$$

the property that convolution with a $\ell$-periodic function $f$ produces the $N^{\text {th }}$ partial trigonometric sum,

$$
\begin{equation*}
\left(D_{N} * f\right)(x)=\frac{1}{\ell} \int_{0}^{\ell} d y D_{N}(x-y) f(y)=\sum_{n=-N}^{N} \tilde{f}(k) e^{i k x} \tag{12.27}
\end{equation*}
$$

12.6. Dirac Measure

Define the Dirac measure (or Shah distribution) $\delta_{\ell \mathbb{Z}}$ on $\mathbb{T}$ as the doubly infinite summation of evenly-spaced Dirac delta (generalized) functions,

$$
\begin{equation*}
\delta_{\ell \mathbb{Z}}(x)=\sum_{n \in \ell \mathbb{Z}} \delta(x-n)=\delta_{\ell \mathbb{Z}}(-x) \tag{12.28}
\end{equation*}
$$

which satisfies the sampling and replicating identities

$$
\begin{equation*}
\left(\delta_{\ell \mathbb{Z}} \cdot f\right)(x):=\delta_{\ell \mathbb{Z}}(x) f(x)=\sum_{n \in \ell \mathbb{Z}} \delta(x-n) f(n) \tag{12.29}
\end{equation*}
$$

and

$$
\begin{equation*}
\left(\delta_{\ell \mathbb{Z}} * g\right)(x)=\sum_{n \in \ell \mathbb{Z}} g(x-n) \tag{12.30}
\end{equation*}
$$

where $f$ and $g$ are well-behaved test functions. In the sense of tempered distributions, $\delta_{\ell \mathbb{Z}}(x)$ vanishes unless $x \in \ell \mathbb{Z}$ and satisfies $\delta_{\ell \mathbb{Z}}(x+\ell)=\delta_{\ell \mathbb{Z}}(x)$ for all $x \in \mathbb{T}$. Since $\delta_{\ell \mathbb{Z}}$ is an $\ell$-periodic tempered distribution, we have the following Fourier representation

$$
\begin{equation*}
\delta_{\ell \mathbb{Z}}(x)=\frac{1}{\sqrt{\ell}} \sum_{n \in \mathbb{Z}} c_{n} e^{2 \pi i n x / \ell} \tag{12.31}
\end{equation*}
$$

which is convergent in the sense of tempered distributions. For each $n \in \mathbb{Z}$, the Fourier coefficient $c_{n}$ has a simple form, which we calculate

$$
\begin{align*}
c_{n} & =\frac{1}{\sqrt{\ell}} \int_{0}^{\ell} \delta_{\ell \mathbb{Z}}(x) e^{-2 \pi i k x / \ell} d x  \tag{12.32}\\
& =\frac{1}{\sqrt{\ell}} \int_{-\ell / 2}^{\ell / 2} \delta(x) e^{-2 \pi i k x / \ell} d x  \tag{12.33}\\
& =\frac{1}{\sqrt{\ell}} . \tag{12.34}
\end{align*}
$$

Introduce the 1 -dimensional simple cubic Bravais lattice $K_{\ell}=\frac{2 \pi}{\ell} \mathbb{Z}=\widehat{\ell \mathbb{Z}}$, the dual (Bravais) lattice $\ell \mathbb{Z}$. Then it follows that

$$
\begin{equation*}
\delta_{K_{\ell}}(x)=\frac{1}{\ell} \sum_{k \in K_{\ell}} e^{i k x}=\delta_{\ell \mathbb{Z}}(x) \quad \text { and } \quad \delta_{K_{\ell}}^{\prime}(x)=\frac{i}{\ell} \sum_{k \in K_{\ell}} k e^{i k x} \tag{12.35}
\end{equation*}
$$

Relate the Dirac measure $\delta_{\ell \mathbb{Z}}$ to the ordinary delta distribution $\delta$ on $\mathbb{R}$ by recalling the identity

$$
\begin{equation*}
(\delta \circ f)(x)=\sum_{\left\{x_{j} \in \mathbb{R} \mid f\left(x_{j}\right)=0\right\}} \frac{\delta\left(x-x_{j}\right)}{\left|f^{\prime}\left(x_{j}\right)\right|} \tag{12.36}
\end{equation*}
$$

Thus,

$$
\begin{align*}
\delta_{\ell \mathbb{Z}}(x) & =\sum_{n \in \ell \mathbb{Z}} \delta(x-n)  \tag{12.37}\\
& =\sum_{n \in \ell \mathbb{Z}} \frac{\delta(x-n)}{\left|\cos \frac{\pi n}{\ell}\right|}  \tag{12.38}\\
& =(\delta \circ \sin )\left(\frac{\pi x}{\ell}\right) \tag{12.39}
\end{align*}
$$

### 12.7. Translated Lattices and Twisted Dirac Measures

For $\chi \in \mathbb{R} \backslash 2 \pi \mathbb{Z}$, define the twisted Dirac measure

$$
\begin{equation*}
\delta_{K_{\ell}^{\chi}}(x)=\frac{1}{\ell} \sum_{k \in K_{\ell}^{\chi}} e^{i k x}=\delta_{K_{\ell}}(x) e^{-i \chi x / \ell} \tag{12.40}
\end{equation*}
$$

where $K_{\ell}^{\chi}=\{k \in \mathbb{R} \mid \ell k+\chi \in 2 \pi \mathbb{Z}\}$ or, equivalently, $K_{\ell}-\frac{\chi}{\ell}=\left\{\left.k-\frac{\chi}{\ell} \in \mathbb{R} \right\rvert\, k \in K_{\ell}\right\}$, a $\frac{\chi}{\ell}$-translate of the dual lattice $K_{\ell}=K_{\ell}^{0}$. Without loss of generality, we may restrict the domain of values of $\chi$ to $(0, \pi)$ by symmetry of the translate $K_{\ell}^{\chi}$. By virtue of the sampling identity in equation (12.29),

$$
\begin{equation*}
\delta_{K_{\ell}^{\chi}}(x)=\sum_{n \in \ell \mathbb{Z}} \delta(x-n) e^{-i n \chi / \ell} . \tag{12.41}
\end{equation*}
$$

Thus, we interpret the tempered distribution $\delta_{K_{\ell}^{\chi}}(x)$ as an $\ell \mathbb{Z}$-sampling of the function $e^{-i \chi x / \ell}$.

### 12.8. Finite, Twisted Dirac Measure

Denote $\kappa_{m, \chi}=\frac{\pi m}{\ell}-\frac{\chi}{\ell}$ for $m \in \mathbb{N}$ and set $\kappa=\kappa_{2 N, \chi}$ for some distinguished integer $N>0$. Consider the finite set $K_{\ell, \kappa}^{\chi}=\left\{k \in K_{\ell}^{\chi}| | k \mid<\kappa \in \mathbb{R}_{>0}\right\}$. Then define the finite Dirac measure on $[0, \ell)$ by the finite sum

$$
\begin{align*}
\delta_{K_{\ell, \kappa}^{\chi}}(x) & =\frac{e^{-i \chi x / \ell}}{\ell} \sum_{n=-N}^{N} e^{2 \pi i n x / \ell}  \tag{12.42}\\
& =\frac{e^{-i \chi x / \ell}}{\ell}\left(\frac{\sin (2 N+1) \frac{\pi x}{\ell}}{\sin \frac{\pi x}{\ell}}\right)  \tag{12.43}\\
& =\frac{e^{-i \chi x / \ell}}{\ell} D_{N}\left(\frac{2 \pi x}{\ell}\right) \tag{12.44}
\end{align*}
$$

Define the projection operator $\mathrm{P}_{K_{\ell, \kappa}^{\chi}}$ acting on the space of tempered distributions $\mathcal{D}^{\prime}(\mathbb{R})$, where

$$
\begin{equation*}
\left(\mathrm{P}_{\mathrm{K}_{\ell, \kappa}^{\chi}} f\right)(x)=\int_{0}^{\ell} d x^{\prime} \delta_{K_{\ell, \kappa}^{\chi}}\left(x-x^{\prime}\right) f\left(x^{\prime}\right) d x=\left(f * \delta_{K_{\ell, \kappa}^{\chi}}\right)(x), \tag{12.45}
\end{equation*}
$$

which is interpreted as a convolution over $[0, \ell]$ or a distribution on $\mathbb{T}$. Set $\mathrm{P}_{K_{\ell}^{\chi}}=\lim _{\kappa \rightarrow \infty} \mathrm{P}_{K_{\ell, \kappa}^{\chi}}$. Although it is generally true that formally $\delta_{K_{\ell}}(-x)=$ $\delta_{K_{\ell}}(x)$, since $\chi_{j}^{b} \neq 0$, it follows that $\delta_{K_{\ell}^{\chi}}(-x) \neq \delta_{K_{\ell}^{\chi}}(x)$. We now explore this curious asymmetry.

Proposition 12.1. Let $f_{ \pm}$be a tempered distribution possessing the Fourier representation

$$
\begin{equation*}
f_{ \pm}(x)=\lim _{\kappa \rightarrow \infty} \frac{1}{\sqrt{\ell}} \sum_{k \in K_{\ell, \kappa}^{\chi}} \tilde{f}_{ \pm}(k) e^{ \pm i k x}=\frac{1}{\sqrt{\ell}} \sum_{k \in K_{\ell}^{\chi}} \tilde{f}_{ \pm}(k) e^{ \pm i k x} \tag{12.46}
\end{equation*}
$$

where the Fourier coefficient $\tilde{f}_{ \pm}(k)$ is given by

$$
\begin{equation*}
\tilde{f}_{ \pm}(k)=\frac{1}{\sqrt{\ell}} \int_{0}^{\ell} d x^{\prime} f_{ \pm}\left(x^{\prime}\right) e^{\mp i k x^{\prime}} \tag{12.47}
\end{equation*}
$$

Then $f_{ \pm}$satisfies the convolution identity $f_{ \pm}=\mathrm{P}_{ \pm K_{\ell}^{\chi}} f_{ \pm}$.
Proof. We calculate the convolution $\left(\mathrm{P}_{ \pm K_{\ell}^{\chi}} f_{ \pm}\right)(x)$,

$$
\begin{align*}
\int_{0}^{\ell} d x^{\prime} \delta_{ \pm K_{\ell}^{\chi}}\left(x-x^{\prime}\right) f_{ \pm}\left(x^{\prime}\right) & =\frac{1}{\ell} \int_{0}^{\ell} d x^{\prime} \sum_{k \in K_{\ell}^{\chi}} e^{ \pm i k\left(x-x^{\prime}\right)} f_{ \pm}\left(x^{\prime}\right)  \tag{12.48}\\
& =\lim _{\kappa \rightarrow \infty} \frac{1}{\ell} \int_{0}^{\ell} d x^{\prime} \sum_{k \in K_{\ell, \kappa}^{\chi}} e^{ \pm i k\left(x-x^{\prime}\right)} f_{ \pm}\left(x^{\prime}\right)  \tag{12.49}\\
& =\lim _{\kappa \rightarrow \infty} \frac{1}{\sqrt{\ell}} \sum_{k \in K_{\ell, \kappa}^{\chi}}\left(\frac{1}{\sqrt{\ell}} \int_{0}^{\ell} d x^{\prime} f_{ \pm}\left(x^{\prime}\right) e^{\mp i k x^{\prime}}\right) e^{ \pm i k x}  \tag{12.50}\\
& =\lim _{\kappa \rightarrow \infty} \frac{1}{\sqrt{\ell}} \sum_{k \in K_{\ell, \kappa}^{\chi}} \tilde{f}_{ \pm}(k) e^{ \pm i k x}  \tag{12.51}\\
& =f_{ \pm}(x) \tag{12.52}
\end{align*}
$$

where the interchange of the summation and integration symbols is justified for finite summations.

Thus, $\delta_{K_{\ell}^{\chi}}(x)$ is the natural Dirac measure for $f_{+}$, whereas $\delta_{-K_{\ell}^{\chi}}(x)=$ $\delta_{K_{j}^{\chi}}(-x)$ is the Dirac measure more suitable for $f_{-}$. The tempered distributions $f_{ \pm}$satisfy $\chi$-twist- $\ell$-periodic boundary conditions

$$
\begin{align*}
f_{ \pm}(x \pm \ell) & =\frac{1}{\sqrt{\ell}} \sum_{k \in K_{\ell}^{\chi}} \tilde{f}_{ \pm}(k) e^{ \pm i k(x \pm \ell)}  \tag{12.53}\\
& =\frac{1}{\sqrt{\ell}} \sum_{k \in K_{\ell}^{\chi}} \tilde{f}_{ \pm}(k) e^{ \pm i k x} e^{i k \ell}  \tag{12.54}\\
& =e^{i \chi} f_{ \pm}(x) \tag{12.55}
\end{align*}
$$

This identity is called a twist relation.
12.8.1. Dirac Measure, Revisited. Fix $\ell, \Lambda \in \mathbb{R}_{>0}$ and $N \in \mathbb{N}$, let $\kappa=\frac{2 \pi}{\ell} N$ and define the symmetric reciprocal lattice $\hat{\mathbb{Z}}=\frac{2 \pi}{\ell} \mathbb{Z}$ and its cut-off counterpart $\hat{\mathbb{Z}}_{\kappa}=\{k \in \hat{\mathbb{Z}}| | k \mid<\kappa\}$. Note that $\hat{\mathbb{Z}}=\bigcup_{N \in \mathbb{N}} \hat{\mathbb{Z}}_{\kappa}$ and we may write without ado $\hat{\mathbb{Z}}=\lim _{\kappa \rightarrow \infty} \hat{\mathbb{Z}}_{\kappa}$. Define the corresponding cut-off lattice $\mathbb{Z}_{N}=\{n \in \mathbb{Z}| | n \mid<$ $N\}$. Clearly, $\left|\hat{\mathbb{Z}}_{\kappa}\right|=\left|\mathbb{Z}_{N}\right|=2 N-1$. Recall the Dirac measure $\delta_{\hat{\mathbb{Z}}}$ on $\mathbb{T}=\mathbb{R} / \ell \mathbb{Z}$ is defined as the fourier series

$$
\begin{equation*}
\delta_{\hat{\mathbb{Z}}}(x)=\frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}} e^{-i k x} \tag{12.56}
\end{equation*}
$$

where the sum converges in the sense of tempered distributions.

### 12.9. Sharp Regularization

Define $\tilde{\chi}_{\kappa}(k)=\theta(\kappa-|k|)$, and observe that $\lim _{\kappa \rightarrow \infty} \tilde{\chi}_{\kappa}(k)=1$. Define the approximate identity $\chi_{\hat{\mathbb{Z}}_{\kappa}}$ on $\mathbb{T}$ as the partial summation

$$
\begin{equation*}
\chi_{\hat{\mathbb{Z}}_{\kappa}}(x)=\frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}} \tilde{\chi}_{\kappa}(k) e^{-i k x}=\frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}_{\kappa}} e^{-i k x}=\delta_{\vec{Z}_{\kappa}}(x) . \tag{12.57}
\end{equation*}
$$

Hence, $\lim _{\kappa \rightarrow \infty} \chi_{\hat{\mathbb{Z}}_{\kappa}}=\delta_{\hat{\mathbb{Z}}}$.

### 12.10. Smooth Regularization

Define the following mollifier, or slow-decrease at infinity distribution,

$$
\begin{align*}
\mu_{\varepsilon, \Lambda}(x) & =\frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{-i k x}  \tag{12.58}\\
& =\frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}}\left(1+\frac{k^{2}}{\Lambda^{2}}\right)^{-\varepsilon / 2} e^{-i k x} \tag{12.59}
\end{align*}
$$

where $\varepsilon \in \mathbb{R}_{\geqslant 0}$. Observe that $\mu_{\varepsilon, \Lambda}$ is an approximate identity as $\lim _{\varepsilon \rightarrow 0^{+}} \mu_{\varepsilon, \Lambda}=$ $\delta_{\hat{\mathbb{Z}}}$ and

$$
\begin{align*}
\lim _{\Lambda \rightarrow \infty} \mu_{\varepsilon, \Lambda}(x) & =\lim _{\Lambda \rightarrow \infty} \frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}}\left(1+\frac{k^{2}}{\Lambda^{2}}\right)^{-\varepsilon / 2} e^{-i k x}  \tag{12.60}\\
& =\lim _{\Lambda \rightarrow \infty} \lim _{k \rightarrow \infty} \frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}_{\kappa}}\left(1+\frac{k^{2}}{\Lambda^{2}}\right)^{-\varepsilon / 2} e^{-i k x}  \tag{12.61}\\
& =\lim _{\kappa \rightarrow \infty} \lim _{\Lambda \rightarrow \infty} \frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}_{\kappa}}\left(1+\frac{k^{2}}{\Lambda^{2}}\right)^{-\varepsilon / 2} e^{-i k x}  \tag{12.62}\\
& =\lim _{\kappa \rightarrow \infty} \frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}_{\kappa}} e^{-i k x}  \tag{12.63}\\
& =\delta_{\hat{\mathbb{Z}}}(x) . \tag{12.64}
\end{align*}
$$

Also, note that $\mu_{\varepsilon, \Lambda}$ is even, real and $\ell$-periodic

$$
\begin{align*}
\mu_{\varepsilon, \Lambda}(x) & =\frac{1}{\ell} \sum_{k \in-\hat{\mathbb{Z}}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{-i k x}=\frac{1}{\ell} \sum_{k \in \hat{Z}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{i k x}=\mu_{\varepsilon, \Lambda}(-x)  \tag{12.65}\\
& =\frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{i k x}=\frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} e^{-i k(-x)}=\mu_{\varepsilon, \Lambda}(x)^{*}  \tag{12.66}\\
& =\frac{1}{\ell} \sum_{n \in \mathbb{Z}} \tilde{\mu}_{\ell \Lambda / 2 \pi}(n)^{-\varepsilon} e^{-2 \pi i n x / \ell} e^{-2 \pi i n}=\mu_{\varepsilon, \Lambda}(x+\ell) \tag{12.67}
\end{align*}
$$

Since $\lim _{\varepsilon \rightarrow \infty} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon}=\delta_{k, 0}$, it follows that $\lim _{\varepsilon \rightarrow \infty} \mu_{\varepsilon, \Lambda}=\ell^{-1}$. Consider the closure satisfied under convolution on $\mathbb{T}$,

$$
\begin{align*}
\left(\mu_{\varepsilon, \Lambda} * \mu_{\varepsilon^{\prime}, \Lambda}\right)(x) & =\int_{0}^{\ell} \mu_{\varepsilon, \Lambda}(x-y) \mu_{\varepsilon^{\prime}, \Lambda}(y) d y  \tag{12.68}\\
& =\frac{1}{\ell} \sum_{k \in \hat{\mathbb{Z}}} \tilde{\mu}_{\Lambda}(k)^{-\left(\varepsilon+\varepsilon^{\prime}\right)} e^{-i k x}  \tag{12.69}\\
& =\mu_{\varepsilon+\varepsilon^{\prime}, \Lambda}(x) . \tag{12.70}
\end{align*}
$$

Define a total order $\mu_{\varepsilon, \Lambda} \leqslant \mu_{\varepsilon^{\prime}, \Lambda}$ if and only if $\varepsilon \leqslant \varepsilon^{\prime}$. Note that $\leqslant$ is translation invariant, i.e., $\mu_{\varepsilon, \Lambda} \leqslant \mu_{\varepsilon^{\prime}, \Lambda}$ implies $\mu_{\varepsilon+\delta, \Lambda} \leqslant \mu_{\varepsilon^{\prime}+\delta, \Lambda}$ for each $\varepsilon, \varepsilon^{\prime}, \delta \in \mathbb{R}_{\geqslant 0}$ and positive in the sense that $\delta_{\hat{Z}} \leqslant \mu_{\varepsilon, \Lambda}$ for each $\varepsilon \in \mathbb{R}_{>0}$.

The next chapter defines the standard operators in the $\mathrm{WZ}_{\theta, \phi}$ model.

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## Chapter 13

## Twist Field Operators

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It is rather curious that a (partial) $\mathbb{Z}_{2}$-graded Lie algebra of physical operators naturally arises from considering quantum fields with twist boundary conditions on $\mathbb{T}$. We describe the construction of these fields and operators from first principles and elucidate some of their properties. The ensuing discussion pertains mainly to the author's critical reading of the manuscript [226], although many of the details can be found in [158], [222], [223] and [225].

### 13.1. Translated Momentum Lattices

For $N_{j}, N_{\alpha, j} \in \mathbb{N}$, introduce the bosonic cutoffs $\kappa_{j}=\frac{2 \pi}{\ell} N_{j}$ and fermionic cutoffs $\kappa_{\alpha, j}=\frac{2 \pi}{\ell} N_{\alpha, j}$. Define $n$ bosonic translated lattices $K_{j}^{b}=\hat{\mathbb{T}}-\frac{\phi}{\ell} \Omega_{j}^{b}$ and
their cutoff analogues $K_{j, \kappa}^{b}=\hat{\mathbb{T}}_{\kappa}-\frac{\phi}{\ell} \Omega_{j}^{b}$, i.e.,

$$
\begin{equation*}
K_{j, \kappa}^{b}=\left\{k \in K_{j}^{b}\left|\ell k+\Omega_{j} \phi \in 2 \pi \mathbb{Z},|k|<\kappa_{j}\right\} .\right. \tag{13.1}
\end{equation*}
$$

Similarly, define $2 n$ translated fermionic lattices $K_{\alpha, j, \kappa}^{f}=\hat{\mathbb{T}}-\frac{\phi}{\ell} \Omega_{\alpha, j}^{f}$ and their finite analogues $K_{\alpha, \kappa}^{f}=\hat{\mathbb{T}}_{\kappa}-\frac{\phi}{\ell} \Omega_{\alpha, j^{\prime}}^{f}$ i.e.,

$$
\begin{equation*}
K_{\alpha, j, \kappa}^{f}=\left\{k \in K_{\alpha, j}^{f}\left|\ell k+\Omega_{\alpha, j}^{f} \phi \in 2 \pi \mathbb{Z},|k|<\kappa_{\alpha, j}\right\}\right. \tag{13.2}
\end{equation*}
$$

for $1 \leqslant \alpha \leqslant 2$. With the above choices for cutoffs, $\left|K_{j, \kappa}^{b}\right|=2 N_{j}$ and $\left|K_{\alpha, j, \kappa}^{f}\right|=2 N_{\alpha, j}$ for $1 \leqslant j \leqslant n$ and $1 \leqslant \alpha \leqslant 2$.

Define the momentum-dependent weight $\Omega_{j}(k)=\theta( \pm k) \Omega_{j}+\theta(\mp k)\left(1-\Omega_{j}\right)$ and momentum-dependent cutoff $\kappa_{ \pm, j}(k)=\theta( \pm k) \kappa_{1, j}+\theta(\mp k) \kappa_{2, j}$. Introduce two more cutoff fermionic lattices,

$$
\begin{equation*}
K_{ \pm, j, k}^{f}=\left\{k\left|\ell k+\Omega_{j}(k) \phi \in 2 \pi \mathbb{Z},|k|<\kappa_{ \pm, j}(k)\right\} .\right. \tag{13.3}
\end{equation*}
$$

Finally, define the constant $\hat{c}=\sum_{j=1}^{n}\left(1-2 \Omega_{j}\right)=n-2 \sum_{j=1}^{n} \Omega_{j}$.
13.2. Sharply and Strongly-Regularized Twist Fields

Consider $n$ independent, time-zero, massless bosonic fields of the form

$$
\begin{align*}
\varphi_{j}(x) & =\frac{1}{\sqrt{\ell}} \sum_{k \in K_{j}^{b}} \tilde{\varphi}_{j}(k) e^{-i k x}  \tag{13.4}\\
& =\frac{1}{\sqrt{2 \ell}} \sum_{k \in K_{j}^{b}}|k|^{-1 / 2}\left(a_{+, j}(k)^{*}+a_{-, j}(-k)\right) e^{-i k x} \tag{13.5}
\end{align*}
$$

for $1 \leqslant j \leqslant n$. Define the function

$$
u_{\alpha, j}(k)=\frac{\sqrt{|k|+k}}{\sqrt{2|k|}}=\sqrt{\frac{1}{2}+\frac{\operatorname{sgn}(k)}{2}}=\left\{\begin{array}{cc}
1 & k>0  \tag{13.6}\\
\frac{1}{\sqrt{2}} & k=0 \\
0 & k<0
\end{array}\right.
$$

Consider $2 n$ independent, time-zero fermionic fields with Fourier representation

$$
\begin{align*}
\psi_{1, j}(x) & =\frac{1}{\sqrt{\ell}} \sum_{k \in K_{1, j}^{f}} \tilde{\psi}_{1, j}(k) e^{-i k x}  \tag{13.7}\\
& =\frac{1}{\sqrt{\ell}} \sum_{k \in K_{1, j}^{f}}\left(u_{1, j}(k) b_{+, j}(k)^{*}+u_{1, j}(-k) b_{-, j}(-k)\right) e^{-i k x} \tag{13.8}
\end{align*}
$$

and

$$
\begin{align*}
\psi_{2, j}(x) & =\frac{1}{\sqrt{\ell}} \sum_{k \in K_{2, j}^{f}} \tilde{\psi}_{2, j}(k) e^{-i k x}  \tag{13.9}\\
& =-\frac{i}{\sqrt{\ell}} \sum_{k \in K_{2, j}^{f}}\left(u_{2, j}(-k) b_{+, j}(k)^{*}-u_{2, j}(k) b_{-, j}(-k)\right) e^{-i k x} . \tag{13.10}
\end{align*}
$$

Define the $4 n$ twisted mollifiers

$$
\begin{align*}
D_{\kappa}^{b}(x) & =e^{i \Omega_{j} \phi x / \ell} D_{\kappa}(x)  \tag{13.11}\\
D_{\alpha, \kappa}^{f}(x) & =e^{i \Omega_{\alpha, j} \phi x / \ell} D_{\kappa}(x)  \tag{13.12}\\
\mu_{1, j, \varepsilon, \Lambda}^{f}(x) & =\mu_{j, \varepsilon, \Lambda}^{b}(x)  \tag{13.13}\\
& =e^{i \Omega_{j} \phi x / \ell} \mu_{\varepsilon, \Lambda}(x)  \tag{13.14}\\
\mu_{2, j, \varepsilon, \Lambda}^{f}(x) & =e^{i\left(1-\Omega_{j}\right) \phi x / \ell} \mu_{\varepsilon, \Lambda}(x) \tag{13.15}
\end{align*}
$$

 $\varphi_{j, \kappa}=D_{\kappa}^{b} * \varphi_{j}$ and $2 n$ sharply-regularized fermionic fields $\psi_{\alpha, j, \kappa}=D_{\alpha, \kappa}^{f} * \psi_{\alpha, j}$ both defined on their corresponding cut-off dual lattices. Introduce $n$ stronglyregularized bosonic fields and $2 n$ strongly-regularized fermionic fields,

$$
\begin{align*}
\varphi_{j, \Lambda, \kappa} & =\mu_{j, \Lambda}^{b} * \varphi_{j, \kappa}  \tag{13.16}\\
\psi_{\alpha, j, \Lambda, \kappa} & =\mu_{\alpha, j, \Lambda}^{f} * \psi_{\alpha, j, \kappa} \tag{13.17}
\end{align*}
$$

### 13.3. Supercharges $Q_{1}$ and $Q_{2}$

13.3.1. Translation Invariance of $D_{1}$ and $D_{2}$. Define the following free derivations

$$
\begin{align*}
& D_{1}=\int_{0}^{\ell} d x D_{1}(x)=\int_{0}^{\ell} d x D_{1,0}(x)+\lambda D_{1, I}(x)  \tag{13.18}\\
& D_{2}=\int_{0}^{\ell} d x D_{2}(x)=\int_{0}^{\ell} d x D_{2,0}(x)+\lambda D_{2, I}(x) \tag{13.19}
\end{align*}
$$

where the free densities are given by

$$
\begin{align*}
& D_{1,0}(x)=i \sum_{j=1}^{n} \psi_{1, j}(x)\left[\pi_{j}(x)-\partial_{x} \varphi_{j}(x)^{*}\right]  \tag{13.20}\\
& D_{2,0}(x)=i \sum_{j=1}^{n} \psi_{2, j}(x)\left[\pi_{j}(x)^{*}+\partial_{x} \varphi_{j}(x)\right] e^{-i \phi x / \ell} \tag{13.21}
\end{align*}
$$

and the interaction densities are given by

$$
\begin{align*}
& D_{1, I}(x)=\sum_{j=1}^{n} \psi_{2, j}(x)\left(V_{j} \circ \varphi\right)(x)^{*}  \tag{13.22}\\
& D_{2, I}(x)=\sum_{j=1}^{n} \psi_{1, j}(x)\left(V_{j} \circ \varphi\right)(x) e^{-i \phi x / \ell} \tag{13.23}
\end{align*}
$$

respectively, where we denote $\left(V_{j} \circ \varphi\right)(x)=\partial_{j} V\left(\varphi_{1}, \ldots, \varphi_{n}\right)$. The exponential factor $e^{-i \phi x / \ell}$ in $D_{2,0}(x)$ is introduced to ensure its $\ell$-periodicity, as will be shown. After a translation along a length $\ell$,

$$
\begin{align*}
D_{1,0}(x+\ell) & =i \sum_{j=1}^{n} \psi_{1, j}(x+\ell)\left[\pi_{j}(x+\ell)-\partial_{x} \varphi_{j}(x+\ell)^{*}\right]  \tag{13.24}\\
& =i \sum_{j=1}^{n} e^{i \chi_{1, j}^{f} \phi} \psi_{1, j}(x)\left[e^{-i \chi_{j}^{b} \phi} \pi_{j}(x)-e^{-i \chi_{j}^{b} \phi} \partial_{x} \varphi_{j}(x)^{*}\right]  \tag{13.25}\\
& =i \sum_{j=1}^{n} e^{i\left(\chi_{1, j}^{f}-\chi_{j}^{b}\right) \phi} \psi_{1, j}(x)\left[\pi_{j}(x)-\partial_{x} \varphi_{j}(x)^{*}\right] . \tag{13.26}
\end{align*}
$$

For $D_{1,0}(x)$ to be translation invariant, i.e., $\ell$-periodic, require that $\chi_{1, j}^{f}=\chi_{j}^{b}$ for $1 \leqslant j \leqslant n$. Similarly,

$$
\begin{align*}
D_{1, I}(x+\ell) & =\sum_{j=1}^{n} \psi_{2, j}(x+\ell)\left(V_{j} \circ \varphi\right)(x+\ell)^{*}  \tag{13.27}\\
& =\sum_{j=1}^{n} e^{i \chi_{2, j}^{f} \phi} \psi_{2, j}(x) V_{j}\left(e^{i \chi_{1}^{b} \phi} \varphi_{1}, \ldots, e^{i \chi_{n}^{b} \phi} \varphi_{n}\right)^{*} . \tag{13.28}
\end{align*}
$$

Thus, to ensure that $D_{1}(x)$ is translation invariant require that $V$ satisfy the relation

$$
\begin{align*}
V_{j}\left(e^{i \chi_{1}^{b} \phi} \varphi_{1}, \ldots, e^{i \chi_{n}^{b} \phi} \varphi_{n}\right)^{*} & =e^{i\left(\chi_{1, j}^{f}-\chi_{2, j}^{f}-\chi_{j}^{b}\right) \phi} V_{j}\left(\varphi_{1}, \ldots, \varphi_{n}\right)^{*}  \tag{13.29}\\
& =e^{-i \chi_{2, j}^{f} \phi} V_{j}\left(\varphi_{1}, \ldots, \varphi_{n}\right)^{*} \tag{13.30}
\end{align*}
$$

or, equivalently, $V_{j}\left(e^{i \chi_{1}^{b} \phi} \varphi_{1}, \ldots, e^{i \chi_{n}^{b} \phi} \varphi_{n}\right)=e^{i \chi_{2, j}^{f} \phi} V_{j}\left(\varphi_{1}, \ldots, \varphi_{n}\right)$. Now impose the translation invariance on $D_{2,0}(x)$,

$$
\begin{align*}
D_{2,0}(x+\ell) & =i \sum_{j=1}^{n} \psi_{2, j}(x+\ell)\left[\pi_{j}(x+\ell)^{*}+\partial_{x} \varphi_{j}(x+\ell)\right] e^{-i \phi(x+\ell) / \ell}  \tag{13.31}\\
& =i \sum_{j=1}^{n} e^{i \chi_{2, j}^{f} \phi} \psi_{2, j}(x)\left[e^{i \chi_{j}^{b} \phi} \pi_{j}(x)^{*}+e^{i \chi_{j}^{b} \phi} \partial_{x} \varphi_{j}(x)\right] e^{-i \phi / \ell} e^{-i \phi x / \ell}  \tag{13.32}\\
& =i \sum_{j=1}^{n} e^{i\left(\chi_{2, j}^{f}+\chi_{j}^{b}-1\right) \phi} \psi_{2, j}(x)\left[\pi_{j}(x)^{*}+\partial_{x} \varphi_{j}(x)\right] e^{-i \phi x / \ell} \tag{13.33}
\end{align*}
$$

To ensure that $D_{2,0}(x)$ is $\ell$-periodic, we must have $\chi_{2, j}^{f}+\chi_{j}^{b}=1$ for $1 \leqslant j \leqslant n$. Similarly,

$$
\begin{align*}
D_{2, I}(x+\ell) & =\sum_{j=1}^{n} \psi_{1, j}(x+\ell)\left(V_{j} \circ \varphi\right)(x+\ell) e^{-i \phi(x+\ell) / \ell}  \tag{13.34}\\
& =\sum_{j=1}^{n} e^{i\left(\chi_{1, j}^{f}-1\right) \phi} \psi_{1, j}(x) V_{j}\left(e^{i \chi_{1}^{b} \phi} \varphi_{1}, \ldots, e^{i \chi_{n}^{b} \phi} \varphi_{n}\right) e^{-i \phi x / \ell}  \tag{13.35}\\
& =\sum_{j=1}^{n} e^{i\left(\chi_{1, j}^{f}+\chi_{2, j}^{f}-1\right) \phi} \psi_{1, j}(x) V_{j}\left(\varphi_{1}, \ldots, \varphi_{n}\right)=D_{2, I}(x) \tag{13.36}
\end{align*}
$$

Let $\zeta=e^{i \chi_{j}^{b} \phi}=e^{i \chi_{2, j}^{f} \phi}$, then $V_{j}$ satisfies the identity,

$$
\begin{equation*}
\zeta V_{j}\left(\varphi_{1}, \ldots, \varphi_{n}\right)=V_{j}\left(\zeta^{v_{1}} \varphi_{1}, \ldots, \zeta^{v_{n}} \varphi_{n}\right), \tag{13.37}
\end{equation*}
$$

where $v_{j}=\chi_{j}^{b} / \chi_{2, j}^{f}=\chi_{1, j}^{f} / \chi_{2, j}^{f}$ and, therefore, $V_{j}$ is a weighted homogeneous polynomial with weights $\left\{v_{1}, \ldots, v_{n}\right\}$. Hence,

$$
\begin{equation*}
\chi_{2, j}^{f}=\frac{1}{1+v_{j}} \quad \text { and } \quad \chi_{1, j}^{f}=\chi_{j}^{b}=1-\frac{1}{1+v_{j}}=\frac{v_{j}}{1+v_{j}} \tag{13.38}
\end{equation*}
$$

Define the rational $\Omega_{j}=\frac{v_{j}}{1+v_{j}}$ and note that $\Omega_{j} \in(0,1)$ for $1 \leqslant j \leqslant n$. One may write $\chi_{1, j}^{f}=\chi_{j}^{b}=\Omega_{j}$ and $\chi_{2, j}^{f}=1-\Omega_{j}$ for $1 \leqslant j \leqslant n$. Suppose $V$ is a weighted homogeneous polynomial with weight $\left\{\Omega_{1}, \ldots, \Omega_{n}\right\}$, i.e., $\lambda V\left(\varphi_{1}, \ldots, \varphi_{n}\right)=V\left(\lambda^{\Omega_{1}} \varphi_{1}, \ldots, \lambda^{\Omega_{n}} \varphi_{n}\right)$ for any $\lambda \in \mathbb{C}^{\times}$, then it must also satisfy
the weighted Euler equation

$$
\begin{equation*}
(V \circ \varphi)(x)=\sum_{i=1}^{n} \Omega_{i} \varphi_{i}(x)\left(V_{i} \circ \varphi\right)(x) \tag{13.39}
\end{equation*}
$$

as shown in previous chapter. It follows that the directional derivative $\partial_{j} V=V_{j}$ satisfies a similar weighted Euler equation,

$$
\begin{align*}
\left(V_{j} \circ \varphi\right)(x) & =\sum_{i=1}^{n}\left(\frac{\Omega_{i}}{1-\Omega_{j}}\right) \varphi_{i}(x)\left(V_{j i} \circ \varphi\right)(x)  \tag{13.40}\\
& =\sum_{i=1}^{n} v_{i} \varphi_{i}(x)\left(V_{i j} \circ \varphi\right)(x) . \tag{13.41}
\end{align*}
$$

In order for both densities $D_{1}(x)$ and $D_{2}(x)$ to be translation invariant, we must assume the following:

1a. (Weak Version) $\partial_{j} V$ is weighted homogeneous with weights $\left\{v_{1}, \ldots, v_{n}\right\}$ for $1 \leqslant j \leqslant n ;$
1 b. (Strong Version $\Longrightarrow$ Weak Version) $V$ is weighted homogeneous with weights $\left\{\Omega_{1}, \ldots, \Omega_{n}\right\}$;
2. $\chi_{1, j}^{f}=\chi_{j}^{b}$ for $1 \leqslant j \leqslant n$;
3. $\chi_{j}^{b}=\Omega_{j}$ for $1 \leqslant j \leqslant n$; and,
4. $\chi_{2, j}^{f}=1-\Omega_{j}$ for $1 \leqslant j \leqslant n$.

In the case that $\chi_{1, j}^{f}=\chi_{2, j}^{f}$, then $\Omega_{j}=1-\Omega_{j}$ and $\Omega_{j}=\frac{1}{2}$ for $1 \leqslant j \leqslant n$. We shall refer to this as a mass-shift, i.e., when $V$ is a Morse polynomial of the form

$$
\begin{equation*}
V(\varphi)=\sum_{j=1}^{n} c_{j} \varphi_{j}(x)^{2} \quad c_{j} \in \mathbb{C}^{\times} \tag{13.42}
\end{equation*}
$$

13.3.2. Nilpotence of $D_{1}$ and $D_{2}$. To show that $D_{\alpha}^{2}(x)=0$ for $\alpha \in\{1,2\}$, it suffices to prove that $D_{\alpha, 0}(x)$ and $D_{\alpha, I}(x)$ are each nilpotent and mutually independent.
13.3.3. Nilpotence of $D_{1,0}$ and $D_{2,0}$. Recall that

$$
\begin{align*}
& D_{1,0}(x)=i \sum_{j=1}^{n} \psi_{1, j}(x)\left[\pi_{j}(x)-\partial_{x} \varphi_{j}(x)^{*}\right]  \tag{13.43}\\
& D_{2,0}(x)=i \sum_{j=1}^{n} \psi_{2, j}(x)\left[\pi_{j}(x)^{*}+\partial_{x} \varphi_{j}(x)\right] e^{-i \phi x / \ell} \tag{13.44}
\end{align*}
$$

Define the following densities

$$
\begin{align*}
& d_{1,0}^{(1)}(x)=i \sum_{j=1}^{n} \psi_{1, j}(x) \pi_{j}(x)  \tag{13.45}\\
& d_{1,0}^{(2)}(x)=-i \sum_{j=1}^{n} \psi_{1, j}(x) \partial_{x} \varphi_{j}(x)^{*}  \tag{13.46}\\
& d_{2,0}^{(1)}(x)=i \sum_{j=1}^{n} \psi_{2, j}(x) \pi_{j}(x)^{*}  \tag{13.47}\\
& d_{2,0}^{(2)}(x)=-i \sum_{j=1}^{n} \psi_{2, j}(x) \partial_{x} \varphi_{j}(x) \tag{13.48}
\end{align*}
$$

Succinctly,

$$
\begin{align*}
& D_{1,0}(x)=d_{1,0}^{(1)}(x)+d_{1,0}^{(2)}(x)  \tag{13.49}\\
& D_{2,0}(x)=\left(d_{2,0}^{(1)}(x)+d_{2,0}^{(2)}(x)\right) e^{-i \phi x / \ell} \tag{13.50}
\end{align*}
$$

We now show that $d_{\alpha, 0}^{\left(\alpha^{\prime}\right)}(x)$ for $\alpha, \alpha^{\prime} \in\{1,2\}$ is nilpotent. To do so we need the following result.

Proposition 13.1. Suppose $A, B, C$ and $D$ are operators on some common domain. If $[A, D]=0=[B, C]$, then $\{A B, C D\}=\{A, C\} B D-C A[B, D]$.

Proof. Observe that

$$
\begin{align*}
\{A B, C D\} & =A B C D+C D A B  \tag{13.51}\\
& =A C B D+A[B, C] D+C A D B+C[D, A] B  \tag{13.52}\\
& =A C B D+(C A B D-C A B D)+C A D B  \tag{13.53}\\
& =\{A, C\} B D+C A[D, B], \tag{13.54}
\end{align*}
$$

as claimed.

Consider the anti-commutator

$$
\begin{align*}
\left\{d_{1,0}^{(1)}(x), d_{1,0}^{(1)}\left(x^{\prime}\right)\right\}= & -\sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x) \pi_{j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right) \pi_{j^{\prime}}\left(x^{\prime}\right)\right\}  \tag{13.55}\\
= & \sum_{j, j^{\prime}=1}^{n} \psi_{1, j^{\prime}}\left(x^{\prime}\right) \psi_{1, j}(x)\left[\pi_{j^{\prime}}\left(x^{\prime}\right), \pi_{j}(x)\right]  \tag{13.56}\\
& \quad-\left\{\psi_{1, j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right)\right\} \pi_{j}(x) \pi_{j^{\prime}}\left(x^{\prime}\right)  \tag{13.57}\\
= & 0 \tag{13.58}
\end{align*}
$$

since $\left[\pi_{j}(x), \pi_{j^{\prime}}\left(x^{\prime}\right)\right]=0=\left\{\psi_{1, j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right)\right\}$. Similarly, consider the anticommutator

$$
\begin{align*}
\left\{d_{1,0}^{(2)}(x), d_{1,0}^{(2)}\left(x^{\prime}\right)\right\} & =-\sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x) \partial_{x} \varphi_{j}(x)^{*}, \psi_{1, j^{\prime}}\left(x^{\prime}\right) \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right\}  \tag{13.59}\\
& =\sum_{j, j^{\prime}=1}^{n} \psi_{1, j^{\prime}}\left(x^{\prime}\right) \psi_{1, j}(x)\left[\partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)^{*}, \partial_{x} \varphi_{j}(x)^{*}\right]  \tag{13.60}\\
& =0 \tag{13.61}
\end{align*}
$$

since $\left[\partial_{x} \varphi_{j}(x)^{*}, \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right]=0$. Finally, consider

$$
\begin{align*}
\left\{d_{1,0}^{(1)}(x), d_{1,0}^{(2)}\left(x^{\prime}\right)\right\}= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x) \pi_{j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right) \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right\}  \tag{13.62}\\
= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right)\right\} \pi_{j}(x) \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)^{*}  \tag{13.63}\\
& \quad-\psi_{1, j^{\prime}}\left(x^{\prime}\right) \psi_{1, j}(x)\left[\pi_{j}(x), \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right] \\
= & 0 \tag{13.64}
\end{align*}
$$

since $\left[\pi_{j}(x), \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right]=0$. Thus, $\left\{d_{1,0}^{(2)}(x), d_{1,0}^{(1)}\left(x^{\prime}\right)\right\}=0$ and therefore

$$
\begin{align*}
\left\{D_{1,0}(x), D_{1,0}\left(x^{\prime}\right)\right\}= & \left\{d_{1,0}^{(1)}(x)+d_{1,0}^{(2)}(x), d_{1,0}^{(1)}\left(x^{\prime}\right)+d_{1,0}^{(2)}\left(x^{\prime}\right)\right\}  \tag{13.65}\\
= & \left\{d_{1,0}^{(1)}(x), d_{1,0}^{(1)}\left(x^{\prime}\right)\right\}+\left\{d_{1,0}^{(2)}(x), d_{1,0}^{(2)}\left(x^{\prime}\right)\right\} \\
& +\left\{d_{1,0}^{(1)}(x), d_{1,0}^{(2)}\left(x^{\prime}\right)\right\}+\left\{d_{1,0}^{(2)}(x), d_{1,0}^{(1)}\left(x^{\prime}\right)\right\}  \tag{13.66}\\
= & 0 \tag{13.67}
\end{align*}
$$

As a corollary, we have $\left\{D_{1,0}(x)^{*}, D_{1,0}\left(x^{\prime}\right)^{*}\right\}=0$. Hence, $D_{1,0}(x)^{2}=0$, as claimed. Now consider

$$
\begin{align*}
\left\{d_{2,0}^{(1)}(x), d_{2,0}^{(1)}\left(x^{\prime}\right)\right\}= & -\sum_{j, j^{\prime}=1}^{n}\left\{\psi_{2, j}(x) \pi_{j}(x)^{*}, \psi_{2, j^{\prime}}\left(x^{\prime}\right) \pi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right\}  \tag{13.68}\\
= & \sum_{j, j^{\prime}=1}^{n} \psi_{2, j^{\prime}}\left(x^{\prime}\right) \psi_{2, j}(x)\left[\pi_{j^{\prime}}\left(x^{\prime}\right)^{*}, \pi_{j}(x)^{*}\right] \\
& \quad-\left\{\psi_{2, j}(x), \psi_{2, j^{\prime}}\left(x^{\prime}\right)\right\} \pi_{j}(x)^{*} \pi_{j^{\prime}}\left(x^{\prime}\right)^{*}  \tag{13.69}\\
= & 0, \tag{13.70}
\end{align*}
$$

since $\left[\pi_{j}(x)^{*}, \pi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right]=0=\left\{\psi_{2, j}(x), \psi_{2, j^{\prime}}\left(x^{\prime}\right)\right\}$. Similarly, consider the anticommutator

$$
\begin{align*}
\left\{d_{2,0}^{(2)}(x), d_{2,0}^{(2)}\left(x^{\prime}\right)\right\} & =-\sum_{j, j^{\prime}=1}^{n}\left\{\psi_{2, j}(x) \partial_{x^{\prime}} \varphi_{j}(x), \psi_{2, j^{\prime}}\left(x^{\prime}\right) \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)\right\}  \tag{13.71}\\
& =\sum_{j, j^{\prime}=1}^{n} \psi_{2, j^{\prime}}\left(x^{\prime}\right) \psi_{2, j}(x)\left[\partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right), \partial_{x} \varphi_{j}(x)\right]  \tag{13.72}\\
& =0 \tag{13.73}
\end{align*}
$$

since $\left[\partial_{x} \varphi_{j}(x), \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)\right]=0$. Finally, consider

$$
\begin{align*}
\left\{d_{2,0}^{(1)}(x), d_{2,0}^{(2)}\left(x^{\prime}\right)\right\}= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{2, j}(x) \pi_{j}(x)^{*}, \psi_{2, j^{\prime}}\left(x^{\prime}\right) \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)\right\}  \tag{13.74}\\
= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{2, j}(x), \psi_{2, j^{\prime}}\left(x^{\prime}\right)\right\} \pi_{j}(x)^{*} \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right) \\
& \quad-\psi_{2, j^{\prime}}\left(x^{\prime}\right) \psi_{2, j}(x)\left[\pi_{j}(x)^{*}, \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)\right]  \tag{13.75}\\
= & 0 . \tag{13.76}
\end{align*}
$$

Similarly, $\left\{d_{2,0}^{(2)}(x), d_{2,0}^{(1)}\left(x^{\prime}\right)\right\}=0$. After summing the four anti-commutators above,

$$
\begin{align*}
\left\{D_{2,0}(x), D_{2,0}\left(x^{\prime}\right)\right\} e^{i\left(x+x^{\prime}\right) \phi / \ell}= & \left\{d_{2,0}^{(1)}(x)+d_{2,0}^{(2)}(x), d_{2,0}^{(1)}\left(x^{\prime}\right)+d_{2,0}^{(2)}\left(x^{\prime}\right)\right\}  \tag{13.77}\\
= & \left\{d_{2,0}^{(1)}(x), d_{2,0}^{(1)}\left(x^{\prime}\right)\right\}+\left\{d_{2,0}^{(2)}(x), d_{2,0}^{(1)}\left(x^{\prime}\right)\right\} \\
& +\left\{d_{2,0}^{(1)}(x), d_{2,0}^{(2)}\left(x^{\prime}\right)\right\}+\left\{d_{2,0}^{(2)}(x), d_{2,0}^{(2)}\left(x^{\prime}\right)\right\}  \tag{13.78}\\
= & 0 \tag{13.79}
\end{align*}
$$

As a corollary, we have $\left\{D_{2,0}(x)^{*}, D_{2,0}\left(x^{\prime}\right)^{*}\right\}=0$. Hence $D_{2,0}(x)^{2}=0$, as claimed.
13.3.4. Independence of $D_{1,0}$ and $D_{2,0}$. We calculate

$$
\begin{align*}
\left\{D_{1,0}(x), D_{2,0}\left(x^{\prime}\right)\right\} e^{i x^{\prime} \phi / \ell}= & \left\{d_{1,0}^{(1)}(x)+d_{1,0}^{(2)}(x), d_{2,0}^{(1)}\left(x^{\prime}\right)+d_{2,0}^{(2)}\left(x^{\prime}\right)\right\}  \tag{13.80}\\
= & \left\{d_{1,0}^{(1)}(x), d_{2,0}^{(1)}\left(x^{\prime}\right)\right\}+\left\{d_{1,0}^{(2)}(x), d_{2,0}^{(2)}\left(x^{\prime}\right)\right\} \\
& +\left\{d_{1,0}^{(1)}(x), d_{2,0}^{(2)}\left(x^{\prime}\right)\right\}+\left\{d_{1,0}^{(2)}(x), d_{2,0}^{(1)}\left(x^{\prime}\right)\right\}  \tag{13.81}\\
= & 0 \tag{13.82}
\end{align*}
$$

Thus, $D_{1,0}(x)$ and $D_{2,0}\left(x^{\prime}\right)$ are independent.
13.3.5. Translation Invariance of $Q_{1,0}$ and $Q_{2,0}$. Define the free supercharge densities

$$
\begin{align*}
& Q_{1,0}(x)=D_{1,0}(x)+D_{1,0}(x)^{*}  \tag{13.83}\\
& Q_{2,0}(x)=D_{2,0}(x)+D_{2,0}(x)^{*} \tag{13.84}
\end{align*}
$$

Since $D_{1,0}(x)$ and $D_{2,0}(x)$ are $\ell$-periodic, then so are $Q_{1,0}(x)$ and $Q_{2,0}(x)$.
13.3.6. Calculation of $Q_{1,0}^{2}$ and $Q_{2,0}^{2}$. In this section, we show how the squares $Q_{1,0}^{2}$ and $Q_{2,0}^{2}$ are related to the free Hamiltonian $H_{0}$ and momentum $P$.

