The Arithmetic of Simple Singularities

Dissertation Advisor: Professor Benedict Gross Jack A. Thorne

The Arithmetic of Simple Singularities

Abstract

We investigate some arithmetic orbit problems in representations of linear algebraic groups arising from Vinberg theory. We aim to give a description of the orbits in these representations using methods with an emphasis on representation theory rather than algebraic geometry, in contrast to previous works of other authors.

It turns out that for the representations we consider, the orbits are related to the arithmetic of the Jacobians of certain algebraic curves, which appear as the smooth nearby fibers of deformations of simple singularities. We calculate these families of algebraic curves, and show that the 2-torsion in their Jacobians is canonically identified with the stabilizers of certain orbits in the corresponding representations.

Contents

Acknowledgements

During my time as a student I have benefited from the kindnesses of many people. It is a pleasure to thank them here.

First and foremost I thank my advisors Benedict Gross and Richard Taylor, who have been extremely generous in sharing both their time and their mathematics. In particular, many of the ideas appearing in this thesis were first explained to me by Dick in his office.

I would also like to thank Manjul Bhargava, Toby Gee, David Geraghty, Florian Herzig, Wei Ho, Tasho Kaletha, Steven Sam and Arul Shankar for useful conversations and for discussing their ideas with me. In particular, I would like to thank Manjul for welcoming me during my time in Princeton.

I must also thank Tadashi Tokieda, who first showed me the true beauty of mathematics, and without whom I would never have set out upon this path.

Finally, this thesis would never have seen the light of day without the unflinching love and support of my family. This thesis is dedicated to them.

1. Introduction

This thesis is a contribution to *arithmetic invariant theory.* Let G be a reductive group over a field k, and let V be a linear representation of G. Then the ring $k[V]^G$ is a k-algebra of finite type, and we can define the quotient $V /\!\!/ G = \operatorname{Spec} k[V]^G$ and a quotient map $\pi : V \to V /\!\!/ G$. The determination of the structure of $k[V]^G$ and the fibers of π falls under the rubric of geometric invariant theory, and is important in algebraic geometry.

In the case where k is not algebraically closed, a further layer of difficulty is obtained by considering the $G(k)$-orbits in the fibers of π over k-points of $V /\!\!/ G$. This problem can be translated into the language in Galois cohomology, and as such often has close ties to arithmetic.

Bhargava has singled out those representations which are coregular, in the sense that $k[V]^G$ is isomorphic to a polynomial ring, as promising candidates for representations which may have interesting connections to arithmetic. For example, he studies the case $G = GL_2$ and $V = \operatorname{Sym}^4(2)^\vee$, the space of binary quartic forms. In this case there are two independent polynomial invariants I and J, and k-rational orbits with given values of I and J are related to classes in the Galois cohomology group $H^1(k, E[2])$ for the elliptic curve

$$E : y^2 = x^3 + Ix + J.$$

These considerations have had very striking applications; see [2], or [23] for a beautiful summary. See also the work of Ho [13] for a variety of similar orbit parameterizations associated to other representations. For each choice of pair (G, V), one makes an ad hoc construction in algebraic geometry which relates orbits in the given representation to algebraic curves, possibly with marked points, given line bundles, or other types of extra data.

By contrast, this thesis represents a first effort to describe some of the phenomena appearing in arithmetic invariant theory through the lens of representation theory. We take as our starting point certain representations arising from Vinberg theory, whose role in arithmetic invariant theory has been emphasized by Gross. If G is a reductive group over k endowed with an automorphism θ of finite order m, then the fixed group G^θ acts on the $\theta = \zeta$ eigenspace $\mathfrak{g}_1 \subset \mathfrak{g}$ for any choice $\zeta \in k^\times$ of primitive m^{th} root of unity. Vinberg theory describes the geometric invariant theory of these representations. In the case when θ is regular and elliptic, in the sense of [12], the generic element of $\mathfrak{g}_1$ will have a finite abelian stabilizer, and orbits in the representation are thus related to interesting Galois cohomology.

If G is a split reductive group over k, then it has a unique $G^{\text{ad}}(k)$-conjugacy class of regular elliptic involutions θ, characterized by the requirement that $\mathfrak{g}_1$ contain a regular nilpotent element. It is the representations associated to these canonical involutions for simple G of type A, D or E that we study in this thesis.

The geometric objects associated to our Vinberg representation are constructed using deformation theory. It turns out that the nilpotent cone in $\mathfrak{g}_1$ (namely, the locus where all invariant polynomials vanish) is smoothly equivalent along the generic point of the singular locus to a plane curve singularity of type A, D or E, the type being that of the group G. (In fact, these are precisely the simple singularities of the title).

Let $X \subset \mathfrak{g}_1$ be a transverse slice to the G^θ-action at such a singular point. Restricting the quotient map $\pi : \mathfrak{g}_1 \to \mathfrak{g}_1 /\!\!/ G^\theta$ to X realizes X as a family of curves deforming the central fiber, the generic fiber of which is smooth. We are thus able to associate to any point $b \in \mathfrak{g}_1 /\!\!/ G^\theta(k)$, at which a suitably defined discriminant does not vanish, a canonical smooth projective curve Y_b. The identification of the families of curves obtained in this manner is given in Theorem 4.8. The families turn out to be universal families of curves of fixed genus and with certain kinds of marked points at infinity.

The curves $X_b \subset Y_b$ come equipped with embeddings $X_b \hookrightarrow \pi^{-1}(b)$. We show that the 2-torsion of the Jacobian J_{Y_b} is canonically isomorphic to the stabilizer $Z_b \subset G^\theta$ of any point in $\pi^{-1}(b)(k)$, and that with these identifications the classifying maps

$$X_b(k) \hookrightarrow J_{Y_b}(k) \to H^1(k, J_{Y_b}[2])$$

and

$$X_b(k) \hookrightarrow \pi^{-1}(b)(k) \to H^1(k, Z_b)$$

coincide. Applying all this to the group $G = PGL_3$, we obtain the orbit correspondence used by Bhargava and Shankar in their work on the 2-Selmer groups of elliptic curves over $\mathbb{Q}$. (We note that we have neglected here the role played by the component group of G^θ, and the variation of G within its isogeny class; for a precise formulation, see Theorem 5.14).

Our methods are inspired primarily by work of Slodowy. Rational double point singularities of surfaces can be classified in terms of the Dynkin diagrams of simply laced simple algebraic groups. Grothendieck conjectured that one could give a representation-theoretic construction of this correspondence, by looking at the generic singularity of the nilpotent cone of the corresponding group G. A proof of this conjecture was announced in a famous ICM lecture of Brieskorn [6], but the first detailed proofs were given by Esnault and Slodowy in the respective works [10] and [28]. Our work is what one obtains on combining the respective ideas of Slodowy and Vinberg.

Let us say a few words about the limits of our methods. Essential to our work is the use of $\mathfrak{sl}_2$-triples, whose existence relies in turn on the Jacobson-Morozov lemma. We must therefore work over a field of sufficiently large characteristic, relative to the Coxeter number of G. In this thesis we choose for simplicity to work over a field of characteristic zero, but to extend to all characteristics will require new methods. (Of course, there are other problems in very small characteristic; for example, Vinberg's description of the invariant polynomials also breaks down). For similar reasons, we cannot say anything about orbits over $\mathbb{Z}$.

More serious is the lack of information we obtain about the image of the map $\pi^{-1}(b)(k) \to H^1(k, J_{Y_b}[2])$ constructed above. It follows from the above considerations that it contains the elements in the image under the 2-descent map δ of $X_b(k)$; we conjecture that it moreover contains the image under δ of the whole group $J_{Y_b}(k)$ of rational points of the Jacobian. In other words, we currently lack a way to *construct* orbits in the representations we study. We hope to return to this question in future work.

Let us now outline the contents of this thesis. In §2, we prove some basic properties of the so-called stable involutions θ, and define the Vinberg representations to which they correspond. An important point here is the calculation of the stabilizers of the regular elements in $\mathfrak{g}_1$ in terms of the root datum of the ambient reductive group G. We also recall some basic facts about Galois cohomology, and the deformation of singularities. In particular, we define the simple curve singularities that appear in this thesis.

In §3, we introduce the subregular nilpotent elements, which appear in the singular locus of the nilpotent cone of $\mathfrak{g}_1$. We then address the question of when $\mathfrak{g}_1$ contains subregular nilpotent elements which are defined over the base field k. In §4, we construct the families of curves mentioned above inside a suitable transverse slice to the subregular nilpotent orbit.

Finally, in §5, we show how to relate the 2-torsion in the Jacobians of our curves and the stabilizers of regular elements, and prove our main theorem relating the 2-descent map to the classifying map for orbits in non-abelian Galois cohomology.

Notation. As mentioned above, we work throughout over a field k of characteristic zero. We assume basic familiarity with the theory of reductive groups over k, as studied for example in [14] or [30]. We assume that reductive groups are connected.

If G is a reductive group acting linearly on a k-vector space V, then the ring of invariants $k[V]^G$ is a k-algebra of finite type (see for example [29], Theorem 2.4.9). We define $V /\!\!/ G = \operatorname{Spec} k[V]^G$ and call it the categorical quotient. It in fact satisfies a universal property, but we will not need this here. We will write $\mathcal{N}(V)$ for the closed subscheme of V cut out by the augmentation ideal of $k[V]^G$.

If $G, H, \dots$ are algebraic groups then we will use gothic letters $\mathfrak{g}, \mathfrak{h}, \dots$ to denote their Lie algebras. Let G be a reductive group, and $T \subset G$ a split maximal torus. Then we shall write $\Phi_{\mathfrak{t}} \subset X^*(T)$ for the set of roots of T in $\mathfrak{g}$, and $\Phi_{\mathfrak{t}}^\vee \subset X_*(T)$ for the set of coroots. The assignment $\alpha \in \Phi_{\mathfrak{t}} \mapsto d\alpha \in \mathfrak{t}^*$ identifies $\Phi_{\mathfrak{t}}$ with the set of roots of $\mathfrak{t}$ in $\mathfrak{g}$, and we will use this identification without comment. We write $W(\mathfrak{t}) = N_G(T)/T$ for the Weyl group of G with respect to $\mathfrak{t}$. We have the Cartan decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in \Phi_{\mathfrak{t}}} \mathfrak{g}^\alpha,$$

where $\dim \mathfrak{g}^\alpha = 1$ for each $\alpha \in \Phi_{\mathfrak{t}}$. We write $U_\alpha \subset G$ for the unique T-invariant closed subgroup with Lie algebra $\mathfrak{g}_\alpha$ (see [14], 26.3, Theorem). The tuple

$$(X^*(T), \Phi_{\mathfrak{t}}, X_*(T), \Phi_{\mathfrak{t}}^\vee)$$

is a root datum in the sense of [30], 7.4.

We will write $L_G = \mathbb{Z}\Phi_{\mathfrak{t}}$ for the root lattice of G and $\Lambda_G \subset L_G \otimes_{\mathbb{Z}} \mathbb{Q}$ for the weight lattice of L_G. (These are the groups Q and P, respectively, of [4], Ch. VI, §1.9). If the group G is clear from the context, we will omit the subscript G. We understand these to depend only on G and not on T, so that L_G and Λ_G are defined up to (non-unique) isomorphism. We write $W_G \subset \operatorname{Aut}(L_G)$ for the corresponding Weyl group.

If $x \in \mathfrak{g}$, we write $Z_G(x)$ for its centralizer in G under the adjoint representation, and $\mathfrak{z}_{\mathfrak{g}}(x)$ for its centralizer in $\mathfrak{g}$. Since we work in characteristic zero, we have $\operatorname{Lie} Z_G(x) = \mathfrak{z}_{\mathfrak{g}}(x)$ and this notation is consistent. We write A_G for the center of G, and $\mathfrak{a}_{\mathfrak{g}}$ for its Lie algebra.

2. Preliminaries

Throughout this section, G is a split reductive group over a field k of characteristic zero. We recall without proof the following basic facts.

Proposition 2.1. *(1) Any $x \in \mathfrak{g}$ can be written uniquely as $x = x_s + x_n$, where x_s is semisimple and x_n is nilpotent. Moreover we have $[x_s, x_n] = 0$.*

(2) If k is algebraically closed then any semisimple element is contained in a Cartan subalgebra and any two Cartan subalgebras of $\mathfrak{g}$ are conjugate by an element of $G(k)$.

Theorem 2.2. *(1) Let $\mathfrak{t} \subset \mathfrak{g}$ be a Cartan subalgebra. Then the restriction map $k[\mathfrak{g}]^G \to k[\mathfrak{t}]^{W(\mathfrak{t})}$ is an isomorphism. Moreover, $k[\mathfrak{t}]^{W(\mathfrak{t})}$ is a polynomial ring over k in $\operatorname{rank} G$ indeterminates.*

(2) Let $p : \mathfrak{g} \to \mathfrak{g}/\!\!/G$ denote the adjoint quotient map. Then p is flat. If k is algebraically closed, then for all $x \in \mathfrak{g}$, $p^{-1}p(x)$ consists of finitely many $G(k)$-orbits.

Proposition 2.3. *Let $x \in \mathfrak{g}$ be semisimple. Then $Z_G(x)$ is reductive (and in particular, connected). Let $T \subset G$ be a maximal torus, and suppose that $x \in \mathfrak{t}$. Then $T \subset Z_G(x)$ is a maximal torus. Let*

$$\Phi_{\mathfrak{t}}(x) = \{\alpha \in \Phi_{\mathfrak{t}} \mid \alpha(x) = 0\} \text{ and } \Phi_{\mathfrak{t}}^\vee(x) = \{\alpha^\vee \in \Phi_{\mathfrak{t}}^\vee \mid \alpha \in \Phi_{\mathfrak{t}}(x)\}.$$

Let $W(x) = Z_{W(\mathfrak{t})}(x)$.

Then the root datum of $Z_G(x)$ is $(X^*(T), \Phi_{\mathfrak{t}}(x), X_*(T), \Phi_{\mathfrak{t}}^\vee(x))$. The Weyl group of $Z_G(x)$ with respect to T can be identified in a natural way with $W(x)$. Finally, $Z_G(x)$ is generated by T and the groups U_α for $\alpha \in \Phi_{\mathfrak{t}}(x)$.

We say that $x \in \mathfrak{g}$ is regular if its centralizer $\mathfrak{z}_{\mathfrak{g}}(x)$ has the minimal possible dimension, namely $\operatorname{rank} G$. Equivalently, the orbit $G \cdot x$ should have the maximal possible dimension.

Lemma 2.4. *Let $x \in \mathfrak{g}$. The following are equivalent:*

(1) x is regular.

(2) If $x = x_s + x_n$ is the Jordan decomposition of x, then x_n is regular in $\mathfrak{z}_{\mathfrak{g}}(x_s)$.

(3) The quotient map $p : \mathfrak{g} \to \mathfrak{g}/\!\!/G$ is smooth at x.

Elements of Vinberg theory. Let $\theta \in \operatorname{Aut}(G)$ be an automorphism of exact order $m > 1$, and let $\zeta \in k$ be a primitive m^{th} root of unity. We will also write θ for the induced automorphism of $\mathfrak{g}$. We associate to θ the grading $\mathfrak{g} = \oplus_{i \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_i$, where by definition we have

$$\mathfrak{g}_i = \{x \in \mathfrak{g} \,|\, \theta(x) = \zeta^i x\}.$$

We write G^θ for the fixed subgroup of θ, and G_0 for its connected component. Then $\operatorname{Lie} G_0 = \mathfrak{g}_0$, so the notation is consistent.

Lemma 2.5. *The action of G^θ on $\mathfrak{g}$ leaves each $\mathfrak{g}_i$ invariant.*

Proof. Immediate. □

In what follows, we shall consider the representation of G_0 on the subspace $\mathfrak{g}_1 \subset \mathfrak{g}$. The study of such representations is what we call Vinberg theory. For the basic facts about Vinberg theory, and in particular for proofs of the unproved assertions in this section, we refer to the papers [33] or [18].

