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Citation	Izard, Véronique, Pierre Pica, Elizabeth S. Spelke, and Stanislas Dehaene. 2008. Exact equality and successor function: two key concepts on the path towards understanding exact numbers. Philosophical Psychology 21(4): 491-505.
Published Version	doi:10.1080/09515080802285354
Accessed	February 19, 2015 10:54:01 AM EST
Citable Link	http://nrs.harvard.edu/urn-3:HUL.InstRepos:10139289
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Exact Equality and Successor Function: Two Key Concepts on the Path towards understanding Exact Numbers

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Abstract

Humans possess two nonverbal systems capable of representing numbers, both limited in their representational power: the first one represents numbers in an approximate fashion, and the second one conveys information about small numbers only. Conception of exact large numbers has therefore been thought to arise from the manipulation of exact numerical symbols. Here, we focus on two fundamental properties of the exact numbers, as prerequisites to the concept of exact numbers: the fact that all numbers can be generated by a successor function, and the fact that equality between numbers can be defined in an exact fashion. We discuss some recent findings assessing how speakers of Mundurucu (an Amazonian

language), and young western children (3-4 years old) understand these fundamental properties of numbers.

1. Roots of Human Numerical Competences

Our numerical competences have been grounded on two core cognitive systems, shared with animals and possibly present from birth. On one hand, animals, and preverbal infants aged at least 4 ½ months represent numbers in the format of analogical internal magnitudes (Dehaene, 1997; Cantlon & Brannon, 2006; Dehaene, 2007). One characteristic of the internal magnitude system is that it represents numbers only in an approximate fashion, and more specifically follows the Weber's law: the extent to which two numbers can be discriminated depends on their ratio (Xu & Spelke, 2000; Feigenson, 2007; Nieder & Miller, 2003). Besides human infants and animals, instances of Weber's law for numbers have also been reported in occidental adults (Piazza et al., 2004; Whalen et al., 1999), and in remote populations, who have not received any formal education in mathematics, and speak a language with a restricted numerical lexicon (Gordon, 2004; Pica et al., 2004).

In addition to the internal magnitude system, infants may also be able to use their attentional resources to extract numerical information from displays containing only a small number of objects. Like adults, infants can track several objects in parallel (up to 3), and they are thought to engage attentional indexes to do so, pointing to the objects to track (Feigenson et al., 2004; Carey, 2004). Although attentional indexes do not encode numerical information in an explicit way, they can convey some numerical information, because the number of indexes engaged corresponds to the number of items in a display. Implicit numerical information of this kind can be sufficient to solve a variety of numerical tasks: for instance, it can support infants' ability to form accurate expectations of how many objects have been hidden behind an occluder, and to update their expectations whenever an object is either added or subtracted (Simon, 1999; Uller et al., 1999). However, the role this system plays in development has been debated (Gelman and Butterworth, 2005 vs LeCorre and Carey, 2007); in particular, it is not clear yet how the implicit numerical informations conveyed by the attentional resources can eventually be integrated with analog numerical representations.

2. Contrasting Natural Integers with Preverbal Representations of Quantity

Although natural integers are the target of world-wide numerical symbol systems (Ifrah & Bellos 2000), and constitute the ground of mathematical theories of numbers, they cannot be grasped by the core systems on which understanding of numbers is rooted. The object tracking system is sensitive to exact numerosities, but it is limited by the number of indexes available, which seems to be around 3 or 4. On the other hand, the internal magnitude system encodes numbers only in an approximate way, such that as quantities increase, representations become more and more confuse, thus making exact discrimination of numbers impossible.

Natural integers possess several properties which distinguish them fundamentally from a representation of approximate quantities. In particular, two different properties of the integers have been highlighted by philosophers and mathematicians as possible starting points to give a formal definition of the integers. The definition of *cardinal* integers, by Frege or Russell, capitalizes on the property of exact equality between numbers, which is instantiated through one-to-one correspondence. On the other hand, Peano derived *ordinal* integers from a successor function, which he applied recursively. Both definitions target the same mathematical object: thus, either property (exact equality or successor function) is sufficient to define the integers, and each of these two properties implies the other. In fact, both properties are equivalent to the existence of a minimal quantity, ONE, which also corresponds to the minimal distance between two numbers.