Observe that the supercharges satisfy the following anti-commutator relations

$$
\begin{align*}
\left\{Q_{1,0}(x), Q_{1,0}\left(x^{\prime}\right)\right\}= & \left\{D_{1,0}(x)+D_{1,0}^{*}(x), D_{1,0}\left(x^{\prime}\right)+D_{1,0}^{*}\left(x^{\prime}\right)\right\}  \tag{13.85}\\
= & \left\{D_{1,0}(x), D_{1,0}\left(x^{\prime}\right)\right\}+\left\{D_{1,0}(x)^{*}, D_{1,0}\left(x^{\prime}\right)\right\}  \tag{13.86}\\
& +\left\{D_{1,0}(x), D_{1,0}\left(x^{\prime}\right)^{*}\right\}+\left\{D_{1,0}(x)^{*}, D_{1,0}\left(x^{\prime}\right)^{*}\right\} \\
= & \left\{D_{1,0}(x)^{*}, D_{1,0}\left(x^{\prime}\right)\right\}+\left\{D_{1,0}(x), D_{1,0}\left(x^{\prime}\right)^{*}\right\} \tag{13.87}
\end{align*}
$$

We calculate this in steps. First consider

$$
\begin{align*}
\left\{D_{1,0}(x), D_{1,0}\left(x^{\prime}\right)^{*}\right\}= & \left\{d_{1,0}^{(1)}(x)+d_{1,0}^{(2)}(x), d_{1,0}^{(1)}\left(x^{\prime}\right)^{*}+d_{1,0}^{(2)}\left(x^{\prime}\right)^{*}\right\}  \tag{13.88}\\
=\{ & \left\{d_{1,0}^{(1)}(x), d_{1,0}^{(1)}\left(x^{\prime}\right)^{*}\right\}+\left\{d_{1,0}^{(1)}(x), d_{1,0}^{(2)}\left(x^{\prime}\right)^{*}\right\} \\
& +\left\{d_{1,0}^{(2)}(x), d_{1,0}^{(1)}\left(x^{\prime}\right)^{*}\right\}+\left\{d_{1,0}^{(2)}(x), d_{1,0}^{(2)}\left(x^{\prime}\right)^{*}\right\} \tag{13.89}
\end{align*}
$$

We calculate the four anti-commutators:

$$
\begin{align*}
\left\{d_{1,0}^{(1)}(x), d_{1,0}^{(1)}\left(x^{\prime}\right)^{*}\right\}= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x) \pi_{j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*} \pi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right\}  \tag{13.90}\\
= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*}\right\} \pi_{j}(x) \pi_{j^{\prime}}\left(x^{\prime}\right)^{*} \\
& \quad-\psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*} \psi_{1, j}(x)\left[\pi_{j}(x), \pi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right]  \tag{13.91}\\
= & \sum_{j=1}^{n} \delta_{K_{1, j}^{f}}\left(x-x^{\prime}\right) \pi_{j}(x) \pi_{j}\left(x^{\prime}\right)^{*} \tag{13.92}
\end{align*}
$$

since $\left\{\psi_{1, j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*}\right\}=\delta_{j j^{\prime}} \delta_{K_{1, j}^{f}}\left(x-x^{\prime}\right)$ and $\left[\pi_{j}(x), \pi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right]=0$.

$$
\begin{align*}
\left\{d_{1,0}^{(2)}(x), d_{1,0}^{(2)}\left(x^{\prime}\right)^{*}\right\}= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x) \partial_{x} \varphi_{j}(x)^{*}, \psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*} \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)\right\}  \tag{13.93}\\
= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*}\right\} \partial_{x} \varphi_{j}(x)^{*} \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right) \\
& \quad-\psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*} \psi_{1, j}(x)\left[\partial_{x} \varphi_{j}(x)^{*}, \partial_{x} \varphi_{j^{\prime}}\left(x^{\prime}\right)\right] \\
= & \sum_{j=1}^{n} \delta_{K_{1, j}^{f}}\left(x-x^{\prime}\right) \partial_{x} \varphi_{j}(x)^{*} \partial_{x} \varphi_{j}\left(x^{\prime}\right) \tag{13.94}
\end{align*}
$$

since $\left[\partial_{x} \varphi_{j}(x), \partial_{x} \varphi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right]=\delta_{j j^{\prime}} \delta_{K_{j}^{b}}\left(x-x^{\prime}\right)$.

$$
\begin{align*}
\left\{d_{1,0}^{(1)}(x), d_{1,0}^{(2)}\left(x^{\prime}\right)^{*}\right\}= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x) \pi_{j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*} \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)\right\}  \tag{13.95}\\
= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*}\right\} \pi_{j}(x) \partial_{x} \varphi_{j^{\prime}}\left(x^{\prime}\right) \\
& \quad-\psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*} \psi_{1, j}(x)\left[\pi_{j}(x), \partial_{x} \varphi_{j^{\prime}}\left(x^{\prime}\right)\right] \\
= & \sum_{j=1}^{n} \delta_{K_{1, j}^{f}}\left(x-x^{\prime}\right) \pi_{j}(x) \partial_{x^{\prime}} \varphi_{j}\left(x^{\prime}\right) \\
& \quad-i \delta_{K_{j}^{b}}^{\prime}\left(x-x^{\prime}\right) \psi_{1, j}\left(x^{\prime}\right)^{*} \psi_{1, j}(x) \tag{13.96}
\end{align*}
$$

since $\left[\pi_{j}(x), \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)\right]=i \delta_{j j^{\prime}} \delta_{K_{j}^{b}}^{\prime}\left(x-x^{\prime}\right)$. Finally, consider

$$
\begin{align*}
\left\{d_{1,0}^{(2)}(x), d_{1,0}^{(1)}\left(x^{\prime}\right)^{*}\right\}= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x) \partial_{x} \varphi_{j}(x)^{*}, \psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*} \pi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right\}  \tag{13.97}\\
= & \sum_{j, j^{\prime}=1}^{n}\left\{\psi_{1, j}(x), \psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*}\right\} \partial_{x} \varphi_{j}(x)^{*} \pi_{j^{\prime}}\left(x^{\prime}\right)^{*} \\
& \quad-\psi_{1, j^{\prime}}\left(x^{\prime}\right)^{*} \psi_{1, j}(x)\left[\partial_{x} \varphi_{j}(x)^{*}, \pi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right]  \tag{13.98}\\
= & \sum_{j=1}^{n} \delta_{K_{1, j}^{f}}\left(x-x^{\prime}\right) \partial_{x} \varphi_{j}(x)^{*} \pi_{j}\left(x^{\prime}\right)^{*} \\
& +i \delta_{K_{j}^{b}}^{\prime}\left(x^{\prime}-x\right) \psi_{1, j}\left(x^{\prime}\right)^{*} \psi_{1, j}(x), \tag{13.99}
\end{align*}
$$

since $\left[\pi_{j}(x)^{*}, \partial_{x^{\prime}} \varphi_{j^{\prime}}\left(x^{\prime}\right)^{*}\right]=i \delta_{j j^{\prime}} \delta_{K_{j}^{b}}^{\prime}\left(x^{\prime}-x\right)$. Summing the four anti-commutators,

$$
\begin{align*}
Q_{1,0}^{2}= & \int_{0}^{\ell} \int_{0}^{\ell} d x d x^{\prime}\left(D_{1,0}(x)+D_{1,0}\left(x^{\prime}\right)^{*}\right)^{2}  \tag{13.100}\\
= & \int_{0}^{\ell} \int_{0}^{\ell} d x d x^{\prime}\left\{D_{1,0}(x), D_{1,0}\left(x^{\prime}\right)^{*}\right\}  \tag{13.101}\\
= & \int_{0}^{\ell} \int_{0}^{\ell} d x d x^{\prime} \sum_{j=1}^{n} \delta_{K_{1, j}^{f}}\left(x-x^{\prime}\right) \pi_{j}(x) \pi_{j}\left(x^{\prime}\right)^{*}+\delta_{K_{1, j}^{f}}\left(x-x^{\prime}\right) \partial_{x} \varphi_{j}(x)^{*} \partial_{x} \varphi_{j}\left(x^{\prime}\right) \\
& +\delta_{K_{1, j}^{f}}\left(x-x^{\prime}\right) \pi_{j}(x) \partial_{x^{\prime}} \varphi_{j}\left(x^{\prime}\right)-i \delta_{K_{j}^{b}}^{\prime}\left(x-x^{\prime}\right) \psi_{1, j}\left(x^{\prime}\right)^{*} \psi_{1, j}(x) \\
& +\delta_{K_{1, j}^{f}}\left(x-x^{\prime}\right) \partial_{x} \varphi_{j}(x)^{*} \pi_{j}\left(x^{\prime}\right)^{*}+i \delta_{K_{j}^{b}}^{\prime}\left(x^{\prime}-x\right) \psi_{1, j}\left(x^{\prime}\right)^{*} \psi_{1, j}(x) \tag{13.102}
\end{align*}
$$

After an integration over $x^{\prime}$ and an integration by parts, we find that

$$
\begin{align*}
Q_{1,0}^{2}= & \int_{0}^{\ell} d x \sum_{j=1}^{n} \pi_{j}(x) \pi_{j}(x)^{*}+\partial_{x} \varphi_{j}(x)^{*} \partial_{x} \varphi_{j}(x) \\
& +\int_{0}^{\ell} d x \sum_{j=1}^{n} \pi_{j}(x) \partial_{x} \varphi_{j}(x)+\partial_{x} \varphi_{j}(x)^{*} \pi_{j}(x)^{*} \\
& -2 i \int_{0}^{\ell} d x \sum_{j=1}^{n} \psi_{1, j}(x)^{*} \partial_{x} \psi_{1, j}\left(x^{\prime}\right)  \tag{13.103}\\
= & H_{0}^{b}+P_{0}^{b}+\left(H_{0}^{f}+P_{0}^{f}\right)=H_{0}+P_{0} \tag{13.104}
\end{align*}
$$

Similarly,

$$
\begin{align*}
Q_{2,0}^{2} & =\int_{0}^{\ell} \int_{0}^{\ell} d x d x^{\prime}\left(D_{2,0}(x)+D_{2,0}^{*}\left(x^{\prime}\right)\right)^{2}  \tag{13.105}\\
& =\int_{0}^{\ell} \int_{0}^{\ell} d x d x^{\prime}\left\{D_{2,0}(x), D_{2,0}\left(x^{\prime}\right)^{*}\right\} \tag{13.106}
\end{align*}
$$

Define the following supercharges

$$
\begin{align*}
Q_{\alpha}(x) & =\int_{0}^{\ell} d x D_{\alpha}(x)+D_{\alpha}(x)^{*}  \tag{13.107}\\
& =Q_{\alpha, 0}+\lambda Q_{\alpha, I} . \tag{13.108}
\end{align*}
$$

Since $D_{\alpha, 0}(x)$ is nilpotent for $\alpha \in\{1,2\}$, we have

$$
\begin{align*}
Q_{\alpha, 0}(x)^{2} & =\int_{0}^{\ell} d x\left(D_{\alpha, 0}(x)+D_{\alpha, 0}(x)^{*}\right)^{2}  \tag{13.109}\\
& =\int_{0}^{\ell} d x\left\{D_{\alpha, 0}(x), D_{\alpha, 0}(x)^{*}\right\} . \tag{13.110}
\end{align*}
$$

Problem 13.3.1. Prove the nilpotence of $D_{1, I}$ and $D_{2, I}$, the independence of $D_{1, I}$ and $D_{2, I}$, the translation Invariance of $Q_{1,0}$ and $Q_{2,0}$, and finally compute of $Q_{1, I}^{2}$ and $Q_{2, I}^{2}$.
13.4. Supercharge Regularization

Define $Q_{+}(\lambda)=D_{1}(\lambda)+D_{1}(\lambda)^{*}$ (not to be confused with the Dirichlet kernel $D_{\kappa}$ ), where

$$
\begin{align*}
D_{1}(\lambda)= & \int_{0}^{\ell} D_{1,0}(x)+D_{1, I}(x) d x  \tag{13.111}\\
= & i \int_{0}^{\ell} \sum_{j=1}^{n} \psi_{1, j}(x)\left[\partial_{x} \varphi_{j}(x)^{*}-\pi_{j}(x)\right] d x \\
& +\lambda \int_{0}^{\ell} \sum_{j=1}^{n} \psi_{2, j}(x)\left(V_{j} \circ \varphi\right)(x)^{*} d x \tag{13.112}
\end{align*}
$$

Since $D_{1}=D_{1}(\lambda)$ is nilpotent, $Q_{+}^{2}=\left(D_{1}+D_{1}^{*}\right)^{2}=\left\{D_{1}, D_{1}^{*}\right\}$ and

$$
\begin{equation*}
\left\{D_{1}, D_{1}^{*}\right\}=\left\{D_{1,0}, D_{1,0}^{*}\right\}+\left\{D_{1,0}, D_{1, I}^{*}\right\}+\left\{D_{1,0}^{*}, D_{1, I}\right\}+\left\{D_{1, I}, D_{1, I}^{*}\right\} \tag{13.113}
\end{equation*}
$$

We will show that $Q_{+}(\lambda)^{2}=H(\lambda)+P$. We calculate

$$
\begin{align*}
\left\{D_{1, I}, D_{1, I}^{*}\right\}= & \int_{0}^{\ell} \int_{0}^{\ell}\left\{D_{1, I}(x), D_{1, I}\left(x^{\prime}\right)^{*}\right\} d x d x^{\prime}  \tag{13.114}\\
= & \lambda^{2} \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell}\left\{\psi_{2, j}(x)\left(V_{j} \circ \varphi\right)(x)^{*}, \psi_{2, j^{\prime}}\left(x^{\prime}\right)^{*}\left(V_{j^{\prime}} \circ \varphi\right)\left(x^{\prime}\right)\right\} d x d x^{\prime} \\
= & \lambda^{2} \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell}\left\{\psi_{2, j}(x), \psi_{2, j^{\prime}}\left(x^{\prime}\right)^{*}\right\}\left(V_{j} \circ \varphi\right)(x)^{*}\left(V_{j^{\prime}} \circ \varphi\right)\left(x^{\prime}\right) d x d x^{\prime} \\
& +\lambda^{2} \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell} \psi_{2, j^{\prime}}\left(x^{\prime}\right)^{*} \psi_{2, j}(x)\left[\left(V_{j^{\prime}} \circ \varphi\right)\left(x^{\prime}\right),\left(V_{j} \circ \varphi\right)(x)^{*}\right] d x d x^{\prime} \\
= & \lambda^{2} \sum_{j=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell} \delta_{K_{2, j}^{f}}\left(x-x^{\prime}\right)\left(V_{j} \circ \varphi\right)(x)^{*}\left(V_{j} \circ \varphi\right)\left(x^{\prime}\right) d x d x^{\prime}  \tag{13.115}\\
= & \lambda^{2} \sum_{j=1}^{n} \int_{0}^{\ell}\left|\left(V_{j} \circ \varphi\right)(x)\right|^{2} d x \tag{13.116}
\end{align*}
$$

since the bosonic fields commute with their adjoints. We calculate

$$
\begin{align*}
\left\{D_{1,0}, D_{1, I}^{*}\right\}= & \int_{0}^{\ell} \int_{0}^{\ell}\left\{D_{1,0}(x), D_{1, I}\left(x^{\prime}\right)^{*}\right\} d x d x^{\prime}  \tag{13.117}\\
= & i \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell}\left\{\psi_{1, j}(x)\left[\partial_{x} \varphi_{j}(x)^{*}-\pi_{j}(x)\right], \psi_{2, j^{\prime}}\left(x^{\prime}\right)^{*}\left(V_{j^{\prime}} \circ \varphi\right)\left(x^{\prime}\right)\right\} d x d x^{\prime} \\
= & i \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell}\left\{\psi_{1, j}(x), \psi_{2, j^{\prime}}\left(x^{\prime}\right)^{*}\right\}\left(\partial_{x} \varphi_{j}(x)^{*}-\pi_{j}(x)\right)\left(V_{j^{\prime}} \circ \varphi\right)\left(x^{\prime}\right) d x d x^{\prime} \\
& +i \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell} \psi_{2, j^{\prime}}\left(x^{\prime}\right)^{*} \psi_{1, j}(x)\left[\left(V_{j^{\prime}} \circ \varphi\right)\left(x^{\prime}\right), \partial_{x} \varphi_{j}(x)^{*}-\pi_{j}(x)\right] d x d x^{\prime} \\
= & i \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell} \psi_{2, j^{\prime}}\left(x^{\prime}\right)^{*} \psi_{1, j}(x)\left[\left(V_{j^{\prime}} \circ \varphi\right)\left(x^{\prime}\right), \partial_{x} \varphi_{j}(x)^{*}-\pi_{j}(x)\right] d x d x^{\prime} \\
= & \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \psi_{2, j^{\prime}}(x)^{*} \psi_{1, j}(x)\left(V_{j j^{\prime}} \circ \varphi\right)(x) d x, \tag{13.118}
\end{align*}
$$

since

$$
\begin{align*}
{\left[\varphi_{j}^{r_{j}}(x), \pi_{j^{\prime}}\left(x^{\prime}\right)\right] } & =\varphi_{j}^{r_{j}-1}(x)\left[\varphi_{j}(x), \pi_{j^{\prime}}\left(x^{\prime}\right)\right]+\left[\varphi_{j}^{r_{j}-1}(x), \pi_{j^{\prime}}\left(x^{\prime}\right)\right] \varphi_{j}(x)  \tag{13.119}\\
& =i \delta_{j j^{\prime}} r_{j} \varphi_{j}^{r_{j}-1}(x) \delta_{K_{j}^{b}}\left(x-x^{\prime}\right) \tag{13.120}
\end{align*}
$$

by induction and, as a result,

$$
\begin{equation*}
\sum_{j^{\prime}=1}^{n}\left[\left(V_{j^{\prime}} \circ \varphi\right)\left(x^{\prime}\right), \pi_{j}(x)\right]=i \sum_{j, j^{\prime}=1}^{n}\left(V_{j j^{\prime}} \circ \varphi\right)\left(x^{\prime}\right) \delta_{K_{j}^{b}}\left(x^{\prime}-x\right) . \tag{13.121}
\end{equation*}
$$

Recall that the bosonic interaction is simply

$$
\begin{equation*}
H_{I}^{b}(\lambda)=\lambda^{2} \sum_{j=1}^{n} \int_{0}^{\ell}\left|\left(V_{j} \circ \varphi\right)(x)\right|^{2} d x \tag{13.122}
\end{equation*}
$$

and the boson-fermion interaction is given by

$$
\begin{align*}
H_{I}^{b f}(\lambda)=\lambda & \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \psi_{1, j}(x) \psi_{2, j^{\prime}}(x)^{*}\left(V_{j j^{\prime}} \circ \varphi\right)(x) \\
& +\psi_{1, j^{\prime}}(x)^{*} \psi_{2, j}(x)\left(V_{j j^{\prime}} \circ \varphi\right)(x)^{*} d x \tag{13.123}
\end{align*}
$$

It follows, then, that

$$
\begin{align*}
H(\lambda) & =H_{0}+H_{I}^{b}(\lambda)+H_{I}^{b f}(\lambda)  \tag{13.124}\\
& =Q_{+}(\lambda)^{2}-P . \tag{13.125}
\end{align*}
$$

Jaffe proves that $Q_{-}(\lambda)$ does not satisfy the dual relation, namely, $Q_{-}(\lambda)^{2}=$ $H(\lambda)-P$, but rather has an error term proportional to $\phi$. Thus, said supersymmetry (algebra) is broken.

### 13.5. Three Regularization Procedures

We now consider three regularization procedures in order of decreasing degree. The first, the most naive, is to cutoff all fields, both bosonic and fermionic, in $H(\lambda)$. The second, one that seems more reasonable, is to cutoff the fields in interaction derivation $D_{1, I}$. The third is to cutoff only those bosonic fields in $D_{1, I}$, namely, through the gradient $(\partial V \circ \varphi)(x)$. We will concern ourselves with the last two procedures, since these are non-trivial.

Consider the fourier representation of fermions on the unshifted lattice $\hat{\mathbb{T}}$,

$$
\begin{equation*}
\psi_{\alpha, j}(x)=\frac{e^{i \Omega_{\alpha, j} \phi x / \ell}}{\sqrt{\ell}} \sum_{k \in \hat{\mathbb{T}}} \tilde{f}_{\alpha, j}(k) e^{-i k x} . \tag{13.126}
\end{equation*}
$$

Set $\tilde{f}_{\alpha, j, \kappa}(k)=\tilde{D}_{\kappa}(k) \tilde{f}_{\alpha, j}(k)$ and denote the corresponding regularized fermionic field $\psi_{\alpha, j, \kappa}$. We apply a sharp cutoff to the fields in $D_{1, I}$,

$$
\begin{equation*}
D_{1, I, \kappa}(\lambda)=\lambda \int_{0}^{\ell} \sum_{j=1}^{n} \psi_{2, j, \kappa}(x)\left(V_{j} \circ \varphi_{\kappa}\right)(x)^{*} d x \tag{13.127}
\end{equation*}
$$

It is easy to see that

$$
\begin{align*}
\left\{D_{1, I, \kappa}, D_{1, I, \kappa}^{*}\right\} & =\lambda^{2} \sum_{j=1}^{n} \int_{0}^{\ell}\left|\left(V_{j} \circ \varphi_{\kappa}\right)(x)\right|^{2} d x  \tag{13.128}\\
& =H_{I, \kappa}^{b}(\lambda) \tag{13.129}
\end{align*}
$$

and

$$
\begin{align*}
\left\{D_{1,0}, D_{1, I, \kappa}^{*}\right\}= & i \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell}\left\{\psi_{1, j}(x)\left[\partial_{x} \varphi_{j}(x)^{*}-\pi_{j}(x)\right], \psi_{2, j^{\prime}, \kappa}\left(x^{\prime}\right)^{*}\left(V_{j^{\prime}} \circ \varphi_{\kappa}\right)\left(x^{\prime}\right)\right\} d x d x^{\prime} \\
= & i \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell}\left\{\psi_{1, j}(x), \psi_{2, j^{\prime}, \kappa}\left(x^{\prime}\right)^{*}\right\}\left(\partial_{x} \varphi_{j}(x)^{*}-\pi_{j}(x)\right)\left(V_{j^{\prime}} \circ \varphi_{\kappa}\right)\left(x^{\prime}\right) d x d x^{\prime} \\
& +i \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell} \psi_{2, j^{\prime}, \kappa}\left(x^{\prime}\right)^{*} \psi_{1, j}(x)\left[\left(V_{j^{\prime}} \circ \varphi_{\kappa}\right)\left(x^{\prime}\right), \partial_{x} \varphi_{j}(x)^{*}-\pi_{j}(x)\right] d x d x^{\prime} \\
= & i \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell} \psi_{2, j^{\prime}, \kappa}\left(x^{\prime}\right)^{*} \psi_{1, j}(x)\left[\left(V_{j^{\prime}} \circ \varphi_{\kappa}\right)\left(x^{\prime}\right), \partial_{x} \varphi_{j}(x)^{*}-\pi_{j}(x)\right] d x d x^{\prime} \\
= & \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \psi_{2, j^{\prime}, \kappa}(x)^{*} \psi_{1, j}(x)\left(V_{j j^{\prime}} \circ \varphi_{\kappa}\right)(x) d x \tag{13.130}
\end{align*}
$$

Hence,

$$
\begin{align*}
\left\{D_{1,0}, D_{1, I, \kappa}^{*}\right\}+\left\{D_{1,0}^{*}, D_{1, I, \kappa}\right\}= & \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \psi_{1, j}(x) \psi_{2, j^{\prime}, \kappa}(x)^{*}\left(V_{j j^{\prime}} \circ \varphi_{\kappa}\right)(x)  \tag{13.131}\\
& +\psi_{1, j}(x)^{*} \psi_{2, j^{\prime}, \kappa}(x)\left(V_{j j^{\prime}} \circ \varphi_{\kappa}\right)(x)^{*} d x \\
= & H_{I, \kappa}^{b f}+H_{I, \kappa}^{b f-\mathrm{high}, 1} \tag{13.132}
\end{align*}
$$

where

$$
\begin{align*}
& H_{I, \kappa}^{b f}(\lambda)=\lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \psi_{1, j, \kappa}(x) \psi_{2, j^{\prime}, \kappa}(x)^{*}\left(V_{j j^{\prime}} \circ \varphi_{\kappa}\right)(x) \\
& +\psi_{1, j^{\prime}, \kappa}(x)^{*} \psi_{2, j, \kappa}(x)\left(V_{j j^{\prime}} \circ \varphi_{\kappa}\right)(x)^{*} d x  \tag{13.133}\\
& =\frac{\lambda}{\sqrt{\ell}} \sum_{j, j^{\prime}=1}^{n} \sum_{k, k^{\prime}, k^{\prime \prime} \in \hat{\mathbb{T}}_{k}} \tilde{f}_{1, j}(k) \tilde{f}_{2, j^{\prime}}\left(k^{\prime}\right)^{*} F_{j, j^{\prime}}\left(k^{\prime \prime}\right) \\
& +\tilde{f}_{1, j}(k)^{*} \tilde{f}_{2, j^{\prime}}\left(k^{\prime}\right) F_{j, j^{\prime}}\left(k^{\prime \prime}\right)^{*} \tag{13.134}
\end{align*}
$$

and

$$
\begin{gather*}
H_{I, \kappa}^{b f-\text { high,1 }}(\lambda)=\frac{\lambda}{\sqrt{\ell}} \sum_{j, j^{\prime}=1}^{n} \sum_{k \in \hat{\mathbb{T}} \backslash \hat{\mathbb{T}}_{k}} \sum_{k^{\prime}, k^{\prime \prime} \in \hat{\mathbb{T}}_{\kappa}} \tilde{f}_{1, j}(k) \tilde{f}_{2, j^{\prime}}\left(k^{\prime}\right)^{*} F_{j, j^{\prime}}\left(k^{\prime \prime}\right) \\
+\tilde{f}_{1, j}(k)^{*} \tilde{f}_{2, j^{\prime}}\left(k^{\prime}\right) F_{j, j^{\prime}}\left(k^{\prime \prime}\right)^{*} . \tag{13.135}
\end{gather*}
$$

Then, it follows that

$$
\begin{equation*}
Q_{\kappa}(\lambda)^{2}=H_{\kappa}(\lambda)+P+H_{I, \kappa}^{b f-\text { high }, 1} \tag{13.136}
\end{equation*}
$$

By a similar analysis, we can just cutoff the bosons in $D_{1, I}$. The result is

$$
\begin{equation*}
Q_{\kappa}(\lambda)^{2}=H_{\kappa}(\lambda)+P+H_{I, \kappa}^{b f-\text { high, } 1,2} \tag{13.137}
\end{equation*}
$$

where

$$
\begin{align*}
H_{I, \kappa}^{b f-\text { high }, 1,2}(\lambda)= & \frac{\lambda}{\sqrt{\ell}} \sum_{j, j^{\prime}=1}^{n} \sum_{k, k^{\prime} \in \hat{\mathbb{T}} \backslash \hat{\mathbb{T}}_{\kappa}} \sum_{k^{\prime \prime} \in \hat{\mathbb{T}}_{\kappa}} \tilde{f}_{1, j}(k) \tilde{f}_{2, j^{\prime}}\left(k^{\prime}\right)^{*} F_{j, j^{\prime}}\left(k^{\prime \prime}\right) \\
& +\tilde{f}_{1, j}(k)^{*} \tilde{f}_{2, j^{\prime}}\left(k^{\prime}\right) F_{j, j^{\prime}}\left(k^{\prime \prime}\right)^{*} . \tag{13.138}
\end{align*}
$$

It is unclear whether or not the operators $H_{I, \kappa}^{b f, \text { high, } 1}$ and $H_{I, \kappa}^{b f, \text { high,1,2 }}$ are positive.

### 13.6. Sharply-Regularized Free Hamiltonian

Introduce $2 n$ bosonic and $2 n$ fermionic number operators

$$
\begin{align*}
& N_{ \pm, j}^{b}(k)=a_{ \pm, j}(k)^{*} a_{ \pm, j}(k)  \tag{13.139}\\
& N_{ \pm, j}^{f}(k)=b_{ \pm, j}(k)^{*} b_{ \pm, j}(k) \tag{13.140}
\end{align*}
$$

respectively. Consider the free, sharply-regularized, free total Hamiltonian

$$
\begin{equation*}
H_{0, \kappa}=H_{0, \kappa}^{b} \otimes I+I \otimes H_{0, \kappa}^{f} \tag{13.141}
\end{equation*}
$$

as the sum involving a sharply-regularized, free bosonic Hamiltonian

$$
\begin{align*}
H_{0, \kappa}^{b} & =\sum_{j=1}^{n}\left\|\pi_{j, \kappa}\right\|_{2}^{2}+\left\|\partial \varphi_{j, \kappa}\right\|_{2}^{2}  \tag{13.142}\\
& =\sum_{j=1}^{n} \int_{0}^{\ell}\left|\pi_{j, \kappa}(x)\right|^{2}+\left|\partial \varphi_{j, \kappa}(x)\right|^{2} d x  \tag{13.143}\\
& =E_{0, \kappa}^{b}(\phi)+\sum_{j=1}^{n} \sum_{k \in K_{j, \kappa}^{b}}|k|\left(N_{+, j}(k)+N_{-, j}(-k)\right) \tag{13.144}
\end{align*}
$$

and the sharply-regularized, free fermionic Hamiltonian

$$
\begin{align*}
H_{0, \kappa}^{f} & =\sum_{j=1}^{n} \int_{0}^{\ell} \bar{\psi}_{j, \kappa}(x)\left(\sigma_{1} \partial\right) \psi_{j, \kappa}(x) d x  \tag{13.145}\\
& =-i \sum_{j=1}^{n} \int_{0}^{\ell} \psi_{1, j, \kappa}(x)^{*} \partial \psi_{1, j, \kappa}(x)+\psi_{2, j, \kappa}(x)^{*} \partial \psi_{2, j, \kappa}(x) d x  \tag{13.146}\\
& \left.=E_{0, \kappa}^{f}(\phi)+\sum_{j=1}^{n}\left(\sum_{k \in K_{+, j, \kappa}^{f}}|k| N_{+, j}(k)+\sum_{k \in K_{-, j, \kappa}^{f}}|k| N_{-, j}(-k)\right)\right) \tag{13.147}
\end{align*}
$$

where $E_{0, \kappa}^{b}(\phi)$ and $E_{0, \kappa}^{f}(\phi)$ are finite for $\kappa<\infty$ but otherwise diverge in the limit $\kappa \rightarrow \infty$. The bosonic zero-point energy is not independent of the twist angle $\phi$. If
$\phi>0$, we sum over $n$ translated lattices,

$$
\begin{align*}
E_{0, \kappa}^{b}(\phi) & =\sum_{j=1}^{n} \sum_{k \in K_{j, k}^{b}}|k|  \tag{13.148}\\
& =\sum_{j=1}^{n}\left(\sum_{n=0}^{N_{j}-1}\left(\frac{2 \pi n}{\ell}+\frac{\Omega_{j} \phi}{\ell}\right)+\sum_{n=1}^{N_{j}}\left(\frac{2 \pi n}{\ell}-\frac{\Omega_{j} \phi}{\ell}\right)\right)  \tag{13.149}\\
& =\frac{2 \pi}{\ell} \sum_{j=1}^{n} \sum_{n=1}^{N_{j}} \frac{2 n}{1+\delta_{n N_{j}}}=\frac{2 \pi}{\ell} \sum_{j=1}^{n} N_{j}^{2}, \tag{13.150}
\end{align*}
$$

which follows from the elementary sum

$$
\begin{equation*}
N+\sum_{n=1}^{N-1} 2 n=N+N(N-1)=N^{2} . \tag{13.151}
\end{equation*}
$$

That is,

$$
\begin{equation*}
E_{0, \kappa}^{b}(\phi)=\frac{\pi}{2 \ell} \sum_{j=1}^{n}\left|K_{j, \kappa}^{b}\right|^{2} . \tag{13.152}
\end{equation*}
$$

If $\phi=0$, then we sum over $n$ symmetric lattices,

$$
\begin{align*}
E_{0, \kappa}^{b}(0) & =\sum_{j=1}^{n} \sum_{k \in \mathbb{T}_{j, k}}|k|  \tag{13.153}\\
& =\frac{2 \pi}{\ell} \sum_{j=1}^{n} \sum_{n=-N_{j}+1}^{N_{j}-1}|n|  \tag{13.154}\\
& =\frac{2 \pi}{\ell} \sum_{j=1}^{n} N_{j}\left(N_{j}-1\right)=\frac{4 \pi}{\ell} \sum_{j=1}^{n}\binom{N_{j}}{2} . \tag{13.155}
\end{align*}
$$

That is,

$$
\begin{equation*}
E_{0, \kappa}^{b}(0)=\frac{\pi}{2 \ell} \sum_{j=1}^{n}\left(\left|\hat{\mathbb{T}}_{j, \kappa}\right|^{2}-1\right) \tag{13.156}
\end{equation*}
$$

Thus, for all non-negative values of the twist angle $\phi$, the bosonic zero-point energy is simply

$$
\begin{equation*}
E_{0, \kappa}^{b}(\phi)=\frac{2 \pi}{\ell} \sum_{j=1}^{n} N_{j}\left(N_{j}-\delta_{\phi, 0}\right) \tag{13.157}
\end{equation*}
$$

Recall that

$$
\begin{equation*}
\sum_{k \in K_{+, j, k}^{f}}|k|=\sum_{k \in K_{1, j, k}^{f}} k \theta(k)-\sum_{k \in K_{2, j, k}^{f}} k \theta(-k) . \tag{13.158}
\end{equation*}
$$

Using the identities $k=k(\theta(k)+\theta(-k))$ and $|k|=k(\theta(k)-\theta(-k))$, we can evaluate the fermionic zero-point energy. If $\phi>0$, we sum over translated
lattices,

$$
\begin{align*}
E_{0, \kappa}^{f}(\phi)= & -\sum_{j=1}^{n} \sum_{k \in K_{+, j, \kappa}^{f}}|k|  \tag{13.159}\\
= & -\frac{1}{2} \sum_{j=1}^{n}\left(\sum_{k \in K_{1, j, k}^{f}} k+|k|-\sum_{k^{\prime} \in K_{2, j, \kappa}^{f}} k^{\prime}-\left|k^{\prime}\right|\right)  \tag{13.160}\\
= & -\frac{1}{2} \sum_{j=1}^{n}\left(\frac{\pi}{2 \ell}\left|K_{1, j, \kappa}^{f}\right|^{2}-\frac{\Omega_{j} \phi}{\ell}\left|K_{1, j, \kappa}^{f}\right|\right) \\
& -\frac{1}{2}\left(\frac{\pi}{2 \ell}\left|K_{2, j, \kappa}^{f}\right|^{2}+\frac{\left(1-\Omega_{j}\right) \phi}{\ell}\left|K_{2, j, \kappa}^{f}\right|\right) \\
=- & \frac{\pi}{4 \ell} \sum_{j=1}^{n}\left(\left|K_{1, j, \kappa}^{f}\right|^{2}+\left|K_{2, j, \kappa}^{f}\right|^{2}\right)+\frac{\phi}{2 \ell} \sum_{j=1}^{n}\left(\Omega_{j}\left|K_{1, j, \kappa}^{f}\right|\right. \\
& \left.-\left(1-\Omega_{j}\right)\left|K_{2, j, \kappa}^{f}\right|\right) . \tag{13.161}
\end{align*}
$$

If $\phi=0$, we sum over symmetric lattices,

$$
\begin{align*}
E_{0, \kappa}^{f}(0) & =-\frac{1}{2} \sum_{j=1}^{n}\left(\sum_{k \in \mathbb{\mathbb { T }}_{1, j, k}} k+|k|-\sum_{k^{\prime} \in \hat{\mathbb{T}}_{2, j, k}} k^{\prime}-\left|k^{\prime}\right|\right)  \tag{13.162}\\
& =-\frac{\pi}{\ell} \sum_{j=1}^{n}\left(\sum_{n \in \mathbb{Z}_{1, j, N}} n+|n|-\sum_{n^{\prime} \in \mathbb{Z}_{2, j, N}} n^{\prime}-\left|n^{\prime}\right|\right)  \tag{13.163}\\
& =-\frac{\pi}{\ell} \sum_{j=1}^{n}\left(\sum_{n \in \mathbb{Z}_{1, j, N}}|n|+\sum_{n^{\prime} \in \mathbb{Z}_{2, j, N}}\left|n^{\prime}\right|\right)  \tag{13.164}\\
& =-\frac{\pi}{\ell} \sum_{j=1}^{n}\left(N_{1, j}\left(N_{1, j}-1\right)+N_{2, j}\left(N_{2, j}-1\right) .\right) \tag{13.165}
\end{align*}
$$

Thus, for all non-negative values of the twist angle $\phi$, the fermionic zero-point energy is simply

$$
\begin{align*}
E_{0, \kappa}^{f}(\phi)=- & \frac{\pi}{\ell} \sum_{j=1}^{n}\left(N_{1, j}\left(N_{1, j}-\delta_{\phi, 0}\right)+N_{2, j}\left(N_{2, j}-\delta_{\phi, 0}\right)\right) \\
& +\frac{\phi}{\ell} \sum_{j=1}^{n}\left(\Omega_{j} N_{1, j}-\left(1-\Omega_{j}\right) N_{2, j}\right) \tag{13.166}
\end{align*}
$$

Hence, the total zero-point energy is the sum

$$
\begin{align*}
E_{0, \kappa}= & E_{0, \kappa}^{b}+E_{0, \kappa}^{f}  \tag{13.167}\\
= & \frac{\pi}{2 \ell} \sum_{j=1}^{n}\left[\left|K_{j, \kappa}^{b}\right|^{2}-\frac{1}{2}\left(\left|K_{1, j, \kappa}^{f}\right|^{2}+\left|K_{2, j, \kappa}^{f}\right|^{2}\right)\right] \\
& +\frac{\phi}{2 \ell} \sum_{j=1}^{n}\left(\Omega_{j}\left|K_{1, j, \kappa}^{f}\right|-\left(1-\Omega_{j}\right)\left|K_{2, j, \kappa}^{f}\right|\right) .
\end{align*}
$$

The discontinuity of the total zero-point energy $E_{0, \kappa}(\phi)$ at $\phi=0$ is a manifestation of a spontaneous breaking of $\mathbb{Z}_{2}$-symmetry of the momentum lattices from a symmetric lattice to a translated lattice when $\phi>0$. We regard this as a twist quantum phase transition.

Suppose $\left|K_{j, \kappa}^{b}\right|=\left|K_{1, j, \kappa}^{f}\right|=\left|K_{2, j, k}^{f}\right|=2 N$ for $1 \leqslant j \leqslant n$, then

$$
\begin{equation*}
E_{0, \kappa}=-\frac{\phi \hat{c}}{\ell} N=-\frac{\phi \hat{c}}{2 \pi} \kappa . \tag{13.168}
\end{equation*}
$$

With $\phi=0$ and the aforementioned choice of cutoffs, the total Hamiltonian $H_{0}=\mathrm{s}-\lim _{\kappa \rightarrow \infty} H_{0, \kappa}$ need not be normal ordered.

### 13.7. Sharply-Regularized Momentum

Consider the total, sharply-regularized momentum operator

$$
\begin{equation*}
P_{\kappa}=P_{\kappa}^{b} \otimes I+I \otimes P_{\kappa}^{f} \tag{13.169}
\end{equation*}
$$

which is the sum involving the sharply-regularized bosonic momentum,

$$
\begin{align*}
P_{\kappa}^{b} & =-\sum_{j=1}^{n} \int_{0}^{\ell} \pi_{j, \kappa}(x) \partial \varphi_{j, \kappa}(x)+\partial \varphi_{j, \kappa}(x)^{*} \pi_{j, \kappa}(x)^{*} d x  \tag{13.170}\\
& =p_{0, \kappa}^{b}+\sum_{j=1}^{n} \sum_{k \in K_{j, \kappa}^{b}} k\left(N_{+, j}^{b}(k)-N_{-, j}^{b}(-k)\right), \tag{13.171}
\end{align*}
$$

and the sharply-regularized fermionic momentum,

$$
\begin{align*}
P_{\kappa}^{f} & =-i \sum_{j=1}^{n} \int_{0}^{\ell} \psi_{1, j, \kappa}(x)^{*} \partial \psi_{1, j, \kappa}(x)-\psi_{2, j, \kappa}(x)^{*} \partial \psi_{2, j, \kappa}(x) d x  \tag{13.172}\\
& =p_{0, \kappa}^{f}+\sum_{j=1}^{n}\left(\sum_{k \in K_{+, j, \kappa}^{f}} k N_{+, j}^{f}(k)-\sum_{k \in K_{-, j, \kappa}^{f}} k N_{-, j}^{f}(-k)\right), \tag{13.173}
\end{align*}
$$

where $p_{0, \kappa}^{b}$ and $p_{0, \kappa}^{f}$ are finite constants for $\kappa<\infty$ but otherwise diverge in the limit $\kappa \rightarrow \infty$. In particular,

$$
\begin{equation*}
p_{0, \kappa}^{b}=\sum_{j=1}^{n} \sum_{k \in K_{j, \kappa}^{b}} k=-\frac{\phi}{\ell} \sum_{j=1}^{n} \Omega_{j}\left|K_{j, \kappa}^{b}\right|, \tag{13.174}
\end{equation*}
$$

and recalling that

$$
\begin{equation*}
\sum_{k \in K_{+, j, \kappa}^{f}} k=\sum_{k \in K_{1, j, k}^{f}} k \theta(k)+\sum_{k \in K_{2, j, k}^{f}} k \theta(-k) \tag{13.175}
\end{equation*}
$$

we find

$$
\begin{align*}
p_{0, \kappa}^{f}= & -\sum_{j=1}^{n} \sum_{k \in K_{+, j, \kappa}^{f}} k  \tag{13.176}\\
= & -\frac{1}{2} \sum_{j=1}^{n}\left(\sum_{k \in K_{1, j, \kappa}^{f}} k+|k|+\sum_{k \in K_{2, j, \kappa}^{f}} k^{\prime}-\left|k^{\prime}\right|\right)  \tag{13.177}\\
= & -\frac{1}{2} \sum_{j=1}^{n}\left(\frac{\pi}{2 \ell}\left|K_{1, j, \kappa}^{f}\right|^{2}-\frac{\Omega_{j} \phi}{\ell}\left|K_{1, j, \kappa}^{f}\right|\right) \\
& +\frac{1}{2}\left(\frac{\pi}{2 \ell}\left|K_{2, j, \kappa}^{f}\right|^{2}+\frac{\left(1-\Omega_{j}\right) \phi}{\ell}\left|K_{2, j, \kappa}^{f}\right|\right)  \tag{13.178}\\
=- & \frac{\pi}{4 \ell} \sum_{j=1}^{n}\left(\left|K_{1, j, \kappa}^{f}\right|^{2}-\left|K_{2, j, \kappa}^{f}\right|^{2}\right) \\
& +\frac{\phi}{2 \ell} \sum_{j=1}^{n}\left(\Omega_{j}\left|K_{1, j, \kappa}^{f}\right|+\left(1-\Omega_{j}\right)\left|K_{2, j, \kappa}^{f}\right|\right) . \tag{13.179}
\end{align*}
$$

Thus, the total zero-point momentum is the sum

$$
\begin{align*}
p_{0, \kappa}= & p_{0, \kappa}^{b}+p_{0, \kappa}^{f}  \tag{13.180}\\
=- & \frac{\pi}{4 \ell} \sum_{j=1}^{n}\left(\left|K_{1, j, \kappa}^{f}\right|^{2}-\left|K_{2, j, \kappa}^{f}\right|^{2}\right) \\
& +\frac{\phi}{2 \ell} \sum_{j=1}^{n}\left(\Omega_{j}\left(\left|K_{1, j, \kappa}^{f}\right|-2\left|K_{j, \kappa}^{b}\right|\right)+\left(1-\Omega_{j}\right)\left|K_{2, j, \kappa}^{f}\right|\right) . \tag{13.181}
\end{align*}
$$

Suppose $\left|K_{j, \kappa}^{b}\right|=\left|K_{1, j, \kappa}^{f}\right|=\left|K_{2, j, \kappa}^{f}\right|=2 N$ for $1 \leqslant j \leqslant n$, then

$$
\begin{align*}
p_{0, \kappa} & =\frac{\phi \hat{c}}{\ell} N  \tag{13.182}\\
& =\frac{\phi \hat{c}}{2 \pi} \kappa . \tag{13.183}
\end{align*}
$$

With $\phi=0$ and the aforementioned choice of the cutoffs, the momentum $P=$ $\mathrm{s}-\lim _{\kappa \rightarrow \infty} P_{\mathcal{K}}$ need not be normal ordered.

### 13.8. Sharply-Regularized Charge Operators

The free bosonic Lagrangian for $n$ complex bosonic fields is given by

$$
\begin{align*}
\mathscr{L}^{b} & =\int_{0}^{\ell} \mathscr{L}_{0}^{b}(x) d x  \tag{13.184}\\
& =\int_{0}^{\ell} \sum_{j=1}^{n} \pi_{j}(x)^{*} \pi_{j}(x)-\partial_{x} \varphi_{j}(x)^{*} \partial_{x} \varphi_{j}(x) d x . \tag{13.185}
\end{align*}
$$

Clearly, $\mathscr{L}_{0}^{b}(x)$ is $\ell$-periodic. The multiplicative group $\mathbb{C}^{\times}$acts on $\left(\varphi_{1}, \ldots, \varphi_{n}\right)$ by multiplying each bosonic field by a phase,

$$
\begin{equation*}
\left(\varphi_{1}, \ldots, \varphi_{n}\right) \mapsto\left(e^{-i q_{1}} \varphi_{1}, \ldots, e^{-i q_{n}} \varphi_{n}\right) \tag{13.186}
\end{equation*}
$$

where $q_{1}, \ldots, q_{n} \in \mathbb{R}$. By Nöther's theorem, there is a conserved charge density $j^{0}(x)$ and therefore an associated charge operator

$$
\begin{align*}
Q_{j}^{b} & =i \int_{0}^{\ell} j_{j}^{0}(x) d x  \tag{13.187}\\
& =-i q_{j} \int_{0}^{\ell} \varphi_{j}(x)^{*} \pi_{j}(x)^{*}-\pi_{j}(x) \varphi_{j}(x) d x \tag{13.188}
\end{align*}
$$

In terms of creation and annihilation operators,

$$
\begin{align*}
Q_{j}^{b} & =\frac{q_{j}}{2} \sum_{k \in K_{j}^{b}} 2 a_{+, j}(k) a_{+, j}(k)^{*}-2 a_{-, j}(-k)^{*} a_{-, j}(-k)  \tag{13.189}\\
& =q_{j}\left(\sum_{k \in K_{j}^{b}} N_{+, j}^{b}(k)-N_{-, j}^{b}(-k)+1\right) \tag{13.190}
\end{align*}
$$

Define the fermionic current

$$
\begin{equation*}
j_{j}^{f}(x)=q_{j} \bar{\psi}_{j}(x) \gamma^{0} \psi_{j}(x) \tag{13.191}
\end{equation*}
$$

and associated charge operator

$$
\begin{align*}
Q_{j}^{f} & =\int_{0}^{\ell} j_{j}^{f}(x) d x  \tag{13.192}\\
& =q_{j} \int_{0}^{\ell} \bar{\psi}_{j}(x) \gamma^{0} \psi_{j}(x) d x  \tag{13.193}\\
& =q_{j} \int_{0}^{\ell} \psi_{1, j}(x)^{*} \psi_{1, j}(x)+\psi_{2, j}(x)^{*} \psi_{2, j}(x) . \tag{13.194}
\end{align*}
$$

Problem 13.8.1. Compute $Q_{j}^{f}$ in terms of fermionic number operators.
13.9. Zero-Point Energy and Momentum Cancellation

Supersymmetry is compatible with the choice $\left|K_{j, \kappa}^{b}\right|=\left|K_{1, j, \kappa}^{f}\right|$ for $1 \leqslant j \leqslant n$, since the sum

$$
\begin{align*}
E_{0, \kappa}+p_{0, \kappa} & =\frac{\pi}{2 \ell} \sum_{j=1}^{n}\left|K_{j, \kappa}^{b}\right|^{2}-\left|K_{1, j, \kappa}^{f}\right|^{2}+\frac{\phi}{\ell} \sum_{j=1}^{n} \Omega_{j}\left(\left|K_{1, j, \kappa}^{f}\right|-\left|K_{j, \kappa}^{b}\right|\right)  \tag{13.195}\\
& =0 \tag{13.196}
\end{align*}
$$

and there is no need to normal order the operator sum

$$
\begin{equation*}
H_{0}+P_{0}=\mathrm{s}-\lim _{\kappa \rightarrow \infty} H_{0, \kappa}+P_{0, \kappa} . \tag{13.197}
\end{equation*}
$$

Unfortunately, the difference $E_{0, \kappa}-p_{0, \kappa}$ does not share a similar cancellation, but by taking $q_{j}\left|K_{j, \kappa}^{b}\right|=\left(d-q_{j}\right)\left|K_{2, j, \kappa}^{f}\right|$ for $1 \leqslant j \leqslant n$, we effectively remove the $\phi$ dependence of the difference,

$$
\begin{align*}
E_{0, \kappa}-p_{0, \kappa} & =\frac{\pi}{2 \ell} \sum_{j=1}^{n}\left|K_{j, \kappa}^{b}\right|^{2}-\left|K_{2, j, \kappa}^{f}\right|^{2}+\frac{\phi}{\ell} \sum_{j=1}^{n} \Omega_{j}\left|K_{j, \kappa}^{b}\right|-\left(1-\Omega_{j}\right)\left|K_{2, j, \kappa}^{f}\right|  \tag{13.198}\\
& =\frac{\pi}{2 \ell} \sum_{j=1}^{n} \Omega_{j}^{-2}\left(1-2 \Omega_{j}\right)\left|K_{2, j, \kappa}^{f}\right|^{2} . \tag{13.199}
\end{align*}
$$

Finally, if $\left|K_{2, j, \kappa}^{f}\right|=b q_{j}$, where $b \in \mathbb{N}$, then $E_{0, \kappa}-p_{0, \kappa}=\frac{\pi(b d)^{2} \hat{c}}{2 \ell}$.

### 13.10. Sharply-Regularized Superpotential

One way to ensure that a holomorphic function $f: \mathbb{C}^{n} \rightarrow \mathbb{C}$ has finitely many zeros forming a compact set is the existence of (finite) constants $\varepsilon, M>0$
such that the following elliptic bounds hold,

$$
\begin{equation*}
\left|\partial^{\alpha} f\right|^{2} \leqslant \varepsilon|\partial f|^{2}+M \quad \text { and } \quad|z|^{2}+|f| \leqslant M\left(|\partial f|^{2}+1\right) \tag{13.200}
\end{equation*}
$$

for any multi-index $\alpha$, where $|z|$ denotes the magnitude of $z=\left(z_{1}, \ldots, z_{n}\right)$ and $|\partial f|^{2}=\sum_{j=1}^{n}\left|\partial_{j} f\right|^{2}$ is the squared magnitude of the gradient of $f$. If $f$ satisfies these, then $f$ is called elliptic.

Recall that if $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a weighted homogeneous polynomial of (weighted) degree $d \in \mathbb{N}$ and integral weights $\left\{q_{0}, \ldots, q_{n}\right\} \in \mathbb{N}^{n+1}$. By definition $f$ satisfies identity

$$
\begin{equation*}
\lambda^{d} f\left(z_{0}, \ldots, z_{n}\right)=f\left(\lambda^{q_{0}} z_{0}, \ldots, \lambda^{q_{n}} z_{n}\right) \tag{13.201}
\end{equation*}
$$

for $\lambda \in \mathbb{C}^{\times}$and, therefore, also satisfies the weighted Euler equation

$$
\begin{align*}
f\left(z_{0}, \ldots, z_{n}\right) & =(\operatorname{deg} f)^{-1} \sum_{j=0}^{n}\left(\operatorname{deg} z_{j}\right) z_{j} \partial_{j} f  \tag{13.202}\\
& =d^{-1} \sum_{j=0}^{n} q_{j} z_{j} \partial_{j} f \tag{13.203}
\end{align*}
$$

Define the reduced weight set $\omega=\left\{\omega_{1}, \ldots, \omega_{n}\right\}$, where $\omega_{j}=\frac{q_{j}}{d}$ for $1 \leqslant j \leqslant n$.
Since weighted homogeneous polynomials have either an isolated critical point at the origin, a continuum of critical points, flat direction, there or none at all, it suffices to prove the existence of constants $\varepsilon, \ell>0$ such that

$$
\begin{equation*}
|\partial f| \geqslant \varepsilon|z|^{\ell} \tag{13.204}
\end{equation*}
$$

in an open neighborhood of the origin, which establishes the existence of an isolated critical point there. This inequality is known in the literature as the (complex) Łojasiewicz inequality. Here, the infimum $\ell_{0}$ of the exponents $\{\ell\}$ is known as the Łojasiewicz exponent and is an invariant of weighted homogeneous polynomials within the same weight equivalence class, as we have described in previous chapters.

PROPOSITION 13.2 (Jaffe, [222], [225], [223]). If the bosonic superpotential $V$ is weighted homogeneous and satisfies the elliptic bounds, then the corresponding heat kernel $e^{-\beta H-i \sigma P-i \theta J}$ of the twist-regularized Wess-Zumino model is trace-class on the boson-fermion Fock space.

We have proven that only the first elliptic bound need be shown, as the latter is redundant and automatically satisfied by virtue of the weighed Euler equation, $q \cdot v$. ., Proposition 3.14 in $\$ 3.3$.

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## Chapter 14

## Bilocal Bounds

> Arguments are to be avoided; they are always vulgar and often convincing.
> - Oscar Wilde

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To study non-trivial interactions in this theory, we consider a superpotential $V\left(\varphi_{\Lambda, \kappa}\right)=V\left(\varphi_{1, \Lambda, \kappa}, \ldots, \varphi_{n, \Lambda, \kappa}\right)$ of the bosonic fields which when considered as function of $\mathbb{C}^{n}$ is a non-degenerate, weighted homogeneous polynomial (with an isolated zero at the origin) with reduced weight set $\Omega$ and satisfies the first of the elliptic bounds. Again, we refer the reader to the following articles [158], [222], [223] and [225] and [226] for a more complete and comprehensive discussion of twist fields.

These calculations represent only a first step toward proving the existence of the partition function. Much more analysis is required.

### 14.1. Strongly-Regularized Interaction Hamiltonian

Define the domain $\mathcal{D}_{\kappa}=\bigcap_{n \in \mathbb{N}} \mathcal{D}\left(H_{0, \kappa}^{n}\right)$. Consider the total, stronglyregularized Hamiltonian

$$
\begin{equation*}
H_{\Lambda, \kappa}(\lambda)=H_{0, \kappa}+H_{I, \Lambda, \kappa}(\lambda) \tag{14.1}
\end{equation*}
$$

where the interaction Hamiltonian

$$
\begin{equation*}
H_{I, \Lambda, \kappa}(\lambda)=H_{I, \Lambda, \kappa}^{b}(\lambda)+H_{\Lambda, \kappa}^{b f}(\lambda) . \tag{14.2}
\end{equation*}
$$

For notational convenience, it is convenient write

$$
\begin{equation*}
H_{\Lambda, \kappa}(\lambda)=H_{0, \kappa}^{f}+H_{\Lambda, \kappa}^{b}(\lambda)+H_{\Lambda, \kappa}^{b f}(\lambda) \tag{14.3}
\end{equation*}
$$

where the bilocal bosonic Hamiltonian $H_{\Lambda, \kappa}^{b}$ is the sum

$$
\begin{align*}
H_{\Lambda, \kappa}^{b}(\lambda) & =H_{0, \kappa}^{b}+H_{I, \Lambda, \kappa}^{b}(\lambda)  \tag{14.4}\\
& =H_{0, \kappa}^{b}+\lambda^{2} \sum_{j=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell} V_{j}\left(\varphi_{\Lambda, \kappa}(x)\right)^{*} v_{j, \Lambda}(x-y) V_{j}\left(\varphi_{\Lambda, \kappa}(y)\right) d x d y \tag{14.5}
\end{align*}
$$

and the local boson-fermion interaction Hamiltonian is given by

$$
\begin{align*}
H_{\Lambda, \kappa}^{b f}(\lambda)= & h_{\Lambda, \kappa}^{b f}(\lambda)+h_{\Lambda, \kappa}^{b f}(\lambda)^{*}  \tag{14.6}\\
= & \lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \psi_{1, j, \Lambda, \kappa}(x) \psi_{2, j^{\prime}, \Lambda, \kappa}(x)^{*} V_{j j^{\prime}}\left(\varphi_{\Lambda, \kappa}\right) \\
& +\psi_{2, j, \Lambda, \kappa}(x) \psi_{1, j^{\prime}, \Lambda, \kappa}(x)^{*} V_{j j^{\prime}}\left(\varphi_{\Lambda, \kappa}\right)^{*} d x \tag{14.7}
\end{align*}
$$

where $V_{j j^{\prime}}\left(\varphi_{\Lambda, \kappa}\right)=V_{j j^{\prime}}\left(\varphi_{\Lambda, \kappa}(x)\right)$. Define the local, cutoff, bosonic interaction Hamiltonian $H_{I, \Lambda, \kappa}^{b l o c}$,

$$
\begin{equation*}
H_{I, \Lambda, \kappa}^{b \mathrm{loc}}(\lambda)=\lambda^{2} \sum_{j=1}^{n} \int_{0}^{\ell}\left|V_{j}\left(\varphi_{\Lambda, \kappa}(x)\right)\right|^{2} d x \geqslant 0 . \tag{14.8}
\end{equation*}
$$

14.1.1. Fourier Representation of the Hamiltonian. Consider the Fourier representation of the directional derivative,

$$
\begin{equation*}
V_{j}\left(\varphi_{\Lambda, \kappa}\right)=\frac{1}{\sqrt{\ell}} \sum_{k \in K_{V_{j}, \kappa}} \tilde{V}_{j}(k) e^{-i k x} \tag{14.9}
\end{equation*}
$$

where $K_{V_{j, \kappa}}=K_{2, j, \kappa}^{f}$. Using the Fourier representations of $V_{j}$, we write $H_{I, \Lambda, \kappa}^{b \text { loc }}$ as a sum over momenta,

$$
\begin{align*}
H_{I, \Lambda, \kappa}^{b \text { loc }} & =\lambda^{2} \sum_{j=1}^{n} \sum_{k, k^{\prime} \in K_{V_{j}, \kappa}} \tilde{V}_{j}(k)^{*} \tilde{V}_{j}\left(k^{\prime}\right)\left(\frac{1}{\ell} \int_{0}^{\ell} e^{i\left(k-k^{\prime}\right) x} d x\right)  \tag{14.10}\\
& =\lambda^{2} \sum_{j=1}^{n} \sum_{k \in K_{V_{j}, \kappa}}\left|\tilde{V}_{j}(k)\right|^{2} \tag{14.11}
\end{align*}
$$

using the fact that $\delta_{k, k^{\prime}}=\frac{1}{\ell} \int_{0}^{\ell} e^{i\left(k-k^{\prime}\right) x} d x$. Similarly, we may write $H_{I, \Lambda, k}^{b}$ as a sum over momenta,

$$
\begin{align*}
H_{I, \Lambda, \kappa}^{b}= & \lambda^{2} \sum_{j=1}^{n} \int_{0}^{\ell} \int_{0}^{\ell} V_{j}\left(\varphi_{\Lambda, \kappa}(x)\right)^{*} v_{j, \Lambda}(x-y) V_{j}\left(\varphi_{\Lambda, \kappa}(y)\right) d x d y  \tag{14.12}\\
= & \frac{\lambda^{2}}{\ell^{2}} \sum_{j=1}^{n} \sum_{k, k^{\prime \prime} \in K_{V_{j}, \kappa}} \sum_{k^{\prime} \in \hat{\mathbb{T}}_{\kappa}} \tilde{V}_{j}(k) \tilde{\mu}_{\Lambda}\left(k^{\prime}\right)^{-2 \varepsilon} \tilde{V}_{j}\left(k^{\prime \prime}\right)^{*} \\
& \cdot \int_{0}^{\ell} \int_{0}^{\ell} e^{-i\left(1-\Omega_{j}\right) \phi(x-y) / \ell} e^{-i\left(k-k^{\prime}\right) x-i\left(k^{\prime}-k^{\prime \prime}\right) y} d x d y \tag{14.13}
\end{align*}
$$

Observe that

$$
\begin{aligned}
& \int_{0}^{\ell} \int_{0}^{\ell} e^{-i\left(1-\Omega_{j}\right) \phi(x-y) / \ell} e^{-i\left(k-k^{\prime}\right) x-i\left(k^{\prime}-k^{\prime \prime}\right) y} d x d y \\
&=\quad \ell^{2} e^{-i\left(k-k^{\prime \prime}\right) \ell / 2} s_{j}^{+}\left(k-k^{\prime}\right) s_{j}^{-}\left(k^{\prime}-k^{\prime \prime}\right)
\end{aligned}
$$

where

$$
\begin{align*}
s_{j}^{ \pm}(k) & =j_{0}\left(\frac{1}{2}\left(k \ell \pm\left(1-\Omega_{j}\right) \phi\right)\right)  \tag{14.14}\\
& =\frac{\sin \left[\frac{1}{2}\left(k \ell \pm\left(1-\Omega_{j}\right) \phi\right)\right]}{\frac{1}{2}\left(k \ell \pm\left(1-\Omega_{j}\right) \phi\right)} \tag{14.15}
\end{align*}
$$

and $j_{0}$ is the zeroth spherical Bessel function. Collecting terms,

$$
\begin{align*}
H_{I, \Lambda, \kappa}^{b}= & \lambda^{2} \sum_{j=1}^{n} \sum_{k, k^{\prime \prime} \in K_{V_{j}, \kappa}} \sum_{k^{\prime} \in \hat{\mathbb{T}}_{\kappa}} \tilde{V}_{j}(k) s_{j}^{+}\left(k-k^{\prime}\right) \tilde{\mu}_{\Lambda}\left(k^{\prime}\right)^{-2 \varepsilon} s_{j}^{-}\left(k^{\prime}-k^{\prime \prime}\right) \tilde{V}_{j}\left(k^{\prime \prime}\right)^{*} e^{-i\left(k-k^{\prime \prime}\right) \ell / 2} \\
= & \lambda^{2} \sum_{j=1}^{n} \sum_{n, n^{\prime}, n^{\prime \prime} \in \mathbb{Z}_{N}}(-1)^{n-n^{\prime \prime}} \tilde{V}_{j}\left(\frac{2 \pi n}{\ell}-\frac{\left(1-\Omega_{j}\right) \phi}{\ell}\right) j_{0}\left(\pi\left(n-n^{\prime}\right)\right) \tilde{\mu}_{\Lambda}\left(\frac{2 \pi n^{\prime}}{\ell}\right)^{-2 \varepsilon} \\
& \cdot j_{0}\left(\pi\left(n^{\prime}-n^{\prime \prime}\right)\right) \tilde{V}_{j}\left(\frac{2 \pi n^{\prime \prime}}{\ell}-\frac{\left(1-\Omega_{j}\right) \phi}{\ell}\right)^{*}  \tag{14.16}\\
= & \lambda^{2} \sum_{j=1}^{n} \sum_{n, n^{\prime}, n^{\prime \prime} \in \mathbb{Z}_{N}}(-1)^{n-n^{\prime \prime}} \delta_{n, n^{\prime}} \delta_{n^{\prime}, n^{\prime \prime}} \tilde{V}_{j}\left(\frac{2 \pi n}{\ell}-\frac{\left(1-\Omega_{j}\right) \phi}{\ell}\right) \tilde{\mu}_{\Lambda}\left(\frac{2 \pi n^{\prime}}{\ell}\right)^{-2 \varepsilon} \\
= & \lambda^{2} \sum_{j=1}^{n} \sum_{k \in \hat{\mathbb{T}}_{\kappa}} \tilde{\mu}_{\Lambda}\left(\frac{2 \pi n^{\prime \prime}}{\ell}-\frac{\left(1-\Omega_{j}\right) \phi}{\ell}\right)^{*}  \tag{14.17}\\
= & \left.\lambda^{2} \sum_{j=1}^{n} \sum_{k \in K_{V_{j}, \kappa}} \tilde{\mu}_{\Lambda}\left(k+\frac{\left(1-\Omega_{j}\right) \phi}{\ell}\right)^{-2 \varepsilon}\left|\tilde{V}_{j}\left(k-\frac{\left(1-\Omega_{j}\right) \phi}{\ell}\right)\right|^{2}(k)\right|^{2} \tag{14.18}
\end{align*}
$$

where we have used the fact that $j_{0}(\pi n)=\delta_{n, 0}$. We claim that $H_{I, \Lambda, \kappa}^{b}$ has a simpler form if we take

$$
\begin{align*}
V_{j}\left(\varphi_{\Lambda, \kappa}(x)\right) & =\frac{e^{i\left(1-\Omega_{j}\right) \phi x / \ell}}{\sqrt{\ell}} \sum_{k \in \hat{\mathbb{T}}_{\kappa}} F_{j}(k) e^{i k x}  \tag{14.20}\\
& =\frac{e^{i\left(1-\Omega_{j}\right) \phi x / \ell}}{\sqrt{\ell}} \sum_{k \in \hat{\mathbb{T}}_{\kappa}} F_{j}(-k) e^{-i k x} . \tag{14.21}
\end{align*}
$$

On the cut-off lattice $K_{V_{j}, \kappa}$,

$$
\begin{align*}
\tilde{V}_{j}(k) & =\frac{1}{\sqrt{\ell}} \int_{0}^{\ell} V_{j}(x) e^{i k x} d x  \tag{14.22}\\
& =\sum_{k^{\prime} \in \hat{\mathbb{T}}_{k}} F_{j}\left(-k^{\prime}\right)\left(\frac{1}{\ell} \int_{0}^{\ell} e^{i\left(k-k^{\prime}\right) x} e^{i\left(1-\Omega_{j}\right) \phi x / \ell}\right) d x  \tag{14.23}\\
& =\sum_{k \in \hat{\mathbb{T}}_{k}} F_{j}\left(-k^{\prime}\right) \delta_{k^{\prime}, k^{\prime \prime}}=F_{j}\left(-k^{\prime \prime}\right) \tag{14.24}
\end{align*}
$$

where $k^{\prime \prime}=k+\frac{\left(1-\Omega_{j}\right) \phi}{\ell} \in \hat{\mathbb{Z}}_{k}$. Hence, $\tilde{V}_{j}(k)^{*}=F_{j}\left(k^{\prime \prime}\right)^{*}$ and, therefore,

$$
\begin{align*}
H_{I, \Lambda, k}^{b} & =\lambda^{2} \sum_{j=1}^{n} \sum_{k \in \hat{\mathbb{T}}_{k}} \tilde{\mu}_{\Lambda}(k)^{-2 \varepsilon}\left|F_{j}(-k)\right|^{2}  \tag{14.25}\\
& =\lambda^{2} \sum_{j=1}^{n} \sum_{k \in \hat{\mathbb{T}}_{k}} \tilde{\mu}_{\Lambda}(k)^{-2 \varepsilon}\left|F_{j}(k)\right|^{2}, \tag{14.26}
\end{align*}
$$

since $\tilde{\mu}_{\Lambda}$ is even.
Equivalently, we could consider a mollified directional derivative $V_{j, \Lambda}=$ $\mu_{2, j, \varepsilon, \Lambda}^{f} * V_{j}$ with Fourier representation

$$
\begin{equation*}
V_{j, \Lambda}\left(\varphi_{\Lambda, k}\right)=\frac{e^{i\left(1-\Omega_{j}\right) \phi x / \ell}}{\sqrt{\ell}} \sum_{k \in \hat{\mathbb{T}}_{k}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon} F_{j}(k) e^{i k x} \tag{14.27}
\end{equation*}
$$

The corresponding local bosonic interaction Hamiltonian with a mollified superpotential $V_{j, \Lambda}$ is exactly the bilocal bosonic interaction with an unmollified superpotential $V_{j}$ since the bilocal interaction Hamiltonian depends only on the Fourier coefficient of its superpotential.