Lemma 2.6. *Let $x \in \mathfrak{g}_1$. Then x can be written uniquely as $x = x_s + x_n$, where x_s, x_n both lie in $\mathfrak{g}_1$ and are respectively semisimple and nilpotent.*

By definition, a Cartan subspace $\mathfrak{c} \subset \mathfrak{g}_1$ is a maximal subalgebra consisting of semisimple elements. Note that $\mathfrak{c}$ is automatically abelian.

Proposition 2.7. *Suppose that k is algebraically closed. Then an element $x \in \mathfrak{g}_1$ is semisimple if and only if it is contained in a Cartan subspace, and all Cartan subspaces are $G_0(k)$-conjugate.*

Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a Cartan subspace, and define $W(\mathfrak{c}, \theta) = N_{G_0}(\mathfrak{c})/Z_{G_0}(\mathfrak{c})$. This is the 'little Weyl group' of the pair (G, θ). We define $\operatorname{rank} \theta = \dim \mathfrak{c}$. This is well-defined by the above proposition.

Theorem 2.8.

(1) *Restriction of functions induces an isomorphism $k[\mathfrak{g}_1]^{G_0} \to k[\mathfrak{c}]^{W(\mathfrak{c},\theta)}$. Moreover, $W(\mathfrak{c}, \theta)$ is a reflection group and $k[\mathfrak{c}]^{W(\mathfrak{c},\theta)}$ is a polynomial ring in $\operatorname{rank} \theta$ indeterminates.*

(2) *Let $\pi : \mathfrak{g}_1 \to \mathfrak{g}_1 /\!\!/ G_0$ denote the quotient map. Then π is flat. If k is algebraically closed, then for all $x \in \mathfrak{g}_1$, $\pi^{-1}\pi(x)$ consists of only finitely many $G(k)$-orbits.*

We say that $v \in \mathfrak{g}_1$ is stable if $G_0 \cdot v$ is closed in $\mathfrak{g}_1$, and $Z_{G_0}(v)$ is finite. We say that θ is stable if $\mathfrak{g}_1$ contains stable elements. The property of being stable is hereditary, in the following sense.

Lemma 2.9. *Suppose that θ is a stable automorphism. Let $x \in \mathfrak{g}_1$ be semisimple. Let $H = Z_G(x)$ and $\mathfrak{h} = \operatorname{Lie} H$. Then $\theta(H) = H$, and $\theta|_H$ is a stable automorphism.*

Proof. Lemma 5.6 of [12] states that $v \in \mathfrak{g}_1$ is stable if and only if it is regular semisimple, and the action of θ on the character group of the unique maximal torus centralizing v is elliptic. Given x as in the lemma, choose a Cartan subspace $\mathfrak{c}$ containing it. Then $\mathfrak{c}$ contains a stable vector, which is also stable when considered as an element of $\mathfrak{h}$; the result follows. $\square$

Stable involutions. In this thesis we shall be particularly interested in the stable involutions.

Lemma 2.10. *Suppose that k is algebraically closed. There is a unique $G(k)$-conjugacy class of stable involutions θ.*

Proof. To show uniqueness, we reduce immediately to the case that G is adjoint. By Lemma 5.6 of [12], any stable vector $v \in \mathfrak{g}_1$ is regular semisimple, and θ acts as -1 on its centralizer $\mathfrak{c} = \mathfrak{z}_{\mathfrak{g}}(v)$. In particular, we have $\mathfrak{c} \subset \mathfrak{g}_1$. It follows that the trace of θ on $\mathfrak{g}$ is equal to $-\dim \mathfrak{c} = -\operatorname{rank} G$, and a well-known theorem of E. Cartan asserts that this determines θ up to $G(k)$-conjugacy. We can also reduce existence to the case of G adjoint. We will prove existence (even when k is not algebraically closed) in this case below. □

Lemma 2.11. *Let θ be a stable involution of G. Then θ satisfies the following.*

(1) $\operatorname{rank} \theta = \operatorname{rank} G$.
(2) There exists a maximal torus C in G on which θ acts by $x \mapsto x^{-1}$.
(3) For all $x \in A_G$, we have $\theta(x) = x^{-1}$.
(4) Let $\mathfrak{c}$ be a Cartan subspace (and hence, a Cartan subalgebra). Then the natural map $W(\mathfrak{c}, \theta) \to W(\mathfrak{c})$ is an isomorphism.

Proof. The first and second properties follow from the proof of the previous lemma. For the third property, we recall that A_G is contained in any maximal torus of G. The final property is [12], Corollary 7.4. □

Suppose for the rest of this section that θ is a stable involution.

Proposition 2.12. *Let $x = x_s + x_n \in \mathfrak{g}_1$ be a regular element. Then $Z_{G^\theta}(x) = A_{Z_G(x_s)}[2]$. In particular, this group is always finite and abelian.*

Proof. We have $Z_G(x) = Z_G(x_s) \cap Z_G(x_n)$, so after replacing G by $Z_G(x_s)$, we may assume that $x = x_n$ is a regular nilpotent element.

Then $Z_G(x) = A_G \cdot Z_U(x)$, a direct product, where U is the unipotent radical of the unique Borel subgroup containing x. Quotienting by A_G, we may suppose that G is adjoint and must show that $Z_U(x)^\theta$ is trivial. But since x is regular, this is a finite unipotent group, so the result follows. □

Corollary 2.13. *Let $x = x_s + x_n$ be a regular element, and let $\mathfrak{c}$ be a Cartan subspace containing x_s. Let $C \subset G$ denote the maximal torus with Lie algebra $\mathfrak{c}$. Then*

$$Z_{G^\theta(x)} \cong \mathrm{Hom}(X^*(C)/2X^*(C) + \mathbb{Z}\Phi_{\mathfrak{c}}(x), \mathbb{G}_m).$$

Proof. For any reductive group G with root datum $(X^*(T), \Phi_{\mathfrak{t}}, X_*(T), \Phi_{\mathfrak{t}}^\vee)$, there is a canonical isomorphism $X^*(Z(G)) \cong X^*(T)/\mathbb{Z}\Phi_{\mathfrak{t}}$. Now apply the previous proposition. □

Corollary 2.14. *Suppose that G is adjoint and that k is algebraically closed. Let $x \in \mathfrak{g}_1$ be a regular semisimple element. Let L denote the root lattice of G, and $\Lambda \subset L \otimes_{\mathbb{Z}} \mathbb{Q}$ the weight lattice. Then there is an isomorphism*

$$Z_{G_0}(x) \cong \mathrm{Hom}(N, \mathbb{G}_m),$$

well defined up to conjugacy by the Weyl group W of L, where N denotes the image of L in $\Lambda/2\Lambda$.

Proof. Let G^{sc} denote the simply connected cover of G. Then θ acts on G^{sc}. A theorem of Steinberg ([22], Chapter 4.4.8, Theorem 9) states that $(G^{\mathrm{sc}})^\theta$ is connected, and hence G_0 is the image of the map $(G^{\mathrm{sc}})^\theta \to G$. The present corollary now follows from the previous one. □

Now suppose that the simple components of G are simply laced (that is, their root systems are all of type A, D, or E), and let L, Λ and W be as in the statement of the corollary. Then there is a W-invariant quadratic form $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}$ uniquely determined by the requirement that $\langle\alpha,\alpha\rangle = 2$ for every root α. The pairing $\langle\cdot,\cdot\rangle$ on L induces a pairing $(\cdot,\cdot) : L/2L \times L/2L \to \mathbb{F}_2$. An easy calculation shows this pairing is alternating. In fact, we have the following:

Lemma 2.15. *The pairing $(\cdot,\cdot)$ descends to a non-degenerate alternating pairing on N.*

Proof. Suppose $x \in L$. Then the image of x in $L/2L$ lies in the radical of $(\cdot,\cdot)$ if and only if $\langle x, L\rangle \subset 2\mathbb{Z}$, if and only if $x \in 2\Lambda$, since Λ is the $\mathbb{Z}$-dual of L with respect to the pairing $\langle\cdot,\cdot\rangle$. (Pairings of this type, associated to regular elliptic elements of Weyl groups, were first considered by Reeder; compare [24]). □

Corollary 2.16. *Suppose that G is an adjoint group, and that the simple components of G are simply laced. Then for any regular semisimple element $x \in \mathfrak{g}_1$, there is a canonical non-degenerate alternating form $(\cdot,\cdot) : Z_{G_0}(x) \times Z_{G_0}(x) \to \mu_2$.*

We now show how to construct a stable involution over an arbitrary field k of characteristic 0. We let G be a simple split adjoint group, and fix a split maximal torus T and a Borel subgroup B containing it. This determines a set $\Phi^+ \subset \Phi = \Phi_{\mathfrak{t}}$ of positive roots, and a root basis $R \subset \Phi^+$. We fix moreover for each $\alpha \in R$ a basis X_α of the one-dimensional vector space $\mathfrak{g}^\alpha \subset \mathfrak{g}$. The tuple $(T, B, \{X_\alpha\}_{\alpha\in R})$ is called a pinning of G.

This choice of data determines a splitting $\mathrm{Aut}(G) = G \rtimes \Sigma$, where Σ is the group of *pinned automorphisms* induced by automorphisms of the Dynkin diagram of G. On the other hand, writing $L = X^*(T) = \mathbb{Z}\Phi$ for the root lattice of $\mathfrak{g}$, the choice of root basis determines a splitting $\mathrm{Aut}(L) = W \rtimes \Sigma$ in a similar manner; see [5], Ch. VIII, §5.2. We write $\sigma \in \Sigma$ for the image of $-1 \in \mathrm{Aut}(L)$, and define $\theta = \rho^\vee(-1) \rtimes \sigma \in \mathrm{Aut}(G)(k)$, where $\rho^\vee \in X_*(T)$ is the sum of the fundamental coweights.

Lemma 2.17. *θ is a stable involution.*

Proof. This follows immediately from Corollary 5.7 of [12]. □

This stable involution has good rationality properties. This is based on the following fact.

Lemma 2.18. *With θ as above, $\mathfrak{g}_1$ contains a regular nilpotent element. All regular nilpotent elements of $\mathfrak{g}_1$ are conjugate by a unique element of $G^\theta(k)$.*

Proof. The element $\sum_{\alpha \in R} X_\alpha$ is regular nilpotent and, by construction, lies in $\mathfrak{g}_1$. Fix a separable closure K of k. If $E, E' \in \mathfrak{g}_1$ are two regular nilpotent elements then they are conjugate by an element of $G^\theta(K)$. (This follows from Theorem 5.16 of [17]).

We saw above that for any such E, $Z_{G^\theta}(E)$ is the trivial group. It follows that E, E' are conjugate by a unique element of $G^\theta(K)$, which must therefore lie in $G^\theta(k)$. □

Corollary 2.19. *There is a unique $G(k)$-conjugacy class of stable involutions θ_1 of G such that there exists a regular nilpotent element $E_1 \in \mathfrak{g}$ with $\theta_1(E_1) = -E_1$.*

Proof. The existence has been shown above. For the uniqueness, fix again a separable closure K of k. We have seen that $G(K)$ acts transitively on pairs (θ_1, E_1). On the other hand, the stabilizer of such a pair in $G(K)$ is trivial. It follows that any two such pairs are conjugate by a unique element of $G(k)$. □

$\mathfrak{sl}_2$-triples. We now suppose again that G is a general split reductive group over k. A tuple (E, H, F) of linearly independent elements of $\mathfrak{g}$ is called an $\mathfrak{sl}_2$-triple if it satisfies the relations

$$[H, E] = 2E, [H, F] = -2F, [E, F] = H.$$

Lemma 2.20. *(1) Any nilpotent element $E \in \mathfrak{g}$ is contained in an $\mathfrak{sl}_2$-triple.*

(2) Any two $\mathfrak{sl}_2$-triples (E, H, F) and (E, H', F') are $Z_G(E)(k)$-conjugate.

Proof. The first part is the well-known Jacobson-Morozov lemma, see [5], Ch. VIII, §11.2, Proposition 2. The second part is [5], Ch. VIII, §11.1, Proposition 2. □

Now suppose that θ is an involution of G. We call a tuple (E, H, F) a normal $\mathfrak{sl}_2$-triple if it is an $\mathfrak{sl}_2$-triple, and moreover we have $E \in \mathfrak{g}_1, H \in \mathfrak{g}_0$, and $F \in \mathfrak{g}_1$. (In particular, the restriction of θ to the subalgebra spanned by these elements is a stable involution).

Lemma 2.21. *(1) Any nilpotent element $E \in \mathfrak{g}_1$ is contained in a normal $\mathfrak{sl}_2$-triple.*

(2) Any two normal $\mathfrak{sl}_2$-triples (E, H, F) and (E, H', F') are $Z_{G_0}(E)(k)$-conjugate.

Proof. Fix a separable closure K of k. For the first part, choose an arbitrary $\mathfrak{sl}_2$-triple (E, h, f) containing E, and decompose $h = h_0 + h_1$ into θ-eigenvectors. The argument of [16], Proposition 4 implies that there is a unique $F \in \mathfrak{g}_1 \otimes_k K$ such that (E, h_0, F) is an $\mathfrak{sl}_2$-triple. But a $\mathfrak{sl}_2$-triple is determined uniquely by any 2 of its 3 elements, so descent implies that $F \in \mathfrak{g}_1$, and (E, h_0, F) is the desired triple.

For the second part, we argue as in the proof of [16], Proposition 4, and apply [5], Ch. VIII, §11.1, Lemma 4 to obtain the desired rationality property. □

Corollary 2.22. *Suppose that G is adjoint. Then $G(k)$ acts simply transitively on the set of pairs $((\theta_1), (E, H, F))$, where θ_1 is a stable involution of G and (E, H, F) is a normal $\mathfrak{sl}_2$-triple with respect to θ_1 in which E is a regular nilpotent element.*

Example 2.23. We illustrate some of the concepts introduced so far in the case where G is a split adjoint group of type A_{2r}. Let V be a vector space of dimension $2r + 1$, with basis $\{e_1, e_2, \dots, e_r, v, f_r, \dots, f_2, f_1\}$. We define an inner product $\langle \cdot, \cdot \rangle$ on V by the formulae

$$\langle e_i, e_j \rangle = 0 = \langle f_i, f_j \rangle = \langle e_i, v \rangle = \langle f_i, v \rangle$$

for all i, j and

$$\langle v, v \rangle = 1, \langle e_i, f_j \rangle = \delta_{ij}.$$

If $T \in \mathrm{End}(V)$, write T^* for the adjoint of T with respect to this inner product. Then we take $G = PGL_{2r+1} = PGL(V)$, and $\theta : \mathfrak{sl}_{2r+1} \to \mathfrak{sl}_{2r+1}$ to be the involution $X \mapsto -X^*$. It is easy to check that $-\theta$ is just reflection in the anti-diagonal. In particular, fixing the standard pinning $(T, B, \{X_\alpha\}_{\alpha\in R})$ of $\mathfrak{sl}_{2r+1}$, this θ is exactly the stable involution constructed above.

Then we see that $G^\theta = G_0 = SO(V)$ is connected, and we have

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1, \mathfrak{g}_0 = \{X \in \mathrm{End}(V) \mid \mathrm{tr}\, X = 0, X = -X^*\} = \mathfrak{so}(V).$$

In particular, $\mathfrak{g}_1 = \{X \in \mathrm{End}(V) \mid \mathrm{tr}\, X = 0, X = X^*\}$ consists of the space of operators self-adjoint with respect to $\langle \cdot, \cdot \rangle$.

The regular nilpotent element determined by the pinning is

$$E = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 0 & 0 \end{pmatrix}.$$

We end this section by recalling some basic facts about Galois cohomology and deformations of singularities.

Galois cohomology. Let K be a separable closure of k. Consider an algebraic group H acting on a variety X over k. Let $x \in X(k)$. Then we can consider the set $\mathcal{S}$ of elements in $X(k)$ which become conjugate to x over K:

$$\mathcal{S} = (H(K) \cdot x) \cap X(k).$$

Proposition 2.24. *There is a bijection, depending only on the $H(k)$-orbit of x:*

$$\mathcal{S}/H(k) \cong \ker(H^1(k, Z_H(x)) \to H^1(k, H)),$$

where the kernel is in the sense of pointed sets. The $H(k)$-orbit of x is mapped to the distinguished element of $H^1(k, Z_H(x))$.

