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Quantum Hall to Insulator Transition in the Bilayer Quantum Hall Ferromagnet

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We describe a new phase transition of the bilayer quantum Hall ferromagnet at filling fraction $\nu = 1$. In the presence of static disorder (modeled by a periodic potential), bosonic $S = 1/2$ spinons can undergo a superfluid-insulator transition while preserving the ferromagnetic order. The Mott insulating phase has an emergent U(1) photon, and the transition is between Higgs and Coulomb phases of this photon. Physical consequences for charge and counterflow conductivity, and for interlayer tunneling conductance in the presence of quenched disorder are discussed.

The quantum Hall effects embody new states of matter, in which two-dimensional electron systems in a perpendicular magnetic field B are incompressible, and exhibit excitations with fractional charge and statistics [1].

Quantum Hall systems with an internal degree of freedom, such as spin, layer index, or valley index, are richer still [2]. Because electron or hole excitations are prohibitive in energy, the low energy states can be characterized by orientation of the vector $\mathbf{n}$, denoting the spin (or pseudospin in the layer index). Exchange interactions lead to ferromagnetism, hence such systems are quantum Hall ferromagnets (QHFM's). Of central importance is the spin-charge relation [3, 4] in the lowest Landau level (LLL), which expresses the Coulomb charge and current $(J^0, \mathbf{J}) = J^\mu$ due to a varying configuration of $\mathbf{n}$

$$J_S^\mu = \frac{e}{8\pi}\epsilon^{\mu\nu\lambda}\mathbf{n} \cdot (\partial_\nu\mathbf{n} \times \partial_\lambda\mathbf{n}) \tag{1}$$

The charge carriers in QHFM's are *spin-textures*, characterized by a topological number. In single-layer $\nu = 1$ systems with spin, they are fermionic skyrmions/antiskyrmions[3] with charge $\pm e$, while in bilayer systems they are quartonic merons/antimerons[4] with charge $\pm e/2$.

While our general framework applies to all quantum Hall ferromagnets, in this paper we focus on the bilayer quantum Hall system [2, 5] at total filling factor $\nu_T = 1$, with an extremely small, but nonzero, tunneling amplitude h between the two layers. We will assume that real spin is frozen (which may not entirely be valid [6]), and spin/peudospin for us will be synonymous with the layer index. Each layer is at half-filling. When the separation between the layers d is of the order of a magnetic length $l_0 = \sqrt{hc/eB}$, the system is an incompressible quantum Hall state [7]. At large enough d/l_0 the system splits up into two weakly interacting, compressible, presumably $\nu = \frac{1}{2}$ Fermi-liquid-like systems [7].

Despite almost two decades of theory [4, 8, 9, 10, 11, 12] and experiments [13, 14], important aspects of the small d/l_0 phase are not understood. Theoretically, the small d/l_0 QHFM breaks symmetry at $T = 0$ even in the absence of an interlayer tunneling[4, 9] (spontaneous interlayer coherence) leading to a Goldstone mode[8].

This state can also be regarded as an excitonic superfluid, and should exhibit a Josephson-like effect, with a finite interlayer current flowing at strictly zero interlayer bias voltage[4, 9]. While there is a peak in the interlayer tunneling conductance G at zero bias [13], the peak has finite width, implying some intrinsic dissipation. Theoretically, interlayer tunneling should take place only within a Josephson length of the contacts [4, 15]. Thus G should be proportional to the length of the contacts, while experimentally it is seen to be proportional to the area [16]. Theoretically, there should be a $T > 0$ Kosterlitz-Thouless transition at which the superfluid stiffness has a universal jump. Experimentally[17], the zero-bias value of G (the closest analog to the superfluid stiffness) vanishes roughly as $(T_c - T)^3$ at the transition. It is believed[10, 18, 19, 20] that quenched disorder is ultimately responsible for these discrepancies, though a detailed understanding is lacking.

Together with H. A. Fertig, one of us has proposed a model[21] where a "coherence network" forms due to the nonperturbative effects of disorder[22]. This model is consistent with several aspects of the experiments, notably the tunneling conductance going as the area[16] rather than the length of the sample, but it is classical.