However, neither of these fundamental properties can be defined on a set of approximate numerical representations. When numbers are represented approximately via internal magnitudes, there is no restriction on the possible values taken by the magnitude representation. Indeed, to explain the occurrence of Weber's law, models of the representation of internal magnitudes have introduced some source of noise into the representation (Dehaene, 2007; Gallistel & Gelman, 2000). As a consequence, the magnitude corresponding to a given number does not always take a fixed value, but is distributed around an average value, with the possibility to take any value in a continuous interval. Consequently, equality between magnitudes can only be defined approximately: magnitudes are declared equal when they are close enough. Also, because there is no closest neighbour to a given value for continuous magnitudes, it is impossible to define the notion of a unique successor number or to order all the values of the magnitudes in a discrete list.

3. Theories of the Development of Exact Numbers

How do children overcome the limitations imposed by their core systems of numeric representations, and finally grasp the concept of exact numbers? Even if the properties of exact equality between numbers, and of the existence of a successor for each number, derive logically from each other, they may not be acquired simultaneously. It is possible that children grasp the notions of exact equality and succession independently, and from different sources, and then integrate these initial pieces of knowledge to derive full-blown exact number concepts.

Different theories have been proposed to account for children's acquisition of exact number concepts. Here, because our goal is to assess the acquisition of the premises of the concept of integer (the successor function and of the notion of exact equality), rather than the concept of integer itself, we focus on the theories that make predictions on these issues.

According to some theorists (Leslie et al., in press; Gelman and Butterworth, 2005; Gelman & Gallistel, 1978), the child possesses an innate system to represent exact numbers. Leslie et al give two proposals for the exact nature of that system. Their first proposal centers on an innate relation of exact equality: first, children know that the individual items of sets can be exchanged without changing the cardinal value of the set; later, they learn that applying the same operation (either addition, subtraction, multiplication, or division) to two sets that are initially equal conserves equality. Moreover, children know that equality is not maintained after additions or subtractions: In particular, addition or subtraction of one or more items changes the cardinal of a set. According to this first proposal, therefore, children have a concept of exact equality to start with; the other properties of exact integers, such as the successor function, are grasped later.

Leslie et al. also discuss an alternative scenario, in which children begin with an intuition about the successor function. Using the successor function, children could construct a mental model of natural integers, akin to a grid, which would have to be calibrated on the internal magnitude system. This proposal capitalizes on the recursion faculty, because the grid is constructed by applying the successor function recursively. Leslie et al. do not specify whether or not the innate exact number representations depend on the language faculty, but their second proposal is reminiscent of a hypothesis originally developed by Chomsky, who postulated that exact numbers come with the language faculty, and in particular the faculty for recursion (Hauser et al. 2002). This hypothesis predicts that only humans should be able to grasp exact numbers, because other animals do not possess the faculty for recursion. Nevertheless, the competence for exact numbers would not depend on the particular language spoken, and in particular it would not depend on the existence of numeral terms in the lexicon. Rather, the apprehension of exact numbers would depend on the presence of universal recursive properties of natural language syntax.

Others have postulated that natural integers do not arise from our innate endowment, but are a construction, driven by cultural inputs. One first set of proposals (the *bootstrapping* theory) emphasizes the role of the linguistic symbols, and in particular the verbal counting list (Carey, 2004; Hurford, 1987; Klahr & Wallace, 1976). According to these theories, the children first discover the meanings of small number words ('one', 'two', 'three'), which they link to the mental models given by their object tracking system. After some practice with the counting list, which they have learned to recite in order, children eventually discover the rule ordering number words in the counting list: two is one more than one, and three is one more than two. At this point, they hypothesize that this rule is valid through the entire counting list, and generalize it to the last counting words as well. On this view, words first serve as placeholders: children learn the list of counting words first, and give them a numerical meaning only afterwards, as they infer the principle of the successor number from the organization of the list. Concepts of exact numbers fully develop later, on the basis of children's earlier-developing knowledge of the successor principle. In this view, the organization of number words in a list is crucial: in the absence of contact with a counting list (be it in a linguistic or non-linguistic format), exact number concepts would not develop.

