Disentangling Neutrino Oscillations

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Abstract

The theory underlying neutrino oscillations has been described at length in the literature. The neutrino state produced by a weak decay is usually portrayed as a linear superposition of mass eigenstates with, variously, equal energies or equal momenta. We point out that such a description is incorrect, that in fact, the neutrino is entangled with the other particle or particles emerging from the decay. We offer an analysis of oscillation phenomena involving neutrinos (applying equally well to neutral mesons) that takes entanglement into account. Thereby we present a theoretically sound proof of the universal validity of the oscillation formulæ ordinarily used. In so doing, we show that the departures from exponential decay reported by the GSI experiment cannot be attributed to neutrino mixing. Furthermore, we demonstrate that the 'Mössbauer' neutrino oscillation experiment proposed by Raghavan, while technically challenging, is correctly and unambiguously describable by means of the usual oscillation formalæ.

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I. INTRODUCTION

Neutrino oscillations are among the most interesting phenomena discovered in particle physics in recent years. Although these oscillations were anticipated long ago [1, 2], their detection was complicated by the small size of the neutrino masses. Today, however, oscillation phenomena have been observed and studied for neutrinos originating from the sun, nuclear reactors, accelerators, and cosmic-ray interactions in the atmosphere. For a review see [3].

Recently, several novel and ingenious experiments have been suggested (and in at least one case carried out) to further explore the physics of neutrino masses. Raghavan has proposed the study of oscillations via the resonant capture of anti-neutrinos produced by the bound-state beta decay of tritium [4, 5, 6, 7]. This suggestion has led to some confusion. Akhmedov et al. [8, 9] agree that oscillations should be expected in this experiment, whilst Bilenky et al. [10, 11] conclude that whether or not oscillations are seen can "test fundamentally different approaches to neutrino oscillations".

In addition, Litvinov et al. [12] report the observation of non-exponential weak decays of hydrogenic ions. Some theoretical analyses interpret these data in terms of neutrino mixing [13, 14, 15, 16] while others refute such an interpretation [17, 18, 19, 20]. In another experiment a stronger bound was set [21] on the amplitude of the oscillatory modulation of the exponential decay of ^{142}Pm at the frequency reported in [12].

Our motivation for this work is to produce a simple and coherent theoretical framework for describing oscillation experiments involving elementary particles. Although a proper treatment of oscillation phenomena may appear (implicitly) in the literature[1], the significant discrepancies and imprecisions in existing approaches to neutrino oscillations suggest the need for such a unified framework.

We use neutrinos as our primary example in the derivation of the oscillation formulæ. As we shall see, our results apply equally well to other types of elementary particle oscillations including those of B and K mesons. We discuss neutral meson oscillations in III.

[1] For example, the work of Nauenberg [22], while not identical to our approach, correctly identifies entanglement as necessary for energy–momentum conservation. Similarly Kayser [23] recognized that sufficiently accurate momentum measurements prevent oscillations.

II. UNIVERSAL OSCILLATIONS

Neutrino oscillations arise because the weak interactions conserve lepton flavor whereas energy eigenstate neutrinos are not flavor eigenstates. Most analyses describe the production of neutrinos (via a weak decay or scattering event) in terms of a flavor eigenstate which is then decomposed as a linear combination of mass eigenstate neutrinos, each of which propagates according to its own dispersion relation. Often an analogy is drawn with a simple two state system (for ease of notation we restrict to two neutrino flavors with mixing angle θ; the generalization to three flavors is straightforward), and frequently one sees formulæ like

$$|\nu_e\rangle = \cos\theta\, |\nu_L\rangle + \sin\theta\, |\nu_H\rangle \tag{1}$$

where we have labeled the mass eigenstates as "H"eavy and "L"ight. This approach is not entirely correct and has led to significant confusion in the literature. For example, the states must depend on the three-momenta of the neutrinos. But because ν_L and ν_H have different masses it is not possible for this superposition to be an eigenstate of both energy and momentum, thus leading some authors to suggest a common energy while others prefer a common momentum. However neither of these suggestions can be correct, because neither can account for simultaneous energy and momentum conservation in the weak process that produces the neutrino.

The resolution to this puzzle is quite simple: the state produced following the weak interaction is *not* of the form (1). Rather, the state produced has the neutrino mass eigenstates entangled with the other particles remaining after the weak process has occurred. Energy and momentum are fully conserved by the process, as must be the case given space-time translation invariance of the underlying interaction.

A simple example serves to illustrate the primary issues. Consider a particle N (the "parent") of mass M which decays to another particle n (the "daughter") of mass M' plus a neutrino.[2] To simplify our discussion we ignore the spins of all particles involved as well as any internal excitations. By assuming the parent to be sufficiently long-lived, we may choose the initial state to have arbitrarily well-defined energy and momentum P and we may

[2] The example of a 2-body decay exhibits all the features of interest, and extension to other processes requires no significant modifications. The particle could be a pion decaying conventionally to a muon or equally well an atom decaying via electron capture.