### 14.2. Local Bound

We establish a bound of the bilocal, bosonic Hamiltonian $H_{\Lambda, \kappa}^{b}$ in terms of the local, bosonic Hamiltonian $H_{\Lambda, \kappa}^{b \text { loc }}$ and a cut-off dependent coupling constant. Write $\lambda_{\Lambda, \kappa}=\lambda \tilde{\mu}_{\Lambda}(\kappa)^{-\varepsilon}$. Since $\inf _{k \in \hat{\mathbb{T}}_{\kappa}} \tilde{\mu}_{\Lambda}(k)^{-\varepsilon}=\tilde{\mu}_{\Lambda}(\kappa)^{-\varepsilon}$, a lower bound, in turn, is then

$$
\begin{align*}
H_{I, \Lambda, \kappa}^{b \operatorname{loc}}\left(\lambda_{\Lambda, \kappa}\right) & =\lambda^{2} \tilde{\mu}_{\Lambda}(\kappa)^{-2 \varepsilon} \sum_{j=1}^{n} \sum_{k \in \hat{\mathbb{T}}_{\kappa}}\left|F_{j}(k)\right|^{2}  \tag{14.28}\\
& \leqslant \lambda^{2} \sum_{j=1}^{n} \sum_{k \in \hat{\mathbb{T}}_{\kappa}} \tilde{\mu}_{\Lambda}(k)^{-2 \varepsilon}\left|F_{j}(k)\right|^{2}  \tag{14.29}\\
& =H_{I, \Lambda, \kappa}^{b}(\lambda) . \tag{14.30}
\end{align*}
$$

The inequalities for the bosonic interactions translate into inequalities for the total bosonic Hamiltonians, so it follows that

$$
\begin{align*}
H_{\Lambda, \kappa}^{b \operatorname{loc}}\left(\lambda_{\Lambda, \kappa}\right) & =H_{0, \kappa}^{b}+H_{I, \Lambda, \kappa}^{b \operatorname{loc}}\left(\lambda_{\Lambda, \kappa}\right)  \tag{14.31}\\
& \leqslant H_{0, \kappa}^{b}+H_{I, \Lambda, \kappa}^{b}(\lambda)  \tag{14.32}\\
& =H_{\Lambda, \kappa}^{b}(\lambda), \tag{14.33}
\end{align*}
$$

which we shall use for establishing both a Divergent Kato bound and a Mass Shift bound.

### 14.3. Divergent Kato Bound

Consider the boson-fermion interaction $H_{\Lambda, \kappa}^{b f}=h_{\Lambda, \kappa}^{b f}+\left(h_{\Lambda, \kappa}^{b f}\right)^{*}$, where

$$
\begin{equation*}
h_{\Lambda, \kappa}^{b f}(\lambda)=\lambda \sum_{j, j^{\prime}=1}^{n} \int_{0}^{\ell} \psi_{1, j, \Lambda, \kappa}(x) \psi_{2, j^{\prime}, \Lambda, \kappa}(x)^{*} V_{j j^{\prime}}\left(\varphi_{\Lambda, \kappa}(x)\right) d x \tag{14.34}
\end{equation*}
$$

As an intermediate step, we show that for each $\varepsilon_{1}>0$ there exists a constant $M_{1}=M_{1}\left(\varepsilon_{0}, \kappa, \Lambda\right)<\infty$ such that the boson-fermion interaction Hamiltonian is a Kato perturbation of the local bosonic Hamiltonian,

$$
\begin{equation*}
\left|H_{\Lambda, \kappa}^{b f}(\lambda)\right| \leqslant \varepsilon_{1} H_{\Lambda, \kappa}^{b \operatorname{loc}}\left(\lambda_{\Lambda, \kappa}\right)+M_{1} \tag{14.35}
\end{equation*}
$$

Suppose the second-order directional derivative $V_{j j^{\prime \prime}}$ has the following Fourier represenation

$$
\begin{equation*}
V_{j j^{\prime}}\left(\varphi_{\Lambda, \kappa}(x)\right)=\frac{e^{i\left(1-\Omega_{j}-\Omega_{j^{\prime}}\right) \phi x / \ell}}{\sqrt{\ell}} \sum_{k \in \hat{\mathbb{T}}_{k}} F_{j, j^{\prime}}(k) e^{i k x} \tag{14.36}
\end{equation*}
$$

Then as a momentum sum, we have

$$
\begin{align*}
h_{\Lambda, \kappa}^{b f}(\lambda)= & \frac{\lambda}{\sqrt{\ell}} \sum_{j, j^{\prime}=1}^{n} \sum_{k \in K_{1, j, \kappa}^{f}} \sum_{k^{\prime} \in K_{2, j^{\prime}, \kappa}^{f}} \sum_{k^{\prime \prime} \in \hat{\mathbb{T}}_{\kappa}} \tilde{\psi}_{1, j, \Lambda, \kappa}(k) \tilde{\psi}_{2, j^{\prime}, \Lambda, \kappa}\left(k^{\prime}\right)^{*} F_{j, j^{\prime}}\left(k^{\prime \prime}\right) \\
& \cdot\left(\frac{1}{\ell} \int_{0}^{\ell} e^{i\left[\left(1-\Omega_{j}-\Omega_{j^{\prime}}\right) \phi / \ell-k+k^{\prime}+k^{\prime \prime}\right] x} d x\right)  \tag{14.37}\\
= & \frac{\lambda}{\sqrt{\ell}} \sum_{j, j^{\prime}=1}^{n} \sum_{k, k^{\prime}, k^{\prime \prime} \in \hat{\mathbb{T}}_{\kappa}} \tilde{\psi}_{1, j, \Lambda, \kappa}\left(k-\frac{\Omega_{j} \phi}{\ell}\right) \tilde{\psi}_{2, j^{\prime}, \Lambda, \kappa}\left(k^{\prime}-\frac{\left(1-\Omega_{j^{\prime}}\right) \phi}{\ell}\right)^{*} F_{j, j^{\prime}}\left(k^{\prime \prime}\right) \\
& \cdot\left(\frac{1}{\ell} \int_{0}^{\ell} e^{i\left[\left(1-\Omega_{j}-\Omega_{j^{\prime}}\right) \phi / \ell-\left(k-\Omega_{j} \phi / \ell\right)+\left(k^{\prime}-\left(1-\Omega_{j^{\prime}}\right) \phi / \ell\right)+k^{\prime \prime}\right] x} d x\right)  \tag{14.38}\\
= & \frac{\lambda}{\sqrt{\ell}} \sum_{j, j^{\prime}=1}^{n} \sum_{k, k^{\prime}, k^{\prime \prime} \in \hat{\mathbb{T}}_{\kappa}} \tilde{\psi}_{1, j, \Lambda, \Lambda}\left(k-\frac{\Omega_{j} \phi}{\ell}\right) \tilde{\psi}_{2, j^{\prime}, \Lambda, \kappa}\left(k^{\prime}-\frac{\left(1-\Omega_{j^{\prime}}\right) \phi}{\ell}\right)^{*} F_{j, j^{\prime}}\left(k^{\prime \prime}\right) \\
& \cdot\left(\frac{1}{\ell} \int_{0}^{\ell} e^{-i\left(k-k^{\prime}-k^{\prime \prime}\right) x} d x\right)  \tag{14.39}\\
= & \frac{\lambda}{\sqrt{\ell}} \sum_{j, j^{\prime}=1}^{n} \sum_{k, k^{\prime} \in \hat{\mathbb{T}}_{\kappa}} \tilde{\psi}_{1, j, \Lambda, \kappa}\left(k-\frac{\Omega_{j} \phi}{\ell}\right) \tilde{\psi}_{2, j^{\prime}, \Lambda, \kappa}\left(k^{\prime}-\frac{\left(1-\Omega_{j^{\prime}}\right) \phi}{\ell}\right)^{*} F_{j, j^{\prime}}\left(k-k^{\prime}\right) \\
= & \frac{\lambda}{\sqrt{\ell}} \sum_{j, j^{\prime}=1}^{n} \sum_{k \in K_{1, j, \kappa}^{f}} \sum_{k^{\prime} \in K_{2, j^{\prime}, \kappa}^{f}} \tilde{\psi}_{1, j, \Lambda, \kappa}(k) \tilde{\psi}_{2, j^{\prime}, \Lambda, \kappa}\left(k^{\prime}\right)^{*} F_{j, j^{\prime}}\left(k-k^{\prime}-\frac{\phi}{\ell}\right) . \tag{14.40}
\end{align*}
$$

Recall the double summation identity

$$
\begin{equation*}
\sum_{n, n^{\prime} \in \mathbb{Z}_{N}}\left|f\left(n-n^{\prime}\right)\right|=\sum_{n \in \mathbb{Z}_{2 N}}(2 N-1-|n|)|f(n)| \tag{14.41}
\end{equation*}
$$

which when applied to functions supported on $\mathbb{Z}_{N}$, simplifies to

$$
\begin{equation*}
\sum_{n, n^{\prime} \in \mathbb{Z}_{N}}\left|f\left(n-n^{\prime}\right)\right|=\sum_{n \in \mathbb{Z}_{N}}(2 N-1-|n|)|f(n)| \tag{14.42}
\end{equation*}
$$

Re-indexing to a summation over the unshifted lattice $\hat{\mathbb{T}}_{k}$, we bound the operator

$$
\begin{align*}
\left|h_{\Lambda, \kappa}^{b f}(\lambda)\right| & \leqslant \frac{\lambda}{\sqrt{\ell}}\left\|\tilde{\psi}_{\Lambda, \kappa}\right\|_{\infty}^{2} \sum_{j, j^{\prime}=1}^{n} \sum_{k, k^{\prime} \in \hat{\mathbb{T}}_{\kappa}}\left|F_{j, j^{\prime}}\left(k-k^{\prime}\right)\right|  \tag{14.43}\\
& =\frac{\lambda}{\sqrt{\ell}}\left\|\tilde{\psi}_{\Lambda, \kappa}\right\|_{\infty}^{2} \sum_{j, j^{\prime}=1}^{n} \sum_{k \in \hat{\mathbb{T}}_{\kappa}}\left(2 N-1-\frac{\ell|k|}{2 \pi}\right)\left|F_{j, j^{\prime}}(k)\right|  \tag{14.44}\\
& \leqslant \frac{\lambda}{\sqrt{\ell}}\left\|\tilde{\psi}_{\Lambda, \kappa}\right\|_{\infty}^{2} \sum_{j, j^{\prime}=1}^{n}\left((2 N-1) \sum_{k \in \hat{\mathbb{T}}_{\kappa}}\left|F_{j, j^{\prime}}(k)\right|-\inf _{k \in \hat{\mathbb{T}}_{\kappa}}\left|F_{j, j^{\prime}}(k)\right| \frac{\ell}{2 \pi} \sum_{k \in \hat{\mathbb{T}}_{k}}|k|\right) .
\end{align*}
$$

Since $\sum_{k \in \hat{\mathbb{T}}_{\kappa}}|k|=\frac{2 \pi}{\ell} N(N-1)$, then

$$
\left|h_{\Lambda, \kappa}^{b f}(\lambda)\right| \leqslant \frac{\lambda}{\sqrt{\ell}}\left\|\tilde{\psi}_{\Lambda, \kappa}\right\|_{\infty}^{2} \sum_{j, j^{\prime}=1}^{n}\left((2 N-1) \sum_{k \in \hat{\mathbb{T}}_{\kappa}}\left|F_{j, j^{\prime}}(k)\right|-N(N-1) \inf _{k \in \mathbb{\mathbb { T }}_{\kappa}}\left|F_{j, j^{\prime}}(k)\right|\right)
$$

where $\left\|\tilde{\psi}_{\alpha, \kappa}\right\|_{\infty}=\sup _{j ; k \in K_{\alpha, j, k}^{f}}\left|\tilde{\psi}_{\alpha, j, \kappa}(k)\right|$ and $\left\|\tilde{\psi}_{\kappa}\right\|_{\infty}=\sup _{\alpha}\left\|\tilde{\psi}_{\alpha, \kappa}\right\|_{\infty}$. Using Parseval's Theorem combined with the elliptic bound, for each $\varepsilon_{0}>0$, there is a
constant $M_{0}=M_{0}\left(\varepsilon_{0}, V\right)<\infty$ such that

$$
\begin{align*}
\sum_{j=1}^{n} \sum_{k \in \hat{\mathbb{Z}}_{\kappa}}\left|F_{j, j^{\prime}}(k)\right|^{2} & =\int_{0}^{\ell} \sum_{j=1}^{n}\left|V_{j j^{\prime}}\left(\varphi_{\Lambda, \kappa}(x)\right)\right|^{2} d x  \tag{14.45}\\
& \leqslant \varepsilon_{0} \int_{0}^{\ell} \sum_{j=1}^{n}\left|V_{j}\left(\varphi_{\Lambda, \kappa}\right)(x)\right|^{2} d x+\ell M_{0} \tag{14.46}
\end{align*}
$$

then recalling that $\lambda_{\Lambda, \kappa}=\lambda \tilde{\mu}_{\Lambda}(\kappa)^{-\varepsilon}$,

$$
\begin{align*}
\left|h_{\Lambda, \kappa}^{b f}(\lambda)\right| & \leqslant(2 N-1) \frac{\lambda n}{\sqrt{\ell}}\left\|\tilde{\psi}_{\kappa}\right\|_{\infty}^{2} \tilde{\mu}_{\Lambda}(\kappa)^{-2 \varepsilon}\left(\varepsilon_{0} \sum_{j=1}^{n} \int_{0}^{\ell}\left|V_{j}\left(\varphi_{\Lambda, \kappa}(x)\right)\right|^{2} d x+\ell M_{0}\right) \\
& =\frac{1}{2}\left(\varepsilon_{1} \lambda^{2} \tilde{\mu}_{\Lambda}(\kappa)^{-2 \varepsilon} \sum_{j=1}^{n} \int_{0}^{\ell}\left|V_{j}\left(\varphi_{\Lambda, \kappa}(x)\right)\right|^{2} d x+M_{1}\right)  \tag{14.47}\\
& =\frac{1}{2}\left(\varepsilon_{1} \lambda_{\Lambda, \kappa}^{2} \sum_{j=1}^{n} \sum_{k \in \hat{\mathbb{Z}}_{\kappa}}\left|F_{j}(k)\right|^{2}+M_{1}\right)  \tag{14.48}\\
& =\frac{1}{2}\left(\varepsilon_{1} H_{I, \Lambda, \kappa}^{b \operatorname{loc}}\left(\lambda_{\Lambda, \kappa}\right)+M_{1}\right)  \tag{14.49}\\
& \leqslant \frac{1}{2}\left(\varepsilon_{1} H_{\Lambda, \kappa}^{b \operatorname{loc}}\left(\lambda_{\Lambda, \kappa}\right)+M_{1}\right) \tag{14.50}
\end{align*}
$$

where

$$
\begin{align*}
M_{1} & =2 n \lambda \sqrt{\ell}(2 N-1)\left\|\tilde{\psi}_{\kappa}\right\|_{\infty}^{2} \tilde{\mu}_{\Lambda}(\kappa)^{-2 \varepsilon} M_{0}  \tag{14.51}\\
\varepsilon_{1} & =\left(\lambda^{2} \ell M_{0}\right)^{-1} M_{1} \varepsilon_{0} \tag{14.52}
\end{align*}
$$

both positive and finite. Note that $\varepsilon_{1}=O\left(\lambda^{-1}\right)$ and $M_{1}=O(\lambda)$. Hence,

$$
\begin{align*}
\left|H_{\Lambda, \kappa}^{b f}(\lambda)\right| & =\left|h_{\Lambda, \kappa}^{b f}(\lambda)+h_{\Lambda, \kappa}^{b f}(\lambda)^{*}\right|  \tag{14.53}\\
& \leqslant 2\left|h_{\Lambda, \kappa}^{b f}(\lambda)\right|  \tag{14.54}\\
& \leqslant \varepsilon_{1} H_{\Lambda, \kappa}^{b \text { loc }}\left(\lambda_{\Lambda, \kappa}\right)+M_{1} \tag{14.55}
\end{align*}
$$

Combining this with the bilocal bound $H_{\Lambda, \kappa}^{b \text { loc }}\left(\lambda_{\Lambda, \kappa}\right) \leqslant H_{\Lambda, \kappa}^{b}(\lambda)$, we establish the bilocal, divergent Kato bound

$$
\begin{equation*}
\left|H_{\Lambda, \kappa}^{b f}(\lambda)\right| \leqslant \varepsilon_{1} H_{\Lambda, \kappa}^{b}(\lambda)+M_{1} \tag{14.56}
\end{equation*}
$$

### 14.4. Mass-Shift Bound

Suppose that $V$ satisfies the Łojasiewicz inequality, then

$$
\begin{align*}
\delta\left|\varphi_{\Lambda, k}\right|^{\theta} & =\delta\left(\sum_{j=1}^{n}\left|\varphi_{j, \Lambda, \kappa}(x)\right|^{2}\right)^{\theta / 2}  \tag{14.57}\\
& \leqslant\left(\sum_{j=1}^{n}\left|V_{j}\left(\varphi_{\Lambda, \kappa}(x)\right)\right|^{2}\right)^{1 / 2}  \tag{14.58}\\
& =\left|\partial V\left(\varphi_{\Lambda, \kappa}\right)\right| \tag{14.59}
\end{align*}
$$

for some $\delta, \theta>0$. Suppose, in particular, $\delta^{2}\left|\varphi_{\Lambda, \kappa}\right|^{2} \leqslant\left|\partial V\left(\varphi_{\Lambda, \kappa}\right)\right|^{2}$. Then

$$
\begin{align*}
H_{I, \Lambda, \kappa}^{b \operatorname{mass}}(\delta \lambda) & =(\delta \lambda)^{2} \sum_{j=1}^{n} \int_{0}^{\ell}\left|\varphi_{j, \Lambda, \kappa}(x)\right|^{2} d x  \tag{14.60}\\
& \leqslant \lambda^{2} \sum_{j=1}^{n} \int_{0}^{\ell}\left|V_{j}\left(\varphi_{\Lambda, \kappa}(x)\right)\right|^{2} d x  \tag{14.61}\\
& =H_{I, \Lambda, \kappa}^{b \operatorname{loc}}(\lambda) \tag{14.62}
\end{align*}
$$

where $H_{I, \Lambda, \kappa}^{b \text { mass }}(m)$ is a mass shift with mass $m$. Combining the above inequalities, we find a stronger lower bound of $H_{\Lambda, \kappa}^{b}$ with a free, but massive bosonic Hamiltonian. Define $m_{\Lambda, \kappa}=\delta \lambda_{\Lambda, \kappa}$. Then

$$
\begin{align*}
H_{0, \kappa}^{b}+H_{I, \Lambda, \kappa}^{b \operatorname{mass}}\left(m_{\Lambda, \kappa}\right) & =H_{0, \kappa}^{b}+\left(\delta \lambda_{\Lambda, \kappa}\right)^{2} \sum_{j=1}^{n} \int_{0}^{\ell}\left|\varphi_{j, \Lambda, \kappa}(x)\right|^{2} d x  \tag{14.63}\\
& \leqslant H_{0, \kappa}^{b}+\lambda_{\Lambda, \kappa}^{2} \sum_{j=1}^{n} \int_{0}^{\ell}\left|V_{j}\left(\varphi_{\Lambda, \kappa}(x)\right)\right|^{2} d x  \tag{14.64}\\
& =H_{\Lambda, \kappa}^{b \operatorname{loc}}\left(\lambda_{\Lambda, \kappa}\right) \tag{14.65}
\end{align*}
$$

Combining this with the local bound $H_{\Lambda, \kappa}^{b l o c}\left(\lambda_{\Lambda, \kappa}\right) \leqslant H_{\Lambda, \kappa}^{b}(\lambda)$, we establish the mass-shift bound,

$$
\begin{equation*}
H_{0, \kappa}^{b}+H_{I, \Lambda, \kappa}^{b \operatorname{mass}}\left(m_{\Lambda, k}\right) \leqslant H_{\Lambda, \kappa}^{b}(\lambda) \tag{14.66}
\end{equation*}
$$

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## Chapter 15

## Twist Partition Function

## Contents

15.1. Trace Class Heat Kernel
15.2. Defining Relation of the Partition Function.............................. . 667


We refer the reader to [158], [222], [223], [225] and [226].
15.1. Trace Class Heat Kernel

Let $\mathscr{F}=\mathscr{F}^{b} \otimes \mathscr{F}^{f}$ denote the tensor product Fock space. Since $H_{\Lambda, \kappa}(\lambda)$ is essentially self-adjoint on $\mathcal{D}_{\kappa}$ for $\lambda \geqslant 0$, then the heat kernel $e^{-\beta H_{\Lambda, \kappa}(\lambda)}$ is well-defined for $\beta>0$ and $\phi, \lambda \geqslant 0$. In this section, we shall prove that the strongly-regularized, total partition function

$$
\begin{equation*}
\mathfrak{Z}_{\Lambda, \kappa}^{\lambda V}(\phi, \beta)=\operatorname{Tr}_{\mathscr{F}} e^{-\beta H_{\Lambda, \kappa}(\lambda)} \tag{15.1}
\end{equation*}
$$

is trace class for $\phi, \lambda \geqslant 0$.

Consider $\lambda=0$ and $\phi>0$, that is, the free strongly-regularized Hamiltonian $H_{0, \kappa}=H_{\Lambda, \kappa}(0)$ with non-zero twist. For now we content ourselves with calculating the partition function of the heat kernel of a normal-ordered Hamiltonian

$$
\begin{align*}
: H_{0, \kappa}: & =H_{0, \kappa}-E_{0, \kappa}  \tag{15.2}\\
& =: H_{0, \kappa}^{b}:+: H_{0, \kappa}^{f}: \tag{15.3}
\end{align*}
$$

which carries no zero-point energy term that might otherwise diverge in the limit $\kappa \rightarrow \infty$. It is a standard calculation to show that the trace of the heat kernel over $\mathscr{F}$, the free twist partition function $\mathfrak{Z}_{\mathcal{K}}^{0}(\phi, \beta)=\operatorname{Tr}_{\mathscr{F}} e^{-\beta: H_{0, \kappa}}$, factors into a product of the bosonic and fermionic partition functions. Define

$$
\begin{equation*}
\mathfrak{Z}_{\kappa}^{0, b}(\phi, \beta)=\operatorname{Tr}_{\mathscr{F} b} e^{-\beta: H_{0, \kappa}^{b}:} \quad \text { and } \quad \mathfrak{Z}_{\kappa}^{0, f}(\phi, \beta)=\operatorname{Tr}_{\mathscr{F} f} e^{-\beta: H_{0, \kappa}^{f}:} . \tag{15.4}
\end{equation*}
$$

Thus, $\mathfrak{Z}_{\kappa}^{0}(\phi, \beta)=\mathfrak{Z}_{\kappa}^{0, b}(\phi, \beta) \mathfrak{Z}_{\kappa}^{0, f}(\phi, \beta)$, where the free bosonic partition function is the product

$$
\begin{equation*}
\mathfrak{Z}_{\kappa}^{0, b}(\phi, \beta)=\prod_{j=1}^{n} \prod_{k \in K_{j, \kappa}^{b}}\left(1-e^{-\beta|k|}\right)^{-2} \tag{15.5}
\end{equation*}
$$

and the free fermionic partition function is the product,

$$
\begin{equation*}
\mathfrak{Z}_{\kappa}^{0, f}(\phi, \beta)=\prod_{k_{-} \in K_{-, j, \kappa}^{f}}\left(1+e^{-\beta\left|k_{-}\right|}\right) \prod_{k_{+} \in K_{+, j, \kappa}^{f}}\left(1+e^{-\beta\left|k_{+}\right|}\right) \tag{15.6}
\end{equation*}
$$

both finite for $\kappa<\infty$. To prove uniform convergence of the two factors in the limit $\kappa \rightarrow \infty$, recall that for $a_{n}, b_{n}>0$,

$$
\begin{equation*}
\exp \left(-2 \sum_{n \leqslant N} a_{n}\right) \leqslant \prod_{n \leqslant N}\left(1-a_{n}\right)^{-2} \leqslant\left(1-\sum_{n \leqslant N} a_{n}\right)^{-2} \tag{15.7}
\end{equation*}
$$

and

$$
\begin{equation*}
1+\sum_{n \leqslant N} b_{n} \leqslant \prod_{n \leqslant N}\left(1+b_{n}\right) \leqslant \exp \left(\sum_{n \leqslant N} b_{n}\right) . \tag{15.8}
\end{equation*}
$$

The products converge uniformly (in the limit $N \rightarrow \infty$ ) if and only if the corresponding summations converge uniformly. First, the bosonic summation is given by

$$
\begin{align*}
\sum_{k \in K_{j, k}^{b}} e^{-\beta|k|} & \leqslant \sum_{k \in K_{j}^{b}} e^{-\beta|k|}  \tag{15.9}\\
& =e^{\beta \Omega_{j} \phi / \ell} \sum_{n \geqslant 1} e^{-2 \pi \beta n / \ell}+e^{-\beta \Omega_{j} \phi / \ell} \sum_{n \geqslant 0} e^{-2 \pi \beta n / \ell}  \tag{15.10}\\
& =\cosh \left(\frac{\beta\left(\pi-\Omega_{j} \phi\right)}{\ell}\right) \operatorname{csch}\left(\frac{\beta \pi}{\ell}\right) \tag{15.11}
\end{align*}
$$

which is finite and less than 1 for $\beta>0$ with the caveat that $0 \leqslant \Omega_{j} \phi \leqslant 2 \pi$ for $1 \leqslant j \leqslant n$. The infinite fermionic summations can be calculated in a similar
fashion, viz.,

$$
\begin{align*}
\sum_{k_{+} \in K_{+, j, k}^{f}} e^{-\beta|k|} & \leqslant \sum_{k_{+} \in K_{+, j}^{f}} e^{-\beta|k|}  \tag{15.12}\\
& =e^{\beta\left(1-\Omega_{j}\right) \phi / \ell} \sum_{n \geqslant 1} e^{-2 \pi \beta n / \ell}+e^{-\beta \Omega_{j} \phi / \ell} \sum_{n \geqslant 0} e^{-2 \pi \beta n / \ell}  \tag{15.13}\\
& =e^{\beta\left(1-2 \Omega_{j}\right) \phi / 2 \ell} \cosh \left(\frac{\beta(2 \pi-\phi)}{\ell}\right) \operatorname{csch}\left(\frac{\beta \pi}{\ell}\right) \tag{15.14}
\end{align*}
$$

and

$$
\begin{align*}
\sum_{k_{-} \in K_{-, j, k}^{f}} e^{-\beta|k|} & \leqslant \sum_{k_{-\in K_{-, j}^{f}}} e^{-\beta|k|}  \tag{15.15}\\
& =e^{\beta \Omega_{j} \phi / \ell} \sum_{n \geqslant 1} e^{-2 \pi \beta n / \ell}+e^{-\beta\left(1-\Omega_{j}\right) \phi / \ell} \sum_{n \geqslant 0} e^{-2 \pi \beta n / \ell}  \tag{15.16}\\
& =e^{-\beta\left(1-2 \Omega_{j}\right) \phi / 2 \ell} \cosh \left(\frac{\beta(2 \pi-\phi)}{\ell}\right) \operatorname{csch}\left(\frac{\beta \pi}{\ell}\right) \tag{15.17}
\end{align*}
$$

respectively, both finite for $\beta, \phi>0$. Thus, $\mathfrak{Z}_{\kappa}^{0}(\phi, \beta)$ converges uniformly in the
 for $\beta, \phi>0$ with $0 \leqslant \Omega_{j} \phi \leqslant 2 \pi$ for $1 \leqslant j \leqslant n$.

Define the regularized, bosonic partition function,

$$
\begin{align*}
\underline{\mathfrak{Z}}_{\kappa}^{0, b}(\phi, \beta) & =\prod_{j=1}^{n} \prod_{k \in \hat{\mathbb{Z}}^{*}}\left(1-e^{-\beta\left|k-\Omega_{j} \phi / \ell\right|}\right)^{-2}  \tag{15.18}\\
& =\mathfrak{Z}_{\kappa}^{0, b}(\phi, \beta) \prod_{j=1}^{n}\left(1-e^{-\beta \Omega_{j} \phi / \ell}\right)^{2}, \tag{15.19}
\end{align*}
$$

which is well-defined and finite for $\phi=0$. To calculate the degree of divergence of the free partition function $\mathfrak{Z}_{K}^{0}(\phi, \beta)$ in the limit $\phi \rightarrow 0^{+}$, observe that for $x>$ 0 ,

$$
\begin{equation*}
x^{-2} \leqslant\left(1-e^{-x}\right)^{-2} \leqslant x^{-2} L(x) \tag{15.20}
\end{equation*}
$$

where $L(x)=1+x+\frac{5 x^{2}}{12}+\frac{x^{3}}{12}+\frac{x^{4}}{240}$, the Laurent expansion of the exponential factor up to fifth order. We use these bounds to estimate the divergent factor $\left(1-e^{-\beta \Omega_{j} \phi / \ell}\right)^{-2}$. Define

$$
\begin{equation*}
\mathscr{L}_{\beta / \ell}^{V}(\phi)=\left(\frac{\ell}{\beta}\right)^{2 n} \prod_{j=1}^{n} \Omega_{j}^{-2} L\left(\frac{\beta \Omega_{j} \phi}{\ell}\right) \tag{15.21}
\end{equation*}
$$

and the constant $\mathscr{L}_{\beta / \ell}^{V}(0)=\left(\frac{\ell}{\beta}\right)^{2 n} \prod_{j=1}^{n} \Omega_{j}^{-2}$. Note that since $V$ has an isolated zero at the origin, none of the reduced weights vanish, hence, $\mathscr{L}_{\beta / \ell}^{V}(\phi)$ is finite for $\phi \geqslant 0$. For $\beta, \phi>0$, we have the lower bound

$$
\begin{equation*}
\phi^{-2 n} \mathscr{L}_{\beta / \ell}^{V}(0) \underline{\mathfrak{Z}}_{\kappa}^{0, b}(\phi, \beta) \leqslant \mathfrak{Z}_{\kappa}^{0, b}(\phi, \beta), \tag{15.22}
\end{equation*}
$$

where the regularized partition function is bounded from above and does not depend on $V$ in the limit $\phi \rightarrow 0^{+}$. We conclude that

$$
\begin{align*}
\mathscr{L}_{\beta / \ell}^{V}(0) \underline{\mathfrak{Z}}_{\kappa}^{0, b}(\phi, \beta) \mathfrak{Z}_{\kappa}^{0, f}(\phi, \beta) & \leqslant \phi^{2 n} \mathfrak{Z}_{\kappa}^{0}(\phi, \beta)  \tag{15.23}\\
& \leqslant \mathscr{L}_{\beta / \ell}^{V}(\phi) \underline{\mathfrak{Z}}_{\kappa}^{0, b}(0, \beta) \mathfrak{Z}_{\kappa}^{0, f}(\phi, \beta), \tag{15.24}
\end{align*}
$$

where the fermionic partition function is bounded from above and does not depend on $V$ in the limit $\phi \rightarrow 0^{+}$. By the squeeze principle, it follows that the divergence of $\mathfrak{Z}_{\mathcal{K}}^{0}(\phi, \beta)=\operatorname{Tr}_{\mathscr{F}} e^{-\beta: H_{0, \kappa}:}$ is $O\left(\phi^{-2 n}\right)$ for $\beta>0$.

Suppose $\lambda>0$ and we take $\varepsilon_{0}$ small enough to ensure $0<\varepsilon_{1}<1$. From the Divergent Kato bound,

$$
\begin{equation*}
\left(1-\varepsilon_{1}\right) H_{\Lambda, \kappa}^{b}(\lambda)+: H_{0, \kappa}^{f}:-M_{1} \leqslant H_{\Lambda, \kappa}(\lambda) \leqslant\left(1+\varepsilon_{1}\right) H_{\Lambda, \kappa}^{b}(\lambda)+: H_{0, \kappa}^{f}:+M_{1} . \tag{15.25}
\end{equation*}
$$

Define $M_{4}=\operatorname{Tr}_{\mathscr{F}} e^{-\beta M_{1}}<\infty$ and $\mathfrak{Z}_{\Lambda, \kappa}^{\lambda V, b}(\phi, \beta)=\operatorname{Tr}_{\mathscr{F} b} e^{-\beta H_{\Lambda, \kappa}^{b}(\lambda)}$. Therefore,

$$
\begin{align*}
M_{4}^{-1} \mathfrak{Z}_{\Lambda, \kappa}^{\lambda V, b}\left(\phi, \beta\left(1+\varepsilon_{1}\right)\right) \mathfrak{Z}_{\kappa}^{0, f}(\phi, \beta) & \leqslant \mathfrak{Z}_{\Lambda, \kappa}^{\lambda V}(\phi, \beta)  \tag{15.26}\\
& \leqslant M_{4} \mathfrak{Z}_{\Lambda, \kappa}^{\lambda V, b}\left(\phi, \beta\left(1-\varepsilon_{1}\right)\right) \mathfrak{Z}_{\kappa}^{0, f}(\phi, \beta) \tag{15.27}
\end{align*}
$$

Since : $H_{0, \kappa}^{b}: \leqslant H_{\Lambda, \kappa}^{b}(\lambda)$, then

$$
\begin{equation*}
\mathfrak{Z}_{\Lambda, \kappa}^{\lambda V}(\phi, \beta) \leqslant M_{4} \mathfrak{Z}_{\kappa}^{0, b}\left(\phi, \beta\left(1-\varepsilon_{1}\right)\right) \mathfrak{Z}_{\kappa}^{0, f}(\phi, \beta), \tag{15.28}
\end{equation*}
$$

which is bounded from above using the same argument for the free partition function with $\phi>0$ and $\beta^{\prime}=\beta\left(1-\varepsilon_{1}\right)>0$. Therefore, $e^{-\beta H_{\Lambda, \kappa}(\lambda)}$ is trace-class for $\beta, \phi>0$ and $\lambda \geqslant 0$.

Now we turn our attention to the case $\phi=0$ with $\lambda>0$. We cannot use the above bounds since $\mathfrak{Z}_{\kappa}^{0, b}\left(\phi, \beta\left(1-\varepsilon_{1}\right)\right)$ is divergent in the limit $\phi \rightarrow 0^{+}$. Instead, consider the free, massive, bosonic Hamiltonian $H_{0, \kappa}^{b}(m)=: H_{0, \kappa}^{b}:+H_{I, \Lambda, \kappa}^{\text {mass }}(m)$
and the corresponding partition function on $\mathscr{F}^{b}$,

$$
\begin{equation*}
\mathfrak{Z}_{\Lambda, \kappa}^{m, b}(\phi, \beta)=\prod_{j=1}^{n} \prod_{k \in \hat{\mathbb{Z}}_{k}}\left(1-e^{-\beta \omega_{m}(k)}\right)^{-2} \tag{15.29}
\end{equation*}
$$

where $\omega_{m}(k)=\left(k^{2}+m^{2}\right)^{1 / 2}=m \mu_{m}(k)$. With this notation, $H_{0, \kappa}^{b}(0)=: H_{0, \kappa}^{b}$ : and $\mathfrak{Z}_{\kappa}^{0, b}(\phi, \beta)$ is the corresponding free, massless bosonic partition function. By the mass shift bound $H_{0, \kappa}^{b}\left(m_{\Lambda, \kappa}\right) \leqslant H_{\Lambda, \kappa}^{b}(\lambda)$, where $m_{\Lambda, \kappa}=\delta \lambda \tilde{\mu}_{\Lambda}(\kappa)^{-\varepsilon}>0$, we can compensate the divergence in the limit $\phi \rightarrow 0^{+}$by incorporating mass that depends on $\lambda, \delta, \Lambda$ and $\kappa$,

$$
\begin{equation*}
\mathfrak{Z}_{\Lambda, \kappa}^{\lambda V}(0, \beta) \leqslant M_{4} \mathfrak{Z}_{\Lambda, \kappa}^{m_{\Lambda, \kappa}, b}\left(0, \beta\left(1-\varepsilon_{1}\right)\right) \mathfrak{Z}_{\kappa}^{0, f}(0, \beta), \tag{15.30}
\end{equation*}
$$

which is bounded from above for $\beta>0$. Hence, $e^{-\beta H_{\Lambda, \kappa}(\lambda)}$ is trace-class for $\beta>0$ and $\lambda, \phi \geqslant 0$ such that not both $\phi$ and $\lambda$ vanish.

We now determine the degree of divergence of $\mathfrak{Z}_{\Lambda, \kappa}^{\lambda V}(\phi, \beta)$ in the limit as $\lambda \rightarrow 0^{+}$with $\phi=0$. Observe that $\omega_{m_{\Lambda, \kappa}}(k) \leqslant \omega_{m_{\Lambda, \kappa}}(\kappa)$ for $k \in \hat{\mathbb{Z}}_{\kappa}$. Define $M_{5}=\left(1-e^{-\beta\left(1-\varepsilon_{1}\right) \omega_{m_{\Lambda, \kappa}}(\kappa)}\right)^{-2 n(2 N-1)}$. Then

$$
\begin{align*}
M_{5}\left(\beta\left(1-\varepsilon_{1}\right) m_{\Lambda, \kappa}\right)^{-2 n} \leqslant & \left(\beta\left(1-\varepsilon_{1}\right) m_{\Lambda, \kappa}\right)^{-2 n} \underline{\underline{Z}}_{\Lambda, \kappa}^{m_{\Lambda, \kappa}, b}\left(0, \beta\left(1-\varepsilon_{1}\right)\right)  \tag{15.31}\\
\leqslant & \mathfrak{Z}_{\Lambda, \kappa}^{m_{\Lambda, \kappa}, b}\left(0, \beta\left(1-\varepsilon_{1}\right)\right)  \tag{15.32}\\
\leqslant & \left(\beta\left(1-\varepsilon_{1}\right) m_{\Lambda, \kappa}\right)^{-2 n} L\left(\beta\left(1-\varepsilon_{1}\right) m_{\Lambda, \kappa}\right)^{n} \\
& \cdot \underline{\mathfrak{Z}}_{\Lambda, \kappa}^{m_{\Lambda, \kappa}, b}\left(0, \beta\left(1-\varepsilon_{1}\right)\right), \tag{15.33}
\end{align*}
$$

which by a squeeze argument shows the degree of divergence is $O((\beta)-$ $\left.\left.\varepsilon_{1}\right) m_{\Lambda, \kappa}\right)^{-2 n}$ ) as $\lambda \rightarrow 0^{+}$with $\phi=0$ for $\beta>0$. That is, there is a constant $M_{6}=M_{4} L\left(\beta\left(1-\varepsilon_{1}\right) m_{\Lambda, \kappa}\right)^{n} \underline{\mathfrak{z}}_{\Lambda, \kappa}^{m_{\Lambda, \kappa}, b}\left(0, \beta\left(1-\varepsilon_{1}\right)\right) \mathfrak{Z}_{\kappa}^{0, f}(0, \beta)<\infty$ such that for $\phi=0$, we have

$$
\begin{align*}
\operatorname{Tr}_{\mathscr{F}} e^{-\beta H_{\Lambda, \kappa}(\lambda)} & \leqslant M_{6}\left(\beta\left(1-\varepsilon_{1}\right) m_{\Lambda, \kappa}\right)^{-2 n}  \tag{15.34}\\
& =M_{6} \tilde{\mu}_{\Lambda}(\kappa)^{2 n \varepsilon}(\beta \delta)^{-2 n}\left(\lambda-(2 N-1) \frac{2 n}{\sqrt{\ell}}\left\|\tilde{\psi}_{\kappa}\right\|_{\infty}^{2} \varepsilon_{0}\right)^{-2 n}
\end{align*}
$$

where $\lambda>(2 N-1) \frac{2 n}{\sqrt{\ell}}\left\|\tilde{\psi}_{\kappa}\right\|_{\infty}^{2} \varepsilon_{0}$.

### 15.2. Defining Relation of the Partition Function

The partition function is defined as $\mathfrak{Z}^{\lambda V}=\operatorname{Tr} \Gamma U e^{-\beta H(\lambda)}$, where $U=$ $e^{-i \theta J-i \sigma P}$ is a symmetry group commuting with $Q_{+}$and anti-commuting with $\Gamma$. According to Jaffe, one may consider $\lambda \in(0,1]$ and

$$
\begin{equation*}
\frac{d \mathfrak{Z}^{\lambda V}}{d \lambda}=\operatorname{Tr} \Gamma U \frac{d}{d \lambda} e^{-H(\lambda)} . \tag{15.36}
\end{equation*}
$$

Observe that

$$
\begin{equation*}
\frac{d}{d \lambda} e^{-H(\lambda)}=-\int_{0}^{1} e^{-s H(\lambda)} H^{\prime}(\lambda) e^{-(1-s) H(\lambda)} d s \tag{15.37}
\end{equation*}
$$

and $H^{\prime}(\lambda)=\left(Q_{+}^{2}-P\right)^{\prime}=\left\{Q_{+}^{\prime}, Q_{+}\right\}$, so

$$
\begin{equation*}
\frac{d}{d \lambda} e^{-H(\lambda)}=-\left\{Q_{+}^{\prime}, Q_{+}\right\} e^{-H(\lambda)} \tag{15.38}
\end{equation*}
$$

Using the fact that $\Gamma U Q=\Gamma Q U=-Q \Gamma U$, it follows that

$$
\begin{align*}
\frac{d \mathfrak{Z}^{\lambda V}}{d \lambda} & =-\operatorname{Tr} \Gamma U\left\{Q_{+}^{\prime}, Q_{+}\right\} e^{-H(\lambda)}  \tag{15.39}\\
& =-\operatorname{Tr}\left(\Gamma U Q_{+}^{\prime} Q_{+}-Q_{+} \Gamma U Q_{+}^{\prime}\right) e^{-H(\lambda)}  \tag{15.40}\\
& =-\operatorname{Tr}\left(\Gamma U Q_{+}^{\prime} Q_{+}-\Gamma U Q_{+}^{\prime} Q_{+}\right) e^{-H(\lambda)}  \tag{15.41}\\
& =0 \tag{15.42}
\end{align*}
$$

by cyclicity of the trace. Jaffe proves the map $\lambda \mapsto \mathfrak{Z}^{\lambda V}$ is differentiable in $\lambda \in$ $(0,1]$. Along with the Hölder continuity of $\mathfrak{Z}^{\lambda V}$ at $\lambda=0$, one concludes that $\mathfrak{Z}^{\lambda}$ is constant in $\lambda$ on $[0,1]$.
15.3. Explicit Evaluation at $\lambda=0$

Define $y=e^{2 \pi i z}$ and $q=e^{2 \pi i \tau}$, where $z=(2 \pi)^{-1}(\theta-\phi \tau), \tau=\ell^{-1}(\sigma+i \beta)$ and $\ell, \beta>0$. Since $\tau \in \mathbb{H}$ and therefore $q \in \Delta$, we may view $\tau$ as a fundamental period of the lattice $\Lambda_{\tau}=\mathbb{Z} \oplus \tau \mathbb{Z}$ and $q$ as the associated nome.

Let $V$ denote a non-degenerate, weighted homogeneous superpotential that satisfies the elliptic bounds. The (twist) partition function is given by the supertrace

$$
\begin{align*}
\mathfrak{Z}^{\lambda V}(z, \tau) & =\operatorname{Tr} \Gamma e^{-\beta H(\lambda)-i \sigma P-i \theta J}  \tag{15.43}\\
& =\operatorname{Str} e^{-\beta H(\lambda)-i \sigma P-i \theta J} \tag{15.44}
\end{align*}
$$

Define $\gamma_{ \pm, j}^{b}( \pm k)=e^{\mp i \theta \omega_{j}-\beta|k|}$ and $\gamma_{ \pm, j}^{f}(k)=e^{\mp \theta \omega_{j}^{f}( \pm k)-\beta|k|}$. Jaffe calculates

$$
\begin{align*}
\mathcal{Z}^{\lambda V}(z, \tau) & =e^{i \theta \hat{c} / 2} \prod_{j=1}^{n}\left(\prod_{k \in K_{j}^{b}} \prod_{k^{\prime} \in K_{1, j}^{f}} \prod_{k^{\prime \prime} \in K_{2, j}^{f}} \frac{\left(1-\gamma_{+, j}^{f}\left(k^{\prime}\right)\right)\left(1-\gamma_{-, j}^{f}\left(k^{\prime \prime}\right)\right)}{\left|1-\gamma_{+, j}^{b}(k)\right|}\right)  \tag{15.45}\\
& =e^{i \theta \hat{c} / 2} \prod_{j=1}^{n} \prod_{k=1}^{\infty}\left(\frac{1-q^{k-1} y^{-\left(1-\omega_{j}\right)}}{1-q^{k-1} y^{-\omega_{j}}}\right)\left(\frac{1-q^{k} y^{1-\omega_{j}}}{1-q^{k} y^{\omega_{j}}}\right) \tag{15.46}
\end{align*}
$$

where $\hat{c}=\sum_{j=1}^{n} 1-2 \omega_{j}$ is the central charge and reduced weight $\boldsymbol{\omega}=$ $\left\{\omega_{1}, \ldots, \omega_{n}\right\} \in \mathbb{Q} \cap\left(0, \frac{1}{2}\right]$.

In the sequel, we study the partition function from the point of view of $q$ calculus and complex geometry. In the next chapter, we prove a $q$-Beta function representation of the elliptic genus. Using this representation, we generalize twist positivity to the supersymmetric case. We refer the reader to Appendix E for a brief review of Ramanujan $q$-series and Jacobi theta functions.

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## Chapter 16

## Supersymmetric Twist Positivity

## Contents

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As the infinite product representation of the elliptic genus resembles a ratio of Ramanujan $q$-series, it should be no surprise that the elliptic genus satisfies interesting $q$-identities. This chapter is devoted to proving that the elliptic genus admits a $q$-representation in terms of well-known $q$-functions, namely, the Jackson integral, $q$-hypergeometric, $q$-Gamma, $q$-Beta functions. A rather surprising consequence of the latter is bosonic twist positivity may be generalized to supersymmetric twist positivity.

### 16.1. The Elliptic Genus as a $q$-Function

Hereafter, we shall assume that $|q|<\left|y^{-\omega_{j}}\right|<1$ for $1 \leqslant j \leqslant n$. The latter inequality is always true, as $\left|y^{-\omega_{j}}\right|=\left|e^{i \theta}\right|^{-\omega_{j}}\left|q^{-\phi / 2 \pi}\right|^{-\omega_{j}}<e^{-\omega_{j}|-i \phi \tau|}$, but the former inequality does not hold in general.

Remark 16.1.1. Consider the square lattice period $\tau=1+i$ and twist angles $\phi=\frac{\pi}{2}, \theta=\frac{\pi}{4}$. The double inequality holds if $\omega \in(0, x)^{n}$ where $x=$ $0.202642367 \ldots \simeq \frac{1}{5}$.

Define $\rho=\tau \mathbf{1}_{n+1}+z \boldsymbol{\omega}$. To ensure that $|q|<\left|y^{-\omega_{j}}\right|<1$ for $0 \leqslant j \leqslant n$, it is sufficient to require that $\rho \in \mathbb{H}^{n}$, which follows from the equality $\left|q y^{\omega_{j}}\right|=$ $\left|e^{2 \pi i\left(\tau+\omega_{j} z\right)}\right|=\left|e^{-2 \pi \operatorname{Im}\left(\rho_{j}\right)}\right|$.

Proposition 16.1. Suppose that $\rho \in \mathbb{H}^{n}$ and define $v=-\frac{z}{\tau}$. The elliptic genus is a product of $q$-series, $\mathfrak{Z}^{\lambda V}=e^{i \hat{c} \theta / 2} \prod_{j=1}^{n} \mathfrak{Z}_{j}^{\lambda V}$, where

$$
\begin{equation*}
\mathfrak{Z}_{j}^{\lambda V}(z, \tau)=\frac{\left(q^{v} ; q\right)_{\infty}}{(q ; q)_{\infty}^{2}} \sum_{k \in \mathbb{Z}}\left(q^{1-v} ; q\right)_{k}\left(q^{k+2-v} ; q\right)_{\infty} q^{k \omega_{j} v} \tag{16.1}
\end{equation*}
$$

Proof. We may write the following infinite product in terms of $q$-products,

$$
\begin{align*}
\prod_{k \geqslant 1}\left(\frac{1-q^{k-1} y^{-\left(1-\omega_{j}\right)}}{1-q^{k-1} y^{-\omega_{j}}}\right)\left(\frac{1-q^{k} y^{1-\omega_{j}}}{1-q^{k} y^{\omega_{j}}}\right) & =\frac{\left(y^{-\left(1-\omega_{j}\right)} ; q\right)_{\infty}\left(y^{\omega_{j}} ; q\right)_{1}\left(y^{1-\omega_{j}} ; q\right)_{\infty}}{\left(y^{-\omega_{j}} ; q\right)_{\infty}\left(y^{1-\omega_{j}} ; q\right)_{1}\left(y^{\omega_{j}} ; q\right)_{\infty}} \\
& =\frac{\left(y^{-\left(1-\omega_{j}\right)} ; q\right)_{\infty}\left(q y^{1-\omega_{j}} ; q\right)_{\infty}}{\left(y^{-\omega_{j}} ; q\right)_{\infty}\left(q y^{\omega_{j}} ; q\right)_{\infty}} . \tag{16.2}
\end{align*}
$$

Setting $\tilde{\zeta}_{j}=y^{-\omega_{j}}, b=q a$ and $a=q y$, and assuming that $|q|<\left|\xi_{j}\right|=\left|y^{-\omega_{j}}\right|<1$, then

$$
\begin{equation*}
\frac{\left(q y \xi_{j} ; q\right)_{\infty}\left(\left(y \xi_{j}\right)^{-1} ; q\right)_{\infty}}{(\xi ; q ; q)_{\infty}(q / \xi ; q)_{\infty}}=\frac{\left(q^{2} y ; q\right)_{\infty}\left(y^{-1} ; q\right)_{\infty}}{(q ; q)_{\infty}^{2}} \sum_{k \in \mathbb{Z}} \frac{(q y ; q)_{k}}{\left(q^{2} y ; q\right)_{k}} \xi_{j}^{k} \tag{16.3}
\end{equation*}
$$

which simplifies to

$$
\begin{align*}
\frac{\left(q^{2} y ; q\right)_{\infty}\left(y^{-1} ; q\right)_{\infty}}{(q ; q)_{\infty}^{2}} \sum_{k \in \mathbb{Z}} \frac{(q y ; q)_{k}}{\left(q^{2} y ; q\right)_{k}} & y^{-k \omega_{j}} \\
& =\frac{\left(y^{-1} ; q\right)_{\infty}}{(q ; q)_{\infty}^{2}} \sum_{k \in \mathbb{Z}}(q y ; q)_{k}\left(q^{k+2} y ; q\right)_{\infty} y^{-k \omega_{j}} \tag{16.4}
\end{align*}
$$

Finally, set $y=q^{z / \tau}=q^{-v}$. This completes the proof.
In terms of the Dirichlet eta function $\eta(\tau)=q^{1 / 24}(q ; q)_{\infty}$,

$$
\begin{equation*}
\mathfrak{Z}^{V}(z, \tau)=q^{n / 12} e^{i \theta \hat{c} / 2} \frac{\left(y^{-1} ; q\right)_{\infty}^{n}}{\eta(\tau)^{2 n}} \prod_{j=1}^{n} \sum_{k \in \mathbb{Z}}(q y ; q)_{k}\left(q^{k+2} y ; q\right)_{\infty} y^{-k \omega_{j}} \tag{16.5}
\end{equation*}
$$

Proposition 16.2. The partition function is a ratio of Jacobi theta functions, $\mathfrak{Z}^{\lambda V}=e^{i \phi \tau \hat{c} / 2} \prod_{j=1}^{n} \mathfrak{Z}_{j}^{\lambda V}$, where

$$
\begin{align*}
\mathfrak{Z}_{j}^{\lambda V}(z, \tau) & =\frac{\theta_{1}\left(\pi\left(1-\omega_{j}\right) z ; q^{1 / 2}\right)}{\theta_{1}\left(\pi \omega_{j} z ; q^{1 / 2}\right)}  \tag{16.6}\\
& =\frac{\vartheta_{1}\left(\left(1-\omega_{j}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{j} z, \tau\right)} \tag{16.7}
\end{align*}
$$

Remark 16.1.2. Appendix $E$ is devoted to Jacobi theta functions.
Proof. Recall that we have defined $z=\frac{\theta-\phi \tau}{2 \pi}, y=e^{2 \pi i z}, q=e^{2 \pi i \tau}$ where $\tau \in \mathbb{H}$. For $a, b \in \mathbb{C}$,

$$
\begin{equation*}
\frac{\theta_{1}\left(\pi a z ; q^{1 / 2}\right)}{\theta_{1}\left(\pi b z ; q^{1 / 2}\right)}=e^{\pi i(a-b) z} \frac{\left(e^{-2 \pi i a z} ; q\right)_{\infty}\left(q e^{2 \pi i a z} ; q\right)_{\infty}}{\left(e^{-2 \pi i b z} ; q\right)_{\infty}\left(q e^{2 \pi i b z} ; q\right)_{\infty}} \tag{16.8}
\end{equation*}
$$

It follows, then, that

$$
\begin{align*}
\mathfrak{Z}_{j}^{\lambda V}(z, \tau) & =e^{i \theta\left(1-2 \omega_{j}\right) / 2} \frac{\left(y^{-\left(1-\omega_{j}\right)} ; q\right)_{\infty}\left(q y^{1-\omega_{j}} ; q\right)_{\infty}}{\left(y^{-\omega_{j}} ; q\right)_{\infty}\left(q y^{\omega_{j}} ; q\right)_{\infty}}  \tag{16.9}\\
& =e^{i \phi\left(1-2 \omega_{j}\right) \tau / 2} \frac{\theta_{1}\left(\pi\left(1-\omega_{j}\right) z ; q^{1 / 2}\right)}{\theta_{1}\left(\pi \omega_{j} z ; q^{1 / 2}\right)} \tag{16.10}
\end{align*}
$$

Finally, note that $\hat{c}=\sum_{j=1}^{n} 1-2 \omega_{j}$.

In terms of the original theta function,

$$
\begin{align*}
\mathfrak{J}^{\lambda V}(z, \tau) & =e^{i \phi \tau \hat{c} / 2} \prod_{j=1}^{n} \frac{\vartheta_{1}\left(\left(1-\omega_{j}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{j} z, \tau\right)}  \tag{16.11}\\
& =e^{i \phi \tau \hat{c} / 2} e^{\pi i \hat{c} z} \prod_{j=1}^{n} \frac{\vartheta\left(\left(1-\omega_{j}\right) z+(1+\tau) / 2, \tau\right)}{\vartheta\left(\omega_{j} z+(1+\tau) / 2, \tau\right)} \tag{16.12}
\end{align*}
$$

Proposition 16.3. The partition function $\mathfrak{Z}^{\lambda V}(z, \tau)$ is a $q$-constant.

Proof. Consider the multivariable $q$-function $f\left(t_{1}, \ldots, t_{n}\right)=\prod_{j=1}^{n} f_{j}\left(t_{j}\right)$, where

$$
\begin{equation*}
f_{j}\left(t_{j}\right)=t_{j}^{a_{j}-b_{j}} \frac{\left(t_{j} q^{a_{j}} ; q\right)_{\infty}\left(t_{j}^{-1} q^{1-a_{j}} ; q\right)_{\infty}}{\left(t_{j} q^{b_{j}} ; q\right)_{\infty}\left(t_{j}^{-1} q^{1-b_{j}} ; q\right)_{\infty}} \tag{16.13}
\end{equation*}
$$

Then $f$ satisfies the identity $f\left(q t_{1}, \ldots, q t_{n}\right)=f\left(t_{1}, \ldots, t_{n}\right)$ and therefore also satisfies the difference equation $D_{q}^{\alpha} f=0$ for any multi-index $\alpha=\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, that is, $D_{q}^{\alpha}=D_{q, 1}^{\alpha_{1}} \cdots D_{q, n}^{\alpha_{n}}$. We now show that the partition function is a specialization of the $q$-constant considered above. Define the following $2 n$ coefficients

$$
\begin{equation*}
a_{j}=\left(1-2 \omega_{j}\right) \frac{\theta}{4 \pi(z+\tau)}-\left(1-\omega_{j}\right) \frac{z}{\tau} \quad \text { and } \quad b_{j}=a_{j}+1+\frac{z}{\tau} \tag{16.14}
\end{equation*}
$$

and the variables $t_{j}=q^{-a_{j}} y^{-\left(1-\omega_{j}\right)}=q^{1-b_{j}} y^{\omega_{j}}$. Then

$$
\begin{align*}
f\left(t_{1}, \ldots, t_{n}\right) & =\prod_{j=1}^{n} t_{j}^{a_{j}-b_{j}} \frac{\left(t_{j} q^{a_{j}} ; q\right)_{\infty}\left(t_{j}^{-1} q^{1-a_{j}} ; q\right)_{\infty}}{\left(t_{j} q^{b_{j}} ; q\right)_{\infty}\left(t_{j}^{-1} q^{1-b_{j}} ; q\right)_{\infty}}  \tag{16.15}\\
& =\prod_{j=1}^{n} e^{i \theta\left(1-2 \omega_{j}\right) / 2} \frac{\left(y^{-\left(1-\omega_{j}\right)} ; q\right)_{\infty}\left(q y^{1-\omega_{j}} ; q\right)_{\infty}}{\left(y^{\omega_{j}} ; q\right)_{\infty}\left(q y^{-\omega_{j}} ; q\right)_{\infty}}  \tag{16.16}\\
& =\mathfrak{Z}^{V}(z, \tau), \tag{16.17}
\end{align*}
$$

which completes the proof.

REMARK 16.1.3. Note that if $\theta=0$, then the constants simplify greatly, namely, $a_{j}=\frac{\left(1-\omega_{j}\right) \phi}{2 \pi}$ and $b_{j}=1-\frac{\omega_{j} \phi}{2 \pi}$.