A third alternative account focuses on the conception of exact equality through one-to-one correspondence (Mix et al., 2002), as a precursor to conceiving exact numbers. Children can manage situations of one-to-one correspondence, for example when distributing objects among people ("one for me, one for you"), or when they use their attentional indexes to track individuals and assess numerical equivalence between sets. According to this account, children are progressively driven to differentiate the quantification of discrete numbers on one hand, and the quantification of continuous amount on the other hand (the only type of quantification made possible by approximate magnitude representations), by realizing that one-to-one correspondence applies to the former but not to the latter. Based on this account, we could imagine exact numbers to derive from the understanding of one-to-one correspondence, with no role for the language. In this case, exact number concepts would be universal among all human beings, and shared by any animal species that can understand one-to-one correspondence.

4. Young Children's Performances in Numeric Tasks

While they still have access to both approximate representations of numerosities and attentional models of sets for small numbers, toddlers start to acquire the numerical symbols for natural integers. Number words are part of children's very early vocabulary. From the age of 2, children learn to recite the beginning of the counting list, and start to count arrays of items, with an increasing proficiency (Fuson, 1988). However, their understanding of the meaning of the number words is limited, as revealed by the Give-A-Number task (Wynn, 1990). At an age when they are able to recite the counting list up to 10 items, children are surprisingly unable to hand a set of two objects to an experimenter, when probed to do so verbally. Instead of conforming to the instructions and produce sets of 2, 3, 4 or more objects, children simply grab a bunch of objects and give them to the experimenter, with no relation between the number given and the number requested. With age, they successively learn to produce sets of 1, 2, 3, and 4 objects with consistency; still they do not use counting to do so. While learning about the small number words, they still give an invariable quantity for all the larger number words. For example, if a child has learned to give sets of 1 and 2 objects accurately, she will give on average the same number of objects for sets of more than 2 objects, but the number of objects given will invariably be larger than 2. Children's pattern of performance at the Give-a-number task is not isolated. The same pattern is observed in a production task, called What's-On-This-Card, where children are asked to describe a series of cards depicting arrays of objects (LeCorre et al., 2006).

These two tasks indicate that younger children seem to attribute an undifferentiated meaning to all the larger number words, as if all number words meant ‘a lot’. Because these children know the meaning of a few small number words only, and do not use counting productively, this state of knowledge is referred to as ‘subset-knower’ or ‘non-counter’ in the literature.

More than one year after they have started to learn counting, at the age of 3 ½, children become able to produce sets for any number in their counting range in the Give-A-Number task, and to quantify arrays correctly in the What's-On-This-Card task. At this age, they systematically apply counting to these situations. Interestingly, after this step, they start to use counting not only for large numbers, but even for the smallest numbers, where they used to succeed by grasping the objects all at once. Because they show a flexible use of the counting procedure, these children are called ‘counters’ or ‘counting principle knowers’ (CP-knowers) in the literature.

The exact nature of the insight that children experience when they reach the state of CP-knower is unclear. In order to understand it better, we need to determine what children know just before this insight, what triggers it, and also what they finally derive from their newly acquired numeric competences. In particular, even if children start using number words in a way that it consistent with exact numbers, they might not have constructed fully consistent integer concepts by then. They understand that counting gives a good answer to the question ‘how many’, because it gives a good answer in the case of small numbers, but still they might not understand that it gives the only possible answer. Children may develop a full understanding of exact number concepts (including understanding of the notion of exact equality, and of succession) in the course of their further practise with counting. Alternatively, it might be necessary for children to develop at least some aspects of the concept of exact numbers before they can get to a fuller understanding of counting and of the number words. In this case, either a prior knowledge of the principles of exact equality, or of the successor function, or both, would be necessary for children to realize how counting gives an assessment of quantity.

4.1 Children's knowledge of the successor principles

Evidence shows that children develop some knowledge of the successor principles just before they become CP-Knowers (Sarnecka & Carey, in prep), therefore understanding the successor function might act as trigger for the counting insight. Sarnecka and Gelman presented children with two trays containing the same quantity of marbles, while announcing them how many marbles were on each plate ("look, here I have 5 marbles, and here I have 5 marbles as well"). Then, in full view of the child, they took one marble on one plate, and added it to the other plate. At the end of this transformation, they asked the child to indicate which plate contained six or four marbles. To succeed at this test, children need to have uncovered at least one aspect of how the successor function is instantiated in the counting list: they need to know that the word that appears just after (or before) a given numeral in the counting list refers to a larger (or smaller) quantity. Both CP-knowers and advanced subset-knowlers choose correctly between the two alternatives. Thus, having a partial knowledge of the successor function might act as a trigger for children to make the step towards becoming a CP-knower. In a second experiment, Sarnecka and Carey studied a different aspect of the successor function: they evaluated whether children know that going to the next word in the count list corresponds to the addition of only one object, as opposed to several objects. Only CP-knowers were able to solve this second task, suggesting that late subset-knowers have only a limited understanding of how the counting list instantiates the successor function.