Here $H^1(k, -) = H^1(\mathrm{Gal}(K/k), -)$ denotes non-abelian continuous Galois cohomology as defined in [25], Ch. 1, §5.

Proof. We construct the maps in either direction. Given an element $y \in (H(K) \cdot x) \cap X(k)$, we can choose $h \in H(K)$ with $h \cdot x = y$. Then we define a 1-cocycle valued in $Z_H(x)(K)$ by the formula $f_\sigma = h^{-1\sigma}h \in Z_H(x)(K)$ for any $\sigma \in \mathrm{Gal}(K/k)$. It is easy to see that a different choice of h changes f_σ by a coboundary, and a choice of $y' \in H(k) \cdot y$ does not change f_σ at all. Furthermore, f_σ is clearly a coboundary when considered as being valued in H.

Conversely, let f_σ be a 1-cocycle representing an element of the above kernel. We can write $f_\sigma = h^{-1\sigma}h$, and then $y = h \cdot x$ defines element of the left hand side depending only on the cohomology class of f_σ. □

Corollary 2.25. *Suppose that $H^1(k, Z_H(x))$ is trivial. Then two points $x, y \in X(k)$ are conjugate by an element of $H(k)$ if and only if they are conjugate by an element of $H(K)$.*

Representations of $\mathbb{G}_m$. Let U be a finite-dimensional k-vector space endowed with a linear action of $\mathbb{G}_m$. Then we can decompose $U = \oplus_{i \in \mathbb{Z}} U(i)$, where $U(i)$ denotes the eigenspace of the character $t \mapsto t^i$ in U. We call those i for which $U(i)$ is non-zero the weights of U.

Now suppose that V, U are as above and let $f : V \to U$ be a regular map intertwining the two $\mathbb{G}_m$-actions. If the weights of V are (with multiplicity) $w_1, \ldots, w_n$ and the weights of U are $d_1, \ldots, d_m$, then we say that f is quasi-homogeneous of type $(d_1, \ldots, d_m; w_1, \ldots, w_n)$. In particular, if $f = f(x_1, \ldots, x_n)$ is a polynomial in n variables, we say that f is quasi-homogeneous of type $(d; w_1, \ldots, w_n)$ if it satisfies the identity

$$f(t^{w_1}x_1, \ldots, t^{w_n}x_n) = t^d f(x_1, \ldots, x_n)$$

for all $t \in \mathbb{G}_m$.

Deformations of quasi-homogeneous singularities. Let $f(x_1, \ldots, x_n)$ be a non-zero quasi-homogeneous polynomial of type $(w; d_1, \ldots, d_n)$. Then the hypersurface $X_0 = f^{-1}(0)$ in $\mathbb{A}^n$ receives an obvious action of $\mathbb{G}_m$. We suppose that it has an isolated singularity at the origin.

By a deformation of X_0, we mean a pair $(R, \mathcal{X})$, where R is a complete Noetherian local k-algebra with residue field k, and $\mathcal{X}$ is a formal scheme, flat over $\operatorname{Spf} R$, equipped with an isomorphism $\iota : \mathcal{X} \otimes_R k \cong X_0$. If moreover $\operatorname{Spf} R$ and $\mathcal{X}$ are equipped with actions of $\mathbb{G}_m$ making the structure morphism $\mathcal{X} \to \operatorname{Spf} R$ equivariant, and compatible with the standard action on X_0, we say that the pair $(R, \mathcal{X})$ is a $\mathbb{G}_m$-deformation. A morphism $(S, \mathcal{Y}) \to (R, \mathcal{X})$ of $\mathbb{G}_m$-deformations is a pair (ϕ, Φ) of morphisms $\phi : \operatorname{Spf} S \to \operatorname{Spf} R$ and $\Phi : \mathcal{Y} \to \mathcal{X}$ such that the following diagram is Cartesian and $\mathbb{G}_m$-equivariant:

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\Phi"] \arrow[d] & \mathcal{X} \arrow[d] \\
\operatorname{Spf} S \arrow[r, "\phi"] & \operatorname{Spf} R.
\end{tikzcd}$$

Let $\mathcal{C}_k$ denote the category of complete Noetherian local k-algebras R with residue field k, equipped with a $\mathbb{G}_m$-action on $\operatorname{Spf} R$. We define a functor $\operatorname{Def}_{X_0} : \mathcal{C}_k \to \mathrm{Sets}$ by taking $\operatorname{Def}_{X_0}(R)$ to be the set of $\mathbb{G}_m$-deformations $(R, \mathcal{X})$ of X_0 up to isomorphism.

We say that a $\mathbb{G}_m$-deformation $(R, \mathcal{X})$ of X_0 is universal if it represents this functor. It is said to be semi-universal if for any other $\mathbb{G}_m$-deformation $(S, \mathcal{Y})$, there exists a morphism $(S, \mathcal{Y}) \to (R, \mathcal{X})$ and moreover the induced map on Zariski tangent spaces $(\mathfrak{m}_S/\mathfrak{m}_S^2)^* \to (\mathfrak{m}_R/\mathfrak{m}_R^2)^*$ is uniquely determined.

Proposition 2.26. *A semi-universal $\mathbb{G}_m$-deformation of X_0 exists.*

Proof. See [28], §2. $\square$

Simple singularities. Let K be an algebraic closure of k, and let X be an algebraic curve over k. We say that X has a simple singularity at $x \in X(k)$ if the completed local ring of $X \otimes_k K$ at x is either regular or isomorphic to $K[\![x, y]\!]/(f(x, y))$, where f is one of the functions appearing in the following table:

Type	$f(x,y)$
$A_r, r \geq 1$	$y^2 - x^{r+1}$
$D_r, r \geq 4$	$xy^2 - x^{r-1}$
E_6	$y^3 - x^4$
E_7	$y^3 - x^3 y$
E_8	$y^3 - x^5$

This gives, by definition, a classification of the simple singularities by simply laced Dynkin diagrams. Simple singularities admit a number of equivalent descriptions. We list a few of them here:

- Suppose that X is Gorenstein. Then X has a simple singularity at $x \in X(k)$ if and only if the completed local ring of $X \otimes_k K$ at x has only finitely many isomorphism classes of indecomposable torsion-free modules ([11]).
- If $k = \mathbb{C}$, it makes sense to talk about the nearby fibers of a semi-universal deformation, as in [1], Ch. 3. Then X has a simple singularity at $x \in X(\mathbb{C})$ if and only if only finitely many isomorphism classes of singularities appear in the nearby fibers of the semi-universal deformation.
- If $k = \mathbb{C}$, then X has a simple singularity at $x \in X(\mathbb{C})$ if and only if the symmetric intersection pairing on the homology of the smooth nearby fibers of the singularity is positive definite ([9]).

3. Subregular elements

We continue to assume that G is a split reductive group. We have seen that if $x \in \mathfrak{g}$, the minimal possible dimension of $\mathfrak{z}_{\mathfrak{g}}(x)$ is $\operatorname{rank} G$ (namely, this occurs when x is a regular element). The next smallest possible dimension for $\mathfrak{z}_{\mathfrak{g}}(x)$ is $\operatorname{rank} G + 2$, by [31], III, 3.25. In this case, we say that x is a subregular element.

Lemma 3.1. *Let $x = x_s + x_n$ be the Jordan decomposition of x. Then x is subregular if and only if x_n is subregular in $\mathfrak{z}_{\mathfrak{g}}(x_s)$.*

Proof. This follows since $Z_G(x_s)$ is reductive and $\operatorname{rank} Z_G(x_s) = \operatorname{rank} G$.

□

Of particular importance will be the subregular nilpotent elements.

Proposition 3.2. *$\mathfrak{g}$ contains subregular nilpotent elements. Suppose that G is simple and that k is algebraically closed. Then there is a unique $G(k)$-orbit of subregular nilpotent elements in $\mathfrak{g}$, and these are dense in the complement of the regular nilpotent orbit in the nilpotent variety of $\mathfrak{g}$.*

Proof. This follows from [32], 3.10, Theorem 1. □

Thus if $\mathfrak{g}$ is simple, then its nilpotent variety has a unique open orbit, consisting of regular nilpotent elements; its complement again has a unique open orbit, consisting of the subregular nilpotents. If $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_s$ is a product of simple Lie algebras, then any nilpotent element n can be written uniquely as a sum $n = n_1 + \cdots + n_s$, where $n_i \in \mathfrak{g}_i$. It is then easy to see that n is regular if and only if each n_i is regular in $\mathfrak{g}$; and n is subregular if and only if some n_i is subregular in $\mathfrak{g}_i$, and all other n_j are regular nilpotent elements. In particular, when k is algebraically closed there are exactly s $G(k)$-orbits of subregular nilpotent elements, and there is a canonical bijection between these and the set of connected components of the Dynkin diagram of $\mathfrak{g}$.

Now suppose that θ is stable involution of G. Before we continue, it is helpful to note the following.

Lemma 3.3. *Let $x \in \mathfrak{g}_1$. Then* $\dim \mathfrak{z}_{\mathfrak{g}_0}(x) = (\dim \mathfrak{z}_{\mathfrak{g}}(x) - \operatorname{rank} G)/2$, *and* $\dim G_0 \cdot x = (\dim G \cdot x)/2$.

Proof. This follows from [16], Proposition 5. □

Our next goal is to show that $\mathfrak{g}_1$ contains subregular nilpotent elements. We use a trick, based on the Kostant-Sekiguchi correspondence.

Theorem 3.4. *Suppose that $k = \mathbb{R}$ and that G is semisimple. Let τ be a Cartan involution of G. Then there are bijections between the following three sets:*

(1) The set of nilpotent $G(\mathbb{R})^0$-orbits in $\mathfrak{g}$.

(2) The set of nilpotent $G^\tau(\mathbb{C})^0$-orbits in $\mathfrak{g}^{\tau=-1} \otimes_{\mathbb{R}} \mathbb{C}$.

(3) The set of nilpotent $G_0(\mathbb{C})$-orbits in $\mathfrak{g}_1 \otimes_{\mathbb{R}} \mathbb{C}$.

The map $G(\mathbb{R})^0 \cdot X \mapsto G^\tau(\mathbb{C})^0 \cdot X'$ satisfies $G(\mathbb{C}) \cdot X = G(\mathbb{C}) \cdot X'$.

Proof. The bijection between the first two sets is constructed in [7], Section 9.5. The existence of the bijection between the latter two follows since τ is a stable involution, and all such are conjugate over $\mathbb{C}$. □

Corollary 3.5. *Suppose that k is algebraically closed. Then $\mathfrak{g}_1$ contains subregular nilpotent elements.*

Proof. This is implied by the above theorem since, if $k = \mathbb{R}$ and $\mathfrak{g}$ is split, all conjugacy classes of nilpotent elements have an element defined over k. □

To obtain more information, we must argue on a case by case basis. For the rest of this section, we assume that G is adjoint, and that $\mathfrak{g}_1$ contains a regular nilpotent element. We first recall the following (see [28], §7.5, Lemma 4).

Proposition 3.6. *Suppose that G is simple and simply laced, and let $x \in \mathfrak{g}$ be a subregular nilpotent element. Then $Z_G(x)$ is the semi-direct product of a unipotent group with either $\mathbb{G}_m$ (if G is type A_r) or the trivial group (if G is of type D_r or E_r). In particular, this centralizer is connected.*

In fact, the connectedness of the centralizer of the subregular nilpotent characterizes the simply laced groups amongst the simple groups. The author does not know a proof of this fact that avoids case-by-case analysis, which suggests why it may be necessary here.

Corollary 3.7. *Suppose that k is algebraically closed, and that G is of type D_r or E_r. Then $(G^\theta/G_0)(k)$ acts simply transitively on the set of $G_0(k)$-orbits of subregular nilpotent elements of $\mathfrak{g}_1$.*

Proof. Let x be a subregular nilpotent element. Then $Z_{G^\theta}(x) = Z_{G_0}(x)$, by the above. It therefore suffices to show that $\#(G^\theta/G_0)(k)$ is equal to the number of real subregular nilpotent orbits. This can be accomplished, for example, by inspection of the tables in [7]. $\square$

Proposition 3.8. *Suppose that k is algebraically closed, and that G is of type A_r. Then there is a unique $G_0(k)$-conjugacy class of subregular nilpotent elements in $\mathfrak{g}_1$.*

Proof. We note that there when $k = \mathbb{R}$, there is a unique real orbit of subregular nilpotents in $\mathfrak{g}$. $\square$

We now treat the case where k is not necessarily algebraically closed.

Proposition 3.9. $\mathfrak{g}_1$ *contains a subregular nilpotent element. In particular, we can find subregular normal $\mathfrak{sl}_2$-triples (e, h, f) in $\mathfrak{g}$.*

Proof. Let K denote a separable closure of k. It suffices to find a normal $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g} \otimes_k K$ such that e is subregular nilpotent and $h \in \mathfrak{g}$. For then the set of subregular elements is Zariski dense in $\mathfrak{g}_1^{\mathrm{ad}\, h=2}$ (see [8], Proposition 7) and our chosen field k is infinite.

Since $\mathfrak{g}_1$ contains a regular nilpotent element, we may assume that G is equipped with a pinning $(T, B, \{X_\alpha\}_{\alpha\in R})$ and that θ is the involution above constructed in terms of this pinning. In particular, $\mathfrak{t}_0 = \mathfrak{t}^\theta \subset \mathfrak{g}_0$ is a split Cartan subalgebra of G_0.

Let (e, h, f) be a subregular normal $\mathfrak{sl}_2$-triple in $\mathfrak{g} \otimes_k K$. After conjugating by an element of $G_0(K)$, we can assume that h lies in $\mathfrak{t}_0 \otimes_k K \subset \mathfrak{t} \otimes_k K$. Now we have $\alpha(h) \in \mathbb{Z}$ for every root α, since h embeds in an $\mathfrak{sl}_2$-triple, and hence h lies in $\mathfrak{t}_0$. The result follows. $\square$

Proposition 3.10. *(1) Suppose that G is of type D_r or E_r. Then all subregular nilpotent elements in $\mathfrak{g}_1$ are $G^\theta(k)$-conjugate.*

(2) Suppose that G is of type A_{2r}. Then there is a bijection between $k^\times/(k^\times)^2$ and the set of $G_0(k)$-orbits of subregular nilpotent elements in $\mathfrak{g}_1$, given by sending $d \cdot (k^\times)^2$ to the orbit of the element (in the notation of Example 2.23 above):

$$(f_1 \mapsto f_2 \mapsto f_3 \mapsto \ldots \mapsto f_n \mapsto de_n, e_n \mapsto e_{n-1} \mapsto \ldots \mapsto e_1, v \mapsto 0).$$

(3) Suppose that G is of type A_{2r+1}. Then all subregular nilpotent elements in $\mathfrak{g}_1$ are $G_0(k)$-conjugate.

Proof. Let $x \in \mathfrak{g}_1$ be a subregular nilpotent element. The first part follows since $Z_{G^\theta}(x)$ is a unipotent group, hence has vanishing first Galois cohomology, and we can apply Corollary 2.25. To prove the second and third parts, we make an explicit calculation using the results of Kawanaka [15]. Briefly, if (e, h, f) is a normal $\mathfrak{sl}_2$-triple, let $\overline{G}_0$ denote the connected subgroup of G with Lie algebra $\mathfrak{g}_0 \cap \mathfrak{g}^{\mathrm{ad}\, h=0}$. Then Kawanaka shows that $Z_{G_0}(e)$ has the form $C \ltimes R$, where R is connected unipotent and $C = Z_{\overline{G}_0}(e)$ has reductive connected component. We summarize the results of this calculation here.

