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Comparison helps students learn to solve equations flexibly and efficiently

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Comparison helps students learn to solve equations flexibly and efficiently

Consider the following, perhaps familiar scenario. A mathematics teacher is circulating around her classroom, looking over the shoulders of students who are busy solving linear equations such as $3x + 2 = 5x + 8$. The teacher notices that one student, Paul, persists in using his own somewhat idiosyncratic and quite inefficient strategy (see Figure 1, below). Although Paul's strategy is not fundamentally incorrect, the extra steps required can lead to more calculation errors and wasted time.

(Insert Figure 1 about here)

Paul is quite good at this strategy and is able to solve problems correctly, but the teacher would prefer that Paul begin to realize *why* his strategy is inefficient and to use other strategies that solve similar equations with less redundancy. The teacher mentions to Paul that he should consider a different, more efficient solution method. Paul answers, "I know that there are better ways than the approach that I use, but this is the way that makes the most sense to me, and it gets me the right answer. Why do I have to learn different ways for solving problems?" How should the teacher respond? Certainly Paul has a point; his strategy, although idiosyncratic and inefficient, does yield the correct answer. Shouldn't this be enough?

The NCTM *Principles and Standards for School Mathematics* state that "by the time students reach the middle grades, they should be skilled at recognizing when various strategies are appropriate to use and should be capable of deciding when and how to use them" (NCTM 2000, 53). Students like Paul will become better problem solvers and have a deeper understanding of mathematical concepts if they can recognize and adapt a

variety of solution strategies. NCTM defines proficiency in mathematics as "the ability to use knowledge flexibly, applying what is learned in one setting appropriately in another." (NCTM 2000, 19) This flexibility in problem solving develops with knowledge of multiple strategies. While Paul might be successful at solving a variety of linear equations using his single inefficient strategy, he will be better equipped to solve more complex and new types of problems as he becomes capable of choosing among a variety of strategies that suit his needs and that make his problem solving more efficient.

The point here is not that Paul should have been discouraged from developing this inefficient strategy in the first place. Rather, a teacher can build on the strategy that (according to Paul) makes sense to him, in order to help him see *why* this strategy is inefficient and what aspects of his strategy can be adapted to make his problem solving more efficient.

One instructional approach that has gained prominence recently and that can help address Paul's situation is the use of comparison. Through comparison of solution strategies, students deliberately consider the similarities and differences among methods and how different strategies can arrive at the same solution. Comparing traditional strategies, more efficient shortcuts, and strategies like the one invented by Paul can "help students develop and use a variety of problem-solving strategies and approaches, and sharing ... methods within the classroom affords students opportunities to assess the strengths and limitations of alternative approaches" (NCTM 2000, 256). According to NCTM, comparison of strategies can enable students to understand what aspects or

features of some strategies make them powerful and efficient while other methods are less so, even if they are easier to understand (NCTM 2000).

Outside of the mathematics classroom, we often learn by comparing and contrasting. Imagine that you are buying a camera online. The best way to learn about the important features of a digital camera is to compare a few of the cameras that are available. Rather than looking at one camera and then the next sequentially, many find it helpful to chose the comparison feature in the online store. Comparing different cameras side by side not only helps us envision the benefits of one camera over another, but it also helps to determine what features are important to have in any good camera and which make the camera suitable for our photography needs.

The importance of comparison for mathematics instruction emerges in international studies of mathematics teachers. Researchers, evaluating the use of comparison in typical mathematics classrooms in the United States, Japan and Hong Kong, found that expert teachers in all three countries frequently used comparison as a tool for teaching mathematics. They compared new mathematical concepts to ideas that were already familiar to students, carefully placing examples side-by-side and using hand gestures to highlight similarities and differences (Richland, Zur, and Holyoak 2007, 1128). According to Richland, comparison "allows students to use commonalities between mathematical representations to help understand new problems or concepts, thereby contributing to integral components of mathematical proficiency" (Richland et al. 2007, 1128).

How can American mathematics teachers more effectively implement comparison into their classrooms? A recent study (Rittle-Johnson and Star 2007) illustrates both the potential benefits of comparison for students' learning and suggests some concrete ways that teachers can make use of comparison in the mathematics classroom.