Proposition 16.4. The partition function $\mathfrak{Z}^{V}$ is a product of $q$-hypergeometric functions, viz.,

$$
\begin{equation*}
\mathfrak{Z}^{V}(z, \tau)=e^{i \hat{c} \theta / 2} \prod_{j=1}^{n}{ }_{2} \Phi_{1}^{\prime}\left[y^{1-2 \omega_{j}},(q y)^{-1} ; y^{-\omega_{j}} ; q, q y^{\omega_{j}}\right] \tag{16.18}
\end{equation*}
$$

Proof. Set $a=y^{1-2 \omega_{j}}, b=(q y)^{-1}, c=y^{-\omega_{j}}$ and $z=c /(a b)$. Since $|z|=$ $\left|q y^{\omega_{j}}\right|<1$, then it follows that

$$
\begin{equation*}
\frac{\left(y^{-\left(1-\omega_{j}\right)} ; q\right)_{\infty}\left(q y^{1-\omega_{j}} ; q\right)_{\infty}}{\left(y^{-\omega_{j}} ; q\right)_{\infty}\left(q y^{\omega_{j}} ; q\right)_{\infty}}={ }_{2} \Phi_{1}^{\prime}\left[y^{1-2 \omega_{j}},(q y)^{-1} ; y^{-\omega_{j}} ; q, q y^{\omega_{j}}\right] \tag{16.19}
\end{equation*}
$$

Since $y=e^{2 \pi i z}=q^{z / \tau}$, we may write

$$
\begin{aligned}
{ }_{2} \Phi_{1}^{\prime}\left[y^{1-2 \omega_{j}},\right. & \left.(q y)^{-1} ; y^{-\omega_{j}} ; q, q y^{\omega_{j}}\right] \\
& ={ }_{2} \Phi_{1}^{\prime}\left[q^{\left(1-2 \omega_{j}\right) z / \tau}, q^{-(1+z / \tau)} ; q^{-\omega_{j} z / \tau} ; q, q^{1+\omega_{j} z / \tau}\right]
\end{aligned}
$$

Proposition 16.5. The partition function $\mathfrak{Z}^{\lambda V}$ is a product of a ratio of $q$-Beta functions,

$$
\begin{equation*}
\mathfrak{Z}^{\lambda V}(z, \tau)=e^{i \hat{\hat{\theta}} \theta / 2} \prod_{j=1}^{n} \frac{B_{q}\left(1+\frac{\omega_{j} z}{\tau},-\frac{\omega_{j} z}{\tau}\right)}{B_{q}\left(1+\frac{\left(1-\omega_{j}\right) z}{\tau},-\frac{\left(1-\omega_{j}\right) z}{\tau}\right)} \tag{16.20}
\end{equation*}
$$

Proof. Define $\alpha_{j}=\frac{\left(1-2 \omega_{j}\right) z}{\tau}, \beta=-\left(1+\frac{z}{\tau}\right)=v-1$, and $\gamma_{j}=-\frac{\omega_{j} z}{\tau}=\omega_{j} v$. Then

$$
\begin{equation*}
\mathfrak{Z}_{j}(z, \tau)={ }_{2} \Phi_{1}^{\prime}\left[q^{\left(1-2 \omega_{j}\right) z / \tau}, q^{-(1+z / \tau)} ; q^{-\omega_{j} z / \tau} ; q, q^{1+\omega_{j} z / \tau}\right] \tag{16.21}
\end{equation*}
$$

With these coefficients, the Jackson integral simplifies to

$$
\begin{align*}
\int_{0}^{1} \frac{t^{\beta-1}(q t ; q)_{\left(1-\omega_{j}\right) z / \tau}}{\left(t q^{1+\omega_{j} z / \tau} ; q\right)_{\left(1-2 \omega_{j}\right) z / \tau}} d_{q} t & =\int_{0}^{1} \frac{t^{\beta-1}(q t ; q)_{\infty}\left(t q^{1+\left(1-\omega_{j}\right) z / \tau} ; q\right)_{\infty}}{\left(t q^{1+\left(1-\omega_{j}\right) z / \tau} ; q\right)_{\infty}\left(t q^{1+\omega_{j} z / \tau} ; q\right)_{\infty}} d_{q} t  \tag{16.22}\\
& =\int_{0}^{1} \frac{t^{\beta-1}(q t ; q)_{\infty}}{\left(t q^{1+\omega_{j} z / \tau} ; q\right)_{\infty}} d_{q} t  \tag{16.23}\\
& =\int_{0}^{1} t^{\beta-1}(q t ; q)_{-\gamma_{j}} d_{q} t  \tag{16.24}\\
& =B_{q}\left(\beta, 1-\gamma_{j}\right) \tag{16.25}
\end{align*}
$$

and, therefore, the right side of equation (16.21) reduces to

$$
\begin{equation*}
{ }_{2} \Phi_{1}^{\prime}\left[q^{\left(1-2 \omega_{j}\right) z / \tau}, q^{-(1+z / \tau)} ; q^{-\omega_{j} z / \tau} ; q, q^{1+\omega_{j} z / \tau}\right]=\frac{B_{q}\left(\beta, 1-\gamma_{j}\right)}{B_{q}\left(\beta, \gamma_{j}-\beta\right)} \tag{16.26}
\end{equation*}
$$

Recalling that $v=-\frac{z}{\tau}$,

$$
\begin{align*}
\frac{B_{q}\left(\beta, 1-\gamma_{j}\right)}{B_{q}\left(\beta, \gamma_{j}-\beta\right)} & =\frac{B_{q}\left(v-1,1-\omega_{j} v\right)}{B_{q}\left(v-1,1-\left(1-\omega_{j}\right) v\right)}  \tag{16.27}\\
& =\frac{B_{q}\left(\omega_{j} v, 1-\omega_{j} v\right)}{B_{q}\left(\left(1-\omega_{j}\right) v, 1-\left(1-\omega_{j}\right) v\right)}  \tag{16.28}\\
& =\frac{B_{q}\left(1+\frac{\omega_{j} z}{\tau},-\frac{\omega_{j} z}{\tau}\right)}{B_{q}\left(1+\frac{\left(1-\omega_{j}\right) z}{\tau},-\frac{\left(1-\omega_{j}\right) z}{\tau}\right)} \tag{16.29}
\end{align*}
$$

from the symmetry $B_{q}(x, y)=B_{q}(y, x)$ and identity

$$
\begin{equation*}
B_{q}(x, y) B_{q}(x+y, z)=B_{q}(y, z) B_{q}(x, y+z) \tag{16.30}
\end{equation*}
$$

16.1.1. Twist-Field Partition Function. Let $H=H(V)$ denote the twist boson-fermion Hamiltonian with elliptic and weighted homogeneous interaction superpotential $V$ with weights $\left\{\omega_{j}\right\}_{j=1}^{n}, P$ denote the boson-fermion momentum operator and $J$ denote the twist operator. Recall that the central charge is simply $\hat{c}=\sum_{j=1}^{n} 1-2 \omega_{j}$. Define the normalized parition function $\mathfrak{Z}^{V}(z, \tau)=\operatorname{Str} e^{-\beta H-i \sigma P-i \theta \bar{J}}$, where $\bar{J}=J-\frac{\hat{c}}{2}$. Jaffe calculates

$$
\begin{equation*}
\mathfrak{Z}^{V}(z, \tau)=\prod_{j=1} \prod_{k \geqslant 0} \frac{\left(y^{-\left(1-\omega_{j}\right)} ; q\right)_{\infty}\left(q y^{\left(1-\omega_{j}\right)} ; q\right)_{\infty}}{\left(y^{-\omega_{j}} ; q\right)_{\infty}\left(q y^{\omega_{j}} ; q\right)_{\infty}} \tag{16.31}
\end{equation*}
$$

Proposition 16.6 (Theta Function Representation). For $(z, \tau) \in \mathbb{C} \times \mathbb{H}$, we have

$$
\begin{equation*}
\mathfrak{Z}^{V}(z, \tau)=y^{-\hat{c} / 2} \prod_{j=1}^{n} \frac{\vartheta_{1}\left(\left(1-\omega_{j}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{j} z, \tau\right)} \tag{16.32}
\end{equation*}
$$

Proof. Define $y=e^{2 \pi i z}$ and $q=e^{2 \pi i \tau}$. Recall that

$$
\begin{equation*}
\frac{\vartheta_{1}(\alpha z, \tau)}{\vartheta_{1}(\beta z, \tau)}=\left(\frac{y^{-\alpha / 2}-y^{\alpha / 2}}{y^{-\beta / 2}-y^{\beta / 2}}\right) \frac{\left(q y^{-\alpha} ; q\right)_{\infty}\left(q y^{\alpha} ; q\right)_{\infty}}{\left(q y^{-\beta} ; q\right)_{\infty}\left(q y^{\beta} ; q\right)_{\infty}} \tag{16.33}
\end{equation*}
$$

However, since $(a q ; q)_{\infty}=(1-a)^{-1}(a ; q)_{\infty}$

$$
\begin{align*}
\frac{y^{-\alpha / 2}-y^{\alpha / 2}}{y^{-\beta / 2}-y^{\beta / 2}} \frac{\left(q y^{-\alpha} ; q\right)_{\infty}\left(q y^{\alpha} ; q\right)_{\infty}}{\left(q y^{-\beta} ; q\right)_{\infty}\left(q y^{\beta} ; q\right)_{\infty}}= & \left(\frac{y^{-\alpha / 2}-y^{\alpha / 2}}{y^{-\beta / 2}-y^{\beta / 2}}\right)\left(\frac{1-y^{-\beta}}{1-y^{-\alpha}}\right) \\
& \cdot \frac{\left(y^{-\alpha} ; q\right)_{\infty}\left(q y^{\alpha} ; q\right)_{\infty}}{\left(y^{-\beta} ; q\right)_{\infty}\left(q y^{\beta} ; q\right)_{\infty}}  \tag{16.34}\\
= & y^{(\alpha-\beta) / 2} \frac{\left(y^{-\alpha} ; q\right)_{\infty}\left(q y^{\alpha} ; q\right)_{\infty}}{\left(y^{-\beta} ; q\right)_{\infty}\left(q y^{\beta} ; q\right)_{\infty}} \tag{16.35}
\end{align*}
$$

With the choice of $\alpha=1-\omega$ and $\beta=\omega$,

$$
\begin{equation*}
\frac{\vartheta_{1}((1-\omega) z, \tau)}{\vartheta_{1}(\omega z, \tau)}=y^{(1-2 \omega) / 2} \frac{\left(y^{-(1-\omega)} ; q\right)_{\infty}\left(q y^{1-\omega} ; q\right)_{\infty}}{\left(y^{-\omega} ; q\right)_{\infty}\left(q y^{\omega} ; q\right)_{\infty}} \tag{16.36}
\end{equation*}
$$

The identity now follows from the definition $\hat{c}=\sum_{j=1}^{n} 1-2 \omega_{j}$.
Given the transformation properties of $\vartheta_{1}$ under the action of the semi-direct product $\mathrm{SL}_{2}(\mathbb{Z}) \rtimes \mathbb{Z}^{2}$, we now uncover another set of symmetries satisfied by the partition function.

Proposition $16.7\left(\mathbb{Z}^{2} \ltimes \mathrm{SL}_{2}(\mathbb{Z})\right.$-Symmetry). For $\gamma_{\delta}=\left((m, n),\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)\right) \in$ $\mathbb{Z}^{2} \ltimes \mathrm{SL}_{2}(\mathbb{Z})$ and $(z, \tau) \in \mathbb{C} \times \mathbb{H}$, we have the transformation law

$$
\begin{equation*}
\left.\mathfrak{Z}^{V}\right|_{\gamma_{\delta}}(z, \tau)=y^{\hat{c} / 2} \mathbf{e}_{c, d}^{\hat{\varepsilon}\left[c z^{2}-(2 m+1) z-a^{\prime} \tau-b^{\prime}\right]} \mathfrak{Z}^{V}(z, \tau), \tag{16.37}
\end{equation*}
$$

where $a^{\prime}=m a+n c$ and $b^{\prime}=m b+n d$.

Proof. Define $\left.y\right|_{\gamma_{\delta}}=\mathbf{e}_{c, d}^{2\left(z+a^{\prime} \tau+b^{\prime}\right)}=e^{2 \pi i\left(z+a^{\prime} \tau+b^{\prime}\right) /(c \tau+d)}$. If $\alpha=1-\omega$ and $\beta=\omega$, then $\alpha-\beta=\alpha^{2}-\beta^{2}=1-2 \omega$ and, therefore,

$$
\begin{align*}
\prod_{j=1}^{n} \frac{\left.\vartheta_{1}\right|_{\gamma_{\delta}}\left(\left(1-\omega_{j}\right) z, \tau\right)}{\left.\vartheta_{1}\right|_{\gamma_{\delta}}\left(\omega_{j} z, \tau\right)} & =\left(\prod_{j=1}^{n} \mathbf{e}_{c, d}^{\left(1-2 \omega_{j}\right) z(c z-2 m)}\right) \prod_{j=1}^{n} \frac{\vartheta_{1}(\alpha z, \tau)}{\vartheta_{1}(\beta z, \tau)}  \tag{16.38}\\
& =\mathbf{e}_{c, d}^{\hat{c z}(c z-2 m)} \prod_{j=1}^{n} \frac{\vartheta_{1}(\alpha z, \tau)}{\vartheta_{1}(\beta z, \tau)} \tag{16.39}
\end{align*}
$$

Thus,

$$
\begin{align*}
\left.\mathfrak{Z}^{V}\right|_{\gamma_{\delta}}(z, \tau) & =\left.y\right|_{\gamma_{\delta}} ^{-\hat{c} / 2} \prod_{j=1}^{n} \frac{\left.\vartheta_{1}\right|_{\gamma_{\delta}}\left(\left(1-\omega_{j}\right) z, \tau\right)}{\left.\vartheta_{1}\right|_{\gamma_{\delta}}\left(\omega_{j} z, \tau\right)}  \tag{16.40}\\
& =\mathbf{e}_{c, d}^{-\hat{c}\left(z+a^{\prime} \tau+b^{\prime}\right)} \mathbf{e}_{c, d}^{\hat{c} z(c z-2 m)} \prod_{j=1}^{n} \frac{\vartheta_{1}\left(\left(1-\omega_{j}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{j} z, \tau\right)}  \tag{16.41}\\
& =\mathbf{e}_{c, d}^{\hat{c} z(c z-2 m-1)-\hat{c}\left(a^{\prime} \tau+b^{\prime}\right)} \prod_{j=1}^{n} \frac{\vartheta_{1}\left(\left(1-\omega_{j}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{j} z, \tau\right)}  \tag{16.42}\\
& =y^{\hat{c} / 2} \mathbf{e}_{c, d}^{\hat{c} z(c z-2 m-1)-\hat{c}\left(a^{\prime} \tau+b^{\prime}\right)} \mathfrak{Z}^{V}(z, \tau), \tag{16.43}
\end{align*}
$$

which is the claim.

Corollary $16.8\left(\mathbb{Z}^{2}\right.$-Symmetry). For $\delta=(m, n) \in \mathbb{Z}^{2}$ and $(z, \tau) \in \mathbb{C} \times \mathbb{H}$, we have the transformation law

$$
\begin{equation*}
\left.\mathfrak{Z}^{V}\right|_{\delta}(z, \tau)=\zeta\left(y q^{1 / 2}\right)^{-\hat{c} m} \mathfrak{Z}^{V}(z, \tau), \tag{16.44}
\end{equation*}
$$

where $\zeta=e^{\pi i \hat{c} n}$.
Corollary $16.9\left(\mathrm{SL}_{2}(\mathbb{Z})\right.$-Symmetry). For $\gamma=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \operatorname{SL}_{2}(\mathbb{Z})$ and $(z, \tau) \in$ $\mathrm{C} \times \mathbb{H}$, we have the transformation law

$$
\begin{equation*}
\left.\mathfrak{Z}^{V}\right|_{\gamma}(z, \tau)=y^{\hat{c} / 2} \mathbf{e}_{c, d}^{\hat{c} z(c z-1)} \mathfrak{Z}^{V}(z, \tau) \tag{16.45}
\end{equation*}
$$

For the convenience of the reader, we replace the original set of variables and find the most general $\mathbb{Z}_{2} \ltimes \mathrm{SL}_{2}(\mathbb{Z})$ transformation law explicitly:

$$
\begin{aligned}
& \mathcal{Z}^{V}\left(\frac{z}{c \tau+d}+m\left(\frac{a \tau+b}{c \tau+d}\right)+n, \frac{a \tau+b}{c \tau+d}\right)= \\
& \quad e^{\pi i \hat{c} z+\pi i \hat{c}\left[c z^{2}-(2 m+1) z-(m a+n c) \tau-(m b+n d)\right] /(c \tau+d)} \mathfrak{Z}^{V}(z, \tau) .
\end{aligned}
$$

In particular, one has the two independent transformation laws:

$$
\begin{align*}
\mathfrak{Z}^{V}\left(\frac{z}{c \tau+d}, \frac{a \tau+b}{c \tau+d}\right) & =e^{\pi i \hat{c} z+\pi i \hat{c}\left(c z^{2}-z\right) /(c \tau+d)} \mathfrak{Z}^{V}(z, \tau)  \tag{16.46}\\
\mathfrak{Z}^{V}(z+m \tau+n, \tau) & =e^{-\pi i \hat{c}(2 m z+m \tau+n)} \mathfrak{Z}^{V}(z, \tau) . \tag{16.47}
\end{align*}
$$

### 16.2. Positivity of the Elliptic Genus

Define $(a ; q)_{n}=\prod_{k=0}^{n-1}\left(1-a q^{k}\right)$, where $n \in \mathbb{N}$ and $|q|<1$. We can analytically continue $n$ to $v \in \mathbb{C} \backslash \mathbb{N}$ by defining $(a ; q)_{v}=\frac{(a ; q)_{\infty}}{\left(a q^{v} ; q\right)_{\infty}}$. For $z \in \mathbb{C} \backslash \mathbb{Z}_{\leqslant 0}$ and $|q|<1$,
define the $q$-Gamma function, $\Gamma_{q}(z)=(1-q)^{1-z} \frac{(q ; q)_{\infty}}{\left(q^{z} ; q\right)_{\infty}}$. The $q$-Beta function is defined as the Jackson integral

$$
\begin{equation*}
B_{q}(x, y)=\int_{0}^{1} t^{x-1}(t q ; q)_{y-1} d_{q} t=\frac{\Gamma_{q}(x) \Gamma_{q}(y)}{\Gamma_{q}(x+y)} \tag{16.48}
\end{equation*}
$$

where $0<|q|<1, \operatorname{Re}(x)>0$ and $\operatorname{Re}(y)>0$ and satisfies the identities

$$
\begin{equation*}
B_{q}(x, y)=B_{q}(y, x) \quad \text { and } \quad B_{q}(x, y) B_{q}(x+y, z)=B_{q}(y, z) B_{q}(y+z, x) \tag{16.49}
\end{equation*}
$$

Moreover, for $q \in(0,1)$ and $x \in(-1,1)$, we have the positivity $\operatorname{sgn}(x) B_{q}(x, 1-$ $x)>0$. For $|z|<1$, define the $q$-hypergeometric function

$$
\begin{equation*}
{ }_{2} \Phi_{1}^{\prime}\left[q^{\alpha}, q^{\beta} ; q^{\gamma} ; q, z\right]=\frac{1}{B_{q}(\beta, \gamma-\beta)} \int_{0}^{1} \frac{t^{\beta-1}(q t ; q)_{\gamma-\beta-1}}{(z t ; q)_{\alpha}} d_{q} t \tag{16.50}
\end{equation*}
$$

The $q$-analogue of the Gauss Hypergeometric Theorem, proved by Jacobi and Heine, is the following identity:

$$
\begin{equation*}
{ }_{2} \Phi_{1}^{\prime}\left[q^{\alpha}, q^{\beta} ; q^{\gamma} ; q, q^{\gamma-\alpha-\beta}\right]=\frac{\left(q^{\gamma-\alpha} ; q\right)_{\infty}\left(q^{\gamma-\beta} ; q\right)_{\infty}}{\left(q^{\gamma} ; q\right)_{\infty}\left(q^{\gamma-\alpha-\beta} ; q\right)_{\infty}} \tag{16.51}
\end{equation*}
$$

16.2.1. Twist Positivity. Jaffe proves that $\operatorname{Tr} U(\theta) e^{-\beta H}$ on the bosonic Fock space $\mathscr{F}_{b}$ is positive for all 1-parameter unitary groups $U(\theta)$ with $U(0)=I$. We extend the analysis to the tensor product Fock space $\mathscr{F}_{b} \otimes \mathscr{F}_{f}$ and prove that when $U(\theta)=\Gamma e^{-i \sigma P-i \theta J}$, the partition function $\mathfrak{Z}^{V}=\mathbf{S t r} e^{-\beta H-i \sigma P-i \theta J}$ is aperiodicially oscillatory about the abscissa. We determine explicitly the intervals where $\mathfrak{Z}^{V}$ is positive.

Proposition 16.10 ( $q$-Beta Representation). The twist partition function $\mathfrak{Z}^{V}(z, \tau)$ admits the following $q$-representation,

$$
\begin{equation*}
\mathfrak{J}^{V}(z, \tau)=e^{i \hat{\imath} \theta / 2} \prod_{j=1}^{n} \frac{B_{q}\left(1+\frac{\omega_{j} z}{\tau},-\frac{\omega_{j} z}{\tau}\right)}{B_{q}\left(1+\frac{\left(1-\omega_{j}\right) z}{\tau},-\frac{\left(1-\omega_{j}\right) z}{\tau}\right)} \tag{16.52}
\end{equation*}
$$

Proof. Write $\mathfrak{Z}^{V}(z, \tau)=e^{i \hat{c} \theta / 2} \prod_{j=1}^{n} \mathfrak{Z}_{j}(z, \tau)$. Define $\alpha_{j}=\frac{\left(1-2 \omega_{j}\right) z}{\tau}, \beta=$ $-\left(1+\frac{z}{\tau}\right)$, and $\gamma_{j}=-\frac{\omega_{j} z}{\tau}$. Then one has the following explicit representation in terms of a $q$-hypergeometric function,

$$
\begin{equation*}
\mathfrak{Z}_{j}(z, \tau)={ }_{2} \Phi_{1}^{\prime}\left[q^{\left(1-2 \omega_{j}\right) z / \tau}, q^{-(1+z / \tau)} ; q^{-\omega_{j} z / \tau} ; q, q^{1+\omega_{j} z / \tau}\right] \tag{16.53}
\end{equation*}
$$

With these exponents, the Jackson integral representation of the $q$ hypergeometric function ${ }_{2} \Phi_{1}^{\prime}$ simplifies

$$
\begin{align*}
\int_{0}^{1} \frac{t^{\beta-1}(q t ; q)_{\left(1-\omega_{j}\right) z / \tau}}{\left(t q^{1+\omega_{j} z / \tau} ; q\right)_{\left(1-2 \omega_{j}\right) z / \tau}} d_{q} t & =\int_{0}^{1} \frac{t^{\beta-1}(q t ; q)_{\infty}\left(t q^{1+\left(1-\omega_{j}\right) z / \tau} ; q\right)_{\infty}}{\left(t q^{1+\left(1-\omega_{j}\right) z / \tau} ; q\right)_{\infty}\left(t q^{1+\omega_{j} z / \tau} ; q\right)_{\infty}} d_{q} t  \tag{16.54}\\
& =\int_{0}^{1} \frac{t^{\beta-1}(q t ; q)_{\infty}}{\left(t q^{1+\omega_{j} z / \tau} ; q\right)_{\infty}} d_{q} t  \tag{16.55}\\
& =\int_{0}^{1} t^{\beta-1}(q t ; q)_{-\gamma_{j}} d_{q} t  \tag{16.56}\\
& =B_{q}\left(\beta, 1-\gamma_{j}\right) \tag{16.57}
\end{align*}
$$

Hence, we may write the $q$-hypergeometric function as a ratio of $q$-beta functions,

$$
\begin{aligned}
{ }_{2} \Phi_{1}^{\prime}\left[q^{\left(1-2 \omega_{j}\right) z / \tau}, q^{-(1+z / \tau)} ; q^{-\omega_{j} z / \tau} ; q, q^{1+\omega_{j} z / \tau}\right] & =\frac{B_{q}\left(\beta, 1-\gamma_{j}\right)}{B_{q}\left(\beta, \gamma_{j}-\beta\right)} \\
& =\frac{B_{q}\left(1+\frac{\omega_{j} z}{\tau},-\frac{\omega_{j} z}{\tau}\right)}{B_{q}\left(1+\frac{\left(1-\omega_{j}\right) z}{\tau},-\frac{\left(1-\omega_{j}\right) z}{\tau}\right)}
\end{aligned}
$$

as claimed.

Proposition 16.11 (Twist Positivity). For all $\beta>0$ the supertrace of the heat kernel $\operatorname{str} e^{-\beta H(\lambda)}$ is positive for any twist angle satisfying $|\phi|<\inf \left\{\frac{2 \pi}{\omega_{i}}, \frac{2 \pi}{1-\omega_{i}}\right\}$.

Proof. The condition $\sigma=\theta=0$ is necessary and sufficient to ensure $q \in(0,1)$ and $\frac{z}{\tau} \in \mathbb{R}$. Assuming further that $|\phi|<\inf \left\{\frac{2 \pi}{\omega_{i}}, \frac{2 \pi}{1-\omega_{i}}\right\}$, we have

$$
\begin{equation*}
\operatorname{Str} e^{-\beta H}=\lim _{\sigma, \theta \rightarrow 0} \mathfrak{Z}^{V}(z, \tau)=\prod_{i=1}^{n} \frac{B_{q}\left(\frac{\omega_{i} \phi}{2 \pi}, 1-\frac{\omega_{i} \phi}{2 \pi}\right)}{B_{q}\left(\frac{\left(1-\omega_{i}\right) \phi}{2 \pi}, 1-\frac{\left(1-\omega_{i}\right) \phi}{2 \pi}\right)}>0 \tag{16.58}
\end{equation*}
$$

as claimed.

As a corollary, one has the positivity of the quantum mechanical index.

### 16.3. Quantum Mechanical Ground States

Recall the space-time parameter $\tau=\frac{\sigma+i \beta}{\ell}$ and twist parameter $z=\frac{\theta-\phi \tau}{2 \pi}$. To further simplify notation, let $y=e^{2 \pi i z}$ and $q=e^{2 \pi i \tau}$, as above. Recall the elliptic
genus is given by

$$
\begin{align*}
\mathfrak{Z}_{V}(z, \tau) & =\operatorname{Str} e^{-\beta H-i \sigma P-i \theta J}  \tag{16.59}\\
& =e^{i \hat{c} \theta / 2} \prod_{i=1}^{n} \prod_{k \geqslant 0} \frac{1-y^{-\left(1-\omega_{j}\right)} q^{k}}{1-y^{-\omega_{j}} q^{k}} \frac{1-y^{1-\omega_{j}} q^{k+1}}{1-y^{\omega_{j}} q^{k+1}} . \tag{16.6o}
\end{align*}
$$

The limit of zero-twist is the Witten (Fredholm) index of $Q_{V}^{+}$, where $\left(Q_{V}^{+}\right)^{2}=$ $H+P$,

$$
\begin{align*}
\operatorname{ind}\left(Q_{V}^{+}\right) & =\operatorname{dim}_{\mathbb{C}} \operatorname{ker} Q_{V}^{+}-\operatorname{dim}_{\mathbb{C}} \operatorname{ker}\left(Q_{V}^{+}\right)^{*}  \tag{16.61a}\\
& =\lim _{\theta, \phi \rightarrow 0} \mathfrak{Z}^{V}(z, \tau)  \tag{16.61b}\\
& =\prod_{i=1}^{n}\left(\frac{1}{\omega_{i}}-1\right)  \tag{16.61c}\\
& =\mu(V), \tag{16.61d}
\end{align*}
$$

which is the Milnor number of the weighted homogeneous polynomial $V$. This integer enumerates the bosonic ground states of the corresponding quantum theory.

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## Chapter 17

## Twist Partition Function as an Elliptic Genus

## Contents

17.1. Atiyah-Singer Index Theorem
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17.1. Atiyah-Singer Index Theorem

Atiyah and Singer proved that the signature $\sigma(M)$ is the Fredholm index of an elliptic operator $D=d^{*}+d$, the signature operator of a compact manifold $M$, where $d$ is the exterior derivative and $D^{2}=d^{*} d+d d^{*}=\Delta$ is the Laplacian restricted to the +1-eigenspace of even forms in the complex bundle $\Omega^{*}(M)=$ $\Omega^{+}(M) \oplus \Omega^{-}(M)$ under a specific $\mathbb{Z}_{2}$-action involving the Hodge star * modulo a normalizing power of $i$ [192]. That is, the index of $D$ is given by

$$
\begin{align*}
\operatorname{ind}(D) & =(-1)^{\ell}\left\langle\operatorname{ch}\left(\Lambda^{+} T_{\mathbb{C}}^{*} M-\Lambda^{-} T_{\mathbb{C}}^{*} M\right) \frac{\operatorname{td}\left(T_{\mathbb{C}} M\right)}{\mathrm{e}(T M)},[M]\right\rangle  \tag{17.1a}\\
& =\left\langle\prod_{i=1}^{2 k} \frac{x_{i}}{\tanh x_{i}},[M]\right\rangle, \tag{17.1b}
\end{align*}
$$

which is precisely $\left\langle L_{k}(M),[M]\right\rangle$, the signature of $M$. Compare these formulas to the Euler characteristic $\chi(M)=\langle\mathrm{e}(T M),[M]\rangle$ and the index of the Dolbeault operator and $\operatorname{ind}(\bar{\partial})=\left\langle\operatorname{td}\left(T_{\mathbb{C}} M\right),[M]\right\rangle$, where e, ch and td denote the Euler,

Chern and Todd classes, respectively. If $M$ is a compact, oriented 4-manifold with a virtual vector bundle $E$, the Atiyah-Singer Index Theorem states that there is a Dirac operator $D_{A}^{+}$corresponding the $\hat{A}$-genus with coefficients in $E$ (Chapter 2, [323]),

$$
\begin{align*}
\operatorname{ind}\left(D_{A}^{+}\right) & =\langle\hat{A}(T M) \operatorname{ch}(E),[M]\rangle  \tag{17.2}\\
& =-\frac{\operatorname{dim} E}{24}\left\langle p_{1}(T M),[M]\right\rangle+\frac{1}{2}\left\langle c_{1}(E)^{2},[M]\right\rangle .
\end{align*}
$$

Combined with the Hirzebruch Signature Theorem, $\sigma(M)=\frac{1}{3}\left\langle p_{1}(T M),[M]\right\rangle$,

$$
\begin{equation*}
\operatorname{ind}\left(D_{A}^{+}\right)=-\frac{1}{8} \sigma(M)+\frac{1}{2}\left\langle c_{1}(E)^{2}-c_{2}(E),[M]\right\rangle \tag{17.4}
\end{equation*}
$$

In particular, if $M$ is smooth spin 4-manifold, then the index of $D_{A}^{+}$is even and the term involving the Chern number is zero.

Proposition 17.1 (Rokhlin). If $M$ is a closed, oriented, smooth spin 4manifold, then $\sigma(M)$ is divisible by 16.

### 17.2. Elliptic Genera

Let $M$ be a closed, oriented, smooth (Riemannian, spin) $4 k$-manifold. An elliptic genus is a character-valued signature on the (free) loop space $\Omega M$ of maps from $S^{1}$ to $M$ [451]. We refer the reader specifically to Chapter 1 in [451] and the articles by Landweber, Ochanine, Ravenel, Witten, Yui and Zagier in [255], where elliptic genera are developed from first principles.

Proposition 17.2. For a smooth, oriented, complex manifold $X$ of dimension $n=\operatorname{dim}_{\mathbb{C}} X$ (which is a smooth, oriented, real manifold of even dimension). Define the series $\mathcal{E} \ell \ell_{y, q}(X)$ given by

$$
\begin{equation*}
y^{\hat{c} / 2} \underset{k \geqslant 1}{\otimes}\left(\bigwedge_{-y^{-d} q^{k-1}} T X \otimes \bigwedge_{-y^{d} q^{k}} T X^{*} \otimes \operatorname{Sym}_{q^{k}} T X \otimes \operatorname{Sym}_{q^{k}} T X^{*}\right) \tag{17.5}
\end{equation*}
$$

where TX and TX* denote the tangent and cotangent bundles of $X$, respectively. The corresponding elliptic genus $\mathfrak{Z}^{V}(X)$ on $X$ is given by

$$
\begin{equation*}
\mathfrak{Z}^{V}(X)=e^{i \phi \tau \hat{c} / 2}\left(\frac{\operatorname{ch}\left(\mathcal{E} \ell \ell_{y, q}(X)\right) \operatorname{td}(X)}{\mathrm{e}(X)}\right) . \tag{17.6}
\end{equation*}
$$

Proof. Define $z=\frac{\theta-\phi \tau}{2 \pi}$. Let $\left\{x_{i}\right\}$ be the formal Chern roots of $T X$, where $x_{i}=2 \pi i z \omega_{i}$ and $\omega$ is the weight of a non-degenerate, weighted homogeneous superpotential $f$ that satisfies the elliptic bounds. Let $y=e^{2 \pi i z}, q=e^{2 \pi i \tau}$ and $\hat{c}=\sum_{i} 1-2 \omega_{i}$. By definition, the Chern class of the elliptic series is the product,

$$
\begin{aligned}
y^{\hat{c} / 2} \prod_{k \geqslant 1} \operatorname{ch}\left(\bigwedge_{-y^{-d} q^{k-1}} T X\right) \cdot \operatorname{ch} & \left(\bigwedge_{-y^{d} q^{k}} T X^{*}\right) \\
& \cdot \operatorname{ch}\left(\operatorname{Sym}_{q^{k}} T X\right) \cdot \operatorname{ch}\left(\operatorname{Sym}_{q^{k}} T X^{*}\right)
\end{aligned}
$$

with

$$
\begin{align*}
\operatorname{ch}\left(\bigwedge_{t} T X\right) & =\prod_{i=1}^{n}\left(1+t e^{x_{i}}\right)  \tag{17.7}\\
\operatorname{ch}\left(\bigwedge_{t} T X^{*}\right) & =\prod_{i=1}^{n}\left(1+t e^{-x_{i}}\right)  \tag{17.8}\\
\operatorname{ch}\left(\mathrm{Sym}_{t} T X\right) & =\prod_{i=1}^{n}\left(1-t e^{x_{i}}\right)^{-1}  \tag{17.9}\\
\operatorname{ch}\left(\operatorname{Sym}_{t} T X^{*}\right) & =\prod_{i=1}^{n}\left(1-t e^{-x_{i}}\right)^{-1} \tag{17.10}
\end{align*}
$$

respectively. The Todd class of $X$ is simply

$$
\begin{equation*}
\operatorname{td}(X)=\prod_{i=1}^{n} \frac{x_{i}}{1-e^{-x_{i}}}, \tag{17.11}
\end{equation*}
$$

and the Euler class of $X$ is

$$
\begin{equation*}
\mathrm{e}(X)=\prod_{i=1}^{n} x_{i} \tag{17.12}
\end{equation*}
$$

Hence, we have

$$
\begin{align*}
\mathfrak{Z}^{V}(X) & =e^{\pi i z \hat{c}} \prod_{i=1}^{n} \prod_{k \geqslant 1} \frac{\left(1-q^{k-1} e^{-2 \pi i z\left(1-\omega_{i}\right)}\right)\left(1-q^{k} e^{2 \pi i z\left(1-\omega_{i}\right)}\right)}{\left(1-q^{k-1} e^{-2 \pi i z \omega_{i}}\right)\left(1-q^{k} e^{2 \pi i z \omega_{i}}\right)}  \tag{17.13a}\\
& =y^{\hat{c} / 2} \prod_{i=1}^{n} \prod_{k \geqslant 1} \frac{\left(1-y q^{k} e^{-x_{i}}\right)\left(1-y^{-1} q^{k-1} e^{x_{i}}\right)}{\left(1-q^{k} e^{-x_{i}}\right)\left(1-q^{k} e^{x_{i}}\right)} \frac{x_{i}}{x_{i}\left(1-e^{-x_{i}}\right)}  \tag{17.13b}\\
& =\frac{\operatorname{ch}\left(\mathcal{E} \ell \ell_{y, q}(X)\right) \operatorname{td}(X)}{\mathrm{e}(X)} \tag{17.13c}
\end{align*}
$$

as claimed.

Remark 17.2.1. One could have defined the elliptic series $\mathcal{E} \ell \ell_{y, q}(X)$ as

$$
y^{\hat{c} / 2} \bigotimes_{k \geqslant 1}^{\otimes}\left(\bigwedge_{-y^{-d} q^{k-1}} T X \otimes \bigwedge_{-y^{d} q^{k}} T X^{*} \otimes \operatorname{Sym}_{q^{k}} T X \otimes \operatorname{Sym}_{q^{k-1}} T X^{*}\right)
$$

which would imply $\mathfrak{Z}^{V}(X)=e^{i \phi \tau \hat{c} / 2} \operatorname{ch}\left(\mathcal{E} \ell \ell_{y, q}(X)\right)$.

The characteristic power series of three well-known elliptic genera [451] are related to the following functions:

$$
\begin{align*}
& E_{Q^{\prime}}(x)=e^{\pi i \tau / 4}\left(e^{x / 2}-e^{-x / 2}\right) \prod_{k \geqslant 1} \frac{\left(1-q^{k} e^{x}\right)\left(1-q^{k} e^{-x}\right)}{\left(1-q^{k} e^{x-\pi i \tau}\right)\left(1-q^{k} e^{-x-\pi i \tau}\right)}  \tag{17.14}\\
& E_{R}(x)=\left(\frac{1-e^{-x}}{1+e^{-x}}\right) \prod_{k \geqslant 1} \frac{\left(1-q^{k} e^{x}\right)\left(1-q^{k} e^{-x}\right)}{\left(1+q^{k} e^{x}\right)\left(1+q^{k} e^{-x}\right)}  \tag{17.15}\\
& E_{Q}(x)=e^{\pi i \tau / 4}\left(e^{x / 2}-e^{-x / 2}\right) \\
&  \tag{17.16}\\
& \quad \prod_{k \geqslant 1} \frac{\left(1-q^{k} e^{x}\right)\left(1-q^{k} e^{-x}\right)}{\left(1-q^{k} e^{x-\pi i( \pm 1+\tau)}\right)\left(1-q^{k} e^{-x-\pi i( \pm 1+\tau)}\right)}
\end{align*}
$$

Note

$$
\frac{x}{E_{R}(x)}=\frac{x}{\tanh \left(\frac{x}{2}\right)} \frac{\left(-e^{x} ; q\right)_{\infty}\left(-e^{-x} ; q\right)_{\infty}}{\left(e^{x} ; q\right)_{\infty}\left(e^{-x} ; q\right)_{\infty}}
$$

Proposition 17.3. The partition function $\mathfrak{Z}^{\lambda V}: \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ satisfies

$$
\begin{equation*}
\lim _{y \rightarrow-1} \mathfrak{J}^{\lambda V}=(-1)^{\hat{c} / 2} \prod_{j=1}^{n} \frac{x_{j}}{E_{J}\left(x_{j}\right)^{\prime}} \tag{17.18}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{x}{E_{J}(x)}=-i e^{x_{j}} \cot \left(\frac{x_{j}}{2}\right) \frac{\left(-e^{x_{j}} ; q\right)_{\infty}\left(-e^{-x_{j}} ; q\right)_{\infty}}{\left(e^{x_{j}} ; q\right)_{\infty}\left(e^{-x_{j}} ; q\right)_{\infty}} \tag{17.19}
\end{equation*}
$$

Proof. The partition function $\mathfrak{Z}^{\lambda V}=e^{i \hat{c} \theta / 2} \prod_{j=1}^{n} \mathfrak{Z}_{j}^{V}$, where

$$
\begin{align*}
\mathfrak{Z}_{j}^{V} & =\prod_{k \geqslant 1} \frac{1-q^{k-1} y^{-\left(1-\omega_{j}\right)}}{1-q^{k-1} y^{-\omega_{j}}} \frac{1-q^{k} y^{1-\omega_{j}}}{1-q^{k} y^{\omega_{j}}}  \tag{17.20}\\
& =\prod_{k \geqslant 1} \frac{1-q^{k-1} e^{-2 \pi i z\left(1-\omega_{j}\right)}}{1-q^{k-1} e^{-2 \pi i z \omega_{j}}} \frac{1-q^{k} e^{2 \pi i z\left(1-\omega_{j}\right)}}{1-q^{k} e^{2 \pi i z \omega_{j}}}  \tag{17.2}\\
& =\left(\frac{1-y^{-1} e^{x_{j}}}{1-e^{-x}}\right) \frac{\left(y^{-1} e^{x_{j}} ; q\right)_{\infty}\left(y e^{-x_{j}} ; q\right)_{\infty}}{\left(e^{x_{j}} ; q\right)_{\infty}\left(e^{-x_{j}} ; q\right)_{\infty}} \tag{17.22}
\end{align*}
$$

where we have set $x_{j}=2 \pi i z \omega_{j}$. In the limit $y \rightarrow-1$, we have

$$
\begin{align*}
\lim _{y \rightarrow-1} \mathfrak{Z}_{j}^{V} & =\frac{e^{x_{j}}}{\tanh \left(\frac{x_{j}}{2}\right)} \frac{\left(-e^{x_{j}} ; q\right)_{\infty}\left(-e^{-x_{j}} ; q\right)_{\infty}}{\left(e^{x_{j}} ; q\right)_{\infty}\left(e^{-x_{j}} ; q\right)_{\infty}}  \tag{17.23}\\
& =-i e^{x_{j}} \cot \left(\frac{x_{j}}{2}\right) \frac{\left(-e^{x_{j}} ; q\right)_{\infty}\left(-e^{-x_{j}} ; q\right)_{\infty}}{\left(e^{x_{j}} ; q\right)_{\infty}\left(e^{-x_{j}} ; q\right)_{\infty}} \tag{17.24}
\end{align*}
$$

Remark 17.2.2. Proposition 17.3 shows that the elliptic genus recovers the algebraic root of the corresponding singularity in the $y \rightarrow-1$ limit. This is rather suggestive of a more general phenomenon.

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## Chapter 18

# Quantum Field Theory and Algebraic Links 

Aut inveniam viam aut faciam.* - Hannibal

## Contents

18.1. Elliptic Genus as a Link Invariant .......................................... 693

### 18.1. Elliptic Genus as a Link Invariant

Given a complex analytic germ $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, define the locus $V_{f, \kappa}=f^{-1}(\kappa)$, where $\kappa \in \mathbb{C}$. There is an $\delta>0$ such that for $0 \leqslant \kappa<\varepsilon<\delta$, the map $\phi_{f}=\frac{f}{\|f\|}: S_{\varepsilon}^{2 n+1} \backslash V_{f, \kappa} \rightarrow S^{1}$ is the projection of a locally trivial fibration over $S^{1}$ [310]. If $f$ is non-degenerate, i.e., $f$ has an isolated critical point at $\{\mathbf{0}\}$, then the intersection $K_{f}=V_{f, 0} \cap S_{\varepsilon}^{2 n+1}$ is an ( $n-2$ )-connected, possibly linked, codimension-two submanifold $\amalg^{r} S^{2 n-1} \subset S_{\varepsilon}^{2 n+1}$ - an algebraic (fibered) link - and the common boundary of a diffeomorphism class of fibers $\left\{F_{f, \vartheta}=\phi_{f}^{-1}\left(e^{i \vartheta}\right)\right\}_{\theta \in S^{1}}$, each with the homotopy-type of a wedge sum of spheres, $\bigvee^{\mu} S^{n}$ (op.cit.). Additionally, if $f$ is weighted homogeneous, i.e., there are rationals $\boldsymbol{\omega}=\left\{\omega_{0}, \ldots, \omega_{n}\right\}$ such that $f=\lambda^{-1} f\left(\lambda^{\omega_{0}} z_{0}, \ldots, \lambda^{\omega_{n}} z_{n}\right)$ for $\lambda \in \mathbb{C}^{\times}$, then each fiber $F_{f, \vartheta}$ is diffeomorphic to $V_{f, 1}$ as a deformation retraction, and the
*I shall either find a way or make one.

Milnor number $\mu=\operatorname{dim}_{\mathbb{C}} \mathbb{C}\left\{z_{0}, \ldots, z_{n}\right\} /\left\langle\partial_{0} f, \ldots, \partial_{n} f\right\rangle=\prod_{i=0}^{n}\left(\frac{1}{\omega_{i}}-1\right)$, where $\omega \subset \mathbb{Q} \cap\left(0, \frac{1}{2}\right][3 \mathbf{1 0}]$.

Definition 18.1. A complex analytic function $g: \mathbb{C}^{m} \rightarrow \mathbb{C}$ satisfies the elliptic bounds if and only if there are positive constants $\varepsilon, M, \rho<\infty$ such that for any non-negative multi-index $\alpha$ and for all $z=\left(z_{1}, \ldots, z_{m}\right)$ satisfying $\|z\|>\rho$, one has $\left\|\partial^{\alpha} g\right\| \leqslant \varepsilon\|\partial g\|^{2}+M$ and $\|z\|^{2}+\|g\| \leqslant M\left(\|\partial g\|^{2}+1\right)$, where $\partial g=$ $\left(\partial_{1} g, \ldots, \partial_{m} g\right)$.

Remark 18.1.1. The second inequality is redundant for non-degenerate, weighted homogeneous polynomials.

Proposition 18.2. Let $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ be a non-degenerate, weighted homogeneous polynomial satisfying the elliptic bounds. The elliptic genus $\mathfrak{Z}^{f}$ determines the reduced Alexander polynomial of the algebraic link $K_{f}$, which is a complete (cobordism and isotopy) invariant if $K_{f} \subset S^{3}$ is a knot.

Proof. Define the space-time-twist parameters $\tau=\frac{\sigma+i \beta}{\ell} \in \mathbb{H}, z=\frac{\theta-\tau \phi}{2 \pi} \in$ C and the associated nomes $q=e^{2 \pi i \tau}$ and $y=e^{2 \pi i z}$. Denote the weights of $f$ by $\boldsymbol{\omega}$, and define $\hat{c}=\sum_{i=0}^{n} 1-2 \omega_{i}$, the central charge. Since $f$ satisfies the standard hypotheses, the corresponding elliptic genus $\mathfrak{Z}^{f}: \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ exists and, assuming $\phi=0$, admits the following exact representation [223],

$$
\begin{align*}
\mathfrak{Z}^{f}(z, \tau) & =y^{\hat{c} / 2} \prod_{i=0}^{n} \prod_{k \geqslant 0} \frac{\left(1-y^{-\left(1-\omega_{i}\right)} q^{k}\right)\left(1-y^{\left(1-\omega_{j}\right)} q^{k+1}\right)}{\left(1-y^{-\omega_{i}} q^{k}\right)\left(1-y^{\omega_{i}} q^{k+1}\right)}  \tag{18.1}\\
& =y^{-(n+1) / 2} \operatorname{Sp}(f ; y)+O(q) \tag{18.2}
\end{align*}
$$

where the Steenbrink series

$$
\begin{align*}
\operatorname{Sp}(f ; y) & =\prod_{i=0}^{n} \frac{y^{1-\omega_{i}}-1}{1-y^{-\omega_{i}}}  \tag{18.3}\\
& =\sum_{j=1}^{\mu} y^{\gamma_{j}} \tag{18.4}
\end{align*}
$$

and $\mu=\operatorname{rank} H_{n}\left(F_{f, 0} ; \mathbb{Z}\right)$ [436]. The spectrum $\operatorname{Sp}(f)=\left\{\gamma_{j}\right\}_{1 \leqslant j \leqslant \mu}$ of the mixed Hodge structure of a generic fiber $F_{f, \vartheta}$ determines the characteristic polynomial $\Delta_{h_{*}}(t)=\operatorname{det}\left(t I-h_{*}\right)$ of the Picard-Lefschetz monodromy $h_{*}: H_{n}\left(F_{f, \vartheta} ; \mathbb{C}\right) \rightarrow$ $H_{n}\left(F_{f, \vartheta+2 \pi} ; \mathbb{C}\right)$ (op. cit.), viz.,

$$
\begin{equation*}
\Delta_{h_{*}}(t)=\prod_{j=1}^{\mu}\left(t-e^{2 \pi i \gamma_{j}}\right), \tag{18.5}
\end{equation*}
$$

the reduced Alexander polynomial of $K_{f}$ [310], viz.,

$$
\begin{equation*}
\Delta_{h_{*}}(t) \doteq(t-1)^{1-\delta_{r, 1}} \Delta_{K_{f}}(t, \ldots, t) \tag{18.6}
\end{equation*}
$$

the Lefschetz zeta function [352],

$$
\begin{align*}
\zeta_{K_{f}}(t) & =\exp \sum_{k \geqslant 0} \Lambda\left(h^{\circ k}\right) \frac{t^{k}}{k}  \tag{18.7}\\
& =\prod_{l \geqslant 0} \operatorname{det}\left(I-t h_{*, l}\right)^{(-1)^{l+1}} \tag{18.8}
\end{align*}
$$

where $h: V_{f, 1} \rightarrow V_{f, 1}$ is the transformation $h(z)=\left(e^{2 \pi i \omega_{0}} z_{0}, \ldots, e^{2 \pi i \omega_{n}} z_{n}\right)$, and the Lefschetz number

$$
\begin{equation*}
\Lambda\left(h^{\circ k}\right)=\sum_{l \geqslant 0}(-1)^{l} \operatorname{Tr}\left(h_{*, l}^{k}: H_{l}\left(V_{f, 1} ; \mathbb{Q}\right) \rightarrow H_{l}\left(V_{f, 1} ; \mathbb{Q}\right)\right) \tag{18.9}
\end{equation*}
$$

equals the Euler characteristic $\chi_{k}=\left\{z \in V_{f, 1} \mid h^{\circ k}(z)=z\right\}$ [310], viz.,

$$
\begin{equation*}
\zeta_{K_{f}}(t)=(-1)^{\mu n}(1-t)^{-1} \Delta_{h_{*}}(t) \tag{18.10}
\end{equation*}
$$

If $n=r=1$, the diffeomorphism-type of the relative pair $\left(S^{3}, K_{f}\right)$ [257]. We have therefore proven the following claim.

Remark 18.1.2. The symmetry $\mathfrak{Z}^{f}(-z, \tau)=\mathfrak{Z}^{f}(z, \tau)$ implies the reflexivity

$$
\begin{equation*}
\operatorname{Sp}(f ; y)=y^{n+1} \operatorname{Sp}\left(f ; \frac{1}{y}\right) \tag{18.11}
\end{equation*}
$$

the reciprocity $\gamma_{\mu+1-j}=n+1-\gamma_{j}$ for $1 \leqslant j \leqslant \mu$, and the functional equation

$$
\begin{equation*}
\Delta_{h_{*}}(t)=(-1)^{\mu n} t^{\mu} \Delta_{h_{*}}\left(\frac{1}{t}\right) . \tag{18.12}
\end{equation*}
$$

Remark 18.1.3. The $q$-linear term of $\mathfrak{Z}^{f}$ is

$$
\begin{equation*}
q y^{-(n+1) / 2} \operatorname{Sp}(f ; y) \sum_{i=0}^{n} y^{\omega_{i}}+y^{1-\omega_{i}}-y^{-\omega_{i}}-y^{-\left(1-\omega_{i}\right)} \tag{18.13}
\end{equation*}
$$

Corollary 18.3. If $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a non-degenerate weighted homogeneous polynomial that satisfies the elliptic bounds, then the number of bosonic ground states of the corresponding quantum theory is the Fredholm index of the supercharge $Q_{f}^{+}$, where $\left(Q_{f}^{+}\right)^{2}=H+P$, and is given by the zero-twist limit

$$
\begin{align*}
\operatorname{ind}\left(Q_{f}^{+}\right) & =\lim _{\theta, \phi \rightarrow 0} \mathfrak{Z}_{f}(z, \tau)  \tag{18.14a}\\
& =\prod_{i=0}^{n}\left(\frac{1}{\omega_{i}}-1\right), \tag{18.14b}
\end{align*}
$$

which coincides with the Milnor number $\mu(f)$.

Corollary $18.4([289])$. If $f:\left(\mathbb{C}^{2}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a squarefree, non-denerate weighted homogeneous polynomial that satisfies the elliptic bounds and the corresponding algebraic link is a knot, then $\operatorname{ind}\left(Q_{f}^{+}\right)=2 \delta(f)$, twice the delta invariant of $f$.

Remark 18.1.4. Given $p, q \in \mathbb{N}_{>1}$, the polynomial $f=x^{p}+y^{q}$ has weights $\left\{\frac{1}{p}, \frac{1}{q}\right\}$ and corresponds to the torus link $T_{p, q}=V_{f, 0} \cap S^{3}$ with $\operatorname{GCD}(p, q)$ components and unknotting number $u\left(T_{p, q}\right)=\frac{1}{2}(p q-p-q+\operatorname{GCD}(p, q))$ [247]. Since
$f$ satisfies the standard hypotheses,

$$
\begin{equation*}
\mathfrak{Z}^{f}=y^{\frac{1}{p}+\frac{1}{q}-1} \sum_{k=0}^{p-2} \sum_{l=0}^{q-2} y^{k / p+l / q}+O\left(e^{2 \pi i \tau}\right) . \tag{18.15}
\end{equation*}
$$

Therefore,

$$
\begin{equation*}
\operatorname{Sp}(f)=\left\{\frac{k}{p}+\frac{l}{q}\right\}_{1 \leqslant k \leqslant p-1,1 \leqslant l \leqslant q-1}=\left\{2-\frac{k}{p}+\frac{l}{q}\right\}_{1 \leqslant k \leqslant p-1,1 \leqslant l \leqslant q-1} . \tag{18.16}
\end{equation*}
$$

Setting $\zeta_{n}=e^{2 \pi i / n}$ for $n \in \mathbb{N}$,

$$
\begin{align*}
\Delta_{h_{*}}(t) & =\prod_{k=1}^{p-1} \prod_{l=1}^{q-1}\left(t-\zeta_{p}^{k} \zeta_{q}^{l}\right)  \tag{18.17}\\
& =\frac{\left(t^{\mathrm{LCM}(p, q)}-1\right)^{\operatorname{GCD}(p, q)}(t-1)}{\left(t^{p}-1\right)\left(t^{q}-1\right)}  \tag{18.18}\\
& =(-t)^{\mu} \Delta_{h_{*}}\left(\frac{1}{t}\right) \tag{18.19}
\end{align*}
$$

where

$$
\begin{equation*}
\mu=\lim _{\theta \rightarrow 0} \mathfrak{Z}^{f}=(p-1)(q-1) \tag{18.20}
\end{equation*}
$$

is the Fredholm index of the supercharge $Q_{+}$, which enumerates the quantummechanical ground states [223]. In particular, if $p$ and $q$ are coprime, then $T_{p, q}$ is a knot, and the index

$$
\begin{equation*}
\mu=2 u\left(T_{p, q}\right)=2 g\left(F_{f, \vartheta}\right) \tag{18.21}
\end{equation*}
$$

twice the genus of a corresponding generic fiber [310, 247].

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## Chapter 19

## Classification of $\mathrm{WZ}_{\theta, \phi}$ Models

> The secret of all victory lies in the organization of the non-obvious. Marcus Aurelius

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19.1. The Category of $\mathbf{W Z}_{\theta, \phi}$ Models
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In this chapter we discuss a classification scheme for the moduli space of the $\mathrm{WZ}_{\theta, \phi}$ models.

Proposition 19.1. The moduli space of the $\mathrm{WZ}_{\theta, \phi}$ models admits a classification by the corresponding spectrum of the Picard-Lefschetz monodromy and the topological type of the corresponding algebraic links. In particular, algebraic knots in $S^{3}$ completely classify said moduli space.

Proof. The claim follows from the Lê and Yamamoto classification of algebraic links [257, 258].
19.1. The Category of $\mathbf{W Z}_{\theta, \phi}$ Models
19.1.1. Elliptic Genera under Sebastiani-Thom Summation. Consider the moduli space of $\mathrm{WZ}_{\theta, \phi}$ models with non-degenerate, weighted homogeneous
superpotentials (that satisfy the elliptic bounds) equipped with the tensor product of underlying Fock spaces.

Proposition 19.2. The elliptic genus of a Sebastiani-Thom summation is defined on a tensor product Fock space and factors as a product of the constituent elliptic genera.

Proof. Computing the elliptic genera in terms of Jacobi theta functions,

$$
\begin{align*}
\mathfrak{J}^{f \boxplus g}(z, \tau) & =y^{-(\hat{c}(f)+\hat{c}(g)) / 2} \prod_{i=1}^{n} \frac{\vartheta_{1}\left(\left(1-\omega_{i}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{i} z, \tau\right)} \prod_{j=1}^{m} \frac{\vartheta_{1}\left(\left(1-v_{j}\right) z, \tau\right)}{\vartheta_{1}\left(v_{j} z, \tau\right)}  \tag{19.1a}\\
& =\left(y^{-\hat{c}(f) / 2} \prod_{j=1}^{n} \frac{\vartheta_{1}\left(\left(1-\omega_{j}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{j} z, \tau\right)}\right)\left(y^{-\hat{c}(g) / 2} \prod_{j=1}^{n} \frac{\vartheta_{1}\left(\left(1-v_{j}\right) z, \tau\right)}{\vartheta_{1}\left(v_{j} z, \tau\right)}\right) \\
& =\mathfrak{J}^{f}(z, \tau) \mathfrak{Z}^{g}(z, \tau) . \tag{19.1b}
\end{align*}
$$

Computing the elliptic genera in terms of heat kernels on Fock space,

$$
\begin{aligned}
\operatorname{Str}_{\mathscr{F}_{f}} e^{-\beta H-i \sigma P-i \theta J} \cdot \operatorname{Str}_{\mathscr{F}_{g}} e^{-\beta H-i \sigma P-i \theta J} & =\operatorname{Str}_{\mathscr{F}} e^{-\beta H-i \sigma P-i \theta J} \otimes e^{-\beta H-i \sigma P-i \theta J} \\
& =\operatorname{Str}_{\mathscr{F}} e^{-\beta H \oplus H-i \sigma P \oplus P-i \theta J \oplus J},
\end{aligned}
$$

where $\mathscr{F}=\mathscr{F}_{f} \otimes \mathscr{F}_{g}$. Since the tensor product is a universal operation, it follows that

$$
\begin{equation*}
\operatorname{Str}_{\mathscr{F}_{f \boxplus g}} e^{-\beta H-i \sigma P-i \theta J}=\operatorname{Str}_{\mathscr{F}_{f} \otimes \mathscr{F}_{g}} e^{-\beta H \oplus H-i \sigma P \oplus P-i \theta J \oplus J}, \tag{19.2}
\end{equation*}
$$

and the claim follows.

Corollary 19.3. The elliptic genus is invariant under iterated stabilization.
Proof. As $\mathfrak{Z}^{z^{2}}=1$,

$$
\begin{align*}
\mathfrak{Z}^{\Sigma f}(z, \tau) & =y^{-\left(\hat{c}(f)+\hat{c}\left(z^{2}\right)\right) / 2} \prod_{i=1}^{n} \frac{\vartheta_{1}\left(\left(1-\omega_{i}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{i} z, \tau\right)}  \tag{19.3}\\
& =\mathfrak{Z}^{f}(z, \tau) \tag{19.4}
\end{align*}
$$

since $\hat{c}\left(z^{2}\right)=0$.

