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Comparison helps students learn to be better estimators

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Abstract: This article describes a recent research study that examined the effectiveness of comparison on students' learning of strategies and concepts for computational estimation.

Comparison helps students learn to be better estimators

The ability to estimate is not only a valuable skill in the area of mathematics but is also a critical life skill. Many adults use estimation in their daily lives, when tipping a waitress, determining the cost of a sale item, or converting units. Within mathematics, the ability to estimate is linked to deeper understanding of place value, mathematical operations, and general number sense (Beishuizen, van Putten, & van Mulken 1997) and allows students to check the reasonableness of their answers to mathematics problems in a variety of contexts.

Estimation has been recognized as a critical mathematical competency by NCTM, as evidenced by its mention in the *Focal Points* at every elementary grade level (NCTM 2006). Pre-kindergarten students are encouraged to estimate quantities such as the number of beans in a jar; after the first grade, students at every grade level are utilizing their estimation skills to approximate computations and measurements. Despite its prominence throughout elementary curricula, estimation has proved to be very difficult for students to learn. Why is this the case?

One answer to this question is that estimation is more difficult and complicated to young learners that it might appear to experts and teachers. When estimating, students must make choices as to how they want to calculate an estimate. For example, many techniques for estimation involve rounding; however, students must choose what number or even place value to round to. This choice, given the wide variety of options, can be daunting.

Furthermore, the reason for the choice of one strategy over the other may not be immediately clear. Not only must students decide among numerous strategies when

estimating but, depending on which strategy they choose, they must also decide whether a given situation merits an estimate that is easy to perform and/or one that is close to the exact value of the calculation. It can be difficult to simultaneously accomplish both of these goals when computing an estimate, as typically an easy to compute estimate is not very close to the exact value (and vice versa).

Given the complexity involved with choosing a strategy for estimating, it makes sense that students have difficulty becoming good estimators. How can we help our students learn to become better estimators? One promising approach that has emerged from research in mathematics education and cognitive psychology emphasizes the role of comparison – comparing and contrasting multiple solution methods – in helping students learn to estimate.

The importance of comparing and contrasting students' strategies has been emphasized by NCTM. The role of comparison in problem solving is noted in PSSM; "In the lower grades, teachers can help children express, categorize, and compare their strategies. … By the time students reach the middle grades, they should be skilled at recognizing when various strategies are appropriate to use and should be capable of deciding when and how to use them. By high school, students should have access to a wide range of strategies, be able to decide which one to use, and be able to adapt and invent strategies" (NCTM 2000 54). This emphasis on comparing multiple strategies for estimation is also present in the recent report from the National Mathematics Advisory Panel (2008), which suggested that

Textbooks need to explicitly explain that the purpose of estimation is to produce an appropriate approximation. Illustrating multiple useful estimation procedures

for a single problem, and explaining how each procedure achieves the goal of accurate estimation, is a useful means for achieving this goal. Contrasting these procedures with others that produce less appropriate estimates is also likely to be helpful. (p. 27)

Despite the central role that comparison plays in the *Standards* and policy documents, there are surprisingly few research studies that show that comparison does indeed help students learn mathematics. Particularly missing are experimental studies (or studies that randomly assign students to receive an instructional intervention) that demonstrate the benefits of comparison. This paper reports on one recently completed study (Star & Rittle-Johnson 2009), where researchers investigated the effectiveness of comparison on students' learning of strategies and concepts for computational estimation. (Our research team has conducted several other studies that confirm and extend the results described here; see Rittle-Johnson & Star 2007, in press.)

The Study

The study took place in two schools; one private urban school and one small rural school. These particular schools were selected both for convenience and because they provided diversity in terms of school size and student demographics. Over the course of a week, researchers worked at each school with a total of 157 fifth- and sixth- graders on estimation of two-digit multiplication problems. After completing a pre-test on Monday, students (in pairs) worked through packets with two-digit multiplication estimation problems on Tuesday, Wednesday, and Thursday as well as received brief whole-class lessons. On Friday, students took a post-test, which was the same as the pre-test. (This same test was given to students two weeks later, as a measure of retention.)