The Study

Over the course of a week, seventy 7th grade students in a private urban school learned to solve linear equations that were similar to the one Paul solved in the example above. All four 7th grade classes in the school participated, including two advanced classes (a total of 36 students) and two regular classes (34 students). All classes were taught by the same teacher.

During each class period, the teacher provided students with a short introduction to a new topic related to solving linear equations. For the rest of the class period, students then (with a partner) worked through a packet where they analyzed hypothetical students' work. The packet contained example problems that were already solved either using a traditional method or a less conventional one. The pairs of students evaluated the problems and answered questions about them. As students worked, the classroom teacher, along with researchers, was available to answer students' questions.

This study was experimental, which means that students within each classroom were randomly assigned to one of two groups (the *comparison* group and the *sequential* group). The example problems presented in the comparison and sequential groups' packets were identical; however, as illustrated in Figure 2, there was a difference in the format in which students saw and analyzed problems and their solutions.

(Insert Figure 2 about here)

The comparison packet example shows hypothetical students' strategies for solving the problem $5(y + 1) = 3(y + 1) + 8$. Mandy solved the problem using a conventional approach. Immediately to the right of Mandy's solution is Erica's more efficient solution to the same problem. The two questions that follow the solutions encourage the students to compare the two solutions; partners were instructed to discuss their answers to these questions and then write their responses in the provided space.

The sequential group packet (see Figure 2) shows the same solution methods; Mandy's problem is identical to the one presented in the comparison packet, while Erica's problem only differs in terms of its coefficients and constants. For this packet, however, Mandy's solution is shown first, on a separate page from Erica's. Students examine and discuss Mandy's solution on its own, then answer a question about it. Erica's solution is on the following page, followed by a question about her solution. Like students using the comparison packet, students using this packet were instructed to discuss their answers to the questions and write their response in the space near the question.

As Figure 2 indicates, the questions asked in the comparison group packet were sometimes different from those asked in the sequential group packet. However, the number of questions was identical between the two groups, and the depth of the questions (e.g., the extent to which questions asked students to analyze, synthesize, and evaluate solution methods) was also the same in the two groups. Comparison packet questions asked students to compare across multiple solution methods, while sequential packet questions did not.

After completing a pre-test on Monday, students worked through packets with pages such as those described above on Tuesday, Wednesday, and Thursday. On Friday, students took a post-test, which was the same as the pre-test.

To test whether comparison was effective in improving students' learning, researchers devised assessments of procedural knowledge, flexibility, and conceptual knowledge. The assessment for procedural knowledge investigated whether students became better equation solvers during the week-long intervention, on both familiar problems such as $5(y - 2) = 3(y - 12) + 20$, as well as those such as $0.25(t + 3) = 0.5$ which were less familiar. The flexibility assessment targeted students' knowledge of multiple solution strategies. For example students were asked to solve the equation $4(x + 2) = 12$ in two different ways; similarly, students were given the equation $2(x + 1) + 4 = 12$ and asked to identify all possible steps that could be done next. Finally, the conceptual knowledge assessment looked at whether students learned more about key concepts involved in equation solving such as equivalence and variable. For example, one question on equivalence showed the equations $213x + 476 = 984$ and $213x + 476 + 4 = 984 + 4$ and asked students to indicate whether the answers to these two equations were the same or not, without solving either equation. (Interested readers can download the packets and assessments used for this study and other related studies on comparison at http://gseacademic.harvard.edu/contrastingcases.)

Did comparison help students learn to be better equation solvers? Figure 3 illustrates the percentage point gains for the comparison and sequential groups on measures of procedural knowledge, flexibility, and conceptual knowledge.

(Insert Figure 3 about here)

The results show that students gained procedural knowledge regardless of which group they participated in: for both groups, the students' ability to solve the equations on the post-test improved as compared with the pre-test. However, the students who actively compared solution strategies for the multi-step problems saw even more procedural knowledge gains than the students in the sequential group. Recall that comparison and sequential packets contained the same problems, yet it appears that seeing the problems side-by-side helped comparison group students more.

In addition, students in both groups also showed improvement in flexibility; however, students who worked by comparing strategies made greater gains as compared with students who worked sequentially. Interestingly, students who had learned by comparing strategies tended to use efficient methods more frequently. This use of the more efficient solution methods also resulted in more accurate answers, possibly due to the reduction in the number of steps the students had to implement.