Proposition 19.4. The moduli space of $W Z_{\theta, \phi}$ models is monoidal.

Proof. It is a straightforward exercise to verify the following identities,

$$
\begin{equation*}
\mathfrak{J}^{f \boxplus g}(z, \tau)=\mathfrak{Z}^{g \boxplus f}(z, \tau) \tag{19.5}
\end{equation*}
$$

and

$$
\begin{align*}
\mathfrak{Z}^{(f \boxplus g) \boxplus h}(z, \tau) & =\left(\mathfrak{Z}^{f} \cdot \mathfrak{Z}^{g} \cdot \mathfrak{Z}^{h}\right)(z, \tau)  \tag{19.6}\\
& =\mathfrak{Z}^{f \boxplus(g \boxplus h)}(z, \tau), \tag{19.7}
\end{align*}
$$

which imply commutativity and associativity under Sebastiani-Thom summation. Thus, the moduli space is a commutative monoid with identity 1 , namely, the elliptic genus of a quasi-Brieskorn-Pham singularity with weights $\left\{\frac{1}{2}, \ldots, \frac{1}{2}\right\}$. Categorify.
19.1.2. Elliptic Genera under Disjoint Union. One may consider a singularity with a discrete set of isolated critical points, in which case the appropriate
topological setting is that of a disjoint union of Milnor fibers and corresponding algebraic links. From the point of view of quantum field theory, one may consider a direct sum of Fock spaces implying an ordinary sum of elliptic genera,

$$
\begin{align*}
\left(\mathfrak{Z}^{f}+\mathfrak{Z}^{g}\right)(z, \tau) & =y^{-\hat{c}(f) / 2} \prod_{i=1}^{n} \frac{\vartheta_{1}\left(\left(1-\omega_{i}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{i} z, \tau\right)}+y^{-\hat{c}(g) / 2} \prod_{i=1}^{n} \frac{\vartheta_{1}\left(\left(1-v_{i}\right) z, \tau\right)}{\vartheta_{1}\left(v_{i} z, \tau\right)} \\
& =\mathfrak{Z}^{f \sqcup g}(z, \tau) . \tag{19.8}
\end{align*}
$$

As the numerical invariants of algebraics links are additive under disjoint union, the number of quantum mechanical grounds states is additive under direct sum.
19.1.3. Elliptic Genera under Kronecker Products. Define the Kronecker product of elliptic genera,

$$
\begin{align*}
\left(\mathfrak{Z}^{f} \otimes \mathfrak{Z}^{g}\right)(z, \tau) & =y^{-\hat{c}(f \boxtimes g) / 2} \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{\vartheta_{1}\left(\left(1-\omega_{i} v_{j}\right) z, \tau\right)}{\vartheta_{1}\left(\omega_{i} v_{j} z, \tau\right)}  \tag{19.9}\\
& =\mathfrak{Z}^{f \boxtimes g}(z, \tau) . \tag{19.10}
\end{align*}
$$

This operation is associative and commutative.

### 19.2. Supersymmetry

The reciprocity of the spectrum, namely, $\gamma_{\mu+1-j}=n+1-\gamma_{j}$ is related to a map of the weights $\omega_{i} \rightarrow 1-\omega_{i}$, which interchanges the bosonic and fermionic degrees of freedom and inverts the elliptic genus. This is supersymmetry, which may equivalently be interpreted as a reciprocity law between the elementary
symmetric and complete homogeneous polynomials and their generating functions. These points are discussed in greater detail in Volume 3.

Lemma 19.5. Supersymmetry is Ehrhart Reciprocity made manifest.

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## Appendices

## Appendix A

Link Data

O time! thou must untangle this, not I;
It is too hard a knot for me to untie!
— William Shakespeare, Twelfth Night (Act II, Scene 2)

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The following tables of prime links (with up to four components) were created using KnotPlot [418]. For the convenience of the reader, we provide all prime knots with no more than ten crossings in Tables A.1, A.2, A. 3 and A.4. The Alexander polynomials of prime knots with no more than seven crossings are given in the Table A.5. All prime links with two components and no more than nine crossings are given in Tables A. 6 and A.7. All prime links with three components and no more than nine crossings are given in Table . Various prime links with four components and zero or eight crossings are given
in Table A.9. The torus links ordered by crossing number are given in Table A.10. The characteristic polynomial of the torus link $T_{p, q}$ for $2 \leqslant p \leqslant q \leqslant 10$ are given in Tables A.11, A. 13 and A.14. Recall the $n^{\text {th }}$-cyclotomic polynomial $\Phi_{n}(t)=\prod_{\mathrm{GCD}(k, n)=1}\left(t-\zeta_{n}^{k}\right)$, where $\zeta_{n}=e^{2 \pi i / n}$.

## A.1. Prime Knots

Table A.1. Prime Knots ( $0_{1}$ to $9_{28}$ ) [418]


Table A.2. Prime Knots (Continued, $9_{29}$ to $10_{43}$ ) [418]


Table A.3. Prime Knots (Continued, $10_{44}$ to $10_{107}$ ) [418]


Table A.4. Prime Knots (Continued, $10_{108}$ to $10_{165}$ ) [418]


## A.2. Alexander Polynomial of Prime Knots

Table A.5. Alexander Polynomials of Prime Knots

| $K$ | $\Delta_{K}(t)$ |
| :---: | :---: |
| $0_{1}$ | 1 |
| $3_{1}$ | $t^{2}-t+1$ |
| $4_{1}$ | $t^{2}-3 t+1$ |
| $5_{1}$ | $t^{4}-t^{3}+t^{2}-t+1$ |
| $5_{2}$ | $2 t^{2}-3 t+2$ |
| $6_{1}$ | $2 t^{2}-5 t+2$ |
| $6_{2}$ | $t^{4}-3 t^{3}+3 t^{2}-3 t+1$ |
| $6_{3}$ | $t^{4}-3 t^{3}+5 t^{2}-3 t+1$ |
| $7_{1}$ | $t^{6}-t^{5}+t^{4}-t^{3}+t^{2}-t+1$ |
| $7_{2}$ | $3 t^{2}-5 t+3$ |
| $7_{3}$ | $2 t^{4}-3 t^{3}+3 t^{2}-3 t+2$ |
| $7_{4}$ | $4 t^{3}-7 t+4$ |
| $7_{5}$ | $2 t^{4}-4 t^{3}+5 t^{2}-4 t+2$ |
| $7_{6}$ | $t^{4}-5 t^{3}+7 t^{2}-5 t+1$ |
| $7_{7}$ | $t^{4}-5 t^{3}+9 t^{2}-5 t+1$ |

## A.3. Prime Links with Two Components

Table A.6. Prime Links with Two Components $\left(0_{1}^{2}\right.$ to $\left.9_{25}^{2}\right)$ [418]


Table A.7. Prime Links with Two Components (Continued, $9_{26}^{2}$ to $9_{61}^{2}$ ) [418]


## A.4. Prime Links with Three Components

Table A.8. Prime Links with Three Components $\left(0_{1}^{3}\right.$ to $9_{21}^{3}$ ) [418]


## A.5. Prime Links with Four Components

Table A.9. Various Prime Links with Four Components [418]


## A.6. Torus Links

Table A.10. Torus Links ( $T_{1,1}$ to $T_{6,7}$ ) [418]


## A.7. Characteristic Polynomials of Torus Links

Table A.11. Characteristic Polynomials of Torus Links

| $L$ | $\Delta_{L}(t)$ |
| :---: | :---: |
| $T_{2,2}$ | $t-1$ |
| $T_{2,3}$ | $t^{2}-t+1$ |
| $T_{2,4}$ | $t^{3}-t^{2}+t-1$ |
| $T_{2,5}$ | $t^{4}-t^{3}+t^{2}-t+1$ |
| $T_{2,6}$ | $t^{5}-t^{4}+t^{3}-t^{2}+t-1$ |
| $T_{2,7}$ | $t^{6}-t^{5}+t^{4}-t^{3}+t^{2}-t+1$ |
| $T_{2,8}$ | $t^{7}-t^{6}+t^{5}-t^{4}+t^{3}-t^{2}+t-1$ |
| $T_{2,9}$ | $t^{8}-t^{7}+t^{6}-t^{5}+t^{4}-t^{3}+t^{2}-t+1$ |
| $T_{2,10}$ | $t^{9}-t^{8}+t^{7}-t^{6}+t^{5}-t^{4}+t^{3}-t^{2}+t-1$ |
| $T_{3,3}$ | $t^{4}-t^{3}-t+1$ |
| $T_{3,4}$ | $t^{6}-t^{5}+t^{3}-t+1$ |
| $T_{3,5}$ | $t^{8}-t^{7}+t^{5}-t^{4}+t^{3}-t+1$ |
| $T_{3,6}$ | $t^{10}-t^{9}+t^{7}-t^{6}-t^{4}+t^{3}-t+1$ |
| $T_{3,7}$ | $t^{12}-t^{11}+t^{9}-t^{8}+t^{6}-t^{4}+t^{3}-t+1$ |
| $T_{3,8}$ | $t^{14}-t^{13}+t^{11}-t^{10}+t^{8}-t^{7}+t^{6}-t^{4}+t^{3}-t+1$ |
| $T_{3,9}$ | $t^{16}-t^{15}+t^{13}-t^{12}+t^{10}-t^{9}-t^{7}+t^{6}-t^{4}+t^{3}-t+1$ |
| $T_{3,10}$ | $t^{18}-t^{17}+t^{15}-t^{14}+t^{12}-t^{11}+t^{9}-t^{7}+t^{6}-t^{4}+t^{3}-t+1$ |
| $T_{4,4}$ | $t^{9}-t^{8}-2 t^{5}+2 t^{4}+t-1$ |
| $T_{4,5}$ | $t^{12}-t^{11}+t^{8}-t^{6}+t^{4}-t+1$ |
| $T_{4,6}$ | $t^{15}-t^{14}+t^{11}-t^{10}+t^{9}-t^{8}+t^{7}-t^{6}+t^{5}-t^{4}+t-1$ |
| $T_{4,7}$ | $t^{18}-t^{17}+t^{14}-t^{13}+t^{11}-t^{9}+t^{7}-t^{5}+t^{4}-t+1$ |
| $T_{4,8}$ | $t^{21}-t^{20}+t^{17}-t^{16}-2 t^{13}+2 t^{12}-2 t^{9}+2 t^{8}+t^{5}-t^{4}+t-1$ |
| $T_{4,9}$ | $t^{24}-t^{23}+t^{20}-t^{19}+t^{16}-t^{14}+t^{12}-t^{10}+t^{8}-t^{5}+t^{4}-t+1$ |
| $T_{4,10}$ | $t^{27}-t^{26}+t^{23}-t^{22}+t^{19}-t^{18}+t^{17}-t^{16}+t^{15}-t^{14}$ |
|  | $+t^{13}-t^{12}+t^{11}-t^{10}+t^{9}-t^{8}+t^{5}-t^{4}+t-1$ |

Table A.12. Characteristic Polynomials of Torus Links (Continued)

| $L$ | $\Delta_{L}(t)$ |
| :---: | :---: |
| $T_{5,5}$ | $t^{16}-t^{15}-3 t^{11}+3 t^{10}+3 t^{6}-3 t^{5}-t+1$ |
| $T_{5,6}$ | $t^{20}-t^{19}+t^{15}-t^{13}+t^{10}-t^{7}+t^{5}-t+1$ |
| $\mathrm{T}_{5,7}$ | $t^{24}-t^{23}+t^{19}-t^{18}+t^{17}-t^{16}+t^{14}-t^{13}+t^{12}-t^{11}+t^{10}-t^{8}+t^{7}-t^{6}+t^{5}-t+1$ |
| $T_{5,8}$ | $t^{28}-t^{27}+t^{23}-t^{22}+t^{20}-t^{19}+t^{18}-t^{17}+t^{15}-t^{14}+t^{13}-t^{11}+t^{10}-t^{9}+t^{8}-t^{6}+t^{5}-t+1$ |
| $T_{5,9}$ | $t^{32}-t^{31}+t^{27}-t^{26}+t^{23}-t^{21}+t^{18}-t^{16}+t^{14}-t^{11}+t^{9}-t^{6}+t^{5}-t+1$ |
| $T_{5,10}$ | $t^{36}-t^{35}+t^{31}-t^{30}-3 t^{26}+3 t^{25}-3 t^{21}+3 t^{20}+3 t^{16}-3 t^{15}+3 t^{11}-3 t^{10}-t^{6}+t^{5}-t+1$ |
| $T_{6,6}$ | $t^{25}-t^{24}-4 t^{19}+4 t^{18}+6 t^{13}-6 t^{12}-4 t^{7}+4 t^{6}+t-1$ |
| $T_{6,7}$ | $t^{30}-t^{29}+t^{24}-t^{22}+t^{18}-t^{15}+t^{12}-t^{8}+t^{6}-t+1$ |
| $\mathrm{T}_{6,8}$ | $t^{35}-t^{34}+t^{29}-t^{28}+t^{27}-t^{26}+t^{23}-t^{22}+t^{21}-t^{20}+t^{19}-t^{18}+t^{17}-t^{16}+t^{15}-t^{14}+t^{13}-t^{12}+t^{9}-t^{8}+t^{7}-t^{6}+t-1$ |
| $T_{6,9}$ | $t^{40}-t^{39}+t^{34}-t^{33}+t^{31}-t^{30}+t^{28}-t^{27}+t^{25}-t^{24}-t^{22}+t^{21}+t^{19}-t^{18}-t^{16}+t^{15}-t^{13}+t^{12}-t^{10}+t^{9}-t^{7}+t^{6}-t+1$ |
| $T_{6,10}$ | $\begin{gathered} t^{45}-t^{44}+t^{39}-t^{38}+t^{35}-t^{34}+t^{33}-t^{32}+t^{29}-t^{28}+t^{27}-t^{26}+t^{25}-t^{24}+t^{23}-t^{22}+t^{21}-t^{20} \\ +t^{19}-t^{18}+t^{17}-t^{16}+t^{13}-t^{12}+t^{11}-t^{10}+t^{7}-t^{6}+t-1 \end{gathered}$ |
| $\mathrm{T}_{7,7}$ | $t^{36}-t^{35}-5 t^{29}+5 t^{28}+10 t^{22}-10 t^{21}-10 t^{15}+10 t^{14}+5 t^{8}-5 t^{7}-t+1$ |
| $\mathrm{T}_{7,8}$ | $t^{42}-t^{41}+t^{35}-t^{33}+t^{28}-t^{25}+t^{21}-t^{17}+t^{14}-t^{9}+t^{7}-t+1$ |
| $\mathrm{T}_{7,9}$ | $\begin{gathered} t^{48}-t^{47}+t^{41}-t^{40}+t^{39}-t^{38}+t^{34}-t^{33}+t^{32}-t^{31}+t^{30}-t^{29}+t^{27}-t^{26}+t^{25}-t^{24}+t^{23}-t^{22}+t^{21} \\ -t^{19}+t^{18}-t^{17}+t^{16}-t^{15}+t^{14}-t^{10}+t^{9}-t^{8}+t^{7}-t+1 \end{gathered}$ |
| $T_{7,10}$ | $\begin{gathered} t^{54}-t^{53}+t^{47}-t^{46}+t^{44}-t^{43}+t^{40}-t^{39}+t^{37}-t^{36}+t^{34}-t^{32}+t^{30}-t^{29}+t^{27}-t^{25}+t^{24}-t^{22}+t^{20} \\ -t^{18}+t^{17}-t^{15}+t^{14}-t^{11}+t^{10}-t^{8}+t^{7}-t+1 \end{gathered}$ |
| $T_{8,8}$ | $t^{49}-t^{48}-6 t^{41}+6 t^{40}+15 t^{33}-15 t^{32}-20 t^{25}+20 t^{24}+15 t^{17}-15 t^{16}-6 t^{9}+6 t^{8}+t-1$ |
| $T_{8,9}$ | $t^{56}-t^{55}+t^{48}-t^{46}+t^{40}-t^{37}+t^{32}-t^{28}+t^{24}-t^{19}+t^{16}-t^{10}+t^{8}-t+1$ |
| $T_{8,10}$ | $\begin{gathered} t^{63}-t^{62}+t^{55}-t^{54}+t^{53}-t^{52}+t^{47}-t^{46}+t^{45}-t^{44}+t^{43}-t^{42}+t^{39}-t^{38}+t^{37}-t^{36}+t^{35}-t^{34}+t^{33}-t^{32}+t^{31}-t^{30} \\ +t^{29}-t^{28}+t^{27}-t^{26}+t^{25}-t^{24}+t^{21}-t^{20}+t^{19}-t^{18}+t^{17}-t^{16}+t^{11}-t^{10}+t^{9}-t^{8}+t-1 \end{gathered}$ |
| $\mathrm{T}_{9,9}$ | $t^{64}-t^{63}-7 t^{55}+7 t^{54}+21 t^{46}-21 t^{45}-35 t^{37}+35 t^{36}+35 t^{28}-35 t^{27}-21 t^{19}+21 t^{18}+7 t^{10}-7 t^{9}-t+1$ |
| $T_{9,10}$ | $t^{72}-t^{71}+t^{63}-t^{61}+t^{54}-t^{51}+t^{45}-t^{41}+t^{36}-t^{31}+t^{27}-t^{21}+t^{18}-t^{11}+t^{9}-t+1$ |
| $T_{10,10}$ | $t^{81}-t^{80}-8 t^{71}+8 t^{70}+28 t^{61}-28 t^{60}-56 t^{51}+56 t^{50}+70 t^{41}-70 t^{40}-56 t^{31}+56 t^{30}+28 t^{21}-28 t^{20}-8 t^{11}+8 t^{10}+t-1$ |

## A.8. Torus Links and Cyclotomic Polynomials

Table A.13. Torus Links and Cyclotomic Polynomials

| $L$ | $\Delta_{L}(t)$ |
| :---: | :---: |
| $T_{2,2}$ | $\Phi_{1}(t)$ |
| $T_{2,3}$ | $\Phi_{6}(t)$ |
| $T_{2,4}$ | $\Phi_{1}(t) \Phi_{4}(t)$ |
| $T_{2,5}$ | $\Phi_{10}(t)$ |
| $T_{2,6}$ | $\Phi_{1}(t) \Phi_{3}(t) \Phi_{6}(t)$ |
| $T_{2,7}$ | $\Phi_{14}(t)$ |
| $T_{2,8}$ | $\Phi_{1}(t) \Phi_{4}(t) \Phi_{8}(t)$ |
| $T_{2,9}$ | $\Phi_{6}(t) \Phi_{18}(t)$ |
| $T_{2,10}$ | $\Phi_{1}(t) \Phi_{5}(t) \Phi_{10}(t)$ |
| $T_{3,3}$ | $\Phi_{1}(t)^{2} \Phi_{3}(t)$ |
| $T_{3,4}$ | $\Phi_{6}(t) \Phi_{12}(t)$ |
| $T_{3,5}$ | $\Phi_{15}(t)$ |
| $T_{3,6}$ | $\Phi_{1}(t)^{2} \Phi_{2}(t)^{2} \Phi_{3}(t) \Phi_{6}(t)^{2}$ |
| $T_{3,7}$ | $\Phi_{21}(t)$ |
| $T_{3,8}$ | $\Phi_{6}(t) \Phi_{12}(t) \Phi_{24}(t)$ |
| $T_{3,9}$ | $\Phi_{1}(t)^{2} \Phi_{3}(t) \Phi_{9}(t)^{2}$ |
| $T_{3,10}$ | $\Phi_{6}(t) \Phi_{15}(t) \Phi_{30}(t)$ |
| $T_{4,4}$ | $\Phi_{1}(t)^{3} \Phi_{2}(t)^{2} \Phi_{4}(t)^{2}$ |
| $T_{4,5}$ | $\Phi_{10}(t) \Phi_{20}(t)$ |
| $T_{4,6}$ | $\Phi_{1}(t) \Phi_{3}(t) \Phi_{4}(t) \Phi_{6}(t) \Phi_{12}(t)^{2}$ |
| $T_{4,7}$ | $\Phi_{14}(t) \Phi_{28}(t)$ |
| $T_{4,8}$ | $\Phi_{1}(t)^{3} \Phi_{2}(t)^{2} \Phi_{4}(t)^{2} \Phi_{8}(t)^{3}$ |
| $T_{4,9}$ | $\Phi_{6}(t) \Phi_{12}(t) \Phi_{18}(t) \Phi_{36}(t)$ |
| $T_{4,10}$ | $\Phi_{1}(t) \Phi_{4}(t) \Phi_{5}(t) \Phi_{10}(t) \Phi_{20}(t)^{2}$ |
|  |  |

Table A.14. Torus Links and Cyclotomic Polynomials (Continued)

| $L$ | $\Delta_{L}(t)$ |
| :---: | :---: |
| $T_{5,5}$ | $\Phi_{1}(t)^{4} \Phi_{5}(t)^{3}$ |
| $T_{5,6}$ | $\Phi_{10}(t) \Phi_{15}(t) \Phi_{30}(t)$ |
| $T_{5,7}$ | $\Phi_{35}(t)$ |
| $T_{5,8}$ | $\Phi_{10}(t) \Phi_{20}(t) \Phi_{40}(t)$ |
| $T_{5,9}$ | $\Phi_{15}(t) \Phi_{45}(t)$ |
| $T_{5,10}$ | $\Phi_{1}(t)^{4} \Phi_{2}(t)^{4} \Phi_{5}(t)^{3} \Phi_{10}(t)^{4}$ |
| $T_{6,6}$ | $\Phi_{1}(t)^{5} \Phi_{2}(t)^{4} \Phi_{3}(t)^{4} \Phi_{6}(t)^{4}$ |
| $T_{6,7}$ | $\Phi_{14}(t) \Phi_{21}(t) \Phi_{42}(t)$ |
| $T_{6,8}$ | $\Phi_{1}(t) \Phi_{3}(t) \Phi_{4}(t) \Phi_{6}(t) \Phi_{8}(t) \Phi_{12}(t)^{2} \Phi_{24}(t)^{2}$ |
| $T_{6,9}$ | $\Phi_{1}(t)^{2} \Phi_{2}(t)^{2} \Phi_{3}(t) \Phi_{6}(t)^{2} \Phi_{9}(t)^{2} \Phi_{18}(t)^{3}$ |
| $T_{6,10}$ | $\Phi_{1}(t) \Phi_{3}(t) \Phi_{5}(t) \Phi_{6}(t) \Phi_{10}(t) \Phi_{15}(t)^{2} \Phi_{30}(t)^{2}$ |
| $T_{7,7}$ | $\Phi_{1}(t)^{6} \Phi_{7}(t)^{5}$ |
| $T_{7,8}$ | $\Phi_{14}(t) \Phi_{28}(t) \Phi_{56}(t)$ |
| $T_{7,9}$ | $\Phi_{21}(t) \Phi_{63}(t)$ |
| $T_{7,10}$ | $\Phi_{14}(t) \Phi_{35}(t) \Phi_{70}(t)$ |
| $T_{8,8}$ | $\Phi_{1}(t)^{7} \Phi_{2}(t)^{6} \Phi_{4}(t)^{6} \Phi_{8}(t)^{6}$ |
| $T_{8,9}$ | $\Phi_{6}(t) \Phi_{12}(t) \Phi_{18}(t) \Phi_{24}(t) \Phi_{36}(t) \Phi_{72}(t)$ |
| $T_{8,10}$ | $\Phi_{1}(t) \Phi_{4}(t) \Phi_{5}(t) \Phi_{8}(t) \Phi_{10}(t) \Phi_{20}(t)^{2} \Phi_{40}(t)^{2}$ |
| $T_{9,9}$ | $\Phi_{1}(t)^{8} \Phi_{3}(t)^{7} \Phi_{9}(t)^{7}$ |
| $T_{9,10}$ | $\Phi_{6}(t) \Phi_{15}(t) \Phi_{18}(t) \Phi_{30}(t) \Phi_{45}(t) \Phi_{90}(t)$ |
| $T_{10,10}$ | $\Phi_{1}(t)^{9} \Phi_{2}(t)^{8} \Phi_{5}(t)^{8} \Phi_{10}(t)^{8}$ |

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## Appendix B

## Classification of Weighted Homogeneous Singularities

> One can easily classify all works of fiction either as descendants of the Iliad or of the Odyssey. - Raymond Queneau

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In this appendix, we discuss the classification of non-degenerate, weighted homogeneous singularities with inner modality between zero and six (inclusive) up to right equivalence.

## B.i. Inner Modality

Let $\mathfrak{m}$ denote the maximal ideal of the ring $\mathbb{C}\left\{z_{1}, \ldots, z_{n}\right\}$. A polynomial $f$ is non-degenerate if there is an $l \in \mathbb{N}$ such that $\mathfrak{m}^{l} \subset J_{\partial f} \subset \mathfrak{m}$. Recall that a monomial $z_{1}^{a_{1}} \cdots z_{n}^{a_{n}}$ has weighted degree $d$ if and only if $\operatorname{deg} z_{i}=q_{i}$ and $\sum_{i=1}^{n} q_{i} a_{i}=d$. Let $f$ be a non-degenerate weighted homogeneous polynomial
with weights $\left\{q_{1}, \ldots, q_{n}\right\}$ and degree $d$. The inner modality of $f$, denoted $m(f)$, is the number of monomials comprising $f$ with weighted degree greater than or equal to $d$, viz., $m(f)=\sum_{i \geqslant d} \mu_{i}=\sum_{i \leqslant C-d} \mu_{i}$.

If $f=\Sigma^{N} f_{0}$ (the iterated stabilization of $f_{0}$ ) for some $N$, then we say that $f_{0}$ is the residual part of $f$. By a theorem of Arnold two non-degenerate, weighted homogeneous singularities $\Sigma^{N} f_{0}$ and $\Sigma^{N} g_{0}$ are analytically equivalent if and only if $f_{0}$ and $g_{0}$ are analytically equivalent [23].

For the convenience of the reader, we tabulate the (corrected) nonequivalent, non-degenerate, residual parts of weighted homogeneous singularities by symbol (with subscript equal to the Milnor number) according to inner modality less than or equal to five, principal weights and non-zero generalized genus spectrum (the sequence of the number of positive lattice points intersecting the corresponding weight polytopes). Arnold classified weighted homogeneous singularities with inner modality equal to zero or one up to right equivalence [20]. Those weighted homogeneous singularities with inner modality between two and five (inclusive) up to right equivalence have been classified by Yoshinaga and Watanabe [494], Yoshinaga and Suzuki [493] and Suzuki [449]. Those weighted homogeneous singularities with inner modality equal to six up to right equivalence have been classified by Sarlabous, Arocha and Fuentes [415], although there remains three questionable singularities.

For those singularities with coefficients, certain conditions must be met to ensure non-degeneracy. Notes of the form $\Delta(a, b) \neq 0$, etc., indicate some (potentially complicated) polynomial relation of the coefficients must not vanish.

## B.2. Inner Modality Zero

Those weighted homogeneous singularities with inner modality equal to zero are right equivalent to simple singularities [20].

Table B.x. The Five Weighted Homogeneous Singularities with Inner Modality Zero [20]

| Symbol | Residual Part | Weights | $p_{g}$ | Notes |
| :---: | :---: | :---: | :---: | :---: |
| $A_{k}$ | $x^{k+1}$ | $\left\{\frac{1}{k+1}\right\}$ | $\left\{k+1,\left\lfloor\frac{k+1}{2}\right\rfloor\right\}$ | $k \geqslant 1$ |
| $D_{k}$ | $x^{k-1}+x y^{2}$ | $\left\{\frac{1}{k-1}, \frac{k-2}{2 k-2}\right\}$ | $\left\{\left\lfloor\frac{k+5}{2}\right\rfloor\right\}$ | $k \geqslant 4$ |
| $E_{6}$ | $x^{4}+y^{3}$ | $\left\{\frac{1}{4}, \frac{1}{3}\right\}$ | $\{3\}$ |  |
| $E_{7}$ | $x^{3}+x y^{3}$ | $\left\{\frac{1}{3}, \frac{2}{9}\right\}$ | $\{4\}$ |  |
| $E_{8}$ | $x^{5}+y^{3}$ | $\left\{\frac{1}{5}, \frac{1}{3}\right\}$ | $\{4\}$ |  |

Remark B.2.1. For example, $A_{k}$ represents the family of singularities $\left\{x^{k+1}+\sum_{i=1}^{n} z_{i}^{2}\right\}$ with weights $\left\{\frac{1}{k+1}, \frac{1}{2} \underline{n}\right\}$, Milnor number $k$ and corresponding generalized genus spectrum $\left\{k+1,\left\lfloor\frac{k+1}{2}\right\rfloor, \overline{0}\right\}$. Similarly, $D_{k \geqslant 4}$ represents the family of singularities $\left\{x^{k-1}+x y^{2}+\sum_{i=2}^{n} z_{i}^{2}\right\}$ with weights $\left\{\frac{1}{k-1}, \frac{k-2}{2 k-2}, \frac{1}{2} \underline{n-1}\right\}$, Milnor number $k-1$ and generalized genus spectrum $\left\{\left\lfloor\frac{k+5}{2}\right\rfloor, \overline{0}\right\}$.

Remark B.2.2. There are four quasi-Brieskorn-Pham families with inner modality equal to zero, namely, $A_{k}$ for $k \geqslant 1, D_{4}, E_{6}$ and $E_{8}$. Three are BrieskornPham, namely, $A_{k}$ for $k \geqslant 1, E_{6}$ and $E_{8}$.

## B.3. Inner Modality One

Table B.2. The Fourteen Weighted Homogeneous Singularities with Inner Modality One [20]

| Symbol | Residual Part | Weights | $p_{g}$ |
| :---: | :---: | :---: | :---: |
| $E_{12}$ | $x^{3}+y^{7}$ | $\left\{\frac{1}{3}, \frac{1}{7}\right\}$ | $\{6,1\}$ |
| $E_{13}$ | $x^{3}+x y^{5}$ | $\left\{\frac{1}{3}, \frac{2}{15}\right\}$ | $\{7,1\}$ |
| $E_{14}$ | $x^{3}+y^{8}$ | $\left\{\frac{1}{3}, \frac{1}{8}\right\}$ | $\{7,1\}$ |
| $Z_{11}$ | $x^{3} y+y^{5}$ | $\left\{\frac{4}{15}, \frac{1}{5}\right\}$ | $\{6,1\}$ |
| $Z_{12}$ | $x^{3} y+x y^{4}$ | $\left\{\frac{3}{11}, \frac{2}{11}\right\}$ | $\{7,1\}$ |
| $Z_{13}$ | $x^{3} y+y^{6}$ | $\left\{\frac{5}{18}, \frac{1}{6}\right\}$ | $\{7,1\}$ |
| $W_{12}$ | $x^{4}+y^{5}$ | $\left\{\frac{1}{4}, \frac{1}{5}\right\}$ | $\{6,1\}$ |
| $W_{13}$ | $x^{4}+x y^{4}$ | $\left\{\frac{1}{4}, \frac{3}{16}\right\}$ | $\{7,1\}$ |
| $Q_{10}$ | $x^{3}+y^{4}+y z^{2}$ | $\left\{\frac{1}{3}, \frac{1}{4}, \frac{3}{8}\right\}$ | $\{1\}$ |
| $Q_{11}$ | $x^{3}+y^{2} z+x z^{3}$ | $\left\{\frac{1}{3}, \frac{7}{1}, \frac{2}{9}\right\}$ | $\{1\}$ |
| $Q_{12}$ | $x^{3}+y^{5}+y z^{2}$ | $\left\{\frac{1}{3}, \frac{1}{5}, \frac{2}{5}\right\}$ | $\{1\}$ |
| $S_{11}$ | $x^{4}+y^{2} z+x z^{2}$ | $\left\{\frac{1}{4}, \frac{5}{16}, \frac{3}{8}\right\}$ | $\{1\}$ |
| $S_{12}$ | $x^{2} y+y^{2} z+x z^{3}$ | $\left\{\frac{4}{13}, \frac{5}{13}, \frac{3}{13}\right\}$ | $\{1\}$ |
| $U_{12}$ | $x^{3}+y^{3}+z^{4}$ | $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{4}\right\}$ | $\{1\}$ |

Remark B.3.1. There are four quasi-Brieskorn-Pham families with inner modality equal to one, namely, $E_{12}, E_{14}, W_{12}$ and $U_{12}$, all of which are BrieskornPham.

## B.4. Inner Modality Two

Table B.3. The Twenty Weighted Homogeneous Singularities with Inner Modality Two [494], [493], [449]

| Symbol | Residual Part | Weights | $p_{g}$ | Notes |
| :---: | :---: | :---: | :---: | :---: |
| $E_{18}$ | $x^{3}+y^{10}$ | $\left\{\frac{1}{3}, \frac{1}{10}\right\}$ | $\{9,1\}$ |  |
| $E_{19}$ | $x^{3}+x y^{7}$ | $\left\{\frac{1}{3}, \frac{2}{2}\right\}$ | $\{10,1\}$ |  |
| $E_{20}$ | $x^{3}+y^{11}$ | $\left\{\frac{1}{3}, \frac{1}{11}\right\}$ | $\{10,1\}$ |  |
| $J_{16}$ | $x^{3}+y^{9}+a x^{2} y^{3}$ | $\left\{\frac{1}{3}, \frac{1}{9}\right\}$ | $\{9,1\}$ | $4 a^{3}+27 \neq 0$ |
| $W_{15}$ | $x^{4}+y^{6}+a x^{2} y^{3}$ | $\left\{\frac{1}{4}, \frac{1}{6}\right\}$ | $\{8,1\}$ | $a^{2}-4 \neq 0$ |
| $W_{17}$ | $x^{4}+x y^{5}$ | $\left\{\frac{1}{4}, \frac{3}{20}\right\}$ | $\{9,1\}$ |  |
| $W_{18}$ | $x^{4}+y^{7}$ | $\left\{\frac{1}{4}, \frac{1}{7}\right\}$ | $\{9,1\}$ |  |
| $Z_{15}$ | $x^{3} y+y^{7}+a x^{2} y^{3}$ | $\left\{\frac{2}{7}, \frac{1}{7}\right\}$ | $\{9,1\}$ | $4 a^{3}+27 \neq 0$ |
| $Z_{17}$ | $x^{3} y+y^{8}$ | $\left\{\frac{7}{24}, \frac{1}{8}\right\}$ | $\{9,1\}$ |  |
| $Z_{18}$ | $x^{3} y+x y^{6}$ | $\left\{\frac{5}{11}, \frac{2}{17}\right\}$ | $\{10,1\}$ |  |
| $Z_{19}$ | $x^{3} y+y^{9}$ | $\left\{\frac{8}{27}, \frac{1}{9}\right\}$ | $\{10,1\}$ |  |
| $Q_{14}$ | $x^{3}+y z^{2}+x y^{4}+a x^{2} y^{2}$ | $\left\{\frac{1}{3}, \frac{1}{6}, \frac{5}{12}\right\}$ | $\{1\}$ | $a^{2}-4 \neq 0$ |
| $Q_{16}$ | $x^{3}+y z^{2}+y^{7}$ | $\left\{\frac{1}{3}, \frac{1}{7}, \frac{3}{7}\right\}$ | $\{1\}$ |  |
| $Q_{17}$ | $x^{3}+y z^{2}+x y^{5}$ | $\left\{\frac{1}{3}, \frac{2}{15}, \frac{13}{30}\right\}$ | $\{1\}$ |  |
| $Q_{18}$ | $x^{3}+y z^{2}+y^{8}$ | $\left\{\frac{1}{3}, \frac{1}{8}, \frac{7}{16}\right\}$ | $\{1\}$ |  |
| $S_{14}$ | $x^{2} z+y z^{2}+y^{5}+a y^{3} z$ | $\left\{\frac{3}{10}, \frac{1}{3}, \frac{2}{5}\right\}$ | $\{1\}$ | $a^{2}-4 \neq 0$ |
| $S_{16}$ | $x^{2} z+y z^{2}+x y^{4}$ | $\left\{\frac{5}{17}, \frac{3}{17}, \frac{7}{17}\right\}$ | $\{1\}$ |  |
| $S_{17}$ | $x^{2} z+y z^{2}+y^{6}$ | $\left\{\frac{7}{24} \frac{1}{6}, \frac{5}{13}\right\}$ | $\{1\}$ |  |
| $U_{14}$ | $x^{3}+x z^{2}+x y^{3}+a y^{3} z$ | $\left\{\frac{1}{3}, \frac{2}{9}, \frac{1}{3}\right\}$ | $\{1\}$ | $a\left(a^{2}+1\right) \neq 0$ |
| $U_{16}$ | $x^{3}+x z^{2}+y^{5}$ | $\left\{\frac{1}{3}, \frac{1}{5}, \frac{1}{3}\right\}$ | $\{1\}$ |  |

Remark B.4.1. There are six quasi-Brieskorn-Pham families with inner modality equal to two, namely, $E_{18}, E_{20}, J_{16}, W_{15}, W_{18}$ and $U_{16}$. Three are Brieskorn-Pham, namely, $E_{18}, E_{20}$ and $W_{18}$.

Remark B.4.2. At least three different versions of $U_{14}$ exist in the literature [493], [494], [449]. In [493], $U_{14}$ is stated as $x^{3}+x z^{2}+r x^{2} y^{3}+s x y^{3} z+t y^{3} z^{2}+y^{9}$ with the explicit assumption that $\Delta(r, s, t) \neq 0$. The corresponding weights are $\left\{\frac{1}{3}, \frac{1}{9}, \frac{1}{3}\right\}$ and Milnor number $\mu=32$. In [494], $U_{14}$ is stated as $x^{3}+x z^{2}+x y^{3}+$ $t y^{3} z$ with the explicit assumption $t\left(t^{2}+1\right) \neq 0$. The corresponding weights are $\left\{\frac{1}{3}, \frac{2}{9}, \frac{1}{3}\right\}$ and Milnor number $\mu=14$. In [449], Suzuki admits that $U_{14}$ is incorrectly stated in [493] and replaces it with $x^{3}+x z^{2}+x y^{3}+s y^{3} z+t y^{4} z$ with the explicit assumption that $\Delta(s, t) \neq 0$. However, the revised singularity is not weighted homogeneous. As the first and third versions appear to be inconsistent with the classification, we state the second version in Table B.3.

## B.5. Inner Modality Three

Table B.4. The Twenty-Four Weighted Homogeneous Singularities with Inner Modality Three [494], [493], [449]

| Symbol | Residual Part | Weights | $p_{g}$ | Notes |
| :---: | :---: | :---: | :---: | :---: |
| $E_{24}$ | $x^{3}+y^{13}$ | $\left\{\frac{1}{3}, \frac{1}{13}\right\}$ | $\{12,2\}$ |  |
| $E_{25}$ | $x^{3}+x y^{9}$ | $\left\{\frac{1}{3}, \frac{2}{27}\right\}$ | $\{13,2\}$ |  |
| $E_{26}$ | $x^{3}+y^{14}$ | $\left\{\frac{1}{3}, \frac{1}{14}\right\}$ | $\{13,2\}$ |  |
| $J_{22}$ | $x^{3}+y^{12}+a x^{2} y^{4}$ | $\left\{\frac{1}{3}, \frac{1}{12}\right\}$ | $\{12,2\}$ | $4 a^{3}+27 \neq 0$ |
| $N_{16}$ | $x^{4} y+x y^{4}+a x^{3} y^{2}+b x^{2} y^{3}$ | $\left\{\frac{1}{5}, \frac{1}{5}\right\}$ | $\{10,1\}$ | $\Delta(a, b) \neq 0$ |
| $Z_{21}$ | $x^{3} y+y^{10}+a x^{2} y^{4}$ | $\left\{\frac{3}{10}, \frac{1}{10}\right\}$ | $\{12,2\}$ | $4 a^{3}+27 \neq 0$ |
| $Z_{23}$ | $x^{3} y+y^{11}$ | $\left\{\frac{10}{33}, \frac{1}{11}\right\}$ | $\{12,2\}$ |  |
| $Z_{24}$ | $x^{3} y+x y^{8}$ | $\left\{\frac{7}{23}, \frac{2}{23}\right\}$ | $\{13,2\}$ |  |
| $Z_{25}$ | $x^{3} y+y^{12}$ | $\left\{\frac{11}{36}, \frac{1}{12}\right\}$ | $\{13,2\}$ |  |
| $N_{19}$ | $x^{4} y+y^{6}$ | $\left\{\frac{5}{24}, \frac{1}{6}\right\}$ | $\{10,1\}$ |  |
| $N_{20}^{1}$ | $x^{4} y+x y^{5}$ | $\left\{\frac{4}{19}, \frac{3}{19}\right\}$ | $\{11,1\}$ |  |
| $N_{20}^{2}$ | $x^{5}+y^{6}$ | $\left\{5, \frac{1}{6}\right\}$ | $\{10,1\}$ |  |
| $N_{21}$ | $x^{5}+x y^{5}$ | $\left\{\frac{4}{25}\right\}$ | $\{11,1\}$ |  |
| $Q_{20}$ | $x^{3}+y z^{2}+y^{9}+a x^{2} y^{3}$ | $\left\{\frac{1}{3}, \frac{1}{,}, \frac{4}{9}\right\}$ | $\{2\}$ | $4 a^{3}+27 \neq 0$ |
| $Q_{22}$ | $x^{3}+y z^{2}+y^{10}$ | $\left\{\frac{1}{3}, \frac{1}{10}, \frac{9}{20}\right\}$ | $\{2\}$ |  |
| $Q_{23}$ | $x^{3}+y z^{2}+x y^{7}$ | $\left\{\frac{1}{3}, \frac{2}{21}, \frac{19}{42}\right\}$ | $\{2\}$ |  |
| $Q_{24}$ | $x^{3}+y z^{2}+y^{11}$ | $\left\{\frac{1}{3}, \frac{1}{11}, \frac{5}{11}\right\}$ | $\{2\}$ |  |
| $V_{15}$ | $x^{2} y+y^{4}+z^{4}+a y^{2} z^{2}+b x^{2} z$ | $\left\{\frac{3}{8}, \frac{1}{4}, \frac{1}{4}\right\}$ | $\{1\}$ | $\Delta(a, b) \neq 0$ |
| $V_{18}^{* 1}$ | $x^{2} y+z^{4}+y^{5}$ | $\left\{5,5, \frac{1}{4}\right\}$ | $\{1\}$ |  |
| $V_{18}^{* 2}$ | $x^{2} y+y^{3} z+x z^{3}$ | $\left\{\frac{7}{19}, \frac{5}{19}, \frac{4}{19}\right\}$ | $\{1\}$ |  |
| $V_{19}^{* 1}$ | $x^{2} y+z^{4}+y^{4} z$ | $\left\{\frac{13}{32}, \frac{3}{16}, \frac{1}{4}\right\}$ | $\{1\}$ |  |
| $V_{19}^{* 2}$ | $x^{2} y+y^{3} z+z^{5}$ | $\left\{\frac{11}{30}, \frac{4}{15}, \frac{1}{5}\right\}$ | $\{1\}$ |  |
| $V_{19}^{* 3}$ | $x^{2} y+x z^{3}+y^{4}$ | $\left\{\frac{3}{8}, \frac{1}{4}, \frac{5}{24}\right\}$ | $\{1\}$ |  |
| $V_{20}^{*}$ | $x^{2} y+y^{4}+z^{5}$ | $\left\{\frac{3}{8}, \frac{1}{4}, \frac{1}{5}\right\}$ | $\{1\}$ |  |

Remark B.5.1. There are five quasi-Brieskorn-Pham families with inner modality equal to three, namely, $E_{24}, E_{26}, J_{22}, N_{16}$ and $N_{20}^{2}$. Four are BrieskornPham, namely, $E_{24}, E_{26}, N_{16}$ and $N_{20}^{2}$.

## B.6. Inner Modality Four

Table B.5. The Twenty-Eight Weighted Homogeneous Singularities with Inner Modality Four [494], [493], [449]

| Symbol | Residual Part | Weights | $p_{g}$ | Notes |
| :---: | :---: | :---: | :---: | :---: |
| $E_{30}$ | $x^{3}+y^{16}$ | $\left\{\frac{1}{3}, \frac{1}{16}\right\}$ | $\{15,2\}$ |  |
| $E_{31}$ | $x^{3}+x y^{11}$ | $\left\{\frac{2}{3}, \frac{2}{33}\right\}$ | $\{16,2\}$ |  |
| $E_{32}$ | $x^{3}+y^{17}$ | $\left\{\frac{1}{3}, \frac{1}{17}\right\}$ | $\{16,2\}$ |  |
| $J_{28}$ | $x^{3}+y^{15}+a x^{2} y^{5}$ | $\left\{\frac{1}{3}, \frac{1}{15}\right\}$ | $\{15,2\}$ | $4 a^{3}+27 \neq 0$ |
| $W_{24}$ | $x^{4}+y^{9}$ | $\left\{\frac{1}{4}, \frac{1}{9}\right\}$ | $\{12,2\}$ |  |
| $W_{25}$ | $x^{4}+x y^{7}$ | $\left\{\frac{1}{4}, \frac{3}{28}\right\}$ | $\{13,2\}$ |  |
| $X_{21}$ | $x^{4}+y^{8}+a x^{3} y^{2}+b x^{2} y^{4}$ | $\left\{\frac{1}{4}, \frac{1}{8}\right\}$ | $\{12,2\}$ | $\Delta(a, b) \neq 0$ |
| $Z_{27}$ | $x^{3} y+y^{13}+a x^{2} y^{5}$ | $\left\{\frac{4}{13}, \frac{1}{13}\right\}$ | $\{15,2\}$ | $4 a^{3}+27 \neq 0$ |
| $Z_{29}$ | $x^{3} y+y^{14}$ | $\left\{\frac{1}{42}, \frac{1}{14}\right\}$ | $\{15,2\}$ |  |
| $Z_{30}$ | $x^{3} y+x y^{10}$ | $\left\{\frac{9}{292}, \frac{2}{29}\right\}$ | $\{16,2\}$ |  |
| $Z_{31}$ | $x^{3} y+y^{15}$ | $\left\{\frac{14}{45}, \frac{1}{15}\right\}$ | $\{16,2\}$ |  |
| $N_{22}$ | $x^{4} y+y^{7}+a x^{2} y^{4}$ | $\left\{\frac{3}{11}, \frac{1}{7}\right\}$ | $\{12,2\}$ | $a^{2}-4 \neq 0$ |
| $N_{24}^{1}$ | $x^{5}+y^{7}$ | $\left\{\frac{1}{5}, \frac{1}{7}\right\}$ | $\{12,2\}$ |  |
| $N_{24}^{2}$ | $x^{4} y+x y^{6}$ | $\left\{\frac{5}{23}, \frac{3}{23}\right\}$ | $\{13,2\}$ |  |
| $N_{25}$ | $x^{4} y+y^{8}$ | $\left\{\frac{7}{32}, \frac{1}{8}\right\}$ | $\{13,2\}$ |  |

Table B.6. The Twenty-Eight Weighted Homogeneous Singularities with Inner Modality Four (Continued) [494], [493], [449]

| Symbol | Residual Part | Weights | $p_{g}$ | Notes |
| :---: | :---: | :---: | :---: | :---: |
| $Q_{26}$ | $x^{3}+y z^{2}+y^{12}+a x^{2} y^{4}$ | $\left\{\frac{1}{3}, \frac{1}{12}, \frac{11}{24}\right\}$ | $\{2\}$ | $4 a^{3}+27 \neq 0$ |
| $Q_{28}$ | $x^{3}+y z^{2}+y^{13}$ | $\left\{\frac{1}{3}, \frac{1}{13}, \frac{6}{13}\right\}$ | $\{2\}$ |  |
| $Q_{29}$ | $x^{3}+y z^{2}+x y^{9}$ | $\left\{\frac{1}{3}, \frac{2}{27}, \frac{55}{54}\right\}$ | $\{2\}$ |  |
| $Q_{30}$ | $x^{3}+y z^{2}+y^{14}$ | $\left\{\frac{1}{3}, \frac{1}{14}, \frac{13}{8}\right\}$ | $\{2\}$ |  |
| $S_{23}$ | $x^{2} z+y z^{2}+y^{8}$ | $\left\{3, \frac{9}{32}, \frac{7}{8}, \frac{7}{16}\right\}$ | $\{2\}$ |  |
| $S_{24}$ | $x^{2} z+y z^{2}+x y^{6}$ | $\left\{\frac{7}{25}, \frac{3}{25}, \frac{11}{25}\right\}$ | $\{2\}$ |  |
| $S_{20}^{*}$ | $x^{2} z+y z^{2}+y^{7}+a x^{3} y+b x^{2} y^{3}$ | $\left\{\frac{2}{7}, \frac{1}{7}, \frac{3}{7}\right\}$ | $\{2\}$ | $\Delta(a, b) \neq 0$ |
| $V_{21}^{*}$ | $x^{2} y+z^{4}+y^{6}+a y^{3} z^{2}$ | $\left\{\frac{5}{12}, \frac{1}{4}, \frac{1}{4}\right\}$ | $\{2\}$ | $a^{2}-4 \neq 0$ |
| $V_{23}^{*}$ | $x^{2} y+z^{4}+y^{5} z$ | $\left\{\frac{17}{40} \frac{3}{20}, \frac{1}{4}\right\}$ | $\{2\}$ |  |
| $V_{24}^{*}$ | $x^{2} y+z^{4}+y^{7}$ | $\left\{\frac{3}{7}, \frac{1}{7}, \frac{1}{4}\right\}$ | $\{2\}$ |  |
| $V_{18}^{\prime}$ | $x^{3}+y^{4}+z^{4}+a y^{2} z^{2}$ | $\left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{4}\right\}$ | $\{1\}$ | $a^{2}-4 \neq 0$ |
| $V_{20}^{\prime}$ | $x^{3}+x y^{3}+y z^{3}$ | $\left\{\frac{1}{3}, \frac{2}{9}, \frac{7}{27}\right\}$ | $\{1\}$ |  |
| $V_{21}^{\prime}$ | $x^{3}+x z^{3}+y^{4}$ | $\left\{\frac{1}{3}, \frac{1}{4}, \frac{2}{9}\right\}$ | $\{1\}$ |  |

Remark B.6.1. There are seven quasi-Brieskorn-Pham families with inner modality equal to four, namely, $E_{30}, E_{32}, J_{28}, W_{24}, X_{21}, N_{24}^{1}$ and $V_{18}^{\prime}$. Four are Brieskorn-Pham, namely, $E_{30}, E_{32}, W_{24}$ and $N_{24}^{1}$.

## B.7. Inner Modality Five

Table B.7. The Thirty-One Weighted Homogeneous Singularities with Inner Modality Five [449]

| Symbol | Residual Part | Weights | $p_{g}$ | Notes |
| :---: | :---: | :---: | :---: | :---: |
| $E_{36}$ | $x^{3}+y^{19}$ | $\left\{\frac{1}{3}, \frac{1}{19}\right\}$ | $\{18,3\}$ |  |
| $E_{37}$ | $x^{3}+x y^{13}$ | $\left\{\frac{1}{3}, \frac{2}{39}\right\}$ | $\{19,3\}$ |  |
| $E_{38}$ | $x^{3}+y^{20}$ | $\left\{\left\{\frac{1}{3}, \frac{1}{20}\right\}\right.$ | $\{19,3\}$ |  |
| $J_{34}$ | $x^{3}+y^{18}+a x^{2} y^{6}$ | $\left\{\frac{1}{3}, \frac{1}{18}\right\}$ | $\{18,3\}$ | $4 a^{3}+27 \neq 0$ |
| $W_{27}$ | $x^{4}+y^{10}+a x^{2} y^{5}$ | $\left\{\frac{1}{4}, \frac{1}{10}\right\}$ | $\{14,2\}$ | $a^{2}-4 \neq 0$ |
| $W_{29}$ | $x^{4}+x y^{8}$ | $\left\{\frac{1}{4}, \frac{3}{32}\right\}$ | $\{15,2\}$ |  |
| $W_{30}$ | $x^{4}+y^{11}$ | $\left\{\frac{1}{4}, \frac{1}{11}\right\}$ | $\{15,2\}$ |  |
| $Z_{33}$ | $x^{3} y+y^{16}+a x^{2} y^{6}$ | $\left\{\left\{\frac{5}{16}, \frac{1}{16}\right\}\right.$ | $\{18,3\}$ | $4 a^{3}+27 \neq 0$ |
| $Z_{35}$ | $x^{3} y+y^{17}$ | $\left\{\frac{16}{51}, \frac{1}{17}\right\}$ | $\{18,3\}$ |  |
| $Z_{36}$ | $x^{3} y+x y^{12}$ | $\left\{\frac{11}{35}, \frac{2}{53}\right\}$ | $\{19,3\}$ |  |
| $Z_{37}$ | $x^{3} y+y^{18}$ | $\left\{\frac{17}{54}, \frac{1}{18}\right\}$ | $\{19,3\}$ |  |
| $N_{26}$ | $x^{5}+x y y^{6}+a x^{3} y^{3}$ | $\left\{\frac{1}{5}, \frac{2}{15}\right\}$ | $\{14,2\}$ | $a^{2}-4 \neq 0$ |
| $N_{28}^{1}$ | $x^{5}+y^{8}$ | $\left\{\frac{1}{5}, \frac{1}{8}\right\}$ | $\{14,2\}$ |  |
| $Q_{32}^{2}$ | $x^{3}+y z^{2}+x y^{10}+a x^{2} y^{5}$ | $\left\{\frac{1}{3}, \frac{1}{15}, \frac{7}{15}\right\}$ | $\{3\}$ | $a^{2}-4 \neq 0$ |
| $Q_{34}$ | $x^{3}+y z^{2}+y^{16}$ | $\left\{\frac{1}{3}, \frac{1}{16}, \frac{1}{32}\right\}$ | $\{3\}$ |  |
| $Q_{35}$ | $x^{3}+y z^{2}+x y^{11}$ | $\left\{\frac{1}{3}, \frac{2}{33}, \frac{, 17}{66}\right\}$ | $\{3\}$ |  |
| $Q_{36}$ | $x^{3}+y z^{2}+y^{17}$ | $\left\{\frac{1}{3}, \frac{1}{17}, \frac{8}{17}\right\}$ | $\{3\}$ |  |

Table B.8. The Thirty-One Weighted Homogeneous Singularities with Inner Modality Five (Continued) [449]

| Symbol | Residual Part | Weights | $p_{g}$ | Notes |
| :---: | :---: | :---: | :---: | :---: |
| $S_{26}$ | $x^{2} z+y z^{2}+y^{9}+a y^{5} z$ | $\left\{\frac{5}{18}, \frac{1}{9}, \frac{4}{9}\right\}$ | $\{2\}$ | $a^{2}-4 \neq 0$ |
| $S_{28}$ | $x^{2} z+y z^{2}+x y^{7}$ | $\left\{\frac{8}{29}, \frac{3}{29}, \frac{13}{29}\right\}$ | $\{2\}$ |  |
| $S_{29}$ | $x^{2} z+y z^{2}+y^{10}$ | $\left\{\frac{11}{40}, \frac{1}{10}, \frac{9}{20}\right\}$ | $\{2\}$ |  |
| $U_{20}^{*}$ | $x^{3}+x z^{2}+y^{6}+a x^{2} y^{2}+b y^{2} z^{2}+c x y^{2} z$ | $\left\{\frac{1}{3}, \frac{1}{6}, \frac{1}{3}\right\}$ | $\{2\}$ | $\Delta(a, b, c) \neq 0$ |
| $U_{24}$ | $x^{3}+x z^{2}+y^{7}$ | $\left\{\frac{1}{3}, \frac{1}{7}, \frac{1}{3}\right\}$ | $\{2\}$ |  |
| $V_{23}^{* 1}$ | $x^{2} z+y z^{3}+y^{6}$ | $\left\{\frac{13}{36}, \frac{1}{6}, \frac{5}{18}\right\}$ | $\{2\}$ |  |
| $V_{24}^{* 1}$ | $x^{2} z+y z^{3}+x y^{4}$ | $\left\{\frac{9}{25}, \frac{4}{25}, \frac{7}{25}\right\}$ | $\{2\}$ |  |
| $V_{22}^{\prime}$ | $x^{3}+y z^{3}+y^{5}+a x y^{2} z$ | $\left\{\frac{1}{3}, \frac{1}{5}, \frac{4}{15}\right\}$ | $\{2\}$ | $\Delta(a) \neq 0$ |
| $V_{24}^{\prime 1}$ | $x^{3}+y^{4}+z^{5}$ | $\left\{\frac{1}{3}, \frac{1}{4}, \frac{1}{5}\right\}$ | $\{2\}$ |  |
| $V_{24}^{\prime 2}$ | $x^{3}+y^{4} z+y z^{3}$ | $\left\{\frac{1}{3}, \frac{2}{11}, \frac{3}{11}\right\}$ | $\{2\}$ |  |
| $O_{16}$ | $x^{3}+y^{3}+z^{3}+w^{3}+(a x+b y+c z+d w)^{3}$ | $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$ | - | $\Delta(a, b, c, d) \neq 0$ |
| $O_{20}$ | $x^{4}+y^{3}+z^{3}+x w^{2}+a y^{2} z$ | $\left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{3}, \frac{3}{8}\right\}$ | - | $4 a^{3}+27 \neq 0$ |
| $O_{21}$ | $x^{2} y+y^{2} z+x w^{2}+z^{4}$ | $\left\{\frac{5}{16}, \frac{3}{8}, \frac{1}{3}, \frac{11}{32}\right\}$ | - |  |
| $O_{22}$ | $x^{3}+y z^{2}+z w^{2}+y^{4}$ | $\left\{\frac{1}{3}, \frac{1}{4}, \frac{3}{8}, \frac{5}{16}\right\}$ | - |  |

Remark B.7.1. There are ten quasi-Brieskorn-Pham families with inner modality equal to five, namely, $E_{36}, E_{38}, J_{34}, W_{27}, W_{30}, N_{28}^{1}, U_{20}^{*}, U_{24}, V_{24}^{\prime}$ and $O_{16}$. Five are Brieskorn-Pham, namely, $E_{36}, E_{38}, W_{30}, N_{28}^{1}$ and $V_{24}^{\prime}$.