If $\mathfrak{g}$ is of type A_{2r}, a choice of subregular nilpotent x is the transformation given by the formula (in the notation of Example 2.23 above):

$$f_1 \mapsto f_2 \mapsto f_3 \mapsto \ldots \mapsto f_n \mapsto e_n \mapsto e_{n-1} \mapsto \ldots \mapsto e_1, v \mapsto 0.$$

If $d \in k^\times$, we define another element x_d by the formula

$$f_1 \mapsto f_2 \mapsto f_3 \mapsto \ldots \mapsto f_n \mapsto de_n, e_n \mapsto e_{n-1} \mapsto \ldots \mapsto e_1, v \mapsto 0.$$

One calculates that $Z_{G_0}(e)$ is a semi-direct product of μ_2 by a connected unipotent group, with Galois cohomology isomorphic (via the Kummer isomorphism) to $k^\times/(k^\times)^2$. With appropriate identifications the element $d \in k^\times/(k^\times)^2$ corresponds to the $G_0(k)$-orbit of the element x_d.

If $\mathfrak{g}$ is of type A_{2r+1}, then one calculates that $Z_{G_0}(e)$ is connected unipotent, so has vanishing first Galois cohomology. $\square$

Proposition 3.11. *Suppose that k is algebraically closed. If G is of type A_r, D_{2r+1} or E_r then the closure of every regular nilpotent $G_0(k)$-orbit in $\mathfrak{g}_1$ contains every subregular nilpotent orbit. If G is of type D_{2r}, then the closure of each regular nilpotent $G_0(k)$-orbit contains exactly 3 subregular nilpotent orbits. Conversely, each subregular nilpotent orbit is contained in the closure of exactly 3 regular nilpotent orbits.*

Proof. The only cases needing proof are A_{2r+1}, D_r, and E_7. The case of A_{2r+1} follows immediately, since $(G^\theta/G_0)(k)$ permutes the regular nilpotent orbits. The cases of D_r and E_7 follow from the descriptions given in the works [21] and [20], respectively. $\square$

4. Subregular curves

For the rest of this thesis, we fix the following notation. We suppose that G is a split simple group over k, of type A_r, D_r or E_r. We fix also a stable involution θ of G and a regular nilpotent element $E \in \mathfrak{g}_1$. We recall that the pair (θ, E) is determined uniquely up to $G^{\mathrm{ad}}(k)$-conjugacy. In this section we construct a family of curves over the categorical quotient $\mathfrak{g}_1/\!\!/G_0$. The construction is based on the notion of transverse slice to the action of an algebraic group, which we now briefly review.

Transverse slices. For the moment, let H be an algebraic group acting on a variety X, both defined over k. Let $x \in X(k)$. By a transverse slice in X to the orbit of x (or more simply, a transverse slice at x), we mean a locally closed subvariety $S \subset X$ satisfying the following:

(1) $x \in S(k)$.

(2) The orbit map $H \times S \to X, (h, s) \mapsto h \cdot s$, is smooth.

(3) S has minimal dimension with respect to the above properties.

It is easy to show that if X is smooth, then transverse slices of the above kind always exist and have dimension equal to the codimension of the orbit $H \cdot x$ in X. (Here we use that k is of characteristic zero; in general, one should assume also that the orbit maps are separable). An important property of transverse slices is the following slight extension of [28], §5.2, Lemma 3:

Proposition 4.1. *Let H, X be as above, and let S_1, S_2 be transverse slices at points x_1, x_2, respectively, where x_1, x_2 lie in the same $H(k)$-orbit of X. Suppose that X is smooth. Let $f : X \to Y$ be a H-equivariant morphism, where H acts trivially on Y. Then:*

- *S_1, S_2 are étale locally isomorphic over Y in the sense that there exists a variety S over Y with a geometric point $\overline{s}$ and étale Y-morphisms $\phi_1 : S \to S_1, \phi_2 : S \to S_2$ with $\phi_1(\overline{s}) = x_1, \phi_2(\overline{s}) = x_2$.*
- *Suppose further that $k = \mathbb{C}$. Then $S_1(\mathbb{C}), S_2(\mathbb{C})$ are locally isomorphic over $Y(\mathbb{C})$ in the analytic topology. Furthermore, there exist arbitrarily small neighborhoods $U_1 \subset S_1(\mathbb{C}), U_2 \subset S_2(\mathbb{C})$ of x and homeomorphisms $\psi : U_1 \to$*

U_2 over $Y(\mathbb{C})$ such that the induced maps $U_1 \hookrightarrow X(\mathbb{C})$, $U_1 \cong U_2 \hookrightarrow X(\mathbb{C})$ are homotopic over $Y(\mathbb{C})$.

Proof. After translating, we can assume that $x_1 = x_2 = x$, say. We can choose a projection $\pi : H \to \mathfrak{h}$ which takes e to 0 and is étale at the identity. Choose a direct sum decomposition $\mathfrak{h} = \mathfrak{z}_{\mathfrak{h}}(x) \oplus V$ and let $C_1 = \pi^{-1}(V)$, $C_2 = C_1^{-1} = \{c^{-1} | c \in C_1\}$. Then the two maps

$$m_1 : C_1 \times S_1 \to X, m_2 : C_2 \times S_2 \to X$$

are étale at (e, x). We set $M_1 = m_2^{-1}(S_1), M_2 = m_1^{-1}(S_2)$. Thus we have Cartesian diagrams

$$\begin{tikzcd} M_1 \arrow[r] \arrow[d] & S_2 \arrow[d] \\ C_1 \times S_1 \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} M_2 \arrow[r] \arrow[d] & S_1 \arrow[d] \\ C_2 \times S_2 \arrow[r] & X, \end{tikzcd}$$

and the arrows in the top row are étale at (e, x). Furthermore, M_1 and M_2 are isomorphic via the map

$$\psi : M_1 \to M_2 : (c, s) \mapsto (c^{-1}, cs).$$

The first part of the proposition follows on taking S to be an open neighborhood of x in $M_1 = M_2$ which is étale over both S_1 and S_2.

We now prove the second part. To simplify notation, let us identify the varieties with their complex points. Let $W \subset C_1 \times S_1$ be a neighborhood of x such that $W \to X$ is an open embedding, and let $U_2 = M_1 \cap W$. We let $U_1 = \psi(U_2)$. After possibly shrinking U_1 and U_2, we can suppose that the map $m_2|_{U_1} : U_1 \to X$ is also an embedding. We can then identify the U_i with their images in the respective S_i.

We may suppose that the projection of W to C_1 is contained in a contractible subset, and let $f_t : W \to C_1$ denote a contraction to the identity. We now define

$g_t : U_2 \to X$ by $g_t(c,s) = m_1(f_t(c), s)$. Then g_1 is equal to the restriction of m_1 to U_2, and $g_0(c,s) = s = m_2(\psi(c,s))$. The result follows. □

An important special case where we can construct transverse slices explicitly is the case of a reductive group H acting via the adjoint representation on its Lie algebra $\mathfrak{h}$. The construction uses the theory of $\mathfrak{sl}_2$-triples. We first recall some basic facts about representations of $\mathfrak{sl}_2$. This simple Lie algebra has a basis $\{x, t, y\}$ consisting of the elements

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; t = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \text{ and } y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In particular, the tuple (x, t, y) is an $\mathfrak{sl}_2$-triple.

Proposition 4.2. *(1) For each integer $m \geq 1$, there is a unique simple $\mathfrak{sl}_2$-module $V(m)$ of dimension m, up to isomorphism. All of the eigenvalues of h on $V(m)$ are integers, and the non-zero weight spaces $V(m)^{t=i}$ are 1-dimensional. The space $V(m)^{t=i}$ is non-zero exactly for $i = 1-m, 3-m, \ldots, m-3, m-1$. We call $V(m)^{t=m-1}$ the highest weight space, and $V(m)^{t=1-m}$ the lowest weight space.*

(2) Suppose that $V(m)^{t=i}$ is non-zero. Then we have $x \cdot V(m)^{t=i} = V(m)^{t=i+2}$ and $y \cdot V(m)^{t=i} = V(m)^{t=i-2}$.

(3) Any finite-dimensional $\mathfrak{sl}_2$-module V decomposes as a direct sum $V = \oplus V_i$ of simple submodules.

Proof. See [5], Ch. VIII, §1. □

Now let us take $e \in \mathfrak{h}$ to be a non-zero nilpotent element, and complete e to an $\mathfrak{sl}_2$-triple (e, h, f). We view $\mathfrak{h}$ as an $\mathfrak{sl}_2$-module by restricting the adjoint action to the subalgebra spanned by these elements, and choose a decomposition $V = \oplus_i V_i$ into a direct sum of simple $\mathfrak{sl}_2$-modules.

Now, the affine tangent space to the orbit $H \cdot e$ at e is given by $e + [e, \mathfrak{h}] \subset \mathfrak{h}$ (see [3], I.3.10). Using the above decomposition, we have $[e, \mathfrak{h}] = \oplus_i [e, V_i]$, and $[e, V_i]$ is just the t-equivariant complement of the lowest weight space in V_i, namely $\ker \operatorname{ad} f : V_i \to V_i$. It follows that $\mathfrak{z}_{\mathfrak{h}}(f)$ is a complement to $[e, \mathfrak{h}]$ in $\mathfrak{h}$, and hence the affine linear subspace $e + \mathfrak{z}_{\mathfrak{h}}(f) \subset \mathfrak{h}$ is a transverse slice at e.

Proposition 4.3. *$S = e + \mathfrak{z}_{\mathfrak{h}}(f)$ is a transverse slice to the action of H at every point of S. In other words, the multiplication map $\mu : H \times S \to \mathfrak{h}$ is everywhere smooth.*

The proof is based on the following construction of Slodowy. We let $\lambda : \mathbb{G}_m \to H$ be the cocharacter with $d\lambda(1) = h$. Let $p_1, \dots, p_r$ be algebraically independent homogeneous polynomials generating the ring of invariants $k[\mathfrak{h}]^H$. We suppose that they have degrees $d_1, \dots, d_r$. We suppose that V_i has dimension m_i, and choose for each i a basis vector v_i of the lowest weight space of V_i.

A general element $v \in S$ can be written in the form $v = e + \sum_i x_i v_i$, and we have

$$\lambda(t)(v) = t^2 e + \sum_i t^{1-m_i} x_i v_i, t \cdot v = te + \sum_i t x_i v_i$$

and

$$p_i(\lambda(t)(v)) = p_i(v), p_i(t \cdot v) = t^{d_i} p_i(v).$$

Defining an action ρ of $\mathbb{G}_m$ on $\mathfrak{h}$ by $\rho(t)(v) = t^2 \lambda(t^{-1}) \cdot v$, we see that S is ρ-invariant, and the ρ-action contracts S to e. If we let $\mathbb{G}_m$ act on $\mathfrak{h} /\!/ H$ by the square of its usual action, then the composite $S \hookrightarrow \mathfrak{h} \to \mathfrak{h} /\!/ H$ becomes $\mathbb{G}_m$-equivariant. In other words, writing $w_1, \dots, w_n$ for the weights of the ρ-action on S, the morphism $S \to \mathfrak{h} /\!/ H$ is quasi-homogeneous of type $(d_1, \dots, d_r; w_1, \dots, w_n)$. The weights w_i are given by the formula $w_i = m_i + 1$.

Proof of Proposition 4.3. Define an action of $\mathbb{G}_m \times H$ on $H \times S$ by $(t, g) \cdot (k, s) = (gk\lambda(t), \rho(t)s)$, and let $\mathbb{G}_m \times H$ act on $\mathfrak{h}$ by $(t, g) \cdot X = t^2 \operatorname{Ad}(g)(X)$. Then the map $\mu : H \times S \to \mathfrak{h}$ is equivariant for these actions, and smooth in a neighborhood of

$H \times \{x\} \subset H \times S$; since the $\mathbb{G}_m$-actions are contracting, it follows that μ is smooth everywhere. $\square$

Corollary 4.4. *The composite $S \hookrightarrow \mathfrak{h} \to \mathfrak{h}/\!\!/H$ is faithfully flat.*

Proof. The composite $H \times S \to S \to \mathfrak{h}/\!\!/H$ is equal to the composite $H \times S \to \mathfrak{h} \to \mathfrak{h}/\!\!/H$, which is a composition of flat morphisms, hence flat ($H \times S \to \mathfrak{h}$ is flat since we have just proved it to be smooth). Since the second projection $H \times S \to S$ is flat, $S \to \mathfrak{h}/\!\!/H$ must also be flat.

The image is a $\mathbb{G}_m$-stable open subset of $\mathfrak{h}/\!\!/H$ containing the origin, hence the whole of $\mathfrak{h}/\!\!/H$. The faithful flatness follows. $\square$

Let us now return to our group G with stable involution θ, and let (e, h, f) now denote a normal $\mathfrak{sl}_2$-triple. From the above, we see that there is a direct sum decomposition $\mathfrak{g} = [e, \mathfrak{g}] \oplus \mathfrak{z}_{\mathfrak{g}}(f)$. Both summands are θ-stable so we deduce that $\mathfrak{g}_1 = [e, \mathfrak{g}_0] \oplus \mathfrak{z}_{\mathfrak{g}}(f)_1$, where by definition $\mathfrak{z}_{\mathfrak{g}}(f)_1 = \mathfrak{z}_{\mathfrak{g}}(f) \cap \mathfrak{g}_1$. It follows that $X = e + \mathfrak{z}_{\mathfrak{g}}(f)_1$ is a transverse slice at $e \in \mathfrak{g}_1$, and identical arguments to those above now prove the following.

Proposition 4.5. *The map $\mu : G_0 \times X \to \mathfrak{g}_1$ is smooth and the composite $X \hookrightarrow \mathfrak{g}_1 \to \mathfrak{g}_1/\!\!/G_0$ is faithfully flat.*

We now examine two special cases of this construction in more detail.

The regular $\mathfrak{sl}_2$ and the Kostant section. Let $d_1, \dots, d_r$ denote the degrees of algebraically independent homogeneous generators of the polynomial ring $k[\mathfrak{g}_1]^{G_0}$. We let (E, H, F) be the unique normal $\mathfrak{sl}_2$-triple containing the element E, and set $\kappa = E + \mathfrak{z}_{\mathfrak{g}}(F)_1$. We call κ the Kostant section. It has the following remarkable properties.

Lemma 4.6. *The composite $\kappa \hookrightarrow \mathfrak{g}_1 \to \mathfrak{g}_1/\!\!/G_0$ is an isomorphism. Every element of κ is regular. In particular, the map $\mathfrak{g}_1(k) \to (\mathfrak{g}_1/\!\!/G_0)(k)$ is surjective, and if k is* *algebraically closed then κ meets every $G_0(k)$-conjugacy class of regular semisimple elements.*

Proof. It is well known that $\mathfrak{g}$ decomposes under the action of a regular $\mathfrak{sl}_2$ as $\mathfrak{g} \cong \oplus_{i=1}^r V(2d_i - 1)$. By the above, the map $\kappa \to \mathfrak{g}_1 /\!\!/ G_0$ is faithfully flat, and quasi-homogeneous of type $(2d_1, \dots, 2d_r; 2d_1, \dots, 2d_r)$. Lemma 4.14 below now implies that it must be an isomorphism. The remaining claims are immediate. $\square$

A subregular $\mathfrak{sl}_2$. Now fix a normal subregular $\mathfrak{sl}_2$-triple (e, h, f), and set $X = e + \mathfrak{z}_{\mathfrak{g}}(f)_1$. (Note that if G is of type A_1, then there is no non-zero subregular nilpotent element, and therefore no subregular $\mathfrak{sl}_2$-triple, since we have defined an $\mathfrak{sl}_2$-triple to consist of 3 linearly independent elements. In this case, we just take $X = \mathfrak{g}_1$). Recall that we have defined an action of $\mathbb{G}_m$ on X.

