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Rational maps and Teichmüller space

Analogies and open problems

Curt McMullen

Let X be a complex manifold and let $f : X \times \widehat{\mathbf{C}} \to \widehat{\mathbf{C}}$ be a holomorphic map. Then f describes a family $f_\lambda(z)$ of rational maps from the Riemann sphere to itself, depending holomorphically on a complex parameter λ ranging in X.

By [MSS], there is an open dense set $X_0 \subset X$ on which the family is structurally stable near the Julia set: in fact f_a and f_b are quasiconformally conjugate on their respective Julia sets whenever a and b lie in the same component U of X_0. The mappings in X_0 are said to be J-stable.

In this note we will record some problems concerning the boundaries of components U, and consequently concerning limits of quasiconformal deformations of a given rational map.

Example I. Quadratic polynomials. The most famous such problem is the following. Let $X = \mathbf{C}$, and let $f_\lambda(z) = z^2 + \lambda$. Then X_0 contains a unique unbounded component U.

Problem. Is the boundary of U locally connected?

This is equivalent to the question:

Is the Mandelbrot set M locally connected?

Indeed, X_0 is just the complement of the boundary of the Mandelbrot set.

The importance of this question is twofold. First, if M is locally connected, then existing work provides detailed information about its combinatorial structure, and one has a good understanding of the "bifurcations" of a quadratic polynomial and many related maps. Secondly, the local connectivity of M implies the density of hyperbolic dynamics ("Axiom A") for degree two polynomials, another well-known conjecture which has eluded proof for many years. For more details see [Dou1], [Dou2], [DH1], [DH2], [Lav], [Th].

Recent progress. Yoccoz has recently made important progress towards the local connectivity of M. He shows that M is locally connected at all λ except perhaps those for which $z^2 + \lambda$ is *infinitely renormalizable*. (The Feigenbaum polynomial is an example of an infinitely renormalizable mapping). Details are available in forthcoming manuscripts of Yoccoz and of Hubbard.

Compactifying the space of proper maps. We now turn to a second example motivated by an analogy with Bers' embedding of Teichmüller space. Let A and B be two proper holomorphic maps of the unit disk Δ to itself, both of degree $n > 1$ and fixing zero. (A and B are finite Blaschke products.) Then it is well-known that A and B are topologically conjugate on the unit circle S^1, and the conjugacy h is unique once we have chosen a pair of fixed points (a, b) for A and B such that $h(a) = b$. Moreover h is quasisymmetric; this is a general property of conjugacies between expanding conformal dynamical systems [Sul].

Now glue two copies of the disk together by h and transport the dynamics of A and B to the resulting Riemann surface, which is a sphere. We obtain in this way an expanding (i.e. hyperbolic) rational map $f(A, B)$. The Julia set J of $f(A, B)$ is a quasicircle, and f is holomorphically conjugate to A and B on the components of the complement of J. The mapping $f(A, B)$ is determined by h up to conformal conjugacy.

We will loosely speak of spaces of mappings as being "the same" if they represent the same conformal conjugacy classes. It is often useful to require that the conjugacy preserves some finite amount of combinatorial data, such as a distinguished fixed point. For simplicity we will gloss over such considerations below.

Example II. Let X be the space of degree n polynomials, X_0 the open dense subset of J-stable polynomials and U the component of X_0 containing z^n. Then U is the same as the set of maps of the form $f(z^n, B)$. Equivalently, U consists of those polynomials with an attracting fixed point with all critical points in its immediate basin.

Let us denote this set of polynomials by $\mathcal{B}(z^n)$. It is easy to see that $\mathcal{B}(z^n)$ is an open set of polynomials with compact closure. Thus this construction supplies both a complex structure for the space of Blaschke products, and a geometric compactification of that space.

Problem. Describe the boundary of $\mathcal{B}(z^n)$ in the space of polynomials of degree n.

For degree $n = 2$ this is easy (the boundary is a circle) but for $n = 3$ it is already subtle.