treat the decay process in perturbation theory. In this approximation we may think of the decay as occurring instantaneously at some time (distributed in accord with the exponential decay law) leaving us in the state

$$|\psi\rangle = \frac{1}{\sqrt{\mathcal{N}}}\Big[\int D_2(k_l, q_l)\ \cos\theta\, |n(k_l)\nu_L(q_l)\rangle + \int D_2(k_h, q_h)\ \sin\theta\, |n(k_h)\nu_H(q_h)\rangle\Big] \tag{2}$$

where $q_i^2 = m_i^2$ and $k_i^2 = M'^2$. The phase space for the two particles $D_2(k, q)$ is

$$D_2(k, q) = \frac{d^3k}{(2\pi)^3 2E_k}\frac{d^3q}{(2\pi)^3 2E_q}(2\pi)^4\delta^4(P - k - q) \tag{3}$$

where the energies E_k, E_q are computed with the appropriate particle masses and, for simplicity, we have assumed an amplitude independent of momenta. The value of the normalization constant $\mathcal{N}$ will not be needed.[3] Note that all particles are on the mass-shell and $|\psi\rangle$ is an eigenstate of energy and momentum with eigenvalue P. This is achieved through the entanglement of the neutrino with the daughter particle and would not be possible if the state were a non-entangled product with the ket of (1) as a factor.

The latter point is worth emphasizing. Flavor-charge operators, such as the electron or muon number operators, remain well-defined in the Standard Model augmented with neutrino mixing but no longer commute with the Hamiltonian. The lepton flavor conserving weak interactions are most simply written in terms of the electron (muon) neutrino field with definite flavor which acts on a state so as to alter the electron (muon) number by one unit. However, since time evolution alters the flavor, it is not very fruitful to consider states of definite flavor. Rather, although the *fields* that create and annihilate mass eigenstates are formed as linear combinations of the *fields* of definite flavor, the corresponding construction for *states* is not helpful. This situation is much like the relation between *chirality* (a useful property of fields) and *helicity* (a measurable property of states).

Having properly identified the final state, how are we to treat oscillations? Most oscillation experiments observe the neutrino as it produces a charged lepton via a weak interaction, and ignore any other particles that accompany the neutrino's production. Because the neutrino is entangled with these other (undetected) particles, we must construct the density matrix for the neutrino by tracing over these other degrees of freedom. Neutrino oscillations

[3] For reference $\mathcal{N} = VT \cdot 2M\Gamma$ where VT is the volume of space-time and Γ is the parent particle decay rate.

arise from an off-diagonal term in this density matrix of the form $|\nu_L\rangle\langle\nu_H|$. Constructing the density matrix from the state (2), we obtain

$$\rho_\nu = \frac{1}{\sqrt{\mathcal{N}}}\Big[\int D_2(k_l,q_l)D_2(\tilde{k}_l,\tilde{q}_l)\cos^2\theta\left\langle n(k_l)\,\middle|\,n(\tilde{k}_l)\right\rangle|\nu_L(q_l)\rangle\langle\nu_L(\tilde{q}_l)|$$
$$+\int D_2(k_l,q_l)D_2(\tilde{k}_h,\tilde{q}_h)\cos\theta\sin\theta\left\langle n(k_l)\,\middle|\,n(\tilde{k}_h)\right\rangle|\nu_L(q_l)\rangle\langle\nu_H(\tilde{q}_h)| + \text{h.c.}$$
$$+\int D_2(k_h,q_h)D_2(\tilde{k}_h,\tilde{q}_h)\sin^2\theta\left\langle n(k_h)\,\middle|\,n(\tilde{k}_h)\right\rangle|\nu_H(q_h)\rangle\langle\nu_H(\tilde{q}_h)|\Big]\ . \quad (4)$$

However the cross terms between $|\nu_L\rangle$ and $|\nu_H\rangle$ on the middle line vanish. A non-zero inner product for the daughter particle ($E'_k \equiv \sqrt{\mathbf{k}^2 + M'^2}$)

$$\left\langle n(k_l)\,\middle|\,n(\tilde{k}_h)\right\rangle = (2\pi)^3 2E'_{k_l}\delta^3(\mathbf{k}_l - \tilde{\mathbf{k}}_h) \quad (5)$$

requires that the two momenta be equal, while the delta functions in D_2 reflecting energy–momentum conservation require that $k_l - \tilde{k}_h = \tilde{q}_h - q_l$. But the two neutrino states have different invariant masses and so this momentum difference can never vanish. Hence these daughter particle states are orthogonal and the neutrino density matrix is diagonal

$$\rho_\nu \propto \int \frac{D_2(k_l,q_l)}{2E^{(\nu_L)}_{q_l}}|\nu_L(q_l)\rangle\langle\nu_L(q_l)|\cos^2\theta + \int\frac{D_2(k_h,q_h)}{2E^{(\nu_H)}_{q_h}}|\nu_H(q_h)\rangle\langle\nu_H(q_h)|\sin^2\theta \quad (6)$$

with probability $\cos^2\theta$ of containing ν_L and probability $\sin^2\theta$ of containing ν_H. Since the amplitude for the detection of ν_L via an electron-implicated weak interaction is $\cos\theta$ and that for ν_H is $\sin\theta$, this leads to a detection probability proportional to $\cos^4\theta + \sin^4\theta$, exactly as we expect in the absence of interference. When the decay products of an initial state of well-defined momentum evolve without further interaction no oscillation phenomena appear.

So how can neutrino oscillations arise? The assumptions of the final sentence of the preceding paragraph must not apply to experiments that exhibit oscillations. In fact, so long as the neutrino remains entangled as in (2), there is no possibility of interference and hence no possibility of oscillation. To realize oscillations the neutrino mass eigenstates must be disentangled.