Proposition 4.7. *We have* $\dim X = r + 1$. *We write $w_1, \dots, w_{r+1}$ for the weights of the $\mathbb{G}_m$-action. After re-ordering, we have $w_i = 2d_i$ for $i = 1, \dots, r-1$. The $2d_i, i = 1, \dots, r-1$ and w_r and w_{r+1} are given in the following table:*

	$2d_1$	$2d_2$	$2d_3$	$\dots$	$\dots$	$2d_{r-2}$	$2d_{r-1}$	$2d_r$	w_r	w_{r+1}
A_r	4	6	8	$\dots$	$\dots$	$2r-2$	$2r$	$2r+2$	2	$r+1$
D_r	4	8	12	$\dots$	$\dots$	$4r-8$	$2r$	$4r-4$	4	$2r-4$
E_6	4	10	12			16	18	24	6	8
E_7	4	12	16		20	24	28	36	8	12
E_8	4	16	24	28	36	40	48	60	12	20

Proof. The proof is by explicit calculation, along similar lines to the proof of [28], §7.4, Proposition 2. We describe the method. If $V \subset \mathfrak{g}$ is a θ-stable simple $\mathfrak{sl}_2$-submodule, then its highest weight space is θ-invariant. Moreover, the eigenvalue of θ on this highest weight space determines the action of θ on every weight space. We can calculate a decomposition of $\mathfrak{g}$ into a direct sum of θ-stable simple $\mathfrak{sl}_2$-modules by calculating the dimension of each weight space of h, and the trace of θ on each weight space. This can be accomplished by using the explicit θ constructed in Lemma 2.17 and a list of the roots of $\mathfrak{g}$. We can then fill in the table by reading off the lowest weight spaces which have θ-eigenvalue equal to -1.

For example, suppose that G is of type A_2. Then a choice of h is

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

in the notation of Example 2.23. We can write the weights of h on $\mathfrak{g}$ with multiplicity as follows:

$$\begin{array}{ccccc} -2 & & 0 & & 2 \\ & -1 & & 1 & \\ & -1 & & 1 & \\ & & 0 & & \end{array}$$

Thus $\mathfrak{g}$ decomposes as a direct sum $V(3) \oplus V(2) \oplus V(2) \oplus V(1)$. In this case -1 is an eigenvalue of θ of multiplicity 1 on each weight space. (Recall that $-\theta$ is reflection in the anti-diagonal). We can now decorate each weight space with a $+$ or $-$, according to its θ-eigenvalue:

$$\begin{array}{ccccc} -2^- & & 0^+ & & 2^- \\ & -1^+ & & 1^- & \\ & -1^- & & 1^+ & \\ & & 0^- & & \end{array}$$

It follows that $\dim \mathfrak{z}_{\mathfrak{g}}(f)_1 = 3$, as expected, and the eigenvalues of h on $\mathfrak{z}_{\mathfrak{g}}(f)_1$ are $-2, -1$ and 0, hence the weights on $e + \mathfrak{z}_{\mathfrak{g}}(f)_1$ are $2, 3$, and 4, as claimed above. $\square$

Henceforth we write $\mathfrak{g}_1 /\!\!/ G_0 = B$, and $\varphi : X \to B$ for the restriction of the quotient map $\pi : \mathfrak{g}_1 \to \mathfrak{g}_1 /\!\!/ G_0$ to X. The main result of this section is the following.

Theorem 4.8. *The fibers of φ are reduced curves. The central fiber $X_0 = \varphi^{-1}(0)$ has a unique singular point which is a simple singularity of type A_r, D_r, E_r, corresponding to that of G. We can choose homogeneous co-ordinates $(p_{d_1}, \ldots, p_{d_r})$ on B and $(p_{d_1}, \ldots, p_{d_{r-1}}, x, y)$ on X such that the family $X \to B$ of curves is as given by the following table:*

G	X
A_r	$y^2 = x^{r+1} + p_2 x^{r-1} + \cdots + p_{r+1}$
D_r	$y(xy + p_r) = x^{r-1} + p_2 x^{r-2} + \cdots + p_{2r-2}$
E_6	$y^3 = x^4 + y(p_2 x^2 + p_5 x + p_8) + p_6 x^2 + p_9 x + p_{12}$
E_7	$y^3 = x^3 y + p_{10} x^2 + x(p_2 y^2 + p_8 y^2 + p_{14}) + p_6 y^2 + p_{12} y + p_{18}$
E_8	$y^3 = x^5 + y(p_2 x^3 + p_8 x^2 + p_{14} x + p_{20}) + p_{12} x^3 + p_{18} x^2 + p_{24} x + p_{30}$.

(This means, for example, that when G is of type A_r, the relation $p_{r+1} = y^2 - (x^{r+1} + p_2 x^{r-1} + \cdots + p_r x)$ holds on X). The proof of Theorem 4.8 follows closely the work of Slodowy [28], with some simplifications due to the fact that we work with curves, rather than surfaces. We begin with some general considerations, and reduce to a case by case calculation using the invariant degrees of G.

The possibility of choosing co-ordinates as above is a consequence of the following general lemma.

Lemma 4.9 ([28], §8.1, Lemma 2)**.** *Let V, U be k-vector spaces of dimensions m, n respectively, on which $\mathbb{G}_m$ acts linearly. Let $\phi : V \to U$ be a morphism equivariant for these actions. Suppose that $d\phi_0$ has rank s and that $\mathbb{G}_m$ acts with strictly positive weights on U and V.*

Then there exist $\mathbb{G}_m$-invariant decompositions $V = V_0 \oplus W$, $U = U_0 \oplus W$, $\dim W =$ *s, and a regular automorphism α of V such that $\phi \circ \alpha$ has the form $(v_0, w) \mapsto (\psi(v_0, w), w)$ for some ψ.*

To apply this to the map $\varphi : X \to B$, we need the following result.

Proposition 4.10. *Let $x \in X$. Then $d\varphi_x$ has maximal rank $r = \operatorname{rank} G$ if and only if x is a regular element. The map $d\varphi_0 : T_e X \to T_0 B$ has rank $r-1$.*

Proof. Let $p : \mathfrak{g} \to \mathfrak{g}/\!/G$ denote the adjoint quotient map. For any $y \in \mathfrak{g}_1$, we have $dp_y(\mathfrak{g}_0) = 0$. This is true if y is regular, since then $\mathfrak{g}_0 = [y, \mathfrak{g}_1] \subset [y, \mathfrak{g}]$ is contained in the tangent space to the orbit $G \cdot y$. It then follows for any $y \in \mathfrak{g}_1$, since the regular elements are dense. In particular, for any $y \in X$, we have $\operatorname{rank} dp_y = \operatorname{rank} d\pi_y = \operatorname{rank} d\varphi_y$. The first part of the proposition now follows, since $y \in \mathfrak{g}_1$ is regular if and only if dp_y has maximal rank.

For the second part, we remark that $\operatorname{rank} dp_e = r - 1$ if e is subregular nilpotent, by [28], §8.3, Proposition 1. □

We thus obtain a decomposition of affine spaces $X = V_0 \oplus W$, $B = U_0 \oplus W$, where $\dim W = r - 1$, $\dim V_0 = 2$, and $\dim U_0 = 1$. With respect to these decompositions we write

$$\varphi : (v_0, w) \mapsto (\psi(v_0, w), w), V_0 \oplus W \to U_0 \oplus W.$$

Recall that φ is $\mathbb{G}_m$-equivariant of type $(2d_1, \ldots, 2d_r; w_1, \ldots, w_{r+1})$. By inspection of the tables above, we have $2d_r > w_i$, each $i = 1, \ldots, r+1$, and hence the weights occurring in W are $2d_1, \ldots, 2d_{r-1}$. Moreover, the unique weight of U_0 is given by $2d_r$ and the weights of V_0 are w_r, w_{r+1}. Let x, y be homogeneous co-ordinates on V_0 of weight w_r and w_{r+1}, respectively. It follows that $X_0 \subset V_0$ is cut out by a quasi-homogeneous polynomial $f(x, y)$ of type $(2d_r; w_r, w_{r+1})$.

Proposition 4.11. *After possibly making a linear change of variables, the polynomial $f(x, y)$ is as given by the following table.*

G	$f(x,y)$
$A_r, r \geq 1$	$y^2 - x^{r+1}$
$D_r, r \geq 4$	$xy^2 - x^{r-1}$
E_6	$y^3 - x^4$
E_7	$y^3 - x^3y$
E_8	$y^3 - x^5$

Proof. We suppose first that k is algebraically closed. Then the induced map $G_0 \times X_0 \to \pi^{-1}(0)$ is smooth, since X is a transverse slice and this property is preserved under passage to fibers (see [28], §5, Lemma 2). Since $\pi^{-1}(0)$ is smooth along the regular locus, X_0 is generically smooth, hence reduced. We now proceed by direct computation. Let us treat for example the case of A_r. Then $f(x,y)$ is quasi-homogeneous of type $(2r+2; 2, r+1)$, where we suppose that the weights of x and y are 2 and $r+1$, respectively.

Since f defines a reduced curve, it must have the form $ay^2 - bx^{r+1}$, with a, b non-zero constants. After rescaling we may assume that f has the form given in the statement of the proposition. The same argument works for the other cases as well.

Now suppose that k is not algebraically closed. The same argument suffices, except in the cases A_{2r+1} and D_{2r}. For example, in case A_{2r+1} one must rule out the possibility $f(x,y) = y^2 - ax^{2r+2}$, where $a \in k^\times$ is a non-square. But the natural action map $G_0 \times X_0 \to \pi^{-1}(0)$ induces an injection on geometric irreducible components, see Lemma 5.13 below. The irreducible components of $\pi^{-1}(0)$ are geometrically irreducible, so it follows that the same must be true for X_0, hence a must be a square. The same argument works for the case of type D_{2r}. $\square$

At this point we have identified the central fiber of φ with the desired curve. We will obtain the identification over the whole of B via a deformation argument. Before doing this, we must determine the singularities appearing in the other fibers of φ.

Proposition 4.12. *Let $t \in \mathfrak{g}_1$ be a semisimple element, and let b denote its image in B. Let D denote the Dynkin diagram of $Z_G(t)$, and write it as a disjoint union $D = D_1 \cup \cdots \cup D_k$ of its connected components.*

Let $y \in \varphi^{-1}(b)(k) = X_b(k)$ be a singular point. Then y is a simple singularity of type D_i for some $i = 1, \ldots, s$.

Proof. We have an isomorphism

$$G_0 \times^{Z_{G_0}(t)} (t + \mathcal{N}(\mathfrak{z}_{\mathfrak{g}}(t)_1)) \cong \pi^{-1}(b),$$

induced by the map $(g, t+n) \mapsto g \cdot (t+n)$. Let y have Jordan decomposition $y = y_s + y_n$. Without loss of generality, we may suppose that k is algebraically closed and that $y_s = t$. Then $y_n \in \mathfrak{z}_{\mathfrak{g}}(t)$ is a subregular nilpotent element. If we decompose $[\mathfrak{z}_{\mathfrak{g}}(t), \mathfrak{z}_{\mathfrak{g}}(t)] = \mathfrak{l}^1 \times \cdots \times \mathfrak{l}^k$ into a product of simple, θ-stable subalgebras then y_n has a decomposition $y_n = y_1 + \cdots + y_k$, where $y_i \in \mathfrak{l}^i$ is a nilpotent element. After re-numbering, we can assume that $y_1 \in \mathfrak{l}^1$ is a subregular nilpotent element, and all of the other $y_i \in \mathfrak{l}^i$ are regular nilpotent. Moreover, the restriction of θ to each $\mathfrak{l}^i$ is a stable involution.

Now fix a transverse slice S_1 to the $Z_{G_0}(t)$-orbit of y_1 in $\mathfrak{l}^1_1$. It then follows that $S_1 + \sum_{j \geq 2} y_j$ is a transverse slice to the $Z_{G_0}(t)$-orbit of y_n in $\mathcal{N}(\mathfrak{l}^1_1)$ and hence $X_1 = t + S_1 + \sum_{j \geq 2} y_j$ is a transverse slice at y to the G_0 action in $\pi^{-1}(b)$, as the above isomorphism makes $\pi^{-1}(b)$ into a fiber bundle over $G_0/Z_{G_0}(t)$ with fiber $\mathcal{N}(\mathfrak{z}_{\mathfrak{g}}(t)_1)$.

On the other hand, we know that X_b is also a transverse slice at y to the G_0 action in $\pi^{-1}(b)$. The result now follows from Proposition 4.1 and Proposition 4.11. $\square$

Semi-universal deformations and the proof of Theorem 4.8. We can now complete the proof of Theorem 4.8. By Proposition 2.26, there exists a semi-universal deformation $\widehat{Z} \to \widehat{D}$ of the central fiber X_0 as a $\mathbb{G}_m$-scheme, where $\widehat{Z} \to \widehat{D}$ is a morphism of formal schemes with underlying reduced schemes given by $X_0 \to \operatorname{Spec} k$.

The proof is based on the fact that, since X_0 is given as the zero set of an explicit polynomial $f(x,y)$, $\widehat{Z} \to \widehat{D}$ admits a canonical algebraization $Z \to D$ which we can calculate explicitly and then compare with $X \to B$.

Proposition 4.13. *Let $f(x,y)$ be a polynomial in two variables, quasi-homogeneous of type $(d; w_1, w_2)$. Let $X_0 \subset \mathbb{A}^2$ denote the closed subscheme defined by f, and suppose that X_0 has an isolated singularity at the origin. Then a semi-universal $\mathbb{G}_m$-deformation of X_0 can be construction as follows: let $J = (\partial f/\partial x, \partial f/\partial y) \subset k[x,y]$ denote the Jacobian ideal of f. Then $k[x,y]/J$ is a finite-dimensional k-vector space, and receives an action of $\mathbb{G}_m$. Choose $\mathbb{G}_m$-invariant polynomials $g_1(x,y), \ldots, g_n(x,y)$ projecting to a k-basis of $\mathbb{G}_m$-eigenvectors of $k[x,y]/J$. Now define*

$$Z = \{f + t_1 g_1 + \cdots + t_n g_n = 0\} \subset \mathbb{A}^2 \times \mathbb{A}^n,$$

and let $\Phi : Z \to D$ denote the natural projection to the $\mathbb{A}^n$ factor.

Suppose that g_i has weight r_i, and let $\mathbb{G}_m$ act on t_i by the character $t \mapsto t^{d-r_i}$. Then Φ is a $\mathbb{G}_m$-equivariant morphism, and the formal completion $\widehat{\Phi} : \widehat{Z} \to \widehat{D}$ of this morphism is a semi-universal $\mathbb{G}_m$-deformation of X_0.

Proof. See [28], §2.4. □

Applying this our fixed polynomial f, we obtain a family of curves $Z \to D$, where D is an affine space of dimension n, and a Cartesian diagram of formal schemes

$$\begin{array}{ccc} \widehat{X} & \longrightarrow & \widehat{Z} \\ \downarrow & & \downarrow \\ \widehat{B} & \longrightarrow & \widehat{D} \end{array} .$$

An elementary (and enjoyable) calculation shows that in each case A_r, D_r, E_r, $n = r$ and $Z \to D$ is the family of curves appearing in the statement of Theorem 4.8. Let us do the example where G is of type E_6 here. Then we can take $f(x,y) =$

$y^3 - x^4$, a quasi-homogeneous polynomial of type $(24; 6, 8)$. The Jacobian ideal is $J = (x^3, y^2)$. A basis of the ring $k[x,y]/J$ can be taken to be the classes of the polynomials $1, x, x^2, y, xy, x^2y$ mod J. Our family of curves thus takes the form

$$y^3 - x^4 + t_{24} + t_{18}x + t_{12}x^2 + y(t_{16} + t_{10}x + t_4x^2) = 0,$$

where $\mathbb{G}_m$ acts on t_i now by the character $t \mapsto t^i$. Renaming the variables, we obtain the family of curves listed above.

The morphism $\widehat{B} \to \widehat{D}$ is given by power series and respects the $\mathbb{G}_m$-actions on either side, which both have strictly positive weights; it follows that these power series are in fact polynomials, so this morphism has a canonical algebraization. We obtain a second Cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z \arrow[d] \\ B \arrow[r] & D. \end{tikzcd}$$

Now the bottom horizontal arrow is a $\mathbb{G}_m$-equivariant polynomial map between affine spaces of the same dimension and the weights on the domain and codomain are the same. We now apply the following lemma.

Lemma 4.14 ([28], §8.1, Lemma 3)**.** *Let $\mathbb{G}_m$ act on affine spaces V, U of dimension n, and let $\phi : V \to U$ be an equivariant morphism. Suppose that:*

- *$\mathbb{G}_m$ acts on V and U with the same strictly positive weights.*
- *The central fiber $\phi^{-1}(0)$ is zero dimensional.*

Then ϕ is an isomorphism.