In addition to working on problems in their packets, students were also given short lessons on new material each day. During the first day of problem solving, the focus of the short lesson and questions that accompanied the worked examples was on ease of computation. Students were asked to consider which strategies, and for which problems, were easiest for computing estimates. On the second and third day of problem solving, the short lessons and packet questions focused on proximity as a criteria for evaluating estimates and estimation strategies.

In order to evaluate the effectiveness of comparison on students' learning of estimation, researchers compared learning from comparing multiple solutions (*compare condition*) to learning from studying individually presented solutions (*sequential condition)*. Students in both conditions studied examples of hypothetical students' estimation strategies and answered questions about the strategies with their partner. Students in the *compare* group studied example pairs for the same problem. These example pairs were presented side-by-side and were followed by questions that asked the students to compare and contrast the solution strategies (see Figure 1). Students in the *sequential* group studied structurally similar pairs of problems, which were presented one after the other on separate pages in their packets. Questions after the problems in the sequential packet typically required students to reflect on strategies individually (see Figure 1).

Practice problems were included in each packet. Each practice problem set asked students to estimate the solution to two problems and then answer one question about their strategy(s) of estimation. In the *compare* packet, students were asked to compute an

estimate to the same problem in two different ways. In the *sequential* packet, students were asked to compute an estimate to two different, but structurally similar problems.

The examples provided to students focused on three estimation strategies for multiplying integers in problems such as 13 x 27. The first two strategies are commonly taught in many texts: *Round both* involves rounding both integers to the nearest multiple of 10 (10 x 30); in *round one*, only one integer is rounded to the nearest multiple of 10 (13 x 30, or 10 x 27). The third strategy was to truncate (or *trunc*) each multiplicand, covering up or ignoring the ones digits and multiplying the tens digits, and subsequently adding two zeros to the resulting product (for 13 x 27, 1█ x 2█ yields 2, and then adding two zeros yields an estimate of 200). This strategy is relative easy and fast and has been advocated for by researchers on computational estimation (Sowder & Wheeler 1989; Reys & Bestgen 1981).

Although researchers (most of whom had classroom teaching experience) taught the brief whole-class lessons, students' classroom teachers participated each day by circulating among student pairs and answering students' questions. More specifically, teachers rephrased and clarified questions in students' packets, provided general encouragement, and gave guidance on problem solving without favoring any given strategy - similar to the help that a teacher would typically give during a chapter test.

Researchers assessed students' learning in three areas: procedural knowledge, flexibility and conceptual knowledge. The assessment for procedural knowledge investigated whether students became better estimators during the week-long intervention and could transfer this knowledge to new problems such as multiplying decimals; the conceptual knowledge assessment looked at whether students learned more about key

concepts involved in estimation; and the flexibility assessment targeted students' knowledge of multiple estimation strategies, and ability to assess the ease of use and accuracy of a strategy.

After one week, the results of the study suggested that comparison played a key role in students' learning of estimation strategies. While all students learned to be better estimators, students in the *compare* condition also made significantly greater gains in their flexible use of estimation strategies (see Figure 2). For example, *compare* students were much better at computing an estimate in more than one way. In addition, *compare* students were better at selecting the estimation strategy that led to the easiest to compute estimate and were also more likely to use the *trunc* strategy (which, for many problems, is the easiest strategy for computing estimates).

In addition to greater gains in flexibility, comparison also helped students improve their conceptual knowledge of estimation. All students made gains in conceptual knowledge; *compare* and *sequential* students did not differ at the post-test. However, for *compare* students who began the study with modest knowledge of one estimation strategy, comparison of multiple strategies helped students to maintain gains in conceptual knowledge weeks after the intervention. Figure 3 shows the conceptual knowledge scores for students who began the study with minimal knowledge of how to estimate and those who began the study with modest knowledge of how to estimate, on the posttest and retention test. The conceptual knowledge of *compare* students who had modest strategy knowledge at pretest was significantly higher on the retention test than the scores of *sequential* students who began the study with modest strategy, even though these students' gains were the same at the end of the intervention.

Figure 4 shows a sample page from a packet that one pair of students in the *compare* condition worked on, along with a brief portion of these students' discussion around the questions on this page. This example is intended to illustrate the ways that students benefited from comparison by discussing the accuracy, efficiency, and constraints of each estimation strategy presented in the worked examples.