We have so far treated the parent particle as an exact energy and momentum eigenstate with an associated unrealistic uniform detection probability throughout spacetime. This is surely not the case in realistic circumstances. Nonetheless, it is instructive to consider this unrealistic state in the situation where the daughter particle is detected in addition to the neutrino. For neutrinos produced in pion decay, for example, the associated muon (or its decay products) might be detected in a state $|\bar{n}\rangle$. Rather then tracing over the unobserved daughter this case requires computation of the joint probability for observation of the daughter in the state $|\bar{n}\rangle$ along with the neutrino. This may be calculated by projecting the state (2) by $|\bar{n}\rangle\,\langle\bar{n}|$. This projection then disentangles the state (2), leaving the neutrino in a simple superposition. The neutrino itself is unaffected by this projection: the two components continue to have the momenta q_l, q_h determined by the decay kinematics.

This projection alters the amplitude of the ν_L and ν_H components in the superposition by the two matrix elements $\langle\bar{n}|n(k_{l,h})\rangle$. The state $|\bar{n}\rangle$ is typically well-localized in space-time, and hence has a substantial spread in momentum. Because the momenta $k_{l,h}$ are nearly the same the matrix elements $\langle\bar{n}|n(k_l)\rangle$ and $\langle\bar{n}|n(k_h)\rangle$ are, for all practical purposes, equal. Hence, subsequent to this projection the neutrino may be treated as a superposition of the two mass eigenstates (as is usually done) with momenta q_l and q_h:

$$|\psi\rangle \sim \cos\theta\,|\nu_L(q_l)\rangle + \sin\theta\,|\nu_H(q_h)\rangle \ . \tag{7}$$

We have restricted the superposition to one spatial dimension, eliminating the integral over the neutrino direction. This is a reasonable approximation because oscillation experiments require the neutrino to propagate far from the production point, hence we detect only those particles traveling in the appropriate direction. As promised in the introduction, the neutrinos are neither equal in energy nor equal in momentum. The detection of the neutrino may be modeled by acting with an operator of electron flavor at the detector space-time location $z \equiv (t, d)$ (as usual we work in the Heisenberg picture) giving a detection amplitude

$$\mathcal{A} \sim \cos^2\!\theta\, e^{iq_l\cdot z} + \sin^2\!\theta\, e^{iq_h\cdot z} \ . \tag{8}$$

The square of this expression contains an interference term between the H and L amplitudes which may produce oscillations. Although the amplitudes in (8) show only complex-exponential dependence on the detection location, realistic experiments involve amplitudes that have an extended space-time support localized around the trajectory $d = vt$. The H and L amplitudes interfere only when they have common support. Because the particles have velocity dispersions with slightly different central values, they separate as they travel towards the detection event. Interference is possible only if this separation is smaller than the localization size of the particle $v\Delta T$, or what is often called the size of the wave-packet.

The condition for interference is

$$\left|\frac{v_h - v_l}{v_h + v_l}\right| = \left|\frac{\sigma\omega\delta q - \delta\omega\sigma q}{\sigma\omega\sigma q - \delta\omega\delta q}\right| \ll \frac{\Delta T}{t} \tag{9}$$

where the sum and difference of the neutrino energies are $\sigma\omega \equiv \omega_h + \omega_l, \delta\omega \equiv \omega_h - \omega_l$ and $\sigma q, \delta q$ are the corresponding expressions for the sum and difference of the magnitudes of the spatial momenta.

The interference term in the square of the amplitude (8) has the phase $\phi \equiv (q_h - q_l) \cdot z$. So far we have made no assumptions about the masses of the particles involved, nor about the momentum of the initial parent that gives rise to the neutrino. This generality allows us to describe oscillations of other particles (such as K and B mesons) as well as neutrinos. The only assumption we make at this stage is that the difference in velocities between the two components is small enough so that the particles may interfere in the detector located at $z \equiv (t, d)$: Eq. (9). This condition applies to K meson oscillations, B meson oscillations and neutrino oscillations under all realistic conditions. We continue to refer to the oscillating particles as neutrinos in the sequel.

Condition (9) ensures that the two components of the state overlap at the detection point, thus allowing them to interfere. For reasonable velocity dispersions this overlap may be evaluated using stationary phase and is dominated when neutrino velocities are $v = d/t$. Thus we may take $\sigma q/\sigma\omega = d/t$. Provided we observe the neutrinos over times such that the two components have not spatially separated, the space-time vector $z \equiv (t, d)$ may then be expressed as

$$z = (t, d) \simeq t\left(1, \frac{\sigma q}{\sigma\omega}\right) = \frac{t}{\sigma\omega}\left(q_h + q_l\right) \ . \tag{10}$$

The oscillation phase is then

$$\phi \equiv (q_h - q_l) \cdot z = \frac{t}{\sigma\omega}(q_h - q_l) \cdot (q_h + q_l) = t\frac{\delta m^2}{\sigma\omega} \ . \tag{11}$$

This is the usual answer for relativistic neutrinos where $t \simeq d$ and $\sigma\omega$ is just twice the neutrino energy. But the same expression applies whenever (9) is satisfied, relativistic or not. For non-relativistic particles, for example, we have $\sigma\omega \simeq m_l + m_h$ and the phase ϕ is then $t\delta m$.