In fact, even CP-knowers do not always use number words in a way that is consistent with an understanding of the successor principle, and its immediate consequence that number words appearing later in the counting list correspond to larger numbers. LeCorre and Carey (2007) asked children to estimate the number of objects in an array, rapidly and without counting. They observed a further distinction among CP-knowers, between *mappers*, whose responses increased with the numerosity of stimuli, and *non-mappers*, who did not. Therefore, even if children start to understand some aspects of the successor principles before they can use counting productively, they do not derive the full consequences of this principle until several months after they become CP-knowers.

4.2 Children's knowledge of the exact equality principles

A pioneer in the investigation of the child's concept of number, Piaget carried out and inspired extensive research on children's understanding of one-to-one correspondence, which indicated that children aged 2 to 5 fail to rely on a formerly established relation of one-to-one correspondence to assess the numerical equality between two sets, as soon as the elements of one set have been moved. Instead, these children rely on some other non-numerical cues such as the extent of the sets (Piaget, 1965). Piaget concluded from his studies that children do not conceive numbers before the age of 5-7. This interpretation of the results have been contested since, based on evidence with children aged 4 and more, who showed that they understood that the one-to-one correspondence relationship was conserved in some slightly modified versions of the task, when the design of the experiment minimized misinterpretations of the conservation question (Mehler & Bever, 1967; McGarrigle & Donaldson, 1974). Furthermore, at the age of 5, children interpret number words as referring to exact numbers, even for numbers beyond their counting range (Lipton & Spelke, 2005). Finally, 5-year-olds recognize that addition and subtraction of the same quantity cancel each other (Gilmore & Spelke, in press). Therefore, a full understanding of the principles of exact equality has developed by the age of five.

Results for younger children are more nuanced. In a first experiment, Sarnecka and Gelman showed that subset-knowers as well as CP-knowers considered that the word "six” no longer applied after the addition of one object (Sarnecka & Gelman, 2004). In this experiment, objects were placed in a box in front of the child, while the experimenter narrated how many objects were in the box ("I am putting 6 moons in this box"). Then a transformation occurred in full view of the child, where the experimenter either removed or added one object. In control trials, the box was shaken. At the end of the transformation, the child was asked how many objects the box contained, with a choice between two number words, one being the initial word, the other being the correct answer on subtraction or addition trials ("Now, how many moons do I have in the box, is it five or is it six?"). Children tended to choose the initial word (e.g. ‘six’) when the box had been just shaken, and the alternative number word (e.g. ‘five’) after an addition or a subtraction. However, two slightly different experiments attempting to address the same problem yielded contradictory results. The first experiment is also reported by Sarnecka and Gelman (2004). Here, children were shown two arrays in one-to-one correspondence, either containing the same number of objects, or a different number of objects, a fact that children easily assessed. Then, the experimenter told them how many objects the first array contained, and asked them about the quantity of objects in the second array. Children who had correctly judged the arrays to be identical or different on the basis of one-to-one correspondence performed at chance when they had to choose between an identical or different number word. Similarly, Condry and Spelke (2008) observed a failure in a slightly more complicated task. Children were given two trays with the same number of objects, announcing how many objects were on each plate (eg five). Then, the experimenter added one object on one of the two plates. Children were subsequently asked to point to the plate containing five objects (the same number word) or six objects (a different number word). Again, subset-knowers performed at chance in this task.