Finally, both groups improved their conceptual knowledge. For this category, the level of improvement was similar for both groups. However, more recent findings using an improved assessment more sensitive to the targeted concepts indicates that comparing solutions also benefits conceptual knowledge (Rittle-Johnson and Star 2009). Furthermore, an additional study in the domain of computational estimation found that comparison supported improved conceptual knowledge, at least for students who began the study with some familiarity with at least one of the to-be-compared estimation strategies (Star and Rittle-Johnson 2009).

Implications

Intuitively, it makes sense that, by using comparison, students can learn more about each of the equation-solving strategies they are studying. Studying strategies through comparison guided students' attention to the possibility of alternate, more efficient solutions as well as to the possibility of greater accuracy with these solutions. As discussed earlier in the camera example, by comparing the different cameras, we inevitably come away with a sense of what makes each one unique, what is important to each, and when it is better to use each type of camera. This study provides evidence, in the form of experimental results, that our intuitions about comparison are correct. When students were given the opportunity to discuss the similarities and differences in strategies, they learned more than those who studied the same strategies one at a time without comparison.

How can teachers implement comparison in the classroom? The most important implication of this study is that teachers should provide students with opportunities to see problems side by side and to engage in discussions about the similarities, differences, advantages, and disadvantages of strategies for solving particular problems. It is important to note that it is *not* merely exposure to multiple strategies that helped students become better equation solvers in this study. Students in the sequential group were exposed to the same, multiple strategies as students in the comparison group. Rather, it was the side-by-side placement of the multiple strategies, as well as the opportunities for comparison conversations, that led to the gains experienced by the comparison group students.

Comparison requires careful support to be effective. Research on comparison, including the study discussed here, provides several suggestions for using comparison effectively in mathematics classrooms. Some important features that promoted learning from comparison were as follows. First, teachers should provide students with a written record of all to-be-compared solution strategies, with the solution steps aligned and consistently labeled, as suggested by the format of Mandy's and Erica's solutions in Figure 2. Second, it seems important for teachers to provide explicit opportunities for students to identify similarities and differences in strategies. For example, teachers may consider asking students questions such as, "How are these two *problems* similar or different?" and "How are these two *strategies* similar or different?". Finally, teachers should use instructional prompts that encourage students to consider the efficiency of the strategies. Questions such as, "Which strategy would you use on a timed test?", "Which strategy is faster to complete?", and "Which strategy is better?" can lead students to engage in critical and thoughtful evaluation of the methods used. In this study, students were asked to respond to questions like these in writing, during partner work on problem packets. However, whole-class discussion around these questions would also be productive.

In conclusion, research, as well as intuition, suggests that comparison is a fundamental learning process. When learning how to solve equations, it pays to compare!

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Figure 1: Paul's solution strategy

$$3x + 2 \ = \ 5x + 8$$
$$3x + 2 - 5x \ = \ 8$$
$$3x + 2 - 5x - 8 \ = \ 0$$
$$3x + 2 - 5x \ = \ 8$$
$$3x - 5x \ = \ 6$$
$$-2x \ = \ 6$$
$$x \ = \ \text{-}3$$

Figure 2: Sample pages from comparison and sequential packets

A. Compare Condition

Mandy's Solution:		Erica's Solution:	
$5(y + 1) = 3(y + 1) + 8$		$5(y + 1) = 3(y + 1) + 8$	
$5y + 5 = 3y + 3 + 8$	*Distribute*	$2(y + 1) = 8$	*Subtract on Both*
$5y + 5 = 3y + 11$	*Combine*	$y + 1 = 4$	*Divide on Both*
$2y + 5 = 11$	*Subtract on Both*	$y = 3$	*Subtract on Both*
$2y = 6$	*Subtract on Both*		
$y = 3$	*Divide on Both*		

　　1. Mandy and Erica solved the problem differently, but they got the same answer. Why?

　　2. Why might you choose to use Erica's way?

B. Sequential Condition

Mandy's Solution:

$$5(y + 1) = 3(y + 1) + 8$$
$$5y + 5 = 3y + 3 + 8 \qquad Distribute$$
$$5y + 5 = 3y + 11 \qquad Combine$$
$$2y + 5 = 11 \qquad Subtract\ on\ Both$$
$$2y = 6 \qquad Subtract\ on\ Both$$
$$y = 3 \qquad Divide\ on\ Both$$