In this argument we used no properties of the vectors $q_{l,h}$ other than the condition (9). The energies and magnitudes of the spatial momenta are fully determined: the two neutrinos are neither equal in energy nor momentum. Nevertheless, use of such incorrect values fortuitously leads to the correct oscillation phase.

Moreover, the detailed properties of the state $|\bar{n}\rangle$, other than the near equality of the matrix elements $\langle\bar{n}|n(k_{l,h})\rangle$, played no role in our analysis. Similarly the mechanism producing the neutrino and any distribution in its momentum are irrelevant to (11) provided (9) is satisfied. In this sense the oscillation phase of (11) is "universal".

Usually the neutrinos are detected without accompanied detection of the daughter particle. In this case we must employ the density matrix for the neutrino after tracing over the daughter Hilbert space. Interference between the ν_L and ν_H components requires a non-vanishing inner product of the daughter states in this trace. In practice this is realized by accounting for a momentum spread arising from the parent. For any realistic experiment, the parent state is not a momentum eigenstate but rather a superposition of momenta in a narrow range. If this range is such that the daughter particle accompanying ν_L can have the same four-momentum as the daughter particle accompanying ν_H, then oscillations become possible. The difference between the daughter particle momenta in the two components is of order $\delta m^2/\sigma\omega$. For realistic neutrino masses and energies, the required momentum difference is exceedingly small, less than 10^{-10} eV. Because realistic experiments always start with an initial state at least slightly localized in space-time (often to a nuclear distance, but surely to within a kilometer or better) this momentum difference always lies within the initial momentum spread.

As an example consider the long-lived parent particle as above but in an initial state which is a superposition of spatial momenta in a narrow band.[4] This may be described by superposing slight boosts Λ_v of the initial particle momentum P in the state (2). Assuming the initial momentum spread is small corresponds to requiring that only $v \ll 1$ appears in this superposition. The final state (2) is then replaced by a similar superposition

$$|\psi\rangle = \frac{1}{\sqrt{\mathcal{N}}}\int d^3v f(\mathbf{v})\Big[\int D_2(k_l,q_l)\ \cos\theta\,|n(\Lambda_v k_l)\nu_L(\Lambda_v q_l)\rangle + \int D_2(k_h,q_h)\ \sin\theta\,|n(\Lambda_v k_h)\nu_H(\Lambda_v q_h)\rangle\Big] \quad (12)$$

where $f(\mathbf{v})$ describes the superposition and we have used the Lorentz invariance of the phase

[4] The finite lifetime of the unstable initial particle produces an additional (Lorentzian) spread in the invariant mass of the daughter plus the neutrino. This effect may be incorporated similarly to our inclusion of the spatial momentum spread.

space. We may simplify as before by restricting to a single direction

$$|\psi\rangle \sim \int dv\, f(v) \Big[\cos\theta\, |n(\Lambda_v k_l)\nu_L(\Lambda_v q_l)\rangle + \sin\theta\, |n(\Lambda_v k_h)\nu_H(\Lambda_v q_h)\rangle\Big] \tag{13}$$

where k_i and q_i are now completely determined in terms of the initial momentum P by conservation of energy and momentum. Although not obvious at first glance, this superposition allows for the disentanglement of the neutrino mass from the daughter momentum.

To see this, note that the vectors k_l and k_h have the same invariant mass, and hence differ from a common four-momentum k by (small) Lorentz boosts: $\Lambda_{v_0} k_h = \Lambda_{-v_0} k_l \equiv k$. The boost velocity v_0 is easily computed in terms of q_l and q_h:

$$v_0 = -\frac{\delta q}{2E_p - \sigma\omega} \; . \tag{14}$$

By shifting the velocity v in the integrals in the two terms of (13) relative to each other we can rewrite the superposition as

$$|\psi\rangle \sim \int dv\, |n(\Lambda_v k)\rangle \otimes \Big[f(v+v_0)\cos\theta\, |\nu_L(\Lambda_v \Lambda_{v_0} q_l)\rangle + f(v-v_0)\sin\theta\, |\nu_H(\Lambda_v \Lambda_{-v_0} q_h)\rangle\Big]. \tag{15}$$

Although this state is still entangled (a sum of products), the neutrino *mass* is not fully entangled with the daughter momentum. The density matrix for the neutrino constructed upon tracing over the daughter states now contains cross terms between neutrino L and neutrino H which give rise to oscillations.

Our previous calculation of the oscillation phase continues to apply, subject to two changes. Firstly the interference term contains a factor of $f(v+v_0)f^*(v-v_0)$ rather than the $|f(v+v_0)|^2$ or $|f(v-v_0)|^2$ factors of the diagonal terms. If, as is generally the case, the function f representing the momentum superposition of the parent does not vary significantly on the scale of the small velocity v_0, these factors are all essentially the same. Secondly the energy $\sigma\omega$ that appears is modified by the boosts:

$$\sigma\omega \to \sigma\omega - v_0 \delta q + v\sigma q \; . \tag{16}$$

If the support of f is such that $v \ll 1$, so that the momentum of the parent is moderately well defined, we may drop the terms proportional to v and v_0. Once again we obtain the familiar oscillation formula, and once again the details of the momentum superpositions involved play no role other than ensuring the presence of the interference term in the neutrino density matrix.