Sarnecka and Gelman's first experiment, where children are successful, differs from the two other ones in a crucial way. Although the whole arrays stay in full view of children throughout the trial in their second experiment as well as in Condry and Spelke's experiment, in their first experiment the transformation occurs while the array is hidden. When the children are never shown the final visual arrays, their responses accord to an exact interpretation of the number words. However, when they can perceive the arrays which they need to contrast, they do not differentiate the number words, even if the difference between the two arrays is evidenced by a one-to-one correspondence setting. In all experiments, the two arrays are nevertheless very close in terms of numerosity: they are separated by a ratio of 5/6, which is hard to discriminate on the basis of approximate internal magnitudes representations. It is possible that, when they have to compute the result of the transformation themselves, e.g. when the array is hidden, children overestimate the effect of this transformation, therefore choose a different number word to qualify the final quantity. Such hypothesis would be in line with the fact that adults show a momentum effect which leads them to overestimate the result of an addition on non-symbolic arrays (and similarly underestimate the result of a subtraction) (McCrink et al., 2007). At least, the fact that children's performances in choosing a same or different number word depend on the visibility of the arrays indicates a limited understanding of which transformation affect number per se, independently of their interpretation of the number words. Therefore, subset-knowers appear to have a poor knowledge, if any, of the concept of exact equality.

4.3. A Study of the Concept of Exact Equality

As stressed before, it is possible that understanding exact equality precedes the acquisition of a full-blown natural integer concept, and that exact equality is acquired independently from the notion of successor number, and at different times. In order to test the knowledge of the principles of exact equality in 32-36-month-olds, we designed a task where children were presented with two sets in a one-to-one correspondence relation, and could use this relation to give a judgment of exact equality (Izard et al., in prep). The task was set as a game, where children were invited to play with finger puppets. At the beginning of each trial, a set of 5 or 6 identical puppets were fixed on a tree with 6 branches, creating a one-to-one correspondence between the branches and the puppets. Then, the puppets disappeared in an opaque box. After some time, puppets were brought back to the branches: the experimenter helped the child to get the first 5 puppets and fix them on the tree, and then we recorded whether the child went back to search in the box for an extra 6th puppet.

In this basic condition, children used the one-to-one correspondence between branches and puppets successfully: they searched more when the box was supposed to contain one extra puppet (when the set contained 6 puppets) than when it was supposed to be empty (a set of 5 puppets). In contrast, in a control condition where the tree contained more branches, so that one-to-one correspondence cues were not available, children did not discriminate between sets of 5 and 6 puppets.

Additional conditions were designed to test for the children's knowledge of principles of exact equality, where some events could happen while the puppets were hidden in the box. In a first condition, one extra puppet was added to the box, or one puppet was removed from the box. In a second condition, one branch was either added or subtracted from the tree. Finally, in a third condition, one puppet was subtracted from the box, and then was replaced by another one. In all these conditions, children performed at chance. Note that, since they searched equivalently in trials where the original setting showed an empty branch and in trials where there was no empty branch, they did not act as if no transformation had occurred, rather they acted as if they suspected that the outcome might be different from the original setting, yet they could not tell what the exact outcome should be.

Two subsequent tests assessed whether the children's failure could not be explained by invoking an extra processing load introduced by the transformation. When tested on small numbers (2-3 puppets and branches), with the exact same addition/subtraction scenarios, children were flawless. They also successfully solved another condition involving large numbers, where one puppet was taken out of the box and then eventually put back in the box. Because these conditions required processing of the same transformations and performance of the same response, their contrasting findings suggest that children's errors in the earlier tasks stem from limits to their representation of exact cardinal values.

In summary, these results show that young children aged 2:8 to 3:0 years have some knowledge of the principles of exact equality, but this knowledge is limited: they know that the cardinal of a set stays exactly the same, as long as the items forming the sets do not change, and they can use one-to-one correspondence to track the cardinal of a given set. Yet, when the identity of the items forming the set is modified by addition, subtraction, or replacement, they suspect the cardinal might have changed, but they can not discriminate within all types of transformations which do or do not change the cardinal of the set. These results reveal a larger understanding of one-to-one correspondence than Piaget's classical conservation task, but less understanding than that proposed by Leslie et al. These children may begin with a first level of knowledge which serves as a trigger to understand the principles of exact equality.

How does knowledge of exact equality relate to the insight enabling a child to become a CP-knower? Most of the children tested here were subset-knowers, however, a few (young) CP-knowers were also tested and they did not perform better than the other participants. Thus, in western children, the acquisition of the principles of exact equality seems to be independent of the discovery of the successor number principle, which is thought to underly the transition from the subset-knower to the CP-knower stage.