Implications

The results of this recent study provide evidence that comparison works – that giving students the opportunity to compare and contrast multiple strategies for computing estimates led to greater gains in students' flexibility, particularly students' knowledge of multiple strategies for estimation and their ability to evaluate multiple strategies for ease and accuracy. In addition, comparison helped many students retain gains in conceptual knowledge.

These gains in flexibility and conceptual knowledge likely came about through two key features of the intervention. First, students in the *compare* group saw problems side-by-side, which afforded them the opportunity to directly compare and contrast strategies. Second, students in the *compare* group were encouraged to evaluate the estimation strategies by discussing and answering questions about the relative utility of each strategy.

Comparison requires careful instructional support to be effective. Research on comparison, including the study described here, provides several suggestions for using comparison effectively in mathematics classrooms. First, teachers should choose problems and solution strategies carefully. The problems to-be-compared should highlight important and meaningful concepts for students to learn and to be solvable

using multiple strategies. In addition, students may need some familiarity with one of the strategies before comparing two different strategies. Second, instructional presentation of compared examples should be carefully designed. Our curriculum materials included 1) a written record of all to-be-compared solution strategies, with the solution steps aligned and consistently labeled; 2) explicit opportunities to identify similarities and differences in strategies; 3) instructional prompts to encourage students to consider the efficiency of the strategies.

How can teachers harness the power of comparison when teaching estimation? First, our results underscore the importance of assessing and then building on students' prior knowledge of estimation strategies. For example, by assessing students' prior knowledge, we learned that many students in our study began as fluent users of the *round both* strategy for computing estimates. Students' comfort with *round both* was a key asset when this strategy was compared to other, less familiar strategies such as *round one* and *trunc*. At the same time, our assessments of students' prior knowledge indicated that some students mistakenly believed that *round both* was the *only* way to compute an estimate. In such cases, comparison both increased students' repertoire of estimation strategies and also encouraged students to evaluate whether a strategy was most appropriate for a given problem.

A second and important way that teachers can use comparison to teach estimation is through classroom discussion. In our study, opportunities for students to identify similarities and differences in strategies, as well as prompts encouraging students to think about the efficiency of strategies, were embedded in our written instructional materials. But we predict that our intervention would have been even more effective had teachers

facilitated a productive and powerful whole-class discussion around these issues. Ideally, such a discussion would have three phases. First, a teacher can prompt students to understand each of the compared problems individually. For example, in Figure 1, students can be asked to describe Allie's way for computing the estimate, and then to describe Claire's way. Second, students can be asked to consider similarities and differences between the compared problems or strategies, such as by identifying how Allie's way is similar or different from Claire's way. Third and finally, it is critical for the discussion to go beyond the mere identification of similarities and differences; teachers should also ask questions that lead students to synthesize, generalize, and evaluate the compared problems and strategies. For example, students can be asked to consider the advantages and disadvantages of Allie's way or Claire's way for a particular problem and for a given estimation goal (e.g., a desire to compute an accurate estimate, or the need to generate an easy estimate).

In conclusion, estimation, like many topics, is a complex, challenging, yet critically important mathematical competency for students. While the importance of comparing and contrasting multiple strategies has been advocated by NCTM for some time, there is little formal research to support this instructional method. The study described here provides evidence of increased student flexibility and conceptual knowledge as a result of comparison; in addition, this study provides a nice example of a concrete, easily implemented technique for implementing comparison, via the side by side presentation of examples.

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Figure 1.

Sample packet page for A) the compare condition and B) the sequential condition.

A. Compare condition

About how much is 27 * 43?

Allie's way:	Claire's way:
27 * 43	27 * 43
My estimate is 800.	My estimate is 1200.
I covered up the ones digits and then multiplied the tens digit like this:	I rounded both numbers.
2█ * 4█ = 8	I rounded 27 up to **30.** I rounded 43 down to **40.**
Then I added two zeros because I covered up two digits and got 800.	Then I multiplied **30 * 40** and got 1200.

3. How is Allie's way similar to Claire's way?

4a. Use Allie's way to estimate 21 * 43.

4b. Would Claire's way give a different estimate for 21 * 43 than Allie's way?