We conclude this section with a brief discussion of the novel oscillation-related experiments mentioned earlier. Consider first the proposal of Raghavan [4, 5, 6, 7] to study the resonant capture of antineutrinos from bound-state tritium decay. The question of whether or not such "Mössbauer neutrino oscillations" are present has been hotly contested. Bilenky et al. [10, 11] conclude that such oscillations may or may not occur and that the Raghavan experiment "provides the unique possibility to discriminate basically different approaches" to neutrino oscillations. Contrariwise, Akhmedov et al. [9] find that "a proper interpretation of the time-energy uncertainty relation is fully consistent with oscillations of Mössbauer neutrinos." The result of our analysis is simple. Condition (9), our unique and simple criterion for the appearance of oscillations is satisfied by the Raghavan experiment: if the Raghavan experiment can be realized, it will be a powerful tool with which to study neutrino oscillations. Furthermore and contrary to Bilenky et al., there is no ambiguity about the approach to neutrino oscillations for the Raghavan experiment to resolve.

Now let us turn to the GSI experiment. An essential feature of this experiment is that the neutrino is not detected: the observed oscillations appear in measurements of the time of disappearance of the parent particle (coincident with the appearance of the daughter particle). As shown below, the arguments we have adduced demonstrate that experiments which do not observe the neutrino cannot display interference. Our discussion so far has not included the production and decay of the parent particle, but this is easily incorporated. We create the parent particle by acting on the vacuum with some (smeared) operator $N_J^\dagger$ producing the parent around $t = 0$, and model the observation of the daughter at a subsequent time by acting with a (smeared) operator n_j around the time t. The neutrino is not observed and remains in the final state. The squared amplitude for this process is represented diagrammatically in Fig. 1. An on-shell neutrino in the final state is represented by the dashed line, corresponding to a cut propagator $\delta(q^2 - m_i^2)\theta(q^0)$. The full squared amplitude is given by a sum over the several neutrino mass eigenstates. Notice that there are no cross terms between these mass eigenstates: because the neutrinos have different invariant masses, there is no possibility of interference between them. Thus the reported oscillation of the decay time cannot be explained in terms of interference between neutrino states.

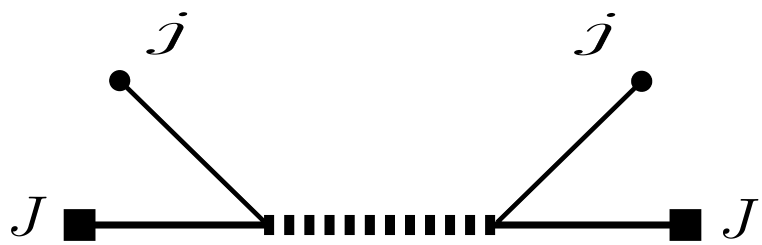

FIG. 1: Feynman diagram representing $|\mathcal{A}|^2$ for the observation of the daughter particle. The square blobs represent the parent source, the round blobs the daughter detector, and the dashed line is the (on-shell) neutrino.

III. NEUTRAL MESON MIXING

We turn to the study of mixing effects of mesons, in particular in the B meson sector. Our discussion is framed to most closely resemble the experiments conducted at B factories, but a similar analysis applies to other cases. Measurements at the B factories observe the B mesons produced in the decay of the $\Upsilon(4S)$. We consider the case where one of the mesons is detected at a space-time location z through its decay to the state $|\mathcal{O}\rangle$ and the other at space-time location $\tilde{z}$ through its decay to the state $|\tilde{\mathcal{O}}\rangle$. The entangled state resulting from the decay of an $\Upsilon(4S)$ into neutral B mesons is

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[\left|B_L(k_l)B_H(\tilde{k}_h)\right\rangle - \left|B_H(k_h)B_L(\tilde{k}_l)\right\rangle\right] \tag{17}$$

where L, H label the light and heavy mass eigenstates and, as in the previous section, we have restricted the momenta to those pointing in the directions of the observation events. By convention the momenta without over-tildes point toward the event z while those with tildes point toward $\tilde{z}$.

The observations of the B mesons may be modeled by taking the matrix element of an appropriate local operator that annihilates the particles in the final states $|\mathcal{O}\rangle\,, |\tilde{\mathcal{O}}\rangle$ between $|\psi\rangle$ and the vacuum:

$$\begin{aligned} S_{\mathcal{O}\tilde{\mathcal{O}}} = \left\langle 0 \,\middle|\, \mathcal{O}(z)\tilde{\mathcal{O}}(\tilde{z}) \,\middle|\, \psi \right\rangle = \frac{1}{\sqrt{2}}\Big[&\langle 0|\mathcal{O}(z)|B_L(k_l)\rangle\,\langle 0|\tilde{\mathcal{O}}(\tilde{z})|B_H(\tilde{k}_h)\rangle \\ &- \langle 0|\mathcal{O}(z)|B_H(k_h)\rangle\,\langle 0|\tilde{\mathcal{O}}(\tilde{z})|B_L(\tilde{k}_l)\rangle\Big]\,. \end{aligned} \tag{18}$$

The space-time dependence of these matrix elements is determined by translation invariance:

$$
\begin{aligned}
\langle 0|\mathcal{O}(z)|B_L(k_l)\rangle =& e^{ik_l\cdot z}\,\langle 0|\mathcal{O}(0)|B_L(k_l)\rangle \equiv e^{ik_l\cdot z}A_L \\
\langle 0|\tilde{\mathcal{O}}(\tilde{z})|B_L(\tilde{k}_l)\rangle =& e^{i\tilde{k}_l\cdot \tilde{z}}\,\langle 0|\tilde{\mathcal{O}}(0)|B_L(\tilde{k}_l)\rangle \equiv e^{i\tilde{k}_l\cdot \tilde{z}}\tilde{A}_L
\end{aligned}
\tag{19}
$$

and similarly for the matrix elements involving B_H.