5. Another case study: Mundurucu speakers

In order to test whether knowledge of exact equality principles could develop in the absence of practice with counting, we adapted the task to test a different population, the Mundurucus. The Mundurucus are an Amazonian indigene group, who for some of them live in relative isolation from occidental civilisation. In the most remote locations, for example upstream from the Cururu Mission, most Mundurucus do not receive any formal education in mathematics (Pica et al., 2004; Dehaene et al., 2006, see also Pica et al. in present volume). Numerical terms in Mundurucu include words corresponding to the small numbers (1-5), although this correspondence is only approximate: for example the word *ebapũg*, which is predominantly used for sets of three objects, can also be applied to sets containing 4 or 2 objects. Larger quantities can sometimes be expressed by combining smaller number words (*pũg pogbi xepxep bodi*, five and two on the other side), or applying a double reduplication to a number word (*ebapũg pũg pũg*: two on both sides, see Pica et al. in this present volume), although comprehension of these phrases is effortful and not shared by all speakers. For complex phrases, it is not clear for now whether they refer to exact or approximate quantities, but some preliminary data seem to favour the hypothesis of an approximate meaning: when shown a given quantity, corresponding to one of these phrases, Mundurucu tended to judge that the number word still applied to the quantity after one dot had been added, or subtracted, and even when the quantity had been halved or doubled. However, such data should be regarded with caution, because the effect might have been mostly driven by the majority of individuals who have difficulties comprehending these utterances.

Most Mundurucus, even monolingual speakers, are also able to recite the counting list in Portuguese. However, only the Mundurucus who showed a high level of proficiency in

Portuguese judged Portuguese number words to refer to exact quantities, whereas monolinguals interpreted the Portuguese number words as referring to approximate quantities: they attributed the same meaning to the Portuguese number words as the Mundurucu number words. Overall, these preliminary results suggest that most Mundurucu speakers (at least the monolingual speakers) do not possess lexical terms to refer to exact quantities..

A computerized version of our one-to-one correspondence task was developed for the Mundurucus (Izard et al., in preparation). In this version, participants were presented with an animation which involved sets of red and black puzzle pieces and a can. At the beginning of each trial, red and black pieces were presented in one-to-one correspondence, and the can was shown to be empty. The black pieces stayed in place during the whole trial, while the red pieces started to move, and disappeared inside the can. At this point, a transformation occurred which affected the hidden set of red pieces: some pieces were added, or subtracted, or pieces were replaced by other ones. At the end, some of the red pieces came back in front of the black pieces, and participants were invited to judge whether the box was empty or not by clicking on one of two alternatives. In the first type of trials, called *exact trials*, the transformations involved only one piece, and the final choice was between an empty can and a can containing one piece. Some other trials, called *approximate trials*, were included in the test in order to measure the global level of difficulty of the task. In these approximate trials, the transformations involved 5 pieces instead of only one, and the final choice was between an empty can and a can containing 5 pieces. Moreover, we also included some extra trials involving only small numbers (up to 4 pieces, *small number* trials), as a second measure of the task difficulty.

Overall, this task was very hard for the Mundurucus; however, all groups, including monolingual speakers and children (aged 5-12 years), performed above chance level in the *exact* trials. Furthermore, performances were identical for the *exact* trials and for the *approximate* or *small number* trials. Therefore, the performance levels must be explained not by a difficulty to handle exact numbers per se, but by the overall difficulty of the task, which required that they track several successive operations over 20 s[1]. Moreover, note that for the Mundurucus, all the different conditions tested incorporated some sort of transformation, and therefore these conditions corresponded to the conditions on which young occidental children failed. We conclude from this test that Mundurucus understand the principles of exact equality: they know which transformations affect numerosities, and they also know that cardinal value of a set is not affected when the identity of the items forming the set is modified. In other words, they know that addition and subtraction compensate each other, so that adding one item and then subtracting one results in no change in the number of objects.