We must verify that the second condition holds. If $b \in B$ is mapped to $0 \in D$, then $X_b \cong X_0$. Proposition 4.12 implies that all singularities in the non-central fibers of φ are simple singularities belonging to simply laced root systems of rank strictly less than r, and so we must have $b = 0$. This completes the proof of Theorem 4.8.

Let $S = e + \mathfrak{z}_{\mathfrak{g}}(f)$, and let τ denote the involution of S induced by $-\theta$. Thus S is an affine space of dimension $r + 2$, and we have $S^\tau = X$.

Lemma 4.15. *We can choose global co-ordinates $z_1, \ldots, z_{r+2}$ on S, $w_1, \ldots, w_r$ on B such that $z_1, \ldots, z_{r+1}$ are fixed by τ, $\tau(z_{r+2}) = -z_{r+2}$, and such that the following holds: the morphism $X \to B$ is given by the formula*

$$(z_1, \ldots, z_{r+1}) \mapsto (z_1, \ldots, z_{r-1}, f(z_1, \ldots, z_{r+1}))$$

for some polynomial function f, and the morphism $S \to B$ is given by the formula

$$(z_1, \ldots, z_{r+2}) \mapsto (z_1, \ldots, z_{r-1}, f(z_1, \ldots, z_{r+1}) + z_{r+2}^2).$$

Proof. We recall that there is a contracting action of $\mathbb{G}_m$ on S, and that this action sends X to itself. Applying Lemma 4.9, we see that we can find $\mathbb{G}_m$ and τ-invariant decompositions $S = V_0 \oplus V_1 \oplus W$, $B = U_0 \oplus W$ such that the map $S \to B$ is given by $(v_0, v_1, w) \mapsto (\psi(v_0, v_1, w), w)$ for some $\mathbb{G}_m$-equivariant morphism ψ. Moreover, τ acts trivially on $V_0 \oplus W$ and as -1 on V_1. We have $\dim V_0 = 2$, $\dim V_1 = \dim U_0 = 1$, $\dim W = r - 1$. Moreover, ψ is quasi-homogeneous of some degree.

We choose co-ordinates as follows: let $z_1, \ldots, z_{r-1}$ be arbitrary co-ordinates on W, z_r, z_{r+1} co-ordinates which are eigenfunctions for the $\mathbb{G}_m$-action, and z_{r+2} an arbitrary linear co-ordinate on V_1. Proposition 2 of [28], §7.4 now implies that z_{r+2} has degree equal to half the degree of ψ. It follows that we must have $\psi(v_0, v_1, w) = \psi(v_0, 0, w) + z_{r+2}^2$, after possibly rescaling co-ordinates. (The coefficient of z_{r+2} must be non-zero since S_0 has a unique isolated singularity). $\square$

Corollary 4.16. *Let $b \in B(k)$, and let $t \in \pi^{-1}(b)(k)$ be a semisimple element. Then there is a bijection between the connected components of the Dynkin diagram of*

$Z_G(t)$ and the singularities of the fiber X_b, which takes each (connected, simply laced) Dynkin diagram to a singularity of corresponding type.

Proof. The above lemma implies that the singular locus of S_b is equal to the singular locus of X_b. We have seen that the singular points of X_b are precisely the subregular elements of X_b. It therefore suffices to show that X_b meets each G-orbit of subregular elements in $p^{-1}(b)$ exactly once, or equivalently that S_b meets each G-orbit of subregular elements in $p^{-1}(b)$ exactly once. This follows immediately from [28], §6.6, Proposition 2 and the remark following.

$\square$

5. Jacobians and stabilizers of regular elements

We continue with the notation of the previous section. Thus G is a split simple group of type A_r, D_r, or E_r, and (θ, E) is a pair of stable involution of G, together with a regular nilpotent element $E \in \mathfrak{g}_1$. The pair (θ, E) is uniquely determined up to $G^{\mathrm{ad}}(k)$-conjugacy. This data determines a regular normal $\mathfrak{sl}_2$-triple (E, H, F). We choose further a subregular normal $\mathfrak{sl}_2$-triple (e, h, f). Our chosen $\mathfrak{sl}_2$-triples give two special transverse slices. First, the Kostant section $\kappa = E + \mathfrak{z}_{\mathfrak{g}}(F)_1$, which is a section of the quotient map $\pi : \mathfrak{g}_1 \to B$ by regular elements. Second, a transverse slice to the G_0-orbit of e, $X = e + \mathfrak{z}_{\mathfrak{g}}(f)_1$. The fibers of the induced map $\varphi : X \to B$ are reduced connected curves.

In this section we shall write $\mathfrak{g}_1^{\mathrm{rs}}$ for the open subvariety of regular semisimple elements, and B^{rs} for its image in B. For any variety $Z \to B$ we will write $Z^{\mathrm{rs}} = Z \times_B B^{\mathrm{rs}}$. Thus the morphism $X^{\mathrm{rs}} \to B^{\mathrm{rs}}$ is a family of smooth curves.

Homology. Fix a separable closure K of k. In the following if X is a k-scheme of finite type, we will write $H_1(X, \mathbb{F}_2)$ for $H^1_{\text{ét}}(X \otimes_k K, \mathbb{F}_2)^*$, the dual of the first étale cohomology of $X \otimes_k K$. This is a finite group, and receives an action of the Galois group $\mathrm{Gal}(K/k)$.

Suppose that A is a finite group scheme over k, and that $Y \to X$ is an A-torsor. This defines a class in $H^1_{\text{ét}}(X \otimes_k K, A \otimes_k K) \cong \mathrm{Hom}(H_1(X, \mathbb{F}_2), A(K))$. Viewing $H_1(X, \mathbb{F}_2)$ as a finite group scheme over k, this class defines a homomorphism $H_1(X, \mathbb{F}_2) \to A$.

Now suppose given an embedding $K \hookrightarrow \mathbb{C}$. Then there is a canonical isomorphism $H_1(X, \mathbb{F}_2) \cong H_1(X(\mathbb{C}), \mathbb{F}_2)$ with the topological homology. If $X(\mathbb{C})$ is connected and $x \in X(\mathbb{C})$, then the homomorphism $\pi_1(X(\mathbb{C}), x) \to A(\mathbb{C})$ factors through the Hurewicz map $\pi_1(X(\mathbb{C}), x) \to H_1(X(\mathbb{C}), \mathbb{F}_2)$ and the induced map $H_1(X(\mathbb{C}), \mathbb{F}_2) \to A(\mathbb{C})$ agrees with the previous one, up to applying the comparison isomorphism. In particular, this map does not depend on the choice of basepoint.

If X is a geometrically connected smooth projective curve over k, then there is a canonical isomorphism $H_1(X, \mathbb{F}_2) \cong J_X[2]$, where J_X denotes the Jacobian of the curve X.

Stabilizers of regular elements. Let $\mathfrak{g}_1^{\mathrm{reg}} \subset \mathfrak{g}_1$ denote the open subset of regular elements. We write $Z \to \mathfrak{g}_1^{\mathrm{reg}}$ for the stabilizer scheme, defined as the equalizer of the following diagram:

$$G_0 \times \mathfrak{g}_1^{\mathrm{reg}} \underset{(g,x)\mapsto x}{\overset{(g,x)\mapsto g\cdot x}{\rightrightarrows}} \mathfrak{g}_1^{\mathrm{reg}}$$

Proposition 5.1. *(1) Z is a commutative group scheme, quasi-finite over $\mathfrak{g}_1^{reg}$.*

(2) Z admits a canonical descent to B. In particular, for any two $x, y \in \mathfrak{g}_1^{reg}$ with the same image in B, there is a canonical isomorphism $Z_{G_0}(x) \cong Z_{G_0}(y)$.

Proof. The first part can be checked on geometric fibers.

For the second part, we show that $\kappa^* Z$ is the sought-after descent. The map $(G^{\mathrm{ad}})^\theta \times \kappa \to \mathfrak{g}_1^{\mathrm{reg}}$ is faithfully flat. In fact, it is étale, and [16], Theorem 7 shows it to be surjective. It is now easy to construct an isomorphism between $\pi^*\kappa^* Z$ and Z over this faithfully flat cover. This defines a morphism of descent data since Z is commutative. $\square$

We will henceforth write Z for the descent to a commutative group scheme over B constructed above. Consider the orbit map $\mu^{\mathrm{rs}} : G_0 \times \kappa^{\mathrm{rs}} \to \mathfrak{g}_1^{\mathrm{rs}}$. This map is finite and étale, and we can form the pullback square:

$$\begin{tikzcd} \Gamma \arrow[r] \arrow[d] & G_0 \times \kappa^{\mathrm{rs}} \arrow[d] \\ X^{\mathrm{rs}} \arrow[r] & \mathfrak{g}_1^{\mathrm{rs}}. \end{tikzcd}$$

Concretely, for $b \in B^{\mathrm{rs}}(k)$, $\Gamma_b \to X_b$ is the Z_b-torsor given by

$$\Gamma_b = \{g \in G_0 \,|\, g \cdot \kappa(b) \in X_b\}.$$

We thus obtain a Galois-equivariant map $H_1(X_b, \mathbb{F}_2) \to Z_b$.

Theorem 5.2. *Suppose that G is simply connected. Then this map is an isomorphism.*

Let us first illustrate the theorem in the case $G = SL_2$. We can take θ to be conjugation by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then we have

$$\mathfrak{h}_0 = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\} \text{ and } \mathfrak{h}_1 = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} \right\}.$$

The regular nilpotents in $\mathfrak{h}_1$ are those with x or y zero but not both, and the only subregular nilpotent element in $\mathfrak{h}_1$ is zero. The quotient map $\mathfrak{h}_1 \to \mathfrak{h}_1 /\!\!/ H_0 \cong \mathbb{A}^1$ sends the above matrix to $xy \in \mathbb{A}^1$. In particular $X = \mathfrak{h}_1$ in this case, with the smooth fibers of the map $\varphi : X \to \mathfrak{h}_1 /\!\!/ H_0$ isomorphic to the punctured affine line.

The group H_0 is isomorphic to $\mathbb{G}_m$, and $t \in \mathbb{G}_m$ acts by

$$t \cdot \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} = \begin{pmatrix} 0 & t^2 x \\ t^{-2} y & 0 \end{pmatrix}.$$

The stabilizer of any regular semisimple element is $\mu_2 \subset \mathbb{G}_m$, and it is clear that for any $b \in \mathbb{A}^1 - \{0\}$, the induced map $H_1(X_b, \mathbb{F}_2) \to \mu_2$ is an isomorphism.

We now consider the proof of the theorem in the general case. It suffices to prove the theorem when $k = \mathbb{C}$, which we now assume. In what follows, we simplify notation by identifying all varieties with their complex points. Fix a choice $\mathfrak{c}$ of Cartan subspace, and let $C \subset G$ denote the corresponding maximal torus.

Now choose $x \in \mathfrak{c}$, and let $b = \pi(x) \in B$. Let $H = Z_G(x)$ and $\mathfrak{h} = \operatorname{Lie} H$. We write H^1 for the derived group of $Z_G(x)$, which is simply connected, since G is. In the following, given $y \in \mathfrak{c}$, we shall write X_y for the fiber of the map $X \times_{\mathfrak{c}/W} \mathfrak{c} \to \mathfrak{c}$ above y, and X^1_y for the fiber of the map $X \times_{\mathfrak{c}/W(x)} \mathfrak{c} \to \mathfrak{c}$ above y. We define $\mathfrak{g}_{1,y}$ and $\mathfrak{h}_{1,y}$ analogously.

Lemma 5.3. *Let $y \in \mathfrak{c}^{rs}$. Then there is a commutative diagram*

$$\begin{tikzcd} H_1(\mathfrak{h}_{1,y}, \mathbb{F}_2) \arrow[r] \arrow[d] & Z_{H^1_0}(y) \arrow[d] \\ H_1(\mathfrak{g}_{1,y}, \mathbb{F}_2) \arrow[r] & Z_{G_0}(y) \end{tikzcd}$$

Proof. This follows from the existence of a commutative diagram

$$\begin{tikzcd} H^1 \arrow[r] \arrow[d] & \mathfrak{h}_{1,y} \arrow[d] \\ G \arrow[r] & \mathfrak{g}_{1,y}. \end{tikzcd}$$

□

Suppose that X_b has a singular point $u = u_s + u_n$. Choose $g \in G_0$ such that $g \cdot u_s = x \in \mathfrak{c}$, and set $v = g \cdot u$. The Jordan decomposition of v is $v = v_s + v_n = x + v_n$. Then $v_n \in \mathfrak{h}_1$ is a subregular nilpotent, corresponding to a connected component $D(v_n)$ of the Dynkin diagram of H. We choose a normal subregular $\mathfrak{sl}_2$-triple (v_n, t, w)

in $\mathfrak{h}$ containing v_n, and define $X^1 = v_n + \mathfrak{z}_{\mathfrak{h}}(w)_1$. X^1 is a transverse slice to the H_0-orbit of v in $\mathfrak{h}_1$, by Proposition 4.5.

Proposition 5.4. *The dimension of X^1 is* $\operatorname{rank} G + 1$. *$X^1 \subset \mathfrak{g}_1$ is a transverse slice to the G_0-orbit of v in $\mathfrak{g}_1$.*

Proof. X^1 has the correct dimension to be a transverse slice to the orbit of a subregular element, so it suffices to check the infinitesimal condition $[v, \mathfrak{g}_0] \cap \mathfrak{z}_{\mathfrak{h}}(w)_1 = 0$. In fact, we show that $[v, \mathfrak{g}] \cap \mathfrak{z}_{\mathfrak{h}}(w) = 0$. Define

$$V = \oplus_{\substack{\alpha \in \Phi_{\mathfrak{c}} \\ \alpha(x) \neq 0}} \mathfrak{g}^{\alpha}.$$

Then V is the orthogonal complement of $\mathfrak{h}$ with respect to the Killing form of $\mathfrak{g}$, and so is $\mathfrak{h}$-invariant. It follows that $[v, \mathfrak{g}] = [v, V] \oplus [v_n, \mathfrak{h}] \subset V \oplus [v_n, \mathfrak{h}] \subset V \oplus \mathfrak{h}$. We thus have $[v, \mathfrak{g}] \cap \mathfrak{z}_{\mathfrak{h}}(w) = [v_n, \mathfrak{h}] \cap \mathfrak{z}_{\mathfrak{h}}(w) = 0$. $\square$

Proposition 5.5. *For all sufficiently small open neighborhoods U of u in X_b, there exists an open neighborhood U_0 of $b \in \mathfrak{c}/W$ such that for all $y \in \pi^{-1}(U_0)$ there is a commutative diagram*

$$\begin{array}{ccc} H_1(X^1_y, \mathbb{F}_2) & \longrightarrow & H_1(\mathfrak{h}_{1,y}, \mathbb{F}_2) \\ \downarrow & & \downarrow \\ H_1(U_y, \mathbb{F}_2) & \longrightarrow & H_1(\mathfrak{g}_{1,y}, \mathbb{F}_2). \end{array}$$

Proof. If U is a sufficiently small open set around u in X, then by Proposition 4.1 we can find an isomorphism ψ between U and an open neighborhood V of v in X^1 over $\mathfrak{c}/W$, such that $\psi(u) = v$ and the two induced maps $V \hookrightarrow \mathfrak{h}_1 \hookrightarrow \mathfrak{g}_1$ and $V \cong U \hookrightarrow \mathfrak{g}_1$ are homotopic over $\mathfrak{c}/W$. After possibly shrinking U, we can assume that the image of V in $\mathfrak{c}/W(x)$ maps injectively to $\mathfrak{c}/W$.