In this paper we argue for a quantum phase transition to an insulating zero-temperature phase of QHFM, which could possibly impact on the experimental issues discussed above. We construct a quantum low energy model at $T = 0$ (neglecting electron and hole excitations and keeping only smooth configurations of $\mathbf{n}$), and mimic the nonperturbative effects of disorder[22] by putting the system on a square lattice. We start with the imaginary time Lagrangian for a two-component quantum Hall system as described by Lee and Kane(LK)[24]

$$\mathcal{L}_{LK} = \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \bar{\Psi}_s\big(\partial_0 - i(a_0 - eA_0)\big)\Psi_s$$
$$+ \frac{1}{2m}|\big(\nabla - i(\mathbf{a} - e\mathbf{A})\big)\Psi_s|^2 + \frac{u}{2}\big(\bar{\Psi}_s\Psi_s - \rho_0\big)^2 \tag{2}$$

Here the Ψ_s are the composite boson (CB) fields[23], obtained by attaching one unit of statistical flux to the electron. The gauge field a_μ with its Chern-Simons term implements this transformation, while the physical (Coulomb) charge $-e$ of the CB leads to minimal cou-

pling to the external electromagnetic field eA_μ. When the filling is exactly $\nu = 1$ ($\rho_0 = 1/2\pi l_0^2$), the statistical and external fields cancel each other on average, and the CB's can Bose condense, which leads to the QH phase with a quantized Hall conductance[23]. One now decomposes[24] the CB field into a single-component charged Higgs field Φ, and a neutral two-component unit length spinor z_s: $\Psi_s = \Phi z_s$ ($\bar{z}_s z_s = 1$). The spin vector emerges as $\mathbf{n} = \bar{z}\sigma z$. We parameterize the spin sector by the variables $\zeta = n_z$ and the angle θ of the xy component of spin, and also add the planar anisotropy energy Γ. After some manipulations, one obtains[24]

$$\frac{i}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda + \bar{\Phi}\big(\partial_0 - i(a_0 + b_0 - eA_0)\big)\Phi$$

$$+\frac{1}{2m}|\big(\nabla - i(\mathbf{a} + \mathbf{b} - e\mathbf{A})\big)\Phi|^2 + \frac{\rho_0}{2m}\big(\nabla\mathbf{n}\big)^2 + \frac{\Gamma}{2}n_z^2 \quad (3)$$

Note that the coupling between the Higgs and spin sectors is only via the gauge-like field

$$b_\mu = i\bar{z}\partial_\mu z = \frac{1}{2}\zeta\partial_\mu\theta \qquad J_S^\mu = \frac{e}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\nu b_\lambda \quad (4)$$

The idea is to allow the periodic potential to drive the Higgs field through a Higgs-Mott Insulator transition, while $\mathbf{n}$ remains ordered. Such a transition without a gauge field is known to exist for two-component bosons on lattices[26]. The Higgs transition in single-component quantum Hall systems has been studied previously in the large-N approximation[27], and yields a second-order quantum phase transition at which the Hall conductance changes discontinuously. We will use the same approach, and assume that our transition is second-order, though the question of the order remains open for the physically relevant case $N = 1$.

Since the transition is at fixed Higgs density, it is described by a relativistic effective theory[25].

$$\mathcal{L} = |\big(\partial_\mu - i\mathcal{A}_\mu\big)\Phi|^2 + M^2|\Phi|^2 + \lambda|\Phi|^4 + \frac{i}{4\pi}\epsilon_{\mu\nu\lambda}a_\mu\partial_\nu a_\lambda +$$

$$\frac{i\rho_0}{2}\zeta\dot{\theta} + \frac{K}{2}\big((1-\zeta^2)(\nabla\theta)^2 + \frac{(\nabla\zeta)^2}{(1-\zeta^2)}\big) + \frac{\Gamma}{2}\zeta^2 - h\cos\theta \quad (5)$$