Further tests confirmed that Mundurucu can use the strategy of one-to-one correspondence spontaneously, even when it is not given to them in the task setting. Hence, we tested them in a match-to-sample task, where they were required to produce a set of seeds to match a number of dots presented to them on a screen, or another set of seeds presented on the table. The sets produced matched the model on average, with some variability. However, the variability was very reduced compared to the predictions of the model of the internal number line, and also the distribution was not Gaussian: exact answers were much more frequent than approximate answers. Very often, the sets produced contained exactly as much seeds as the model set, providing evidence that Mundurucus used some kind of exact strategy to solve the task. Hence, when they were performing the task, they seemed to be silently scanning the items of the model set one by one, and grabbing a seed each time they switched to a new item. Similar results have been obtained recently with speakers of Piraha, a language with a number lexicon even more restricted than Mundurucu (Frank et al., in press), and children from indigene populations in Australia (Butterworth & Reese, this volume), who adopted a particular strategy of matching the configuration of the objects in order to achieve one-to-one correspondence.

Together, these two tests converge to show that Mundurucus can conceive exact equality between numbers, even in the absence of experience with exact number words or of practice with counting. However, since the acquisition of the successor function seems to be triggered or accelerated by experience with the counting list in western children, it is possible that Munducuru speakers do not understand this property. We did not test a full understanding of the successor number principle with the Mundurucus, but studied whether the participants who could recite the counting list in Portuguese (but still were monolingual in Munducuru) could compare number words given in Portuguese. The results show no understanding of the rule ordering the counting list: whenever one of the two words to be compared was larger than 5, Mundurucus performed at chance. This result shows that they have no understanding of the fact that number words that come later in the counting list refers to larger numbers. Their knowledge of the property of successor numbers must be, at best, limited.

Conclusion

From infancy, two nonverbal systems are available to encode numeric information. A first system represents approximate numbers in the form of internal magnitudes. Furthermore, infants possess an attentional mechanism to track objects in sets containing up to 3 items, and this second system can give them access to the number of items present in small sets. Natural integers transcend these two types of representations: although internal magnitude are approximate, integers are defined exactly, up to the precision of one unit; and although object tracking mechanisms are limited to sets of up to 3 objects, natural integers have no limit.

We focused on two properties of the natural integers, which distinguish them from a system of approximate numerosities, and whose understanding constitutes a prerequisite to a full understanding of the concept of exact number. First, equality between integers can be defined exactly. Understanding exact equality requires realizing the two following properties: first, that addition or subtraction operations, even minimal (involving only one item) change the value of an integer; second, that the number of objects in a set is invariant over changes in the identity of the objects: in other words, that addition and subtraction can cancel each other. A second fundamental property of natural integers, which can not be defined on a system of approximate magnitudes, is that each integer has a successor, obtained by the addition of ONE; and that by listing all the successive successors, one can enumerate any of the integers.

Here, we propose that these two aspects of natural integers are acquired independently, and from different sources. Manipulation of exact number words, organized as a counting list, seems to play a role in occidental children’s understanding the notion of a successor number. In young occidental children, the acquisition of a concept of exact equality seems to happen later, and independently from, children's experience with symbols for exact numbers. Understanding exact equality could emerge from experience with one-to-one correspondence settings, or it could develop as an abstraction of the faculty of recursion. In line with this proposal, the Mundurucus, speakers of a natural, recursive language with no exact numerical symbols, possess a good understanding of exact equality between numbers, but their knowledge of the successor principle is at best limited.

Acknowledgments

The work with Mundurucu speakers would not have been possible without the help of André Ramos, Celso Tawé, Felix Tawé, André Tawé, and of the Department of Studies and

Research of Funai. It is based on data collected by P. P. within a larger project on the nature of quantification and functional categories developed jointly by the linguistic section of the Departement of anthropology of the National museum of Rio and the Unité Mixte de Recherche 7023 of the CNRS, with the agreement of Funai and CNPQ. It was supported by INSERM, the SHS department of CNRS (P.P.), NIH (E.S.), and a McDonnell Foundation centennial fellowship (S.D.). The work with American children was supported by a NIH grant to E.S. and a grant of the Fyssen Foundation to V.I.. We thank all the research assistants who helped collecting data, and in particular Ariel Grace and Kate Ellison. We also thank all the families who kindly participated to the study.

Notes

[1] It is possible that the difficulty of the task was different here from the task used with children, because of the difference in the methods. For the Mundurucus, stimuli were presented as animations on a computer screen, instead of real objects. Furthermore, they had more trials to solve, and also these trials were shorter, with events succeeding to each other at a faster paste.

[2] The expressions constructed by double reduplication are used only in very specific contexts, in activities which can be assimilated to word games.

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