Taking the absolute-value squared of S, we obtain

$$
\begin{aligned}
|S_{\mathcal{O}\tilde{\mathcal{O}}}|^2 = \frac{1}{2}\Big\{ & |A_L|^2|\tilde{A}_H|^2 e^{i(k_l-k_l^*)\cdot z}e^{i(\tilde{k}_h-\tilde{k}_h^*)\cdot\tilde{z}} \\
& - A_L A_H^* \tilde{A}_H \tilde{A}_L^* e^{i(k_l-k_h^*)\cdot z}e^{i(\tilde{k}_h-\tilde{k}_l^*)\cdot\tilde{z}} - \text{c.c.} \\
& + |A_H|^2|\tilde{B}_L|^2 e^{i(k_h-k_h^*)\cdot z}e^{i(\tilde{k}_l-\tilde{k}_l^*)\cdot\tilde{z}}\Big\} \ .
\end{aligned}
\tag{20}
$$

We have uncharacteristically kept the complex conjugation on the momenta of the B mesons. This is to keep track of the finite lifetime of the mesons that may be incorporated as an imaginary part for the energy.[5] Using the formula derived in the previous section, we have

$$
\begin{aligned}
i(k_l-k_l^*)\cdot z = -\Gamma_L t \qquad & i(k_h-k_h^*)\cdot z = -\Gamma_H t \\
i(\tilde{k}_l-\tilde{k}_l^*)\cdot \tilde{z} = -\tilde{\Gamma}_L \tilde{t} \qquad & i(\tilde{k}_h-\tilde{k}_h^*)\cdot \tilde{z} = -\tilde{\Gamma}_H \tilde{t} \\
i(k_l-k_h^*)\cdot z = -\frac{\Gamma_L+\Gamma_H}{2}t - it\frac{\delta m^2}{\sigma\omega} & \\
i(\tilde{k}_h-\tilde{k}_l^*)\cdot \tilde{z} = -\frac{\tilde{\Gamma}_L+\tilde{\Gamma}_H}{2}\tilde{t} + i\tilde{t}\frac{\delta m^2}{\sigma\tilde{\omega}} & \ .
\end{aligned}
\tag{21}
$$

In the laboratory frame the two mesons generally have (slightly) different velocities. For completeness, we have kept the difference between the widths $\Gamma, \tilde{\Gamma}$ and energies $\sigma\omega, \sigma\tilde{\omega}$ of these mesons. (In the $\Upsilon(4S)$ rest frame B_H and B_L, which are produced back-to-back, have (nearly) identical velocities and in this frame we have $\Gamma_H = \tilde{\Gamma}_H, \Gamma_L = \tilde{\Gamma}_L, \sigma\omega = \sigma\tilde{\omega}$.) Thus (20) becomes

$$
|S_{\mathcal{O}\tilde{\mathcal{O}}}|^2 = \frac{1}{2}e^{-\Gamma t-\tilde{\Gamma}\tilde{t}}\Big\{|A_L|^2|\tilde{A}_H|^2 + |A_H|^2|\tilde{A}_L|^2 - A_L A_H^*\tilde{A}_H\tilde{A}_L^* e^{i\xi} - A_L^* A_H \tilde{A}_H^* \tilde{A}_L e^{-i\xi}\Big\}
\tag{22}
$$

where

$$
\xi \equiv \delta m^2(\tilde{t}/\sigma\tilde{\omega} - t/\sigma\omega) \ .
\tag{23}
$$

[5] We are being slightly sloppy. A proper treatment would use a local operator to create the B meson from the vacuum and then follow its propagation. For a width small compared to the mass this propagator is dominated by a simple pole that is not on the real axis but rather on the second sheet, with an imaginary part given by the decay width. The net result is the complex exponential in (20).

We can equally well express ξ in terms of the laboratory frame distances the B mesons travel, $d, \tilde{d}$: $\xi = \delta m^2(\tilde{d}/\sigma\tilde{p} - d/\sigma p)$. Alternatively, we may use the B meson decay times $T, \tilde{T}$ evaluated in the $\Upsilon(4S)$ rest frame, where $\sigma\omega = \sigma\tilde{\omega} \simeq \sigma m$ and $\xi = \delta m(\tilde{T} - T)$. To simplify our results we have ignored the difference in widths between the heavy and light B mesons, taking $\Gamma_L = \Gamma_H \equiv \Gamma$. Only slightly more effort is required to keep track of this effect.