Here we have defined $\mathcal{A}_\mu = a_\mu + b_\mu - eA_\mu$, and allowed the spin stiffness K to be renormalized down to its lowest Landau level (LLL) value (as obtained by Moon et al[4]). We have also introduced the interlayer tunneling h, which we will take to be much smaller than any other energy scale, and will therefore neglect unless it is essential for some physical quantity. Note that we have used the $\bar{\Phi}\Phi\bar{z}\partial_0 z$ term of the LK action (Eq. (3)) with $\langle\bar{\Phi}\Phi\rangle = \rho_0$ to obtain the $\zeta\dot{\theta}$ term in the lagrangian. This is not only the most relevant time-derivative term, but has the correct dynamics for the density-density correlations due to spin-textures to be of order q^4 at small q, a requirement of being in the LLL. The higher time derivative terms coming from integrating out high-energy modes will appear in such a combination as to not violate the LLL property. In this LLL QHFM, the 2×2 correlator of $\langle b_\mu b_\nu\rangle$ (with $b_0(k)$ really standing for $\tilde{b}_0(k) = b_0(k) - \frac{\omega}{q^2}\mathbf{q} \cdot \mathbf{b}(k)$, see below), for small q, ω is

$$G_b^{(0)}(k) = \begin{bmatrix} D & 0 \\ 0 & Fq^2 \end{bmatrix} \quad (6)$$

where D, F are constants.

Now we integrate out Φ to obtain

$$\mathcal{S}_{eff} = \mathcal{S}_{CS} + \int dk\big(\mathcal{A}_\mu(k)\Pi_\Phi^{\mu\nu}(k)\mathcal{A}_\nu(-k)\big) \quad (7)$$

Here $k = (\omega, \mathbf{q})$ and the polarization tensor Π_Φ has the gauge-invariant form

$$\Pi_\Phi^{\mu\nu}(k) = \frac{f(k^2)}{4\pi k^2}(k^2\delta^{\mu\nu} - k^\mu k^\nu) \quad (8)$$

In the Higgs phase f is a constant, while in the Mott phase $f \simeq k^2$ for $k \ll M$. Going to Coulomb gauge for a_μ ($\nabla \cdot \mathbf{a} = 0$), one can rewrite the gauge action in a 2×2 matrix form acting on $a_0(k)$ and $a_T(k) = i\hat{\mathbf{q}} \times \mathbf{a}(k)$, where $\hat{\mathbf{q}} = \mathbf{q}/q$. We will make a similar decomposition for b_μ and A_μ. Define $\tilde{b}_0(k) = b_0(k) - \frac{i\omega}{q}b_L(k)$, with $b_T(k) = i\hat{\mathbf{q}} \times \mathbf{b}(k)$ and $b_L(k) = -i\hat{\mathbf{q}} \cdot \mathbf{b}(k)$. Then the entries corresponding to a_0 and a_T are $\tilde{b}_0$ and b_T. In this 2×2 language we have

$$\Pi_\Phi = \frac{1}{4\pi}\begin{bmatrix} f\frac{q^2}{k^2} & 0 \\ 0 & f \end{bmatrix} \qquad \Pi_{CS} = \frac{1}{4\pi}\begin{bmatrix} 0 & iq \\ iq & 0 \end{bmatrix} \quad (9)$$

To proceed, we define $\Pi = \Pi_\Phi + \Pi_{CS}$ and integrate out the gauge fields to obtain an effective action for $\mathcal{B}_0 = \tilde{b}_0 - e\tilde{A}_0$ and $\mathcal{B}_T = b_T - eA_T$:

$$\int dk \big(\mathcal{B}_0(k) \ \mathcal{B}_T(k) \big) [\Pi_\Phi - \Pi_\Phi\Pi^{-1}\Pi_\Phi] \begin{pmatrix} \mathcal{B}_0(-k) \\ \mathcal{B}_T(-k) \end{pmatrix} \quad (10)$$

where the matrix $Q = \Pi_\Phi - \Pi_\Phi\Pi^{-1}\Pi_\Phi$ has the form

$$Q = \frac{1}{4\pi}\begin{bmatrix} \frac{fq^2}{f^2+k^2} & \frac{iqf^2}{f^2+k^2} \\ \frac{iqf^2}{f^2+k^2} & \frac{fk^2}{f^2+k^2} \end{bmatrix} \quad (11)$$