## B.8. Inner Modality Six

Table B.9. The Thirty-Seven Weighted Homogeneous Singularities with Inner Modality Six [415]

| Symbol | Residual Part | Weights | $p_{g}$ | Notes |
| :---: | :---: | :---: | :---: | :---: |
| $E_{42}$ | $x^{3}+y^{22}$ | $\left\{\frac{1}{3}, \frac{1}{22}\right\}$ | $\{21,3\}$ |  |
| $E_{43}$ | $x^{3}+x y^{15}$ | $\left\{\frac{1}{3}, \frac{2}{45}\right\}$ | $\{22,3\}$ |  |
| $E_{44}$ | $x^{3}+y^{23}$ | $\left\{\frac{1}{3}, \frac{1}{23}\right\}$ | $\{22,3\}$ |  |
| $J_{40}$ | $x^{3}+a x^{2} y^{7}+y^{21}$ | $\left\{\frac{1}{3}, \frac{1}{21}\right\}$ | $\{21,3\}$ | $4 a^{3}+27 \neq 0$ |
| $Z_{39}$ | $x^{3} y+a x^{2} y^{7}+y^{19}$ | $\left\{\frac{6}{19}, \frac{1}{19}\right\}$ | $\{21,3\}$ | $4 a^{3}+27 \neq 0$ |
| $Z_{41}$ | $x^{3} y+y^{20}$ | $\left\{\frac{19}{60}, \frac{1}{20}\right\}$ | $\{21,3\}$ |  |
| $Z_{42}$ | $x^{3} y+x y^{14}$ | $\left\{\frac{13}{41}, \frac{2}{41}\right\}$ | $\{22,3\}$ |  |
| $Z_{43}$ | $x^{3} y+y^{21}$ | $\left\{\frac{20}{63}, \frac{1}{21}\right\}$ | $\{22,3\}$ |  |
| $N_{25}$ | $x^{5} y+a x^{4} y^{2}+b x^{3} y^{3}+c x^{2} y^{4}+x y^{5}$ | $\left\{\frac{1}{6}, \frac{1}{6}\right\}$ | $\{15,3\}$ | $\Delta(a, b, c) \neq 0$ |
| $N_{28}^{*}$ | $x^{4} y+a x^{2} y^{5}+b x^{3} y^{3}+y^{9}$ | $\left\{\frac{2}{9}, \frac{1}{9}\right\}$ | $\{16,2\}$ | $\Delta(a, b) \neq 0$ |
| $N_{29}$ | $x^{5} y+y^{7}$ | $\left\{\frac{6}{35}, \frac{1}{7}\right\}$ | $\{15,3\}$ |  |
| $N_{30}^{1}$ | $x^{5} y+x y^{6}$ | $\left\{\frac{5}{29}, \frac{4}{29}\right\}$ | $\{16,3\}$ |  |
| $N_{30}^{2}$ | $x^{6}+y^{7}$ | $\left\{\frac{1}{6}, \frac{1}{7}\right\}$ | $\{15,3\}$ |  |
| $N_{31}^{1}$ | $x^{4} y+y^{10}$ | $\left\{\frac{9}{40}, \frac{1}{10}\right\}$ | $\{16,2\}$ |  |
| $N_{31}^{2}$ | $x^{6} y+y^{6}$ | $\left\{\frac{5}{36}, \frac{1}{6}\right\}$ | $\{16,3\}$ |  |
| $N_{31}^{3}$ | $x^{5}+x y^{7}$ | $\left\{\frac{1}{5}, \frac{4}{35}\right\}$ | $\{16,2\}$ |  |
| $N_{32}^{1}$ | $x^{4} y+x y^{8}$ | $\left\{\frac{7}{31}, \frac{3}{31}\right\}$ | $\{17,2\}$ |  |
| $N_{32}^{2}$ | $x^{5}+y^{9}$ | $\left\{\frac{1}{5}, \frac{1}{9}\right\}$ | $\{16,2\}$ |  |

Table B.10. The Thirty-Seven Weighted Homogeneous Singularities with Inner Modality Six (Continued) [415]

| Symbol | Residual Part | Weights | $p_{g}$ | Notes |
| :---: | :---: | :---: | :---: | :---: |
| $Q_{38}$ | $x^{3}+y z^{2}+x y^{12}+a x^{2} y^{6}$ | $\left\{\frac{1}{3}, \frac{1}{18}, \frac{17}{36}\right\}$ | $\{3\}$ | $a \neq 0$ |
| $Q_{40}$ | $x^{3}+y z^{2}+y^{19}$ | $\left\{\frac{1}{3}, \frac{1}{19}, \frac{9}{19}\right\}$ | $\{3\}$ |  |
| $Q_{41}$ | $x^{3}+y z^{2}+x y^{13}$ | $\left\{\frac{1}{3}, \frac{2}{39}, \frac{37}{78}\right\}$ | $\{3\}$ |  |
| $U_{26}$ | $x^{3}+x z^{2}+x y^{5}+a y^{5} z$ | $\left\{\frac{1}{3}, \frac{2}{15}, \frac{1}{3}\right\}$ | $\{2\}$ | $a \neq 0$ |
| $U_{28}$ | $x^{3}+x z^{2}+y^{8}$ | $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right\}$ | $\{2\}$ |  |
| $V_{25}^{*}$ | $x^{2} y+y^{4}+z^{6}+a y^{2} z^{3}$ | $\left\{\frac{3}{8}, \frac{1}{4}, \frac{1}{6}\right\}$ | $\{2\}$ | $a \neq 0$ |
| $V_{27}^{* 1}$ | $x^{2} y+y^{6} z+z^{4}+a y^{4} z^{2}+b y^{2} z^{3}$ | $\left\{\frac{7}{16}, \frac{1}{8}, \frac{1}{4}\right\}$ | $\{2\}$ | $\Delta(a, b) \neq 0$ |
| $V_{30}^{*}$ | $x^{2} y+z^{4}+a x y^{5}+b y^{9}$ | $\left\{\frac{4}{9}, \frac{1}{9}, \frac{1}{4}\right\}$ | $\{2\}$ | $\Delta(a, b) \neq 0$ |
| $V_{27}^{* 2}$ | $x^{2} y+y^{4}+x z^{4}$ | $\left\{\frac{3}{8}, \frac{1}{4}, \frac{5}{32}\right\}$ | $\{2\}$ |  |
| $V_{31}^{*}$ | $x^{2} y+z^{4}+y^{7} z$ | $\left\{\frac{25}{56}, \frac{3}{28}, \frac{1}{4}\right\}$ | $\{2\}$ |  |
| $V_{26}^{1}$ | $x^{3}+y^{3} z+x z^{4}+a x^{2} y^{2}$ | $\left\{\frac{1}{3}, \frac{1}{4}, \frac{3}{16}\right\}$ | $\{2\}$ | $4 a^{3}+27 \neq 0$ |
| $V_{26}^{\prime 2}$ | $x^{3}+y^{4}+y z^{4}$ | $\left\{\frac{1}{19}, \frac{1}{3}, \frac{9}{19}\right\}$ | $\{3\}$ | $\Delta(a, b, c, d) \neq 0$ |
| $V_{40}^{\prime}$ | $a x^{10} z+b x z^{2}+c x^{19}+d y^{3}$ |  |  | $\Delta(a) \neq 0$ |
| $O_{22}^{2}$ | $x+x y+a x y+y z+z w$ | $\left\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{4}\right\}$ | - | $4 a^{3}+27 \neq 0$ |
| $O_{24}^{1}$ | $x^{2} z+y^{3}+a y^{2} z+z^{3}+w^{4}$ |  |  |  |
| $O_{24}^{2}$ | $x w^{2}+y z^{2}+y^{2} w+x^{2} z$ | $\left\{\frac{1}{5}, \frac{1}{3}, \frac{1}{3}, \frac{2}{5}\right\}$ | - | $\Delta(a, b, c, d, e, f, g) \neq 0$ |
| $O_{24}^{3}$ | $a x^{5}+b x z^{2}+c x^{3} w+d y^{3}+e y^{2} z+f y z^{2}+g z^{3}$ | $\left\{\frac{3}{13}, \frac{4}{13}, \frac{1}{3}, \frac{5}{13}\right\}$ | - |  |
| $O_{24}^{4}$ | $x^{3} y+x w^{2}+y^{2} w+z^{3}$ | $\left\{\frac{2}{9}, \frac{11}{36}, \frac{1}{3}, \frac{7}{18}\right\}$ | - |  |
| $O_{25}$ | $x^{3} z+x w^{2}+y^{2} w+z^{3}$ |  |  |  |

REmARK B.8.1. The singularities $V_{26}^{\prime 1}, O_{22}^{2}$ and $O_{24}^{2}$ appear erroneous as written in [415], as $V_{26}^{\prime 1}$ does not appear to be weighted homogeneous, $\mu\left(O_{22}^{2}\right)=0$ and $\mu\left(O_{24}^{2}\right)=16$.

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## Appendix C

## Fractional Part Summations


#### Abstract

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator, the smaller the fraction. - Leo Tolstoy


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C.1. Non-modular Residue Summations743

In this short appendix, I discuss a method of simplification of a certain fractional part summation. Our primary aim is to simplify the fractional summation

$$
\begin{equation*}
\sum_{i=1}^{\left\lfloor 1 / \omega_{2}\right\rfloor}\left\{\frac{1-i \omega_{2}}{\omega_{1}}\right\}=\sum_{i=1}^{\left\lfloor d / q_{2}\right\rfloor}\left\{\frac{d-i q_{2}}{q_{1}}\right\} \tag{C.1}
\end{equation*}
$$

where $d \omega_{i}=q_{i}$, where $q_{i}, d \in \mathbb{N}$. The ultimate aim is to compute the sum in equation (C.1) for $q_{1}, q_{2}, d \in \mathbb{N}$, and give exact, non-summatory representations for the delta invariant $\delta$ and branch number $r$ of an arbitrary weighted homogeneous plane curve singularity with weights $\left\{q_{1}, q_{2}\right\}$ and weighted degree $d$. Unfortunately, the following identity [245] is not immediately applicable: For
$m \in \mathbb{Z}, n \in \mathbb{N}$ and $\alpha \in \mathbb{R}$,

$$
\begin{align*}
\sum_{k=0}^{n}\left\lfloor\frac{\alpha+m k}{n}\right\rfloor= & \frac{1}{2}(m-1)(n-1)+\frac{1}{2}(\operatorname{GCD}(m, n)+1) \\
& +\operatorname{GCD}(m, n)\left\lfloor\frac{\alpha}{\operatorname{GCD}(m, n)}\right\rfloor \tag{C.2}
\end{align*}
$$

which reduces to a related summation identity involving the fractional part function. However, there is an approach which may simplify the fractional part summation to one involving substantially fewer terms. By the identity $x-y\left\lfloor\frac{x}{y}\right\rfloor=x \bmod y$, one has

$$
\begin{equation*}
\sum_{i=0}^{\left\lfloor d / q_{2}\right\rfloor}\left\{\frac{d-i q_{2}}{q_{1}}\right\}=\frac{1}{q_{1}}\left(\left(d \bmod q_{2}\right)\left\lfloor\frac{d}{q_{2}}\right\rfloor-\sum_{i=1}^{\left\lfloor d / q_{2}\right\rfloor}\left(i q_{2} \bmod q_{1}\right)\right) \tag{C.3}
\end{equation*}
$$

where only the summands are reduced modulo $q_{1}$ but not the entire summation.

## C.1. Non-modular Residue Summations

For $a, b, n \in \mathbb{N}$, define the non-modular residue summation

$$
\begin{equation*}
S(a, b, n)=\sum_{i=1}^{n}(a i \bmod b) . \tag{C.4}
\end{equation*}
$$

Clearly, if $b$ divides $a$, then $S(a, b, n)=0$. Define $\bar{a}=\frac{a}{\operatorname{GCD}(a, b)}$ and $\bar{b}=\frac{a}{\operatorname{GCD}(a, b)}$.

Proposition C.1. Given $a, b \in \mathbb{N}$ and $c=\operatorname{GCD}(a, b)$, if $b \nmid a$, then for $n \in \mathbb{N}$,

$$
\begin{align*}
S(a, b, n)=c & \left(c\left\lfloor\frac{n}{b}\right\rfloor+\left\lfloor\frac{(n \bmod b)}{\bar{b}}\right\rfloor+1\right)\binom{\bar{b}}{2} \\
& +c \sum_{k=1}^{\bar{b}-1}(k \bar{a} \bmod \bar{b})\left\lfloor\frac{(n \bmod \bar{b})-k}{\bar{b}}\right\rfloor . \tag{C.5}
\end{align*}
$$

Proof. If $a$ and $b$ are coprime, then one can split the sum $S(a, b, n)$ into $b-1$ non-trivial sums involving the disjoint multiplicative residue classes $[b],[2 b], \ldots,[(b-1) a]$ modulo $b$, where each class has $\left\lfloor\frac{n-1}{b}\right\rfloor, \ldots,\left\lfloor\frac{n-b+1}{b}\right\rfloor$ elements, respectively. Summing over these classes yields

$$
\begin{equation*}
S(a, b, n)=\binom{b}{2}+\sum_{k=1}^{b-1}(k a \bmod b)\left\lfloor\frac{n-k}{b}\right\rfloor . \tag{C.6}
\end{equation*}
$$

For $a$ and $b$ not necessarily coprime, the identity $S(a, b, n)=c S(\bar{a}, \bar{b}, n)$, where $c=\operatorname{GCD}(a, b)$, implies

$$
\begin{equation*}
S(a, b, n)=c\left(\binom{\bar{b}}{2}+\sum_{k=1}^{\bar{b}-1}(k \bar{a} \bmod \bar{b})\left\lfloor\frac{n-k}{\bar{b}}\right\rfloor\right) \tag{C.7}
\end{equation*}
$$

Combining the transformation identities*,

$$
\begin{align*}
S(a, b, n)= & c\left\lfloor\frac{n}{b}\right\rfloor S(\bar{a}, \bar{b}, b)+c S(\bar{a}, \bar{b}, n \bmod b)  \tag{C.8a}\\
= & c\left\lfloor\frac{n}{b}\right\rfloor\left(\left\lfloor\frac{b}{b}\right\rfloor S(\bar{a}, \bar{b}, \bar{b})+S(\bar{a}, \bar{b}, b \bmod \bar{b})\right) \\
& +c\left(\left\lfloor\frac{(n \bmod b)}{\bar{b}}\right\rfloor S(\bar{a}, \bar{b}, \bar{b})+S(\bar{a}, \bar{b},(n \bmod b) \bmod \bar{b})\right.  \tag{C.8b}\\
= & \left.c^{2}\left\lfloor\frac{n}{b}\right\rfloor S(\bar{a}, \bar{b}, \bar{b})+c\left\lfloor\frac{n}{b}\right\rfloor S(\bar{a}, \bar{b}, b \bmod \bar{b})\right) \\
& +c\left\lfloor\frac{(n \bmod b)}{\bar{b}}\right\rfloor S(\bar{a}, \bar{b}, \bar{b})+c S(\bar{a}, \bar{b},(n \bmod b) \bmod \bar{b})  \tag{C.8c}\\
= & \left(c^{2}\left\lfloor\frac{n}{b}\right\rfloor+c\left\lfloor\frac{(n \bmod b)}{\bar{b}}\right\rfloor\right)\binom{\bar{b}}{2}+c S(\bar{a}, \bar{b}, n \bmod \bar{b}), \tag{C.8d}
\end{align*}
$$

where we have used the identities $b \equiv 0 \bmod \bar{b},(n \bmod b) \bmod \bar{b} \equiv n \bmod \bar{b}$, $S(\cdot, \cdot, 0)=0$ and $S(a, b, b)=\binom{b}{2}$ if $a$ and $b$ are coprime. Applying equation (C.7) to equation (C.8d) yields the claim.

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## Appendix D

## Quasi-Brieskorn-Pham Surface Singularities

There is geometry in the humming of the strings, there is music in the spacing of the spheres. - Pythagorus of Samos

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D.4. Quasi-Brieskorn-Pham Surface Singularities by Milnor Number ..... 776

Recall that a polynomial is Brieskorn-Pham if and only if it is of the form $f=\sum_{i=0}^{n} z_{i}^{a_{i}}$, where $a_{i} \geqslant 1$, with weights $\left\{\frac{1}{a_{0}}, \ldots, \frac{1}{a_{n}}\right\}$. Similarly, a polynomial is quasi-Brieskorn-Pham if and only if it is a non-degenerate weighted homogeneous polynomial with inverse (positive) integer weights. In either case, the corresponding Milnor number is the product $\prod_{i=0}^{n}\left(a_{0}-1\right)$.

## D.1. Geometric Genera by Lattice Point Enumeration

Let $a, b$ and $c$ be positive integers with no common factor, i.e., $\operatorname{GCD}(a, b, c)=$ 1. Define $a^{\prime}=\operatorname{GCD}(b, c), b^{\prime}=\operatorname{GCD}(c, a), c^{\prime}=\operatorname{GCD}(a, b), d=a^{\prime} b^{\prime} c^{\prime}, l=a+b+c$ and $l^{\prime}=a^{\prime}+b^{\prime}+c^{\prime}$. In Volume 2 of this work, we prove that the number of positive lattice points in the (integral) $t$-dilate of the lattice tetrahedron $\mathcal{W}=$
$\operatorname{conv}\left\{0, a \mathbf{e}_{1}, b \mathbf{e}_{2}, c \mathbf{e}_{3}\right\}$ is the degree 3 polynomial ${ }^{*}$,

$$
\begin{gather*}
\mathscr{L}_{\mathcal{W},+}(t)=\frac{a b c}{6} t^{3}-\frac{(a+b) c}{4} t^{2}+\left(\frac{c}{4}+\frac{b c}{12 a}+\frac{c\left(a+c^{\prime}\right)\left(a-c^{\prime}\right)}{12 a b}\right) t \\
-\sum_{i=1}^{a t} \sum_{j=1}^{\lfloor b(t-i / a)\rfloor}\left\{c\left(t-\frac{i}{a}-\frac{j}{b}\right)\right\}, \tag{D.1}
\end{gather*}
$$

which is precisely the geometric genus $p_{g}\left(f_{t}\right)$ of the $t$-dilate of the quasi-Brieskorn-Pham surface singularity $f$ with weights $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$, which includes, in particular, the Brieskorn-Pham singularity $f=x^{a}+y^{b}+z^{c}$. We compute the fractional part summation

$$
\begin{aligned}
& \sum_{i=1}^{a t} \sum_{j=1}^{\lfloor b(t-i / a)\rfloor}\left\{c\left(t-\frac{i}{a}-\frac{j}{b}\right)\right\}=\frac{a b-d}{4} t^{2} \\
&+\left(a^{\prime} \mathfrak{s}\left(\frac{b c}{d}, \frac{a a^{\prime}}{d}\right)+b^{\prime} \mathfrak{s}\left(\frac{a c}{d}, \frac{b b^{\prime}}{d}\right)+c^{\prime} \mathfrak{s}\left(\frac{a b}{d}, \frac{c c^{\prime}}{d}\right)+\gamma\right) t
\end{aligned}
$$

where $\gamma=\frac{1}{4}\left(a^{\prime}+b^{\prime}+c^{\prime}-a-b\right)-\frac{a^{2} b^{2}+c^{2}\left(c^{\prime}\right)^{2}+d^{2}}{12 a b c}$ and

$$
\begin{equation*}
\mathfrak{s}(p, q)=\frac{1}{4 q} \sum_{k=1}^{q-1} \cot \left(\frac{\pi k}{q}\right) \cot \left(\frac{\pi k p}{q}\right) \tag{D.2}
\end{equation*}
$$

denotes the standard Dedekind sum on $\mathbb{N}^{2}$. By generalizing the previous computation to arbitrary triples of positive integers, with no constraint on their greatest common divisor, one can show the following, q.v., Proposition 6.25.

[^46]Proposition D.1. Given $a, b, c, t \in \mathbb{N}$, the geometric genus of the $t$-dilate of the quasi-Brieskorn-Pham polynomial $f$ with inverse weights $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$ is the degree 3 polynomial

$$
\begin{align*}
p_{g}\left(f_{t}\right)=\frac{a b c}{6} & t^{3}-\frac{1}{4}\left(a b+b c+c a-\frac{d}{\tau}\right) t^{2} \\
& +\left(\frac{1}{4}\left(l-l^{\prime}\right)+\frac{1}{12}\left(\frac{a b}{c}+\frac{b c}{a}+\frac{c a}{b}+\frac{d^{2}}{a b c \tau^{2}}\right)-\mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)\right) t \tag{D.3}
\end{align*}
$$

where $\tau=\operatorname{GCD}(a, b, c), l=a+b+c, a^{\prime}=\operatorname{GCD}(b, c), b^{\prime}=\operatorname{GCD}(a, c), c^{\prime}=$ $\operatorname{GCD}(a, b), l^{\prime}=a^{\prime}+b^{\prime}+c^{\prime}, d=a^{\prime} b^{\prime} c^{\prime}$ and

$$
\begin{equation*}
\mathfrak{S}\left(a, b, c ; \frac{d}{\tau}\right)=a^{\prime} \mathfrak{s}\left(\frac{b c \tau}{d}, \frac{a a^{\prime} \tau}{d}\right)+b^{\prime} \mathfrak{s}\left(\frac{a c \tau}{d}, \frac{b b^{\prime} \tau}{d}\right)+c^{\prime} \mathfrak{s}\left(\frac{a b \tau}{d}, \frac{c c^{\prime} \tau}{d}\right) . \tag{D.4}
\end{equation*}
$$

## D.2. Quasi-Brieskorn-Pham Surface Singularities

Let $\{a, b, c\}$ denote the quasi-Brieskorn-Pham surface singularity with weights $\left\{\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right\}$, which include the Brieskorn-Pham singularities of the form $f=x^{a}+y^{b}+z^{c}$. Without loss of generality, we may assume $1 \leqslant a \leqslant b \leqslant c$.

By an exhaustive numerical search we determine the exponents $\{a, b, c\}$ satisfying $a \leqslant b \leqslant c \leqslant 150$, with geometric genera $p_{g} \leqslant 25$. The inverse weights have been grouped into equivalence classes by their geometric genus $p_{g}$ and each have a size according to the sequence*:

$$
\begin{gathered}
\{\infty, 16,17,22,27,22,28,22,28,34,39,22,35,30, \\
33,33,38,28,38,27,43,46,33,27,55,33 \ldots\}
\end{gathered}
$$

[^47]
## D.3. Quasi-Brieskorn-Pham Surface Singularities by Geometric Genus

Those triples in red indicate an integral homology 3-sphere for the corresponding Brieskorn-Pham 3-manifold.

Table D.1. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=0$

$$
0(\infty) \mid\{1, l, k\} \quad\{2,2, c\} \quad\{2,3,3\} \quad\{2,3,4\} \quad\{2,3,5\}
$$

Table D.2. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=1$

1 (16) | $\{2,3,6\}$ | $\{2,3,7\}$ | $\{2,3,8\}$ | $\{2,3,9\}$ | $\{2,3,10\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,11\}$ | $\{2,4,4\}$ | $\{2,4,5\}$ | $\{2,4,6\}$ | $\{2,4,7\}$ |  |
|  | $\{2,5,5\}$ | $\{2,5,6\}$ | $\{3,3,3\}$ | $\{3,3,4\}$ | $\{3,3,5\}$ |
| $\{3,4,4\}$ |  |  |  |  |  |

Table D.3. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=2$

$2(17) |$| $\{2,3,12\}$ | $\{2,3,13\}$ | $\{2,3,14\}$ | $\{2,3,15\}$ | $\{2,3,16\}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,17\}$ | $\{2,4,8\}$ | $\{2,4,9\}$ | $\{2,4,10\}$ | $\{2,4,11\}$ |
| $\{2,5,7\}$ | $\{2,5,8\}$ | $\{2,5,9\}$ | $\{3,3,6\}$ | $\{3,3,7\}$ |
| $\{3,3,8\}$ | $\{3,4,5\}$ |  |  |  |

Table D.4. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=3$

$3(22) |$| $\{2,3,18\}$ | $\{2,3,19\}$ | $\{2,3,20\}$ | $\{2,3,21\}$ | $\{2,3,22\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,23\}$ | $\{2,4,12\}$ | $\{2,4,13\}$ | $\{2,4,14\}$ | $\{2,4,15\}$ |  |
| $\{2,6,6\}$ | $\{2,6,7\}$ | $\{2,6,8\}$ | $\{2,7,7\}$ | $\{2,7,8\}$ |  |
| $\{3,3,9\}$ | $\{3,3,10\}$ | $\{3,3,11\}$ | $\{3,4,6\}$ | $\{3,4,7\}$ |  |
|  | $\{3,5,5\}$ | $\{3,5,6\}$ |  |  |  |

Table D.5. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=4$

$$
4(27) \left\lvert\, \begin{array}{cccccc}
\{2,3,24\} & \{2,3,25\} & \{2,3,26\} & \{2,3,27\} & \{2,3,28\} \\
& \{2,3,29\} & \{2,4,16\} & \{2,4,17\} & \{2,4,18\} & \{2,4,19\} \\
\{2,5,10\} & \{2,5,11\} & \{2,5,12\} & \{2,5,13\} & \{2,6,9\} \\
\{2,6,10\} & \{2,6,11\} & \{2,7,9\} & \{3,3,12\} & \{3,3,13\}
\end{array}\right.
$$

Table D.6. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=5$

$5(22) |$| $\{2,3,30\}$ | $\{2,3,31\}$ | $\{2,3,32\}$ | $\{2,3,33\}$ | $\{2,3,34\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,35\}$ | $\{2,4,20\}$ | $\{2,4,21\}$ | $\{2,4,22\}$ | $\{2,4,23\}$ |  |
| $\{2,5,14\}$ | $\{2,5,15\}$ | $\{2,5,16\}$ | $\{2,7,10\}$ | $\{2,7,11\}$ |  |
| $\{3,3,15\}$ | $\{3,3,16\}$ | $\{3,3,17\}$ | $\{3,4,10\}$ | $\{3,4,11\}$ |  |
|  | $\{4,4,6\}$ | $\{4,4,7\}$ |  |  |  |

Table D.7. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=6$

$6(28) |$| $\{2,3,36\}$ | $\{2,3,37\}$ | $\{2,3,38\}$ | $\{2,3,39\}$ | $\{2,3,40\}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,41\}$ | $\{2,4,24\}$ | $\{2,4,25\}$ | $\{2,4,26\}$ | $\{2,4,27\}$ |
| $\{2,5,17\}$ | $\{2,5,18\}$ | $\{2,5,19\}$ | $\{2,6,12\}$ | $\{2,6,13\}$ |
| $\{2,6,14\}$ | $\{2,7,12\}$ | $\{2,7,13\}$ | $\{2,8,8\}$ | $\{2,8,9\}$ |
| $\{2,8,10\}$ | $\{2,9,9\}$ | $\{2,9,10\}$ | $\{3,3,18\}$ | $\{3,3,19\}$ |
| $\{3,3,20\}$ | $\{3,5,8\}$ | $\{4,5,6\}$ |  |  |

Table D.8. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=7$

$7(22) |$| $\{2,3,42\}$ | $\{2,3,43\}$ | $\{2,3,44\}$ | $\{2,3,45\}$ | $\{2,3,46\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,47\}$ | $\{2,4,28\}$ | $\{2,4,29\}$ | $\{2,4,30\}$ | $\{2,4,31\}$ |  |
| $\{2,6,15\}$ | $\{2,6,16\}$ | $\{2,6,17\}$ | $\{2,8,11\}$ | $\{3,3,21\}$ |  |
|  | $\{3,3,22\}$ | $\{3,3,23\}$ | $\{3,5,9\}$ | $\{3,5,10\}$ | $\{3,6,6\}$ |
|  | $\{3,6,7\}$ | $\{3,7,7\}$ |  |  |  |

Table D.9. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=8$

8 (28) $|$| $\{2,3,48\}$ | $\{2,3,49\}$ | $\{2,3,50\}$ | $\{2,3,51\}$ | $\{2,3,52\}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\{2,3,53\}$ | $\{2,4,32\}$ | $\{2,4,33\}$ | $\{2,4,34\}$ | $\{2,4,35\}$ |
| $\{2,5,20\}$ | $\{2,5,21\}$ | $\{2,5,22\}$ | $\{2,5,23\}$ | $\{2,8,12\}$ |
| $\{2,8,13\}$ | $\{2,9,11\}$ | $\{3,3,24\}$ | $\{3,3,25\}$ | $\{3,3,26\}$ |
|  | $\{3,4,12\}$ | $\{3,4,13\}$ | $\{3,4,14\}$ | $\{3,5,11\}$ |
|  | $\{3,6,8\}$ |  |  |  |
|  | $\{4,4,8\}$ | $\{4,4,9\}$ | $\{4,5,7\}$ |  |

Table D.10. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=9$

$9(34) |$| $\{2,3,54\}$ | $\{2,3,55\}$ | $\{2,3,56\}$ | $\{2,3,57\}$ | $\{2,3,58\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,59\}$ | $\{2,4,36\}$ | $\{2,4,37\}$ | $\{2,4,38\}$ | $\{2,4,39\}$ |  |
| $\{2,5,24\}$ | $\{2,5,25\}$ | $\{2,5,26\}$ | $\{2,6,18\}$ | $\{2,6,19\}$ |  |
| $\{2,6,20\}$ | $\{2,7,14\}$ | $\{2,7,15\}$ | $\{2,7,16\}$ | $\{2,8,14\}$ |  |
|  | $\{2,8,15\}$ | $\{2,9,12\}$ | $\{3,3,27\}$ | $\{3,3,28\}$ | $\{3,3,29\}$ |
| $\{3,4,15\}$ | $\{3,4,16\}$ | $\{3,5,12\}$ | $\{3,6,9\}$ | $\{3,7,8\}$ |  |
| $\{4,4,10\}$ | $\{4,4,11\}$ | $\{4,5,8\}$ | $\{4,6,6\}$ |  |  |

Table D.11. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=10$

10 (39) | $\{2,3,60\}$ | $\{2,3,61\}$ | $\{2,3,62\}$ | $\{2,3,63\}$ | $\{2,3,64\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,65\}$ | $\{2,4,40\}$ | $\{2,4,41\}$ | $\{2,4,42\}$ | $\{2,4,43\}$ |  |
|  | $\{2,5,27\}$ | $\{2,5,28\}$ | $\{2,5,29\}$ | $\{2,6,21\}$ | $\{2,6,22\}$ |
| $\{2,6,23\}$ | $\{2,7,17\}$ | $\{2,7,18\}$ | $\{2,9,13\}$ | $\{2,9,14\}$ |  |
|  | $\{2,10,10\}$ | $\{2,10,11\}$ | $\{2,10,12\}$ | $\{2,11,11\}$ | $\{2,11,12\}$ |
|  | $\{3,3,30\}$ | $\{3,3,31\}$ | $\{3,3,32\}$ | $\{3,4,17\}$ | $\{3,5,13\}$ |
|  | $\{3,5,14\}$ | $\{3,6,10\}$ | $\{3,6,11\}$ | $\{3,7,9\}$ | $\{4,5,9\}$ |
|  | $\{4,6,7\}$ | $\{5,5,5\}$ | $\{5,5,6\}$ | $\{5,6,6\}$ |  |

Table D.12. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=11$

$11(22) |$| $\{2,3,66\}$ | $\{2,3,67\}$ | $\{2,3,68\}$ | $\{2,3,69\}$ | $\{2,3,70\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,71\}$ | $\{2,4,44\}$ | $\{2,4,45\}$ | $\{2,4,46\}$ | $\{2,4,47\}$ |  |
| $\{2,7,19\}$ | $\{2,9,15\}$ | $\{2,10,13\}$ | $\{3,3,33\}$ | $\{3,3,34\}$ |  |
|  | $\{3,3,35\}$ | $\{3,4,18\}$ | $\{3,4,19\}$ | $\{3,7,10\}$ | $\{3,8,8\}$ |
|  | $\{3,8,9\}$ | $\{5,5,7\}$ |  |  |  |

Table D.13. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=12$

12 (35) | $\{2,3,72\}$ | $\{2,3,73\}$ | $\{2,3,74\}$ | $\{2,3,75\}$ | $\{2,3,76\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\{2,3,77\}$ | $\{2,4,48\}$ | $\{2,4,49\}$ | $\{2,4,50\}$ | $\{2,4,51\}$ |
| $\{2,5,30\}$ | $\{2,5,31\}$ | $\{2,5,32\}$ | $\{2,5,33\}$ | $\{2,6,24\}$ |  |
|  | $\{2,6,25\}$ | $\{2,6,26\}$ | $\{2,7,20\}$ | $\{2,7,21\}$ | $\{2,7,22\}$ |
|  | $\{2,8,16\}$ | $\{2,8,17\}$ | $\{2,8,18\}$ | $\{2,9,16\}$ | $\{2,9,17\}$ |
|  | $\{2,10,14\}$ | $\{2,11,13\}$ | $\{3,3,36\}$ | $\{3,3,37\}$ | $\{3,3,38\}$ |
|  | $\{3,4,20\}$ | $\{3,4,21\}$ | $\{4,4,12\}$ | $\{4,4,13\}$ | $\{4,6,8\}$ |

Table D.14. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=13$

13 (30) $|$| $\{2,3,78\}$ | $\{2,3,79\}$ | $\{2,3,80\}$ | $\{2,3,81\}$ | $\{2,3,82\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,83\}$ | $\{2,4,52\}$ | $\{2,4,53\}$ | $\{2,4,54\}$ | $\{2,4,55\}$ |  |
| $\{2,5,34\}$ | $\{2,5,35\}$ | $\{2,5,36\}$ | $\{2,6,27\}$ | $\{2,6,28\}$ |  |
| $\{2,6,29\}$ | $\{2,7,23\}$ | $\{2,8,19\}$ | $\{2,11,14\}$ | $\{3,3,39\}$ |  |
|  | $\{3,3,40\}$ | $\{3,3,41\}$ | $\{3,4,22\}$ | $\{3,4,23\}$ | $\{4,4,14\}$ |
| $\{4,4,15\}$ | $\{4,5,10\}$ | $\{4,7,7\}$ | $\{4,7,8\}$ | $\{5,5,8\}$ |  |

Table D.15. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=14$

14 (33) | $\{2,3,84\}$ | $\{2,3,85\}$ | $\{2,3,86\}$ | $\{2,3,87\}$ | $\{2,3,88\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\{2,3,89\}$ | $\{2,4,56\}$ | $\{2,4,57\}$ | $\{2,4,58\}$ | $\{2,4,59\}$ |
|  | $\{2,5,37\}$ | $\{2,5,38\}$ | $\{2,5,39\}$ | $\{2,7,24\}$ | $\{2,7,25\}$ |
|  | $\{2,8,20\}$ | $\{2,8,21\}$ | $\{2,10,15\}$ | $\{2,10,16\}$ | $\{3,3,42\}$ |
|  | $\{3,3,43\}$ | $\{3,3,44\}$ | $\{3,5,15\}$ | $\{3,5,16\}$ | $\{3,5,17\}$ |
|  | $\{3,6,12\}$ | $\{3,6,13\}$ | $\{3,7,11\}$ | $\{3,8,10\}$ | $\{4,5,11\}$ |
|  | $\{4,6,9\}$ | $\{5,5,9\}$ | $\{5,6,7\}$ |  |  |

Table D.16. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=15$
15 (33) $\left.\left\lvert\, \begin{array}{ccccc}\{2,3,90\} & \{2,3,91\} & \{2,3,92\} & \{2,3,93\} & \{2,3,94\} \\ & \{2,3,95\} & \{2,4,60\} & \{2,4,61\} & \{2,4,62\}\end{array}\{22,4,63\}\right.\right\}$

Table D.17. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=16$

16 (38) $|$| $\{2,3,96\}$ | $\{2,3,97\}$ | $\{2,3,98\}$ | $\{2,3,99\}$ | $\{2,3,100\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,101\}$ | $\{2,4,64\}$ | $\{2,4,65\}$ | $\{2,4,66\}$ | $\{2,4,67\}$ |  |
| $\{2,5,40\}$ | $\{2,5,41\}$ | $\{2,5,42\}$ | $\{2,5,43\}$ | $\{2,6,33\}$ |  |
|  | $\{2,6,34\}$ | $\{2,6,35\}$ | $\{2,9,18\}$ | $\{2,9,19\}$ | $\{2,9,20\}$ |
| $\{2,10,18\}$ | $\{2,10,19\}$ | $\{2,11,16\}$ | $\{2,11,17\}$ | $\{3,3,48\}$ |  |
|  | $3,3,49\}$ | $\{3,3,50\}$ | $\{3,4,24\}$ | $\{3,4,25\}$ | $\{3,4,26\}$ |
|  | $\{3,5,19\}$ | $\{3,6,15\}$ | $\{3,7,13\}$ | $\{4,4,16\}$ | $\{4,4,17\}$ |
|  | $\{4,5,13\}$ | $\{4,6,11\}$ | $\{4,7,9\}$ |  |  |

Table D.18. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=17$

17 (28) $|$| $\{2,3,102\}$ | $\{2,3,103\}$ | $\{2,3,104\}$ | $\{2,3,105\}$ | $\{2,3,106\}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\{2,3,107\}$ | $\{2,4,68\}$ | $\{2,4,69\}$ | $\{2,4,70\}$ | $\{2,4,71\}$ |
| $\{2,5,44\}$ | $\{2,5,45\}$ | $\{2,5,46\}$ | $\{2,9,21\}$ | $\{2,12,15\}$ |
| $\{3,3,51\}$ | $\{3,3,52\}$ | $\{3,3,53\}$ | $\{3,4,27\}$ | $\{3,4,28\}$ |
| $\{3,5,20\}$ | $\{3,5,21\}$ | $\{3,6,16\}$ | $\{3,6,17\}$ | $\{4,4,18\}$ |
|  | $44,4,19\}$ | $\{5,6,8\}$ | $\{5,7,7\}$ |  |

Table D.19. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=18$

$18(38) |$| $\{2,3,108\}$ | $\{2,3,109\}$ | $\{2,3,110\}$ | $\{2,3,111\}$ | $\{2,3,112\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,113\}$ | $\{2,4,72\}$ | $\{2,4,73\}$ | $\{2,4,74\}$ | $\{2,4,75\}$ |  |
|  | $\{2,5,47\}$ | $\{2,5,48\}$ | $\{2,5,49\}$ | $\{2,6,36\}$ | $\{2,6,37\}$ |
|  | $\{2,6,38\}$ | $\{2,7,28\}$ | $\{2,7,29\}$ | $\{2,7,30\}$ | $\{2,8,24\}$ |
|  | $\{2,8,25\}$ | $\{2,8,26\}$ | $\{2,9,22\}$ | $\{2,9,23\}$ | $\{2,11,18\}$ |
| $\{2,12,16\}$ | $\{2,13,15\}$ | $\{3,3,54\}$ | $\{3,3,55\}$ | $\{3,3,56\}$ |  |
|  | $\{3,4,29\}$ | $\{3,5,22\}$ | $\{3,7,14\}$ | $\{3,7,15\}$ | $\{3,9,9\}$ |
|  | $\{3,9,10\}$ | $\{3,10,10\}$ | $\{4,5,14\}$ |  |  |

Table D.20. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=19$

19 (27) $|$| $\{2,3,114\}$ | $\{2,3,115\}$ | $\{2,3,116\}$ | $\{2,3,117\}$ | $\{2,3,118\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,119\}$ | $\{2,4,76\}$ | $\{2,4,77\}$ | $\{2,4,78\}$ | $\{2,4,79\}$ |  |
| $\{2,6,39\}$ | $\{2,6,40\}$ | $\{2,6,41\}$ | $\{2,7,31\}$ | $\{2,7,32\}$ |  |
| $\{2,8,27\}$ | $\{2,11,19\}$ | $\{2,12,17\}$ | $\{2,13,16\}$ | $\{3,3,57\}$ |  |
| $\{3,3,58\}$ | $\{3,3,59\}$ | $\{3,4,30\}$ | $\{3,4,31\}$ | $\{3,8,12\}$ |  |
|  | $\{3,9,11\}$ | $\{5,6,9\}$ |  |  |  |

Table D.21. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=20$

20 (43) $|$| $\{2,3,120\}$ | $\{2,3,121\}$ | $\{2,3,122\}$ | $\{2,3,123\}$ | $\{2,3,124\}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\{2,3,125\}$ | $\{2,4,80\}$ | $\{2,4,81\}$ | $\{2,4,82\}$ | $\{2,4,83\}$ |
| $\{2,5,50\}$ | $\{2,5,51\}$ | $\{2,5,52\}$ | $\{2,5,53\}$ | $\{2,7,33\}$ |
| $\{2,8,28\}$ | $\{2,8,29\}$ | $\{2,9,24\}$ | $\{2,9,25\}$ | $\{2,10,20\}$ |
| $\{2,10,21\}$ | $\{2,10,22\}$ | $\{2,11,20\}$ | $\{2,11,21\}$ | $\{2,13,17\}$ |
| $\{3,3,60\}$ | $\{3,3,61\}$ | $\{3,3,62\}$ | $\{3,4,32\}$ | $\{3,4,33\}$ |
|  | $\{3,5,23\}$ | $\{3,8,13\}$ | $\{4,4,20\}$ | $\{4,4,21\}$ |
| $\{4,5,15\}$ |  |  |  |  |
|  | $\{4,5,16\}$ | $\{4,7,10\}$ | $\{5,5,10\}$ | $\{5,5,11\}$ |
|  | $\{6,6,6\}$ | $\{6,6,7\}$ | $\{6,7,7\}$ |  |

Table D.22. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=21$

$21(46) |$| $\{2,3,126\}$ | $\{2,3,127\}$ | $\{2,3,128\}$ | $\{2,3,129\}$ | $\{2,3,130\}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\{2,3,131\}$ | $\{2,4,84\}$ | $\{2,4,85\}$ | $\{2,4,86\}$ | $\{2,4,87\}$ |
| $\{2,5,54\}$ | $\{2,5,55\}$ | $\{2,5,56\}$ | $\{2,6,42\}$ | $\{2,6,43\}$ |
| $\{2,6,44\}$ | $\{2,7,34\}$ | $\{2,7,35\}$ | $\{2,7,36\}$ | $\{2,8,30\}$ |
| $\{2,8,31\}$ | $\{2,10,23\}$ | $\{2,12,18\}$ | $\{2,12,19\}$ | $\{2,14,14\}$ |
| $\{2,14,15\}$ | $\{2,14,16\}$ | $\{2,15,15\}$ | $\{2,15,16\}$ | $\{3,3,63\}$ |
| $\{3,3,64\}$ | $\{3,3,65\}$ | $\{3,4,34\}$ | $\{3,4,35\}$ | $\{3,5,24\}$ |
| $\{3,5,25\}$ | $\{3,6,18\}$ | $\{3,6,19\}$ | $\{3,7,16\}$ | $\{3,8,14\}$ |
| $\{3,9,12\}$ | $\{3,10,11\}$ | $\{4,4,22\}$ | $\{4,4,23\}$ | $\{4,7,11\}$ |
|  | $\{5,5,12\}$ |  |  |  |

Table D.23. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=22$

22 (33) $|$| $\{2,3,132\}$ | $\{2,3,133\}$ | $\{2,3,134\}$ | $\{2,3,135\}$ | $\{2,3,136\}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\{2,3,137\}$ | $\{2,4,88\}$ | $\{2,4,89\}$ | $\{2,4,90\}$ | $\{2,4,91\}$ |
| $\{2,5,57\}$ | $\{2,5,58\}$ | $\{2,5,59\}$ | $\{2,6,45\}$ | $\{2,6,46\}$ |
| $\{2,6,47\}$ | $\{2,7,37\}$ | $\{2,9,26\}$ | $\{2,9,27\}$ | $\{2,9,28\}$ |
| $\{2,10,24\}$ | $\{2,13,18\}$ | $\{3,3,66\}$ | $\{3,3,67\}$ | $\{3,3,68\}$ |
| $\{3,5,26\}$ | $\{3,6,20\}$ | $\{3,7,17\}$ | $\{3,9,13\}$ | $\{4,5,17\}$ |
|  | $\{4,8,8\}$ | $\{4,8,9\}$ | $\{4,9,9\}$ |  |

Table D.24. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=23$

$23(27) |$| $\{2,3,138\}$ | $\{2,3,139\}$ | $\{2,3,140\}$ | $\{2,3,141\}$ | $\{2,3,142\}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\{2,3,143\}$ | $\{2,4,92\}$ | $\{2,4,93\}$ | $\{2,4,94\}$ | $\{2,4,95\}$ |
| $\{2,7,38\}$ | $\{2,7,39\}$ | $\{2,12,20\}$ | $\{2,14,17\}$ | $\{3,3,69\}$ |
| $\{3,3,70\}$ | $\{3,3,71\}$ | $\{3,5,27\}$ | $\{3,6,21\}$ | $\{3,7,18\}$ |
| $\{3,10,12\}$ | $\{4,5,18\}$ | $\{4,6,12\}$ | $\{4,6,13\}$ | $\{5,5,13\}$ |
|  | $\{5,6,10\}$ | $\{6,6,8\}$ |  |  |

Table D.25. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=24$

| 24 (55) | \{2,3,144\} | \{2, 3,145$\}$ | \{2, 3, 146\} | $\{2,3,147\}$ | $\{2,3,148\}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | \{2,3,149\} | \{2, 4,96$\}$ | $\{2,4,97\}$ | \{2,4,98\} | \{2,4,99\} |
|  | $\{2,5,60\}$ | \{2,5,61\} | \{2,5,62\} | $\{2,5,63\}$ | \{2,6,48\} |
|  | \{2, 6, 49\} | $\{2,6,50\}$ | \{2, 7, 40\} | $\{2,7,41\}$ | \{2, 8, 32\} |
|  | \{2, 8, 33 \} | \{2, 8, 34\} | $\{2,9,29\}$ | \{2, 10, 25 \} | \{2,10,26\} |
|  | $\{2,12,21\}$ | \{2, 13, 19\} | \{2, 13, 20 $\}$ | $\{2,14,18\}$ | $\{2,15,17\}$ |
|  | $\{3,3,72\}$ | \{3,3,73\} | \{3,3,74\} | \{3, 4, 36\} | $\{3,4,37\}$ |
|  | $\{3,4,38\}$ | \{3, 5, 28\} | $\{3,5,29\}$ | $\{3,6,22\}$ | $\{3,6,23\}$ |
|  | $\{3,7,19\}$ | \{3, 8,15$\}$ | $\{3,8,16\}$ | \{3, 11, 11 $\}$ | $\{3,11,12\}$ |
|  | \{4, 4, 24\} | \{4, 4, 25\} | $\{4,5,19\}$ | $\{4,6,14\}$ | $\{4,7,12\}$ |
|  | $\{4,8,10\}$ | \{5,5,14\} | $\{5,6,11\}$ | $\{5,7,9\}$ |  |
|  | $\{5,8,8\}$ |  |  |  |  |

Table D.26. Quasi-Brieskorn-Pham Surface Singularities with $p_{g}=25$

25 (33) $|$| $\{2,3,150\}$ | $\{2,3,151\}$ | $\{2,3,152\}$ | $\{2,3,153\}$ | $\{2,3,154\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,3,155\}$ | $\{2,4,100\}$ | $\{2,4,101\}$ | $\{2,4,102\}$ | $\{2,4,103\}$ |  |
| $\{2,5,64\}$ | $\{2,5,65\}$ | $\{2,5,66\}$ | $\{2,6,51\}$ | $\{2,6,52\}$ |  |
| $\{2,6,53\}$ | $\{2,8,35\}$ | $\{2,9,30\}$ | $\{2,10,27\}$ | $\{2,11,22\}$ |  |
| $\{2,11,23\}$ | $\{2,11,24\}$ | $\{2,12,22\}$ | $\{2,12,23\}$ | $\{3,3,75\}$ |  |
| $\{3,3,76\}$ | $\{3,3,77\}$ | $\{3,4,39\}$ | $\{3,4,40\}$ | $\{3,7,20\}$ |  |
|  | $\{3,9,14\}$ | $\{4,4,26\}$ | $\{4,4,27\}$ |  |  |

## D.4. Quasi-Brieskorn-Pham Surface Singularities by Milnor Number

Proposition D.2. The number of equivalence classes of quasi-Brieskorn-Pham surface singularities with Milnor number $\mu$ is the backward difference

$$
\begin{equation*}
\nabla \sum_{i=1}^{\mu} \sum_{j=i}^{\lfloor\mu / i\rfloor} \sum_{k=j}^{\lfloor\mu /(i j)\rfloor} 1 \tag{D.5}
\end{equation*}
$$

Proof. The number of factorizations of $n$ into a product of three positive integers is the backward difference

$$
\begin{equation*}
\nabla \sum_{i=1}^{n} \sum_{j=i}^{\lfloor n / i\rfloor} \sum_{k=j}^{\lfloor n /(i j)\rfloor} 1 \tag{D.6}
\end{equation*}
$$

which can be proved by properly counting the number of positive integral solutions of the equation $n=i j k$, which is equal to that of the inequality $n \leqslant i j k$ minus that of $n-1 \leqslant i j k$. The difference above now follows.

In this section, we compile those quasi-Brieskorn-Pham surface singularities by Milnor number less than or equal to 300.

Table D.27. Quasi-Brieskorn-Pham Surface Singularities with $\mu=0$

$$
0(\infty) \mid\{1, l, k\}
$$

Table D.28. Quasi-Brieskorn-Pham Surface Singularities with $\mu=1$
$1(1) \mid\{2,2,2\}$

Table D.29. Quasi-Brieskorn-Pham Surface Singularities with $\mu=p$ Prime
Prime $p(1) \mid\{2,2, p+1\}$

Table D.30. Quasi-Brieskorn-Pham Surface Singularities with $\mu=4$

$$
4(2) \mid\{2,2,5\} \quad\{2,3,3\}
$$

Table D.31. Quasi-Brieskorn-Pham Surface Singularities with $\mu=6$

$$
6(2) \mid\{2,2,7\} \quad\{2,3,4\}
$$

Table D.32. Quasi-Brieskorn-Pham Surface Singularities with $\mu=8$

$$
8(3) \mid\{2,2,9\} \quad\{2,3,5\} \quad\{3,3,3\}
$$

Table D.33. Quasi-Brieskorn-Pham Surface Singularities with $\mu=9$

$$
9(2) \mid\{2,2,10\} \quad\{2,4,4\}
$$

Table D.34. Quasi-Brieskorn-Pham Surface Singularities with $\mu=10$

$$
10(2) \mid\{2,2,11\} \quad\{2,3,6\}
$$

Table D.35. Quasi-Brieskorn-Pham Surface Singularities with $\mu=12$

$$
12(4) \mid\{2,2,13\} \quad\{2,3,7\} \quad\{2,4,5\} \quad\{3,3,4\}
$$

Table D.36. Quasi-Brieskorn-Pham Surface Singularities with $\mu=14$
$14(2) \mid\{2,2,15\} \quad\{2,3,8\}$

Table D.37. Quasi-Brieskorn-Pham Surface Singularities with $\mu=15$
$15(2) \mid\{2,2,16\} \quad\{2,4,6\}$

Table D.38. Quasi-Brieskorn-Pham Surface Singularities with $\mu=16$

$$
16(4) \mid\{2,2,17\} \quad\{2,3,9\} \quad\{2,5,5\} \quad\{3,3,5\}
$$

Table D.39. Quasi-Brieskorn-Pham Surface Singularities with $\mu=18$ $18(4) \mid\{2,2,19\} \quad\{2,3,10\} \quad\{2,4,7\} \quad\{3,4,4\}$

Table D.40. Quasi-Brieskorn-Pham Surface Singularities with $\mu=20$

$$
20(4) \mid\{2,2,21\} \quad\{2,3,11\} \quad\{2,5,6\} \quad\{3,3,6\}
$$

Table D.41. Quasi-Brieskorn-Pham Surface Singularities with $\mu=21$
$21(2) \mid\{2,2,22\} \quad\{2,4,8\}$

Table D.42. Quasi-Brieskorn-Pham Surface Singularities with $\mu=22$

$$
22(2) \mid\{2,2,23\} \quad\{2,3,12\}
$$

Table D.43. Quasi-Brieskorn-Pham Surface Singularities with $\mu=24$

$$
24(6) \left\lvert\, \begin{aligned}
& \{2,2,25\} \\
& \{3,4,5\}
\end{aligned} \quad\{2,3,13\} \quad\{2,4,9\} \quad\{2,5,7\} \quad\{3,3,7\}\right.
$$

Table D.44. Quasi-Brieskorn-Pham Surface Singularities with $\mu=25$
$25(2) \mid\{2,2,26\} \quad\{2,6,6\}$

Table D.45. Quasi-Brieskorn-Pham Surface Singularities with $\mu=26$

$$
26(2) \mid\{2,2,27\} \quad\{2,3,14\}
$$

Table D.46. Quasi-Brieskorn-Pham Surface Singularities with $\mu=27$

$$
27(3) \mid\{2,2,28\} \quad\{2,4,10\} \quad\{4,4,4\}
$$

Table D.47. Quasi-Brieskorn-Pham Surface Singularities with $\mu=28$

$$
28(4) \mid\{2,2,29\} \quad\{2,3,15\} \quad\{2,5,8\} \quad\{3,3,8\}
$$

Table D.48. Quasi-Brieskorn-Pham Surface Singularities with $\mu=30$

$$
30(5) \mid\{2,2,31\} \quad\{2,3,16\} \quad\{2,4,11\} \quad\{2,6,7\} \quad\{3,4,6\}
$$

Table D.49. Quasi-Brieskorn-Pham Surface Singularities with $\mu=32$

$$
32(5) \mid\{2,2,33\} \quad\{2,3,17\} \quad\{2,5,9\} \quad\{3,3,9\} \quad\{3,5,5\}
$$

Table D.50. Quasi-Brieskorn-Pham Surface Singularities with $\mu=33$

$$
33(2) \mid\{2,2,34\} \quad\{2,4,12\}
$$

Table D.51. Quasi-Brieskorn-Pham Surface Singularities with $\mu=34$ $34(2) \mid\{2,2,35\} \quad\{2,3,18\}$

Table D.52. Quasi-Brieskorn-Pham Surface Singularities with $\mu=35$
$35(2) \mid\{2,2,36\} \quad\{2,6,8\}$

Table D.53. Quasi-Brieskorn-Pham Surface Singularities with $\mu=36$

$$
36(8) \left\lvert\, \begin{array}{ccccc}
\{2,2,37\} & \{2,3,19\} & \{2,4,13\} & \{2,5,10\} & \{2,7,7\} \\
& \{3,3,10\} & \{3,4,7\} & \{4,4,5\} &
\end{array}\right.
$$

Table D.54. Quasi-Brieskorn-Pham Surface Singularities with $\mu=38$

$$
38(2) \mid\{2,2,39\} \quad\{2,3,20\}
$$

Table D.55. Quasi-Brieskorn-Pham Surface Singularities with $\mu=39$

$$
39(2) \mid\{2,2,40\} \quad\{2,4,14\}
$$

Table D.56. Quasi-Brieskorn-Pham Surface Singularities with $\mu=40$

$$
40(6) \left\lvert\, \begin{gathered}
\{2,2,41\} \\
\{3,5,6\}
\end{gathered} \quad\{2,3,21\} \quad\{2,5,11\} \quad\{2,6,9\} \quad\{3,3,11\}\right.
$$

Table D.57. Quasi-Brieskorn-Pham Surface Singularities with $\mu=42$

$$
42(5) \mid\{2,2,43\} \quad\{2,3,22\} \quad\{2,4,15\} \quad\{2,7,8\} \quad\{3,4,8\}
$$

Table D.58. Quasi-Brieskorn-Pham Surface Singularities with $\mu=44$ $44(4) \mid\{2,2,45\} \quad\{2,3,23\} \quad\{2,5,12\} \quad\{3,3,12\}$

Table D.59. Quasi-Brieskorn-Pham Surface Singularities with $\mu=45$

$$
45(4) \mid\{2,2,46\} \quad\{2,4,16\} \quad\{2,6,10\} \quad\{4,4,6\}
$$

Table D.60. Quasi-Brieskorn-Pham Surface Singularities with $\mu=46$ $46(2) \mid\{2,2,47\} \quad\{2,3,24\}$

Table D.61. Quasi-Brieskorn-Pham Surface Singularities with $\mu=48$

$$
48(9) \left\lvert\, \begin{array}{ccccc}
\{2,2,49\} & \{2,3,25\} & \{2,4,17\} & \{2,5,13\} & \{2,7,9\} \\
& \{3,3,13\} & \{3,4,9\} & \{3,5,7\} & \{4,5,5\} \\
\end{array}\right.
$$

Table D.62. Quasi-Brieskorn-Pham Surface Singularities with $\mu=49$
$49(2) \mid\{2,2,50\} \quad\{2,8,8\}$

Table D.63. Quasi-Brieskorn-Pham Surface Singularities with $\mu=50$

$$
50(4) \mid\{2,2,51\} \quad\{2,3,26\} \quad\{2,6,11\} \quad\{3,6,6\}
$$

Table D.64. Quasi-Brieskorn-Pham Surface Singularities with $\mu=51$

$$
51(2) \mid\{2,2,52\} \quad\{2,4,18\}
$$

Table D.65. Quasi-Brieskorn-Pham Surface Singularities with $\mu=52$

$$
52(4) \mid\{2,2,53\} \quad\{2,3,27\} \quad\{2,5,14\} \quad\{3,3,14\}
$$

Table D.66. Quasi-Brieskorn-Pham Surface Singularities with $\mu=54$
$54(6) \mid\{2,2,55\} \quad\{2,3,28\} \quad\{2,4,19\} \quad\{2,7,10\} \quad\{3,4,10\}$ $\{4,4,7\}$

Table D.67. Quasi-Brieskorn-Pham Surface Singularities with $\mu=55$
$55(2) \mid\{2,2,56\} \quad\{2,6,12\}$

Table D.68. Quasi-Brieskorn-Pham Surface Singularities with $\mu=56$

$$
56(6) \left\lvert\, \begin{gathered}
\{2,2,57\} \\
\{3,5,8\}
\end{gathered} \quad\{2,3,29\} \quad\{2,5,15\} \quad\{2,8,9\} \quad\{3,3,15\}\right.
$$

Table D.69. Quasi-Brieskorn-Pham Surface Singularities with $\mu=57$

$$
57(2) \mid\{2,2,58\} \quad\{2,4,20\}
$$

Table D.70. Quasi-Brieskorn-Pham Surface Singularities with $\mu=58$ $58(2) \mid\{2,2,59\} \quad\{2,3,30\}$

Table D.71. Quasi-Brieskorn-Pham Surface Singularities with $\mu=60$

$$
60 \text { (10) } \begin{array}{cccccc}
\{2,2,61\} & \{2,3,31\} & \{2,4,21\} & \{2,5,16\} & \{2,6,13\} \\
& \{2,7,11\} & \{3,3,16\} & \{3,4,11\} & \{3,6,7\} & \{4,5,6\}
\end{array}
$$

Table D.72. Quasi-Brieskorn-Pham Surface Singularities with $\mu=62$ $62(2) \mid\{2,2,63\} \quad\{2,3,32\}$

Table D.73. Quasi-Brieskorn-Pham Surface Singularities with $\mu=63$ $63(4) \mid\{2,2,64\} \quad\{2,4,22\} \quad\{2,8,10\} \quad\{4,4,8\}$

Table D.74. Quasi-Brieskorn-Pham Surface Singularities with $\mu=64$

$$
64(7) \left\lvert\, \begin{array}{ccccc}
\{2,2,65\} & \{2,3,33\} & \{2,5,17\} & \{2,9,9\} & \{3,3,17\} \\
\{3,5,9\} & \{5,5,5\}
\end{array}\right.
$$

Table D.75. Quasi-Brieskorn-Pham Surface Singularities with $\mu=65$ $65(2) \mid\{2,2,66\} \quad\{2,6,14\}$

Table D.76. Quasi-Brieskorn-Pham Surface Singularities with $\mu=66$ $66(5) \mid\{2,2,67\} \quad\{2,3,34\} \quad\{2,4,23\} \quad\{2,7,12\} \quad\{3,4,12\}$