As our first example we evaluate the mixing probability obtained from measurements in which we observe one B meson decaying into a final state $|\mathcal{O}^{\pm}\rangle$ and the other into a final state $|\tilde{\mathcal{O}}^{\pm}\rangle$, each containing a charged lepton. Because there is negligible direct CP violation in these B decays the various amplitudes are related. The B_L meson in (19) may be created by a local operator of the form $p(\bar{d}b) + q(\bar{b}d)$ where p and q are constants determined by the requirement that this operator does not also create the B_H meson. This leads to the usual expressions for p and q (with the usual phase freedom). A similar argument shows that the operator $p(\bar{d}b) - q(\bar{b}d)$ creates only the B_H meson. Imposing CP invariance in the time-development of the operators $\mathcal{O}^{\pm}$, $(CP)\mathcal{O}^{\pm}(CP) = \mathcal{O}^{\mp}$, yields the relations:

$$
\begin{aligned}
A_L^+ &= A_H^+ = pA \qquad & A_L^- &= -A_H^- = qA \\
\tilde{A}_L^+ &= \tilde{A}_H^+ = p\tilde{A} \qquad & \tilde{A}_L^- &= -\tilde{A}_H^- = q\tilde{A}\ .
\end{aligned}
\tag{24}
$$

Using these relations in (22) we obtain

$$
\begin{aligned}
|S_{++}|^2 &= |A|^2|\tilde{A}|^2 e^{-\Gamma t - \tilde{\Gamma}\tilde{t}}|p|^4 \sin^2\frac{\xi}{2} \\
|S_{--}|^2 &= |A|^2|\tilde{A}|^2 e^{-\Gamma t - \tilde{\Gamma}\tilde{t}}|q|^4 \sin^2\frac{\xi}{2} \\
|S_{-+}|^2 &= |A|^2|\tilde{A}|^2 e^{-\Gamma t - \tilde{\Gamma}\tilde{t}}|p|^2|q|^2 \cos^2\frac{\xi}{2}\ .
\end{aligned}
\tag{25}
$$

In the absence of CP violation $|q/p| = 1$. The mixing probability χ is then

$$
\chi \equiv \frac{\iint_0^\infty dt d\tilde{t}(|S_{++}|^2 + |S_{--}|^2)/2}{\iint_0^\infty dt d\tilde{t}(|S_{++}|^2/2 + |S_{--}|^2/2 + |S_{-+}|^2)} = \frac{x^2}{2(1+x^2)}\ , \tag{26}
$$

where $x \equiv \delta m^2/(\sigma\omega\,\Gamma)$ and we have divided by 2 when integrating to avoid double counting identical final states. In evaluating this integral we have used the fact that $\Gamma\,\sigma\omega$ is Lorentz invariant so that $\Gamma\,\sigma\omega = \tilde{\Gamma}\,\sigma\tilde{\omega}$. Further, this Lorentz invariance allows the evaluation of x in the rest frame where $\delta m^2/\sigma\omega = (M_H^2 - M_L^2)/(M_H + M_L) = \delta m$ and $\Gamma = \Gamma_0$. Therefore $x = \delta m/\Gamma_0$, and χ is seen to be the usual expression. If the difference in widths of the two states is taken into account, we obtain $\chi = (x^2 + (\delta\Gamma/2\Gamma)^2)/(2(1+x^2))$.

We turn to the time-dependent CP asymmetries in B meson decay. In this case we tag one of the B mesons via a decay to a charged lepton as before, but then observe the decay of the other B into a CP eigenstate, f. Our previous analysis continues to apply: the A amplitudes continue to refer to measurements involving a charged lepton and are still given by (24). The other amplitudes, now denoted as $A^f_{L,H}$, refer to the detection of a CP eigenstate f.

The two amplitudes A^f_L, A^f_H are in general independent. It is conventional to define

$$A^f_{L,H} \equiv pA_f \pm q\bar{A}_f = pA_f(1 \pm \lambda_f)\,, \tag{27}$$

where $\lambda_f = (q/p)(\bar{A}_f/A_f)$. The tagged rates are

$$\begin{aligned} |S_{+f}|^2 &\propto e^{-\Gamma t - \tilde{\Gamma}\tilde{t}}\,|p^2 A A_f|^2 \left[|\lambda_f|^2 \cos^2\frac{\xi}{2} + \sin^2\frac{\xi}{2} + \mathrm{Im}\lambda_f \sin\xi \right] \\ |S_{-f}|^2 &\propto e^{-\Gamma t - \tilde{\Gamma}\tilde{t}}\,|pqAA_f|^2 \left[\cos^2\frac{\xi}{2} + |\lambda_f|^2 \sin^2\frac{\xi}{2} - \mathrm{Im}\lambda_f \sin\xi \right]\,, \end{aligned} \tag{28}$$

where the tilded quantites refer to the observation of the B meson decaying into the CP eigenstate f and ξ is given by (23). For the B meson system $|p| \simeq |q|$ and the time-depenent asymmetry is

$$a_f \equiv \frac{|S_{+f}|^2 - |S_{-f}|^2}{|S_{+f}|^2 + |S_{-f}|^2} = \frac{(|\lambda_f|^2 - 1)\cos\xi + 2\,\mathrm{Im}\,\lambda_f \sin\xi}{1 + |\lambda_f|^2}\,. \tag{29}$$

Using the value of ξ in the $\Upsilon(4S)$ rest frame this is seen to be the standard expression [24].

Other examples of neutral meson mixing may be treated similarly. The universal formula (11) allows a ready treatment of all pertinent cases.