In order to obtain the electromagnetic response, one should now integrate out b_μ, that is, ζ, θ. We take account of the $b_\mu Q_{\mu\nu}b_\nu$ term in the random phase approximation (RPA) to obtain the "full" correlator (as opposed to the "bare" correlator of Eq. (6)).

$$\langle b_\mu(k)b_\nu(-k)\rangle = G_{b,\mu\nu}^{RPA} = \big((G_b^{(0)})^{-1} + Q\big)_{\mu\nu}^{-1} \quad (12)$$

We now integrate out b to obtain the final effective action for the electromagnetic potential A_μ, which takes the form

$$S_{eff}[A_\mu] = \int dk\frac{e^2}{4\pi}A_\mu(k)\big(Q - QG_b^{RPA}Q\big)_{\mu\nu}A_\nu(-k) \quad (13)$$

Now we can read off the conductivity matrix from

$$P_{\mu\nu} = 4\pi\big(Q - QG_b^{RPA}Q\big) = 4\pi\big(G_b^{(0)} + Q^{-1}\big)_{\mu\nu}^{-1} \quad (14)$$

Going to the real frequency domain, we obtain

$$Re(\sigma_{xx}(q,\omega)) = \frac{e^2}{2\pi}\frac{\omega}{q^2}Im(P_{00}) \quad (15)$$

$$\sigma_{xy}(q,\omega) = \frac{e^2}{2\pi}\frac{1}{q}P_{01} \quad (16)$$

In the Higgs phase, $Re(\sigma_{xx})$ vanishes below the gap f and σ_{xy} approaches the quantized value of $e^2/2\pi$ for $\omega,\ q \ll f$ denoting a quantized Hall phase. In the Mott phase the entire conductivity matrix vanishes for $\omega,\ q \ll M$, denoting an insulating phase. At the critical point $f = \frac{\pi}{8}\sqrt{q^2 - \omega^2}$, and the system has the critical conductivities

$$\sigma_{xx}^*(0,\omega) = \frac{e^2}{2\pi}\frac{\pi/8}{1+(\pi/8)^2} \qquad \sigma_{xy}^* = \frac{\pi}{8}\sigma_{xx}^* \qquad (17)$$

Going back to Eq. (12), one can find propagating charge modes as poles of G_b^{RPA}. In the Higgs phase, one finds a propagating mode with a dispersion

$$\omega = \sqrt{q^2(1 + Df) + f^2/(1 + Ffq^2)} \qquad (18)$$

In the Mott phase, no propagating charge modes exist (there are only branch cuts in the charge correlator).

Let us now turn to the spin sector. The most interesting quantity is the self-energy matrix (in ζ, θ space) Σ_{ab} near the critical point, which can be related to measurable quantities. For example $Im(\Sigma_{\theta\theta})$ is related to the dissipative counterflow conductivity via

$$Re(\sigma_{CF}(\mathbf{q},\omega)) \simeq K^2 q^2 \frac{Im(\Sigma_{\theta\theta}(\mathbf{q},\omega))}{\omega[(\omega - c_\theta q)^2 + (Im(\Sigma_{\theta\theta}))^2]} \qquad (19)$$

and to the interlayer conductance via

$$G_{interlayer} = 2\rho_0\omega\frac{Im(\Sigma_{\theta\theta}(\mathbf{q},\omega))}{[(\omega - c_\theta q)^2 + (Im(\Sigma_{\theta\theta}))^2]} \qquad (20)$$

where we have absorbed the real part of Σ_{ab} into a renormalization of the spin-wave velocity.

In the Higgs phase we find for small q close to threshold $\omega - f \ll f$

$$Im(\Sigma_{\theta\theta})(\omega,q) \simeq q^2 f^2(K(\omega-f)^2 + h)\Theta(\omega-f)$$
$$Im((\Sigma_{\zeta\zeta})(\omega,q) \simeq q^2 f^2(K(\omega-f)^2 + \Gamma)\Theta(\omega-f)$$
$$Im(\Sigma_{\zeta\theta})(\omega,q) \simeq q^2 f^2\Theta(\omega-f) \qquad (21)$$

where we define the off-diagonal term in the quadratic term of the effective action of ζ, θ in real frequency as $(1 + \Sigma_{\zeta\theta})i\omega\rho_0/2$. The main features are the presence of a threshold frequency (the gap to charge excitations) which vanishes linearly as the Higgs field approaches criticality $f \to 0$, as well as a coupling which also vanishes as a power of f. Note the qualitative difference made in $Im(\Sigma_{\theta\theta})$ by the presence of h.