B. Sequential condition

About how much is 57 * 23?

```
                        Claire's way:

57 * 23

My estimate is 1200.

I rounded both numbers.

I rounded 57 up to 60.
I rounded 23 down to 20.

Then I multiplied 60 * 20 and got 1200.
```

2a. Is this an OK estimate?   _____Yes   _____No

2b. How do you decide if an estimate is OK?

-------------------------------------- *NEXT PAGE* --------------------------------------

About how much is 27 * 43?

```
                        Allie's way:

27 * 43

My estimate is 800.

I covered up the ones digits and then multiplied the tens
digits like this:

2█ * 4█ = 8

Then I added two zeros because I covered up two digits
and got 800.
```

4a. Use Allie's way to estimate the answer to 21 * 43.

4b. Did Allie get the same estimate for these two number problems?

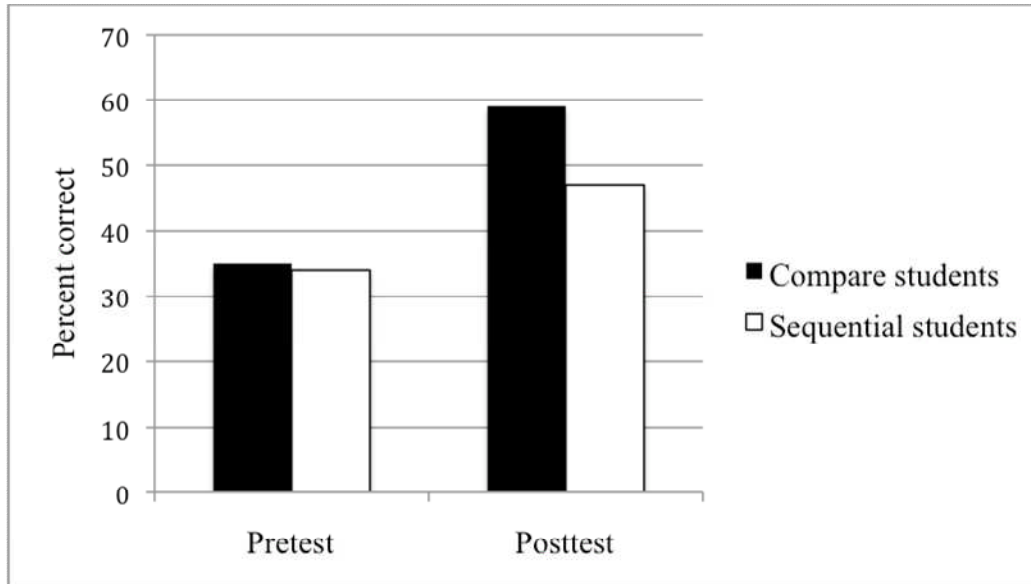*Figure 2.*

Student performance, Flexibility assessment

Figure 3.