Table D.77. Quasi-Brieskorn-Pham Surface Singularities with $\mu=68$ $68(4) \mid\{2,2,69\} \quad\{2,3,35\} \quad\{2,5,18\} \quad\{3,3,18\}$

Table D.78. Quasi-Brieskorn-Pham Surface Singularities with $\mu=69$ $69(2) \mid\{2,2,70\} \quad\{2,4,24\}$

Table D.79. Quasi-Brieskorn-Pham Surface Singularities with $\mu=70$ $70(5) \mid\{2,2,71\} \quad\{2,3,36\} \quad\{2,6,15\} \quad\{2,8,11\} \quad\{3,6,8\}$

Table D.8o. Quasi-Brieskorn-Pham Surface Singularities with $\mu=72$

$$
72 \text { (12) } \begin{array}{cccccc}
\{2,2,73\} & \{2,3,37\} & \{2,4,25\} & \{2,5,19\} & \{2,7,13\} \\
& \{2,9,10\} & \{3,3,19\} & \{3,4,13\} & \{3,5,10\} & \{3,7,7\} \\
& \{4,4,9\} & \{4,5,7\} & & &
\end{array}
$$

Table D.81. Quasi-Brieskorn-Pham Surface Singularities with $\mu=74$

$$
74(2) \mid\{2,2,75\} \quad\{2,3,38\}
$$

Table D.82. Quasi-Brieskorn-Pham Surface Singularities with $\mu=75$

$$
75(4) \mid\{2,2,76\} \quad\{2,4,26\} \quad\{2,6,16\} \quad\{4,6,6\}
$$

Table D.83. Quasi-Brieskorn-Pham Surface Singularities with $\mu=76$

$$
76(4) \mid\{2,2,77\} \quad\{2,3,39\} \quad\{2,5,20\} \quad\{3,3,20\}
$$

Table D.84. Quasi-Brieskorn-Pham Surface Singularities with $\mu=77$

$$
77(2) \mid\{2,2,78\} \quad\{2,8,12\}
$$

Table D.85. Quasi-Brieskorn-Pham Surface Singularities with $\mu=78$
$78(5) \mid\{2,2,79\} \quad\{2,3,40\} \quad\{2,4,27\} \quad\{2,7,14\} \quad\{3,4,14\}$

Table D.86. Quasi-Brieskorn-Pham Surface Singularities with $\mu=80$

$$
80(9) \left\lvert\, \begin{array}{ccccc}
\{2,2,81\} & \{2,3,41\} & \{2,5,21\} & \{2,6,17\} & \{2,9,11\} \\
& \{3,3,21\} & \{3,5,11\} & \{3,6,9\} & \{5,5,6\}
\end{array}\right.
$$

Table D.87. Quasi-Brieskorn-Pham Surface Singularities with $\mu=81$

$$
81(4) \mid\{2,2,82\} \quad\{2,4,28\} \quad\{2,10,10\} \quad\{4,4,10\}
$$

Table D.88. Quasi-Brieskorn-Pham Surface Singularities with $\mu=82$ $82(2) \mid\{2,2,83\} \quad\{2,3,42\}$

Table D.89. Quasi-Brieskorn-Pham Surface Singularities with $\mu=84$

$$
84 \text { (10) } \begin{array}{cccccc}
\{2,2,85\} & \{2,3,43\} & \{2,4,29\} & \{2,5,22\} & \{2,7,15\} \\
& \{2,8,13\} & \{3,3,22\} & \{3,4,15\} & \{3,7,8\} & \{4,5,8\}
\end{array}
$$

Table D.90. Quasi-Brieskorn-Pham Surface Singularities with $\mu=85$

$$
85(2) \mid\{2,2,86\} \quad\{2,6,18\}
$$

Table D.91. Quasi-Brieskorn-Pham Surface Singularities with $\mu=86$ $86(2) \mid\{2,2,87\} \quad\{2,3,44\}$

Table D.92. Quasi-Brieskorn-Pham Surface Singularities with $\mu=87$ $87(2) \mid\{2,2,88\} \quad\{2,4,30\}$

Table D.93. Quasi-Brieskorn-Pham Surface Singularities with $\mu=88$
$88(6) \mid\{2,2,89\}$
$\{2,3,45\}$
$\{2,5,23\}$
$\{2,9,12\} \quad\{3,3,23\}$ $\{3,5,12\}$

Table D.94. Quasi-Brieskorn-Pham Surface Singularities with $\mu=90$

$$
90 \text { (10) } \begin{array}{cccccc}
\{2,2,91\} & \{2,3,46\} & \{2,4,31\} & \{2,6,19\} & \{2,7,16\} \\
\{2,10,11\} & \{3,4,16\} & \{3,6,10\} & \{4,4,11\} & \{4,6,7\}
\end{array}
$$

Table D.95. Quasi-Brieskorn-Pham Surface Singularities with $\mu=91$

$$
91(2) \mid\{2,2,92\} \quad\{2,8,14\}
$$

Table D.96. Quasi-Brieskorn-Pham Surface Singularities with $\mu=92$
$92(4) \mid\{2,2,93\} \quad\{2,3,47\} \quad\{2,5,24\} \quad\{3,3,24\}$

Table D.97. Quasi-Brieskorn-Pham Surface Singularities with $\mu=93$

$$
93(2) \mid\{2,2,94\} \quad\{2,4,32\}
$$

Table D.98. Quasi-Brieskorn-Pham Surface Singularities with $\mu=94$

$$
94(2) \mid\{2,2,95\} \quad\{2,3,48\}
$$

Table D.99. Quasi-Brieskorn-Pham Surface Singularities with $\mu=95$

$$
95(2) \mid\{2,2,96\} \quad\{2,6,20\}
$$

Table D.100. Quasi-Brieskorn-Pham Surface Singularities with $\mu=96$

$$
96 \text { (12) } \begin{array}{cccccc}
\{2,2,97\} & \{2,3,49\} & \{2,4,33\} & \{2,5,25\} & \{2,7,17\} \\
& \{2,9,13\} & \{3,3,25\} & \{3,4,17\} & \{3,5,13\} & \{3,7,9\} \\
& \{4,5,9\} & \{5,5,7\} & & &
\end{array}
$$

Table D.101. Quasi-Brieskorn-Pham Surface Singularities with $\mu=98$

$$
98(4) \mid\{2,2,99\} \quad\{2,3,50\} \quad\{2,8,15\} \quad\{3,8,8\}
$$

Table D.102. Quasi-Brieskorn-Pham Surface Singularities with $\mu=99$

$$
99(4) \mid\{2,2,100\} \quad\{2,4,34\} \quad\{2,10,12\} \quad\{4,4,12\}
$$

Table D.103. Quasi-Brieskorn-Pham Surface Singularities with $\mu=100$

$$
100(8) \left\lvert\, \begin{array}{ccccc}
\{2,2,101\} & \{2,3,51\} & \{2,5,26\} & \{2,6,21\} & \{2,11,11\} \\
& \{3,3,26\} & \{3,6,11\} & \{5,6,6\} &
\end{array}\right.
$$

Table D.104. Quasi-Brieskorn-Pham Surface Singularities with $\mu=102$

$$
102(5) \mid\{2,2,103\} \quad\{2,3,52\} \quad\{2,4,35\} \quad\{2,7,18\} \quad\{3,4,18\}
$$

Table D.105. Quasi-Brieskorn-Pham Surface Singularities with $\mu=104$
$104(6) \mid\{2,2,105\} \quad\{2,3,53\} \quad\{2,5,27\} \quad\{2,9,14\} \quad\{3,3,27\}$ $\{3,5,14\}$

Table D.106. Quasi-Brieskorn-Pham Surface Singularities with $\mu=105$
$105(5) \mid\{2,2,106\} \quad\{2,4,36\} \quad\{2,6,22\} \quad\{2,8,16\} \quad\{4,6,8\}$

Table D.107. Quasi-Brieskorn-Pham Surface Singularities with $\mu=106$

$$
106(2) \mid\{2,2,107\} \quad\{2,3,54\}
$$

Table D.108. Quasi-Brieskorn-Pham Surface Singularities with $\mu=108$

$$
108 \text { (12) } \begin{array}{cccccc}
\{2,2,109\} & \{2,3,55\} & \{2,4,37\} & \{2,5,28\} & \{2,7,19\} \\
& \{2,10,13\} & \{3,3,28\} & \{3,4,19\} & \{3,7,10\} & \{4,4,13\} \\
& \{4,5,10\} & \{4,7,7\} & & &
\end{array}
$$

Table D.109. Quasi-Brieskorn-Pham Surface Singularities with $\mu=110$ $110(5) \mid\{2,2,111\} \quad\{2,3,56\} \quad\{2,6,23\} \quad\{2,11,12\} \quad\{3,6,12\}$

Table D.110. Quasi-Brieskorn-Pham Surface Singularities with $\mu=111$

$$
111(2) \mid\{2,2,112\} \quad\{2,4,38\}
$$

Table D.111. Quasi-Brieskorn-Pham Surface Singularities with $\mu=112$

$112(9) |$| $\{2,2,113\}$ | $\{2,3,57\}$ | $\{2,5,29\}$ | $\{2,8,17\}$ | $\{2,9,15\}$ |
| :---: | :---: | :---: | :---: | :---: |
|  | $\{3,3,29\}$ | $\{3,5,15\}$ | $\{3,8,9\}$ | $\{5,5,8\}$ |

Table D.112. Quasi-Brieskorn-Pham Surface Singularities with $\mu=114$
$114(5) \mid\{2,2,115\} \quad\{2,3,58\}$
$\{2,4,39\} \quad\{2$
$\{2,7,20\}$
$\{3,4,20\}$

Table D.113. Quasi-Brieskorn-Pham Surface Singularities with $\mu=115$

$$
115(2) \mid\{2,2,116\} \quad\{2,6,24\}
$$

Table D.114. Quasi-Brieskorn-Pham Surface Singularities with $\mu=116$ $116(4) \mid\{2,2,117\} \quad\{2,3,59\} \quad\{2,5,30\} \quad\{3,3,30\}$

Table D.115. Quasi-Brieskorn-Pham Surface Singularities with $\mu=117$
$117(4) \mid\{2,2,118\} \quad\{2,4,40\} \quad\{2,10,14\} \quad\{4,4,14\}$

Table D.116. Quasi-Brieskorn-Pham Surface Singularities with $\mu=118$

$$
118(2) \mid\{2,2,119\} \quad\{2,3,60\}
$$

Table D.117. Quasi-Brieskorn-Pham Surface Singularities with $\mu=119$

$$
119(2) \mid\{2,2,120\} \quad\{2,8,18\}
$$

Table D.118. Quasi-Brieskorn-Pham Surface Singularities with $\mu=120$

120 (16) $|$| $\{2,2,121\}$ | $\{2,3,61\}$ | $\{2,4,41\}$ | $\{2,5,31\}$ | $\{2,6,25\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,7,21\}$ | $\{2,9,16\}$ | $\{2,11,13\}$ | $\{3,3,31\}$ | $\{3,4,21\}$ |  |
| $\{3,5,16\}$ | $\{3,6,13\}$ | $\{3,7,11\}$ | $\{4,5,11\}$ | $\{4,6,9\}$ |  |
|  | $\{5,6,7\}$ |  |  |  |  |

Table D.119. Quasi-Brieskorn-Pham Surface Singularities with $\mu=121$ $121(2) \mid\{2,2,122\} \quad\{2,12,12\}$

Table D.120. Quasi-Brieskorn-Pham Surface Singularities with $\mu=122$

$$
122(2) \mid\{2,2,123\} \quad\{2,3,62\}
$$

Table D.121. Quasi-Brieskorn-Pham Surface Singularities with $\mu=123$

$$
123(2) \mid\{2,2,124\} \quad\{2,4,42\}
$$

Table D.122. Quasi-Brieskorn-Pham Surface Singularities with $\mu=124$

$$
124(4) \mid\{2,2,125\} \quad\{2,3,63\} \quad\{2,5,32\} \quad\{3,3,32\}
$$

Table D.123. Quasi-Brieskorn-Pham Surface Singularities with $\mu=125$

$$
125(3) \mid\{2,2,126\} \quad\{2,6,26\} \quad\{6,6,6\}
$$

Table D.124. Quasi-Brieskorn-Pham Surface Singularities with $\mu=126$
126 (10)
$\begin{array}{ccccc}\{2,2,127\} & \{2,3,64\} & \{2,4,43\} & \{2,7,22\} & \{2,8,19\} \\ \{2,10,15\} & \{3,4,22\} & \{3,8,10\} & \{4,4,15\} & \{4,7,8\}\end{array}$

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Table D.125. Quasi-Brieskorn-Pham Surface Singularities with $\mu=128$
$128(8) \mid\{2,2,129\} \quad\{2,3,65\} \quad\{2,5,33\} \quad\{2,9,17\} \quad\{3,3,33\}$ $\{3,5,17\} \quad\{3,9,9\} \quad\{5,5,9\}$

Table D.126. Quasi-Brieskorn-Pham Surface Singularities with $\mu=129$

$$
129(2) \mid\{2,2,130\} \quad\{2,4,44\}
$$

Table D.127. Quasi-Brieskorn-Pham Surface Singularities with $\mu=130$

$$
130(5) \mid\{2,2,131\} \quad\{2,3,66\} \quad\{2,6,27\} \quad\{2,11,14\} \quad\{3,6,14\}
$$

Table D.128. Quasi-Brieskorn-Pham Surface Singularities with $\mu=132$

$$
132 \text { (10) } \begin{array}{llllll}
\{2,2,133\} & \{2,3,67\} & \{2,4,45\} & \{2,5,34\} & \{2,7,23\} \\
& \{2,12,13\} & \{3,3,34\} & \{3,4,23\} & \{3,7,12\} & \{4,5,12\}
\end{array}
$$

Table D.129. Quasi-Brieskorn-Pham Surface Singularities with $\mu=133$

$$
133(2) \mid\{2,2,134\} \quad\{2,8,20\}
$$

Table D.130. Quasi-Brieskorn-Pham Surface Singularities with $\mu=134$

$$
134(2) \mid\{2,2,135\} \quad\{2,3,68\}
$$

Table D.131. Quasi-Brieskorn-Pham Surface Singularities with $\mu=135$
135 (6)
(6) $\mid\{2,2,136\}$
$\{4,6,10\}$

Table D.132. Quasi-Brieskorn-Pham Surface Singularities with $\mu=136$

$136(6) \left\lvert\,$| $\{2,2,137\}$ |
| :---: |
| $\{3,5,18\}$ |$\quad\{2,3,69\} \quad\{2,5,35\} \quad\{2,9,18\} \quad\{3,3,35\}\right.$

Table D.133. Quasi-Brieskorn-Pham Surface Singularities with $\mu=138$

$$
138(5) \mid\{2,2,139\} \quad\{2,3,70\} \quad\{2,4,47\} \quad\{2,7,24\} \quad\{3,4,24\}
$$

Table D.134. Quasi-Brieskorn-Pham Surface Singularities with $\mu=140$

$$
140 \text { (10) } \begin{array}{cccccc}
\{2,2,141\} & \{2,3,71\} & \{2,5,36\} & \{2,6,29\} & \{2,8,21\} \\
& \{2,11,15\} & \{3,3,36\} & \{3,6,15\} & \{3,8,11\} & \{5,6,8\}
\end{array}
$$

Table D.135. Quasi-Brieskorn-Pham Surface Singularities with $\mu=141$

$$
141(2) \mid\{2,2,142\} \quad\{2,4,48\}
$$

Table D.136. Quasi-Brieskorn-Pham Surface Singularities with $\mu=142$

$$
142(2) \mid\{2,2,143\} \quad\{2,3,72\}
$$

Table D.137. Quasi-Brieskorn-Pham Surface Singularities with $\mu=143$
$143(2) \mid\{2,2,144\} \quad\{2,12,14\}$

Table D.138. Quasi-Brieskorn-Pham Surface Singularities with $\mu=144$

$144(18) |$| $\{2,2,145\}$ | $\{2,3,73\}$ | $\{2,4,49\}$ | $\{2,5,37\}$ | $\{2,7,25\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\{2,9,19\}$ | $\{2,10,17\}$ | $\{2,13,13\}$ | $\{3,3,37\}$ | $\{3,4,25\}$ |
| $\{3,5,19\}$ | $\{3,7,13\}$ | $\{3,9,10\}$ | $\{4,4,17\}$ | $\{4,5,13\}$ |  |
|  | $\{4,7,9\}$ | $\{5,5,10\}$ | $\{5,7,7\}$ |  |  |

Table D.139. Quasi-Brieskorn-Pham Surface Singularities with $\mu=145$

$$
145(2) \mid\{2,2,146\} \quad\{2,6,30\}
$$

Table D.140. Quasi-Brieskorn-Pham Surface Singularities with $\mu=146$ $146(2) \mid\{2,2,147\} \quad\{2,3,74\}$

Table D.141. Quasi-Brieskorn-Pham Surface Singularities with $\mu=147$
$147(4) \mid\{2,2,148\} \quad\{2,4,50\} \quad\{2,8,22\} \quad\{4,8,8\}$

Table D.142. Quasi-Brieskorn-Pham Surface Singularities with $\mu=148$ $148(4) \mid\{2,2,149\} \quad\{2,3,75\} \quad\{2,5,38\} \quad\{3,3,38\}$

Table D.143. Quasi-Brieskorn-Pham Surface Singularities with $\mu=150$

$$
150 \text { (10) } \begin{array}{cccccc}
\{2,2,151\} & \{2,3,76\} & \{2,4,51\} & \{2,6,31\} & \{2,7,26\} \\
& \{2,11,16\} & \{3,4,26\} & \{3,6,16\} & \{4,6,11\} & \{6,6,7\}
\end{array}
$$

Table D.144. Quasi-Brieskorn-Pham Surface Singularities with $\mu=150$

$$
150 \text { (10) } \begin{array}{cccccc}
\{2,2,151\} & \{2,3,76\} & \{2,4,51\} & \{2,6,31\} & \{2,7,26\} \\
& \{2,11,16\} & \{3,4,26\} & \{3,6,16\} & \{4,6,11\} & \{6,6,7\}
\end{array}
$$

Table D.145. Quasi-Brieskorn-Pham Surface Singularities with $\mu=152$

$$
152(6) \left\lvert\, \begin{gathered}
\{2,2,153\} \\
\{3,5,20\}
\end{gathered} \quad\{2,3,77\} \quad\{2,5,39\} \quad\{2,9,20\} \quad\{3,3,39\}\right.
$$

Table D.146. Quasi-Brieskorn-Pham Surface Singularities with $\mu=153$

$$
153(4) \mid\{2,2,154\} \quad\{2,4,52\} \quad\{2,10,18\} \quad\{4,4,18\}
$$

Table D.147. Quasi-Brieskorn-Pham Surface Singularities with $\mu=154$ $154(5) \mid\{2,2,155\} \quad\{2,3,78\} \quad\{2,8,23\} \quad\{2,12,15\} \quad\{3,8,12\}$

Table D.148. Quasi-Brieskorn-Pham Surface Singularities with $\mu=155$

$$
155(2) \mid\{2,2,156\} \quad\{2,6,32\}
$$

Table D.149. Quasi-Brieskorn-Pham Surface Singularities with $\mu=156$
156 (10) $\mid\{2,2,157\} \quad\{2,3,79\} \quad\{2,4,53\} \quad\{2,5,40\} \quad\{2,7,27\}$
$\{2,13,14\} \quad\{3,3,40\} \quad\{3,4,27\} \quad\{3,7,14\} \quad\{4,5,14\}$

Table D.150. Quasi-Brieskorn-Pham Surface Singularities with $\mu=158$ $158(2) \mid\{2,2,159\} \quad\{2,3,80\}$

Table D.151. Quasi-Brieskorn-Pham Surface Singularities with $\mu=159$

$$
159(2) \mid\{2,2,160\} \quad\{2,4,54\}
$$

Table D.152. Quasi-Brieskorn-Pham Surface Singularities with $\mu=160$

$$
160 \text { (12) } \begin{array}{cccccc}
\{2,2,161\} & \{2,3,81\} & \{2,5,41\} & \{2,6,33\} & \{2,9,21\} \\
& \{2,11,17\} & \{3,3,41\} & \{3,5,21\} & \{3,6,17\} & \{3,9,11\} \\
& \{5,5,11\} & \{5,6,9\} & & &
\end{array}
$$

Table D.153. Quasi-Brieskorn-Pham Surface Singularities with $\mu=161$ $161(2) \mid\{2,2,162\} \quad\{2,8,24\}$

Table D.154. Quasi-Brieskorn-Pham Surface Singularities with $\mu=162$

$$
162(9) \left\lvert\, \begin{array}{ccccc}
\{2,2,163\} & \{2,3,82\} & \{2,4,55\} & \{2,7,28\} & \{2,10,19\} \\
\{3,4,28\} & \{3,10,10\} & \{4,4,19\} & \{4,7,10\} &
\end{array}\right.
$$

Table D.155. Quasi-Brieskorn-Pham Surface Singularities with $\mu=164$

$$
164(4) \mid\{2,2,165\} \quad\{2,3,83\} \quad\{2,5,42\} \quad\{3,3,42\}
$$

Table D.156. Quasi-Brieskorn-Pham Surface Singularities with $\mu=165$
$165(5) \mid\{2,2,166\} \quad\{2,4,56\} \quad\{2,6,34\} \quad\{2,12,16\} \quad\{4,6,12\}$

Table D.157. Quasi-Brieskorn-Pham Surface Singularities with $\mu=166$

$$
166(2) \mid\{2,2,167\} \quad\{2,3,84\}
$$

Table D.158. Quasi-Brieskorn-Pham Surface Singularities with $\mu=168$

$$
168 \text { (16) } \left\lvert\, \begin{array}{ccccc}
\{2,2,169\} & \{2,3,85\} & \{2,4,57\} & \{2,5,43\} & \{2,7,29\} \\
\{2,8,25\} & \{2,9,22\} & \{2,13,15\} & \{3,3,43\} & \{3,4,29\} \\
\{3,5,22\} & \{3,7,15\} & \{3,8,13\} & \{4,5,15\} & \{4,8,9\} \\
\{5,7,8\} & & & &
\end{array}\right.
$$

Table D.159. Quasi-Brieskorn-Pham Surface Singularities with $\mu=169$ $169(2) \mid\{2,2,170\} \quad\{2,14,14\}$

Table D.160. Quasi-Brieskorn-Pham Surface Singularities with $\mu=170$
$170(5) \mid\{2,2,171\}$
$\{2,3,86\}$
$\{2,6,35\}$
$\{2,11,18\} \quad\{3,6,18\}$

Table D.161. Quasi-Brieskorn-Pham Surface Singularities with $\mu=171$
$171(4) \mid\{2,2,172\} \quad\{2,4,58\} \quad\{2,10,20\} \quad\{4,4,20\}$

Table D.162. Quasi-Brieskorn-Pham Surface Singularities with $\mu=172$ $172(4) \mid\{2,2,173\} \quad\{2,3,87\} \quad\{2,5,44\} \quad\{3,3,44\}$

Table D.163. Quasi-Brieskorn-Pham Surface Singularities with $\mu=174$
$174(5) \mid\{2,2,175\}$
$\{2,3,88\}$
$\{2,4,59\}$
$\{2,7,30\}$
$\{3,4,30\}$

Table D.164. Quasi-Brieskorn-Pham Surface Singularities with $\mu=175$
$175(4) \mid\{2,2,176\} \quad\{2,6,36\} \quad\{2,8,26\} \quad\{6,6,8\}$

Table D.165. Quasi-Brieskorn-Pham Surface Singularities with $\mu=176$

$$
176(9) \left\lvert\, \begin{array}{ccccc}
\{2,2,177\} & \{2,3,89\} & \{2,5,45\} & \{2,9,23\} & \{2,12,17\} \\
& \{3,3,45\} & \{3,5,23\} & \{3,9,12\} & \{5,5,12\}
\end{array}\right.
$$

Table D.166. Quasi-Brieskorn-Pham Surface Singularities with $\mu=177$
$177(2) \mid\{2,2,178\} \quad\{2,4,60\}$

Table D.167. Quasi-Brieskorn-Pham Surface Singularities with $\mu=178$

$$
178(2) \mid\{2,2,179\} \quad\{2,3,90\}
$$

Table D.168. Quasi-Brieskorn-Pham Surface Singularities with $\mu=180$

180 (20) $|$| $\{2,2,181\}$ | $\{2,3,91\}$ | $\{2,4,61\}$ | $\{2,5,46\}$ | $\{2,6,37\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\{2,7,31\}$ | $\{2,10,21\}$ | $\{2,11,19\}$ | $\{2,13,16\}$ | $\{3,3,46\}$ |
| $\{3,4,31\}$ | $\{3,6,19\}$ | $\{3,7,16\}$ | $\{3,10,11\}$ | $\{4,4,21\}$ |  |
|  | $\{4,5,16\}$ | $\{4,6,13\}$ | $\{4,7,11\}$ | $\{5,6,10\}$ | $\{6,7,7\}$ |

Table D.169. Quasi-Brieskorn-Pham Surface Singularities with $\mu=182$

$$
182(5) \mid\{2,2,183\} \quad\{2,3,92\} \quad\{2,8,27\} \quad\{2,14,15\} \quad\{3,8,14\}
$$

Table D.170. Quasi-Brieskorn-Pham Surface Singularities with $\mu=183$ $183(2) \mid\{2,2,184\} \quad\{2,4,62\}$

Table D.171. Quasi-Brieskorn-Pham Surface Singularities with $\mu=184$
$184(6) \mid\{2,2,185\}$
$\{2,3,93\}$
$\{2,5,47\}$
$\{2,9,24\} \quad\{3,3,47\}$ $\{3,5,24\}$

Table D.172. Quasi-Brieskorn-Pham Surface Singularities with $\mu=185$ $185(2) \mid\{2,2,186\} \quad\{2,6,38\}$

Table D.173. Quasi-Brieskorn-Pham Surface Singularities with $\mu=186$

$$
186(5) \mid\{2,2,187\} \quad\{2,3,94\} \quad\{2,4,63\} \quad\{2,7,32\} \quad\{3,4,32\}
$$

Table D.174. Quasi-Brieskorn-Pham Surface Singularities with $\mu=187$ $187(2) \mid\{2,2,188\} \quad\{2,12,18\}$

Table D.175. Quasi-Brieskorn-Pham Surface Singularities with $\mu=188$

$$
188(4) \mid\{2,2,189\} \quad\{2,3,95\} \quad\{2,5,48\} \quad\{3,3,48\}
$$

Table D.176. Quasi-Brieskorn-Pham Surface Singularities with $\mu=189$
$189(6) \mid\{2,2,190\}$
$\{4,8,10\}$

Table D.177. Quasi-Brieskorn-Pham Surface Singularities with $\mu=190$

$$
190(5) \mid\{2,2,191\} \quad\{2,3,96\} \quad\{2,6,39\} \quad\{2,11,20\} \quad\{3,6,20\}
$$

Table D.178. Quasi-Brieskorn-Pham Surface Singularities with $\mu=192$

192 (16) $|$| $\{2,2,193\}$ | $\{2,3,97\}$ | $\{2,4,65\}$ | $\{2,5,49\}$ | $\{2,7,33\}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\{2,9,25\}$ | $\{2,13,17\}$ | $\{3,3,49\}$ | $\{3,4,33\}$ | $\{3,5,25\}$ |
| $\{3,7,17\}$ | $\{3,9,13\}$ | $\{4,5,17\}$ | $\{4,9,9\}$ | $\{5,5,13\}$ |
|  | $\{5,7,9\}$ |  |  |  |

Table D.179. Quasi-Brieskorn-Pham Surface Singularities with $\mu=194$

$$
194(2) \mid\{2,2,195\} \quad\{2,3,98\}
$$

Table D.180. Quasi-Brieskorn-Pham Surface Singularities with $\mu=195$
$195(5) \mid\{2,2,196\}$
$\{2,4,66\}$
$\{2,6,40\}$
$\{2,14,16\} \quad\{4,6,14\}$

Table D.181. Quasi-Brieskorn-Pham Surface Singularities with $\mu=196$

$$
196(8) \left\lvert\, \begin{array}{ccccc}
\{2,2,197\} & \{2,3,99\} & \{2,5,50\} & \{2,8,29\} & \{2,15,15\} \\
& \{3,3,50\} & \{3,8,15\} & \{5,8,8\} &
\end{array}\right.
$$

Table D.182. Quasi-Brieskorn-Pham Surface Singularities with $\mu=198$
198 (10) $\left.\left\lvert\, \begin{array}{ccccc}\{2,2,199\} & \{2,3,100\} & \{2,4,67\} & \{2,7,34\} & \{2,10,23\} \\ & \{2,12,19\} & \{3,4,34\} & \{3,10,12\} & \{4,4,23\}\end{array}\right.\right\}\{4,7,12\}$

Table D.183. Quasi-Brieskorn-Pham Surface Singularities with $\mu=200$

$$
200 \text { (12) } \begin{array}{cccccc}
\{2,2,201\} & \{2,3,101\} & \{2,5,51\} & \{2,6,41\} & \{2,9,26\} \\
& \{2,11,21\} & \{3,3,51\} & \{3,5,26\} & \{3,6,21\} & \{3,11,11\} \\
& \{5,6,11\} & \{6,6,9\} & & &
\end{array}
$$

Table D.184. Quasi-Brieskorn-Pham Surface Singularities with $\mu=201$

$$
201(2) \mid\{2,2,202\} \quad\{2,4,68\}
$$

Table D.185. Quasi-Brieskorn-Pham Surface Singularities with $\mu=202$ $202(2) \mid\{2,2,203\} \quad\{2,3,102\}$

Table D.186. Quasi-Brieskorn-Pham Surface Singularities with $\mu=203$ $203(2) \mid\{2,2,204\} \quad\{2,8,30\}$

Table D.187. Quasi-Brieskorn-Pham Surface Singularities with $\mu=204$

$$
\left.\left.204 \text { (10) } \left\lvert\, \begin{array}{ccccc}
\{2,2,205\} & \{2,3,103\} & \{2,4,69\} & \{2,5,52\} & \{2,7,35\} \\
& \{2,13,18\} & \{3,3,52\} & \{3,4,35\} & \{3,7,18\}
\end{array}\right.\right\} 4,5,18\right\}
$$

Table D.188. Quasi-Brieskorn-Pham Surface Singularities with $\mu=205$

$$
205(2) \mid\{2,2,206\} \quad\{2,6,42\}
$$

Table D.189. Quasi-Brieskorn-Pham Surface Singularities with $\mu=206$

$$
206(2) \mid\{2,2,207\} \quad\{2,3,104\}
$$

Table D.190. Quasi-Brieskorn-Pham Surface Singularities with $\mu=207$
$207(4) \mid\{2,2,208\}$
$\{2,4,70\}$
$\{2,10,24\}$
$\{4,4,24\}$

Table D.191. Quasi-Brieskorn-Pham Surface Singularities with $\mu=208$
208 (9) $\begin{array}{cccc}\{2,2,209\} & \{2,3,105\} & \{2,5,53\} & \{2,9,27\} \\ \{3,3,53\} & \{3,5,27\} & \{3,9,14\} & \{5,5,14\}\end{array}$

Table D.192. Quasi-Brieskorn-Pham Surface Singularities with $\mu=209$ $209(2) \mid\{2,2,210\} \quad\{2,12,20\}$

Table D.193. Quasi-Brieskorn-Pham Surface Singularities with $\mu=210$

$$
210 \text { (14) } \begin{array}{cccccc}
\{2,2,211\} & \{2,3,106\} & \{2,4,71\} & \{2,6,43\} & \{2,7,36\} \\
& \{2,8,31\} & \{2,11,22\} & \{2,15,16\} & \{3,4,36\} & \{3,6,22\} \\
& \{3,8,16\} & \{4,6,15\} & \{4,8,11\} & \{6,7,8\} &
\end{array}
$$

Table D.194. Quasi-Brieskorn-Pham Surface Singularities with $\mu=212$

$$
212(4) \mid\{2,2,213\} \quad\{2,3,107\} \quad\{2,5,54\} \quad\{3,3,54\}
$$

Table D.195. Quasi-Brieskorn-Pham Surface Singularities with $\mu=213$

$$
213(2) \mid\{2,2,214\} \quad\{2,4,72\}
$$

Table D.196. Quasi-Brieskorn-Pham Surface Singularities with $\mu=214$ $214(2) \mid\{2,2,215\} \quad\{2,3,108\}$

Table D.197. Quasi-Brieskorn-Pham Surface Singularities with $\mu=215$

$$
215(2) \mid\{2,2,216\} \quad\{2,6,44\}
$$

Table D.198. Quasi-Brieskorn-Pham Surface Singularities with $\mu=216$

216 (19) | $\{2,2,217\}$ | $\{2,3,109\}$ | $\{2,4,73\}$ | $\{2,5,55\}$ | $\{2,7,37\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\{2,9,28\}$ | $\{2,10,25\}$ | $\{2,13,19\}$ | $\{3,3,55\}$ | $\{3,4,37\}$ |
|  | $\{3,5,28\}$ | $\{3,7,19\}$ | $\{3,10,13\}$ | $\{4,4,25\}$ | $\{4,5,19\}$ |
|  | $\{4,7,13\}$ | $\{4,9,10\}$ | $\{5,7,10\}$ | $\{7,7,7\}$ |  |

Table D.199. Quasi-Brieskorn-Pham Surface Singularities with $\mu=217$

$$
217(2) \mid\{2,2,218\} \quad\{2,8,32\}
$$

Table D.200. Quasi-Brieskorn-Pham Surface Singularities with $\mu=218$ $218(2) \mid\{2,2,219\} \quad\{2,3,110\}$

Table D.201. Quasi-Brieskorn-Pham Surface Singularities with $\mu=219$

$$
219(2) \mid\{2,2,220\} \quad\{2,4,74\}
$$

Table D.202. Quasi-Brieskorn-Pham Surface Singularities with $\mu=220$

$$
220 \text { (10) } \left\lvert\, \begin{array}{cccccc}
\{2,2,221\} & \{2,3,111\} & \{2,5,56\} & \{2,6,45\} & \{2,11,23\} \\
& \{2,12,21\} & \{3,3,56\} & \{3,6,23\} & \{3,11,12\} & \{5,6,12\}
\end{array}\right.
$$

Table D.203. Quasi-Brieskorn-Pham Surface Singularities with $\mu=221$

$$
221(2) \mid\{2,2,222\} \quad\{2,14,18\}
$$

Table D.204. Quasi-Brieskorn-Pham Surface Singularities with $\mu=222$
$222(5) \mid\{2,2,223\} \quad\{2,3,112\} \quad\{2,4,75\} \quad\{2,7,38\} \quad\{3,4,38\}$

Table D.205. Quasi-Brieskorn-Pham Surface Singularities with $\mu=224$

224 (12) | $\{2,2,225\}$ | $\{2,3,113\}$ | $\{2,5,57\}$ | $\{2,8,33\}$ | $\{2,9,29\}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,15,17\}$ | $\{3,3,57\}$ | $\{3,5,29\}$ | $\{3,8,17\}$ | $\{3,9,15\}$ |
| $\{5,5,15\}$ | $\{5,8,9\}$ |  |  |  |

Table D.206. Quasi-Brieskorn-Pham Surface Singularities with $\mu=225$

$225(8) |$| $\{2,2,226\}$ | $\{2,4,76\}$ | $\{2,6,46\}$ | $\{2,10,26\}$ | $\{2,16,16\}$ |
| :---: | :---: | :---: | :---: | :---: |
|  | $\{4,4,26\}$ | $\{4,6,16\}$ | $\{6,6,10\}$ |  |

Table D.207. Quasi-Brieskorn-Pham Surface Singularities with $\mu=226$

$$
226(2) \mid\{2,2,227\} \quad\{2,3,114\}
$$

Table D.208. Quasi-Brieskorn-Pham Surface Singularities with $\mu=228$

$$
228 \text { (10) } \left\lvert\, \begin{array}{ccccc}
\{2,2,229\} & \{2,3,115\} & \{2,4,77\} & \{2,5,58\} & \{2,7,39\} \\
& \{2,13,20\} & \{3,3,58\} & \{3,4,39\} & \{3,7,20\}
\end{array}\{4,5,20\}\right.
$$

Table D.209. Quasi-Brieskorn-Pham Surface Singularities with $\mu=230$

$$
230(5) \mid\{2,2,231\} \quad\{2,3,116\} \quad\{2,6,47\} \quad\{2,11,24\} \quad\{3,6,24\}
$$

Table D.210. Quasi-Brieskorn-Pham Surface Singularities with $\mu=231$
$231(5) \mid\{2,2,232\}$
$\{2,4,78\}$
$\{2,8,34\}$
$\{2,12,22\} \quad\{4,8,12\}$

Table D.211. Quasi-Brieskorn-Pham Surface Singularities with $\mu=231$
232 (6)
$\{2,2,233\}$
$\{2,3,117\}$
$\{2,5,59\} \quad\{2,9,30\}$
$\{3,3,59\}$ $\{3,5,30\}$

Table D.212. Quasi-Brieskorn-Pham Surface Singularities with $\mu=234$

$$
234 \text { (10) } \begin{array}{cccccc}
\{2,2,235\} & \{2,3,118\} & \{2,4,79\} & \{2,7,40\} & \{2,10,27\} \\
& \{2,14,19\} & \{3,4,40\} & \{3,10,14\} & \{4,4,27\} & \{4,7,14\}
\end{array}
$$

Table D.213. Quasi-Brieskorn-Pham Surface Singularities with $\mu=235$

$$
235(2) \mid\{2,2,236\} \quad\{2,6,48\}
$$

Table D.214. Quasi-Brieskorn-Pham Surface Singularities with $\mu=236$

$$
236(4) \mid\{2,2,237\} \quad\{2,3,119\} \quad\{2,5,60\} \quad\{3,3,60\}
$$

Table D.215. Quasi-Brieskorn-Pham Surface Singularities with $\mu=237$

$$
237(2) \mid\{2,2,238\} \quad\{2,4,80\}
$$

Table D.216. Quasi-Brieskorn-Pham Surface Singularities with $\mu=238$
$238(5) \mid\{2,2,239\} \quad\{2,3,120\}$
$\{2,8,35\}$
$\{2,15,18\}$
$\{3,8,18\}$

Table D.217. Quasi-Brieskorn-Pham Surface Singularities with $\mu=240$

$$
240(24) \left\lvert\, \begin{array}{cccccc}
\{2,2,241\} & \{2,3,121\} & \{2,4,81\} & \{2,5,61\} & \{2,6,49\} \\
\{2,7,41\} & \{2,9,31\} & \{2,11,25\} & \{2,13,21\} & \{2,16,17\} \\
\{3,3,61\} & \{3,4,41\} & \{3,5,31\} & \{3,6,25\} & \{3,7,21\} \\
\{3,9,16\} & \{3,11,13\} & \{4,5,21\} & \{4,6,17\} & \{4,9,11\} \\
\{5,5,16\} & \{5,6,13\} & \{5,7,11\} & \{6,7,9\} &
\end{array}\right.
$$

Table D.218. Quasi-Brieskorn-Pham Surface Singularities with $\mu=242$

$$
242(4) \mid\{2,2,243\} \quad\{2,3,122\} \quad\{2,12,23\} \quad\{3,12,12\}
$$

Table D.219. Quasi-Brieskorn-Pham Surface Singularities with $\mu=243$
$243(5) \mid\{2,2,244\}$
$\{2,4,82\}$
$\{2,10,28$
$\{4,4,28\}$
$\{4,10,10\}$

Table D.220. Quasi-Brieskorn-Pham Surface Singularities with $\mu=244$

$$
244(4) \mid\{2,2,245\} \quad\{2,3,123\} \quad\{2,5,62\} \quad\{3,3,62\}
$$

Table D.221. Quasi-Brieskorn-Pham Surface Singularities with $\mu=245$

$$
245(4) \mid\{2,2,246\} \quad\{2,6,50\} \quad\{2,8,36\} \quad\{6,8,8\}
$$

Table D.222. Quasi-Brieskorn-Pham Surface Singularities with $\mu=246$ $246(5) \mid\{2,2,247\} \quad\{2,3,124\} \quad\{2,4,83\} \quad\{2,7,42\} \quad\{3,4,42\}$

Table D.223. Quasi-Brieskorn-Pham Surface Singularities with $\mu=247$ $247(2) \mid\{2,2,248\} \quad\{2,14,20\}$

Table D.224. Quasi-Brieskorn-Pham Surface Singularities with $\mu=248$
248 (6)
(6) $\begin{gathered}\{2,2,249\} \\ \{3,5,32\}\end{gathered}$

Table D.225. Quasi-Brieskorn-Pham Surface Singularities with $\mu=249$

$$
249(2) \mid\{2,2,250\} \quad\{2,4,84\}
$$

Table D.226. Quasi-Brieskorn-Pham Surface Singularities with $\mu=250$

$250(6) \left\lvert\,$| $\{2,2,251\}$ |
| :--- |
| $\{6,6,11\}$ |$\quad\{2,3,126\} \quad\{2,6,51\} \quad\{2,11,26\} \quad\{3,6,26\}\right.$

Table D.227. Quasi-Brieskorn-Pham Surface Singularities with $\mu=252$

$252(20) |$| $\{2,2,253\}$ | $\{2,3,127\}$ | $\{2,4,85\}$ | $\{2,5,64\}$ | $\{2,7,43\}$ |
| :---: | :---: | :---: | :---: | :---: | :---: |
| $\{2,8,37\}$ | $\{2,10,29\}$ | $\{2,13,22\}$ | $\{2,15,19\}$ | $\{3,3,64\}$ |
| $\{3,4,43\}$ | $\{3,7,22\}$ | $\{3,8,19\}$ | $\{3,10,15\}$ | $\{4,4,29\}$ |
| $\{4,5,22\}$ | $\{4,7,15\}$ | $\{4,8,13\}$ | $\{5,8,10\}$ | $\{7,7,8\}$ |

Table D.228. Quasi-Brieskorn-Pham Surface Singularities with $\mu=253$
$253(2) \mid\{2,2,254\} \quad\{2,12,24\}$

Table D.229. Quasi-Brieskorn-Pham Surface Singularities with $\mu=254$

$$
254(2) \mid\{2,2,255\} \quad\{2,3,128\}
$$

Table D.230. Quasi-Brieskorn-Pham Surface Singularities with $\mu=255$

$$
255(5) \mid\{2,2,256\} \quad\{2,4,86\} \quad\{2,6,52\} \quad\{2,16,18\} \quad\{4,6,18\}
$$

Table D.231. Quasi-Brieskorn-Pham Surface Singularities with $\mu=256$
256 (10)
$\{2,2,257\} \quad\{2,3,129\}$
$\{2,5,65\}$
$\{2,9,33\}$
$\{2,17,17\}$
$\{3,3,65\} \quad\{3,5,33\} \quad\{3,9,17\} \quad\{5,5,17\} \quad\{5,9,9\}$

Table D.232. Quasi-Brieskorn-Pham Surface Singularities with $\mu=258$
$258(5) \mid\{2,2,259\}$
$\{2,3,130\}$
$\{2,4,87\}$
$\{2,7,44\}$
$\{3,4,44\}$

Table D.233. Quasi-Brieskorn-Pham Surface Singularities with $\mu=259$

$$
259(2) \mid\{2,2,260\} \quad\{2,8,38\}
$$

Table D.234. Quasi-Brieskorn-Pham Surface Singularities with $\mu=260$

$$
\left.\left.260(10) \left\lvert\, \begin{array}{ccccc}
\{2,2,261\} & \{2,3,131\} & \{2,5,66\} & \{2,6,53\} & \{2,11,27\} \\
& \{2,14,21\} & \{3,3,66\} & \{3,6,27\} & \{3,11,14\}
\end{array}\right.\right\} 5,6,14\right\}
$$

Table D.235. Quasi-Brieskorn-Pham Surface Singularities with $\mu=261$

$$
261(4) \mid\{2,2,262\} \quad\{2,4,88\} \quad\{2,10,30\} \quad\{4,4,30\}
$$

Table D.236. Quasi-Brieskorn-Pham Surface Singularities with $\mu=262$ $262(2) \mid\{2,2,263\} \quad\{2,3,132\}$

Table D.237. Quasi-Brieskorn-Pham Surface Singularities with $\mu=264$

$264(16) |$| $\{2,2,265\}$ | $\{2,3,133\}$ | $\{2,4,89\}$ | $\{2,5,67\}$ | $\{2,7,45\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\{2,9,34\}$ | $\{2,12,25\}$ | $\{2,13,23\}$ | $\{3,3,67\}$ | $\{3,4,45\}$ |
|  | $\{3,5,34\}$ | $\{3,7,23\}$ | $\{3,12,13\}$ | $\{4,5,23\}$ | $\{4,9,12\}$ |
|  | $\{5,7,12\}$ |  |  |  |  |

Table D.238. Quasi-Brieskorn-Pham Surface Singularities with $\mu=265$

$$
265(2) \mid\{2,2,266\} \quad\{2,6,54\}
$$

Table D.239. Quasi-Brieskorn-Pham Surface Singularities with $\mu=266$
$266(5) \mid\{2,2,267\}$
$\{2,3,134\}$
$\{2,8,39\} \quad\{$
$\{2,15,20\} \quad\{3,8,20\}$

Table D.240. Quasi-Brieskorn-Pham Surface Singularities with $\mu=267$

$$
267(2) \mid\{2,2,268\} \quad\{2,4,90\}
$$

Table D.241. Quasi-Brieskorn-Pham Surface Singularities with $\mu=268$

$$
268(4) \mid\{2,2,269\} \quad\{2,3,135\} \quad\{2,5,68\} \quad\{3,3,68\}
$$

Table D.242. Quasi-Brieskorn-Pham Surface Singularities with $\mu=270$

$270(16) |$| $\{2,2,271\}$ | $\{2,3,136\}$ | $\{2,4,91\}$ | $\{2,6,55\}$ | $\{2,7,46\}$ |
| :---: | :---: | :---: | :---: | :---: |
| $\{2,10,31\}$ | $\{2,11,28\}$ | $\{2,16,19\}$ | $\{3,4,46\}$ | $\{3,6,28\}$ |
|  | $\{3,10,16\}$ | $\{4,4,31\}$ | $\{4,6,19\}$ | $\{4,7,16\}$ |
| $\{4,10,11\}$ |  |  |  |  |

Table D.243. Quasi-Brieskorn-Pham Surface Singularities with $\mu=272$

$$
272(9) \left\lvert\, \begin{array}{cccccc}
\{2,2,273\} & \{2,3,137\} & \{2,5,69\} & \{2,9,35\} & \{2,17,18\} \\
& \{3,3,69\} & \{3,5,35\} & \{3,9,18\} & \{5,5,18\} &
\end{array}\right.
$$

Table D.244. Quasi-Brieskorn-Pham Surface Singularities with $\mu=273$

$$
273(5) \mid\{2,2,274\} \quad\{2,4,92\} \quad\{2,8,40\} \quad\{2,14,22\} \quad\{4,8,14\}
$$

Table D.245. Quasi-Brieskorn-Pham Surface Singularities with $\mu=274$
$274(2) \mid\{2,2,275\} \quad\{2,3,138\}$

Table D.246. Quasi-Brieskorn-Pham Surface Singularities with $\mu=275$

$$
275(4) \mid\{2,2,276\} \quad\{2,6,56\} \quad\{2,12,26\} \quad\{6,6,12\}
$$

Table D.247. Quasi-Brieskorn-Pham Surface Singularities with $\mu=276$

$$
\left.\left.276 \text { (10) } \left\lvert\, \begin{array}{ccccc}
\{2,2,277\} & \{2,3,139\} & \{2,4,93\} & \{2,5,70\} & \{2,7,47\} \\
& \{2,13,24\} & \{3,3,70\} & \{3,4,47\} & \{3,7,24\}
\end{array}\right.\right\} 4,5,24\right\}
$$

Table D.248. Quasi-Brieskorn-Pham Surface Singularities with $\mu=278$ $278(2) \mid\{2,2,279\} \quad\{2,3,140\}$

Table D.249. Quasi-Brieskorn-Pham Surface Singularities with $\mu=279$

$$
279(4) \mid\{2,2,280\} \quad\{2,4,94\} \quad\{2,10,32\} \quad\{4,4,32\}
$$

Table D.250. Quasi-Brieskorn-Pham Surface Singularities with $\mu=280$

$$
280 \text { (16) } \left\lvert\, \begin{array}{cccccc}
\{2,2,281\} & \{2,3,141\} & \{2,5,71\} & \{2,6,57\} & \{2,8,41\} \\
\{2,9,36\} & \{2,11,29\} & \{2,15,21\} & \{3,3,71\} & \{3,5,36\} \\
\{3,6,29\} & \{3,8,21\} & \{3,11,15\} & \{5,6,15\} & \{5,8,11\} \\
\{6,8,9\} & & & &
\end{array}\right.
$$

1000

Table D.251. Quasi-Brieskorn-Pham Surface Singularities with $\mu=282$
$282(5) \mid\{2,2,283\}$
$\{2,3,142\}$
$\{2,4,95\}$
$\{2,7,48\} \quad\{3,4,48\}$

1001

Table D.252. Quasi-Brieskorn-Pham Surface Singularities with $\mu=284$

$$
284(4) \mid\{2,2,285\} \quad\{2,3,143\} \quad\{2,5,72\} \quad\{3,3,72\}
$$

Table D.253. Quasi-Brieskorn-Pham Surface Singularities with $\mu=285$
$285(5) \mid\{2,2,286\}$
$\{2,4,96\}$
$\{2,6,58\}$
$\{2,16,20\} \quad\{4,6,20\}$

1003

Table D.254. Quasi-Brieskorn-Pham Surface Singularities with $\mu=286$ $286(5) \mid\{2,2,287\} \quad\{2,3,144\} \quad\{2,12,27\} \quad\{2,14,23\} \quad\{3,12,14\}$

1004

Table D.255. Quasi-Brieskorn-Pham Surface Singularities with $\mu=287$

$$
287(2) \mid\{2,2,288\} \quad\{2,8,42\}
$$

Table D.256. Quasi-Brieskorn-Pham Surface Singularities with $\mu=288$

$288(24) |$| $\{2,2,289\}$ | $\{2,3,145\}$ | $\{2,4,97\}$ | $\{2,5,73\}$ | $\{2,7,49\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\{2,9,37\}$ | $\{2,10,33\}$ | $\{2,13,25\}$ | $\{2,17,19\}$ | $\{3,3,73\}$ |
|  | $\{3,4,49\}$ | $\{3,5,37\}$ | $\{3,7,25\}$ | $\{3,9,19\}$ | $\{3,10,17\}$ |
|  | $\{3,13,13\}$ | $\{4,4,33\}$ | $\{4,5,25\}$ | $\{4,7,17\}$ | $\{4,9,13\}$ |
|  | $\{5,5,19\}$ | $\{5,7,13\}$ | $\{5,9,10\}$ | $\{7,7,9\}$ |  |

1006

Table D.257. Quasi-Brieskorn-Pham Surface Singularities with $\mu=289$ $289(2) \mid\{2,2,290\} \quad\{2,18,18\}$

Table D.258. Quasi-Brieskorn-Pham Surface Singularities with $\mu=290$

$$
290(5) \mid\{2,2,291\} \quad\{2,3,146\} \quad\{2,6,59\} \quad\{2,11,30\} \quad\{3,6,30\}
$$

1008

Table D.259. Quasi-Brieskorn-Pham Surface Singularities with $\mu=291$

$$
291(2) \mid\{2,2,292\} \quad\{2,4,98\}
$$

Table D.260. Quasi-Brieskorn-Pham Surface Singularities with $\mu=292$

$$
292(4) \mid\{2,2,293\} \quad\{2,3,147\} \quad\{2,5,74\} \quad\{3,3,74\}
$$

1010

Table D.261. Quasi-Brieskorn-Pham Surface Singularities with $\mu=294$

$$
294(10) \left\lvert\, \begin{array}{ccccc}
\{2,2,295\} & \{2,3,148\} & \{2,4,99\} & \{2,7,50\} & \{2,8,43\} \\
\{2,15,22\} & \{3,4,50\} & \{3,8,22\} & \{4,8,15\} & \{7,8,8\}
\end{array}\right.
$$

Table D.262. Quasi-Brieskorn-Pham Surface Singularities with $\mu=295$

$$
295(2) \mid\{2,2,296\} \quad\{2,6,60\}
$$

Table D.263. Quasi-Brieskorn-Pham Surface Singularities with $\mu=296$
296 (6)
$\{2,3,149\}$
$\{2,5,75\}$
$\{2,9,38\} \quad\{3,3,75\}$
$\{3,5,38\}$

1013

Table D.264. Quasi-Brieskorn-Pham Surface Singularities with $\mu=297$
297 (6)
(6) $\begin{aligned} & \{2,2,298\} \\ & \{4,10,12\}\end{aligned}$
$\{2,4,100\} \quad\{2,10,34\} \quad\{2,12,28\} \quad\{4,4,34\}$

Table D.265. Quasi-Brieskorn-Pham Surface Singularities with $\mu=298$

$$
298(2) \mid\{2,2,299\} \quad\{2,3,150\}
$$

Table D.266. Quasi-Brieskorn-Pham Surface Singularities with $\mu=299$

$$
299(2) \mid\{2,2,300\} \quad\{2,14,24\}
$$

Table D.267. Quasi-Brieskorn-Pham Surface Singularities with $\mu=300$

$300(20) |$| $\{2,2,301\}$ | $\{2,3,151\}$ | $\{2,4,101\}$ | $\{2,5,76\}$ | $\{2,6,61\}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: |
|  | $\{2,7,51\}$ | $\{2,11,31\}$ | $\{2,13,26\}$ | $\{2,16,21\}$ | $\{3,3,76\}$ |
| $\{3,4,51\}$ | $\{3,6,31\}$ | $\{3,7,26\}$ | $\{3,11,16\}$ | $\{4,5,26\}$ |  |
|  | $\{4,6,21\}$ | $\{4,11,11\}$ | $\{5,6,16\}$ | $\{6,6,13\}$ | $\{6,7,11\}$ |

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## Appendix E

## Jacobi Theta Functions


#### Abstract

It is true that Fourier had the opinion that the principal aim of mathematics was public utility and explanation of natural phenomena; but a philosopher like him should have known that the sole end of science is the honor of the human mind, and that under this title a question about numbers is worth as much as a question about the system of the world.


- Carl Gustav Jacob Jacobi


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In this appendix, we discuss a class of functions which are naturally associated with complex torus, the Jacobi theta functions. For more details, the readers is encouraged to consult $[18,328]$ and [329], from which we shall quote freely.

## E.1. Jacobi Theta Functions

Let $\mathbb{H}$ denote the upper-half plane, $\{\eta \in \mathbb{C} \mid \operatorname{lm}(\eta)>0\}$. On $\mathbb{C} \times \mathbb{H}$, we define the (third of four) theta function $\vartheta_{3}$ by the infinite series

$$
\begin{equation*}
\vartheta_{3}(z, \tau)=\sum_{k \in \mathbb{Z}} e^{\pi i k^{2} \tau+2 \pi i k z} \tag{E.1}
\end{equation*}
$$

Proposition E. 1 (Mumford, [328]). The Jacobi theta function $\vartheta_{3}(z, \tau)$ is the unique holomorphic function $f: \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ such that $\lim _{\operatorname{Im}(\tau) \rightarrow+\infty} f(z, \tau)=1$ and the following transformations hold

1. $f(z+1, \tau)=f(z, \tau)$,
2. $f(z+\tau, \tau)=e^{-\pi i \tau-2 \pi i z} f(z, \tau)$,
3. $f\left(z+\frac{1}{2}, \tau+1\right)=f(z, \tau)$ and
4. $f\left(\frac{z}{\tau},-\frac{1}{\tau}\right)=(-i \tau)^{1 / 2} e^{\pi i z^{2} / \tau} f(z, \tau)$.

Using the transformation properties of $\vartheta_{3}(0, i \tau)$, Riemann proved the following functional identity satisfied by $\zeta(s)=\sum_{k \geqslant 1} k^{-s}$ (appropriately analytically continued), the so called reflection formula, which yields deep insights into the distribution of the primes.

Proposition E. 2 (Riemann, 1859). Let $\xi(s)=\pi^{-s / 2} \Gamma\left(\frac{s}{2}\right) \zeta(s)$. Then $\xi$ has a meromorphic continuation on $\mathbb{C} \backslash\{1\}$ and satisfies the functional equation

$$
\begin{equation*}
\xi(s)=\xi(1-s) . \tag{E.2}
\end{equation*}
$$

The Jacobi theta function $\vartheta_{3}$ is not limited to only analytic investigations, but finds applications in combinatorial number theory.
E.1.1. Sums-of-Squares Functions. Let $r_{n}(k)$ denote the number of ways to represent a positive integer $k$ as a sum of $n$ squares, i.e., $k=k_{1}^{2}+\cdots k_{n}^{2}$. For example, $r_{2}(5)=8$ since $5=( \pm 1)^{2}+( \pm 2)^{2}=( \pm 2)^{2}+( \pm 1)^{2}$. We can understand the behavior of $r_{n}(k)$ by appealing to the study of the function $\varphi(q):=\vartheta_{3}(0, \tau)=\sum_{k \in \mathbb{Z}} q^{k^{2}}$, Jacobi noticed the deep connection between the Fourier coefficients of powers of the theta function and sums of squares,

$$
\begin{equation*}
\varphi(q)^{n}=\sum_{k_{1} \in \mathbb{Z}} \cdots \sum_{k_{n} \in \mathbb{Z}} q^{k_{1}^{2}+\cdots+k_{n}^{2}}=\sum_{k \geqslant 0} r_{n}(k) q^{k} \tag{E.3}
\end{equation*}
$$

where $q=e^{\pi i \tau}$. He subsequently proved that

$$
\begin{equation*}
r_{2}(k)=4 \sum_{d \mid k}(-1)^{(d-1) / 2}=4\left(d_{1,4}(k)-d_{3,4}(k)\right) \tag{E.4}
\end{equation*}
$$

where $d_{a, b}(k)$ enumerates the positive divisors $d$ of $k$ such that $d$ is congruent to $a$ modulo $b$, written as $(a, b) \equiv d$. From this it may be concluded that each prime congruent to 1 modulo 4 can be written as a sum of two squares.