At the large-N critical point we find

$$Im\Sigma_{\theta\theta} \simeq (\omega^7, q^7): \quad h = 0 \qquad (22)$$
$$\simeq (\omega^5, q^5): \quad h \neq 0 \qquad (23)$$

Here we see that powers of f have been replaced by powers of ω, q, with one extra power appearing due to the branch cut.

Let us discuss finite-T properties briefly. Starting in the Higgs phase, if one goes to nozero $T \gg f$ the system is quantum critical[28] and we can expect to see the above behavior with $\omega \to T$. Also, in the presence of quenched disorder, we can expect (in addition to the activated contribution due to the tunneling of merons[21]) a power-law contribution in T to all physical quantities in the quantum critical regime.

Consider now the qualitative behavior of the interlayer conductance G for $T \gg h$, believed to be true of experimental samples at millikelvin temperatures. For $T \gg f$ we expect the interlayer tunneling to be incoherent, which is consistent with the finite width of the zero-bias peak seen in experiments. With quenched disorder there may be a Higgs glass phase (see below) with gapless charge fluctuations, which would imply incoherent interlayer tunneling for any nonzero T.

Finally, deep in the Mott phase, there is again a threshold frequency $\omega_{th}(\mathbf{q}) = \sqrt{4M^2 + q^2}$.

$$Im\Sigma_{\theta\theta}(\omega,q) \approx (\omega - \omega_{th})^4 q^2\Theta(\omega - \omega_{th}) \qquad (24)$$

The addition of the long-range Coulomb interaction to the action makes no qualitative difference to the above.

Let us now comment on previous related work. Some authors have argued for an XY spin-glass phase at sufficient disorder[21, 29] but these arguments are for the classical model. There are also proposals that the ground state at $T = 0$ is a gauge-glass[10, 19] which is still a quantum Hall phase, but with power-law XY order. Other authors have argued for a *spontaneous* breaking of translation invariance[30]: Like the quantum Hall-Wigner crystal transition this likely occurs at large imbalance[30]. Finally, some authors have argued for a translation-invariant QH phase with no long-range order in $\mathbf{n}$[31]. In our model, we assume that the XY ferromagnetism is robust across the Higgs transition which is driven by a periodic potential (a proxy for disorder), which differs from all the above proposals.

In the presence of static disorder, a key feature of our model is that the dissipation arises not directly from disorder coupled to the XY order parameter[32], but rather from the disorder inducing a phase transition which creates a phase with *dynamical* low-energy spinon and charge excitations coupled to the XY modes. We call this phase a Higgs glass, in analogy to the Bose glass[25] and gauge glass[10, 19] phases. The Higgs glass differs from the gauge glass in having *dynamical, gapless gauge fluctuations*. We expect that there will be an imperfect charge-flux relation in the Higgs glass phase, which will allow us to infer gapless charge fluctuations with perhaps a vanishing density of states at vanishing energy.

In conclusion, we have studied a lowest Landau level model for the $\nu = 1$ bilayer quantum Hall ferromagnet which displays a Higgs→Mott transtion of spinons in the presence of a periodic potential. Our understanding of the Mott phase of the spinons, which is also a ferromagnetic insulator, remains incomplete. The standard expectation is that the large d/l_0 bilayer system is best

described in terms of two species of weakly interacting Composite Fermions (CF's) [11]. It is possible that the transition we have described on the lattice is preempted by a first-order transition (reverting to second-order with quenched disorder) to a phase adiabatically connected to two decoupled species of CF's.

The framework we have sketched should apply to the single layer QHFM with spin, where the existence of charge fluctuations below the Zeeman energy is not understood[33].

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