To explain the kind of answer one might expect, we consider not one boundary but many. More precisely, let $\mathcal{B}(A)$ denote the space of rational maps $f(A, B)$ for some other *fixed* A and varying B. This space also inherits a complex structure and the map $f(z^n, B) \mapsto f(A, B)$ gives an biholomorphic map

$$F : \mathcal{B}(z^n) \to \mathcal{B}(A).$$

The closure of $\mathcal{B}(A)$ in the space of rational maps provides another geometric compactification of this complex manifold.

Problem. Show that for $n > 2$ and $A \neq z^n$, F does *not* extend to a homeomorphism between the boundaries of $\mathcal{B}(z^n)$ and $\mathcal{B}(A)$.

Thus we expect that the complex space $\mathcal{B}$ (whose complex structure is independent of A) has many natural geometric boundaries. But perhaps the lack of uniqueness can be accounted for by the presence of *complex submanifolds* of the boundary, i.e. by the presence of rational maps in the compactification which admit quasiconformal deformations.

To make this precise, let $\partial(A)$ denote the quotient of the boundary of $\mathcal{B}(A)$ by the equivalence relation $f \sim g$ if f and g are quasiconformally conjugate (this implies f and g lie in a connected complex submanifold of the boundary). The resulting space (in the quotient topology) still forms a boundary for $\mathcal{B}(A)$, but it is non-Hausdorff when $n > 2$.

Conjecture. The holomorphic isomorphism $F : \mathcal{B}(z^n) \to \mathcal{B}(A)$ extends to a homeomorphism from $\partial(z^n)$ to $\partial(A)$.

Problem. Give a combinatorial description of the topological space $\partial(z^n)$.

Such a description may involve laminations, as discussed in [Th].

An analogy with Teichmüller theory. The "mating" of A and B has many similarities with the mating of Fuchsian groups uniformizing a pair of compact genus g Riemann surfaces X and Y. Such a mating is provided by Bers' simultaneous uniformization theorem [Bers]. The result is a Kleinian group $\Gamma(X, Y)$ whose limit set is a quasicircle. Moreover, fixing X, the map $Y \mapsto \Gamma(X, Y)$ provides a holomorphic embedding of the Teichmüller space of genus g into the space of Kleinian groups. One can then form a boundary for Teichmüller space by taking the closure.

It has recently been shown that *this* boundary does indeed *depend* on the base point X [KT]. However Thurston has conjectured that the space $\partial(X)$, obtained by identifying quasiconformally conjugate groups on the boundary, is a (non-Hausdorff) boundary which is independent of X.

Moreover a combinatorial model for $\partial(X)$ is conjecturally constructed as follows. Let $\mathbf{P}\mathcal{ML}$ denote the space of projective measured laminations on a surface of genus g; then $\partial(X)$ is homeomorphic to the quotient of $\mathbf{P}\mathcal{ML}$ by the equivalence relation which forgets the measure. (See [FLP] for a discussion of $\mathbf{P}\mathcal{ML}$ as a boundary for Teichmüller space.)

Remarks.

1. We do not expect that one can give a combinatorial description of the "actual" boundary of $\mathcal{B}(z^n)$ (in the space of polynomials). For similar reasons, we believe it unlikely that one can describe the uniform structure induced on the space of critically finite rational maps by inclusion into the space of all rational maps.

2. It is known that Teichmüller space is a domain of holomorphy. So it is natural to ask the following intrinsic:

Question. Is $\mathcal{B}(z^n)$ a domain of holomorphy? More generally, is every component of the space of expanding rational maps (or polynomials) a domain of holomorphy?

Density of cusps. The preceding discussion becomes interesting only when the space of rational maps under consideration has two or more (complex) dimensions. We conclude with two concrete questions about boundaries in a one-parameter family of rational maps.

Example III. Let

$$f_\lambda(z) = \lambda z^2 + z^3$$

where λ ranges in $X = \mathbf{C}$, and let U denote the component of X_0 containing the origin. That is, U is the set of λ for which both finite critical points are in the immediate basin of zero.

A *cusp* on ∂U is an f_λ with a parabolic periodic cycle.

Conjecture. Cusps are dense in ∂U.

This conjecture is motivated by the density of cusps on the boundary of Teichmüller space [Mc]. It is not hard to show that it is implied by the following:

Conjecture. The boundary of U is a Jordan curve.

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