Proposition E. 3 (Ramanujan).

$$
\begin{align*}
& \varphi(q)^{4}=1+8 \sum_{k \geqslant 1} \frac{k q^{k}}{1+(-1)^{k} q^{k}}  \tag{E.5}\\
& \varphi(q)^{6}=1-4 \sum_{k \geqslant 0} \frac{(-1)^{k}(2 k+1)^{2} q^{2 k+1}}{1-q^{2 k+1}}+16 \sum_{k \geqslant 1} \frac{k^{2} q^{k}}{1+q^{2 k}} \tag{E.6}
\end{align*}
$$

and

$$
\begin{equation*}
\varphi(q)^{8}=1+16 \sum_{k \geqslant 1} \frac{k^{3} q^{k}}{1-(-1)^{k} q^{k}} \tag{E.7}
\end{equation*}
$$

Introduce the sum-divisor function $\sigma_{\ell}(k)=\sum_{d \mid k} d^{\ell}$. The sum-of-foursquares function admits the representation,

$$
r_{4}(k)=8 \sum_{d \mid k, 4 \nmid d} d= \begin{cases}\sigma_{1}(k) & d \bmod 4 \not \equiv 0  \tag{E.8}\\ \sigma_{1}(k)-4 \sigma_{1}\left(\frac{d}{4}\right) & d \bmod 4 \equiv 0 .\end{cases}
$$

Hence, every positive integer is the sum of four squares, since the divisor functions (or difference of divisor functions) above are never zero. Through similar arguments, Ramanujan gave the rather complicated formula,

$$
\begin{align*}
r_{6}(k)=4 & \left(\sum_{d \mid k,(d, 4) \equiv 3} d^{2}-\sum_{d \mid k,(d, 4) \equiv 1} d^{2}\right) \\
& +16\left(\sum_{d \mid k,\left(\frac{k}{d}, 4\right) \equiv 1} d^{2}-\sum_{d \mid k,\left(\frac{k}{d}, 4\right) \equiv 1} d^{2}\right) \tag{E.9}
\end{align*}
$$

as well as the remarkably simple formula,

$$
\begin{equation*}
r_{8}(k)=16(-1)^{k} \sum_{d \mid k}(-1)^{d} d^{3} \tag{E.10}
\end{equation*}
$$

## E.2. Properties of the Jacobi Theta Functions

Jacobi introduced four functions $\theta_{i}: \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$. Following the notational convention of Whittaker and Watson, introduce the nome $q=e^{\pi i \tau}$ with $\tau \in \mathbb{H}$
(hence, $|q|<1$ ) and define the first of four theta functions as the infinite series

$$
\begin{equation*}
\theta_{1}(z, \tau)=-i \sum_{k \in \mathbb{Z}}(-1)^{k} e^{i\left(\pi(2 k+1)^{2} \tau / 4+(2 k+1) z\right)} \tag{E.11}
\end{equation*}
$$

When the explicit dependence on $q$ is needed, instead of $\theta_{1}(z, \tau)$, we will write $\theta_{1}(z ; q)$. There is a second and extremely useful summation representation of $\theta_{1}$, namely,

$$
\begin{align*}
\sum_{k \in \mathbb{Z}}(-1)^{k} e^{\pi i(2 k+1)^{2} \tau / 4} e^{i(2 k+1) z}= & \sum_{k \geqslant 0}(-1)^{k} e^{\pi i(2 k+1)^{2} \tau / 4} e^{i(2 k+1) z} \\
& +\sum_{k>0}(-1)^{-k} e^{\pi i(1-2 k)^{2} \tau / 4} e^{i(1-2 k) z}  \tag{E.12}\\
= & \sum_{k \geqslant 0}(-1)^{k} e^{\pi i(2 k+1)^{2} \tau / 4} e^{i(2 k+1) z} \\
& -\sum_{k \geqslant 0}(-1)^{k} e^{\pi i(2 k+1)^{2} \tau / 4} e^{-i(2 k+1) z}  \tag{E.13}\\
= & 2 i \sum_{k \geqslant 0}(-1)^{k} q^{(k+1 / 2)^{2}} \sin (2 k+1) z \tag{E.14}
\end{align*}
$$

Hence,

$$
\begin{equation*}
\theta_{1}(z ; q)=2 \sum_{k \geqslant 0}(-1)^{k} q^{(k+1 / 2)^{2}} \sin (2 k+1) z . \tag{E.15}
\end{equation*}
$$

Proposition E.4. The function $\theta_{1}$ satisfies the second-order partial differential equation

$$
\begin{equation*}
\frac{\partial^{2} \theta_{1}}{\partial z^{2}}=-4 q \frac{\partial \theta_{1}}{\partial q} \tag{E.16}
\end{equation*}
$$

Proof. Since the infinite summation of equation (E.15) converges absolutely, we can differentiate term-wise,

$$
\begin{align*}
4 q \frac{\partial \theta_{1}}{\partial q}(z ; q) & =4 \sum_{k \geqslant 0}(-1)^{k}\left(k+\frac{1}{2}\right)^{2} q^{(k+1 / 2)^{2}} \sin (2 k+1) z  \tag{E.17}\\
& =2 \sum_{k \geqslant 0}(-1)^{k} q^{(k+1 / 2)^{2}}(2 k+1)^{2} \sin (2 k+1) z  \tag{E.18}\\
& =-\frac{\partial^{2} \theta_{1}}{\partial z^{2}}(z ; q) \tag{E.19}
\end{align*}
$$

which proves the claim.

Remark E.2.1. The Jacobi theta function $\theta_{1}(0, i \tau)$ satisfies the heat equation.
E.2.1. Ramanujan $q$-Products and $q$-Series. Define the following functions,

$$
\begin{equation*}
(a ; q)_{n}:=\prod_{k=0}^{n-1}\left(1-a q^{k}\right) \quad \text { and } \quad(a ; q)_{\infty}:=\lim _{n \rightarrow \infty}(a ; q)_{n} \tag{E.20}
\end{equation*}
$$

where $n \in \mathbb{N}$ and $|q|<1$. For $n \in \mathbb{C} \backslash \mathbb{N}$, we define

$$
\begin{equation*}
(a ; q)_{n}=\frac{(a ; q)_{\infty}}{\left(a q^{n} ; q\right)_{\infty}} \tag{E.21}
\end{equation*}
$$

Observe that for $|a|<1$, we have

$$
\begin{equation*}
(a q ; q)_{\infty}=(1-a)^{-1}(a ; q)_{\infty} \tag{E.22}
\end{equation*}
$$

These functions are known as Ramanujan $q$-products or $q$-Pochhammer symbols as they generalize the Pochammer symbol, i.e., the rising factorial function,

$$
\begin{equation*}
(n)_{k}:=\prod_{j=0}^{k-1}(n+j)=\frac{\Gamma(n+k)}{\Gamma(n)}=\lim _{q \rightarrow 1^{-}} \frac{\left(q^{n} ; q\right)_{k}}{(1-q)^{k}} \tag{E.23}
\end{equation*}
$$

A Ramanujan $q$-series is an infinite series of the form

$$
\begin{equation*}
\sum_{k \in \mathbb{Z}} a_{k} q^{k} \tag{E.24}
\end{equation*}
$$

where the coefficients $\left\{a_{k}\right\}_{k \in \mathbb{Z}}$ are not all zero and $|q|<1$.

Proposition E.5. For $|q|<1$ and $z \in \mathbb{C}$, we have the following product representations

$$
\begin{align*}
\theta_{1}(z ; q) & =-i q^{1 / 4}\left(q^{2} ; q^{2}\right)_{\infty} e^{i z}\left(e^{-2 i z} ; q^{2}\right)_{\infty}\left(q^{2} e^{2 i z} ; q^{2}\right)_{\infty}  \tag{E.25}\\
& =2 q^{1 / 4}\left(q^{2} ; q^{2}\right)_{\infty} \sin z\left(q^{2} e^{-2 i z} ; q^{2}\right)_{\infty}\left(q^{2} e^{2 i z} ; q^{2}\right)_{\infty} \tag{E.26}
\end{align*}
$$

Proof. The equivalence of the two expressions follows from the identity

$$
\begin{equation*}
\sin z=\frac{e^{i z}\left(1-e^{-2 i z}\right)}{2 i}=\frac{e^{i z}\left(e^{-2 i z} ; q^{2}\right)_{\infty}}{2 i\left(q^{2} e^{-2 i z} ; q^{2}\right)_{\infty}} \tag{E.27}
\end{equation*}
$$

Finally, the Jacobi triple product implies

$$
\begin{align*}
\left(e^{-2 i z} ; q^{2}\right)_{\infty}\left(q^{2} e^{2 i z} ; q^{2}\right)_{\infty}\left(q^{2} ; q^{2}\right)_{\infty} & =\sum_{k \in \mathbb{Z}}(-1)^{k} q^{k(k+1)} e^{2 i k z}  \tag{E.28}\\
& =q^{-1 / 4} e^{-i z} \sum_{k \in \mathbb{Z}}(-1)^{k} q^{(2 k+1)^{2} / 4} e^{i(2 k+1) z} \tag{E.29}
\end{align*}
$$

which proves the claim.
As a result of absolute convergence of the infinite products, we can multiply the linear factors of each and write

$$
\begin{equation*}
\theta_{1}(z ; q)=2 q^{1 / 4}\left(q^{2} ; q^{2}\right)_{\infty} \sin z \prod_{k \geqslant 1}\left(1-2 q^{2 k} \cos 2 z+q^{4 k}\right) \tag{E.30}
\end{equation*}
$$

We now prove two important symmetries enjoyed by $\theta_{1}$.

Proposition E.6. The following identities hold,

$$
\begin{align*}
\theta_{1}(z+\pi, \tau) & =-\theta_{1}(z, \tau)  \tag{E.31}\\
\theta_{1}(z+\pi \tau, \tau) & =-e^{-\pi i \tau-2 i z} \theta_{1}(z, \tau) \tag{E.32}
\end{align*}
$$

Proof. Consider

$$
\begin{align*}
\theta_{1}(z+\pi ; q) & =-i \sum_{k \in \mathbb{Z}}(-1)^{k} q^{(k+1 / 2)^{2}} e^{(2 k+1) i(z+\pi)}  \tag{E.33}\\
& =-i \sum_{k \in \mathbb{Z}}(-1)^{k+2 k+1} q^{(k+1 / 2)^{2}} e^{(2 k+1) i z}  \tag{E.34}\\
& =-\theta_{1}(z ; q) \tag{E.35}
\end{align*}
$$

which establishes a $2 \pi$-periodicity in the first argument. To establish the second symmetry, note that

$$
\begin{align*}
\theta_{1}(z+\pi \tau ; q) & =-i q^{1 / 4} e^{i(z+\pi \tau)}\left(q^{2} ; q^{2}\right)_{\infty}\left(e^{-2 i(z+\pi \tau)} ; q^{2}\right)_{\infty}\left(q^{2} e^{2 i(z+\pi \tau)} ; q^{2}\right)_{\infty}  \tag{E.36}\\
& =-i q^{5 / 4} e^{i z}\left(q^{2} ; q^{2}\right)_{\infty}\left(q^{-2} e^{-2 i z} ; q^{2}\right)_{\infty}\left(q^{4} e^{2 i z} ; q^{2}\right)_{\infty} \tag{E.37}
\end{align*}
$$

Then

$$
\begin{align*}
\frac{\theta_{1}(z+\pi \tau ; q)}{\theta_{1}(z ; q)} & =q \frac{\left(q^{-2} e^{-2 i z} ; q^{2}\right)_{\infty}\left(q^{4} e^{2 i z} ; q^{2}\right)_{\infty}}{\left(e^{-2 i z} ; q^{2}\right)_{\infty}\left(q^{2} e^{2 i z} ; q^{2}\right)_{\infty}}  \tag{E.38}\\
& =q \frac{\left(1-q^{-2} e^{-2 i z}\right)\left(q^{-2} e^{-2 i z} ; q^{2}\right)_{\infty}\left(q^{2} e^{2 i z} ; q^{2}\right)_{\infty}}{\left(1-q^{2} e^{2 i z}\right)\left(q^{-2} e^{-2 i z} ; q^{2}\right)_{\infty}\left(q^{2} e^{2 i z} ; q^{2}\right)_{\infty}}  \tag{E.39}\\
& =-q^{-1} e^{-2 i z}, \tag{E.40}
\end{align*}
$$

which completes the proof.
Write $\left.f\right|_{g}=f \circ g$.

Proposition E.7. For $\gamma=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \operatorname{SL}(2, \mathbb{Z})$ and $(z, \tau) \in \mathbb{C} \times \mathbb{H}$, we have identity

$$
\begin{equation*}
\left.\theta_{1}\right|_{\gamma}(z, \tau)=\zeta_{8}(c, d)(c \tau+d)^{1 / 2} e^{i c z^{2} / \pi(c \tau+d)} \theta_{1}(z, \tau) \tag{E.41}
\end{equation*}
$$

where $\zeta_{8}(c, d)$ is an eight-root of unity depending on $c$ and $d$.

Proof. See [18].

Proposition E.8. For $\alpha, \beta \in \mathbb{C}^{\times}$,

$$
\begin{equation*}
\lim _{z \rightarrow 0} \frac{\theta_{1}(\alpha z ; q)}{\theta_{1}(\beta z ; q)}=\frac{\alpha}{\beta} \tag{E.42}
\end{equation*}
$$

Proof. Observe that

$$
\begin{align*}
\lim _{z \rightarrow 0} \frac{\theta_{1}(\alpha z ; q)}{\theta_{1}(\beta z ; q)} & =\lim _{z \rightarrow 0} \frac{\sin (\alpha z)}{\sin (\beta z)} \prod_{k=1}^{\infty} \frac{1-2 q^{2 k} \cos (2 \alpha z)+q^{4 k}}{1-2 q^{2 k} \cos (2 \beta z)+q^{4 k}}  \tag{E.43}\\
& =\lim _{z \rightarrow 0} \frac{\sin (\alpha z)}{\sin (\beta z)}=\frac{\alpha}{\beta^{\prime}} \tag{E.44}
\end{align*}
$$

where the last equality involves L'Hospital's rule.
The number of identities and symmetries satisfied by the Jacobi theta functions is rivaled only by the myriad of notations used to define them. To avoid confusion, we shall respect the notational conventions of Whittaker and Watson and subsequently introduce our own scaled variant. As before, let $q=e^{\pi i \tau}$. Introduce three more Jacobi theta functions $\theta_{i}: \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$, where

$$
\begin{align*}
& \theta_{2}(z, \tau)=\sum_{k \in \mathbb{Z}} e^{\pi i(k+1 / 2)^{2} \tau+(2 k+1) i z}  \tag{E.45}\\
& \theta_{3}(z, \tau)=\sum_{k \in \mathbb{Z}} e^{\pi i k^{2} \tau+2 i k z}  \tag{E.46}\\
& \theta_{4}(z, \tau)=\sum_{k \in \mathbb{Z}}(-1)^{k} e^{\pi i k^{2} \tau+2 i k z} \tag{E.47}
\end{align*}
$$

When the explicit dependence on $q$ is needed, instead of $\theta_{i}(z, \tau)$, we will write $\theta_{i}(z ; q)$. The four theta functions are related through the following identities.

Proposition E.9. The following identities hold:

$$
\begin{align*}
& \theta_{1}(z, \tau)=-i e^{i z+\pi i \tau / 4} \theta_{4}\left(z+\frac{\pi \tau}{2}, \tau\right)  \tag{E.48}\\
& \theta_{2}(z ; q)=\theta_{1}\left(z+\frac{\pi}{2}, \tau\right)  \tag{E.49}\\
& \theta_{3}(z, \tau)=\theta_{4}\left(z+\frac{\pi}{2}, \tau\right) \tag{E.50}
\end{align*}
$$

Proof. We prove the first identity. Observe that

$$
\begin{align*}
\theta_{4}\left(z+\frac{\pi \tau}{2}, \tau\right) & =\sum_{k \in \mathbb{Z}}(-1)^{k} e^{\pi i k^{2} \tau+2 i k(z+\pi \tau / 2)}  \tag{E.51}\\
& =e^{\pi i \tau / 4-i z} \sum_{k \in \mathbb{Z}}(-1)^{k} e^{i\left(\pi(2 k+1)^{2} \tau / 4+(2 k+1) z\right)} \tag{E.52}
\end{align*}
$$

which is $i e^{-i z-\pi i \tau / 4} \theta_{1}(z, \tau)$. The last two identities follow similarly.
As a result, we have $\theta_{4}(z ; q)=\theta_{3}(z ;-q)$. Other interesting representations include the following,

Proposition E.10. For $q=e^{\pi i \tau} \in \Delta$ and $z \in \mathbb{C}$, we have the equivalent representations

$$
\begin{align*}
\theta_{2}(z, \tau) & =2 \sum_{k \geqslant 0} q^{(k+1 / 2)^{2}} \cos (2 k+1) z  \tag{E.53}\\
& =2 q^{1 / 4}\left(q^{2} ; q^{2}\right)_{\infty} \cos z\left(-q^{2} e^{-2 i z} ; q^{2}\right)_{\infty}\left(-q^{2} e^{2 i z} ; q^{2}\right)_{\infty} \tag{E.54}
\end{align*}
$$

Before proceeding to the next result, we need the following identity.

Proposition E.11. Let $\zeta_{n}=e^{2 \pi i / n}$. Then

$$
\begin{equation*}
(q ; q)_{\infty}=\prod_{k=0}^{n-1}\left(q^{1 / n} \zeta_{n}^{k} ; q^{1 / n}\right)_{\infty} \tag{E.55}
\end{equation*}
$$

Corollary E.12. For $q=e^{\pi i \tau} \in \Delta$, we have the identity

$$
\begin{equation*}
(q ; q)_{\infty}=\frac{q^{-1 / 24}}{\sqrt{3}} \theta_{2}\left(\frac{\pi}{6} ; q^{1 / 6}\right) \tag{E.56}
\end{equation*}
$$

Proof. The $q$-product representation of the $\theta_{2}$ implies

$$
\begin{equation*}
\theta_{2}\left(\frac{\pi}{6} ; q^{1 / 3}\right)=\sqrt{3} q^{1 / 12}\left(q^{2 / 3} ; q^{2 / 3}\right)_{\infty}\left(-q^{2 / 3} \zeta_{3} ; q^{2 / 3}\right)_{\infty}\left(-q^{2 / 3} \zeta_{3}^{-1} ; q^{2 / 3}\right)_{\infty} \tag{E.57}
\end{equation*}
$$

The identity

$$
\begin{align*}
\left(q^{2} ; q^{2}\right)_{\infty} & =\left(q^{2 / 3} ; q^{2 / 3}\right)_{\infty}\left(q^{2 / 3} \zeta_{3}^{-1} ; q^{2 / 3}\right)_{\infty}\left(q^{2 / 3} \zeta_{3} ; q^{2 / 3}\right)_{\infty}  \tag{E.58}\\
& =\left(q^{2 / 3} ; q^{2 / 3}\right)_{\infty}\left(-q^{2 / 3} \zeta_{3} ; q^{2 / 3}\right)_{\infty}\left(-q^{2 / 3} \zeta_{3}^{-1} ; q^{2 / 3}\right)_{\infty} \tag{E.59}
\end{align*}
$$

and scaling $q^{2} \rightarrow q$ implies the claim.

Proposition E.13. For $q=e^{\pi i \tau} \in \Delta$ and $z \in \mathbb{C}$, we have the equivalent representations

$$
\begin{align*}
\theta_{3}(z, \tau) & =1+2 \sum_{k \geqslant 1} q^{k^{2}} \cos (2 k z)  \tag{E.6o}\\
& =\left(q^{2} ; q^{2}\right)_{\infty}\left(-q e^{-2 i z} ; q^{2}\right)_{\infty}\left(-q e^{2 i z} ; q^{2}\right)_{\infty} \tag{E.61}
\end{align*}
$$

Proposition E.14. For $q=e^{\pi i \tau} \in \Delta$ and $z \in \mathbb{C}$, we have the equivalent representations

$$
\begin{align*}
\theta_{4}(z, \tau) & =1+2 \sum_{k \geqslant 1}(-1)^{k} q^{k^{2}} \cos (2 k z)  \tag{E.62}\\
& =\left(q^{2} ; q^{2}\right)_{\infty}\left(q e^{-2 i z} ; q^{2}\right)_{\infty}\left(q e^{2 i z} ; q^{2}\right)_{\infty} \tag{E.63}
\end{align*}
$$

Thus, we also have

$$
\begin{aligned}
& \theta_{2}(z, \tau)=2 q^{1 / 4}\left(q^{2} ; q^{2}\right)_{\infty} \cos z \prod_{k \geqslant 1}\left(1+2 q^{2 k} \cos (2 z)+q^{4 k}\right) \\
& \theta_{3}(z, \tau)=\left(q^{2} ; q^{2}\right)_{\infty} \prod_{k \geqslant 1}\left(1+2 q^{2 k-1} \cos (2 z)+q^{4 k-2}\right) \\
& \theta_{4}(z, \tau)=\left(q^{2} ; q^{2}\right)_{\infty} \prod_{k \geqslant 1}\left(1-2 q^{2 k-1} \cos (2 z)+q^{4 k-2}\right)
\end{aligned}
$$

Proposition E.15. The following q-product identities hold:

$$
\begin{align*}
(-q,-q)_{\infty} & =(q ;-q)_{\infty}\left(q^{2} ; q^{2}\right)_{\infty}\left(-q ; q^{2}\right)_{\infty}^{2}  \tag{E.64}\\
\left(q^{4} ; q^{4}\right)_{\infty} & =\left(q^{2} ; q^{2}\right)_{\infty}\left(-q^{2} ; q^{2}\right)_{\infty}  \tag{E.65}\\
(q ; q)_{\infty} & =\left(q^{2} ; q^{2}\right)_{\infty}\left(q ; q^{2}\right)_{\infty}  \tag{E.66}\\
1 & =\left(-q^{2} ; q^{2}\right)_{\infty}\left(q ; q^{2}\right)_{\infty}\left(-q ; q^{2}\right)_{\infty} \tag{E.67}
\end{align*}
$$

The last identity is equivalent to the Jacobi Triple Product.

Corollary E.16. The following identity holds,

$$
\begin{equation*}
2^{4} q\left(-q^{2} ; q^{2}\right)_{\infty}^{8}+\left(q ; q^{2}\right)_{\infty}^{8}=\left(-q ; q^{2}\right)_{\infty}^{8} \tag{E.68}
\end{equation*}
$$

Define the last three nullwerts, $\theta_{i}(q)=\theta_{i}(0, \tau)$.

Corollary E.17. The non-trivial theta nullwerts are given by

$$
\begin{equation*}
\theta_{2}(q)=2 q^{1 / 4} \frac{\left(q^{4} ; q^{4}\right)_{\infty}^{2}}{\left(q^{2} ; q^{2}\right)_{\infty}}, \quad \theta_{3}(q)=\frac{(-q,-q)_{\infty}}{(q ;-q)_{\infty}} \quad \text { and } \quad \theta_{4}(q)=\frac{(q ; q)_{\infty}^{2}}{\left(q^{2} ; q^{2}\right)_{\infty}} \tag{E.69}
\end{equation*}
$$

Proof. Since $\theta_{3}(q)=\left(q^{2} ; q^{2}\right)_{\infty}\left(-q ; q^{2}\right)_{\infty}^{2}$, it suffices to prove the identity

$$
\begin{equation*}
(-q,-q)_{\infty}=(q ;-q)_{\infty}\left(q^{2} ; q^{2}\right)_{\infty}\left(-q ; q^{2}\right)_{\infty}^{2} \tag{E.70}
\end{equation*}
$$

which we leave as an exercise. By combining

$$
\begin{align*}
& \theta_{2}(q)=2 q^{1 / 4}\left(q^{2} ; q^{2}\right)_{\infty}\left(-q^{2} ; q^{2}\right)_{\infty}^{2}  \tag{E.71}\\
& \theta_{4}(q)=\left(q^{2} ; q^{2}\right)_{\infty}\left(q ; q^{2}\right)_{\infty}^{2} \tag{E.72}
\end{align*}
$$

with the identities

$$
\begin{align*}
\left(q^{4} ; q^{4}\right)_{\infty} & =\left(q^{2} ; q^{2}\right)_{\infty}\left(-q^{2} ; q^{2}\right)_{\infty}  \tag{E.73}\\
(q ; q)_{\infty} & =\left(q^{2} ; q^{2}\right)_{\infty}\left(q ; q^{2}\right)_{\infty} \tag{E.74}
\end{align*}
$$

respectively, the other two identities follow.

Corollary E. 18 (Jacobi). The following identity holds:

$$
\begin{equation*}
\theta_{3}^{4}(q)=\theta_{2}^{4}(q)+\theta_{4}^{4}(q) \tag{E.75}
\end{equation*}
$$

Proof. The identity follows from Proposition E. 17 and the $q$-series identity

$$
\begin{align*}
2^{4} q\left(q^{4} ; q^{4}\right)_{\infty}^{8}+(q ; q)_{\infty}^{8} & =\left(\frac{(-q,-q)_{\infty}\left(q^{2} ; q^{2}\right)_{\infty}}{(q ;-q)_{\infty}}\right)^{4}  \tag{E.76}\\
& =\left(q^{2} ; q^{2}\right)_{\infty}^{8}\left(-q ; q^{2}\right)_{\infty}^{8} \tag{E.77}
\end{align*}
$$

which is equivalent to $2^{4} q\left(-q^{2} ; q^{2}\right)_{\infty}^{8}+\left(q ; q^{2}\right)_{\infty}^{8}=\left(-q ; q^{2}\right)_{\infty}^{8}$.
Given the myriad of conventions, it should be no surprise that we now introduce yet another version of the theta functions suitable for our purpose. Since the modern definition of the nome is $q=e^{2 \pi i \tau} \in \Delta$, and we wish for 2-periodicity and $\tau$-quasiperioidicity in the first argument of $\vartheta_{1}$, we make the substitution $\tau \rightarrow 2 \tau$ and $z \rightarrow \pi z$ in $\theta$ above and define the variant

$$
\begin{align*}
\vartheta_{1}(z, \tau) & =-i q^{1 / 8} e^{\pi i z}(q ; q)_{\infty}\left(e^{-2 \pi i z} ; q\right)_{\infty}\left(q e^{2 \pi i z} ; q\right)_{\infty}  \tag{E.78}\\
& =2 q^{1 / 8}(q ; q)_{\infty}\left(q e^{-2 \pi i z} ; q\right)_{\infty}\left(q e^{2 \pi i z} ; q\right)_{\infty} \sin \pi z  \tag{E.79}\\
& =\theta_{1}\left(\pi z ; e^{\pi i \tau}\right) \tag{E.8o}
\end{align*}
$$

and the three other variants, $\vartheta_{i}(z, \tau)=\theta_{i}\left(\pi z ; e^{\pi i \tau}\right)$ for $i=2,3$ and 4 . Alternatively, if we wish to focus on the dependence of $q=e^{2 \pi i \tau}$, we write $\vartheta_{i}(z ; q)$. That is, $\vartheta_{i}\left(z ; e^{2 \pi i \tau}\right)=\theta_{i}\left(\pi z ; e^{\pi i z}\right)$.

Proposition E.19. The following identities hold:

$$
\begin{align*}
\vartheta_{1}(\pi z, \tau) & =-i e^{i \pi(z+\tau / 2)} \vartheta_{4}(z+\tau, \tau)  \tag{E.81}\\
\vartheta_{2}(z ; \tau) & =\vartheta_{1}(z+\tau ; \tau)  \tag{E.82}\\
\vartheta_{3}(z, \tau) & =\vartheta_{4}(z+\tau, \tau) \tag{E.83}
\end{align*}
$$

Introduce a second nome $y=e^{2 \pi i z} \in \mathbb{C}^{\times}$. Then we see

$$
\begin{align*}
\vartheta_{1}(z, \tau) & =-i q^{1 / 8} y^{1 / 2}(q ; q)_{\infty}\left(y^{-1} ; q\right)_{\infty}(q y ; q)_{\infty}  \tag{E.84}\\
& =i q^{1 / 8}(q ; q)_{\infty}\left(y^{-1 / 2}-y^{1 / 2}\right)\left(q y^{-1} ; q\right)_{\infty}(q y ; q)_{\infty} \tag{E.85}
\end{align*}
$$

which descends to a function on $\mathbb{C}^{\times} \times \Delta$. The involution $\vartheta_{1}(-z, \tau)=-\vartheta_{1}(z, \tau)$ is clear from this representation. By construction,

Proposition E.20. We have the symmetries

$$
\begin{equation*}
\vartheta_{1}(z+1, \tau)=-\vartheta_{1}(z, \tau) \quad \text { and } \quad \vartheta_{1}(z+\tau, \tau)=-e^{-\pi i \tau-2 \pi i z} \vartheta_{1}(z, \tau) . \tag{E.86}
\end{equation*}
$$

Proof. Observe that

$$
\begin{align*}
\frac{\vartheta_{1}(z+\tau, \tau)}{\vartheta_{1}(z, \tau)} & =\frac{\left(q^{-1 / 2} y^{-1 / 2}-q^{1 / 2} y^{1 / 2}\right)\left(y^{-1} ; q\right)_{\infty}\left(q^{2} y ; q\right)_{\infty}}{\left(y^{-1 / 2}-y^{1 / 2}\right)\left(q y^{-1} ; q\right)_{\infty}(q y ; q)_{\infty}}  \tag{E.87}\\
& =\frac{\left(1-y^{-1}\right)\left(q^{-1 / 2} y^{-1 / 2}-q^{1 / 2} y^{1 / 2}\right)\left(y^{-1} ; q\right)_{\infty}(q y ; q)_{\infty}}{(1-q y)\left(y^{-1 / 2}-y^{1 / 2}\right)\left(y^{-1} ; q\right)_{\infty}(q y ; q)_{\infty}}  \tag{E.88}\\
& =-q^{-1 / 2} y^{-1} . \tag{E.89}
\end{align*}
$$

Also, $\frac{\vartheta_{1}(z+1, \tau)}{\vartheta_{1}(z, \tau)}=\frac{\theta_{1}\left(\pi z+\pi ; e^{\pi i \tau}\right)}{\theta_{1}\left(\pi z ; e^{\pi i \tau}\right)}=-1$, which completes the proof.

Let $\delta=(m, n) \in \mathbb{Z}^{2}$ act on $(z, \tau) \in \mathbb{C} \times \mathbb{H}$ by $\delta(z, \tau)=(z+m \tau+n, \tau)$. More generally, we have the following functional transformation identity. Note that $\delta$ corresponds to a point on the lattice $\Lambda_{\tau}=\mathbb{Z} \oplus \tau \mathbb{Z}$.

Proposition E.21. For $\delta=(m, n) \in \mathbb{Z}^{2}$ and $(z, \tau) \in \mathbb{C} \times \mathbb{H}$, we have identity

$$
\begin{equation*}
\left.\vartheta_{1}\right|_{\delta}(z, \tau)=(-1)^{n+m} e^{-\pi i m^{2} \tau-2 \pi i m z} \vartheta_{1}(z, \tau) \tag{E.90}
\end{equation*}
$$

Proof. The identity $(a ; q)_{-v}=(-1)^{v} q^{v(v-1) / 2} \frac{(q / a)^{v}}{(q / a ; q)_{v}}$ implies the equality

$$
\begin{equation*}
(a q ; q)_{v}(q / a ; q)_{-v}=(-1)^{v} q^{v^{2} / 2} a^{v} \frac{\left(a q^{v}\right)^{-1 / 2}-\left(a q^{v}\right)^{1 / 2}}{a^{-1 / 2}-a^{1 / 2}} \tag{E.91}
\end{equation*}
$$

for $v \in \mathbb{C}$. Translating $z$ to $z+n$ gives the factor $(-1)^{n}$, so we need only consider the ratio

$$
\begin{align*}
\frac{\vartheta_{1}(z+m \tau, \tau)}{\vartheta_{1}(z, \tau)} & =\frac{\left(q^{-m / 2} y^{-1 / 2}-q^{m / 2} y^{1 / 2}\right)\left(q^{1-m} y^{-1} ; q\right)_{\infty}\left(q^{1+m} y ; q\right)_{\infty}}{\left(y^{-1 / 2}-y^{1 / 2}\right)\left(q y^{-1} ; q\right)_{\infty}(q y ; q)_{\infty}}  \tag{E.92}\\
& =\frac{q^{-m / 2} y^{-1 / 2}-q^{m / 2} y^{1 / 2}}{\left(y^{-1 / 2}-y^{1 / 2}\right)\left(q y^{-1} ; q\right)_{-m}(q y ; q)_{m}}  \tag{E.93}\\
& =(-1)^{m} q^{-m^{2} / 2} y^{-m}, \tag{E.94}
\end{align*}
$$

where we have used the identity $\frac{\left(a q^{v} ; q\right)_{\infty}}{(a ; q)_{\infty}}=(a ; q)_{v}$.
We have thusfar been concerned with transformation identities of $\vartheta_{1}(z, \tau)$ in the first argument. Transformation identities in the second argument are more geometrically rich and provide a bridge to the study of modular functions and forms. Let $\gamma=\left(\begin{array}{l}a b \\ c \\ d\end{array}\right) \in \operatorname{SL}(2, \mathbb{Z})$ act on $(z, \tau) \in \mathbb{C} \times \mathbb{H}$ by $\gamma(z, \tau)=\left(\frac{z}{c \tau+d}, \frac{a \tau+b}{c \tau+d}\right)$.

Recall the Jacobi symbol of $a \in \mathbb{Z}$ with respect to $k \in \mathbb{N}$ is the product of Legendre symbols,

$$
\begin{equation*}
\left(\frac{a}{k}\right)=\left(\frac{a}{p_{1}}\right)^{r_{1}} \cdots\left(\frac{a}{p_{n}}\right)^{r_{n}} \tag{E.95}
\end{equation*}
$$

where the integer $k$ is given by the prime decomposition $p_{1}^{r_{1}} \cdots p_{n}^{r_{n}}$. Recall that we have defined the following theta functions,

$$
\begin{align*}
& \vartheta_{1}(z, \tau)=-i \sum_{k \in \mathbb{Z}}(-1)^{k} e^{\pi i(k+1 / 2)^{2} \tau+\pi i(2 k+1) z}  \tag{E.96}\\
& \vartheta_{3}(z, \tau)=\sum_{k \in \mathbb{Z}} e^{\pi i k^{2} \tau+2 \pi i k z} \tag{E.97}
\end{align*}
$$

Proposition E.22. Let $\vartheta$ denote either $\vartheta_{1}$ or $\vartheta_{3}$. For $\gamma=\left(\begin{array}{ll}a & b \\ c & d\end{array}\right) \in \operatorname{SL}(2, \mathbb{Z})$ with $a b$ and $c d$ even, and $(z, \tau) \in \mathbb{C} \times \mathbb{H}$, we have identity

$$
\begin{equation*}
\left.\vartheta\right|_{\gamma}(z, \tau)=\zeta_{8}(c, d)(c \tau+d)^{1 / 2} e^{\pi i c z^{2} /(c \tau+d)} \vartheta(z, \tau) \tag{E.98}
\end{equation*}
$$

where the eight-root of unity $\zeta_{8}(c, d)$ is such that

$$
\zeta_{8}(c, d)=\left\{\begin{array}{cl}
i^{(d-1) / 2}\left(\frac{c}{|d|}\right) & c \text { even and } d \text { odd }  \tag{E.99}\\
e^{-\pi i c / 4}\left(\frac{d}{c}\right) & c \text { odd and } d \text { even. }
\end{array}\right.
$$

Thus, the action of $\gamma_{\delta}=\left((m, n),\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)\right) \in \mathbb{Z}_{2} \ltimes \mathrm{SL}_{2}(\mathbb{Z})$ on $(z, \tau) \in \mathbb{C} \times \mathbb{H}$ is given by

$$
\begin{align*}
\gamma_{\delta}(z, \tau) & =(\delta \circ \gamma)(z, \tau)  \tag{E.100}\\
& =\delta\left(\frac{z}{c \tau+d^{\prime}}, \frac{a \tau+b}{c \tau+d}\right)  \tag{E.101}\\
& =\left(\frac{z}{c \tau+d}+m\left(\frac{a \tau+b}{c \tau+d}\right)+n, \frac{a \tau+b}{c \tau+d}\right)  \tag{E.102}\\
& =\left(\frac{z+a^{\prime} \tau+b^{\prime}}{c \tau+d}, \frac{a \tau+b}{c \tau+d}\right) \tag{E.103}
\end{align*}
$$

where $a^{\prime}=m a+n c$ and $b^{\prime}=m b+n d$. Note that this notation allows us to write $\gamma_{0}$ as simply $\gamma$, where $0=(0,0)$ is the additive identity in $\mathbb{Z}^{2}$.
E.2.2. Jacobi Forms. We use the notation and convention of [328, 329]. Let $y=e^{2 \pi i z}$ and $q=e^{2 \pi i \tau}$. Define $\mathbf{e}_{c, d}^{z}=e^{\pi i z /(c \tau+d)}$.

Definition E.23. A weak Jacobi form of level 1 , weight $k \in \mathbb{Z}$ and index $v \in \mathbb{Q}$ is a holomorphic function $f: \mathbb{C} \times \mathbb{H} \rightarrow \mathbb{C}$ satisfying the following:

1. $\left.f\right|_{\gamma}(z, \tau)=(c \tau+d)^{k} \mathbf{e}_{c, d}^{2 v c z^{2}} f(z, \tau)$ for all $\gamma=\binom{a b}{c d} \in \mathrm{SL}_{2}(\mathbb{Z})$; and,
2. $\left.f\right|_{\delta}(z, \tau)=\left(y^{2} q^{m}\right)^{-m v} f(z, \tau)$ for all $\delta=(m, n) \in \mathbb{Z}_{2}$.

By the transformation identities satisfied by $\vartheta_{1}(z, \tau)$, we conclude the following.

Proposition E.24. The Jacobi Theta Function $\vartheta_{1}(z, \tau)$ is a weak Jacobi form of level 1, weight $\frac{1}{2}$ and index $\frac{1}{2}$.

Corollary E.25. For $\alpha, \beta \in \mathbb{C}^{\times}$and $(z, \tau) \in \mathbb{C} \times \mathbb{H}$,

$$
\begin{equation*}
\frac{\left.\vartheta_{1}\right|_{\gamma}(\alpha z, \tau)}{\left.\vartheta_{1}\right|_{\gamma}(\beta z, \tau)}=e^{\pi i c\left(\alpha^{2}-\beta^{2}\right) z^{2} /(c \tau+d)} \frac{\vartheta_{1}(\alpha z, \tau)}{\vartheta_{1}(\beta z, \tau)} . \tag{E.104}
\end{equation*}
$$

Corollary E.26. For $\alpha, \beta \in \mathbb{C}^{\times}$and $(m, n) \in \mathbb{Z}^{2}$, we have

$$
\begin{equation*}
\frac{\vartheta_{1}(\alpha z+m \tau+n, \tau)}{\vartheta_{1}(\beta z+m \tau+n, \tau)}=e^{-2 \pi i m(\alpha-\beta) z} \frac{\vartheta_{1}(\alpha z, \tau)}{\vartheta_{1}(\beta z, \tau)} \tag{E.105}
\end{equation*}
$$

Corollary E.27. For $\alpha, \beta \in \mathbb{Q}^{\times}$, the ratio $\frac{\vartheta_{1}(\alpha z, \tau)}{\vartheta_{1}(\beta z, \tau)}$ is a Jacobi form of level 1, weight $o$ and index $\frac{1}{2}\left(\alpha^{2}-\beta^{2}\right)$.

Proof. Suppose $f$ and $g$ are Jacobi forms of level 1 , weight $k$ with different indices $v_{1}=v(f)$ and $v_{2}=v(g)$. Then we have the symmetries

$$
\begin{align*}
& \frac{\left.f\right|_{\gamma}(z, \tau)}{\left.g\right|_{\gamma}(z, \tau)}=\mathbf{e}_{c, d}^{2 c\left(v_{1}-v_{2}\right) z^{2}} \frac{f(z, \tau)}{g(z, \tau)}  \tag{E.106}\\
& \frac{\left.f\right|_{\delta}(z, \tau)}{\left.g\right|_{\delta}(z, \tau)}=\left(y^{2} q^{m}\right)^{-m\left(v_{1}-v_{2}\right)} \frac{f(z, \tau)}{g(z, \tau)} \tag{E.107}
\end{align*}
$$

and therefore conclude that $h=\frac{f}{g}$ is a Jacobi form of level 1 , weight $o$ and index $v=v_{1}-v_{2}$. For $\alpha, \beta \in \mathbb{C}^{\times}, \gamma_{\delta}=\left((m, n),\left(\begin{array}{ll}a & b \\ c & d\end{array}\right)\right) \in \mathbb{Z}_{2} \ltimes \operatorname{SL}_{2}(\mathbb{Z})$ and $(z, \tau) \in \mathbb{C} \times \mathbb{H}$, we have the identity

$$
\begin{equation*}
\frac{\left.\vartheta_{1}\right|_{\gamma_{\delta}}(\alpha z, \tau)}{\left.\vartheta_{1}\right|_{\gamma_{\delta}}(\beta z, \tau)}=\mathbf{e}_{c, d}^{-2 m(\alpha-\beta) z+c\left(\alpha^{2}-\beta^{2}\right) z^{2}} \frac{\vartheta_{1}(\alpha z, \tau)}{\vartheta_{1}(\beta z, \tau)} . \tag{E.108}
\end{equation*}
$$

Thus, the claim follows immediately.

## E.3. Theta Functions with Rational Characteristics

For $r, s \in \mathbb{Q}$, define the following two-parameter theta function

$$
\begin{equation*}
\vartheta_{r, s}(z, \tau)=\sum_{k \in \mathbb{Z}} e^{\pi i(k+r)^{2} \tau} e^{2 \pi i(k+r)(z+s)} \tag{E.109}
\end{equation*}
$$

Taking $r=s=\frac{1}{2}$ [328], we have

$$
\begin{align*}
\vartheta_{\frac{1}{2}, \frac{1}{2}}(z, \tau) & =\sum_{k \in \mathbb{Z}} e^{\pi i(k+1 / 2)^{2} \tau} e^{2 \pi i(k+1 / 2)(z+1 / 2)}  \tag{E.110}\\
& =i \sum_{k \in \mathbb{Z}}(-1)^{k} e^{\pi i(k+1 / 2)^{2} \tau} e^{2 \pi i(k+1 / 2) z}  \tag{E.111}\\
& =-\vartheta_{1}(z, \tau) \tag{E.112}
\end{align*}
$$

## E.4. $q$-Theta Function

For $|q|<1$ and $w \in \mathbb{C}$, define the $q$-theta function*

$$
\begin{equation*}
\theta_{q}(w)=(w ; q)_{\infty}\left(q w^{-1} ; q\right)_{\infty} \tag{E.113}
\end{equation*}
$$

Note that $\theta_{q}(w)=\theta_{q}\left(q w^{-1}\right)=-w \theta_{q}\left(w^{-1}\right)$ and

$$
\begin{align*}
\vartheta_{1}(z, \tau) & =-i q^{1 / 8} y^{1 / 2}(q ; q)_{\infty} \theta_{q}\left(y^{-1}\right)  \tag{E.114}\\
& =i q^{1 / 8} y^{-1 / 2}(q ; q)_{\infty} \theta_{q}(y) \tag{E.115}
\end{align*}
$$

where $y=e^{2 \pi i z}$ and $q=e^{2 \pi i \tau}$.
${ }^{*}$ The notation $\theta_{q}$ should not be confused with $\theta$ defined at the beginning of the section.

Proposition E.28. For $\alpha, \beta \in \mathbb{C}^{\times}$, we have the limits

$$
\begin{equation*}
\lim _{z \rightarrow 0} \frac{\vartheta_{1}(\alpha z, \tau)}{\vartheta_{1}(\beta z, \tau)}=\frac{\alpha}{\beta}=\lim _{z \rightarrow 0} \frac{\theta_{q}\left(e^{2 \pi i \alpha z}\right)}{\theta_{q}\left(e^{2 \pi i \beta z}\right)} \quad q=e^{2 \pi i \tau} \tag{E.116}
\end{equation*}
$$

Proof. The first equality follows from the identity

$$
\begin{equation*}
\frac{\vartheta_{1}(\alpha z, \tau)}{\vartheta_{1}(\beta z, \tau)}=\frac{\theta_{1}\left(\alpha \pi z ; e^{\pi i \tau}\right)}{\theta_{1}\left(\beta \pi z ; e^{\pi i \tau}\right)} \tag{E.117}
\end{equation*}
$$

The second equality follows from the identity

$$
\begin{equation*}
\frac{\theta_{q}\left(e^{2 \pi i \alpha z}\right)}{\theta_{q}\left(e^{2 \pi i \beta z}\right)}=e^{\pi i(\alpha-\beta) z} \frac{\vartheta_{1}(\alpha z, \tau)}{\vartheta_{1}(\beta z, \tau)} \tag{E.118}
\end{equation*}
$$

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## Postface

What is best in mathematics deserves not merely to be learned as a task, but to be assimilated as a part of daily thought, and brought again and again before the mind with ever-renewed encouragement. Real life is, to most men, a long second-best, a perpetual compromise between the real and the possible; but the world of pure reason knows no compromise, no practical limitations, no barrier to the creative activity embodying in splendid edifices the passionate aspiration after the perfect from which all great work springs. Remote from human passions, remote even from the pitiful facts of nature, the generations have gradually created an ordered cosmos, where pure thought can dwell as in its natural home, and where one, at least, of our nobler impulses can escape from the dreary exile of the natural world. - B. Russell
E.4.1. Errata. In an attempt to write a self-contained and interdisciplinary treatise, I have included substantial background material and an extensive bibliography (covering all three volumes). I have taken great care (and even greater grief) to establish a smooth, connected and relatively elementary presentation. Such an enormous undertaking requires equally daunting, aggressive and continued revision often yielding innumerably distinct versions. Continuing in this manner with the aim of utter perfection is not only futile but also ultimately incompatible with the timely completion of a doctorate. The present work is therefore a draft composition that should be read critically and with caution.

This draft version is dated: March 2013.

Errata, typographical or otherwise, are to be expected, so a corrigendum shall be written. Should the dear reader be so moved to action by the discovery of such abominable creatures, contacting the author is well-advised, welcomed and certainly appreciated. The most current version of this work can be obtained directly from the author.
E.4.2. Colophon. This document was prepared, compiled and formatted by the author using $\operatorname{LAT}_{\mathrm{E}} \mathrm{X}$ and its derivatives AMS-EATEX and XeTeX (via TeX Live 2010-2012 for Mac OS X) with the amsart document class and mathpazo (Palatino) font type with fncychap (modified Lenny) chapter style. For easier navigation, we use the hyperref package for both internal and external hyperlinking of table of contents, indices, parts, chapters, sections, subsections, propositions, corollaries, equations, tables, figures, citations and URLS, etc., to their respective labels. The formatting of this work conforms to the Form of the PhD Dissertation (January 2013) written by the Graduate School of Arts and Sciences (GSAS) at Harvard University, including those additional guidelines mandated by the Registrar's Office.

The author has made extensive use of the technical computing software Mathematica for numerical calculations and the computer algebra system Singular for computing singularity type, Jacobi ideals, local algebras and related numerical invariants.

The ornate border on the front (volume) cover page is a modified reproduction from W. R. Tymm and Sir M. Digby Wyatt's The Art of Illuminating
as practised in Eurofe from the Earliest Times: Illustrated by Borders, Initial Letters, and Alphabets (1860). The rose image in the preface is an unmodified reproduction from Blackie's Little Ones' Annual: Stories and Poems for Little People (1890). Most of the knot and link diagrams in Volume 1 can be found in Wikimedia Commons and are considered non-copyrighted, within the public domain and subject to fair use [389].

The figure of torus links ordered by crossing number was drawn using the knot/links graphing suite KnotPlot [418]. The figure of the five platonic solids was drawn using Polyhedra Stellation Applet [73]. Unless stated otherwise, the remaining figures were drawn by the author using a combination of Mathematica, Omnigraffle, and LATEX $X^{i} T$.
E.4.3. Veritate. Despite what is written on the official title page-a requirement to receive the doctoral degree-the subject matter of this work is clearly Mathematics and Mathematical Physics. Moreover, it was written in the Lyman Physical Laboratory in the Department of Physics at Harvard University.
E.4.4. Tractus. The dear reader may well wonder-rightly so-why this work is rather lengthy. I offer the following Aesopean fable as my reply.

## The Boy and the Nettles*

While gathering berries from a hedge, a boy's hand was stung by a nettle. He ran to his mother, and with great agony said to her, "I touched a nettle ever so slightly! Why does it hurt so?" "This, my dear boy, is why you got stung," she replied. "Had you grasped it boldly, with all of your might, it wouldn't have hurt you in the least."


Figure E.1. A Common Nettle (Urtica dioica) [389]

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[^0]:    *I dare say even impudent curio, for one often finds its mysterious silhouette cast on the most unlikely of mathematical problems without reason or cause or so much as any hint of an invitation.

[^1]:    *Bitter are the roots of study, but how sweet their fruit.

[^2]:    *A conjectured spin-2 massless boson, the graviton, may mediate the gravitational force, and dark matter and weakly interacting massive particles (WIMPS) may account for the remaining unobserved mass in the universe.

[^3]:    *As is customary in quantum field theory, we assume natural units, e.g., $\hbar=c=1$.
    ${ }^{\dagger}$ More precisely, the GWSM postulates an $\mathrm{SU}(2)_{L} \times \mathrm{U}(1)_{Y}$ invariant Lagrangian containing four massless (precursor) scalar fields $A_{1}, A_{2}, A_{3}$ and $B$ and a single complex (Higgs) doublet $\Phi$. The Higgs field is a left-handed doublet with weak isospin $+\frac{1}{2}$ and hypercharge +1 that preserves $U(1)_{E M}$ but spontaneously breaks $S U(2)_{L} \times U(1)_{Y}$, resulting in a non-zero vacuum expectation value of the Higgs field $v_{h}$, two charged, massive vector bosons, $W^{+}$and $W^{-}$(from linear combinations of $A_{1}$ and $A_{2}$ ) and a neutral, massive vector boson, $Z$ (from linear combinations of $A_{3}$ and $B$ ), a massless photon, $\gamma$ (from linear combinations of $A_{3}$ and $B$ ), and a massive scalar $h$, the Higgs boson.

[^4]:    *Exotic baryons (e.g., tetraquark and pentaquark bound states) should exist but have not yet been definitively observed.
    ${ }^{\dagger}$ The representation theory of the Poincaré (spacetime symmetry) group and the internal symmetry groups (isospin, flavor, etc.) including their Lie algebras, govern transformations and mass spectra of the Standard Model.

[^5]:    *By enhancing further still the Standard Model to a Two-Higgs-Doublet Model (THDM) [59], the Lee-Quigg-Thacker bound of the lightest Higgs boson can be improved to $m_{h} \leqslant 411$ GeV [238].

[^6]:    *As there is no notion of spin in less than three dimensions, the existence of a graded Lie algebra of field operators suffices to define a two-dimensional, supersymmetric quantum theory.

[^7]:    *In the MSSM, superpartners also share gauge numbers (viz., color charge, weak isospin charge, hypercharge).
    ${ }^{\dagger}$ This is the lowest order in perturbation theory and considers only interactions with loopless Feynman diagrams, hence the name.
    $\ddagger$ The mass of the CP-odd Higgs $A$ is assumed to be significantly larger than that of $Z$.

[^8]:    *By late 2012, CERN measurements had been improved to $126.0 \pm 0.4$ (stat) $\pm 0.4$ (syst) GeV .

[^9]:    *The equality above suggests that the Fredholm index depends on the singularity structure of $f=\partial V$ at infinity, since $\operatorname{deg} f=\lim \sup _{r \rightarrow \infty} \frac{\log f(r)}{\log r}$, where $f(r)=\max \{|f(z)|| | z \mid=r\}$.

[^10]:    ${ }^{*}$ The elliptic genus is a graded invariant arising from the categorification of the $\mathrm{WZ}_{\theta, \phi}$ model in much the same way that the Jones polynomial is regarded as the graded Euler characteristic Khovanov Homology of the corresponding knot.

[^11]:    *Knots of the form $T_{p, 2} \simeq T_{2, p}$ are often denoted by the Alexander-Briggs notation $p_{1}$, e.g, $T_{2,3}$ is $3_{1}$.

[^12]:    * This is one example of Jaffe's Unreasonable Effectiveness of Physics in Mathematics, a lecture in some sense dual to Wigner's Unreasonable Effectiveness of Mathematics in the Natural Sciences.

[^13]:    *The conjecture states that there are no integral solutions of $a^{d}+b^{d}=c^{d}$ for $d>2$ save the trivial solutions satisfying $a b c=0$. In 1995, Andrew Wiles published a proof of the TaniyamaShimura Conjecture, which implied the Fermat Conjecture, based on the work of Hellegouarch, Frey, Serre, Ribet, Flach and Kalyvagin.

[^14]:    *That is, singular homology with $\mathbb{Z}$-coefficients.

[^15]:    * Note the distinction between homotopy invariance and homotopy equivalence (or homotopy-type equivalence). The former is a statement about maps; the latter concerns topological spaces.

[^16]:    *The origin plays no special part other than one of convention. This analysis may take place at any singular point of $f$.

[^17]:    *In this special case, one says that the complex analytic germ $f$ is non-degenerate at the origin, or simply non-degenerate when the context is clear.

[^18]:    *A simply connected, closed 3-manifold is a topological 3-sphere, i.e., homeomorphic to $S^{3}$.

[^19]:    *Arising from, say, $f=x^{2}+y^{2}$ over $\mathbb{C}^{2}$.

[^20]:    * Of course, such an invariant can be defined in a neighborhood about any singular point.
    ${ }^{\dagger}$ That is, an undirected, unweighted, finite, connected, loop-free graph.

[^21]:    *The Euler characteristic does not satisfy the Excision Property for general CW complexes.

[^22]:    ${ }^{*}$ Given a set of non-degenerate, complex analytic germs $f_{i}:\left(\mathbb{C}^{n_{i}+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$, denote the corresponding fibers by $F_{f_{i}}=F_{f_{i}, 0}$ and topological indices by $\mu_{i}=\mu_{\mathrm{top}}\left(f_{i}\right)$. The wedge sum $\bigvee_{i} F_{i}$ would necessarily have the homotopy-type of a wedge sum of mixed spheres, $\bigvee_{i} \bigvee^{\mu_{i}} S^{n_{i}}$. Such a homotopy-type cannot be realized by Milnor's construction as a single composite fiber unless, of course, the integers $n_{i}$ are identical or if the complex analytic germs are degenerate.
    ${ }^{\dagger}$ This is not to be confused with set complement.

[^23]:    ${ }^{*}$ Moreover, $P_{\bigvee_{i=1}^{r} S^{n} \star \bigvee_{j=1}^{s} s^{m}}(t)=P_{\bigvee^{r s} S^{n+m+1}}(t)=r s t^{n+m+1}+1$.

[^24]:    * We use a superscript rather than a subscript to denote the Grothendieck group $K^{p}$ since it is a cohomology group stemming from an exceptional cohomology theory.

[^25]:    *Hereafter, weights shall refer to either integral weights and weighted degree or reduced weights depending on context or unless otherwise specified.

[^26]:    *The weight $\omega$ is not a vector or a set rather a multiset, wherein multiplicity matters but order does not.

[^27]:    *The Kronecker summation of matrices is not to be confused with the standard matrix direct summation that invariably takes a block diagonal form.

[^28]:    ${ }^{*}$ A complex analytic map $f:\left(\mathbb{C}^{n+1}, \mathbf{0}\right) \rightarrow(\mathbb{C}, 0)$ is a real map $f:\left(\mathbb{R}^{2 n+2}, \mathbf{0}\right) \rightarrow\left(\mathbb{R}^{2}, \mathbf{0}\right)$.

[^29]:    *Presumably, there is a higher-dimensional generalization of this claim. However, the author is not aware of a such a generalization at the time of writing.

[^30]:    *It is important to recall that there are non-Brieskorn-Pham singularities with the same weight set, $\left\{\frac{1}{\bar{p}}, \frac{1}{q}\right\}$. Such polynomials are quasi-Brieskorn-Pham since their weights are inverse integers. If $p$ divides $q$ and $0 \leqslant a, c \leqslant p$, then $f=x^{a} y^{b}+x^{c} y^{d}$ is one such example provided that $b=\frac{q}{p}(p-a)$ and $d=\frac{q}{p}(p-c)$. For instance, $f=x^{2}+x y^{2}$ is non-degenerate with weights $\left\{\frac{1}{2}, \frac{1}{4}\right\}$.

[^31]:    ${ }^{*}$ Recall that a weak homotopy equivalence of (path-connected, pointed) spaces (i.e., $\pi_{i}(X) \cong$ $\pi_{i}(Y)$ for $i \geqslant 0$ ) does not imply homotopy equivalence (i.e., $X \simeq Y$ ) of said spaces, although the converse is true.

[^32]:    *The standard notation for the inner modality is $\mu_{0}$, but we use $\bar{\mu}_{0}$ to avoid conflict with the $0^{\text {th }}$-coefficient of the Hilbert-Poincaré series of a local algebra, $\mu_{0}=\operatorname{dim}_{\mathbb{C}} \mathbb{C}=1$.

[^33]:    ${ }^{*}$ Here, $\{\cdot\}$ is used to denote the fractional part function and the delimiters of a set. The context should clearly differentiate the two.

[^34]:    *Mathematics is the Queen of Sciences and Number Theory is the Queen of Mathematics.

[^35]:    *A magma is a closed set under a binary operation.

[^36]:    *Here, one is essentially computing the homology of the fiber with coefficients in $\mathbb{Z}^{d}$.

[^37]:    *A classical result states that the row and column ranks are equal for any matrix. We use the terminology full row/column rank simply to distinguish the proper rectangular matrices from the square matrices.

[^38]:    *A conjectured spin-2 massless boson, the graviton, may mediate the gravitational force, and dark matter and weakly interacting massive particles (WIMPS) may account for the remaining unobserved mass in the universe.

[^39]:    *Exotic baryons (e.g., tetraquark and pentaquark bound states) should exist but have not yet been definitively observed.
    ${ }^{\dagger}$ The representation theory of the Poincaré (spacetime symmetry) group and the internal symmetry groups (isospin, flavor, etc.) including their Lie algebras, govern transformations and mass spectra of the SM.

[^40]:    *By enhancing further still the SM to a Two-Higgs-Doublet Model (THDM) [59], the Lee-Quigg-Thacker bound of the lightest Higgs boson can be improved to $m_{h} \leqslant 411 \mathrm{GeV}$ [238].

[^41]:    *As there is no notion of spin in less than three dimensions, the existence of a graded Lie algebra of field operators suffices to define a two-dimensional, supersymmetric quantum theory.

[^42]:    *The mass of the CP-odd Higgs $A$ is assumed to be significantly larger than that of $Z$.
    ${ }^{\dagger}$ By late 2012, CERN measurements had been improved to $126.0 \pm 0.4$ (stat) $\pm 0.4$ (syst) GeV .

[^43]:    *The equality above suggests that the Witten index depends on the singularity structure of $f=\partial V$ at infinity, since $\operatorname{deg} f=\lim \sup _{r \rightarrow \infty} \frac{\log f(r)}{\log r}$, where $f(r)=\max \{|f(z)|| | z \mid=r\}$.

[^44]:    ${ }^{*}$ The elliptic genus is a graded invariant arising from the categorification of the $\mathrm{WZ}_{\theta, \phi}$ model in much the same way that the Jones polynomial is regarded as the graded Euler characteristic Khovanov Homology of the corresponding knot.

[^45]:    *The author would like to thank the user anon on Math.SE for providing these two transformation identities.

[^46]:    *Here, $\{\cdot\}$ is used to denote the fractional part function and the delimiters of a set. The context should clearly differentiate the two.

[^47]:    ${ }^{*}$ The notation $a_{b}$ is to be interpreted as the integer $a$ repeated $b$ times.

[^48]:    *Hyperlinked pages (linking to pages in the text where a given citation appears) can be found immediately after the corresponding entry in the Bibliography.

[^49]:    *Adapted from Aesor's Fables as translated by V. S. Vernon Jones.