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　　1. Would you choose to use Mandy's way to solve problems like this? Why or why not?
-----NEXT PAGE-----

**Erica's Solution:**

$$10(x + 3) = 6(x + 3) + 16$$
$$4(x + 3) = 16 \qquad Subtract\ on\ Both$$
$$x + 3 = 4 \qquad Divide\ on\ Both$$
$$x = 1 \qquad Subtract\ on\ Both$$

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　　1. Check Erica's solution by substituting her answer into the equation. Did Erica get the right answer?

Note. From "Does Comparing Methods Facilitate Conceptual and Procedural Knowledge? An Experimental Study on Learning to Solve Equations," by B. Rittle-Johnson and J.R. Star, 2007, *Journal of Educational Psychology*, 99, 561-574. Copyright by APA. Reprinted with permission.

Figure 3: Gains for students in the comparison and sequential groups, from pre- to post-test

Percentage Point Gains from Pre- to Post-test
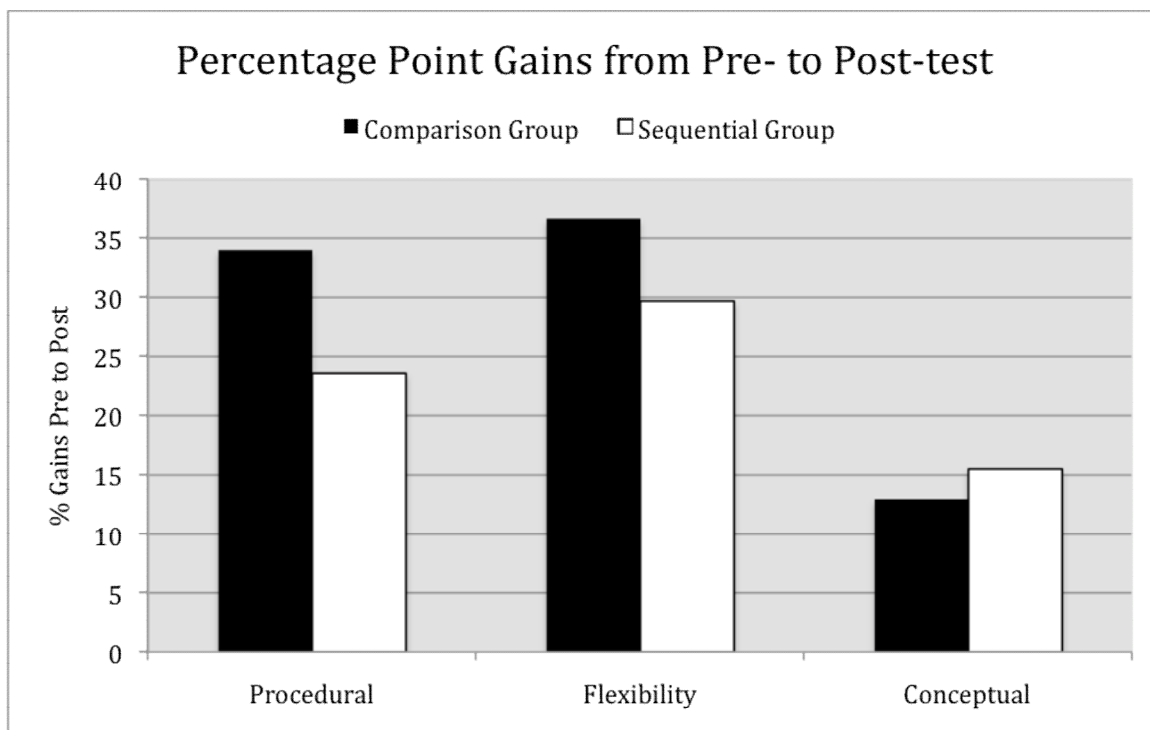
Note. Comparison group students' gains in procedural knowledge and flexibility were significantly higher than sequential group students. The gains experienced by students in conceptual knowledge were comparable across the two groups.