IV. CONCLUSIONS

Oscillation phenomena, whether involving neutral mesons or neutrinos, have been widely studied experimentally. Although attempts to describe the underlying theoretical formalism are rife in the literature, the arguments used are often obscure, confusing, or simply wrong. The starting point of many such analyses is a "flavor eigenstate" which is neither an energy eigenstate nor takes into account the entanglement of the neutrino with other final state particles. This leads to equal-momentum versus equal-energy controversies, to inappropriate appeals to "energy–time uncertainty," and to alleged ambiguities related to the oscillation phase that are somehow to be resolved by future experiments.

In this paper, we present a theoretical analysis of oscillation phenomena in a fashion that is simple, entirely general, and free of ambiguities. The oscillation phase is unambiguously given by (11), an expression equally applicable for neutrinos and mesons of any energy, relativistic or not. The occurrence of oscillations requires simply that 1.) the oscillating particles be observed and 2.) condition (9) be satisfied thus ensuring the overlap of these particles at the time of their detection.

Our approach to oscillations shows that the variations in decay times observed in the GSI experiment (where neutrinos from electron capture are not observed) cannot be attributed to neutrino mass mixing. Furthermore, we point out that our universal criterion is satisfied by the proposed Raghavan experiment which, if it proves feasible, should enable the observation of neutrino oscillations.

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[1] B. Pontecorvo, “Neutrino experiments and the question of leptonic-charge conservation,” *Sov. Phys. JETP* **26** (1968) 984–988.

[2] V. N. Gribov and B. Pontecorvo, “Neutrino astronomy and lepton charge,” *Phys. Lett.* **B28** (1969) 493.

[3] S. M. Bilenky, “The history of neutrino oscillations,” *Phys. Scripta* **T121** (2005) 17–22, `arXiv:hep-ph/0410090`.

[4] R. S. Raghavan, “Recoilless resonant capture of antineutrinos,” `arXiv:hep-ph/0511191`.

[5] R. S. Raghavan, “Recoilless resonant capture of antineutrinos from tritium decay,” `arXiv:hep-ph/0601079`.

[6] R. S. Raghavan, “Hypersharp Resonant Capture of Anti-Neutrinos,”

arXiv:0806.0839 [hep-ph].

[7] R. S. Raghavan, "Hypersharp Neutrino Lines," arXiv:0805.4155 [hep-ph].

[8] E. K. Akhmedov, J. Kopp, and M. Lindner, "Oscillations of Mossbauer neutrinos," *JHEP* **05** (2008) 005, arXiv:0802.2513 [hep-ph].

[9] E. K. Akhmedov, J. Kopp, and M. Lindner, "On application of the time-energy uncertainty relation to Mossbauer neutrino experiments," arXiv:0803.1424 [hep-ph].

[10] S. M. Bilenky, F. von Feilitzsch, and W. Potzel, "Time-Energy Uncertainty Relations for Neutrino Oscillation and Mössbauer Neutrino Experiment," *J. Phys.* **G35** (2008) 095003, arXiv:0803.0527 [hep-ph].

[11] S. M. Bilenky, F. von Feilitzsch, and W. Potzel, "Different Schemes of Neutrino Oscillations in Mossbauer Neutrino Experiment," arXiv:0804.3409 [hep-ph].

[12] Y. A. Litvinov *et al.*, "Observation of Non-Exponential Orbital Electron Capture Decays of Hydrogen-Like ^{140}Pr and ^{142}Pm Ions," *Phys. Lett.* **B664** (2008) 162–168, arXiv:0801.2079 [nucl-ex].

[13] H. J. Lipkin, "New method for studying neutrino mixing and mass differences," arXiv:0801.1465 [hep-ph].

[14] A. N. Ivanov, R. Reda, and P. Kienle, "On the time–modulation of the K–shell electron capture decay of H-like $^{140}\mathrm{Pr}^{58+}$ ions produced by neutrino–flavour mixing," arXiv:0801.2121 [nucl-th].

[15] H. Kleinert and P. Kienle, "Neutrino-Pulsating Vacuum and Neutrino Mass Difference," arXiv:0803.2938 [nucl-th].

[16] H. J. Lipkin, "The GSI method for studying neutrino mass differences - For Pedestrians," arXiv:0805.0435 [hep-ph].

[17] C. Giunti, "Comment on the Neutrino-Mixing Interpretation of the GSI Time Anomaly," arXiv:0801.4639 [hep-ph].

[18] M. Peshkin, "Comment on 'New method for studying neutrino mixing and mass differences'," arXiv:0804.4891 [hep-ph].

[19] H. Kienert, J. Kopp, M. Lindner, and A. Merle, "The GSI anomaly," *J. Phys. Conf. Ser.* **136** (2008) 022049, arXiv:0808.2389 [hep-ph].

[20] A. Gal, "Neutrinos do not oscillate yet at GSI," arXiv:0809.1213 [nucl-th].

[21] P. A. Vetter *et al.*, "Search for Oscillation of the Electron-Capture Decay Probability of

^{142}Pm," *Phys. Lett.* **B670** (2008) 196–199, arXiv:0807.0649 [nucl-ex].

[22] M. Nauenberg, "Correlated wave packet treatment of neutrino and neutral meson oscillations," *Phys. Lett.* **B447** (1999) 23–30, arXiv:hep-ph/9812441.

[23] B. Kayser, "On the Quantum Mechanics of Neutrino Oscillation," *Phys. Rev.* **D24** (1981) 110.

[24] P. F. Harrison and H. R. Quinn (eds.), "The BaBar physics book: Physics at an asymmetric B factory," *SLAC-R-504* (1998) .