In particular, for c sufficiently close to b we have a commutative diagram

$$\begin{tikzcd} H_1(V_c, \mathbb{F}_2) \arrow[r] \arrow[d] & H_1(\mathfrak{h}_{1,c}, \mathbb{F}_2) \arrow[d] \\ H_1(U_c, \mathbb{F}_2) \arrow[r] & H_1(\mathfrak{g}_{1,c}, \mathbb{F}_2). \end{tikzcd}$$

To obtain the statement in the proposition, we note that for c sufficiently close to b and $y \in \pi^{-1}(c)$, we can find an open subset $V_c' \subset V_c$ such that the inclusion $V_c' \subset X_y^1$ induces an isomorphism on H_1. (Use the contracting $\mathbb{G}_m$-action). This completes the proof. □

Corollary 5.6. *With hypotheses as above, suppose in addition that $y \in \mathfrak{c}^{rs}$. Let $C(x) \subset H^1$ be the maximal torus with Lie algebra $\mathfrak{c} \cap \mathfrak{h}^1$. Then there is a commutative diagram:*

$$\begin{tikzcd} H_1(X_y^1, \mathbb{F}_2) \arrow[r] \arrow[d] & X_*(C(x))/2X_*(C(x)) \arrow[d] \\ H_1(X_y, \mathbb{F}_2) \arrow[r] & X_*(C)/2X_*(C). \end{tikzcd}$$

Proof. It suffices to note that there is an isomorphism

$$Z_{G_0}(y) \cong X_*(C)/2X_*(C),$$

and similarly for $Z_{H_0^1}(y)$. □

To go further, it is helpful to compare this with another description of the homology of the curves X_y.

Theorem 5.7.
- *The map $X^{rs} \to \mathfrak{c}^{rs}/W$ is a locally trivial fibration (in the analytic topology), and so the homology groups $H_1(X_c, \mathbb{F}_2)$ for $c \in B^{rs}$ fit into a local system $\mathcal{H}_1(X)$ over $\mathfrak{c}^{rs}/W$. The pullback of this local system to $\mathfrak{c}^{rs}$ is constant.*

- *Suppose $x \in \mathfrak{c}$ has been chosen so that $\alpha(x) = 0$ for some $\alpha \in \Phi_{\mathfrak{c}}$, and the only roots vanishing on x are $\pm\alpha$. Then for each $y \in \mathfrak{c}^{rs}$ there is a vanishing cycle $\gamma_\alpha \in H_1(X_y, \mathbb{F}_2)$, associated to the specialization $X_y \to X_x$. This element defines a global section of the pullback of $\mathcal{H}_1(X)$ to $\mathfrak{c}^{rs}$.*
- *Let $R_{\mathfrak{c}} \subset \Phi_{\mathfrak{c}}$ denote a choice of root basis. Then for each $y \in \mathfrak{c}^{rs}$ the set $\{\gamma_\alpha \,|\, \alpha \in R_{\mathfrak{c}}\}$ is a basis of $H_1(X_y, \mathbb{F}_2)$.*

It seems likely that this description of the local system $\mathcal{H}_1(X)$ is well-known to experts, but we have not been able to find an adequate reference in the literature. For the definition of the vanishing cycle γ_α, and for the proof of this theorem, we refer to the final section below.

Now suppose $x \in \mathfrak{c}$ has been chosen so that $\alpha(x) = 0$ for some $\alpha \in \Phi_{\mathfrak{c}}$, and the only roots vanishing on x are $\pm\alpha$. Then the derived group of H is isomorphic to SL_2. By Corollary 4.16 above, the fiber X_x has a unique singularity of type A_1. For $y \in \mathfrak{c}^{\mathrm{rs}}$ sufficiently close to x, we have a diagram

$$\begin{tikzcd} H_1(X_y^1, \mathbb{F}_2) \arrow[r] \arrow[d] & X_*(C(x))/2X_*(C(x)) \arrow[d] \\ H_1(X_y, \mathbb{F}_2) \arrow[r] & X_*(C)/2X_*(C). \end{tikzcd}$$

It follows from our above calculation for $G = SL_2$ that the top arrow is an isomorphism, while the right vertical arrow has image equal to the image of the set $\{0, \alpha^\vee\}$ in $X_*(C)/2X_*(C)$. Moreover, it is clear from the proof of the proposition above that the image of the non-trivial element of $H_1(X_y^1, \mathbb{F}_2)$ in $H_1(X_y, \mathbb{F}_2)$ is exactly the vanishing cycle γ_α. Applying the commutativity of the above diagram, we deduce that the image of γ_α in $X_*(C)/2X_*(C)$ is just $\alpha^\vee \bmod 2X_*(C)$. Since γ_α comes from a global section of the local system $\mathcal{H}_1(X)$, we deduce the result for any $y \in \mathfrak{c}^{\mathrm{rs}}$, not just y sufficiently close to x.

It follows that for any $y \in \mathfrak{c}^{\mathrm{rs}}$, the map $H_1(X_y, \mathbb{F}_2) \to Z_{G_0}(y) \cong X_*(C)/2X_*(C)$ takes a basis of $H_1(X_y, \mathbb{F}_2)$, namely the set of γ_α as α ranges over a set of simple roots, to a basis of $X_*(C)/2X_*(C)$, namely the corresponding set of simple coroots. This completes the proof of the theorem.

The case of G adjoint. We now introduce a compactification of the family $X \to B$ of affine curves.

Lemma 5.8. *$\varphi : X \to B$ admits a compactification to a family $Y \to B$ of projective curves. Endow $Y \setminus X$ with its reduced closed subscheme structure. Then $Y \setminus X$ is a disjoint union of smooth non-intersecting open subschemes $P_1, \dots, P_s$, each of which maps isomorphically onto B. Moreover, $Y \to B$ is smooth in a Zariski neighborhood of each P_i. For each $b \in B^{rs}(k)$, Y_b is the unique smooth projective curve containing X_b as a dense open subset. Each irreducible component of Y_0 meets exactly one of the sections P_i.*

Proof. We take the projective closure of the equations given in Theorem 4.8, and blow up any singularities at infinity. An easy calculation shows in each case that the induced family $Y \to B$ satisfies the above properties. □

Let us now suppose that G is adjoint, and let $G^{\mathrm{sc}} \to G$ denote its simply connected cover. We write Z^{sc} for the stabilizer scheme of G^{sc} over B. The natural map $Z^{\mathrm{sc}} \to Z$ is fiberwise surjective. Fix $b \in B^{\mathrm{rs}}(k)$. In the previous section, we saw that the inclusion $X_b \hookrightarrow \mathfrak{g}_{1,b}$ induces an isomorphism $H_1(X_b, \mathbb{F}_2) \to Z^{\mathrm{sc}}_b$ of finite k-groups. On the other hand, we have a surjection $H_1(X_b, \mathbb{F}_2) \to H_1(Y_b, \mathbb{F}_2)$.

Theorem 5.9. *The composite*

$$H_1(X_b, \mathbb{F}_2) \to Z_b^{sc} \to Z_b$$

factors through this surjection, and induces an isomorphism $H_1(Y_b, \mathbb{F}_2) \cong Z_b$.

By Corollary 2.16, there is a canonical alternating pairing on Z_b^{sc}, with radical equal to the kernel of the map $Z_b^{\mathrm{sc}} \to Z_b$. On the other hand, there is a pairing $(\cdot, \cdot)$ on $H_1(X_b, \mathbb{F}_2)$, namely the intersection product, whose radical is exactly the kernel of the map $H_1(X_b, \mathbb{F}_2) \to H_1(Y_b, \mathbb{F}_2)$. The theorem is therefore a consequence of the following result.

Theorem 5.10. *The isomorphism $H_1(X_b, \mathbb{F}_2) \cong Z_b^{sc}$ preserves these alternating pairings.*

Corollary 5.11. *There is an isomorphism $J_{Y_b}[2] \cong Z_b$ of finite k-groups, that takes the Weil pairing to the pairing on Z_b defined above.*

Proof of Theorem 5.10. We can again reduce to the case $k = \mathbb{C}$. Fix a choice of Cartan subspace $\mathfrak{c}$, and let $C \subset G^{\mathrm{sc}}$ be the corresponding maximal torus. Choose $y \in \mathfrak{c}^{\mathrm{rs}}$. Let $\gamma_\alpha \in H_1(X_y, \mathbb{F}_2)$ be the element defined in the previous section. The theorem will follow from the following statement: fix a root basis $R_\mathfrak{c}$ of $\Phi_\mathfrak{c}$, and let $\alpha, \beta \in R_\mathfrak{c}$. Then $(\gamma_\alpha, \gamma_\beta) = 0$ if $\alpha = \beta$ or if α, β are not adjacent in the Dynkin diagram of $\mathfrak{g}$, and $(\gamma_\alpha, \gamma_\beta) = 1$ if they are adjacent.

Suppose first that α, β are distinct but adjacent in the Dynkin diagram of $\mathfrak{g}$. Then we can choose $x \in \mathfrak{c}$ such that the elements of $\Phi_\mathfrak{c}$ vanishing on x are exactly the linear combinations of α and β. Let $H = Z_{G^{\mathrm{sc}}}(x)$ and $H^1 = H^{\mathrm{der}}$. Then $H^1 \cong SL_3$, and the root system $\Phi_\mathfrak{c}(x) \subset \Phi_\mathfrak{c}$ is spanned by α and β. Moreover, we have by Corollary 5.6 for all $y \in \mathfrak{c}^{\mathrm{rs}}$ sufficiently close to x a commutative diagram

$$\begin{tikzcd}
H_1(X_y^1, \mathbb{F}_2) \arrow[r] \arrow[d] & X_*(C(x))/2X_*(C(x)) \arrow[d] \\
H_1(X_y, \mathbb{F}_2) \arrow[r] & X_*(C)/2X_*(C),
\end{tikzcd}$$

where $C(x) \subset H^1$ is the maximal torus with Lie algebra $\mathfrak{c} \cap \mathfrak{h}^1$. We know that the horizontal arrows are isomorphisms, and the vertical arrows are injective. The vertical arrows preserve the corresponding intersection pairings.

Now, both of the objects in the top row of the above diagram are 2-dimensional $\mathbb{F}_2$-vector spaces, and their corresponding pairings are non-degenerate. (This is easy to see: the curve X^1_y is a smooth affine curve of the form $y^2 = x^3 + ax + b$). There is a unique non-degenerate alternating pairing on any 2-dimensional $\mathbb{F}_2$-vector space, so we deduce that $(\gamma_\alpha, \gamma_\beta) = 1$.

Now let $\alpha, \beta \in R_\mathfrak{c}$ be distinct roots which are not adjacent in the Dynkin diagram of $\mathfrak{g}$. We can again choose $x \in \mathfrak{c}$ such that the roots vanishing on x are exactly the linear combinations of α and β. Let $H = Z_G(x)$ and $H^1 = H^{\text{der}}$. Then $H^1 \cong SL_2 \times SL_2$, and X_y has exactly two singularities, each of type A_1. We can choose disjoint open neighborhoods U_1, U_2 of these singularities in X such that for each $y \in \mathfrak{c}^{\text{rs}}$ suffciently close to x, the map $H_1(U_{1,y} \cup U_{2,y}, \mathbb{F}_2) \to H_1(X_y, \mathbb{F}_2)$ is injective and has image equal to the span of γ_α and γ_β. We see that these homology classes can be represented by cycles contained inside disjoint open sets of X_y. Therefore their intersection pairing is zero, and the theorem follows. $\square$

A parameterization of orbits. We suppose again that k is a general field of characteristic 0. Before stating our last main theorem, we summarize our hypotheses. We fix the following data:

- A split simple adjoint group G over k, of type A_r, D_r, or E_r.
- A stable involution θ of G and a regular nilpotent element $E \in \mathfrak{g}_1$.
- A choice of subregular normal $\mathfrak{sl}_2$-triple (e, h, f).

In terms of these data, we have defined:

- The categorical quotient $B = \mathfrak{g}_1 /\!\!/ G_0$.
- A family of reduced connected curves $X \to B$.

- A family of projective curves $Y \to B$ containing X as a fiberwise dense open subset.
- A stabilizer scheme $Z \to B$ whose fiber over $b \in B(k)$ is isomorphic to the stabilizer of any regular element in $\mathfrak{g}_{1,b}$.
- For each $b \in B^{\mathrm{rs}}(k)$, a natural isomorphism $J_{Y_b}[2] \cong Z_b$, which takes the Weil pairing to the non-degenerate alternating pairing on Z_b defined above.

Proposition 5.12. *For each $b \in B^{rs}(k)$, there is a canonical injection*

$$\mathfrak{g}_{1,b}(k)/G_0(k) \to H^1(k, J_{Y_b}[2]).$$

Proof. There is a canonical bijection $\mathfrak{g}_{1,b}(k)/G_0(k) \cong \ker(H^1(k, Z_b) \to H^1(k, G_0))$ (see Proposition 2.24), which takes the $G_0(k)$-orbit of κ_b to the distinguished element of $H^1(k, Z_b)$. Combining this with the isomorphism $Z_b \cong J_{Y_b}[2]$ gives the statement in the proposition. □

To go further we want to interpret the relative position of the nilpotent elements E and e geometrically.

Lemma 5.13. *There are canonical bijections between the following sets:*

(1) The set of irreducible components of X_0.

(2) The set of G_0-orbits of regular nilpotent elements in $\mathfrak{g}_1$ containing the G_0-orbit of e in their closure.

(3) The set of connected components of $Y \setminus X$.

Proof. The map $\mu_0 : G_0 \times X_0 \to \mathcal{N}(\mathfrak{g}_1)$ is flat, and so has open image. This image therefore contains all regular nilpotent G_0-orbits whose closure meets e. On the other hand, one checks using Proposition 3.11 that in each case that the number of regular nilpotent G_0-orbits containing e in their closure is equal to the number of irreducible components of X_0. We can therefore define a bijection between the first two sets by taking an irreducible component of X_0 to the G_0-orbit of any point on its smooth locus.

We write $Y \setminus X = P_1 \cup \cdots \cup P_s$ as a disjoint union of open subschemes, each of which maps isomorphically onto B. By Lemma 5.8, each irreducible component of Y_0 meets a unique section P_i. We define a bijection between the first and third sets above by taking an irreducible component of X_0 to the unique section P_i meeting its closure in Y_0. $\square$

We choose a section $P \cong B$ inside $Y \setminus X$, and we suppose that E corresponds under the bijection of Lemma 5.13 to the unique component of X_0 whose closure in Y_0 meets P. For each $b \in B^{\mathrm{rs}}(k)$, $P_b \in Y_b(k)$ defines an Abel-Jacobi map $f^{P_b} : Y_b \hookrightarrow J_{Y_b}$. (For the definition of this map, see [19], §2).

Theorem 5.14. *For every $b \in B^{rs}(k)$, there is a commutative diagram, functorial in k, and depending only on e up to $G_0(k)$-conjugacy:*

$$\begin{tikzcd} X_b(k) \arrow[r, "\iota"] \arrow[d, "g"] & \mathfrak{g}_{1,b}(k)/G_0(k) \arrow[d, "\gamma"] \\ J_{Y_b}(k) \arrow[r, "\delta"] & H^1(k, J_{Y_b}[2]). \end{tikzcd}$$

The arrows in this diagram are defined as follows:

- *ι is induced by the inclusion $X_b \hookrightarrow \mathfrak{g}_{1,b}$.*
- *g is the restriction of the Abel-Jacobi map f^{P_b} to $X_b \subset Y_b$.*
- *δ is the usual 2-descent map in Galois cohomology associated to the exact sequence*

$$0 \longrightarrow J_{Y_b}[2] \longrightarrow J_{Y_b} \xrightarrow{[2]} J_{Y_b} \longrightarrow 0.$$

- *γ is the classifying map of Proposition 5.12.*

Proof. We think of the group $H^1(k, J_{Y_b}[2])$ as classifying $J_{Y_b}[2]$-torsors over k. With b as in the theorem, let $E_b = [2]^{-1} f^{P_b}(Y_b) \subset J_{Y_b}$. We write $j_b : E_b \to Y_b$ for the natural map. This is a $J_{Y_b}[2]$-torsor over Y_b, and the composite $\delta \circ g$ sends a point $Q \in X_b(k)$ to the class of the torsor $j_b^{-1}(Q) \subset E_b$.