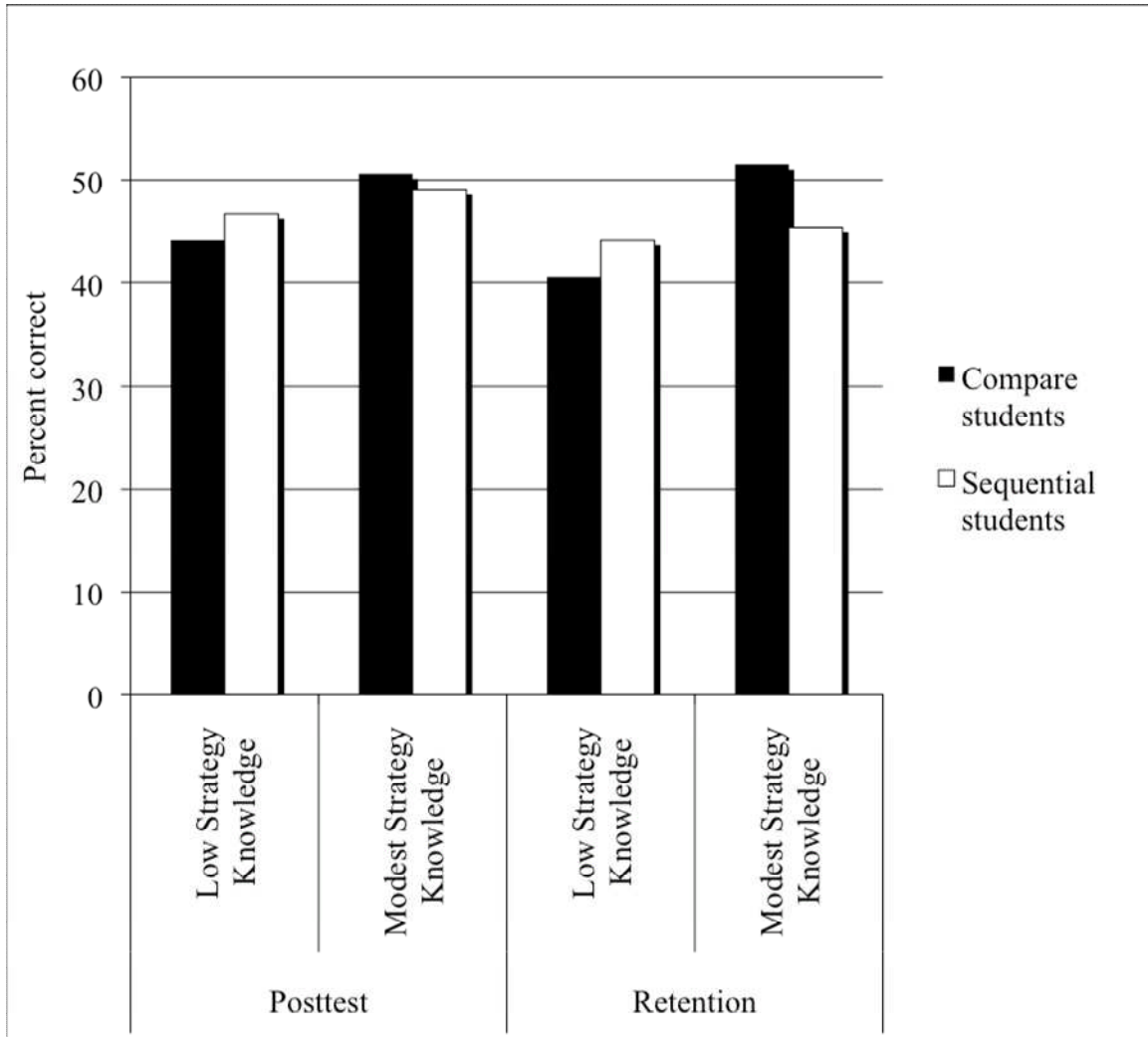Student performance, conceptual knowledge assessment

Figure 4.

Sample of student work and student discussion in the *compare* condition

About how much is 9 * 78?

Darius' way:	Joshua's way:
9 * 78	9 * 78
My estimate is 780.	My estimate is 800.
I rounded one number to the nearest ten.	I rounded both numbers to the nearest ten.
I rounded 9 up to **10**.	I rounded 9 up to **10**.
	I rounded 78 up to **80**.
Then I multiplied **10 * 78** and got 780.	
	Then I multiplied **10 * 80** and got 800.

21a. Use Darius' way to estimate 11 * 78. $10 \times 780 = 780$

21b. Use Joshua's way to estimate 11 * 78. $10 \times 80 = 800$

22a. Ask a teacher what the exact values are for each number problem, and decide whose estimate is closer to the exact value for each number problem.

9 * 78 = 702 Whose way was closer? ✓ Darius ___ Joshua

11 * 78 = 858 Whose way was closer? ___ Darius ✓ Joshua

22b. Do you think rounding one number always gives you a closer estimate than rounding both numbers?

_____ Yes   ✓ No

> It could, it could.
>
> So sometimes this might give a better answer because it makes it quicker.
>
> Well, I wrote sometimes it may give a better answer but not always. It may be better to round both numbers instead of just one because Joshua rounded both and got a better answer.

> No.
>
> Okay well I said I don't think rounding one number always gets the closer number. For example, Joshua rounded both numbers and got the closer number.

22c. **Explain your answer**.

alway I don't think rounding one number get the closer number for example Joshua rounded both number and got the closer numer.