On the other hand, we have constructed a $J_{Y_b}[2]$-torsor $\Gamma_b \to X_b$ above, which extends uniquely to a torsor $h_b : D_b \to Y_b$, by Theorem 5.9. The composite $\gamma \circ \iota$ sends a point $Q \in X_b(k)$ to the class of $h_b^{-1}(Q)$. It follows from [19], Proposition 9.1 that the two covers $D_b \to Y_b$ and $E_b \to Y_b$ become isomorphic as $J_{Y_b}[2]$-torsors after extending scalars to a separable closure of k. To prove the theorem, it therefore suffices to prove that D_b and E_b are isomorphic as $J_{Y_b}[2]$-torsors over Y_b, before extending scalars. It even suffices to prove that $h_b^{-1}(P_b)$ is always the split torsor, or in other words that $h_b^{-1}(P_b)(k)$ is not the empty set.

Let $\mu : G_0 \times \kappa \to \mathfrak{g}_1$ denote the orbit map, and let X' denote the intersection of X with the image of μ. Because of the compatibility between E and P, the subset $X' \cup P$ of the underlying topological space of Y is open; let Y' denote the corresponding open subscheme. Then Y' contains a Zariski open neighborhood of P in Y.

Let $\Gamma' = \mu^{-1}(X')$; this is a Z-torsor over X'. We show that Γ' extends to Z-torsor over Y'. In fact, there is a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & H^1_{\text{ét}}(Y', Z) & \longrightarrow & H^1_{\text{ét}}(X', Z) & \longrightarrow & H^0_{\text{ét}}(Y', R^1 j_* Z) \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & H^1_{\text{ét}}(Y'_K, Z) & \longrightarrow & H^1_{\text{ét}}(X'_K, Z) & \longrightarrow & H^0_{\text{ét}}(Y'_K, R^1 j_{K,*} Z),
\end{array}$$

where $j : X' \to Y'$ is the obvious open immersion, and $(\cdot)_K$ denotes base change to the separable closure K/k. Let $i : P \hookrightarrow Y'$ denote the complementary closed immersion. There is an isomorphism $R^1 j_{K,*}(Z) \cong i_{K,*} Z$, and hence $H^0_{\text{ét}}(Y'_K, R^1 j_{K,*}(Z)) = H^0_{\text{ét}}(B_K, Z) = 0$. The rightmost vertical arrow in the above diagram is injective, and so the class of Γ' in $H^1_{\text{ét}}(X', Z)$ lifts to $H^1_{\text{ét}}(Y', Z)$. We write $D' \to Y'$ for the corresponding torsor.

Let $F' \to B$ denote the pullback of D' to $B \cong P \hookrightarrow Y'$. We must show that for b as in the theorem, F'_b is the trivial Z-torsor over k. We claim that in fact, F' is trivial. For we can choose a Zariski open neighborhood U_0 of $0 \in B$ and a Galois finite étale cover $U \to U_0$ such that $F' \times_B U$ has a trivialization as a Z-torsor. If U is sufficiently small, then $Z(U) \hookrightarrow Z_0 = 0$ is trivial, so there is a unique such trivialization. By descent, there exists a unique trivialization of F' over U_0. The existence of the contracting $\mathbb{G}_m$-action on $X \to B$ now implies that F' must be globally trivial, as required. This completes the proof of the theorem. □

The proof of Theorem 5.7. In this section we prove Theorem 5.7. Thus G is a simple simply connected group over $k = \mathbb{C}$, θ a stable involution, and $\mathfrak{c} \subset \mathfrak{g}_1$ a Cartan subspace. We fix a normal subregular $\mathfrak{sl}_2$-triple (e, h, f) in $\mathfrak{g}$, and define $S = e + \mathfrak{z}_{\mathfrak{g}}(f)$, $X = e + \mathfrak{z}_{\mathfrak{g}}(f)_1 = S \cap \mathfrak{g}_1$. Let τ denote the automorphism of S induced by $-\theta$; we then have $S^\tau = X$. In what follows we identify all varieties with their complex points.

Lemma 5.15. *Both S^{rs} and X^{rs} are locally trivial fibrations (in the analytic topology) over $\mathfrak{c}^{rs}/W$.*

Proof. We combine the Ehresmann fibration theorem and the existence of a good compactification for X^{rs} to see that it is a locally trivial fibration over $\mathfrak{c}^{\mathrm{rs}}/W$. The corresponding result for S follows from the simple relationship between S and X, see Lemma 4.15. □

Corollary 5.16. *The homology groups $H_2(S_b, \mathbb{F}_2)$ and $H_1(X_b, \mathbb{F}_2)$ for $b \in \mathfrak{c}^{rs}/W$ form local systems $\mathcal{H}_2(S)$ and $\mathcal{H}_1(X)$. Moreover, these local systems are canonically isomorphic.*

Proof. Only the second part needs proof. It follows either from a sheaf-theoretic argument, or from the assertion that suspension does not change the monodromy representation of a singularity, at least when one is working modulo 2; see [1], Theorem 2.14. □

Given $y \in \mathfrak{c}$ we write X_y and S_y for the respective fibers over y of the maps $X \times_{\mathfrak{c}/W} \mathfrak{c} \to \mathfrak{c}$ and $S \times_{\mathfrak{c}/W} \mathfrak{c} \to \mathfrak{c}$.

Lemma 5.17. *The local systems $\mathcal{H}_1(X)$ and $\mathcal{H}_2(S)$ become trivial after pullback to $\mathfrak{c}^{rs}$.*

Proof. In light of the corollary, it suffices to prove this assertion for $\mathcal{H}_2(S)$. The existence of the Springer resolution implies the existence of a proper morphism $\widetilde{S} \to S \times_{\mathfrak{c}/W} \mathfrak{c}$ such that for every $y \in \mathfrak{c}$, the induced map $\widetilde{S} \to S_y$ is a minimal resolution of singularities. Moreover, $\widetilde{S} \to \mathfrak{c}$ is a locally trivial fiber bundle and $\widetilde{S} \times_{\mathfrak{c}} \mathfrak{c}^{\mathrm{rs}} \to S \times_{\mathfrak{c}/W} \mathfrak{c}^{\mathrm{rs}}$ is an isomorphism. See [27] for more details. These facts imply the lemma. $\square$

It follows that for any $y, z \in \mathfrak{c}^{\mathrm{rs}}$, the groups $H_1(X_y, \mathbb{F}_2)$ and $H_1(X_z, \mathbb{F}_2)$ are canonically isomorphic.

It is a consequence of Lemma 4.15 that given $b \in \mathfrak{c}/W$, a fiber X_b has a unique non-degenerate critical point if and only if S_b does. Let $\gamma : [0, 1] \to \mathfrak{c}$ be a path such that $\gamma(t)$ is regular semisimple for $0 \leq t < 1$, but such that a unique pair of roots $\pm\alpha$ vanishes on $\gamma(1) = x$. Then X_x (or S_x) has a unique non-degenerate critical point, by Corollary 4.16. Let $y = \gamma(0)$. We define a homology class (that we call a vanishing cycle) $[\gamma]_1 \in H_1(X_y, \mathbb{F}_2)$ as follows.

We can find local holomorphic co-ordinates $z_1, \ldots, z_{r+1}$ on X centered at the critical point of X_b and local holomorphic co-ordinates $w_1, \ldots, w_r$ on $\mathfrak{c}/W$ centered at b such that the map $X \to \mathfrak{c}/W$ is locally of the form $(z_1, \ldots, z_{r+1}) \mapsto (z_1, \ldots, z_{r-1}, z_r^2 + z_{r+1}^2)$. For t close to 1, we can choose co-ordinates as above and define a sphere (for a suitable continuous choice of branch of $\sqrt{w_r(t)}$ near $t = 1$):

$$S^1(t) = \cdot\{(w_1(t), \ldots, w_{r-1}(t), \sqrt{w_r(t)}z_r, \sqrt{w_r(t)}z_{r+1}) \mid z_r^2 + z_{r+1}^2 = 1, \Im z_i = 0\}.$$

We define a homology class in $H_1(X_y, \mathbb{F}_2)$ by transporting the class of $S^1(t)$ for t close to 1 along the image of the path γ in $\mathfrak{c}/W$. An entirely analogous procedure defines $[\gamma]_2 \in H_2(S_x, \mathbb{F}_2)$.

Lemma 5.18. *The homology class of the cycle $[\gamma]_1 \in H_1(X_y, \mathbb{F}_2)$ (respectively, $[\gamma]_2 \in H_2(S_y, \mathbb{F}_2)$) is well-defined and depends only on α. Moreover, these classes correspond under the isomorphism $H_1(X_y, \mathbb{F}_2) \cong H_2(S_y, \mathbb{F}_2)$ of the previous corollary.*

Proof. It is well-known that the $[\gamma]_i$ are well-defined and depend only on the path γ up to homotopy. It follows from the previous corollary that the $[\gamma]_i$ depend only on the endpoint $x = \gamma(1)$ and not on the choice of path. To prove the lemma it suffices to show that $[\gamma]_2$ depends only on α. In fact $[\gamma]_2$ is, by construction, the unique non-trivial element in the kernel of the map $H_2(S_y, \mathbb{F}_2) = H_2(\widetilde{S}_y, \mathbb{F}_2) \cong H_2(\widetilde{S}_x, \mathbb{F}_2) \to H_2(S_x, \mathbb{F}_2)$. The proof of [26], Theorem 3.4 implies that there is an isomorphism of local systems $\mathcal{H}_2(\widetilde{S}) \cong X_*(C)/2X_*(C)$ over $\mathfrak{c}$, and that the kernel of the map $H_2(\widetilde{S}_x, \mathbb{F}_2) \to H_2(S_x, \mathbb{F}_2)$ corresponds under this isomorphism to span in $X_*(C)/2X_*(C)$ of $\alpha^\vee$. $\square$

We can therefore define for each $\alpha \in \Phi_{\mathfrak{c}}$ a global section γ_α of the pull-back of the local system $\mathcal{H}_1(X)$ to $\mathfrak{c}^{\mathrm{rs}}$, namely the class $[\gamma]_1$ constructed above. Theorem 5.7 now follows from the above facts and the following result.

Lemma 5.19. *Let $R_{\mathfrak{c}} \subset \Phi_{\mathfrak{c}}$ be a choice of root basis, and let $x \in \mathfrak{c}^{rs}$. Then the set $\{\gamma_\alpha \,|\, \alpha \in R_{\mathfrak{c}}\}$ is a basis of $H_1(X_x, \mathbb{F}_2)$ as $\mathbb{F}_2$-vector space.*

Proof. This follows immediately from the corresponding fact for the simple coroots $\{\alpha^\vee \,|\, \alpha \in R_{\mathfrak{c}}\}$. $\square$

References

[1] V. I. Arnol′d, V. A. Vasil′ev, V. V. Goryunov, and O. V. Lyashko. Singularities. I. Local and global theory. In *Current problems in mathematics. Fundamental directions, Vol. 6*, Itogi Nauki i Tekhniki, pages 5–257. Akad. Nauk SSSR Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1988.

[2] M. Bhargava and A. Shankar. Binary quartic forms having bounded invariants, and the boundedness of the average rank of elliptic curves, June 2010. Preprint, to appear in *Annals of Math.*

[3] A. Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.

[4] N. Bourbaki. *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*. Actualités Scientifiques et Industrielles, No. 1337. Hermann, Paris, 1968.

[5] N. Bourbaki. *Lie groups and Lie algebras. Chapters 7–9*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2005. Translated from the 1975 and 1982 French originals by Andrew Pressley.

[6] E. Brieskorn. Singular elements of semi-simple algebraic groups. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 279–284. Gauthier-Villars, Paris, 1971.

[7] D. H. Collingwood and W. M. McGovern. *Nilpotent orbits in semisimple Lie algebras*. Van Nostrand Reinhold Mathematics Series. Van Nostrand Reinhold Co., New York, 1993.

[8] W. A. de Graaf. Computing representatives of nilpotent orbits of θ-groups. *J. Symbolic Comput.*, 46(4):438–458, 2011.

[9] M. Entov. On the A-D-E classification of the simple singularities of functions. *Bull. Sci. Math.*, 121(1):37–60, 1997.

[10] H. Esnault. Sur l'identification de singularités apparaissant dans des groupes algébriques complexes. In *Seminar on Singularities (Paris, 1976/1977)*, volume 7 of *Publ. Math. Univ. Paris VII*, pages 31–59. Univ. Paris VII, Paris, 1980.

[11] G.-M. Greuel and H. Knörrer. Einfache Kurvensingularitäten und torsionsfreie Moduln. *Math. Ann.*, 270(3):417–425, 1985.

[12] B. H. Gross, P. Levy, M. Reeder, and J.-K. Yu. Gradings of positive rank on simple Lie algebras. Available at `http://www.math.harvard.edu/~gross/preprints/PosRank.pdf`. Preprint.

[13] W. Ho. *Orbit parametrizations of curves*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–Princeton University.

[14] J. E. Humphreys. *Linear algebraic groups*. Springer-Verlag, New York, 1975. Graduate Texts in Mathematics, No. 21.

[15] N. Kawanaka. Orbits and stabilizers of nilpotent elements of a graded semisimple Lie algebra. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(3):573–597, 1987.

[16] B. Kostant and S. Rallis. Orbits and representations associated with symmetric spaces. *Amer. J. Math.*, 93:753–809, 1971.

[17] P. Levy. Involutions of reductive Lie algebras in positive characteristic. *Adv. Math.*, 210(2):505–559, 2007.

[18] P. Levy. Vinberg's θ-groups in positive characteristic and Kostant-Weierstrass slices. *Transform. Groups*, 14(2):417–461, 2009.

[19] J. S. Milne. Jacobian varieties. In *Arithmetic geometry (Storrs, Conn., 1984)*, pages 167–212. Springer, New York, 1986.

[20] D. Ž. Đoković. The closure diagram for nilpotent orbits of the split real form of E_7. *Represent. Theory*, 5:284–316 (electronic), 2001.

[21] D. Ž. Đoković and M. Litvinov. The closure ordering of nilpotent orbits of the complex symmetric pair $(\mathrm{SO}_{p+q}, \mathrm{SO}_p \times \mathrm{SO}_q)$. *Canad. J. Math.*, 55(6):1155–1190, 2003.

[22] A. L. Onishchik and È. B. Vinberg. *Lie groups and algebraic groups*. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1990. Translated from the Russian and with a preface by D. A. Leites.

[23] B. Poonen. Average rank of elliptic curves [after Manjul Bhargava and Arul Shankar]. In *Séminaire Bourbaki, Vol. 2011/12, 64ème année, Exp. No. 1049*. January 2012.

[24] M. Reeder. Elliptic centralizers in Weyl groups and their coinvariant representations. *Represent. Theory*, 15:63–111, 2011.

[25] J.-P. Serre. *Cohomologie galoisienne*, volume 5 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, fifth edition, 1994.

[26] N. I. Shepherd-Barron. On simple groups and simple singularities. *Israel J. Math.*, 123:179–188, 2001.

[27] P. Slodowy. *Four lectures on simple groups and singularities*, volume 11 of *Communications of the Mathematical Institute, Rijksuniversiteit Utrecht*. Rijksuniversiteit Utrecht Mathematical Institute, Utrecht, 1980.

[28] P. Slodowy. *Simple singularities and simple algebraic groups*, volume 815 of *Lecture Notes in Mathematics*. Springer, Berlin, 1980.

[29] T. A. Springer. *Invariant theory*. Lecture Notes in Mathematics, Vol. 585. Springer-Verlag, Berlin, 1977.

[30] T. A. Springer. *Linear algebraic groups*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, second edition, 2009.

[31] T. A. Springer and R. Steinberg. Conjugacy classes. In *Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69)*, Lecture Notes in Mathematics, Vol. 131, pages 167–266. Springer, Berlin, 1970.

[32] R. Steinberg. *Conjugacy classes in algebraic groups*. Lecture Notes in Mathematics, Vol. 366. Springer-Verlag, Berlin, 1974. Notes by Vinay V. Deodhar.

[33] È. B. Vinberg. The Weyl group of a graded Lie algebra. *Izv. Akad. Nauk SSSR Ser. Mat.*, 40(3):488–526, 709, 1976.